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## WG 7 REPORT

### FROM GEOMETRICAL THINKING TO GEOMETRICAL WORK

Alain Kuzniak

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The Cerme 5 Working Group on Geometrical Thinking worked within the continuity of Cerme 3 and 4. During these former sessions, some main points were considered within a first common theoretical point of view on geometry with regard to epistemology, psychology and semiotic. Before presenting topics debated during our last session, we are looking back to the common background built and discussed before this session (see also Dorier *et al.* (2003) and Straesser *et al.* (2005)).

#### THEORETICAL BACKGROUND OF THE GROUP

##### Paradigms in Geometry

Traditionally, Geometry has different, somewhere contradictory, trends which roughly said refer on one hand to reality and suitable applications in future life and on the other hand to a more axiomatic and logical perspective. To take into account the variety of geometrical approaches, a paradigmatic perspective was introduced by Houdement and Kuzniak [2003, 2006]. Based on Kuhn's and Gonseth's works, three main coherent paradigms were brought out to explain various purposes aimed by Geometry. In this view, Geometry I (Natural Geometry with source of validation closely related to intuition and reality with eventually the use of measurement and or construction by real tools) differs deeply from Geometry II (Natural and axiomatic Geometry based on hypothetical deductive laws related to a set of axioms close as possible on the sensory reality). The last paradigm Geometry III (formal and axiomatic geometry) is today of a least importance in the compulsory school but it determines the horizon of mathematics at the university: in this case the set of axioms is independent of reality and should be complete in the formal sense. Naturally, people coming from countries where the Euclidean Geometry is traditionally taught recognize here a common problem they are faced with: How to manage transition from Geometry I to Geometry II. For the other, this frame could appear somehow exotic but it has been shown that it enables to compare different institutions with different aims and thought about the role of Geometry. Naturally, these paradigms are not explicitly taught and there are to be seen as useful tools for explaining some misunderstandings common in the classroom when the teacher's working horizon differs from the student's ones.

## **Geometrical Thinking and Geometrical Work**

Once the main aim of geometry is accepted, the problem remains of knowing how students can be successful at geometrical tasks. At the birth of the group, in 1999, the influence of psychologist scholars led to focus on geometrical thinking with studies and references to development stages using frameworks like Piaget's ones or the famous Van Hiele's levels. Naturally, these frames are helpful especially in the first access to Geometry by young children, but they do not seem to fit with geometry taught at High school or University levels. Otherwise, it appeared that rather than focusing on thinking first, it would be more efficient to define and study what kind of "geometrical work" was at stake in geometry teaching and learning. In this trend, studying geometrical thinking remains a basic and fundamental problem but drawn by geometry understanding in a school context rather than in a laboratory environment. In this view, we need a clear borderline (even it could change during the schooling) between Geometry and Pregeometry: We could accept that this line passes through an existing justification based upon a logical and articulate discourse.

## **Semiotics and Registers of representation**

The semiotic perspective is nowadays a living trend existing in didactics of geometry for a long time as it could be seen in the difference that authors made between drawing and figure and which partially refers to the signifier/signified pair. The Duval's registers of representation (1998-2006) are also used as a support of analysis with regard to the deductive entrance in Geometry. In this case, it is useful to work with several registers especially the discursive and figural. More widely, the semiotic approach could give a rich look on the various characterizations of objects used in Geometry which can be seen as supports of knowledge, description or perception.

## **SOME TOPICS TACKLED BY THE GROUP**

### **Spatial abilities and Geometrical tasks.**

This topic was at the heart of numerous papers accepted in the group and gave birth to an interesting discussion. As Panaoura-Gagatsis pointed it, quoting Weathley (1998), we can agree that one unified and wide accepted definition of spatial abilities does not exist: the way this term has been defined and the instruments used to collect data are nearly as varied as the number of studies using them. The concept of space in itself does not allow a unique definition. As Speranza (1997) points out, we can enlighten what 'space' could be only by using contrapositions: he shows at least ten possible conceptual couples useful for the articulation of spatial understanding. Therefore it is very important that authors precise their definition of spatial abilities before beginning a study on relations between these kind of abilities and those useful to solve geometrical problems. In some case, it seems that we can paraphrase the famous definition attributed to Binet: What is intelligence? It is what my test measures.

Based upon traditional tests like ETS (Pittalis), some authors tried to find a relationship between spatial and geometrical abilities. But the question is turning to: How shall we evaluate geometrical abilities? Such abilities could be defined as combination of general intelligence applied to the geometrical context. That supposes a definition of the context and we are coming back to our problem. During the meeting, different proposals were given to solve this question, more or less persuasive. In fact, a tight task analysis is requested to support the results presented. Some tasks used are not clearly related to geometry especially tasks situated at the visual level from Van Hiele.

In their paper, Panaoura-Gagatsis introduces 2D and 3D geometrical tasks clearly related to the syllabus and they measure spatial abilities using Demetriou and Kyriades (2006) model. In this model, the spatial-imaginal system of the human being is organized upon three components: Image Manipulation, Mental Rotation and Coordination of Perspectives. They give some interesting results on the relation between students' performances to each category of tests. If the majority of the students who performed high scores in geometry belong to high spatial ability group, there are some students with high spatial abilities and who do not performed high in geometry. At the same time, they show that spatial intuitions remain active even if geometrical topics have been taught and that performances on geometric task depend closely on the age of the students and the dimension (2D or 3D) of the space where the question is posed.

### **Knowing young initial pupils' geometric knowledge**

One of the more important stake for researchers in mathematics didactics is certainly to gain a better understanding of pupils' abilities in the classroom rather than in a laboratory. This point was developed by Marchini-Vighi and Markopoulos who have been working with young pupils (5 to 8 years old). They follow a rather similar approach to deal with this question: they gave open and fuzzy tasks to catch initial conceptions of their students. Inspired by Swoboda (2005) and having given tiles to pupils, Marchini and Vighi asked them to build 'from these tiles as beautiful floor as possible'

This ambiguous way of giving a problem was naturally questioned by the rest of the group, but authors argue that it is probably the best way to let enter young people in a task and to obtain information about their initial knowledge. Their results show a great variety of "déjà-là" (set-before) knowledge and it is probably possible to manage a real geometry teaching based upon it. That leads to an outstanding question: Which is the status and the place of "spatial knowledge" into the curriculum?

### **Do exist epistemological obstacles in Geometry?**

Based on the seminal work of Bachelard, Brousseau (1997) has introduced the notion of epistemological obstacle in didactics. An obstacle is made apparent by

reproducible errors not due to chance. When the origin of the error could be explainable by reasons based upon history and epistemology, it will be talked about epistemological obstacle, other kind of obstacles exist related to the ontological child development or to teaching methods.

Papers from Modestou-Iliada, Kratka and Bulf were respectively dealing with some initial conception like “linear model”, “infinity horizon”, “principle of symmetry” which could sometimes appear as obstacle in new knowledge building. Deciding if the former difficulties are or not obstacle and of what kind is not easy and depend clearly on each item.

Related to proportionality, ‘linear model’ seems to appear as an obstacle when geometry deals with area and volume. Infinity case is less clear, Kratka argues that horizon could explain some problem related to infinity. When do appear infinity and horizon in Elementary Geometry? Perhaps, the transition from meso-space to macro-space (in the sense of Brousseau) rests on this point.

At least, the ‘symmetry principle’ exhibited by works in cognitive science (see Palmer, 1985) belongs to tools helping the students to reach a first stage in geometry but it seems that it can act against the development of a more abstract vision of the figure. We find again the opposition between knowing and seeing.

### **On possible uses of geometrical paradigms**

Since Cerme 3, the theme of geometrical paradigms is taken into account by the group and new participants have questioned this point: Which is the real benefit of this approach? Two papers gave some perspectives in this way. Houdement shows how she uses these tools to explore the comparison between curricula in different countries, here France and Chile. With help of the notion of Geometrical Working Space (GWS), she had studied the place of Geometry I and II in these countries and she could word some general questions into this theoretical framework:

Do we need to teach Geometry II?

and if so

Which is the best way to enter into Geometry II?

Is it by teaching Geometry I longer?

What is a coherent and rich approach of Geometry I?

In a second paper, Bulf deals with the question of the link between Geometrical knowledge and the reality. She is studying the use of everyday objects and situations in the teaching of symmetry at secondary school level. A double play occurs between the couples GeometryI/GeometryII on one side and Reality/Theory on the other side. She observed that knowledge used in everyday life context are not very useful in the context of the theoretical approach and vice versa.

In Cerme4, Kuzniak-Rauscher (2005) have shown a possible use of paradigms in teachers' training by making them aware about some difficulties related to these different approaches of Geometry. In this direction, we could interpret some results of Kospentaris' paper. In his study, the author confirms results already presented at Cerme 3 and 4 about old student's geometrical thinking. He shows that students at the end of secondary school and with good knowledge in Euclidean geometry solve some geometrical problems by using visual strategies or "measurement by compass and straightedge" in contradiction with their supposed Van Hiele's level. He explains this by the fact that "they think in another context". It's another way to say that the problem depends on its paradigm's horizon (Geometry I or II) and not on a developmental approach not appropriated to aged students.

### **Artefacts and Geometry**

Nowadays, it is impossible to think about geometry without looking at DGS (Dynamical Geometrical Software) which have deeply changed the nature of constructions and proofs in the domain. If few papers were concerned by this trend (due to Working Groups on proof and on technology at Cerme), the way they took in charge the problem seems interesting and gives a new look on ancient problems by revisiting them.

#### *Geometrical paradoxes revisited.*

Based upon a tangram-software, Vighi built an example of a jigsaw possible to solve with six or seven pieces. This spectacular paradox depends on how approximation is controlled by the software. It did not appear as a paradox for young pupils who find natural that two different configurations of the pieces recover a different area.

#### *Prototypic images revisited.*

Unfortunately absent from the meeting, Larios gave to solve the problem of midpoints configuration in a polygon to 14 old students with DGS. He observed that, even in this environment, students tried to build prototypic drawings that allow them to see some results better.

Due to Mann-Ludwig's paper, the relationships between media and didactic instruments were touched. Every year, new electronic and interactive tools enter in the classroom (like video, internet or interactive whiteboards). How can we turn this media into effective teaching and learning tools? In a preliminary study, Mann-Ludwig have observed students using a DGS enriched by video-facilities. They propose an interesting 'Learner model' showing the link between different approaches, traditional or not. Using this frame, the question becomes : How can we include the advantage of the usual learning in the classroom into video environments?

## PERSPECTIVES

If we look at didactics as a science turned to applications, the proposals made during the present work session focused more on description of problems encountered in the classroom than to prospective use in geometry teaching and learning. Discussions over teaching training that were intense during the former sessions did not emerge during the present Working Group. We could perhaps regret this and equally the relative weakness of task analysis based upon the traditional tools developed in didactics.

Nowadays, a semiotic approach allows to work on geometrical objects as drawing and figural concepts. Some specific components of the geometrical work like visualisation, construction and reasoning are deeply studied into the cognitive approach. During, this session few papers were based on these aspects and we expect that the group's future work will be nourished by specific studies on these points.

By focusing the debate on geometric work, we hope to lead the group to precise the existing theoretical tools helpful to explore and describe the nature and the construction of the Geometrical Working Space used by students and teachers. Do we need new tools or are the existing ones sufficient?

All the people participating to the group – except two – were coming from Mediterranean countries. Does it mean that Geometry is a 'Mediterranean cultures state of affairs'?

We conclude this paper by some ideas of collaboration between participants and some suggestions for the future.

Collaborations are envisaged about the transition from primary to secondary school. A common framework to work out such kind of studies could be based on some tools discussed by the group, this session and before, like paradigms, geometrical workspace, spatial abilities and conception about the figure. Some geometrical tasks presented during the meeting could give good common supports to progress in this international cooperation.

And for the future meeting, if we hope that this group could continue we suggest changing its name into Research in Teaching and Learning of Geometry (closer to the ICMI approach). That will allow a great variety of approaches less centred on the student's way of thinking but on its work and also on the teachers' work. Equally, to work out theoretical approaches it would be useful to invite authors or request papers presenting the state of art on various points related to geometrical working and thinking.

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# THE USE OF EVERYDAY OBJECTS AND SITUATIONS IN MATHEMATICS TEACHING: THE SYMMETRY CASE IN FRENCH GEOMETRY TEACHING

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*This thesis work is concerned by the use of everyday objects or situations in teaching a new mathematical concept. The concept of symmetry and in particular its meaning both familiar and mathematical, is explored in two different directions: school and vocational contexts. Only a part of our investigation in school is presented here: analysis of interviews with pupils from 11 to 15 years old and productions. Using this data, a phenomenon that we chose to call “transformations’ exclusiveness” is brought out. The paper shows how this phenomenon could come from an adaptation of schemes to initial perception.*

## GENERAL PRESENTATION OF THE RESEARCH QUESTION

Integrating the real world into mathematics education is not a new issue. However, there exist many forms to integrate it. Consequently, the impact of the real world in mathematics learning and in the conceptualization of a new mathematical concept can be very different too. It is interesting to realize how such-and-such references can definitely orientate or not the conceptualization, in order to control them and foresee the understanding of pupils. There is a quite strong pressure from French curriculum to use real situation through “activities” to introduce a new concept.

“Architecture, piece of art, natural element, usual objects... we can bring out from these links some universal feature of geometrical object connected to their natural or synthetic achievement. (...) difficulties from usual vocabulary and previous representations own to pupils (...) to work on these primal conceptions (...) the teacher’s management does not have to occult these primal conceptions but at the opposite use them to make questions.”[1]

“Activities” from the real world, proposed in classroom are inspired by this wave and are used to support the mathematical concept. Bachelard (see Bachelard, 1938) points out that “nothing is done, all is building”, he adds the notion of obstacles “to set down the problem of scientific knowledge”. In particular, he mentions “the excessive using of familiar images”, and suggests how orientating in a wrong way can be the way of thinking. Then, there is a question about the result of teaching the reflection through a line supported by folding or familiar references. Does it help or not to work out mathematical thinking, in particular to understand the new transformations of the plan?

## REFERENCES AND THEORETICAL FRAMEWORK SUITABLE TO GEOMETRY

The real world is precisely used in mathematics teaching to grasp a new concept. “Real” is used in a very large and common meaning: what is immediately effective or concrete, and can be submitted to our sense and build experimentation. Thus the concept of symmetry is the subject of this research because it is everywhere in real life and it is a cross concept in school too. Until the beginning of secondary school, symmetry is only viewed as reflection through a line which is the usual conception too. Then on 5<sup>th</sup> grade (12 y.o.) pupils learn the reflection through a point, and on 4<sup>th</sup> grade (13 y.o.) they learn translation and finally on 3<sup>rd</sup> grade (15 y.o) rotation.

Naturally, the first distinction between familiar concept and scientific concept recalls Vygotski’s framework (see Vygotski, 1997), but it appeared this strict dichotomy was not so relevant for the study despite of his suitable definition. So, it has been decided not to be involved in Vygotski’s position even if its definitions of familiar and scientific concept were suitable: “it is living in action (perception, folding) without being compared or differentiated. Its characteristics and properties are neither necessarily aware nor put into words”: the concept of invariance, for example is not mentioned in a familiar context. On the other hand, the mathematical concept of orthogonal symmetry gets in the isometric category. It is used through “symbolical representation” as for example:  $s$  is an involutive transformation, that means:  $s^2=id$ . Some “general and useful result” can be formulated as *any isometry can be decomposed into orthogonal symmetries*.

« In mathematics, it seems necessary to distinguish clearly mathematical objects (such as numbers, functions, spaces, etc.) and concepts we use to characterize the former with its own properties » [2]. Vergnaud’s theory analyses the human component of a concept in action, which seems to be a relevant description as regards the familiar component of the concept of symmetry. Vergnaud defines a concept with  $(S, I, s)$  (see Vergnaud, 1991) where  $S$  is the set of different “situations of references” which make sense to the concept. The meaning of familiar is not the same for everyone (it depends on education, culture, and so on.) but the “operational invariants”  $I$  which are acting in different situations  $S$ , are actually defined by the concept-in-action (“relevant or irrelevant notion naturally involved in the mathematics at stake”), theorem-in-action (“proposition assumed right or wrong, used instinctively in the mathematics at stake”). I add the notion of principle-in-action (I define it as a theoretical general rule which is at the basis of concept and theorem-in-action). The set of theses invariants  $I$  make the schemes (notion inspired by Piaget) operate. A scheme is the “invariant organisation of behaviour for a class of given situation. The scheme is acting as a whole: it is a functional and dynamical whole, a kind of module finalized by the subject’s intention and organized by the way used to reach his goal”.  $s$ , the “signifiers” (according to Pressmeg’s translation of Saussure’s meaning, 2006) is the set of representations of the concept, its

properties, and its ways of treatment (language, signs, diagrams, etc.). Finally, this kind of “conceptual field” of symmetry according to Vergnaud is one of the aims of this research.

Since our research is focused on the interaction between familiar conception, living in action in “real” space, and the mathematical conception living in an axiomatic form through mathematical model (Euclidian one), the Houdement and Kuzniak’s theoretical framework of Paradigm of Geometry I, II, III and the notion of *Geometric Working Space* (see Houdement and Kuzniak, 2006) have been chosen for this study. Geometry I (GI) is the naive and natural geometry and its validity is the real and sensible world. The deduction operates mainly on material objects through perception and experimentation. Geometry II (GII) is the natural and axiomatic geometry, and its validity operates on an axiomatic system (Euclid). This geometry is modelling reality. Geometry III (GIII) is the formal axiomatic geometry which is completely apart from reality and is just a logical reasoning from an axiomatic system. The notion of *Geometric Working Space (GWS)* is the study of the environment, organized on a suitable way to articulate these three components: the real and local space, the artefacts (eg: tools and schemes), and the theoretical references (organized on a model). This *GWS* is used by people who organise it into different aims: The *reference GWS* is seen as the institutional *GWS* from the community of mathematicians. The *idoine GWS* is the efficient one in order to reach a definite goal. And the *personal GWS* is the one built with its own knowledge and personal experiments. Thus this framework takes into account the double side of our concept with a mathematical point of view. The focus of study deals with the crossing GI-GII aimed at secondary school.

The notion of didactical contract designed by Brousseau has been chosen as a theoretical reference (see Brousseau, 1997): “Then a relationship is formed which determines - explicitly to some extent, but mainly implicitly - what each partner, the teacher and the student, will have the responsibility for managing and, in some way or other, be responsible to the other person for managing and, in some way or other, be responsible to the other person for. This system of reciprocal obligation resembles a contract”. It is an important dimension as regards the school factor to analyse the nature of the implicit interactions at stake during the crossing GI-GII.

**Finally, the research question focuses on: how are schemes adapted to the crossing GI-GII in secondary school? And how do teachers and pupils take into account these interactions and adaptations?**

This diagram below summarizes the articulation of our different theoretical frameworks:

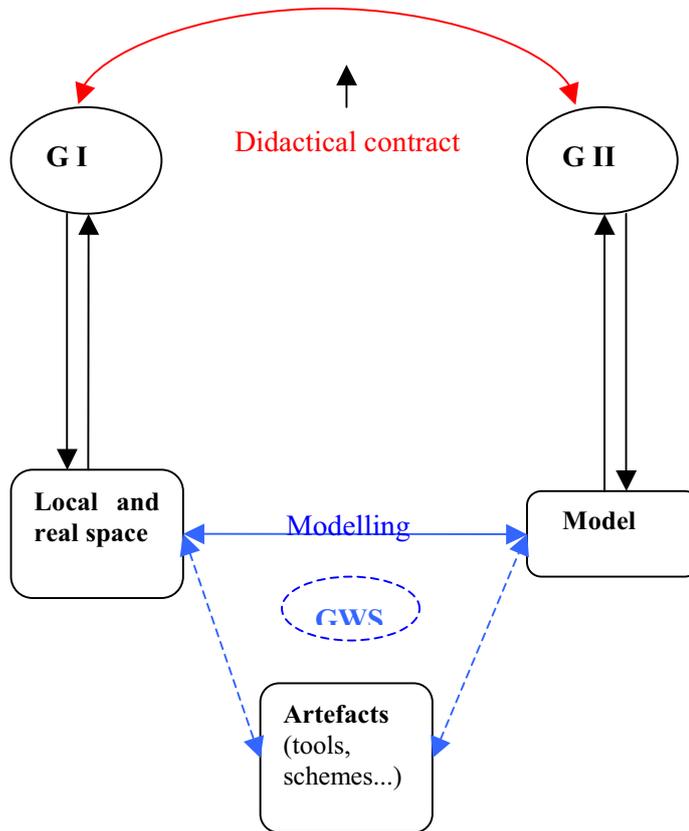


Diagram 1: Theoretical framework suited to geometry.

## THE PHENOMENON OF “TRANSFORMATIONS’ EXCLUSIVENESS”

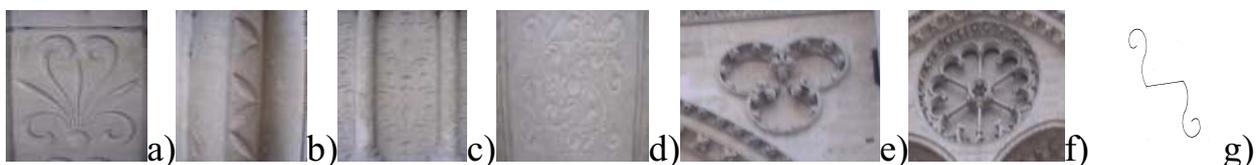
### A partial look on the investigation

The aim of this part is to give some results about one kind of *situation*: symmetry recognition task and by extension the others transformations.

Twenty eight pupils from 11 to 15 years old have been interviewed:

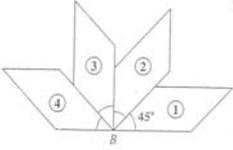
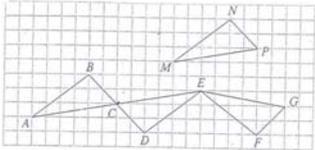
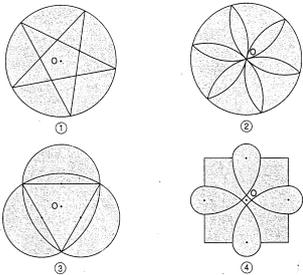
- 9 pupils from 6<sup>th</sup> grade (11-12 y.o.) before the lesson about the reflection through an axis.
- The same 9 pupils a few months after the lesson.
- 9 pupils from 5<sup>th</sup> grade (12-13 y.o.) a few months after the lesson about the reflection through a point.
- 10 pupils from 3<sup>rd</sup> (14-15 y.o) grade just after the lesson about rotation.

First, I asked them open questions: have you already heard about symmetry? What is symmetry for you and how do you recognize it? Then, I asked them to group together pictures (see below) with their own criteria:



And finally, I asked them if some symmetrical pictures were among these pictures and if they can explain why.

Advanced productions of 3<sup>rd</sup> grade: A test was given to 10 pupils from 3<sup>rd</sup> grade with various levels, at the end of the school year. Only three of the five exercises posed (still on the *transformations recognition situation*) are studied here. This protocol was worked out *a priori* in order to evaluate the inference of the reflection through a line among the other transformations at the end of secondary school (variable on global perception and punctual perception). So, the task in these exercises is to recognize and define the transformations of the plan. Two figures (one is the image of the other one) are given in the exercise 1 and 2, whereas the exercise 4 is based on four different global invariant figures:

<p><u>Exercice 1</u></p> <p>Cette figure est composée de quatre parallélogrammes (numérotés de 1 à 4) superposables.</p>  <p>Indiquez comment (de toutes les manières possibles) et Justifiez :</p> <p>1 → 2 1 → 3 1 → 4 2 → 4</p>	<p><i>This figure is composed of four parallelograms (from 1 to 4 on the figure) which can be superimposed.</i></p> <p><i>Please show how (with all the possible way) and Justify:</i></p> <p>1 → 2 1 → 3 1 → 4 2 → 4</p>
<p><u>Exercice 2</u></p> <p>Dans chacun des cas suivants, dites laquelle des translations ou des symétries ou des rotations :</p> <p>a) transforme ABC en EDC b) transforme CDE en GFE c) transforme ABC en MNP</p> <p>Justifiez vos réponses.</p> 	<p><i>In each following case, which translations or symmetries or rotations :</i></p> <p>a) transform ABC on EDG b) transform CDE on GFE c) transform ABC on MNP</p> <p><i>Please justify your answers.</i></p>
<p><u>Exercice 4</u></p> <p>Indiquez les symétries, translation ou rotations qui transforment la rosace en elles-mêmes. Justifiez les réponses.</p> 	<p><i>Show the symmetries, translations, or rotations which transform the rose on itself.</i></p> <p><i>Please justify your answers.</i></p>

## Results

The reflection through a centre seems clearly differentiated during the interview from the reflection through an axis by **8/10** pupils of 3<sup>rd</sup> grade, and **5/10** pupils

mention rotation or translation too (though I do not mention these transformations). To the last question on the symmetrical pictures, 9/10 associate exclusively one transformation to one figure. Let's see below which transformation is associated to the pictures (see p.5 the pictures) according to the level:

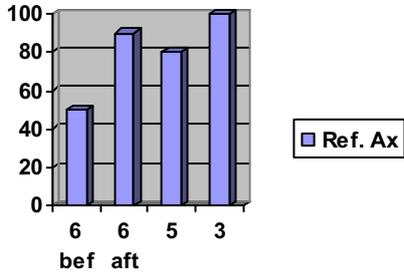


Photo a)

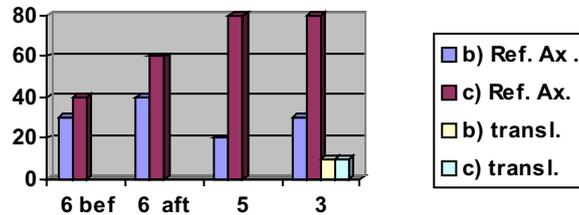


Photo b) and c)

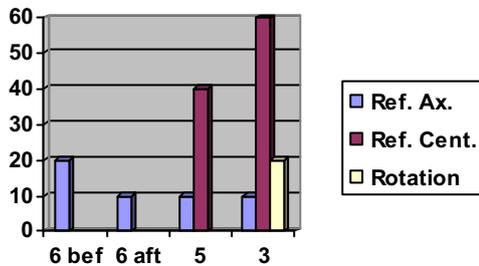


Photo d)

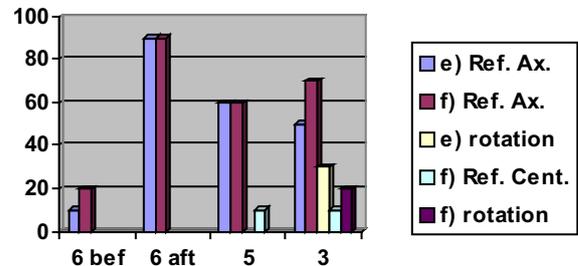


Photo e) and f)

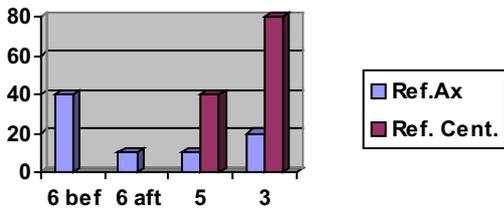


Photo g)

Diagrams 2: Transformations recognition during interview with pupils from 6<sup>th</sup> to 3<sup>rd</sup> grade. They answered the same to exercise 4: only 2 among 10 associated more than one transformation with one figure. Let's see now the distribution of the transformations in exercise 4 on 3<sup>rd</sup> grade:

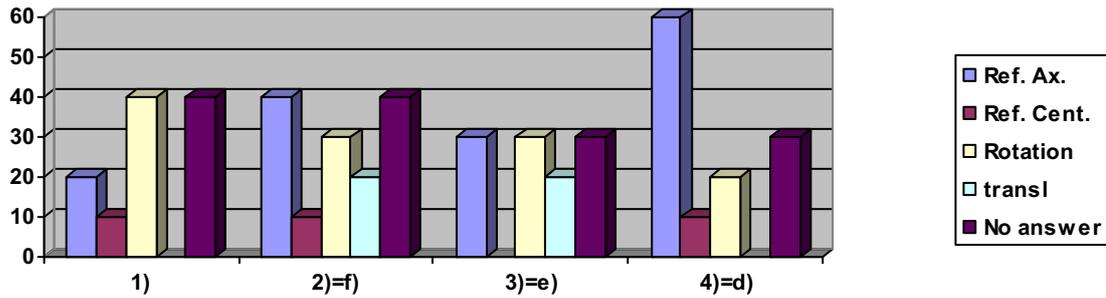


Diagram 3: Distribution of the transformations on exercise 4 from test on 3<sup>rd</sup> grade.

According to these data, we assume the existence of a phenomenon we call “transformations exclusiveness” which consists in associating one transformation to one figure.

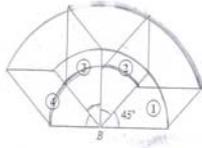
Furthermore, in exercise 4 all the figures are invariant by rotation but it appears a difference depending on the parity of the rotation. If the rotation is even pupils recognize a reflection through an axis more than a rotation (diagram 3 stick 2 and 4) and if the rotation is odd, the pupil recognizes a rotation more than a reflection (diagram 3 stick 1 and equal for 3). Interviews clearly show it: according to the diagram about Picture e) and f), the rate of Ref. Ax. of e) (odd) is falling whereas the rate of rotation is increasing and the rate of Ref. Ax of f) (even) is still high. Moreover, pupils hardly ever recognize the reflection through a point in exercise 4 and the picture f) whereas during the interview, most of pupils recognize a reflection through a centre at the picture d) or g) (see diagram 2 photo d), g) and f) and diagram 3). The diagram of a) and c) confirms that pupils recognize straightforwardly a reflection through a vertical axis.

#### **An interpretation of these results: inhibiting schemes**

According to our study, we assume the following hypothesis: schemes associated to an even or odd rotation are different and they seem work as one was inhibiting the other. In exercise 1 and 2, an interesting theorem-in-action appears to check a rotation: pupils use their compasses to check if the image-points and starting points of the figure belong to the same circle:

Exercice 1

Cette figure est composée de quatre parallélogrammes (numérotés de 1 à 4) superposables.



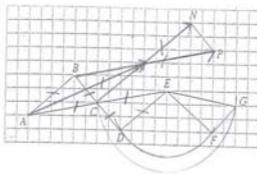
Indiquez comment (de toutes les manières possibles) et Justifiez :

- 1 → 2 rotations (B, 45°, +)
- 1 → 3 rotations (B, 90°, +)
- 1 → 4 rotations (B, 135°, +)
- 2 → 4 rotations (B, 135°, +)

Exercice 2

Dans chacun des cas suivants, dites laquelle des translations ou des symétries ou des rotations :

- a) transforme ABC en EDC ⇒ symétrie de centre C
  - b) transforme CDE en GFE ⇒ rotation de centre E
  - c) transforme ABC en MNP ⇒ translation de vecteur  $\vec{AN}$
- Justifiez vos réponses.



In particular, some pupils use this theorem-in-action to recognize a rotation (exercise 2 case c) when it is actually a reflection through an axis. It is acting as if the schemes of reflection were inhibited by rotation ones. Then, we assume that the same happens in exercise 4. This cocyclicity action can be seen as a *signifier* of one more general principle-in-action that we called principle-in-action of the **application point by point**. Then, according to interviews and exercise 4, we suppose this principle-in-action inhibits the ones associated to the reflection through an axis. The interviews with the rest of the pupils (from 11 y.o to 15 y.o.) show that the first schemes used by pupils when they look for a reflection through an axis is the **principle-in-action to divide in two half planes or two equal parts** (drawing an imaginary or real axis is one of its *signifier*). This principle-in-action implies the **global invariant principle-in-action**: a figure is globally invariant (and not point by point) under the action of a transformation (folding is one of its *signifier*), which is actually a different way of thinking from the principle-in-action of the application point by point. Thus this could explain the difference between the rate of recognition between odd and even rotations. In the last case, schemes of the reflection through an axis are implied by the fact that the rotation is even and the schemes from rotation (point by point) looks inhibited.

Moreover, pupils make a reflection through an axis different from the reflection through a centre by using the concept-in-action of orientation. That could explain why they see easily the reflection through a centre when the rotation is of order 2 or 4 (photo d) and g)). Indeed they can easily recognize if the orientation is different, whereas on photo f) (order is 12) or exercise 4, some figures are invariant by reflections through an axis and through a centre, then the concept-in-action of the orientation can not be efficient to discern them.

Thus some perceptive signs which orientate the schemes chains are just brought out. What is known about the other hints based on perception as points or axis already drawn? Any axis of symmetry is drawn on exercise 4 but the centre O is mentioned on each figure. However, pupils tend all to look for a particular axis more than they do for the centre. Besides, we assume that the perception of some typical geometrical objects might orientate pupils' behaviour: as for example the square in exercise 4 figure 4, most of pupils recognize a reflection through an axis (see diagram 3 stick 4) but on the picture d) from the interview where a square is drawn too, most of 3<sup>rd</sup> grade pupils are right: they recognize a reflection through a centre without saying a reflection through an axis.

### ANOTHER ADAPTATION: FROM GI TO GII

According to the interviews and the written exercises previously presented, pupil's *personal Geometrical Working Space (GWS)* is built on a natural geometry GI. They mention real space through experiments or movements with some operational invariants based on global perception (as *folding* or *half-turn*). Afterwards, this global perception is enriched with the punctual vision and GI works out to GII, and then pupils can make first reasoning using mathematical properties coming from the mathematical model about transformations (length, angle, points lie along a line, parallel, middle, orthogonal, etc.). Let's see for example Martin's theorem-in-action of invariance point by point in exercise 2 (see p.5):

a) is a rotation  $R(180^\circ; C; -)$  because  $B \rightarrow D, A \rightarrow E$  and  $C$  is still the same (rotation point).

b) is a reflection through an axis because  $C \rightarrow G, D \rightarrow F$  and  $E$  still the same (it is on the axis)

Thus the schemes are also working out. The resolution of Exercise 1 and Exercise 2 gives an interesting example of adaptation of these schemes. In exercise 2, *personal GWS* seems close to the *idoine GWS*. Indeed, this exercise explicitly requires a punctual perception to justify how transformations are used with mathematical properties, and nine pupils among ten recognize the right transformations (including those who do not write all the punctual characteristics). In exercise 1, the geometrical contract expected is less explicit and a global perception is suggested to solve the problem and pupils adapt their behaviour and recognize reflection through an axis or make mistakes: The parallelogram is seen as a rhombus. Only **4/10** recognize a rotation and only **2/10** mention the right rotation centre although it is the only point written on the whole figure. According to this result, we conclude that pupils go back to GI and use schemes based on perception and adapted to the contract at stake.

### CONCLUSION AND DISCUSSION

The *recognition of transformations situation* shows the phenomenon of "transformations exclusiveness". The action of inhibiting schemes based on perception could explain it. Furthermore, pupils' schemes contribute to build the *personal Geometrical Working Space* which seems unbalanced between GI and GII. Now, we focus our investigation on the reflection through a point which is exactly

situated at this interplay between GI and GII. The aim is to better understand the adaptation of pupils' schemes to different situations and to see how this transformation can contribute to unbalance the way to GII.

#### Notes

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2. definition of *concept* in Encyclopedia Universalis.

3. <http://smf.emath.fr/Enseignement/TribuneLibre/EnseignementPrimaire/ConfMontrealmai2001.pdf#search=%22VERGNAUD%20sym%C3%A9trie%20sch%C3%A8me%22> p. 14

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## GEOMETRICAL WORKING SPACE, A TOOL FOR COMPARISON

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*This theoretical text is nourished by a comparison project (ECOS program 2003-2005) on mathematical curricula between Chile and France. How and what to look at in curricula? What tools could help to produce fruitful comparison? Following the presentation of our theoretical framework, Geometrical Paradigms, the study of an exercise about the determination of inaccessible magnitude, from French and Chilean point de view will lead on to the definition of Geometrical Working Space. With these concepts, we will precise important differences between French and Chilean intended and available curricula, what concerns Geometry between 8<sup>th</sup> and 10<sup>th</sup> grade.*

### INTRODUCTION

Within the context of education research cooperation between Chile and France (aiming at mathematical curriculum comparison) we chose elementary geometry as field of study. We think that Geometry is a good mathematical subject for comparison:

- it is a field studied from infancy to the end of statutory curriculum;
- it is a field in which models are produced with different degrees of complexity: geometric education usually begins by studying and using real material objects (cuboids... graphic lines on a paper sheet or a computer screen), but more stylised than real objects; then it progressively deals with intellectual objects: the mathematician's square is not the child's square, it is a construction of the mind which includes an infinite number of points and exists only through its own properties;
- it is a field particularly connected with logical thought, deductive reasoning and proof, a characteristic property of Mathematics.

Our study (Castela & al. 2006) has been carried out on four levels.

- The first level we have studied corresponds to the statutory contents of the syllabus (knowledge, skills and understanding), which international comparison surveys call the *intended curriculum*.
- The second level that is generally described by what we call *accompanying texts* concerns the context, activities and areas of study through which the statutory contents should be taught. According to the countries these texts are mandatory or just pieces of advice.

- The third level is composed of text books that offer an organized list of classroom activities and exercises ready for teaching. We note that the second and third level both concern a part of the *available curriculum*.
- The last level is composed of practise of some teachers from either country and of students' performances confronted to the same geometrical problem.

We (Houdement & Kuzniak 1999, 2002, 2003) have worked on Geometry as it is taught in France and produced a theoretical framework to understand and describe the different meanings determined by the same term of Geometry.

The aim of this text is to show how Geometrical Paradigms and Geometrical Working Space can help to organize a comparative analysis; particularly what concerns *intended curriculum* and *available curriculum* about determination of an inaccessible magnitude.

Let us present Geometrical Paradigms.

## **GEOMETRICAL PARADIGMS**

Our research (Houdement and Kuzniak 1999) following Gonseth (1945-1955) shows how three different paradigms could explain the different forms of geometry. We keep the idea of paradigm from Kuhn (1962; 1970) who used it to explain the development of science. A paradigm is composed of a theory to guide observation, activity and judgement and to permit new knowledge production. A paradigm is shared by a community; the scientific activity of a researcher is guided by the paradigm on which he is working. We made the following hypothesis: Kuhn's analysis of the development of science can be imported into Mathematics, precisely into Elementary Geometry.

We distinguish three paradigms whose names would be easily remembered: Geometry 1, Geometry 2 and Geometry 3. Let us now precise some properties of each paradigm.

### **Geometry 1**

The objects of Geometry 1 are material objects, graphic lines on a paper sheet or virtual lines on a computer screen. Even material, the lines are always consecutive to a first representation of reality. Objects of the sensitive space can be schematised in a micro-space (Berthelot and Salin 1998) by a network of lines. The straight line is a model thus it refuses bumps; the circle is perfect, all its points are at the same distance of the centre. The chosen graphic objects (and their properties) are often in a first time the most convenient to describe reality, hence the name of Natural Geometry for Geometry 1. The objects of Geometry 1 are already the consequences of a first classification that gathers all the objects related by an isometric transformation.

In this paradigm the ordinary techniques are the drawing techniques with ordinary geometrical tools: ruler, set square, compasses but also folding, cutting, superposing...

To produce new knowledge in this paradigm, all methods are allowed: evidence, real or virtual experience and of course reasoning. The backward and forward motion between the model and the real is permanent and enables to prove the assertions: the most important thing is to convince.

### Geometry 2

In Natural Axiomatic Geometry (one model is Euclid's Geometry) the objects are no more material but ideal. Definitions and axioms are necessary to create the objects, but in this paradigm they are as close as possible to the intuition of the sensitive space, therefore the name of Natural Axiomatic Geometry. Geometry 2 stays a model of reality. But, once the axioms fixed, demonstrations inside the system are requested to progress and to reach certainty. In this paradigm the text takes a great importance, all the objects should be defined by texts, drawings are only illustrations, accompaniments of textual propositions. As it is convenient the expert works with drawings, but he knows how to read these drawing and how all the indications he puts on the drawing are validated by the text.

### Geometry 3

Lastly we have Formalist Axiomatic Geometry (Geometry III): in this paradigm the system of axioms itself has no relation with reality, it is complete and independent of its possible applications to the world. This paradigm is not very present in statutory curriculum.

### Relationships between the two main paradigms, Geometry 1 and Geometry 2

The true question of geometrical teaching concerns Geometry 1 and Geometry 2. Here a table that resumes the main differences between the two paradigms.

	<b>Geometry 1</b>	<b>Geometry 2</b>
<b>Space</b>	Intuitive and physical space	Geometrical Euclidian space
<b>Objects</b>	<i>Material</i> objects (or digital ones). <i>Drawings, models</i> , products of instrumental activity	<i>Ideal</i> objects without dimension Figures ( <i>some areas of space, some relations</i> ). <i>Definitions, theorems</i>
<b>Artefacts</b>	Various tools (ruler, set square, template, paper folding....). Dynamic Software.	Physical tools (ruler, compass) with use theoretically justified "Logical-deductive reasoning"
<b>Proof</b>	Evidence, checking by instrument (f.i dragging) OR effective construction	Properties and "pieces of demonstration" (formal proof) Partial of axiomatic
<b>Measuring</b>	Licit: it products knowledge	Non licit for production of knowledge, but licit for heuristics

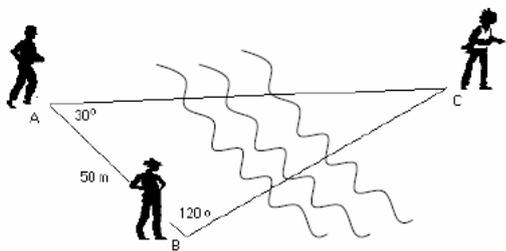
<b>Status of drawing</b>	Object of study and object of validation	Heuristic tool, support of reasoning and “figural concept” (Fischbein 1993)
<b>Privileged aspect</b>	Self-evidence and construction	Properties et demonstration

**Table 1: Differences between Geometry 1 and Geometry 2**

One paradigm is not superior to the other in their relation to space as shown by the study of the following exercise.

## HOW DO GEOMETRICAL PARADIGMS WORK?

### A particular study



The drawing shows André and Bernard standing on the same river bank at a distance of 50 meters from each other. Camille stands on the opposite bank.

How far away is André from Camille?

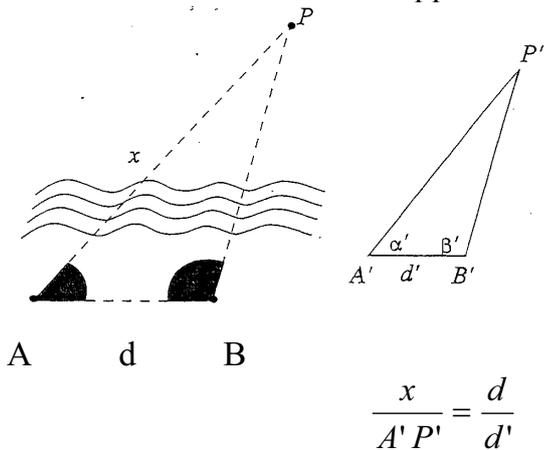
**Figure 1: Excerpt coming from *Matemática 2° Medio*. Chile: Arrayan Editores (2001),**

Why did we choose this exercise (from a Chilean text book for 10th grade -15-16 old students)? First it evokes a real problem through a representation of the situation. But the representation is not transparent; it must be read with geometrical knowledge: the given triangle is isosceles, which can not be seen immediately. To be informed of the nature of the triangle it is necessary to deduce it from the information provided by the angles. This first part of geometrical activity is important and related to the “education of sight” in geometrical teaching.

How could it be solved? A first method consists in constructing a similar triangle  $A'B'C'$  on another scale, measuring  $A'C'$  and deducing  $AC$  through calculation. In the French curriculum this method would be accessible in the 7<sup>th</sup> grade, but rejected in upper grades. Another method, more formal, consists in first deducing from the angle magnitudes that the triangle is isosceles (using the theorem of the sum of three angles in a triangle) and then trying to calculate the unknown length: this calculation requires the drawing of further lines like the right bisector of  $AC$  or the perpendicular height from  $B$  -to obtain two right angled triangles) and the use of theorems like Pythagoras or cosine. In the French curriculum these methods are expected from 8<sup>th</sup> to 10<sup>th</sup> grade.

What does the Chilean text book of the 10<sup>th</sup> grade suggest? We can deduce it from the study of another activity in the same book, just before the preceding river exercise.

If you want to calculate the distance between a point A that is situated on the river bank and a tree that is situated on the opposite bank, you can act this way:



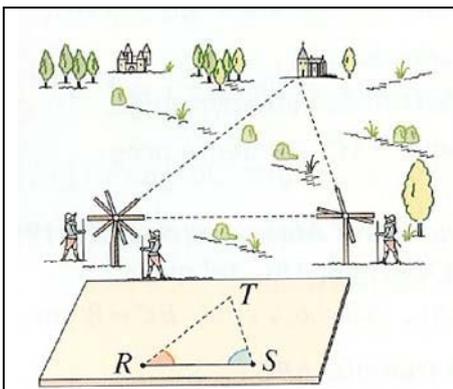
- 1- situate a point B at a determined distance from A;
- 2- measure off the angles PAB and ABP taking line of sight;
- 3- measure off the distance AB;
- 4- construct a scale drawing of a triangle A'B'P' similar to the triangle ABP (angular criteria for similarity);
- 5- measure with a ruler the length of A'P';
- 6- calculate the length of AP taking into account the similarity ratio of the scale d/d'.

Figure 2: Excerpt coming from *Matemática 2º Medio*. Chile: Arrayan Editores (2001)

The heart of the solution is propositions 4-5-6; the former one helps to transform a space question (to calculate a real distance) into a geometric question.

It is remarkable that the Chilean textbook recommends to draw and to measure on the drawing. The drawing is an object of study and permits to obtain the unknown length by effective measuring.

It would be inconceivable at the same age group in France: the unknown length could only be deduced from given textual information in a way as independent as possible from the drawing in most French text books of 9<sup>th</sup> grade where no other method is suggested, as it is shown below.



*To determine inaccessible magnitude...*

A precise point T is taken as sight from situated points R and S whose distance as the crow flies is known.

Then the angles of the triangle RST are measured, which allows to determine the distances with convenient approximation, **because of** :

$$RT = RS \frac{\sin \hat{S}}{\sin \hat{T}} \text{ et } ST = RS \frac{\sin \hat{R}}{\sin \hat{T}}$$

Figure 3: Excerpt coming from *Maths 3º. Cinq sur Cinq*. France : Hachette (2003)

Already in most of the 8<sup>th</sup> grade (13-14 years old students) French text books there is the assertion « *Seeing or measuring on a drawing is not enough to prove that a geometrical phrase is true* » (*Triangle 5<sup>ème</sup>* Editions Hatier 2001 page 127, *Triangle 4<sup>ème</sup>* Editions Hatier 2002 page 94...).

Thus the Chilean curriculum accepts and expects a method that is refused at the same grade in the French curriculum. These methods would be accepted in France in lower

grades, but such problems whose first work consists in thinking how (and why) to schematize reality (propositions 1-2-3-4) are generally not proposed in lower grades text books.

Consequently in similar questions 10<sup>th</sup> grade French students prefer not to answer rather than to propose an answer by making a drawing and measuring it.

### **An analysis with Geometrical Paradigms**

In 10<sup>th</sup> grade even if students are confronted to the same river problem, the answers are not the same: France considers that a treatment in Geometry 1, with the effective use of measures is not convenient. On the contrary in Chile a treatment in Geometry 1 is convenient and recommended by text books as we have seen above.

To solve practically the problem, the first method, drawing at scale that takes place in Geometry 1 is sufficient and effective. The other methods, in Geometry 2 because they don't depend on the drawing, consider ideal situations and use conceptual results: they bring more precision and allow generalisation without new drawings. But precision and generalisation are not required in the river problem. The other methods enable to solve other questions than the determination of that distance only.

It looks as though in France, Geometry 2 takes the place of Geometry 1 and makes it disappear, whereas it is easy to see how complimentary both paradigms are.

Knowledge and practise of Geometry 1 is always necessary first to realise a convenient drawing (see the first exercise), more generally to treat space professional problems with drawing as schematisation; secondly to visualize specific configurations in this drawing (add right further lines to divide the first triangle into two right angled triangles): Duval (1998) already studied the importance of visualization.

Geometry 2 often permits generalisation and logical justification of action in Geometry 1. Geometry 1 is necessary to Geometry 2 as an experience field (Boero 1994), but could not be reduced to an application of Geometry 2.

We now need a new concept to conciliate Geometry 1 and Geometry 2, *Geometrical Working Space* (Kuzniak 2004).

### **GEOMETRICAL WORKING SPACE: GWS**

The Geometrical Working Space (GWS) is the place organized to ensure the geometrical work and to integrate the play between both paradigms. It puts the three following components in a network:

- the objects whose nature depends on the geometrical paradigm,
- the artifacts like drawings tools, computers but also rules of deduction used by the geometrician,

- a theoretical system of reference possibly organized in a theoretical model depending on the geometrical paradigm.

The Geometrical Working Space becomes manageable only when its user can link and master the three components above mentioned. An expert solving a problem of geometry creates a *suitable GWS* to work. This GWS must comply with two conditions: its components should be sufficiently powerful to handle the problem in the right geometrical paradigm and its various components should be mastered and used in a valid way. When the expert has decided what geometrical paradigm is convenient for the problem, s/he can organize the use of artifacts and the type of reasoning thanks to the GWS which suits this paradigm.

When a person (student or professor) is confronted to a problem, this person handles the problem with his/her *personal GWS*. This *personal GWS* generally depends on the knowledge of the person but also on the institution where the person works: what kind of geometrical productions are accepted or valorised by the institution at any time?

Through the organization of the geometrical different contents by grade, the teaching recommendations to the teachers and the notes about how a student can learn geometry, the curricula define specific geometrical environments that can also be seen as GWS: we will call them *institutional GWS*.

### **THE INSTITUTIONAL GWS OF A PARTICULAR THEME**

Taking an example “figures of same shapes”, it is easy to make clear the difference between Chile and France, only through a syllabus reading.

In France the different notions: enlargement-reduction (4<sup>th</sup> and 5<sup>th</sup>), scale representation and lengths (7<sup>th</sup>), Thales theorem (8<sup>th</sup> and 9<sup>th</sup>), similar triangles (10<sup>th</sup>), enlargement transformation (11<sup>th</sup> in speciality) are successively taught in different grades with a perspective strongly focused on Geometry 2 from 8<sup>th</sup> (following syllabus and textbooks). Thus scale representation (and plan reading) could not be functional either in mathematical activities (it becomes fast forbidden to measure on drawing) or in practical problems (not practised in classrooms).

In Chile students meet enlargement-reduction activities first in 6<sup>th</sup> grade, similar triangle and scale representation in 8<sup>th</sup> with a Geometry 1 perspective on lengths and angles and in relation to proportionality. But in 10<sup>th</sup> grade all these notions are taught again in a network with mathematics’ complements (Thales theorem, enlargement transformations) and also history and arts complements about the theory of proportions. The main perspective is always Geometry 1 to create relationships between different notions of a same theme and construct the students’ practical culture, even nourished by some theoretical results of Geometry 2 (like Thales theorem).

We think that relating mathematical teaching to reality including in the succession of the different notions of a same theme is also a way to define *institutional GWS*. The Chilean curriculum permits a play between both paradigms from 10<sup>th</sup> grade; the French curriculum does not officially permit that different ways to solve a problem meet, for it officially rejects Geometry 1 already from 8<sup>th</sup> grade.

The study of the *institutional GWS* has become our first work to precise the difference between both curricula.

## **BACK TO GENERAL COMPARISON THROUGH INSTITUTIONAL GWS**

We will try to precise particularly the crucial differences between Chile and France for the period between 8<sup>th</sup> and 10<sup>th</sup> grade.

### **The system of reference**

Both curricula don't act with the same *institutional GWS*. The **French** reference is Geometry 2: the unique authorized public reasoning concerns ideal objects and even conceptual objects and logic deduction. Geometry 1 is not a suitable paradigm in French 10<sup>th</sup> grade curriculum; it is not officially integrated in the institutional GWS; it must stay private. **In Chile** Geometry 1 is an assumed reference and plays a public role in the institutional GWS. Geometry 2 can exist too, but it is entirely under the teacher's responsibility.

### **The place of drawing**

**In Chile** the drawing is taken as a field of experience (Boero 1994) and also a validation object: a field of experience because students are taught to experiment on drawings, to look for reasons of regularities on drawings, to extend validity of observed regularities on drawing; a validation object because constructing a drawing allows to check regularities and to convince of the plausibility of an assertion.

The drawing with usual geometrical tools is considered as a prime model of reality: for example the triangle is introduced as the simplest non deformable structure to show its interest for construction.

A special teaching time is dedicated to techniques of drawing and construction drills (not directly but through various activities).

**In France** geometrical drawing has no official place; it must stay private and only serve as a support for a conjecture. But how it can serve for geometrical thinking is not taught, thus it can not constitute an experience field. Out of the private mind, drawing is simply and purely forbidden.

Construction activity (for example with ruler and compasses) is not emphasized (it disappears in France from 6<sup>th</sup> grade) and in the textbooks each spatial problem is immediately illustrated by a drawing, so that students are always in front of a schematised situation. The construction act appears as not very important for geometrical thinking in French curricula.

## Validation

**In France** the only recognized validation is that which verifies the non contradiction inside mathematics; a new proposition is accepted as valid only if it can be logically deduced from other accepted propositions.

**In Chile** two levels of validation are accepted and distinguished: first conformity to reality, reality of the sensible world, the graphic line on paper; this conformity can be a pretext for a declaration that is recognized and accepted as ‘plausible’; this declaration must be demonstrated to become true in mathematics.

## The geometrical objects

From French 8<sup>th</sup> grade, licit geometrical objects are definitions and theorems, hence only textual declarations that can be accompanied by drawing (as ‘figural concept’ Fischbein 1993). Thus all objects are conceptual, that means ideal but coherent with and inside a theory (Bunge 1983). There is no recognized place for other objects (material or virtual), even if they are used inside the classroom.

In Chile all the objects are accepted, material (like drawings), ideal, but the quality of the declaration made about the drawing does not have the same conceptual quality as that made by the teacher quoting mathematics.

## CONCLUSION

For our comparison we have studied syllabus, accompanying texts and text books through a particular filter: *institutional GWS*. GWS organizes different components of geometrical activity: what objects, what licit tools and what licit validation, what play between both paradigms? Let us resume the main differences.

The study of the nature of objects and the validation precise what paradigm is referent and what type of reasoning is valid inside the *institutional GWS*. **Chile** accepts explicitly two levels of reasoning, thus implicitly two paradigms (Geometry 1 and Geometry 2). **France** only considers a deductive organisation of discourse (reference Geometry 2) as licit to produce valid declarations.

The study of drawing is related to licit tools (and the use of these tools and the teaching of the use of these tools); the given status of drawing contributes to define the institutional GWS. In **Chile** Geometry 1 and all the work on drawing is considered as the heart of geometry, the experience field on what the students could constitute their prime experience and confront their declarations. In **France** Geometry 1 is considered as a perturbation of geometrical teaching that must be forgotten to access to “true geometry”.

Our very few effective class practices seem to confirm these differences but a larger survey would be necessary to take a sight of *implemented curriculum* and *attained curriculum*.

We hope our readers will be convinced that an entry through the *institutional GWS* in different grades of curricula could produce rich comparison at least in *intended curriculum* and *available curriculum* and open new perspectives for geometrical teaching in his/her own country.

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# COMPARISON OF OBSERVATION OF NEW SPACE AND ITS OBJECTS BY SIGHTED AND NON-SIGHTED PUPILS <sup>1</sup>

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*It is almost a commonplace to state that the visually impaired people use different methods than people who can see to receive information about features of the objects and spatial localization. In this paper we briefly present research that was realized in order to better understand the perceiving the space and its objects by non-sighted pupils. Three non-sighted and four sighted pupils participated in the experiment and results of the qualitative analysis offer some proposals and ideas how to improve the teaching of space geometry to non-sighted and also sighted pupils.*

## INTRODUCTION

The changes in society, inflow of liberty and humanism, caused the integration of handicapped people (Slovakia in 1993) have become an actual problem and one can partly speak about it as fashion trend that is carrying its advantage and limitations. Nowadays, we notice use of mathematics in lot of disciplines, the serious mathematical grounding is necessary not only for prospective mathematicians, but it begins to be popular also at humane sciences as sociology, psychology, linguistics or philology. We are also witnesses to rapid expansion of information technologies that require new technicians all the time, whose education is based on mathematics as well. So we cannot wonder about the attendance of visually impaired people who would like to engage in study of mathematics. These facts, as well as author's experience with working with visually impaired pupils lead us to pay more attention to study of mathematics of visually impaired people. The other remarkable thing is the question of limit. Since in Slovakia there is no standard for teaching mathematics to integrated visually impaired students on the secondary level (the standards for common students are valid), the teacher has to determine requirements on these students by his own, on his subjective opinion.

This paper is based upon previous research that was realized in academic year 2004/2005 (Kohanová, 2005). The non-sighted people (students and adults) were asked to solve four mathematical problems that concerned algebra, mathematics of common life, Euclidean geometry and analytic geometry. We find out that the visually impaired people use personal geometrical instruments and strong

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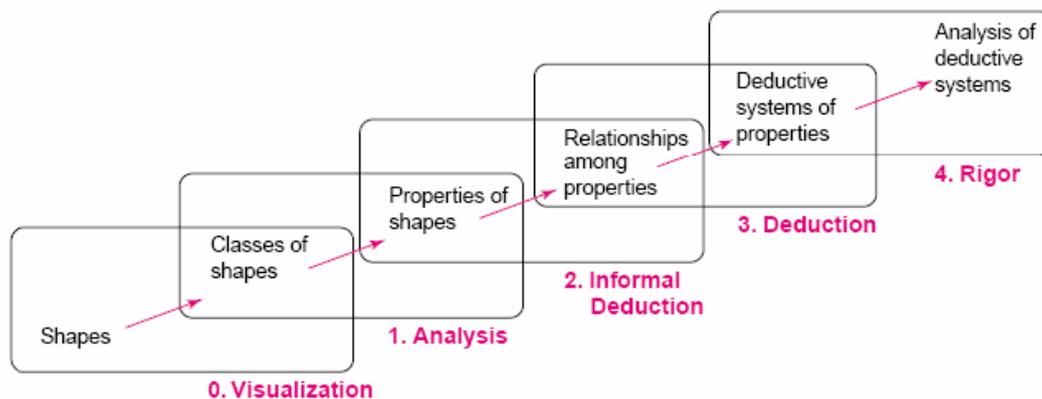
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imagination, all object (solids and plane figures) are first touched and then stored. Geometry is for them kind of adaptation to the environment. So we think this adaptation is dynamic in sense that they continually change the system of operations of environment that explores. Since every environment is a new environment s/he has to store all information (tactile, auditory, olphactive, etc.) and so make mental images. All that has inspired us to study more in the field of space geometry, to see how non-sighted people are adapted to various environments, what are their personal tools, since geometry can provide a more complete appreciation of the world. Results, interviews, remarks and observations of this research will act as propaedeutic of teaching solid geometry at the secondary school. But not only for teaching visually impaired students, but also for sighted ones, since there does not exist any methodical guide for teachers of mathematics at special primary and secondary schools. Last, but not least, it might be methodical guide for teachers at schools, which integrate visually impaired students and do not have any experience of working with non-sighted students.

## THEORETICAL FRAMEWORK

### Understanding of geometric figures

Van Hiele (1986) published a theory in which he classified five levels of understanding spatial concepts through which children move sequentially on their way to geometric thinking. Different numbering systems are found in the literature but the van Hiele's spoke of levels 0 through 4. At each level of geometric thought, the ideas created become the focus or object of thought at the next level as shown in Fig. 1 (Van de Walle, 2001).



**Fig. 1: Van Hiele's levels.**

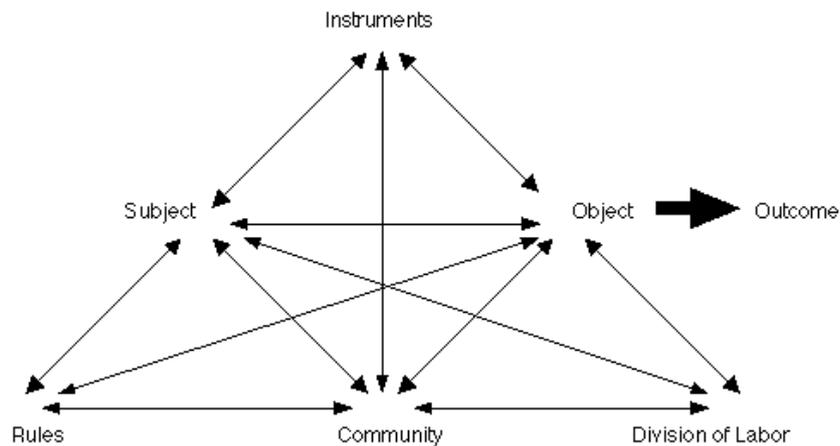
According to Jirotková (2001) there are three levels of the quality of the mental picture of a perceived solid:

1. the solid is a 'personality' for the pupil,
2. the solid is unknown to the pupil, however, the pupil perceives some relationship between the considered solid and another solid which is a 'personality' for him/her,

3. the solid is entirely new for the pupil

### Analysis of the activity

Activity theory originated in the former Soviet Union as a part of the cultural-historical school of psychology founded by Vygotsky, Leontiev and Luria. Its unit of analysis is an activity that is being composed of a subject, and an object, mediated by a tool. In following model (see Fig.2) of an activity system, the subject refers to the individual or group whose point of view is taken in the analysis of the activity.



**Fig. 2: Model of activity system.**

The object (or objective) is the target of the activity within the system. Instruments refer to internal or external mediating artifacts, which help to achieve the outcomes of the activity. The community is comprised of one or more people who share the objective with the subject. Rules regulate actions and interactions within the activity system. The division of labor discusses how tasks are divided horizontally between community members as well as referring to any vertical division of power and status. We have used this model as a tool for description and analysis of realized experiment.

### THE RESEARCH AND DETERMINATION OF THE HYPOTHESES

We have placed various subjects of different shapes in the room. Except of typical office subjects (table, chairs, PC, cabinets) we put in the room the fit ball, air freshener, the clock of pyramid shape and flowers as well. The lamp on the table was on as well as the PC; water in the sink, which is in the closet, flow. That all because we wanted to observe what sense the person in the room will use while exploring the room. Before realizing the experiment, we consulted about the location of subjects in the room with visually impaired university student, who is experienced in exploration of new places. Final arrangement is shown in following figures.



**Fig.3: Arrangement of the objects in prepared room.**

Consequently, seven pupils took part in the experiment, the sighted pupils (SP) were selected at random and all pupils were of 7<sup>th</sup> - 9<sup>th</sup> grade. Pupils of these grades know 2-D and 3-D shapes and their characteristics; they have their personal experience and they have learned it also in the school. However, the problem was the number of pupils who took part in experiment. We wanted to form pairs of all possible combination of sighted and non-sighted pupils (NSP), which means 4 pairs. It is needed to say we concentrated only on pupils who are non-sighted since birth and so do not have any visual imagination. That is why we were able to find only 3 non-sighted pupils (age 13-14) attending the special primary school for visually impaired children in Bratislava. Then we changed pairs for trinities and pairs as follows:

$$\text{NSP}_1\text{-NSP}_2\text{-SP}_1 \quad \text{NSP}_3\text{-NSP}_2\text{-SP}_2 \quad \text{SP}_3\text{-SP}_4$$

where always the first one of the trinity/pair went in to the room and verbally described what s/he sees and the others of the trinity/pair built the model of the room on the basis of audio record. The first one of the trinity/pair built the model of the room as well, but on the basis of her/his memory. In the first and second trinity is the same person (NSP<sub>2</sub>) and we are conscious that it might influence the results, but NSP<sub>2</sub> was not told that she is building the model of the same room in both cases.

During the experiment we observed:

- the orientation in the space
- the way of description of the room and objects
- the relationship between the image in the pupil's mind and the vocabulary s/he uses in the communication
- what is the dominant attribute by description of the room
- perception of the shapes, positions and dimensions
- what senses s/he uses

- what way s/he builds the model of the room
- differentiation of the shapes and characteristics of the objects

Consequently we have set following hypothesis:

- H1: The sighted and non-sighted pupils perceive the space and its objects in different way. The point of view on geometry of the space of visually impaired people is point of perception and it is dynamic. The point of view on geometry of sighted people is static.
- H2: Based on the senses the non-sighted pupils are able to differentiate and name basic geometric figures and solids.
- H3: When exploring new room and objects in it, the non-sighted are using several senses; sense of touch, smell and ear; while sighted rely only on sight.
- H4: The non-sighted pupils will describe objects in the space (shape and position) better and more exact as sighted pupils.
- H5: The non-sighted pupils have better imagination about position of objects in the space as sighted pupils and so they build more precise scale model of the room, even if they build it on the basis of given audio record.

### **Method and description of the experiment**

As written above we have divided children into the trinities and pair. We call the one who goes into the room pupil A, pupil B is the one who doesn't go into the room. The tasks for the pupils were as follows:

#### Task 1

Pupil A: Enter the room. Within the twenty minutes explore it and tell me exactly what do you see. Tell me about everything, about all objects, their characteristics and their localization.

#### Task 2a

Pupil B: By using these packages and stuff try to build the model of the room on the basis of audio record of Pupil A. The caps of plastic bottles represent the chairs. Later on you can ask for more information, but only by asking questions to which Pupil A can only answer 'Yes' or 'No'.

#### Task 2b

Pupil A: By using these packages and stuff try to build the model of the room on the basis of your memory, on the basis of what you have seen. The caps of plastic bottles represent the chairs.

Applying the Activity theory we described two activities, one that has been carried out in the room (Task 1) and the second activity that has been realized out of the room (Task 2). In Task 1 we made audio records of Pupils A descriptions of the

room. In Task 2a we made audio records of dialogs between Pupil B and Pupil A, in Task 2 pictures of all 8 models of the room.

The exploring and describing the prepared room is the activity that refers to the subject of Pupil A, who goes into the room. The object of her/his activity is the room and all objects in it. The expected outcome is the as precise verbal description of the room as possible; consequently we are going to analyze this description in sense of perceiving the space and its objects. There were no seted rules concerning the progressing activity, just one restriction regarding the time was given. It has an implication that Pupil A can proceed as s/he wants, in the way s/he likes, so there are no horizontally segmented tasks of division of labor. Anyway, with respect to action of university student M. and our experience we have expected the following possible actions which Pupil A could make in the room:

- to specify the shape of the ground plan and verify the dimensions of the room;
- to seek points of the reference by means of the echo of the windows, of the doors, of the voice, etc.;
- to individuate and memorize every possible obstacle;
- to look for references in the noises and vibrations or in the odours;
- to clapp one's hands to grasp the dimensions and the volume of a room;
- to move with the white stick and perceive the space, objects and obstacles;
- to perceive the obstacles by air pressure on the face;
- to touch all objects and describe them.

Mentioned possible actions could be done by using the white stick, all senses, language, imagination, etc. and these are mediating tools or instruments by which the Pupil A can achieve the outcome of the activity. There is also no vertical division of status and power concerning the division of labor, since the community of this activity consists only of researcher who is present in the room in order to record the description and assist if necessary. It is needed to mention that the whole environment in which the experiment was realized, as well as the researcher was new for pupils, so that is the reason why we are conscious of pupils's doubtful and sometimes reserved behaviour. All that might influence the objectivity of the experiment.

The second activity was realized out of the prepared room and its outcome is to interpret the room by building the model, which is also kind of description of the room and we can analyze it in the frame of perceiving and recognition of the space and its objects. The model of the room built by us is shown in following picture.

This activity has to be distinguished with respect to the pupil who is building the model (Task 2a, Task 2b). In both cases the object of the activity is the prepared room and the rest changes.

In case of Pupil A, who is the subject of the activity, the rules are given only by saying that pupil should build the room by using given packages and stuff, moreover the bottle caps have to be used as chairs. In case of Pupil B we have two more rules about the building the model according to record and about the way of asking questions to Pupil A. All given packages and stuff of different shapes and sizes (playing cubes, packages of tea, matches, medicaments and cosmetics; tennis's and squash's balls, buttons, batteries, eraser, carton models) are for Pupil A and B instruments to build the model. The difference between Pupil A and Pupil B is that other instrument of Pupil A is her/his internal model of the room stored in her/his memory, while Pupil B has audio record of Pupil A at disposal. Pupil B can ask for more information that is becoming also his/her instruments. The community in both cases consists of researcher and her assistant and other pupils who took part in experiment. In case of Pupil A all community except of researcher is just side, unimportant effect; they were just observers, no interfering into the process of building the model. On the other hand, important role of community plays in case of Pupil B the researcher who moderates the conversation and Pupil A, who answers to the questions. Since the instructions of Task 2a say to Pupil B first to build the model of the room on the basis of audio record and later on to ask the supplementary questions, here we have horizontally segmented actions of division of labor (which is actually given by the rules). Also the succession: question, answer, and potential change of model represent partial horizontal division of the actions.

## RESULTS OF THE QUALITATIVE ANALYSIS

The sighted pupil really showed expected behaviour, right she entered the room she stated what is in there (sometimes very inexactly), while the non-sighted pupils detected the space gradually. So here we have development and dynamics of detection, which are actually facilitating the subsequent better description. If we would be able to bring the sighted pupils to such a dynamics, then the certain superficiality can be eliminated and hence also the superficial perception of the space.

In Task 1 pupils should describe the room and its objects, their characteristics and localization so pupil B in Task 2a can build the model of the room. We had seen that non-sighted pupils recognized and named many objects of different shapes (cube, cuboid, pyramid, cylinder, triangle, circle, trapezoid, square, and rectangle), so the second hypothesis seems to be true, although in some cases they used wrong terminology.

R2: Then there is cabinet, also shape of rectangle, classic cabinet with rectangular shelves.

M27: So, in the middle of the room is the table in shape of rectangle.

The hypothesis H3 has been confirmed only partially because the non-sighted pupils didn't use sense of smell, neither by finding the air freshener nor by flowers. The

sense of touch is their leading analyzer and sense of ear is complementary analyzer. We can illustrate the usage of sense of ear by demonstrations from the protocols:

R22: I have heard water and then I went to see...

M39: You can hear here the whirr of computer and like [...] as the water flows [...] or like that.

Also in case of sighted pupil SP<sub>3</sub> we cannot claim she only relied on sight. The true is she also didn't notice the air freshener, she saw the flowers, but she mentioned the sink in the cabinet even she couldn't see it since the door was closed; on the other side she didn't say anything about hearing.

J6: At the door are cabinets, where is for example the sink, in one there are books.

Since non-sighted pupils had to go over the whole room and touch everything, they described continuously and more exactly the objects in the room than sighted pupil, who stand in one point and described what she saw. Sighted pupil didn't mention lot of things, she didn't find it as necessary, even she was told to describe it precise. On the other hand, when building the scale model of the room, she did it very exact, which says about her strong visual memory. Based on these facts we can confirm hypothesis H4.

The fifth hypothesis wasn't neither acknowledged nor disproved since all Pupils A (sighted and non-sighted as well) built almost exact model of the room. In the case of pupils B we had noticed the ability to interpret the verbal description of the space and ability to create an image of solids and their location in the space. We cannot compare the results of sighted and non-sighted pupils who participated in Task 2 since there was the same non-sighted person participating two times in experiment. Anyway, regarding the mental representation of the space, the world of non-sighted is not different in comparison with that one of people who are sighted.

Except of determined hypotheses we came also to following conclusions that are applicable in pedagogic practice of the teacher.

- Right in the experiment, concretely at Task 2, the visiting math's teachers from special school for visually impaired children pointed out that the same or similar tasks have considerable value as educational tools. They could be used for the diagnosis and assessment of pupils' levels of understanding of three-dimensional solids (van Hiele's levels) and metrics of the space and to develop their communicative skills about the solids. The Task 1 required the pupils to describe new space and its objects. This gave a very clear indication of level of vocabulary of the pupils and the communicative skills.

R48: ...this one side [...] front [...] If I hold it like this [...] it is longer than the other side. Actually, the horizontal side is longer than vertical. It depends how you hold it. I have it along, horizontally to me ...

R49: And under it is bigger packet which has shape of [...] it is also not the shape of cube [...] but it is shape of [...] what can I compare it to? It is shape of cuboid. Also the upper packet has had this shape. Yes [...] it is cuboid.

According to some similar experiments (Littler, Jirotková 2004) when authors observed sighted children in process of tactile manipulation with solids and their verbal communication, this analysis help us also to construct the process of building structure of geometrical knowledge or even the process of creating new knowledge by extending the existing structure or its restructuring.

- At the first glance we saw the difference, while models of sighted pupils were large, the models of non-sighted were “small”, tight, all objects were close to each other. The reason for it might be on one side the necessity of the control of the model by hands, on the other side also the lack of experience with metrics. The other point is related to the estimation of distance and measure. It is shown in the protocols that non-sighted pupils compared the measures to their body.

R7: That cabinet is high about [...] something more than knees or like my thighs.

It could be meaningful to think about the usage and application of English system of measurements instead of metric system in their case.

- Both sighted and non-sighted pupils built quite exact model of the explored room and thus, as regards the mental representation of the space, the world of non-sighted is not different in comparison with that one of people who are sighted. The difference is the way one gets information about the space. Through the sense of sight, one can obtain an overall knowledge of the environment, whereas one can achieve it through an analytic way, if s/he employs the haptic perception.
- Except of some above mentioned proposals for future phase of the research we consider as interesting to observe the perception of the space and its object in connection with language as an individual tool. In what way the language and exactness of expression might influence the knowledge, but not only in the case of non-sighted pupils. The other improvement might be done in connection with realization of similar experiment with more pupils. However, we cannot influence the number of non-sighted pupils who will participate.

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# ASSESSING THE ATTAINMENT OF ANALYTIC-DESCRIPTIVE GEOMETRICAL THINKING WITH NEW TOOLS

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*Transition to Van Hiele level 2 is characterized by a gradual primacy of geometrical structures upon the gestalt unanalyzed visual forms and application of geometric properties of shapes. Some special test items have been constructed to clarify some aspects of this transitional process. Perceptual strategies seem to persist even in university students, suggesting complementation of typical tests with items focusing on this issue.*

## INTRODUCTION

Van Hiele (1987), describing the evolution of his theory since 1955, regrets the fact that initially he “had not seen the importance of visual level”, but finds that “nowadays the appreciation of the first level has improved” (p.41). However little research has been made to analyse more systematically levels 1 and 2. As Hershkowitz (1990) nicely puts it: “Visualization and visual processes have a very complex role in geometrical processes...More work is needed to understand better the positive and negative contributions of visual processes” (p.94). The results reported in this paper are part of a wider research attempting to elucidate certain aspects of this contribution (and its inverse also: the effect of geometry learning to the visual processes) and are related exclusively to the problem of the transition from level 1 to 2. Assuming in principle Van Hiele’s theoretical framework the main questions posited were:

-To what degree secondary education students have substantially progressed from the “visual” level 1 to the “descriptive-analytic” level 2 and particularly: do they *apply* the geometric structures of the second level in a visually differentiated context?

Or more specifically: Do secondary students tend to use “visual” (level 1) or “analytic” (level 2) strategies to solve tasks which allow both procedures?

## THEORETICAL CONSIDERATIONS

### Some remarks upon Van Hiele level 2

Certainly, Van Hiele considered as main characteristic of level 2 the fact that the visual figure recedes to the background and the shape is represented by the totality of its properties. He stressed however that the discovery of these properties should be made by the pupils themselves and not be offered ready-made by the teacher (Van Hiele,1986;p. 54,62,63). But this is not sufficient: level 2 is attained when the pupil is able to “*apply* operative properties of well-known figures” (ibid. p. 41, see also p.

43). We should keep in mind that attainment of level 2 is not simply recollection of learned properties of shapes, but possibly a more active state: a mode of mental activity that tries to find new properties and *apply* the already known ones.

### **Type and content of test items**

Reviewing the research literature on Van Hiele levels we find out that the test items specific to level 2 are of the following kind: a simple, basic geometric shape (e.g. a rhombus) is shown to the student and he/she is asked to “list its properties” (Gutiérrez & Jaime, 1994, 1998) or to identify a particular quadrilateral in a set including a variety of different types (Burger & Shagnessy, 1986) or to select among propositions referring to known properties of basic shapes (Usiskin, 1982). Considering what have been said above about the application of properties it is clear that this kind of task puts a rather one-sided weight upon the recollection of properties instead of application of them in novel situations. The skill of “applicability of properties” has been taken previously into account by Hoffer (1983,1986) and Fuys et al. (1988), who set additional criteria like “discovering of new properties by deduction” and “solving problems by using known properties of figures” (level 1!). Another matter of concern is the one-sided dealing with the concept of “congruence” (of line segments or angles) and neglecting other topics, a point already mentioned by Senk (1989).

### **METHOD**

Under the above considerations some special geometrical tasks have been constructed focusing on three essential concepts: congruence, similarity and area. The main idea is to present a problem but in a visual context different of that of a usual geometry textbook. The correct answer could be found either by some geometrical reasoning pertaining to level 2 or by a visual estimate leading, with the higher degree of certainty attainable, to perceptual misjudgement, due to the well-known limits of the human visual system’s capacities. So the deliberate aim of the problem’s set was to test the students’ *choice* concerning the appropriate strategy and not of course the latter’s efficiency or exactness. There were two tasks for each topic and two alternative versions (to prevent students cheating, depending on the classroom conditions), six in all for each student. Fig.1 shows two examples of the test items related to congruence of figures (C1) and line segments (C4) (Application of property of circle and rectangle). Task C2 was a more difficult one about congruence of triangles and C3 the known Müller-Lyer optical illusion with two equal line segments, one double arrowed the other tailed, against a background of parallel lines and circles that provided geometrical cues for reasoning. Tasks S1 and S2 (shown in Fig.2), and their alternate versions S3 and S4 with exactly the same underlying geometrical idea but different visual context, had to do with the concept of similarity. Finally tasks Ar1 and Ar2 (Ar2 is the same task by means of which J.Piaget (1960) tested whether the relation between length and area has been established), and their corresponding variations Ar3 and Ar4, were about the concept of area. The paper and

pencil test battery included 11 items in all (the remaining five tested other spatial capacities) and total process time allowed was 25 min.

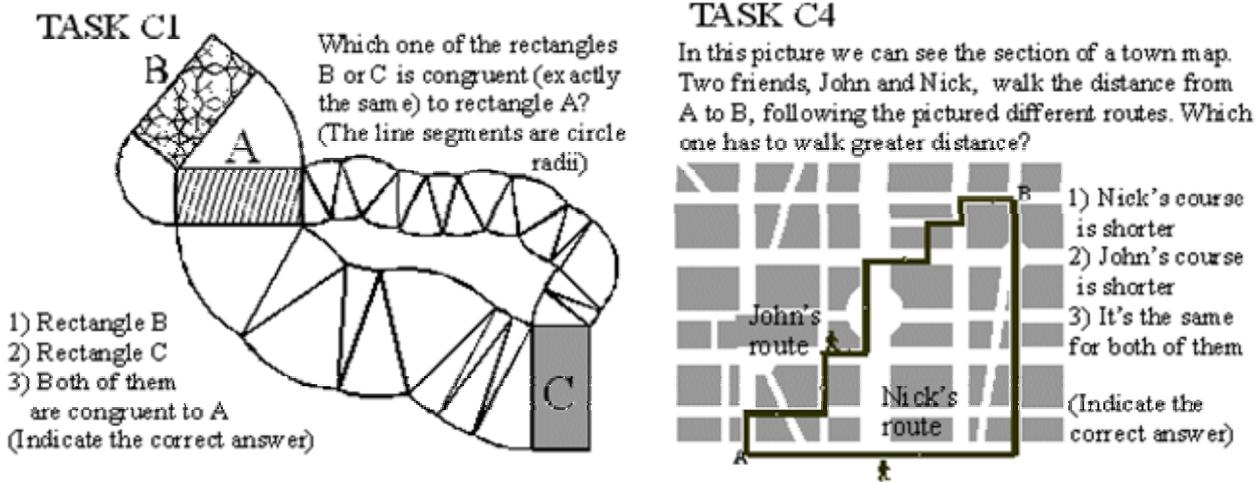


Figure 1: Congruency (The pictures are scaled-down to one half of the original)

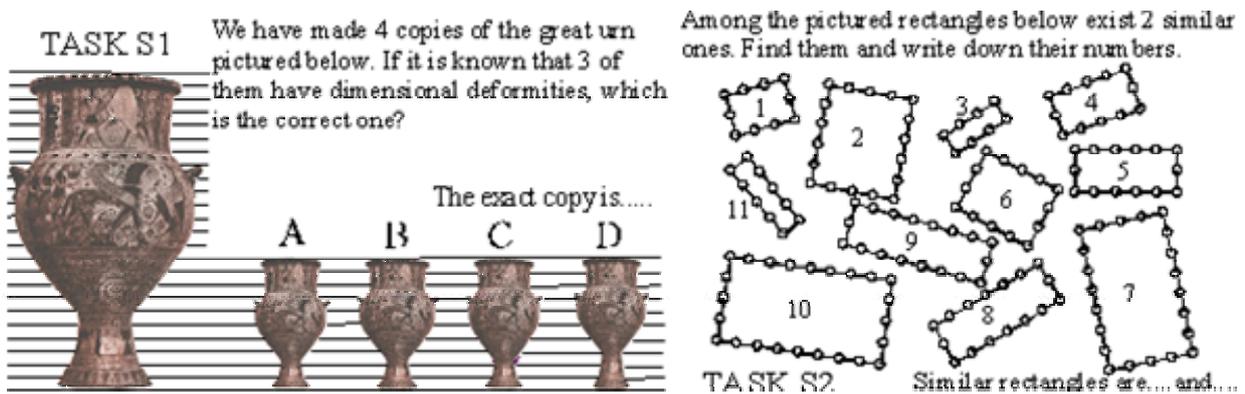


Figure 2: Similarity

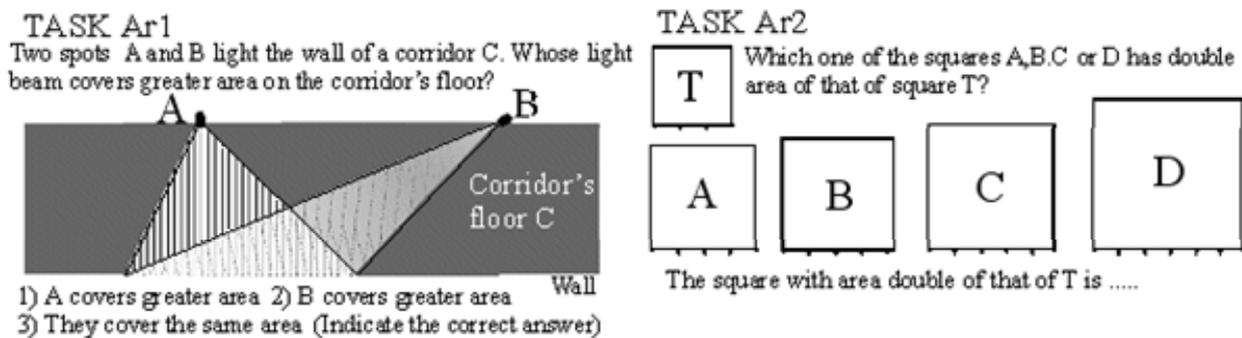


Figure 3: Area

This test questionnaire was administered to 478 students (ages ranging 15 to 23). This sample was composed of two main groups. Firstly, we sought for a population in this age range undergone the minimum possible geometrical instruction (another research matter was to elucidate the pattern of the effect of various types and contents of the

formal geometrical education upon the process of the development towards level 2) to compare the effect of formal education; a likely candidate was a group (labelled A1) of 80 students attending Bakery-Pastry Practical Apprenticeship School (under the supervision of Greek State Organization for the Employment of the Working Force). These students had received some basic geometry instruction in elementary and lower secondary school, but possibly their involvement in the educational process was limited. A second subgroup (A2) consisted of the typical tenth graders (154 in number) entering the upper secondary school and ready to attend the Euclidean Geometry syllabus (mandatory for all students of this level in Greece). So members of group A were young adolescents with a non-systematic instruction in Geometry. Group B consisted of two subgroups; 150 upper secondary twelfth graders (B1; age 17-18) and 94 Mathematics Department students in Athens University (B2; age 20-23), both subject to substantial and systematic geometrical instruction. The test battery was administered in the period between the 4/2004 and the 10/2005 in the corresponding classrooms.

We assumed that wrong answers mainly implied either unsuccessful visual estimates or defective geometrical reasoning. However, in case of a correct answer there was the possibility that student might have used the visual estimate strategy and this particular difference mattered for the transition to level 2. We considered that a written instruction asking “How you worked it out?” shouldn’t be included in the paper test for the following reasons: the aim of the study was to test the student’s spontaneous reaction and immediate choice without any clue relating the task to geometrical reasoning; the reply “By the eye” doesn’t necessarily precludes another more analytic strategy at her/his disposal as an alternate, second choice; it would be of considerable interest to check whether this questionnaire could serve as reliable, convenient and independent instrument for level 2 assessment; and , finally, for general methodological reasons (triangulation). So we interviewed a number of students of the subgroups A1 (14), A2 (101) and B1 (42). The interview protocol was based on three questions; first: “How did you obtain the answer to this question?” (for correct answers only); in case the student answered “By the eye”, we proceeded to the second: “Could you imagine a different, more secure, way to solve it?”; in case of a negative answer we framed the third: “What about using some property of the shapes you see, for instance this is a circle etc.”.

To compare the performance of the students in a more typical Van Hiele assessment instrument, we composed two variations of Usiskin’s test (1982), each including ten tasks aiming at levels 1 and 2, and administered it to a sample from A2 (70) and B1 (50). Finally, for group A1 we had at our disposal each student’s marks in mathematics lesson for his/her three years in lower secondary school (of which we took the average). For group B1 the marks in Geometry lesson for the two years it is taught (again we took the average). This mark has been taken as an indicator of student’s formal education competence.

## RESULTS

The alternate versions C3, C4 (about congruence), S3, S4 (about similarity) and Ar3, Ar4 (about area) were administered only to a number of students of subgroups A2 and B1; differences in performance for these groups across tasks C1, C3 and C4 were insignificant ( $X^2=1.273$ , d.f.=2, n.s. and  $X^2=1,226$ , d.f.=2, n.s, correspondingly), so we pooled these data, under a general label C1. This was possible for similarity and area tasks except tasks S1 and S3 for subgroup B1 ( $X^2=11,97232$ , d.f.=1,  $p<0.005$ ). Task C2 demanded more difficult reasoning, so we present the corresponding results separately. Group B2 outperformed significantly the other three in task C1 ( $X^2=41.62$ , d.f.=3,  $p<0.0001$ ), S2 ( $X^2=61,97$ , d.f.=3,  $p<0.0001$ ), Ar1 ( $X^2=43.23$ , d.f.=3,  $p<0.0001$ ) and Ar2 ( $X^2=71$ , d.f.=3,  $p<0.0001$ ), all other differences between subgroups A1, A2 and B1 being insignificant, except in task S2 ( $X^2=20,54$ , d.f.=2,  $p<0.005$ ). The correct rate (%) for each group and task is presented in Table 1.

Group \ Task	A1	A2	B1		B2
C1 (C3,C4)	47.5	58.46	65.53		90.42
C2	68.75	71.79	54.25		59.57
S1 (S3)	50	37.01	55.31	16.07	51.06
S2 (S4)	1.25	2.59	14		32.97
Ar1 (Ar3)	43.75	44.80	57.33		81.91
Ar2 (Ar4)	21.25	19.48	30		68.08

**Table 1: Correct rate (%) of the subgroups in the six tasks.**

Rating one point to a correct answer each student's total score is composed. ANOVA single factor analysis upon these score data showed significant difference between groups ( $F(4,474)=33.77$ ,  $p<0.0001$ ). Post hoc comparison test between groups limited this difference to B2 relative to the other three and to B1 relative to A1 and A2 (in the A2-B1 comparison using Bonferonni and Scheffé test a significant difference was found ( $t(474)=-2.68$ ,  $p<0.0083$  and  $F(1,474)=7.228$ ,  $p<0.01$ , correspondingly), but applying Tukey HSD test this proved insignificant ( $q=3.237$ , n.s.)).

In the following paragraph some examples of the interview procedure are presented (the questions concern right answers only): Student G.L. (group B1, task C1):

Interviewer: So how did you worked it out?

G.L.: By the eye!

Interviewer: Could you imagine a more secure way to solve it? By the eye you are not so certain, are you?

G.L.: Eh... eeh....No...I think I cannot find something.

Interviewer: Look, how about using some geometrical properties that you know? Here, for example it says something circle radii...What do you know about them?

G.L.: They are all equal...So, ...eeh.....all these line segments are equal...hence the rectangles are congruent?

Interviewer: Does it suffice? ....To have two sides equal , I mean?

G.L.: I think so... Yes.

Interviewer: O.K. You're right!

This student was classified as one having of course an initial response pertaining to visual strategy, but after probing during the interview as one that could attain geometrical reasoning. Student M.L. (group B1, task A2):

Interviewer: And this one how did you worked it out?

M.L.: I put the small one [T] inside the big ones [she has sketched on the test paper a T square inside each one of the bigger squares] and estimated by the eye the remaining area to be equal to that of T...

Interviewer: And in the case of B you found that the remaining area is the same as that of T? Well, this seems quite difficult to me! ... Isn't there some other, more certain and easier way to find it?

M.L.: What else....nothing comes into my mind.

Interviewer: Something that has to do with the area of certain shapes?

G.L.: Oh, sir, geometry has never been my strong point !

This student was classified as one having the same response before and after the interview. Student S.K. (group B1, task S1):

Interviewer: And how did you find the answer to this one?

S. K: I found the ratio of the two sides and compared these ratios ... similar rectangles have the same one.

This was of course a clear case classified as one implementing geometrical reasoning in the initial response. The results of the interviews are shown in Table 2. We included in the correct geometrical reasoning answers to tasks C1-C2 the "measurement by

	A1	A2				B1						
Geometrical reasoning in	0 task	0 task	1 task	2 tasks	3 tasks	0 task	1 task	2 tasks	3 tasks	4 tasks	5 tasks	6 tasks
Initial response	100	84.15	12.87	2.97	0	54	24	4	2	2	8	6
During the interview	100	60.39	25.74	10.89	2.97	40	30	12	0	2	6	10

**Table 2 Interviews' results (percentage)**

straightedge, compass, pencil, etc.”. For example, as came up during the interview, 24% of group B1 had an initial response involving some form of geometrical reasoning in only one task and this percentage became 30% after the interviewer’s help (second row labelled “during the interview”). Or 2.37% of the A2 gave geometric solutions in 2 tasks in their initial response and this changed to 10.98% during the interview.

According to our version of the typical Van Hiele test and using the “strict criterion” (4 correct answers in 5 questions) the 42.85% of the sample of group A2 and 61.64% of the sample of group B1 had acquired already level 2.

The Pearson product moment correlation coefficient between performance in formal mathematical education (specifically geometrical for group B1) and the six tasks of our test found  $r=0,381$  for group A1 and  $r=0,284$  for group B1.

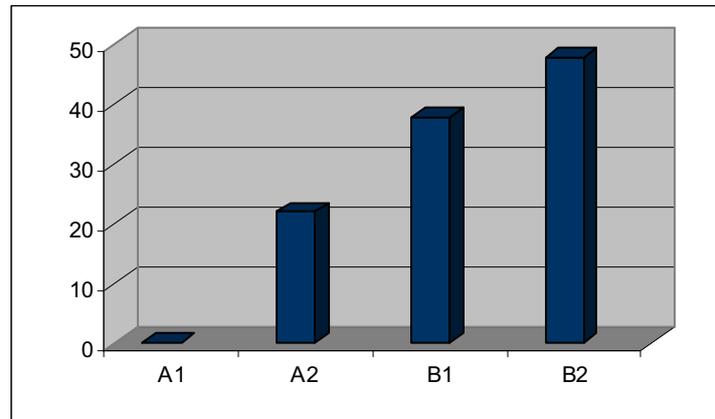
## **DISCUSSION-CONCLUSIONS**

Level 2 seems to be critical for the subsequent progress of a student in more abstract geometrical education (Senk, 1989). In the second section we argued for a widening of the test items both in type and content, specifically for level 2 contrasted to the “visual” level 1. It is evident that the choice of the criteria and the corresponding tests should have substantial effect upon the assignment of a Van Hiele level to a student. In the traditional Van Hiele levels research have been developed two standard versions: the, so called, “strict criterion”, that is: a particular level is assigned to a student if she/he answers correctly to 4 out of 5 questions pertaining to this level, and the “lax criterion”, where we have 3 out of 5 correct answers. The choice of the success rate that should be considered as the appropriate qualifier for a student is clearly a matter of discussion.

Indicatively we could accept as a “lax criterion” of level 2 attainment, considering the difficultness of the tasks compared to the traditional instruments as well, the 3 correct answers out of 6 (50% success rate). Taking additionally into account the fact that performance in congruence tasks (C1-C4) can be based quite efficiently on visual strategies (as is well established in cognitive science experimental research and actually confirmed by the above results) we can set as criterion the following: C: “At least 3 correct answers, but in case of only 3 (correct answers) not 2 of them pertaining both to congruence”.

Applying this criterion only 22% of A2 could be classified in level 2 compared to 44% of the traditional test (applying the “lax criterion”), and only 38% of B1 compared to 62% of the traditional test. These percentages are closer to those revealed by the interviews, where 15% and 40% of groups A2 and B1, respectively, managed to find some form of geometrical reasoning in more than 2 problems. Almost 100% of A1 sample, 60% of A2 and 40% of B1, insisting on visual strategies in all tasks even after probing during the interview, we might say that are still in level

1. In Figure 4 the percentages for all groups concerning our written test are shown. As already mentioned above, setting another criterion would of course result in quite different percentages.



**Figure 4: Percentage of students that have attained level 2 according to criterion C.**

The results imply that the typical tests putting exclusive stress on recollection of properties of figures or formal definitions (especially Usiskin's test questions pertaining to level 2 have not easy quantifiers and figures do not play any role except in one) rather fail to capture the above mentioned processes substantial for attainment of level 2. Therefore traditional test instruments might be complemented with items like the ones presented above.

The overall performance of group B2, as was reasonably expected, is higher than that of the others. But the fact that a not negligible percentage of Mathematics students have not yet attained some form of the *geometrical* concept of similarity (see also Kospentaris & Spyrou, 2005) or the relation between square's side and its area, should not pass unnoticed. The extensive use of visual-perceptual strategies may also be verified by the results in the more demanding task C2 or S1. In these tasks this led to loss of whatever geometrical advantage they might have relative to the other groups. Van Hiele explicitly states (1987, p.63) that a person after having attained level 2 even in *visual thinking* the formed structures of this level are always at his/her disposal, but with one exception: if he thinks in another context. But this effect of the visual set seems not to be evenly distributed among tasks and may related to the task's content (i.e. inadequate or ineffective instruction in similarity and area). The question arising is: have these students really attained level 2 and have a trouble to apply simple geometrical structures of level 2 in a visually differentiated context or they are still acting in level 1, at least in these particular topics? Previous research (Mayberry, 1983; Gutiérrez & Jaime, 1987) in fact revealed that preservice elementary school teachers usually act in level 1 or 2, but here we have to do with a mathematically sophisticated population.

Another result worth noticing is the marginal difference between B1 and A2, despite the great amount of geometrical instruction the former have received (3 or 2 hours per

week for 2 years in Geometry and 5 in Analytic Geometry for a year for approximately 80% of them) and the observed low correlation to school geometry performance. A likely explanation seems to have to do with the presentation of Geometry as axiomatic deductive system and the method of instruction; as Van Hiele warns (1986,p.63)

If pupils do not find the network of relations of a given level by themselves, when starting from a concrete situation, will have difficulties returning to the corresponding signification in the developed network of relations, unless the concrete situation happens to be that of the teacher's original situation.

It would be useful to make use here of the notions (and the corresponding terminology) of Kuziak et al. (2007) about the different paradigms or “geometrical worlds”, inside which each student thinks and behaves: Geometry I is the domain of space relations of the real world discovered and validated mostly by experiment, measurement, visual estimations, etc. Geometry II is the domain of abstracted relations and deductions based on fixed axioms and logical rules and its archetype is classic Euclidean Geometry. The objects and the basic axioms are modeled according to the real world experiential relations, but with a great degree of idealization.

So we can restate the problem as follows: why these students cannot transfer the procedures of Geometry II in a context that strongly induces Geometry I situations? One likely explanation is that formal geometry curriculum in lower secondary school does not provide enough time and adequate learning activities related to Geometry I and jumps abruptly to a Geometry II style.

Furthermore, insistence to present Euclidean Geometry in more abstract, logic deductive manner, that is: moving mainly inside a Geometry II environment, as things continue to happen until today in Greece, implies that we take for granted that students entering upper secondary education have mastered level 2, that is: they are in position to act appropriately in a Geometry II environment. The above results posit serious questions about this assumption.

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# **HORIZON AS EPISTEMOLOGICAL OBSTACLE TO UNDERSTANDING INFINITY**

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*The paper refers to a horizon as a main principle of an understanding of infinity, more specifically as the process from 'very big' towards potential infinity and actual infinity. The role of the horizon will be maintained by the results of interviews, supplemented with illustrations, and some ideas of historical development of the infinity concept (particularly in a geometrical context). We are able to formulate conclusions about the characteristics defining the phenomenon of the horizon in the development of the understanding of infinity.*

## **INTRODUCTION: CONCEPT OF INFINITY**

Infinity as a mathematical concept is highly abstract and at the same time very attractive. It is related to the amount of mathematical and non-mathematical concepts such as basic geometrical objects, functions and their behaviours, set cardinality, size of the universe etc. Thanks to the richness of context where we can meet the concept of infinity, it was necessary to focus attention only on some of them. I have chosen the geometrical one. One reason is that children meet infinity in this context at an early age (implicitly first). Another reason is that a main part of mathematics (above European mathematics) is built on geometry.

This article is focused on one phenomenon – the phenomenon of the horizon – one of the significant elements present in analysis of interviews with students. All the mentioned interviews were conducted as a part of wider research comparing the phylogenetic and ontogenetic development of the concept of infinity in a geometrical context. The interviews were focused on the understanding of concepts such as the point, the line, their parts and their mutual relationships.

## **THEORETICAL BACKGROUND: EPISTEMOLOGICAL OBSTACLES**

In this part, we outline the key thoughts of the theory enabling us to compare ontogenetic and phylogenetic development of mathematical concepts in general. Following, we compare possible attributes of the phenomenon of infinity and suggest possible obstacle of a process of understanding it.

### **Why we can find connection between ontogeny and phylogeny?**

We can find different reasons to accept and reject the idea of 'genetic parallel' (Radford, Boero a Vasco, 2000; Radford, 1997; Rogers, 2000). I have chosen the Brousseau's theory of epistemological obstacle. The theory proved to be the most

suitable for our aim – to find reasonable and non-random connections between ontogenetic and phylogenetic development of the conception of mathematical notion.

G. Brousseau defines *an obstacle* [1] as a set of mistakes related to previous knowledge. These mistakes are not unstable, but on the contrary, they are recurrent and permanent. The obstacle can be

Knowledge (there is a domain where particular knowledge is used fruitfully, this domain is usually investigated, the knowledge is verified by many experiences).

But there is a domain where the knowledge fails and produces wrong results, the knowledge is not able to transfer to another context (because there is a different viewpoint of thought or/and it is thought in a more general context).

Knowledge, which resists contradictions and discrepancies with which it is confronted and hence, does not lead to a creation of ‘better’ knowledge. (It is the difference between an obstacle and a difficulty.) Knowledge presents itself in the same way whenever there is a repetition of such a situation.

Knowledge establishes itself after its integration into a system of cognition. Because there are other notions connecting to original knowledge – obstacle. (Brousseau, 1997)

The guiding idea of the theory is that an obstacle does not represent an absence of knowledge, the knowledge is there but it does not succeed in a particular situation. Knowledge, as an obstacle, is resistant to rejection. It has a tendency to adapt locally, to modify itself with a change as small as possible, and to optimize within a narrower domain. The reason is that an obstacle is knowledge related to a concept, that means a mathematical notion with a set of situations – problems which give to the notion a sense, a set of meanings which an individual is able to connect with the notion, and a set of tools, theorems and algorithms which an individual is able to use in working with the notion. We can classify the three following sources of obstacles: *an ontogenetic source; a didactic source; an epistemological source*. Each of these sources is connected with a different system, which enters into a pedagogical interaction. The most important for us, an epistemological source relates to the process of gaining the knowledge itself. These are obstacles, which we cannot and we should not divest ourselves of, because they are fundamental and essential for a formation of target knowledge. We can meet these obstacles within a history of notions themselves (Radford, 1997; Spagnolo and Čižmár, 2003).

### **Attributes of the phenomenon of infinity**

We are not able to see the infinity itself, in its absolute pureness. We acquaint ourselves with it, understand it, and interpret it by help of such attributes, which appear always when meet it. Already Rodrigo de Arriaga [2] distinguished between the possible manifestations of the phenomenon of infinity: *infinity as far as size is concerned; infinity as far as the number of elements is concerned; and infinity as far as intensity is concerned*. We do not move very far away from his ideas when we consider following attributes: cardinality (of sets), orderliness (discrete ord. and

continuum); limitedness or boundedness; measure (an object of zero-measure); infinite process; and limit, convergence, and supremum/infimum. We do not assume that this list is fully exhaustive. From a different point of view, we can get new important determinant attributes of ‘infinite objects’. For example, we do not mention uncertainty, which was an inseparable phenomenon for the ancient Greek philosophers but unacceptable for European mathematics.

We can consider the horizon as fundamental phenomenon for each of the attribute. Crossing the horizon, rediscovery of the horizon and a hypothesis that the world beyond the horizon is similar to the world in front of it, or on the other hand, expecting fantastic things beyond the horizon, is an impetus to a process of understanding of the infinity – from ‘big’ or ‘very big’, over ‘potentially infinite’, after as much as ‘actually infinite’ – in all of its attributes

The questions in interviews, which are introduced in the next paragraph, were formulated with an aim to cover these attributes and with regard to expected obstacles, which are closely connected to them.

## **INTERVIEW: METHODOLOGY AND RESULTS**

I carried out 22 semi-structures interview with students aged from 9 to 19 years. The aim was to discover how the students understand the concept of infinitely long line, the concept of a point and the concept of an arrangement of points on a line [3].

Key questions of the interview where: (Q1) *We have a straight line and a half line. Which one is the longer one?* (Q1a) *We have two half-lines. Which one is the longer one?* (Q2) *We have given line  $d$  and point  $A$  not lying on the line. Construct square  $ABCD$ , where point  $D$  is on line  $d$  and its area is large as possible.* (Q3) *We have a square  $ABCD$ . Find such point  $X$  on the side  $BC$  that the area of the triangle  $ABX$  is small as possible.* When interviewing the students, I used questions which are not legitimate from mathematical point of view but legitimate from the respondents’ point of view, as they help to create a conflict in the students’ knowledge of infinity. As an illustration, I chose only interesting parts out of the interviews [4] to demonstrate the main ideas:

### **Interview 1: Jan, a good grammar school student, 17 year old boy**

Interviewer: ... which one is longer, a line or a half-line?

Jan: it cannot be determined as the line as well as the half-line are infinite... or they do not have an end ... if we have given, given the infinity, we could say that the half-line is shorter ...

Interviewer: There are two half-lines. Is any of them longer? If yes, which one?

Jan: ... again, if we have infinity, simply, where the infinity is or ends (smiling) the infinity ends, simply, where the infinity is placed, ... However, infinity is not given so it is not possible to determine it ...

Interviewer: [see Q2].

Jan: Well, as the line is indefinite, I could take another line to the line we have, well, line  $AD$  could be parallel with  $d$ , so they would intersect somewhere in infinity, so they would be almost parallel ... Actually, in infinity we could see they are not parallel .. and the square would be infinitely large.

Interviewer: Where would be point  $B$  and  $C$  in this case?

Jan: Points  $B$  and  $C$  would be in an infinite distance from point  $A$ . They would be dependent on point  $D$ , so if I have pint  $D$  somewhere in infinity in this direction, there would be in the same distance point  $B$ ... perpendicular on line  $AD$ . (pointing to it)

Analysing the interview, we can perceive many different and interesting phenomenon important (not only) for his conception of a straight line and parallels. But our aim is the role of horizon, so focus just on the underlined parts of the answers from this point of view. He answers very often “it cannot be determined” or something similar. These answers are common in all interviews where there is a conflict of the concept of actual and potential infinity concerning students’ knowledge. A dynamic using with the horizon is typical for potential approach and the argument of moving is very telling. But we need to break trough and to make a clean break with horizon for truly understanding of actual approach.

Jan also uses formulation “if we have given infinity” or “where ‘the’ infinity is” and we can come across to similar expression in most of the interviews. I understand these formulations in the way that Jan unintentionally assign the role of actual infinity to the horizon. In the concept of potential infinity it is possible to prolong a line in the direction of the horizon, in the concept of actual infinity we have already reach the horizon and it is not possible to go further. The reason is that the way of thinking concerning the potential concept can not be usefully applied in the actual concept. The typical argument of ‘more and more’, or ‘further and further’ fails if we want to understand objects as actually infinite.

Jan’s answer “[parallel lines] intersect somewhere in infinity” and “not until we reach the infinity we could see they are not parallel” or “in infinite distance” express that actual infinity has been replaced by the horizon. The expression “go up to infinity” can be understood as ‘go to the horizon’. Therefore the horizon is the place where infinity is!

### **Interview 2: Martin and Vasek, good grammar school students, 15 year old boys**

Martin: The line is, by me, just a series of such points, the minimal ones, which is not seen and we only magnify it that would be seen.

Vasek: It is just for us.

Interviewer: So, do you thing that the line has some width?

Martin: Some minimal of the one undefined point.

Vasek: I thing that it do not have. In this case it wouldn’t be a straight line.

Students have bigger problems with ‘small’ infinity than ‘big’ infinity. We can see it very often. The small infinity – e.g. a point – is everything before us and still behind

the horizon. Martin works with the minimal measurement corresponding to a point. On his way from ‘a small point’ over ‘a very small point’ to ‘a minimal point’, he shifts his horizon many times. But his point is still on his last horizon. On the other hand, we guess Vasek was able to accept the idea of absolute exceeding the horizon, because he rejects the minimal width of a line. But he answers differently in another place of the interview:

Interviewer: [Q3]

Martin: Tightly over the point  $B$ .

Vasek: Possibly closely to the point  $B$ , but different.

Interviewer: Is there something between the points  $B$  and  $X$ ?

Vasek: I thing, there is nothing.

Vasek: But the point will probably have some measurement, otherwise a set of points could not be a straight line.

Vasek, in fact, works with points as with a minimal atom, similarly to Martin. The reason is that something what does not have any measurement is not able produce an object with nonzero measurement. This hypothesis does not allow him to break the horizon.

### **Interview 3: Lada, a good grammar school student, 15 year-old girl**

Interviewer: [see Q2]

Lada: So, the point could be, maybe, here. (pointing on the end of drawn line on the paper)

Interviewer: But you said the line continues on.

Lada: So, I do not know. Somewhere, just there the line finishes.

The reason why I included this interview is to show that 15 years-old good grammar student is able to work slightly in front of the visible horizon. She is also able to shift her horizon ahead, to some finite distance, however the principle of crossing the horizon is not still handled.

### **Interview 4: Marek, a good primary school student, 10 year old boy**

On the other hand, younger students are able to work with infinite lines or an infinite process:

Interviewer: [see Q1a]

Marek: (5 sec) Not a single one.

Interviewer: Why? (10 sec) One girl said that both of them have the same length. Do you agree with her?

Marek: Yes, I do.

Interviewer: But another girl said that the half-line  $b$  is longer.

Marek: No, the half line  $a$  can finish farther.

Marek’s answers do not differ from Lada’s. But he is able to repeat the imaginary movement – he prolongs a half-line in his mind whenever he needs it and uses it for

speculation about (not yet actual) an infinite line. Thus, using the horizon (even if implicitly) is a fruitful tool for handling infinite objects.

To end this part, we should mention attributes, which occurred in discussing the situations. Questions about lines (Q1 and Q2) focus on the attribute of unlimitedness or unboundedness; we can see unsurprisingly the attribute of measure and infinite process; but also the attribute of supreme. The Question Q3 regarding the distribution of points on a line refers to following attributes: orderliness, measure, infinite process, and supreme.

In the following part, we try to find connection with ideas from historical development of conception of infinity.

### **LINKS WITH HISTORICAL DEVELOPMENT: COMPARISON OF INTERVIEWS ANALYSIS**

The aim of this paper is not to analyse deeply the historical development. We merely mention some important ideas or events, which could be useful for our intended comparison with students' answers. We outline very briefly the main approaches to geometrical phenomenon. Then, we try to point out some ideas, where we can see how mathematicians coped with the problems of infinity focusing on the horizon. We refer to the works of Petr Vopěnka, a significant Czech mathematician, historian and philosopher of mathematics, who profoundly studies problems of infinity [5].

Geometrical object can be seen (in simple) from two viewpoints. If we understand them in the potential concept then there exist only those we are able to make (for example to draw them or to imagine them). The number of these objects is finite even though it is always possible to make another one. This concept has its own consequences. For instance, a line is a unique object, not a set of points. On the contrary, if we understand geometrical object from the point of view of actual infinity all such objects are existing and existed before. They had existed before we started to work with them (Vopěnka, 2004a). Both concepts are possible. Let's explain that Euclid approaches the geometrical object from the potential point of view. This fact can be proved by his postulates formulated as tasks (Eukleidés). On the other hand, Hilbert armed with modern logic and the set theory approaches geometrical objects from the actual point of view seeing the space as if it is filled with all the existing objects. "The principle of creator" is the original approach to the geometry for all students. It could bring obstacle of picture – as a model, for example. Overcoming the principle connects with the horizon again. The horizon is tightly linked with creating person. So, leaving the principle of creator connects with breaking the horizon [6]. Even the translator of Bernard Bolzano (1781–1848) noticed that Bolzano's function cannot exist, because it is not possible to draw it (Vopěnka, 2004b).

The ancient mathematicians were afraid of stepping over the horizon. They dared to do it only in case they needed to ensure themselves about something which was in front of the horizon. Illustration of such insight beyond the horizon is can be for

example Euclid's fifth postulate. Why did so many people want to 'prove' the fifth postulate? The problem is that some truth about our world in front of horizon is based on something behind the horizon (Vopěnka, 2000).

Euclid of Alexandria (about 450-370 B.C.) was very careful as regards infinity. In his work *Elements* (*Stoicheia*) accepts the theory of Aristotle [7]. In definition XIV of the first book says that all shapes are limited [8]. It is essential for us to see his concept of a line (*eutheia*), our concept is represented by any part of a line (straight line). This limited line (*eutheia peperasmenes*) can be, according to postulate II, „prolong without a bound... to produce a limited straight line in a straight line“ [9], he regards such a line as *eutheia grammé*, and if necessary the line can be prolonged *ep apeiron*. We are reminded of the way how Lada and Marek worked with the line. Lada explicitly worked only with limited objects. Marek spoke about arbitrary prolongation, but still works with *eutheia*, not with a straight line in contemporary meaning.

The introduced problem of the conflict between potential and actual infinity continued until the beginning of 20<sup>th</sup> century, when the set theory was formulated. We encounter attempts to overcome horizon in an incorrect way during the whole history of mathematics. For instance, H. Schumacher used the circle in an infinite diameter to 'prove' the fifth axiom in his letter to Gauss (Vopěnka, 2004b).

Zeno of Elea (about 490–430 B.C.) in his paradoxes presents in a mastery way the conflict between sensual perception and abstraction of thoughts and idealisation. In his arguments „ad absurdum“, he comes to a conclusion that movement is impossible (Dichotomy, Achilles and the Tortoise, The Arrow, The Stadium). Zeno comes to the paradoxical connection of the world of pure, unchangeable ancient mathematics and infinity which steps over the horizon in its own essence. Thanks to the works of Zeno and his teacher Parmenides the Greek mathematics started to be attracted by the problem of infinity (Trlifajová, 2001). We can see the conflict between the potential and actual approach in the interviews, too. Clearly, it is evident in Vasek's answers (Interview 2), but also in the Jan's interview (1). It is typical common obstacle to understanding of actual infinity, as it is described in many studies, e.g. (Eisenmann 2002, Jirotková 1997, Jirotková and Littler, 2003, Monaghan, 2001, Tall, 2001).

The problem explicitly formulated by Zeno was not satisfactorily solved till Aristotle of Stagira (384–322 B.C.). Aristotle excluded actual infinity from the pure ancient science and left the space only for potential infinity. In his conception, a line is arbitrarily long, it means infinite in a possibility.

Our conception of limitlessness does not remove the ideas of mathematics saying that it denies that the enlargement of a line would be unlimited in real as it cannot be brought to the very end; mathematicians do not need limitlessness in real and do not use it, they are satisfied with the fact that an unlimited line is arbitrarily long; ... For their proofs it is indifferent how it is with existing sizes. [Aristotelés, book III, chapter 7, p. 89]

Before Aristotle, Democritus of Abdera (460–360 B.C.) set out a different way of solution to Zeno’s problem. He formulated well known theory of atoms (Patočka, 1996). The analogous way of thinking is evident in many interviews, markedly e.g. in the interview with Martin and Vasek. From young children till university students, all the students tend to stop on ‘their atom’. The concept could be ‘a 1mm–point’, ‘a very, very small point’, or ‘a minimal point’, but always the horizon plays important role – my point is as small as I am able to shift my horizon. We encounter similar answers about numbers and number line. For example, students of every age answer that the smallest positive (real, rational) number is ‘0.1’, ‘0.00...01’ or ‘ $1/\infty$ ’. We can again observe shifting the horizon, stopping on it, and replacing objects for infinite ones on it. To add, René Descartes (1596–1650) used the idea of neighboring points in his physics, though it was he, who created the number representation of geometrical objects. From this point of view, remark that identifying the number line and a straight line is a nontrivial, highly abstract process.

## CONCLUSIONS

As was already mentioned, the obstacle is a knowledge, which can produce incorrect answers when is used in a new context. This knowledge must be familiar to its owner and must be many times successfully used. These requirements are met by the concept of the horizon as a tool for understanding infinity.

As we can see from the interviews, the horizon is something which can help significantly to clarify infinity – enables us to understand it better. The answers of the students such as “go to infinity” can be interpreted as ‘go to the horizon’. Therefore the horizon presents more palpable position for infinity, as it seems to be less abstract – ‘the horizon is the place where infinity is’. The horizon is a tool we use to understand actual infinity. However, it is an obstacle for a full grasp of actual infinity with all its consequences. For this reason we can conclude that the horizon plays a role of an epistemological obstacle in Brousseau’s concept of the transition from potential to actual infinity.

The last note is devoted to the relationship of these hypotheses with the learning process. One of the essential thoughts and ideas for the theory of obstacles is that they should not be avoided but broken through. It requires teachers to be conscious of these obstacles and be prepared for them. In this case, it means they should expect incorrect answers (which are quite common) as well as elaborate and organise the situations built on carefully chosen problems, which will challenge the student's previous conceptions of (potential) infinity and allow to overcome other and other horizons. A possible starting point could be the mentioned interview questions.

## NOTES

1. The idea of an epistemological obstacle was firstly used by G. Bachelard in his work *La formation de l'esprit scientifique* (Bachelard, 1993).
2. The Spanish Jesuit and professor of Prague University; lived from 1592 to 1667.
- 3., See (Krátká, 2005; Krátká, in press) for more details.
4. The interviews were video-recorded. All the answers were accompanied by pictures.
5. Relevant works are (Vopěnka, 1989, 2000, 2004a, 2004b). Unfortunately just one book was translated to English: P. Vopěnka, *Mathematics in the alternative set theory*, Teubner (1979).
6. The horizon itself is unchangeable. If we set off toward the horizon, it moves in the same direction and stays unreachable. Today's geometry is fully embedded in the classic world of geometry and not in the natural world of geometry. It means that all geometrical objects are placed on the horizon and are regarded as absolutely infinite (Vopěnka, 1989).
7. We can see a strong influence of the philosophy of Plato, for example the definition of the point, but it is not true in case of infinity.
8. Definition XIII: "Limit is something which is a border of something." Definition XIV: "Shape is something which is limited by borders." It is also confirmed by principle VIII: "The whole is greater than the part." (Eukleidés)
9. Translation (Artmann, 2001).

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# GEOMETRICAL RIGIDITY AND THE USE OF DRAGGING IN A DYNAMIC GEOMETRY ENVIRONMENT

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***Abstract.** Dynamic Geometry offers the opportunity to approach Geometry with a dynamic handling of geometric objects; this allows possibilities not available so far for high school students. However, some cognitive phenomena are still present, such as geometric rigidity, and the fact of preferring some geometric properties which are visually evident over some others. These phenomena are influenced by the geometric object perception which, in turn, is influenced by the dynamic feature of software, particularly the dragging operation. To go deeper on this direction, we carried out a research with secondary school students in Mexico; we used the Theory of Figural Concepts in order to study these phenomena on a Dynamic Geometry environment.*

## INTRODUCTION

The presence of *Dynamic Geometry Software* (DGS) in Mathematics Education field has motivated the creation of teaching proposals of Geometry using it, but it also rises the necessity of research about its teaching consequences in order to prepare the teacher for a suitable using as semiotic mediator between knowledge and student (Vygotski, 1979).

We know, according to scientific literature about it, that DGS allows the design of useful learning environments as experimentation fields of geometric objects representations, but the user (i.e., the student) also has to realize the software main features, like the dynamic aspect of constructions, and the teacher must take into account the possible difficulties which appear during its use in a Geometry course.

Therefore, in this paper some research results and remarks are reported. The research was made with 14-15 years old students in Mexico (see Larios, 2005). Here I will focus only on geometrical rigidity -which happens when individual cannot mentally manage a geometrical figure when its orientation is not standard or cannot imagine the result of one transformation-, and the influence of dragging as part of a semiotic mediator.

With this in mind, I posed the following questions: *Which phenomena regarding visualization arise when geometric facts are watched and when justifications are constructed in dynamic geometry environment? In particular, which is the influence of “geometrical rigidity” phenomenon over the identification of figures and geometrical properties? and, which is the influence of the perception of the main feature of DGS called ‘dragging’?*

According to these ideas, we should have some references about aspects concerning geometrical objects and their representations. This I will do in the next section.

## FIGURAL AND CONCEPTUAL ASPECTS OF GEOMETRICAL OBJECTS

In research, when considering geometrical objects handling we have to consider its nature and how students conceive them. According to this idea, we can explain students' behaviour based on the fact that geometrical objects have a *figural* component and a *conceptual* one, according to the 'Theory of Figural Concepts' of Fischbein (1993). Both components are closely related, and they force us to distinguish between *figures* and *drawing* (Parzysz, 1988; Laborde & Capponi, 1994; Hölzl, 1995; Goldenberg & Cuoco, 1998; Maracci, 2001). This distinction "is strongly emphasised by programs like Cabri" (Hölzl, 1995, p. 118).

*Drawing* is a graphical, material representation which refers to a *geometrical object*, which in fact has its own theoretical reference and it is restricted or "controlled" by definitions and logical restrictions. Drawings correspond to the *figural* aspect of geometrical objects according to Fischbein (1993) and contain information that sometimes is not necessary, because perhaps it includes colour, thickness, or orientation. Now then, as we cannot access to geometrical objects directly, we represent them by drawings and we assign them *meanings*, which are the relations between objects and its representations assigned by an individual.

This meanings correspond to *figural concepts* (in Fischbein's theory) or to *figures* since they are considered as "the agent of an objects' class which shares the set of geometrical properties the figure was built with" (Sánchez, 2003, p. 31).

However, both aspects (figural and conceptual) influence an individual according to his or her cognitive development. Indeed, it is even necessary a fusion between both kinds of aspects to be successful in the appropriate management of geometrical objects; this situation seems ideal and extreme (Maracci, 2001). Fischbein says: "What happens is that conceptual and figural properties remain under the influence of the respective systems, the conceptual and figural ones" (1993, p. 150).

Nevertheless, it seems that students look for that fusion in several ways when they try to build *satisfactory* drawings, that is the drawings must have a good gestalt that convince students about their correctness (Maracci, 2001). But this satisfaction is not always related to logical or conceptual restrictions, since some conditions are more related to figural restrictions (orientation, shape, etc.).

On the part of DGS, its dynamic feature creates the need of evidencing the difference between *drawing* and *figure*, because in this environment the geometrical constructions are built according to logical relationships between objects, not according to figural aspect. This is because *dragging*, the operation that allows the user to directly manipulate the objects in screen, is a tool student can reach and use. With dragging he or she can modify possible configurations of one geometrical construction, but ever preserving the geometrical properties the objects was created with. In this way DGS is converted in a semiotic mediator that influences the Geometry's perception.

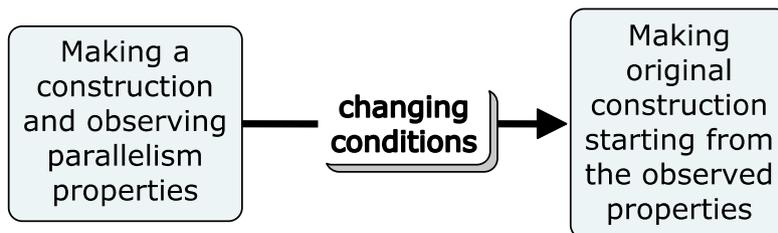
In general, this skill of direct manipulation over geometrical constructions might allow students to begin to differentiate between *drawing* and *figure*, which is related with individual's skill to "see" beyond graphical representation and acquire a higher level of abstraction (Hoyles & Jones, 1998, p.124). However, the user has to realize about this difference, as well as the dependence between geometrical objects in a construction, to exploit the dynamic feature of software in an effective way.

In this sense, Cabri-Géomètre may be turned into a environment that allows exploration in Geometry, but it also may result in new situations in teaching, as well as in the research field of Mathematics Education.

## METHODOLOGY

The research process was carried out with teams of two 14 and 15 years old students, in a suburban town near to Querétaro city, Mexico. The teaching experiment lasted two weeks.

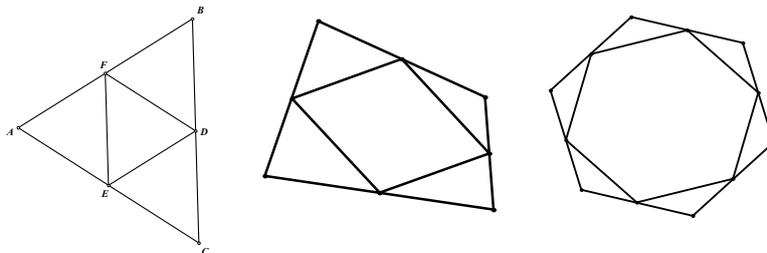
Students made twelve activities grouped in three sets: five with triangles, five with quadrilaterals, and two with hexagons. In all these the tasks were designed according to:



**Figure 1:**

That is, at the beginning of teaching experiment's each part students were asked to make geometrical construction in order to observe parallelism between sides of constructed figures. Later on the conditions were changed since students were asked to make inverse constructions: using the observed properties they had to reconstruct the original figure.

The activities were based on Acuña (n.d.) using the *middle points polygons* [1]. We just used triangles, quadrilaterals, and hexagons, with their respective *middle points polygons* (see Figure 2 for some examples).



**Figure 2: Triangle, quadrilateral, and hexagon with their middle points polygon**

During this process, students had to measure their constructions, to make an exploration through dragging, to observe geometrical properties, and to provide arguments in order to justify their observations. These observations must regard to the geometrical relations between original figure and its *middle points polygon*. Students also must provide conjectures, which allow students to detect necessary, sufficient conditions for the geometrical constructions.

Triangles' activities were:

- T1. Students are asked to write down their conceptions about triangles in order to know about it.
- T2. Students construct one triangle and its *middle points triangle* through middle points in triangle's sides, as well as observe parallelism between sides of both triangles. Students used only the software.
- T3. Students construct a *middle points triangle* through a reciprocal situation than above. Observation of parallelism is emphasized.
- T4. Students are asked to propose a procedure to construct the original triangle starting from its *middle points triangle*.
- T5. Students perform their procedure with a scalene triangle provided by researcher in a cabri file. They check it and provide a logical justification. If the construction (and proposed procedure) fails the researcher encourage students to figure out the mistakes and look for suitable properties.

Quadrilateral's activities were:

- C1. Students construct one quadrilateral, its *middle points quadrilateral*, and observe the properties. We wanted to students realize the kind of quadrilateral the middle points quadrilateral is. It is a parallelogram.
- C2. Students were asked to explore relations between the quadrilateral's diagonals and the middle points quadrilateral. They might justify the parallelism with observations made in T2 activity.
- C2a y C3. Students were asked to figure out the properties of a quadrilateral in order to its *middle points quadrilateral* be a rectangle. Properties observed before have to be taken in count.
- C4. Students were asked to propose one procedure to build a quadrilateral starting from its *middle points quadrilateral*. One cabri file with a parallelogram was provided to them by researcher.

Hexagons' activities were:

- H1. One regular hexagon was provided to students in a cabri file. They had to observe parallelism in opposite sides using the diagonal between them, and extending adjacent sides. Students had to relate this activity with properties observed in the second and third activities with triangles (T2 y T3).

H2. Starting with one regular hexagon provided in a cabri file, students were asked to construct its *middle points hexagon* and to observe properties in order to determinate whether the last one is a regular hexagon too.

In each task students were asked to observe and justify using geometrical properties observed in former activities.

The software was a window on students thinking (Noss & Hoyles, 1996) and we used it (through cabri files), as well as protocols (worksheets), and some talks recorded to get information of students' work.

## RESULTS AND DISCUSSION

We identified students' teams thanks to their names, so we got 22 protocols' sets and cabri files. Next we shall show some examples of students' responses and their analysis.

### On preference of geometrical properties

During the activities students were asked to observe some properties about parallelism. However, it happened that students didn't consider them. It seems students "see" properties closer than their direct experience.

By example, in the third activity of triangles we asked them:

Question: a) How are sides BC and EF?

Nine teams gave as a reference that segments are equal, six gave references regarding the sizes, and two gave references taking in count the shape. 80% of the references regard to the shape or the sides' size, and only 20% says something about parallelism.

In the next question, we asked to students to justify their descriptions. Just one team uses parallelism:

Question: a) How are sides BC and EF?

Answer: They are two equal lines.

Question: Why?

Answer: Because they are parallel.

In the next activity (T4) students were asked to propose a procedure to get triangle *ABC* starting from its *middle points triangle*. They should take in count parallelism, but six teams used the sides' sizes, others six said something about the location of vertex, four said something about the necessity to locate the middle points, and three gave references to shape (Catalina & Patricia said: "we put points *d e f* and as the little one is pointed down we joint vertex, then we put the points for the bigger triangle and we got three big triangles and one little"). In other words, 77% of references are related to movement or visual properties, meanwhile only five percent mentions parallelism.

In the fifth triangles' activity (*T5*) we asked to students:

Question: How do remain the sides of triangle *ABC* regarding those of triangle *DEF*?

The majority gave references of sides' size and just three of them said something about parallelism. Some teams mentioned the change of position, or the sides are "symmetrical", or the sides "are going in opposites directions". All of these responses are related to visual aspects.

According to this idea, we might say that parallelism is not a relevant, evident property for students. It seems that for students are more evident other properties like length (in segments) and shape (in polygons). It appears that students prefer properties based in figural aspects rather than conceptual ones.

### Geometrical rigidity and constructions' orientation

Another aspect we studied was the "geometrical rigidity". This phenomenon is related with visualization of geometrical figures, and it is strongly influenced by orientation of graphical representations because is very common individual cannot mentally manage a geometrical figure when its orientation is not standard or cannot imagine the result of one transformation. Next there are some examples.

Bibiana & Mariana's team made direct references to geometrical objects' shape when were asked about the properties of *middle points quadrilateral*:

Question: *g*) The inner quadrilateral has a special property?

Answer: Yes. Even although the big one is **bent** [2], the little one still have two equal sides.

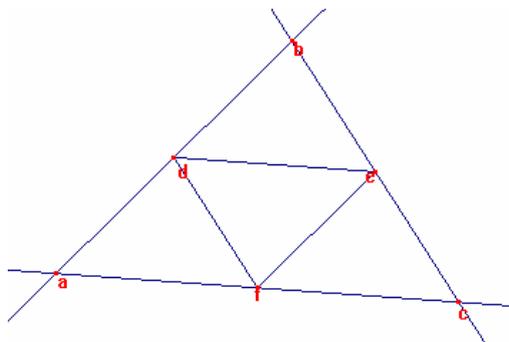
In the cabri files is evident that they prefer to use constructions with standard position and shape. In the third triangles' activity (*T3*) they even anticipated the final position of triangle *ABC* and built a "suitable" triangle *DEF* (see figure below):

Question: *b*) About the little triangle, it has have some special property?

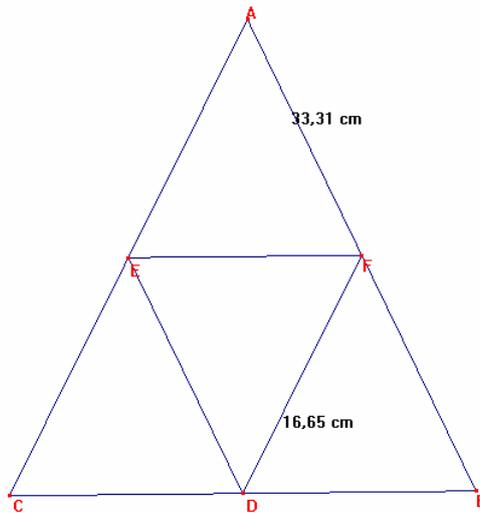
Answer: Yes Because it has to have the sides of the same length.

Question: *c*) Can you use any triangle as the *middle points triangle*?

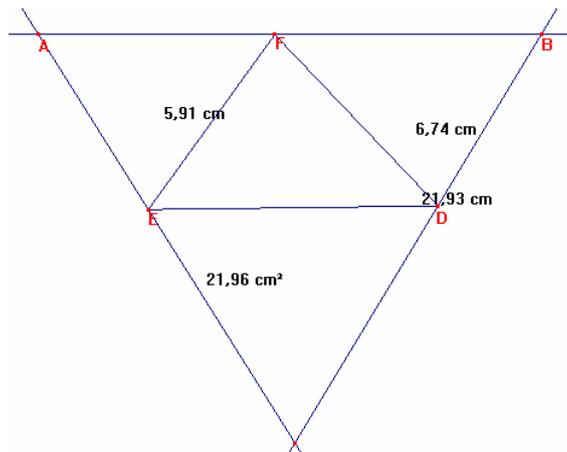
Answer: Yes.



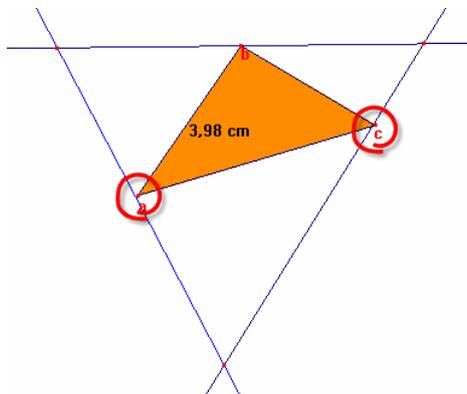
On its own, Celene & Marissa’s team also showed the same tendency in using figures with standard position and shape. In the second activity of triangles they used one isosceles triangle pointing up with a horizontal side:



Even if it is necessary, they modified the provided construction. In the fifth triangles’ activity they changed the shape of the provided triangle in order to get one more “comfortable” triangle (one horizontal side and almost an isosceles one):



In the same activity Luis & Fernando made a cabri file whose picture is below:



You may note that points *a*, and *c* are not over the sides of the biggest triangle. Despite little triangle’s shape, the big one’s shape clearly looks like an isosceles

triangle with one horizontal side. Furthermore, we note that students switched the vertexes' labels since those of the little triangle should be  $D$ ,  $E$ , and  $F$ , but now they are  $a$ ,  $b$ , and  $c$ , while the biggest triangle's vertexes have not labels.

## FINAL REMARKS

As I said above, the cognitive phenomena related to figural aspects in students' responses are geometrical rigidity, using of empirical justifications, and using dragging as an external tool.

Geometrical rigidity is a phenomena related to visualization of geometrical figures. It happens when individual is not able to mentally manage geometrical figures when they are not in standard positions or to imagine when a figure is translated or deformed (Larios, 2005). This phenomenon showed up at different times with students on this research.

This phenomenon may be related to conflicts existing between figural and conceptual aspects of geometrical objects, as well as the need of satisfactory drawings (Maracci, 2001). Indeed, some students need to use drawings in standard positions, otherwise they cannot "see" the figures, their relationships, and their movements. By example, Celene & Marissa's team changed the shape of the provided triangle in activity  $T5$  and made it isosceles with its base in horizontal position. In this way, they recovered a known, rigid, and "right" figure to be managed.

Therefore the figure orientation is considered by students as an important figure attribute, despite of the fact that it is not shown in objects' definitions.

On other hand, sometimes when some students used the *dragging*, arose an inability to visualize the whole process of transformation of figure. Those students couldn't visualize intermediate steps and consider them as particulars cases. I observed that students perceived something I called *start-end dragging* (Larios, 2005) and Olivero (2003) called *photo-dragging*. Students just considered two cases: the starting and the ending constructions. We think this phenomenon is another form of geometrical rigidity (Larios, 2005), because that one reported in scientific literature is in regard to inability imaging the figures' movement, but here we talk about the inability to "see" as a figure itself the intermediate steps during the transformation.

In general we perceive the preponderance of figural aspects and therefore the observation of geometrical properties is affected. Indeed, using figures in standard positions and shapes limit the possibilities during exploration and, therefore, the number of observed properties is quite less. When students use this kind of configurations in their constructions they miss the opportunity to observe some necessary conditions and, therefore, some properties.

In our view, these phenomena show us that figural concepts have not been appropriated by students, since figural aspect is not used as heuristic resource but as referential one, while the conceptual aspect is restricted in its performance because

students don't see the necessity of using geometrical properties. This means that the fusion between both aspects cannot happen.

Other phenomenon, although linked with the above-mentioned, is the fact of considering some properties and ignoring others. We mentioned this above, at the beginning of previous section. We noted that there are properties more evident to students, and they "deserve" to be considered, while others don't. In this sense, some observed properties or situations about size of geometrical objects, or about their shape, or the concurrence of curves, seem to have more possibilities to be taken in count than other properties less evident as, in this case, parallelism.

## NOTES

1. The *middle points polygon* of one polygon is obtained when the sides' middle points of the original polygon are joined by segments.
2. Emphasis added.

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# THE UTILISATION OF VIDEO ENRICHED MICROWORLDS BASED ON DYNAMIC GEOMETRY ENVIRONMENTS

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**Abstract.** *Dynamic Geometry Environments (DGEs) are more and more used in today's classrooms. Their potentials to teach and learn Geometry were evaluated by many researchers. On the other hand more and more students have internet access. So they have many opportunities for learning provided over the world wide web. For our studies we developed and evaluated a learning environment that includes geometrical microworlds that are based on a DGE. These microworlds were enriched with screen recording videos to support students in their individual learning process. Up to now we could observe 32 students working with the environment. Our findings indicate that there are potentials for learning success in the student's individual and self-directed learning process.*

## INTRODUCTION

In this paper we report about experiences we made with the assignment of interactive worksheets to learn Geometry. These worksheets are web-based (i.e. available everywhere and anytime) and with a high degree of interactivity. The essential parts of them can easily be generated with the Dynamic Geometry Environment *Cinderella* (Richter-Gebert and Kortenkamp, 1999). The worksheets are mathematical microworlds that provide different kinds of multimedia assistance to students, namely hints in text form, construction videos and other aids. So learners are able to work individually and self-directed on construction tasks.

Our study took place with 8<sup>th</sup> grade students. Our findings expose different strategies and patterns of computational activity developed and used by students who worked on geometrical tasks. A future goal will be the development of learning sequences (Arzarello, 2002) setting up on a learning environment that is described in this paper. As Boon (2006) argues: "For development researchers the challenge lies in the design of convincing learning trajectories that integrate these new tools."

## THEORETICAL FRAMEWORK

The substantial elements of our research are video enriched, interactive worksheets. In short an interactive worksheet is a webpage including a Java applet that offers a geometrical task, for instance a construction task or a manipulative task. So it combines a number of features and potentials of DGEs with some basic ideas of mathematical microworlds. Hence our theoretical framework relies on these two major ideas. It is partly based on the ideas of microworlds as described by Hoyles & Noss (1992), Hoyles & Noss (2002) and Boon (2006) as well as on substantial results of research on Dynamic Geometry Environments (e.g. Laborde et al., 2006, Jones, 2002) and their use in the classroom (e.g. Gawlick, 2002).

As we are particularly interested in individual and self-directed learning we will need to have a look at the role of feedback. If there is no teacher supervising the learning process the feedback will be in electronic form and has to be seen under aspects of interactivity. We will see that the role of feedback for students is essential.

### **The Microworld idea.**

What is a microworld? Hoyles and Noss (2002) describe it as follows.

Thus microworlds are environments where people can explore and learn from what they receive back from the computer in return for their exploration. It follows, therefore, that a microworld has its own set of tools and operations that are open for inspection and change.

So microworlds can be very open constructs with a variety of possibilities for its users. But they can also be more close or instructive, leading learners on clear, given paths. What the learner receives ‘from the computer’ is at first any kind of feedback. Learners are expected to learn from this feedback (Hoyles & Noss, 2002). This feedback is often in a graphical or iconic form: “while the user is manipulating objects directly, there is also graphical feedback as to the results of their actions” (*ibid.*). Of course other kinds of feedback are thinkable.

Microworlds differ not only depending on the kind of feedback, but also on the kind of task . So Boon (2006) categorises three kinds of microworlds, because “this classification helps to structure the underlying design choices”. The third category is called “Applets that offer a mathematical microworld. In these applets mathematical objects like formulas, equations and graphs can be constructed and transformed.” So it “contains applets that work on formal mathematical objects, etc. In this sense they are comparable with mathematical tools like the graphic or symbolic calculator or CAS.” But these ideas can be assigned for DGEs as well, e.g. in the form of interactive worksheets.

### **Construction tasks in Dynamic Geometry Environments.**

DGEs in their ‘natural’ form are very open tools. They “offer ways of teaching and learning Geometry, which are not available in a traditional paper-pencil-environment” (Strässer, 2006). Because of its variety of features and potentials research on DGEs concentrates on a number of different aspects. For us the following are of importance. First there is the drag mode. You have to consider if learners used this mode they need it for adjusting in construction tasks (Laborde, et al., 2006). Laborde, Kynigos, Hollebrands and Strässer (2006) report on the solution of construction tasks in DGEs. They stress that a “construction by eye or by manual adjustment fails” in a DGE, since it must be preserved by the drag mode. This adjustment describes a typical behaviour of students if working with a DGE. A problem that arises from this is the need for feedback. To see their failure students must receive adequate feedback.

When Jones (2002) summarises research on the use of Dynamic Geometry Software (DGS), he also discusses the importance of the feedback. He argues research “suggests that DGS cannot provide a self-contained environment, but that other activities are needed for students to make progress in mathematics”. Jones (2002) suggests that the teacher plays an “important role in guiding students to theoretical thinking”. With these aspects in mind we developed a learning environment for individual, self-directed learning. This means at first we had to ask, if students cannot learn individually with a DGE at all. Or the other way around: What can they learn on their own? And how can they learn it? Furthermore we have to evaluate what other activities are needed and if these activities can be provided by a web-based, interactive learning environment. A special point is the role of the teacher. What role does he take and can this role be occupied by an “electronic teacher”? In particular this means that the feedback, which is as we have seen essential for learners, can be given by a computational environment. Another aspect emphasised by Jones (2002) is the fact that it matters *how* DGEs are used. As consequence following questions arose for us: How *is* the software used? Where are problems and how to face them? And can these problems be eliminated by an environment that is not as complex as a DGE and which provides more aids? So we created different kinds of assistance, worked with an instructional design and always observed the interaction between learner and environment, too.

### **The Role of Feedback and Interaction.**

As we have emphasised, the role of feedback is essential for the learners in electronic learning environments. DGEs provide graphical feedback indeed by direct manipulation but it is no didactical feedback. This has to be given by experts (e.g. teachers). Laborde, Kynigos, Hollebrands and Strässer (2006) stress the role of feedback, such as the potential of feedback in the learning process and on the impact on learning. The feedback in electronic environments is closely linked to interactivity or to the interaction between user and environment. As the term of interactivity is understood in different ways, we reference to Schulmeister (2002). He classifies multimedia elements in his “Taxonomy of the Interactivity of Multimedia” according to the degree of interaction between a user and a multimedia component. According to this taxonomy interactive worksheets are on the highest level (Mann et al. 2004). We took this into account for the development of the interactive worksheets and for the analyse of our data.

### **DESIGN OF THE LEARNING ENVIRONMENT**

Our research focus is on self-directed, individual learning. For individual learning inside or outside the classroom with computational learning environments students at first need appropriate tasks. Further it is necessary for them to get assistance for the solution of their tasks. And as mentioned before feedback is of importance. These aspects in mind we developed interactive, web-based worksheets.

### **Interactive Worksheets: Integrating a DGE in a mathematical microworld.**

The Dynamic Geometry Environment *Cinderella* in its handling does not differ significantly from other representatives for DGEs such as *Cabri (Cabri II plus)*, *DynaGeo* or *Sketchpad*. With *Cinderella* it is possible to export interactive, web-based worksheets very fast, easily and comfortably by One-Klick-Export. This way two different types of interactive worksheets can be created. The first one is the *interactive webpage*, the second type the *interactive exercise*. An *interactive webpage* in this case is a webpage including a Dynamic Geometry applet which is created and designed by the developer, e.g. a teacher or a didactician. Interactive webpages allow direct manipulation, so the learner can use the drag mode and experiment with ready-made geometrical constructions (manipulative tasks). With an *interactive exercise* the learner can do a lot more. The developer of the exercise can provide construction tools, so that the user can construct geometrical constructions on his own. The developer decides, which of the DGEs construction tools are available for the student working on the task. Moreover hints can be created and provided in textform or in the form of insertion of construction elements. In the process of development of an interactive worksheet certain design and development criteria have to be considered. In addition general criteria for the design of websites (e.g. see <http://psychology.wichita.edu/>) and special mathematical aspects (e.g. the tool selection) have to be taken into account. Furthermore didactic criteria must be fulfilled in the creation of an assistance.

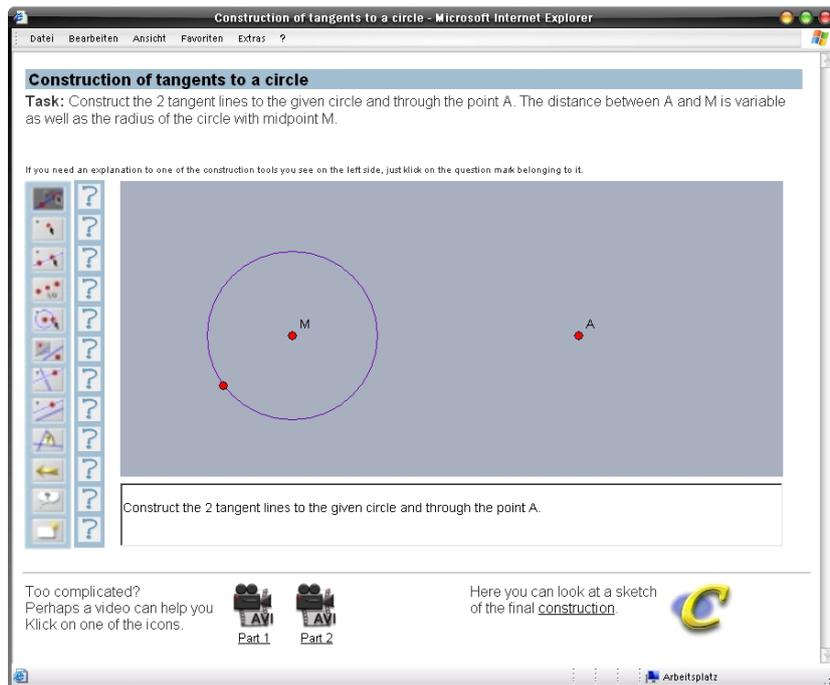
The advantages of interactive, web-based worksheets are obvious. Learners do not need to be familiar with all the different tools, features and icons of a ‘traditional’ DGE. They just need to know how to use the tools needed for their task. In case that they are not familiar with those tools we integrated different kinds of assistance. Finally every interactive, web-based worksheet includes a Java applet, support for the use of the construction tools and different kinds of assistance for the solution of a construction task, namely textual hints, construction elements, a manipulative construction and a video recording of the solution process. To sum up the worksheets are produced with a DGE and include the potentials of DGEs. They are mathematical microworlds in the form of applets including graphical and symbolic feedback. And as they are web-based they can be “distributed over the world wide web, which makes them accessible quite easily” (Boon, 2006).

### **Video enriched interactive worksheets.**

Interactive worksheets so far can relieve of some of the students’ problems with the use of DGEs and the corresponding tasks. But some problems remain. How do students have to use special tools, which are unfamiliar to them? And how can they solve construction tasks without an expert’s help, i.e. in the form of individual, self-directed learning. We suggest that for this individual work assistance is essential. Therefore we offer different kinds of so-called construction assistance. As construction assistance a whole set of multimedia components is conceivable. In our learning environment the following kinds of construction assistance are available:

- Help with the construction tools: name, description and video description.
- Hints in text form, colouring and display (insertion) of construction elements.
- (Dynamic) sketch of the construction.
- A “dynamic constructional description” in form of videos.

All kinds of assistance are available anytime (see illustration 1). Sometimes the user has to wait for a defined period, before he may call it. The videos can be viewed with a common (Windows) media player. So all features of this software can be used, namely Play, Stop, Pause and Seek.



**Illustration 1: Translated example of an Interactive Worksheet**

## METHOD

Our methodology is partly influenced by the approach of Arzarello (2006) as well as by experiences with log file analyses (e.g. Priemer, 2004 and Degenhard, 2001). Furthermore we use a classical experimental design as described e.g. by Kerlinger (1964) consisting of pre-test, treatment, post-test and delayed post-test. In this paper we report of the first phase of the empirical study. This first phase has already taken place. Goals of the second phase which is currently in progress will be discussed in the conclusion.

## Overview

32 students participated in the first phase. They all were students of the 8<sup>th</sup> grade of a German secondary school and between 13 and 14 years of age. All students participated voluntarily. The study was divided into two parts hence our results are obtained by two groups. The first group consisted of 13 students. They were given

five tasks concerning the Theorem of Thales. Students first had to construct a right triangle, then a square, a kite, a rectangle and the tangents to a circle.

An example for one of the tasks was as follows:

Given the diagonal AC. Construct a square ABCD by using the Theorem of Thales. The distance AC is variable.

No student had ever worked with Geometry software before, but all of them had some experiences with ICT (Information and Communication Technologies). So every student had the possibility to use a PC at home, eleven of them had internet access at home. Ten of them had used a media player before (to watch videos). This means that they knew the basic functions of a media player. Only three female students had never used a media player before. But they did not have any technical problems with the use of video assistance.

The second group included 19 students. They also were given tasks around the Theorem of Thales like the construction of a right triangle, the construction of the midpoint and the construction of the tangent lines to a circle. All students of this group had the possibility to use a PC at home, 17 of them had internet access at home and 17 of them had used a media player before (to watch videos). In this group 17 students had worked with Geometry software at school formerly.

### **Data Collection and Analysis.**

We wanted the students to work through a sequence of specially designed geometrical construction tasks provided in the form of interactive worksheets (see illustration 1) in our learning environment. When working with the environment the students were observed and recorded in the following ways:

- Their face and parts of their torso were recorded by a Webcam.
- A directional microphone recorded their statements (when thinking loudly or in communication with a partner).
- By means of screen recording software the screen and all mouse activities were recorded.

Similar to a log file analysis (Priemer, 2004, Degenhardt, 2001) the collected data could be synchronised and analysed with a special software. A questionnaire, the pre-test, the post-test and the delayed post-test provided information about the mathematical knowledge before and after the treatment. And it provided information about the students' experiences with digital media. In that way we wanted to identify a possible increase of mathematical knowledge and if ICT competences influence it (cp. Lagrange, 2003). In the first phase we obtained 26 sets of data out of the 32 students. 20 of them learned individually, 12 in pairs. Each set of data consisted of a video recording (Webcam and screen recording, cp. illustration 2), log file data and the tests. The lengths of the recordings reach from about 25 up to 50 minutes.

Konstruktion eines rechtwinkligen Dreiecks

**Aufgabe:** Konstruiere ein rechtwinkliges Dreieck mit der Hypotenuse AB. Der Abstand zwischen den Punkten A und B ist frei wählbar.

**Hinweis:** Erstelle dir eine Planfigur auf Papier, wenn du meinst, sie könnte dir helfen.

Wiss du eine Erklärung zu einem der Konstruktionswerkzeuge auf der linken Seite benötigst, dann klicke auf das Fragezeichen daneben.

**Construction of a Right Triangle**

Konstruiere ein rechtwinkliges Dreieck mit der Hypotenuse AB.

Zu kompliziert? Vielleicht hilft dir ein Video. Klicke auf eines der Bilder.

Du kannst dir hier auch eine fertige Konstruktion ansehen.

Applet de cinderella. OndyApplet started

**Illustration 2: A student working with an interactive worksheet.**

## OBSERVATIONS ON STUDENTS' ACTIVITIES

Three striking results will be discussed in the following: firstly the intuitive utilisation of the learning environment and its integrated videos, secondly different patterns or strategies of use and finally some test results.

### Intuitive use of instructional designs and the utilization of videos

A striking observation was the way students intuitively used our learning environment, especially the interactive worksheet and the given assistance. The entrance in the environment in most cases took place very effortless and impartial. The students sat down in front of their notebooks and started working immediately. Also the usage of the videos took place very smoothly. Many students used the pause- and play-buttons when watching a video clip for the first time. Afterwards they used this function for several times. We assume that with the design of our learning environment an introduction to the usage of a microworld like this is no longer necessary.

In the questionnaire filled out after the intervention we asked about self-assessment of the assistance received by the video of a construction. In the first group all participants declared, that they have used the assistance "video of a construction". And 12 out of the 13 students estimated it as helpful for them, only one person rated it as partially helpful. In the second group, where we received 16 questionnaires, for 12 of them the video assistance was seen as helpful. Three students rated it as

partially helpful, only one as not helpful for him. Thus we argue that students accepted the videos and the assistance given by them very well.

### **Utilization patterns**

In the first group we analysed the student's work with the interactive worksheets and in particular with the assistance given with a focus on patterns of use. We were able to observe and describe seven different patterns or strategies developed by the 16 students. Some worked with different strategies depending on the task they had to solve. Surprisingly there were some special patterns, which were developed independently by different students. For example the pattern we called "video strategy" was used systematically by four students and sporadic by all other students as well. A systematic implementation of the so called "sketch-strategy" we could discover by three other students.

For example the "video strategy" proceeds as follows. First the student tries to get an overview over the interactive worksheet. In doing so he moves his mouse over striking images and notices if something happens. Then he tries to construct the required figure. When a problem with a tool appears, he calls the explanation in text form and skims through the text. If this does not help, he watches a video which describes the tool or reads the text more concentrated. When he has difficulties with the construction, he starts the video description for the construction, watches it for some seconds and presses the pause-button. Then he imitates the viewed steps of the construction on his worksheet, goes back to the video, watches the next step and so on. Instead of using a video the students pursuing the "sketch strategy" made use of the dynamic sketch of the construction [1].

But there were also completely different strategies, which for example made more use of the assistance in text form. It could also be observed that students revised or adapted their strategies in their working process.

### **Results of the delayed post-test**

The video strategy and the sketch strategy are founded on instructional ideas. Since most students used one of these strategies it was interesting for us to see how effective the strategies were. I.e. we were interested in their learning success. We wanted to know what the students had learned, what they could reproduce and what they could remember after four weeks. The delayed post-test could give us some answers. Between treatment and delayed post-test both groups did not learn anything in their regular classes about the mathematical contents covered by the worksheets. The result was that quite a remarkable number could solve the construction tasks.

The most interesting cases were the students that could not solve the construction task in the pre-test but in the post-test and the delayed post-test. I.e. they could at least reproduce the construction. It was striking that there were some students who could not solve the tasks in the pre-test but both in the post-test and the delayed post-test and who only used the video strategy. That is one reason why we assume that our

instructional design can be successful under certain conditions. To uncover these conditions in detail will be one of our main future aims.

## CONCLUDING REMARKS

With the second phase of our study we pursue this and other goals. In this phase the students will work with an optimised version of the learning environment. Another goal is to identify the dependencies between learning success and the kind of assistance that is used by the learner. After identifying that nearly all students were developing their individual utilisation patterns another interesting topic is how students react, if they fail with a self-developed strategy although this has been successful before. And we will have an eye on the relationship between the effectiveness of any assistance and the complexity of a task.

If we are able to uncover the important variables we will develop learning sequences (or learning trajectories) for geometrical contents based on our findings. These sequences will be based on the described microworlds. And they will implement the best suited kinds of assistance.

## NOTES

1. No student really used the drag mode with such a construction', which was only seen as a static one. Otherwise using the drag mode was neither necessary nor requested for solving the task.

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# GEOMETRICAL TILES AS A TOOLS FOR REVEALING STRUCTURES [1]

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*Abstract. In School Year 2005/2006, we extended, in Italy, a part of a research initiated by Ewa Swoboda in Poland. Our research used simple tools but early results indicate that certain geometric aspects may depend on factors such as competency, psychology and gender. In this paper we present a preliminary analysis of the protocols.*

## **THEORETICAL FRAMEWORK.**

This work can be considered in between spatial and geometrical understanding. We ask the pupils to perform a (free) drawing, with the aim of investigating their spatial abilities by working with direct manipulation. Nevertheless we investigate the protocols from geometrical point of view: connexions, continuity, isometries and so on, which can be settled in Geometry 1 paradigm, following Houdement & Kuzniak (1999). The starting point of the work is an interesting research carried out by Swoboda. She puts forward an interpretation of children's protocols, based on van Hiele's theory (van Hiele, 1986), a theory studying geometrical thinking and understanding. This theory divides the educational processes in geometry into different levels. In particular, the analysis of three initial levels shows a very important common aspect: manipulation. Van Hiele also distinguishes between rigid or feeble structures. Regarding the last point, the Polish researcher notices:

“In his opinion feeble structures are worth noticing, they fill out the majority of our everyday life. They come from a non-verbal, intuitional way of thinking, but mathematical thinking is not superior to the intuitional one. Feeble structures may be a beginning of knowledge on a higher level of thinking where we may have something to do with, ex. rigid structures or still a feeble one” (Swoboda, 2005b).

This intuitional way of thinking can be considered spatial rather than geometrical (Panoura *et al.*, 2007; Pittalis *et al.*, 2007). Moreover Swoboda shows that feeble structures can be used to study and to analyse an activity based on the creation of a floor. Feeble structures are very important for educational research in order to detect the child's thought. They can reveal the process of early geometric knowledge appropriation. They are expressions of spatial intuitions that cannot be expressed by word, but only by graphical language, but they are the first steps in geometric understanding (Bishop, A., 1980, 1983). Feeble structures are characterized by presence of connections, rotations, parallel translations, symmetries, applied only locally in the drawing. Rigid structures reveal the presence of a mental project using both geometric shapes or isometries or the sake of regularity. Pupil could pass through feeble structures to rigid ones by awareness of the 'regularity' and of

isometric transformations, using sight. Swoboda (2006) shows that there is a relation between mastery of rigid structures and school success.

We give to pupils four kinds of tiles as a sort of alphabet for a language, which can be considered the first step of a future expression by words. Therefore, following Vygotsky (Vygotskji, 1992), we helped the coming into existence of geometrical concepts.

We ask pupils to pave an A4 paper sheet with these tiles with the aim of constructing a ‘floor’. In this way we arrange a milieu (Perrin-Glorian & Hersant, 2003), from which we can explore, geometrical aspects and children’s ability and potentiality relating with spatial and geometrical thinking:

“... activities such as these described here give the opportunity to bring out many intuitions that can be treated as a basis for developing not very simple geometrical notions” (Swoboda, 2005b).

This sort of activity is customary in school for introducing the concept of area, nevertheless Rozek & Urbanska (1998) has shown children have different levels of awareness of ‘horizontal’ and ‘vertical’ organisation.

The tiles we use have inherent symmetry; therefore they can suggest constructing regular tessellations, based on symmetries. Following Nùñez *et al.* (1999), the metaphor of balance:

“is so basic that all homo sapiens – no matter when and where they live on earth – have experienced it [...] this experience [the balance], it’s working out in cultural expressions such as language, art, dance, science”.

This embodied cognition must affect the pupils’ products, thence there must be a relevant presence of symmetries in their protocols.

Moreover the act of constructing a tessellation requires a long sequence of elementary acts: observation, ordering, copying, and repeating. Swoboda (2005a) shows that drawing a pattern is not a mere perceptual copying, but it is a deep thinking process which involves body and gestures (Marchini & Vighi, 2005).

“Of the domains of knowledge where children must enter, geometry is the one needing the fullest cognitive activity, as it requires gesture, language and looking. It requires the child to construct, to reason and to see, each activity indissoluble from the others.” (Duval, 2005).

Arzarello (2004) emphasises the role of the body movements and gestures in learning. Gesture expressiveness can be considered a sort of language useful to understand pupils’ thoughts taking in account of the poor language competencies of children of these ages.

For other aspects of the theoretical framework of this research we refer to (Swoboda 2005b) and the references therein.

## THE RESEARCH.

Our research uses exactly the same setting as (Swoboda 2005b), in order to compare Polish and Italian results. The tiles are proposed by Kuřina (1995):

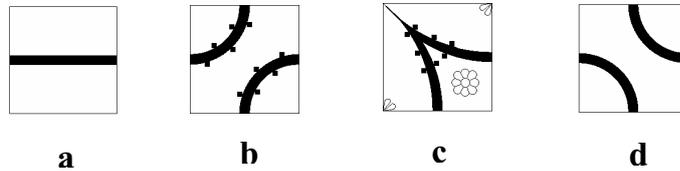


Fig. 1: Kuřina (1995) tiles.

Pupils suggested name for tiles: 1.a: *straight*, 1.b: *branch* 1.c: *flowery*, and 1.d: *swallows*; the names, in themselves, reveal a naturalistic interpretation, not a geometric one. The same happened with many titles children gave to protocols. Remark static (1.a – 1.c) and dynamic interpretations of tiles (1.d: *swallows* as trajectories of the birds). The task is to pave a floor, but the pupils tended to see it as an opportunity for self expression. They produced gardens or flowers (girls) and streets or racing tracks (boys).

The first author requested permission and obtained it to conduct the experiment in the Kindergartens and Primary Schools of his home town, Viadana (MN) in the province of Mantova, Northern Italy [2]. Viadana is a small town with agriculture and artisan industry. There are many immigrants from other Italian regions and from abroad, but there was no statistically significant difference between the protocols of Italian and non-Italian children, nevertheless, some specific protocols seem to be influenced by the familiar culture of these foreign pupils. In this research were involved 212 pupils (97 - last year Kindergarten, 68 - first year Primary School, 47 - second year Primary School) worked singly in classroom environment.

The first phase of the research consists of manipulation activities. The task is: “Create from these tiles as beautiful floor as possible” (Swoboda, 2005b).

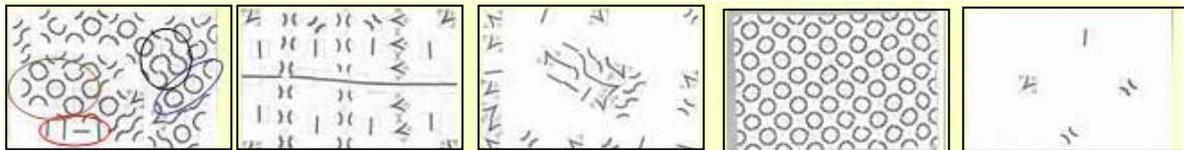
We can discuss this requirement, since it seems too ambiguous, but it allows the children express themselves in a good way. In other words, the children are completely free to choose which, how, where, how many tiles, and how many times, in order to obtain the most beautiful flooring that they can, so that individual artistic taste and choice of design are what determines the choice of tile. Looking for their intuitions, we are interested in some geometrical order; other more precise statement of the task would be more difficult to be interpreted by these young pupils.

The task offers an approach very different from the customary one in the school, the one introducing ‘standard’ geometric figures. Our request can be presented very early and it is motivating and well accepted also by kindergarten pupils; the time that pupils spent in constructing their works supports this conclusion. Moreover it offers an occasion for free explorations of the space (the blank sheet of paper). The geometric tile structure limits degrees of freedom, but, as protocols shown, children were free enough in expressing their own intuitions. Furthermore the mind activity

required to construct and to colour the drawing is, in our opinion, a suitable, right task of spatial - geometric activity necessary to prepare the next more formal treatment of geometry.

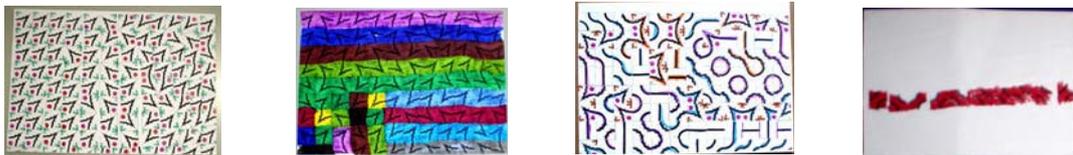
The ‘floor’ consists of an A4 blank sheet (21 cm × 29.7 cm); it must be paved gluing tiles on. Sides of the square tiles measure 2.5 cm; therefore they do not fit the sides of the paper. The children have not scissors, and therefore they have to face problems regarding their conception of space (for an investigation of this issue using different tools, see Marchini, 2004) in filling the space as actual tiles do with an actual floor. The problem was worsened because our 14,700 tiles were slightly irregular having been photocopied and cut up by hand.

The whole experimental activity took one school year (2005-2006). The activity of each child was video-recorded, in order to allow a deeper of deep thought processes manifested by body and gestures. Figure 2 presents some examples of protocols.



**Fig. 2** a b c d e

The second phase is centred on the use of colour: each protocol was photocopied, the pupil coloured his/her protocol and gave it a title. The introduction of colour and title for protocols is the main distinction in the methodology of the research respect to original Swoboda one (Swoboda, 2005b). In this way we get each protocol in black and white and in colour. We show here some examples:



**Fig. 3.** a b c d

The colour could afford new information about pupil’s aims. The colour can be also a kind of language, therefore we asked children to colour the ‘paving’ tessellation, as this might reveal the criteria the learners base their design on. The colour and the title given to the protocol, take place of semantics for the black and white drawing.

### THE PROTOCOLS.

Choosing and gluing each tile carefully it is possible to ‘save’ 4 mm and to cover exactly 27.5 cm of the ‘long’ side of the sheet. Another way is to place tiles not contiguously, leaving small regular gaps between them. In this way, the number of tiles that can be glued onto the sheet without going over the edges is 88. We consider this number 88 as the theoretical covering index (in the following it is assumed as 1, by normalization). We find only 11 pupils using exactly this numbers of tiles. On the other hand, 32 children choose to extend the paving beyond the sheet edges and use

96 tiles (constructing an  $8 \times 12$  floor, they cover a hypothetical sheet of  $20 \text{ cm} \times 30 \text{ cm}$ ). We obtain 119 protocols using less than 88 tiles, 58 protocols (more than 27.35%) which used from 88 to 96 tiles and 35 protocols using more than 96 tiles.

By a rough analysis, protocols we obtained are of the same types of Polish pupils' products. We obtained works showing the presence of feeble structures (fig. 2.a) [3], some others are revealing the presence of rigid structures (figg. 2.b - 2.e), as in Poland. In Polish experience, protocols where tiles are placed in one 'horizontal' row, in the middle of the sheet, approximately (fig. 3.d), were absent.

## THE QUANTITATIVE ANALYSIS OF RESULTS.

We recorded the number of tiles of each kind, in each protocol. We also introduced a diversity index, borrowed from biological research. This kind of protocols investigation is absent in Polish research.

The first type of analysis is obtained by counting the number and type of tile each child used. It is remarkable that it yields interesting information.

*Covering index.* It is possible that the number of tiles the children glue on the sheet is determined by their attention span, by their manual coordination, by their commitment to the task, and by their interest in producing their own design, and also it can be related with age and teacher's practice. Since the number of tiles is simple to calculate it can be used as a rough indicator of all these aspects.

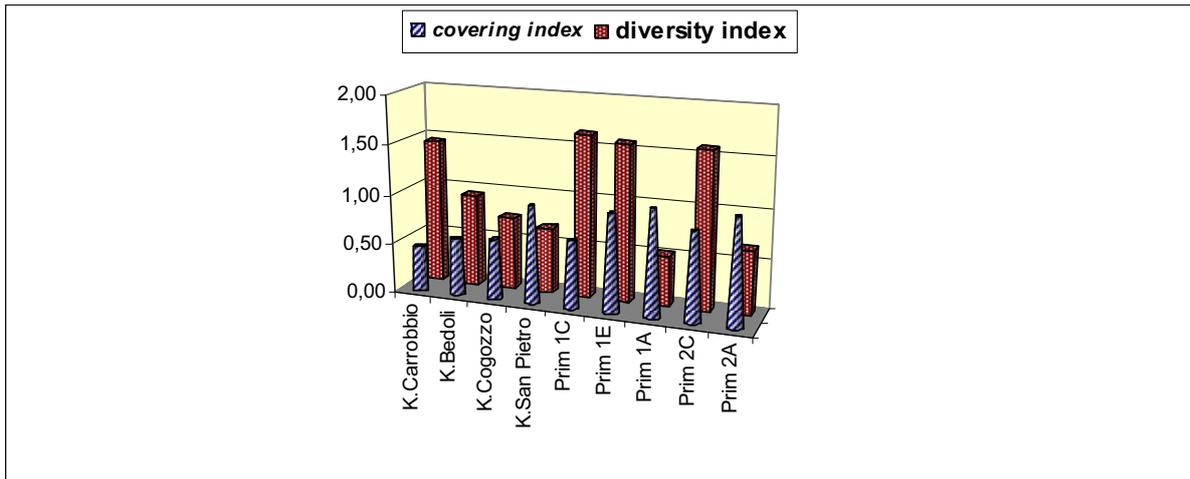
Table 1 shows the average covering index values. It is clear that the average covering index increases with the years of schooling. The presence of high scores for particular classes could be explained by pupils' possible independent experience of a similar activity or by different teaching practices.

School	no. pupils	Sample	Male	Female
Kindergarten	97	53.5	47.9	59.8
1 <sup>st</sup> grade Prim.	68	77.1	79.1	74.0
2 <sup>nd</sup> grade Prim.	47	86.5	87.2	85.4

**Table 1: Average Covering Index**

In more details, in Bedoli and San Pietro Kindergartens, in 1E and 2A of Primary School, the males' average covering index is greater than the females' one, in the remaining classes it is conversely. The following figure 4 shows the average covering index for each class, together with diversity index.

*Diversity index.* It is the Shannon's entropy measure. It varies between 0 (every tile in the protocol of the same kind) and 2 (equal number of tiles of each kind in the protocol). In Biology, the good ecological 'health' of an environment is measured by a diversity index near to 2. In our work, we can consider the high diversity index as a measure of the great teacher's respect for the propensities of pupils.



**Figure 4: Average Covering and Diversity Indexes for classes.**

*Average use of tiles.* Table 2 shows the number of tiles children used in paving the ‘floor’. This Table is of immediate interest in that it shows a clear difference between males and females, which is evident at Kindergarten and Primary School, both. In a sense the tiles have a gender connotation. The way girls and boys use the flowery tiles is particularly striking. The ‘monopolization’ of the flowery tiles by girls lowers the diversity index. These facts are thence connected.

	<i>Flowery</i> rate of use	<i>Branch</i> rate of use	<i>Straight</i> rate of use	<i>Swallows</i> rate of use	Total n. of tiles
Sample Males	15.77	16.18	14.12	<b>21.91</b>	8,294
Sample Fem.	<b>36.21</b>	10.12	13.23	12.06	6,446
K. Males	18.17	17.27	17.15	<b>29.82</b>	5,851
K. Females	<b>46.25</b>	12.66	13.50	11.61	3,697
1 <sup>st</sup> grade Males	14.69	14.36	12.55	<b>37.50</b>	3,332
1 <sup>st</sup> grade Fem.	<b>35.88</b>	5.27	16.46	16.35	1,923
2 <sup>nd</sup> grade Mal.	23.21	21.48	<b>23.83</b>	18.69	2,529
2 <sup>nd</sup> grade Fem.	<b>61.22</b>	9.33	8.78	6.06	1,537

**Table 2: Rates of tile use.**

We think that a quantitative investigation of this kind can give some interesting information to teachers. It seems us a tool which is enough simple to apply. Other more sophisticated analyses, e.g. recognition of feeble and rigid structures or the interpretation of gesture, require more competencies.

**QUALITATIVE ANALYSYS OF THE PROTOCOLS.**

Firstly, protocols reveal different concepts of space. It can be intra-figural or inter-figural (fig. 2.b, 2.e). The space can be limited (fig. 2.c, 2.e) or unlimited (fig. 2.d) and

“Greek thought ...tried to escape from the unlimited, considered a form of imperfection. For Aristotle there is no space above the sky of fixed stars” (Speranza, 1997).

This is the same conception which leads some learners in our activity not to stick on tiles which would go over the edge of the sheet of paper. Other learners conceive of space as unlimited, and have thus an ‘in act’ conception of infinity (Marchini, 2004).

Moreover colour gives us much more information whether the pupil’s attention is on local features (fig. 3.a) or it concerns the drawing as a whole (fig. 3.b).

The second intuition is connected with important concepts of area and plane. An actual floor is designed by covering a plane space without gaps and without superimposing tiles. A child who leaves “space” between the sides of the tiles does not still have the idea of covering. Usually there is a privileged reference frame, namely, the edges of the sheet, which makes ‘horizontal’ and ‘vertical’ array more likely, but some protocols use only ‘horizontal’ lines, others only ‘vertical’ ones. They lack the idea of bi-dimensional distribution and the concept of array that underlies multiplication, according to (Rozek & Urbanska, 1998).

Using square tiles with drawings which make creating some “whole” or patterns possible, children can feel a need to arrange tiles one close to another, sometimes also in row-column order (fig. 3.a).

Thirdly, the regular size of the tiles and the drawing on them (fig. 1) can focus the children’s attention to the connection in order to construct continuous patterns. The colouring of drawings confirms that continuity is present in pupils’ mind even if, from the real gluing of tiles, the connection is not complete (fig. 3.c).

Fourthly, on the basis of our experience, we cannot confirm the hypothesis of Nùñez *et al.* (1999). The embodied cognition of ‘balance metaphor’, did not affect the pupils’ products, since there was not a relevant presence of symmetries. We suggest that symmetry is only a learning object.

An analysis of protocols allows making an initial classification based on the criteria used by the pupils in the construction of the floor. We distinguish the following kinds of criteria:

0) *random*: pupils glue the tiles as they pick them up at random, without observing the drawing on them;

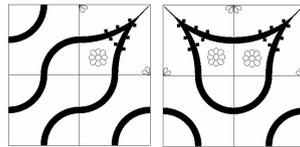
1) *taking account of the drawing on the tile*: on the *straight* tile the line is parallel to the edge; so the children tend glue the tile with the line ‘horizontal’ or ‘vertical’. Other tiles do not have a preferred direction, although tiles tend to be placed with their sides parallel to the edges of the paper as far as this is possible. There are few cases in which *swallows* and *branch* tiles are placed with tile diagonal parallels to sheet edges, and in these case the ‘non-intersecting’ diagonal is ‘vertical’ (fig. 2.b);

2) *influenced by and based on neighbouring tiles*: construction of a route, translation, symmetry; construction of a flower in the case of *flowery*;

- 3) *regular*: an iterative and regular tessellation (fig. 2.d and left side of fig. 3.a);
- 4) *progressive conquest of regularity*: initially the pupils glue tiles at random and subsequently choose regular tessellation;
- 5) *project*: pupils first ‘see’ a mental representation, and then proceed to the concrete manipulation (fig. 2.d and 2.e).

In case 2) and 4) there are feeble structures, generally; in case 3) and 5) the structure is generally rigid, when the project is realized in a complete way or the regularity is observed in all the work. Sometimes pupils are unable to concretise their mental spatial image obtaining feeble structures (fig. 2.a; 3.c; 3.d).

The drawings on the tiles are such that when the same type of tiles is placed next to one another, the lines fit together perfectly. This feature leads pupils to imagine concrete things drawing. For example, the swallows or straight tiles might give the idea of a road; the branch might give the idea of a scene from nature such as a thorny lawn, the *flowery* tile a garden. But pupils’ imagination is even more fertile than this; they ‘see’ for example a chick and a cat in the following arrangements of tiles.



**Figure 5: Chick and cat.**

Geometry of tiles also influenced the children’s work. Their drawings are of three kinds from point of view of inherited symmetries:

- 1) *straight*: two reflections with orthogonal axis, the medians and consequently a central symmetry;
- 2) *branch* and *swallows*: two reflections with orthogonal axis, the diagonals and consequently a central symmetry;
- 3) *flowery*: one reflection on a diagonal line.

The *flowery* tile allows for different ‘interpretations’. The tiles can be glued so that the wider lines connect, or to compose whole flowers instead, without connection of wider lines. The arrangements with more symmetries followed fit both, see Appendix. The protocols show all these arrangements, often locally, only; we think that it could be a result obtained by chance, but it can be a good occasion given to teacher.

It may be the case that it is more mature pupils who create a greater number of basic motives (Budden, 1972). Some of the older Primary School children showed the presence of strong structures and pupils there tended to opt for the flowery tile. Girls are naturally more mature than boys and this may be the reason they chose the flowery tile (Arnheim, 2005). There was however a considerable increase in the use of the flowery tile in the second grade.

## NOTES

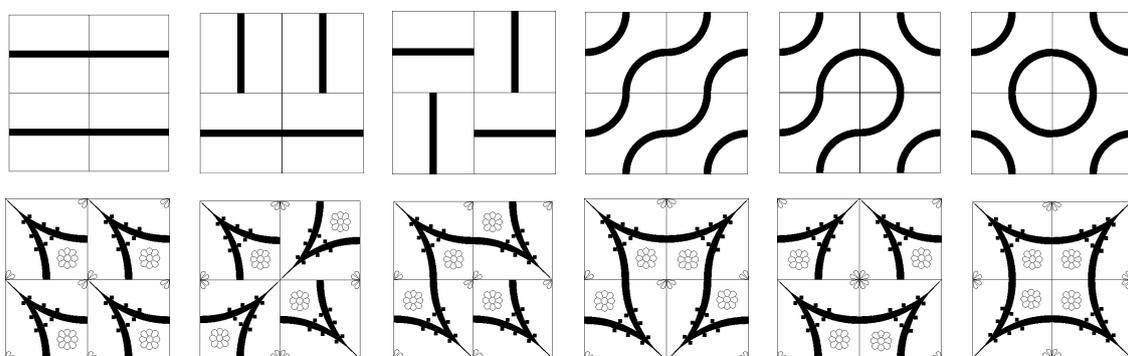
1. Work done in the sphere of Local Research Unit into Mathematics Education, Parma University, Italy.
2. We would like thank the School Heads and the teachers of Bedoli, Carrobbio, Cogozzo, and San Pietro Kindergartens, and Classes 1A, 1C, 1E, 2A and 2C of Primary School, for permitting and collaborating with our work.
3. We superimpose some oval on the reproduction of original protocol in Fig. 2.a, in order to draw attention to feeble structures.

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APPENDIX



# **THE PROCESS OF COMPOSITION AND DECOMPOSITION OF GEOMETRIC FIGURES WITHIN THE FRAME OF DYNAMIC TRANSFORMATIONS**

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*In this paper, we investigate children's strategies while they decompose and compose geometrical shapes and we study these strategies within the frame of dynamic transformations. Twelve children, in the context of a clinical interview, faced six tasks that involved the above processes and which varied in terms of the kind of the decomposed shape and the elements of the transformation that were given. The results indicate that children developed different strategies and built certain relations. By reflecting on the process of transformation (decomposition – composition) the children acted intentionally and considered the shapes as geometrical objects.*

## **INTRODUCTION**

The process of composing and decomposing quantities is considered as the basis for constructing mathematical concepts. For example, the concepts of number and area are approached by defining the unit, iterating it, and creating composite units (Steffe and Cobb, 1988). Moreover, the composition and decomposition of shape is related to the development of visual reasoning, an important ability in problem solving (Ferrari, 1992). In geometry, there are research examples that focus on children's ability to compose and decompose geometrical figures that lead to a hierarchical description of children's strategies during this process (Clements, Sarama & Wilson, 2001).

In our study, we consider these two processes as types of dynamic transformations of geometrical shapes. This frame allows us to see these two processes as interrelated, as one is the inverse function of the other, and consequently focus on their effect on children's understanding of geometrical shape.

## **THEORETICAL FRAMEWORK**

A dynamic transformation of a geometrical shape is defined as the process in which the shape changes its form through the variation of some of its elements and the conservation of others. This seems to be related to the "concept of invariance" which promotes intuitive reasoning (Otte, 1997). Different types of dynamic transformations have been described in our previous work, where the transformation is concrete (eg. Markopoulos and Potari, 2000) or mental (Markopoulos and Potari, 2005). In these transformations the focus was on the dynamic manipulation of geometrical solids, a process where "the solid changes its form through the variation of some of its elements and the conservation of others" (Markopoulos & Potari, 2000). These

contexts supported children's transition from physical to visual and finally to mental actions.

In this study, the process of decomposing a shape by cutting it and dividing it into two parts and the rearrangement and composition of these parts defines a transformation. The process of this transformation is a function where an initial shape corresponds to another one of the same area. An infinite number of transformations can be defined. For example, if the initial shape is an isosceles triangle and it is divided by the height into two equal right-angled triangles, then there is an infinite number of rearrangements that can be performed and an infinite number of potential shapes. The transformation involves three elements: an initial shape, the process of transformation and the produced shape. A number of tasks can be developed by varying which of those elements are given. For example, the initial shape and the produced can be given while the process has to be defined or the initial shape and the process can be given while examples of produced shapes are required. In this context, the composition and decomposition of shapes are interrelated as they are parts of the same transformation that provides students with the opportunity to experience a variety of forms realize the concept of geometrical shape, its properties and relations.

Duval (1995) distinguishes four hierarchical levels in the way that children look at a geometrical figure: perceptual, sequential, discursive and operative apprehension. Composing and decomposing a geometrical figure is characterized by Duval as the mereologic way of modifying a given figure. This operation constitutes a specific figural processing which provides figures with a heuristic function and is an expression of operative apprehension. To conceive figures dynamically with this heuristic function supports problem solving in geometry. However, to create such flexible images is not an automatic process and requires children's involvement in situations which encourage such dynamic transformations of the figure. Examples of research in this area are the work of Clements, Sarama & Wilson (2001) and Carter and Ferruci (2003). The first example offers a hypothetical learning trajectory consisting of six levels of thinking in the domain of composing geometric figures. The second example is a survey of paper cutting and folding in mathematics textbooks. In this study, we focus on the processes of cutting a shape and reassembling its parts to produce other shapes. In particular, we investigate children's strategies during these two processes; the reasoning they developed while comparing the initial and the produced shapes and the emerged relations between the different elements of the transformation.

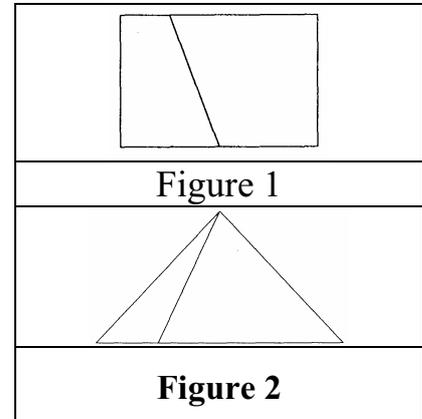
## **METHODOLOGY**

### **The participants**

The methodology adopted in this study is the *clinical interview* (Hunting, 1997). Twelve children participated in the main research process, six of the 4<sup>th</sup> grade and six of the 6<sup>th</sup> grade of a primary school in Patras. The choice of these twelve children was

based on their responses in written geometrical tasks. In this pre- test phase of the research, 31 children participated, 18 children of the 4<sup>th</sup> grade and 13 children of the 6<sup>th</sup> grade. The children worked in pairs in a 30-minute session. Each pair was given two different figures (figure 1 and figure 2) and asked to write a description of each figure which would be given to one of their classmates.

The description should be as accurate as possible, so that their classmate could reconstruct each figure based only on their description. From the analysis of their responses, we identified three groups of children in each classroom according to the way they referred to the properties of each figure. The first group involved the children that saw the shapes on a holistic level. Children of the second group could identify some of the properties of the shape while the third group of children could relate the shape to its properties. For the main research we chose one pair of children of each of the three categories for each classroom. So, we formed a group of three pairs of children from the 4<sup>th</sup> grade and three pairs from the 6<sup>th</sup> grade accordingly.



### The process

In the main research process children had to face six tasks involving the process of transformation. As we mentioned above the transformation involves three elements: the initial shape, the process of transformation and the produced shape. The six tasks varied in the form of the initial shape and the produced one as well as in the kind of appropriate cutting and assembling processes. Children were interviewed in pairs for about one hour.

### The tasks

Task 1: Children were given a rectangle and asked to cut it and reassemble the dissected parts in order to construct as many different shapes as possible. The children were asked to compare each of the produced shapes with the initial one. In this task only the initial shape was given. The mathematical concepts that are implicit in this task are the conservation of area and the concept of polygon.

Task 2: The difference of this task from the previous was that the line section was given and it was the diagonal of the rectangle. Two elements of the transformation were given: the initial shape and the line section. The transformation is realised under certain conditions. Possible mathematical issues that can emerge from this task are the equality of the two dissecting areas, the conservation of the area, and the equality of the sides.

Task 3: In this task, the produced shape was given, a triangle, while the initial was again the rectangle. Children had to anticipate the result of the cutting – assembling process in order to construct the triangle. In particular, they had to construct a mental

image of the produced shape and mentally relate the cutting and the assembling process with the initial shape and the produced one. In terms of mathematics the children had to consider the conservation of area and the equality of sides and angles.

Task 4: This task is the same as task 1 but in this case the initial shape was an unfamiliar shape, a concave quadrilateral.

Task 5: The difference of this task from task 4 was that the line section was given.

Task 6: In the last task children were given an isosceles triangle and its height as the line section. They were asked to imagine the assembling process and produce as many different shapes as they could. They had to draw the produced shapes and make comparisons between each of the produced shapes and the initial one. In this task children actually had to anticipate the effect of the transformation process and perform the whole process mentally. Here, the children had to consider the properties of the isosceles triangle and through the comparisons to consider certain relations between the properties of the two shapes. Moreover, they had to realise the difference between the equality of areas and the equality of shapes.

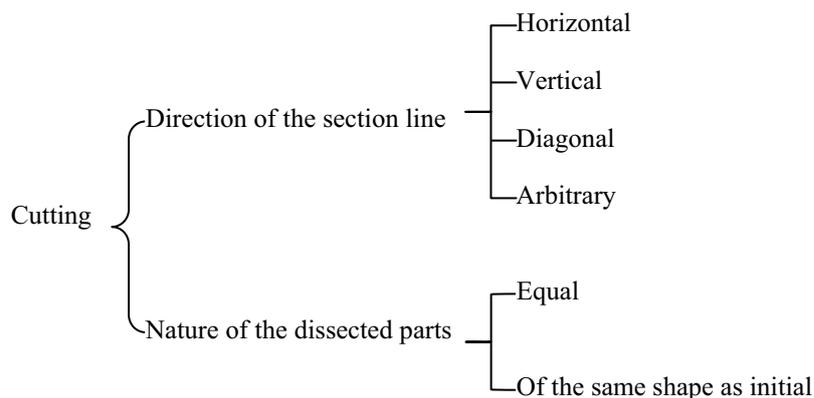
### Analysis of the data

The data consists of the six transcribed video recordings and written responses to the six tasks. Two levels of analysis were implemented. In the first one we tried to chart the strategies that children followed through the cutting and assembling processes as well as their strategies for comparing. In the second level we attempted to identify relations between the different elements of transformations that children built during the experiment.

## RESULTS

### The process of transformation

#### Cutting

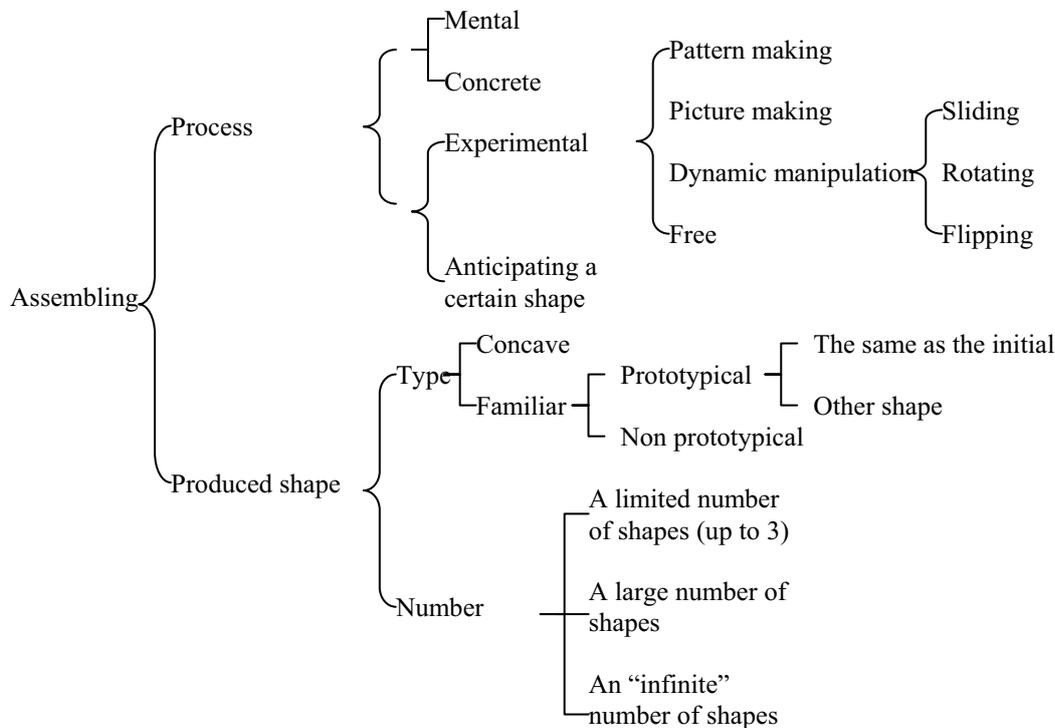


**Figure 3: The process of dissecting the initial shape**

The cutting process that children followed throughout the tasks could be represented using the systemic network in figure 3. In particular, children’s strategies of cutting the initial shape varied in the direction of the section line and in the nature of the dissected parts. The direction of the line was horizontal, vertical, diagonal or arbitrary. By horizontal or vertical we mean the relation of the section line to the orientation of the initial shape. The diagonal direction of the section line was the diagonal of the rectangle. Most children tended to cut the initial shape in order to keep its symmetry.

Assembling

Children’s strategies while assembling the dissected parts could be characterized by the process and/or the produced shape (Figure 4). In particular, the process of assembling was either mental or concrete and could be characterized either as experimental and/or as a process that involved the anticipation of a certain produced shape.



**Figure 4: The process of assembling the produced shape**

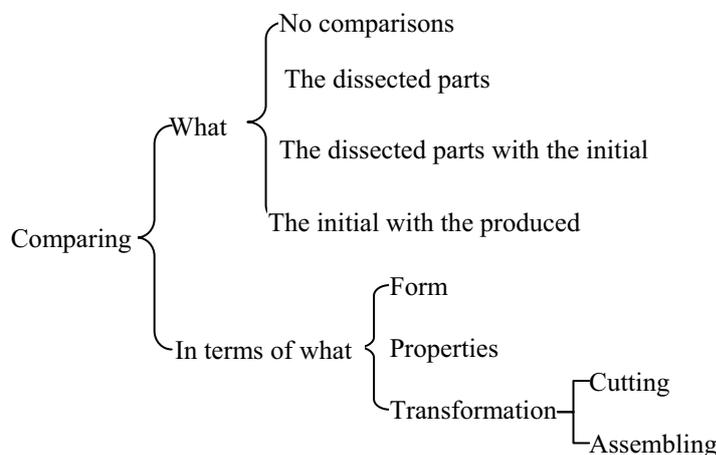
The experimental process of reassembling the dissected part could be characterized as pattern making, picture making, dynamic manipulation or free exploration. By pattern making we mean the re-composition of the two parts by rearranging its position in a rather systematic way. The picture making experimental process of reassembling the two parts referred to the children’s tendency to look for familiar shapes that formed unintentionally. The dynamic manipulation involved children’s strategies of re-composition by sliding, flipping or rotating the two parts. Finally, the free exploration

process of assembling referred to those children’s strategies that seemed to be arbitrary.

The shape that was produced through the assembling process was either a concave or a familiar one. Children tended to construct familiar shapes with a prototypical form. This prototypical form was either the same as the initial shape or a different one which was known to the children. In most cases these shapes were either rectangles or triangles. Finally, children’s reassembling strategies varied in terms of the number of the produced shapes. There seems to be a relation between the number of the produced shapes and the process of assembling that had been used. Children who used an experimental pattern making process of re-composing the dissected parts, produced a large number of shapes, while two children that manipulated the two parts dynamically, referred to the possibility of producing an “infinite” number of shapes.

### Comparing

Children’s comparison strategies varied according to the objects of the comparison and the criteria by which the comparison was made. Some children could not make any comparisons. Others were limited to comparisons between the dissected parts. Those children could not see the composition of two parts into a whole shape. Some others compared the dissected parts with the initial shape. Most children in the last two categories compared the shapes in terms of their form. The children who referred to the properties of the shapes were those that compared the initial shape with the produced one. Finally, some of them based their reasoning on the process of transformation (cutting – assembling).

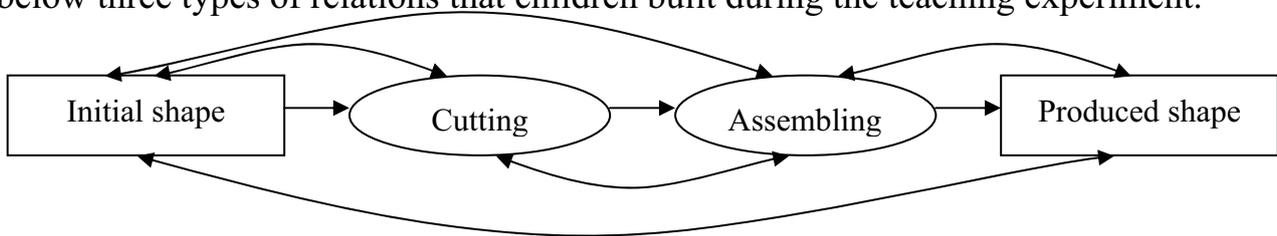


**Figure 5: The comparing strategies**

### **Relations within the transformation**

From children’s work through the tasks, certain relations among the elements of the transformation emerged. Figure 6 shows possible relationships that can exist between these elements. As the arrows illustrate, these relationships can be of one direction or of the inverse. For example, the arrow from the initial shape to the process of cutting

shows that the children cut the initial shape physically or mentally. The inverse denotes that the children consider the part – whole relationship as they realize that the two dissected parts form the initial shape. Children’s flexibility in building complex relations among the elements of the transformation as well as their ability to justify these relations adequately indicate children’s geometrical understanding. We present below three types of relations that children built during the teaching experiment.



**Figure 6: The possible relations in the process of transformation**

### Building oneway relations

The children “moved” along the transformation from left to right without reflecting on previous steps. In particular, they cut the initial shape by drawing a section or following a given. Then they reassembled the dissected parts experimentally and produced pictures of different shapes. They tended to produce concave shapes but also prototypical ones. They did not conceive of any relation between the initial and the produced shapes. When the initial and the produced shapes were given, the process of transformation (cutting and reassembling) could not be realized. The above actions could be performed only in a concrete context. All the children managed to follow this linear path. A typical case is Stavros (fourth grade) who seemed to consider geometrical shapes at a holistic level. Stavros cut the rectangle horizontally dividing it into two almost equal rectangles (Figure 7a). He rearranged the two pieces, placed them side by side and made a different rectangle (Figure 7b). While the researcher asked him to compare the produced shape with the initial, he referred to the dissected parts and named them “rectangles”. In his attempts to make other shapes he formed a concave shape (Figure 7c) and he recognized it as a picture: “It is the letter T”. Again when he was asked to make comparisons he only referred to the two parts “The top one is fatter that the lower one”. The only way that he brought the initial shape into the discussion was by trying to identify its image as a part of the produced one. In task 3, where he was asked to cut a rectangle to make a triangle, he could not make any cut. He believed that he could form a triangle only if he drew two line sections. To face the above task he had to build a relation between the produced and the initial shapes by considering the cutting and assembling processes simultaneously and anticipating the effect of these processes. In the last task, he was also unable to perform the process of transformation mentally.



Figure 7a



Figure 7b

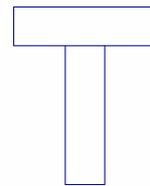
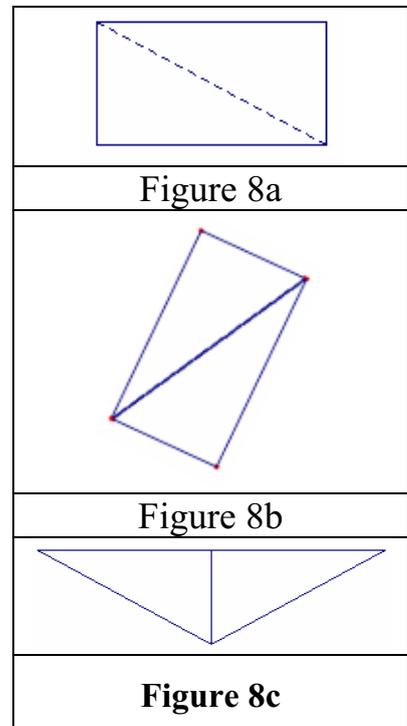


Figure 7c

Building relations by reflecting on the process of transformation

In this case, the children moved along the transformation but could also reflect on the previous steps. They seemed to be aware of the effect of the transformation on the initial shape. In particular, they saw relations between the initial and the produced shape by considering both the cutting and the assembling processes. While comparing the initial and the produced shape they used the inverse transformation process to justify their claims. They could perform the above actions in both a concrete and imaginary context. Four children managed to build these kinds of relations (two of the 6<sup>th</sup> grade and two of the 4<sup>th</sup> grade).

An interesting example is the case of Nikos (fourth grade) who could recognize properties of the geometrical shapes. Nikos cut the rectangle diagonally (figure 8a) and produced convex and concave shapes. He used both geometrical terms and similes to name these shapes: “it makes a rhombus” (figure 8b), “it looks like a bird, an airplane, an arrow” (figure 8c). While he compared the produced shapes with the initial one, he blended intuitive and formal knowledge by both referring to shapes’ geometrical attributes (number of angles, dimensions) and its perceptual features like “it seems bigger, it is also longer”. He not only compared the two geometrical shapes as different entities, but he also reflected on the transformation process to indicate the conservation of areas of the shapes. By referring to figure 8a he reconstructed the initial shape (rectangle) and he suggested that “the shape (figure 8a) is different because it is a different shape but when we join these parts together it will be the same as the initial”. His ability to reflect on the transformation process was also indicated in task 3 where he could cut and assemble the dissected parts by anticipating the result of his actions. In the last task, he acted at a mental level and produced a number of different shapes.



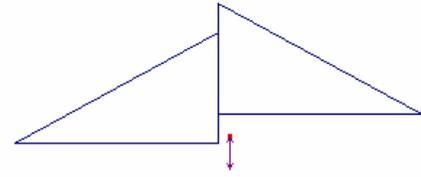
Building intermediate relations

The above two relations indicate different levels of awareness of the process of transformation. In the first relation the children implemented the transformation but without being aware of its role in the relation between the initial and the produced

shapes. In the second, the children could reflect on the process and were aware of its effect. However, most children moved between these two extremes.

For example, Christos (sixth grade), who could identify some geometrical properties, produced a large number of different shapes and discovered a dynamic way (figure 9)

of assembling the parts to form these shapes. He said that he could make “thousands of shapes” and he rotated and slid the dissected pieces to produce both concave and convex shapes. In his comparisons, he reflected on the cutting and assembling processes and related the initial shape to the two dissected parts: “the two parts are similar to the initial shape because they can make it (the initial,)”, “one part stayed the same and I placed the other in another place”. However, he could not see the produced shape as an independent entity. In task 3 he did not anticipate the result of the cutting- assembling process but rather he acted intuitively. He drew the diagonal and then he applied the transformation mentally to check if he could get a triangle.



**Figure 9**

However, he could not see the produced shape as an independent entity. In task 3 he did not anticipate the result of the cutting- assembling process but rather he acted intuitively. He drew the diagonal and then he applied the transformation mentally to check if he could get a triangle.

Another case which can be characterized as an example of intermediate relations is Fay’s (sixth grade) contributions. Fay used the transformation as a tool to produce only convex shapes but there was no reference during her work to the process of transformation. She could see the initial and the produced shapes as different geometrical entities and she compared them in terms of their angles and sides. However, her knowledge remained at a typical level and in the case of the concave initial shape (task 4) she could not produce any shape.

## CONCLUDING REMARKS

Children developed different strategies in their attempts to face the designed tasks. In the cutting process, children tended to keep the symmetry. The process of assembling was mental or concrete, experimental or intentional. Pattern making, picture making, dynamic manipulation or free exploration were some strategies that produced a large number of different shapes. These strategies of recomposition of the shape seem to be related to the development of children’s mathematical actions-on-geometrical objects proposed by Clements, Sarama & Wilson (2001). Certain conceptions of geometrical shape also emerged, including prototypical examples, geometrical objects and real life pictures. In terms of the comparison between the initial shape and the produced one, children’s strategies were either independent or related to the transformation process. In cases where the children reflected on the cutting – assembling processes, they seemed to have as the basis for their comparisons the shapes’ properties.

Considering the role of transformation on understanding the concept of geometrical shape, the children built certain relations. Some moved “along” the transformation and performed the various tasks at a concrete and experimental level. In this case their understanding was limited to the recognition of the form of the shapes. Those

who could reflect on the process of transformation and build relations among its elements acted rather in an intentional way, considering the shapes as geometrical objects and comparing them in terms of their properties.

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## **PROBLEM SOLVING IN GEOMETRY: THE CASE OF THE ILLUSION OF PROPORTIONALITY**

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*The study explores different aspects of students' abilities in problem solving concerning area and volume and their interrelation across two grades. Students in grades 9 and 10 were given a test involving three types of problems: usual computation problems, pseudo-proportional problems and unusual ones. The results suggest that the type of reasoning involved in the tasks has an effect on students' problem solving processes. This effect, though, varies across grades. For the younger students the pseudo-proportional problems were of a similar nature as the usual problems. Older students approached the pseudo-proportional problems differently from the usual ones, indicating a weaker impact of the linear model on their reasoning compared to younger students' thinking.*

### **INTRODUCTION**

Geometry has always been a privileged domain of research among psychologists and researchers of Mathematics Education. The continual interaction between theory and space and between text and figure gives to geometry learning a distinctive character. A topic of particular interest in geometry is measurement of length (Nuhrenborger, 2001) and area (Frade, 2005). Students' abilities in problem solving concerning the concepts of length, area and volume have been studied extensively in the recent years (De Bock, Verschaffel & Janssens, 1998; Modestou & Gagatsis, 2007, 2006) under the perspective of the phenomenon of the illusion of linearity (pseudo-proportionality). This phenomenon refers to students' tendency to apply the linear model in non-proportional situations of area and volume, which involve an enlargement or reduction of the figure's size in relation to its side length. The objective of the present paper is to illuminate this phenomenon by articulating a structural model related to geometry problem solving and more specifically to the abilities involved in solving area and volume problems. A main concern is also to compare the structure of the aforementioned model between students in 9th grade and in 10th grade. The two grades examined were deliberately chosen based on the fact that grade 9 and grade 10 belong to two different educational levels in Cyprus with different approaches in the teaching of geometry. In particular, in grade 10 the Euclidean geometry is taught in a more systematic manner as a continuation and extension of the teaching of the particular topic in grade 9.

## THEORETICAL CONSIDERATIONS

Linear relations constitute the easiest way for getting access to the world of functions. Therefore, they have been given a special attention and status, starting from the early years of age. Linear or proportional relations refer to the function of the form  $f(x) = ax$  (with  $a \neq 0$ ) and are represented graphically by a straight line passing through the origin (De Bock, Van Dooren, Janssens, & Verschaffel, 2002). The basic linguistic structure for problems involving proportionality includes four quantities (a, b, c, d), of which, in most cases, three are known and one unknown, and an implication that the same multiplicative relationship that links a with b, links c with d. Consider for example the following problem case: “A pianist needs 5 minutes to perform 2 musical themes. How much time does he need to execute 3 themes of the same duration as the first ones?” In this case a true proportionality exists as the relationship between the terms is a fixed ratio.

However, there are cases where problems match this general linguistic structure without being proportional ones. In these cases the problems are considered “pseudo-proportional”, because of the strong impression they create for the application of the linear model. For example, in the case of the constant problem: “A pianist needs 5 minutes to execute a musical theme. How much time do 3 pianists need to execute the same theme in the same orchestra?”, students spontaneously answer that the pianists need 15 minutes, falling in this way to the pseudo-proportionality trap; that is they do not consider the fact that the 3 piano players perform the theme simultaneously. Therefore, if a problem matches the general linguistic structure of proportionality, the tendency to evoke direct proportionality can be extremely strong even if it does not benefit these problems (Verschaffel, Greer & De Corte, 2000).

Freudenthal (1983) focuses on the appropriateness of the linear relation as a phenomenal tool of description and indicates that there are cases in which this primitive phenomenology fails. One of these cases, which is the focus of the present study, is the case of the non-linear behaviour of area and volume under linear enlargement or reduction. Students’ former real life practices with enlarging and reducing operations do not necessarily make them aware of the different growth rates of lengths, areas and volumes (De Bock et al., 2002). Students in fact fail to see the non-linear character of the increase and handle the relations between length and area or between length and volume as linear instead of quadratic and cubic (Modestou & Gagatsis, 2007). Consequently, they apply the linear scale factor instead of its square or cube to determine the area or volume of an enlarged or reduced figure.

In recent years, researchers (De Bock et al., 1998, 2002; Modestou & Gagatsis, 2007) have examined students’ tendency to deal linearly with non-proportional tasks, and have suggested ways of overcoming it. In particular, De Bock et al. (1998, 2002) showed an alarmingly strong tendency among 12-16 year old students to apply proportional reasoning to problem situations concerning areas, for which it is not suited. Furthermore, the use of a number of different experimental scaffoldings did

not yield the expected results. The inclusion of visual support at the non-proportional problems, like self-made or given drawings, did not have a beneficial effect on students' performance, as students most often relied on formal strategies such as using formulas (De Bock et al., 1998). Students in some cases even discarded the results given from well-used formulas for finding the area and volume of a figure, in favour of the application of the linear model (Modestou & Gagatsis, 2007). In some cases where the improvement of students' success rates at the non-proportional tasks was achieved, drawbacks at the proportional tasks were observed as students started to apply non proportional methods at these tasks (De Bock et al., 2002).

The actual processes and the mechanisms used by students while solving non-proportional problems were unravelled by means of interviews (De Bock et al., 2002; Van Dooren, De Bock, Janssens & Verschafel, in press). It appears that explanatory elements of the phenomenon of the "illusion of linearity" (i.e., an explicit belief in a linear relation between lengths, areas and volumes of similarly enlarged figures) can also be found in the intuitive and heuristic nature of the linear model, shortcomings in geometrical knowledge and inadequate habits and beliefs about solving word problems. Linearity appears to be deeply rooted in students' intuitive knowledge and is used in a spontaneous way, which makes the linear approach quite natural, unquestionable, and to certain extents inaccessible to introspection (De Bock et al., 2002).

The originality of our study lies in fact in the structural model that we develop in an attempt to give insight into different aspects of problem solving abilities on area and volume and their interrelations for each of the two age groups of students: 14 and 15 years of age. The present study intends also to investigate the variance of students' problem solving abilities in geometry and their structure across grades: 9th grade and 10th grade.

## **METHOD**

The sample of this study consisted of 653 students of grade 9 and 10 (14 and 15 year olds) of 13 different gymnasiums and 10 lyceums in Cyprus. In particular, 348 students attended the 9th grade and 305 students the 10th grade. These two grades were chosen as suitable for the study as the test consisted of tasks of geometrical nature that required the use of mathematical formulas for their solution. Therefore, 14 and 15-year old students could more easily handle such tasks.

The students were administered a 40 minutes test that consisted of 9 geometrical word problems concerning the perimeter, area and volume of different figures, offered in groups of three. Each group of problems was accompanied by a given number. Students were first asked to solve all of the three problems of each group and then to choose the problem that was appropriate for the given number, i.e. the *one* problem that had the same solution as the number given at the beginning of each group of word problems. Each group of problems consisted of the usual problem which was Appropriate for the given number (A1, A2 & A3), of one Pseudo-

proportional problem, where the application of the linear model would give the given number as an answer (Pa1, Pa2, Pv3), and one “Unusual” problem (Un1, Un3), which had many solutions. Any sensible solution for the unusual problems could not result to the given number, whereas any attempt to solve them, using only the syntax of the problem ignoring semantic implications, would give the given number as an answer. The responses that were considered as appropriate for the unusual problems were the ones that involved explicitly the realization that they did not have only one answer or even that they could not be solved. The first group’s problems are given in Table 1. The other two groups were formed accordingly.

As an exception to the formulation of the groups, a perimeter pseudo-proportional problem (Pl2) was included in the place of the unusual problem, in the second group of problems. The particular problem was as follows: “Consider two equal semicircles. The perimeter of each semicircle is  $9\pi$  cm. If the two semicircles are joined together in such way that they form a circle, find the perimeter of the circle”. On one hand the geometric nature of this problem yields the application of the linear process. On the other hand, the fact that the problem incorporates the configuration of the two semicircles makes it similar to the structure of one of the unusual problems in the test (see Problem C in Table 1 - Un1).

50	
A.	Mr. Ben emptied all the water of an open cubic tank, in order to paint it. If he needs 10L of paint to paint the bottom of the tank, how much paint will he need for the entire tank? <i>(Appropriate-Usual - A1)</i>
B.	George measured the surface of his classroom floor and found that its area is $25\text{m}^2$ . The gym’s floor has double the dimensions of the classroom. What is the area of the gym’s floor? <i>(Pseudo-proportional - Pa1)</i>
C.	A classroom has two rectangular blackboards joint together with a common width. The first blackboard’s perimeter is 30m and the second one’s 20m. How many meters of ribbon are needed in order to frame both blackboards together? <i>(Unusual - Un1)</i>

**Table 1: Example of the problem formulation in the first group of problems**

The particular structure of the test is justified by the fact that this study is a part of a larger research project, which aimed not only at exploring and comparing students’ performance in the three problem categories, but also at making students question the spontaneous and uncritical application of linearity (Modestou & Gagatsis, 2006). However, the focus of this study is not to investigate students’ choices of the problems that corresponded to the given number, but to examine and compare their problem solving abilities in the three different types of tasks and their problem solving reasoning behaviour across grades.

## RESULTS

A 2 (the two age groups) X 3 (appropriate-usual vs pseudo-proportional vs unusual problems) multivariate analysis of variance (MANOVA) was performed to specify the possible influence of the task variable, that is the type of the problems and the subjects' variable, that is age, on problem solving.

The effect of age  $F_{(1,651)}=3.121$ ,  $p=.078$ ,  $\eta^2=0.005$  was not significant, indicating that mean performances of the two age groups were not significantly different. The main effect of the type of tasks was very strong  $F_{(2,650)}=1086.906$ ,  $p<0.0005$ ,  $\eta^2=0.770$ . Mean scores in the three types of tasks by the students of the two grade levels are shown in Table 2. Decomposing this effect by means of a univariate analysis revealed that the appropriate-usual problems were significantly easier than the pseudo-proportional problems, which in turn were easier than the unusual problems. The relative difficulty of the three types of tasks applied in both age groups, as no interaction between age and type of task appeared  $F_{(2,650)}=.924$ ,  $p=.397$ ,  $\eta^2=.003$ .

Type of tasks	Whole sample		Grade 9		Grade 10	
	Mean	SD	Mean	SD	Mean	SD
Usual	1.859	.952	1.793	.944	1.934	.957
Pseudo-proportional	1.155	1.081	1.118	1.069	1.197	1.095
Unusual	.107	.447	.089	.403	.128	.493

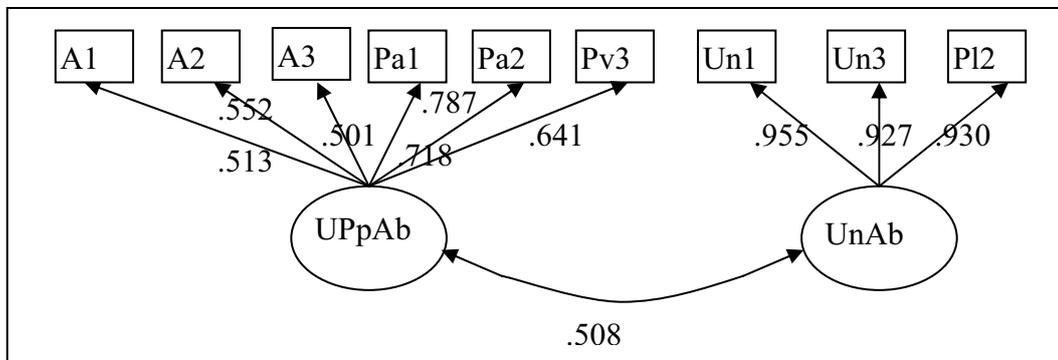
**Table 2: Mean scores and standard deviations in each type of tasks for the whole sample and by grade**

Confirmatory Factor Analysis (CFA), has been employed to explore the structural organization of the various dimensions of geometrical problem solving, examined here, in each age group. Bentler's (1995) EQS program was used for the analysis. The tenability of a model can be determined by using the following measures of goodness-of-fit:  $X^2/df$ , CFI (Comparative Fit Index), RMSEA (Root Mean Square Error of Approximation) and SRMR (Standardized Root Mean Square Residual) (Bentler, 1995). The following values of the three indices are needed to hold true for supporting an adequate fit of the model: The observed values for  $X^2/df$  should be less than 2, the values for CFI should be higher than .9, the values of RMSEA need to be lower than .06 and the values of SRMR should be lower than .10.

The a priori model hypothesized that the variables of the test would be explained by two main first-order factors. In particular, we assumed that one of the first-order factors would be measured by the usual tasks involving calculations of area, perimeter and volume, i.e. A1, A2 and A3, as well as by the pseudo-proportional tasks, i.e. Pa1, Pa2, Pv3 and Pl2. The formulation of the assumption for the first

factor was based on well documented research findings suggesting students' strong tendency to use similar processes in solving usual and non-proportional tasks, such as employing the linear model and formal strategies (e.g. using formulas) (De Bock et al., 1998, 2002; Modestou & Gagatsis, 2007). Concerning the other factor we assumed that it would correspond to the scores of the unusual problems, i.e. Un1 and Un3, as they seem to have a different geometric character from the other problems.

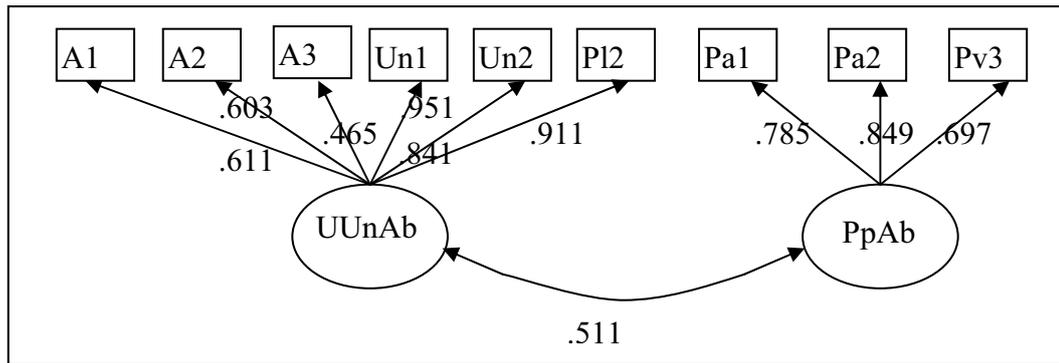
However, the fit of this model was poor in both grades [Grade 9:  $X^2(14) = 45.685$ ; CFI=.908; RMSEA=.081, SRMR=.174; Grade 10:  $X^2(15) = 38.317$ ; CFI=.956; RMSEA=.071, SRMR=.127]. In particular, due to problem's P12 difference from the other pseudo-proportional problems as regards the solution process it required, its increased difficulty level and the resemblance of its context with the context of an unusual problem, the model was modified. In particular, the solution of this problem depended on the understanding of the configuration of two figures and on the processing of the rather complex algebraic relations that emerge. As a result, students may have treated it as an unusual problem. Therefore, the observed variable P12 was removed from the first factor and was added to the second factor's indicators. The fit of the model (see Figure 1) was good [ $X^2(14) = 15.985$ ; CFI=.997; RMSEA=.014; SRMR=.085] in the group of the 9th grade students. This was not, however, the case in the group of the older students [ $X^2(15) = 38.510$ ; CFI=.956; RMSEA=.072; SRMR=.115].



**Figure 1: The model for the two aspects of problem solving ability in geometry in grade 9 [1],[2],[3]**

Exploratory factor analysis in the group of 10th graders showed that two factors are also needed to explain the intercorrelations of the nine observed variables [ $X^2(12) = 11.9$ ,  $p=.45$ ]. However, the problems used to measure each of the factors in the group of 10th grade students were different from the ones in the group of 9th grade students, indicating that the ways of understanding and solving the problems differed between the two grade levels. According to the outcomes of the exploratory factor analysis, a CFA model of a different structure was tested in the group of the older students. The model (see Figure 2) involved two first-order factors, the first one of which was measured by the usual tasks as well as the unusual problems. The second

factor was comprised by the pseudo-proportional tasks. The fit of the model was good [ $\chi^2(15) = 15.985$ ; CFI=.998; RMSEA=.015; SRMR=.085].



**Figure 2: The model for the two aspects of problem solving ability in geometry in grade 10 [1], [4], [5]**

The results of the CFA in both groups indicate that the one-level architecture captures accurately the data. The models in both groups involve two first-order factors that although different in each age group, they are intercorrelated. This suggests that the type of reasoning that the different type of tasks require, does have an effect on students' problem solving performance; an effect that varies across the two age groups. The significant correlation of the two latent factors in both groups however, can be seen as an indication that the two dimensions of the variables investigated here, although distinct, are interconnected and may contribute to the overall problem solving ability in geometry.

## DISCUSSION

The majority of the researchers that have worked on students' reasoning in problem solving concerning area and volume and specifically on the tendency to deal linearly with non-proportional tasks agree that the obstacle of linearity is very difficult to overcome by the students (De Bock et al., 2002; Modestou & Gagatsis, 2006). Other studies showed that there are not any differences in students' solutions of these problems across different grades (Modestou & Gagatsis, 2007). However, the findings of this study illustrate that despite the invariance of the students' mean performance in problem solving with respect to grade level, the structure of a model involving problem solving of pseudo-proportional tasks in combination with problem solving of unusual and typical tasks on area and volume does show variance between grade 9 and 10. Students of the two different grades responded to the given set of tasks in a manner that resulted in different dimensions of geometrical problem solving. This provides support for the idea that instruction and maturity may have a role in developing the understanding of spatial-organized quantities and in problem solving that requires different types of reasoning.

The tendency of the 9th grade students to apply the proportional model was strong and for these students the pseudo-proportional problems were almost of the same nature as the usual geometrical problems. For this reason the three usual problems

and the three pseudo-proportional ones constituted one factor. On the other hand, the present study indicates that students of 15 years of age started to differentiate their ways of interpreting and understanding this kind of problems. They appeared to approach the pseudo-proportional problems by activating different processes from the ones they used in usual problems. This change may be due to 10th grade's mathematics curriculum which, unlike 9th grade curriculum, gives special attention to inductive reasoning, on one hand, and to definitions and theorems, on the other hand. Inductive reasoning is directly associated with analogical reasoning (NCTM, 2000) and consequently can help students realize the structural similarity between situations and not focus on perceptual similarities (Gonswami, 1992). Moreover, the definitions and theorems of Euclidean geometry that are emphasized in grade 10 may have helped students to better conceptualize the situation described in the problems.

Therefore, 10th grade students may have confronted the non-proportional problems in a less superficial way relatively to the younger students, indicating the weaker impact of the linear model on their reasoning. In other words, they started to question to some degree the deep-rooted linear model's applicability in all the types of geometry problems. However, further qualitative research is needed to substantiate this inference. Interviewing students of the same mathematical ability level across the two grades while solving non-proportional problems may unravel the actual processes and the mechanisms used by them and allow their in-depth and more analytic comparison.

The unusual problems were also handled differently across the two grades. Students of grade 9 dealt with these problems differently from the other two types of problems, as they formed a distinct factor. However, the abilities of 10th grade students to tackle the unusual problems and to resolve the usual tasks established a common factor. A hypothetical explanation for this finding is that older students were more familiar with the structure of the unusual problems, because of their more systematic involvement with problems of Euclidean geometry.

The above results suggest that the type of reasoning that the particular geometrical tasks require does have an effect on students' problem solving processes. Despite the significant variation of these effects across the two age groups, certain commonalities appeared in the models of the groups, revealing that some aspects of their ways of thinking in problem solving remained invariant with development. A particular problem, although pseudo-proportional and solvable (P12), was confronted as an unusual one by the students of the two age groups, as in both models it belonged to the same factor as the unusual problems. The similarity of its context and its syntax with the unusual problems and the quite advanced algebraic reasoning that it involved made students treat it like the "unusual" problems.

Another common feature in the models of the two groups concerns the lower factor loadings of the solutions of usual computation tasks relatively to the approaches in dealing with the pseudo-proportional problems and with the unusual ones. The different nature and reasoning requirements of the typical tasks compared to the other

two types of tasks may provide an explanation for this difference. Even though good geometrical knowledge and competence in employing formal strategies such as using formulas met the requirements of the usual computation tasks, they were not sufficient for the solution of the pseudo-proportional or the unusual problems. Solving the pseudo-proportional problems required students' overcoming of the illusion of linearity, while tackling the unusual problems required students' sensible and realistic considerations for the interpretation of the situations involved, understanding of the semantic implications of the problems and breach of inadequate habits and beliefs about solving word problems, such as that they are obliged to provide only one answer to all the problems given to them.

The research directed towards finding ways to develop students' flexibility in dealing with geometrical problems of different reasoning requirements should continue so as to provide explanations for the variation of the structure of students' problem solving abilities across grades and to determine those factors concerning the students themselves, i.e. intuitive ideas, level of geometrical knowledge, habits and beliefs towards solving word problems, students' experiences in the mathematics classrooms, that may interact with age in the development of these abilities. The results of such attempts may help teachers at the high school levels to place emphasis on certain dimensions of geometrical problem solving and use more appropriate approaches to teaching them. By these means students can be assisted in constructing a solid and deep understanding of length, area and volume and their different growth rates, in interpreting different types of problems on these notions appropriately and in employing the solution processes that correspond to their reasoning requirements adequately and flexibly.

## NOTES

1. A1, A2, A3, Pa1, Pa2, Pv3, Un1, Un3 and Pl2 stand for the observed variables corresponding to students' performance to the tasks
2. "UPpAb" stands for the ability to solve Usual problems and Pseudo-proportional problems in geometry
3. "UnAb" stands for the ability to solve Unusual problems in geometry
4. "UUnAb" stands for the ability to solve Usual problems and Unusual problems in geometry
5. "PpAb" stands for the ability to solve Pseudo-proportional problems in geometry

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# **SPATIAL ABILITIES IN RELATION TO PERFORMANCE IN GEOMETRY TASKS**

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*The aim of this paper was to investigate how different subcomponents of spatial ability are related to the performance of primary (grades 4 and 6) and secondary school students (grade 8) on geometry tasks involving 2D figures, 3D figures or nets of geometrical solids. The results suggest that image manipulation, mental rotation and coordination of perspectives are predicting factors of students' geometry performance. The similarity analysis reveals that, while students of all three age groups generally confronted spatial abilities tasks and geometry tasks involving 2D and 3D figures in a different way, the older students realize that the same cognitive processes underlie image manipulation tasks and mental rotation tasks on one hand, and manipulating net-representations of 3D geometrical figures on the other hand.*

## **INTRODUCTION**

Geometry and spatial reasoning are important as a way to interpret and reflect on the physical environment. As Bishop (1983) has noted, geometry is the mathematics of space. Mathematics educators, therefore, are concerned with helping pupils gain knowledge and skills in the mathematical interpretations of space.

The research in geometry and spatial thinking has evolved from studies in psychology, when in the 1970s some researchers were interested in the relationship of spatial abilities to mathematical learning and problem solving (Owens & Outhred, 2006). Research on spatial ability as a single component has indicated that it has a strong connection with achievement in mathematics (Clements and Battista, 1992). However, it is not clear how certain subcomponents of spatial ability are related to students' geometry performance. Additionally, no pieces of research investigated whether students confront spatial ability tasks and geometry tasks in a similar or in a different way. This is what this study aims to do.

## **THEORETICAL BACKGROUND AND RESEARCH QUESTIONS**

### **Spatial Abilities**

Generally speaking, the concept of spatial ability is used for the abilities related to the use of space. Psychologists as well as mathematics educators have contributed to the discussion of how spatial ability may be understood. But, as Wheatley (1998) has noted, the way the term spatial ability (and other related terms) have been defined and the instruments used to collect data are nearly as varied as the number of studies

using this term. For example, spatial ability may be defined as “the ability to generate, retain, retrieve and transform well-structured visual images” (Lohman, 1996). Despite the fact that there is no agreement on the definition of the concept, researchers agree that spatial ability is not a unitary construct. Different components of “spatial ability” have been identified, each emphasizing different aspects of the process of image generation, storage, retrieval, and transformation. The three major dimensions of spatial ability that are commonly addressed are spatial visualization, spatial orientation and spatial relations. In the present study we follow Demetriou and Kyriakides (2006) suggesting three components related to the spatial-imaginal specialized structural system of the human mind: image manipulation, mental rotation and coordination of perspectives.

### **Geometry and Spatial Abilities**

When mathematics educators consider geometry from a theoretical perspective, the key role of spatial abilities is universally accepted, even though spatial knowledge is not thought of as a synonym for geometric knowledge (Gorgorió, 1998). The development and improvement of spatial ability is regarded to be one of the basic aims of geometry in elementary school. On the other hand, for many mathematics educators, spatial ability is regarded an important prerequisite for geometry problem solving in particular, and by some researchers even for mathematics learning in general. High levels of spatial abilities have frequently been linked to high performance in mathematics in general, or especially in geometry. For instance, Battista (1990) indicated in his study that spatial visualization is an important factor in geometry learning, while Tartre (1990) suggested that spatial orientation skill is related with mathematical problem solving behaviour. In more recent studies spatial ability and declarative knowledge of geometric concepts were reported among the variables which correlated with scores on geometry questions (Reiss, 1999). However, it would be interesting to examine the relation of different subcomponents of spatial ability and different aspects of geometry performance, as this is related to the different dimensions of the geometrical figures involved in geometry tasks.

### **The present study**

This paper is based upon a larger research project which investigates the transition from elementary to secondary school geometry in Cyprus, gathering data concerning students’ performance in tasks involving 2D shapes, 3D shapes and nets of geometrical solids, as well as the students’ spatial abilities. The aim of the present paper was to investigate whether and to what extent primary and secondary school students’ spatial abilities are related to their performance on geometry tasks involving 2D figures, 3D figures, or nets of solids. The research questions are:

- Which spatial ability subcomponents are more likely to predict students’ performance in tasks involving geometrical figures?

- How is the level of students' spatial ability related to their performance in geometry tasks?
- Do the primary and secondary school students confront spatial ability tasks and geometry items involving (a) two-dimensional figures, (b) three-dimensional figures and (c) nets in a similar or in a different way?
- Which implicative relations, if any, exist amongst spatial ability tasks and geometry tasks?

So, in this study we investigate the relation between students' spatial abilities and their geometry performance, trying to extend the research on geometry and spatial thinking in three ways: First, we accept that spatial ability is not a unitary construct and we examine the role of distinct spatial abilities (image manipulation, mental rotation, and coordination of perspectives) on geometry performance. Second, we investigate the role of spatial abilities on geometry performance that is differentiated based on the geometrical figures involved in the geometry tasks (2D figures, 3D figures and nets of geometrical solids). Third, we make a further step towards the relation of spatial abilities and geometry performance trying to gather information on the tasks' level, investigating the existence of similarity and implicative relations amongst spatial ability tasks and different geometry tasks.

## **METHOD**

### **Participants**

The participants were 1000 primary and secondary school students (488 males and 512 females). Specifically, 332 were 4<sup>th</sup> graders (10 years old), 333 were 6<sup>th</sup> graders (12 years old) and 335 were 8<sup>th</sup> graders (14 years old).

### **Material and Procedure**

Data were collected through a written test which was administered to all students of the three age groups and consisted of spatial ability tasks and geometry items. The test was administered in two parts during normal teaching, either by the first author or by students in Mathematics Education at the University of Cyprus, who followed specific instructions concerning the test administration. The first part of the test was administered to all schools in the same week, while the second part of the test was administered one week later.

The geometry tasks presented to the students were chosen taking into consideration the geometry curriculum (part of the mathematics curriculum) in Cyprus and the geometry tasks presented in mathematics books at the primary education level. The geometry test consisted of tasks involving 2D geometric figures, 3D figures and nets of geometrical solids. It mainly included recognition items where students had to identify different representations of geometrical shapes (a) in simple geometrical figures and (b) in complicated geometrical figures, problem solving tasks which involved the use of geometrical reasoning to be solved and some multiple choice

questions examining declarative knowledge of geometric concepts. Examples of the geometry items used can be found in the Appendix. The spatial ability battery test administered consisted of tasks used by Demetriou and his associates in their studies of mind (for full description of the tasks, see Demetriou & Kyriakides, 2006). It included five tasks addressed to image manipulation (*paper folding* task), mental rotation (*cubes* task and *clock* task), and coordination of perspectives (*tilted bottle* task and *car* task). Each item involved was scored on a pass (1) / fail (0) basis. The total task score equaled the number of items passed by the participant.

### Statistical Analyses

With the use of the Extended Logistic Model of Rasch (Rasch, 1980), an interval scale presenting both item difficulties and students' performance was created (a) for the geometry test and (b) for the spatial abilities test. The analysis of data revealed that the two batteries of tests had satisfactory psychometric properties, namely construct validity and reliability.

For the analysis and processing of the data collected the statistical package of SPSS was used, as well as Gras's similarity and implicative statistical analysis by using the computer software CHIC (Classification Hiérarchique Implicative et Cohésitive) (Bodin, Couturier, & Gras, 2000).

## RESULTS

The main findings of this study are presented in three sections. The first one refers to the results of regression analyses using students' geometry performance as the dependent variable. In the second section we present the results of crosstabs analyses examining how the level of students' spatial ability is related to their geometry performance and vice versa, while the third section elaborates on the similarity and implicative statistical analysis conducted based on students' performance in spatial ability and geometry tasks.

### Regression Analyses Results

Regarding the prediction of performance in geometry tasks including geometrical figures, Stepwise Regression analysis was first performed for the entire sample. The results indicated that the statistically significant predictive factors are, in order of significance: students' performance in the spatial abilities test (Beta=0.449,  $t=15.660$ ,  $p<0.01$ ) and students' age (Beta=0.191,  $t=6.676$ ,  $p<0.001$ ). These factors account for up to 30% of the variation of students' performance in geometry tasks. On the contrary, students' gender (Beta=0.046,  $t=1.731$ ,  $p=0.084$ ) does not contribute to the performance of students in the specific geometry tasks.

Considering the fact that performance in the spatial abilities test and students' age appeared to be predictors of geometry performance, it would be valuable to examine whether there is a pattern of variables referring to distinct spatial abilities which predict geometry performance for each age group. Table 1 presents the summary of

stepwise multiple regression analyses conducted separately for each age group, using as independent variables the scores for (a) image manipulation, (b) mental rotation, and (c) coordination of perspectives, for predicting students' performance in geometry tasks involving geometrical figures.

Predictors	4 <sup>th</sup> graders			6 <sup>th</sup> graders			8 <sup>th</sup> graders		
	Beta	t	p	Beta	t	p	Beta	t	p
I. M. *	0.281	5.603	0.001	0.352	7.250	0.001	0.326	6.564	0.001
M.R. *	0.271	5.414	0.001	0.284	5.848	0.001	0.342	6.820	0.001
C.P.*	0.194	3.872	0.001	0.222	4.587	0.001	0.176	3.504	0.001
Model*	0.181			0.232			0.202		

(\*I.M.=image manipulation, M.R.=mental rotation, C.P.=coordination of perspectives, Model=Adjusted R-Square of Model)

**Table 1: Summary of stepwise multiple regression analyses of geometry performance by age group**

The most important conclusion of the regression analysis is that in the case of 4<sup>th</sup> and 6<sup>th</sup> graders image manipulation, mental rotation, and coordination of perspectives, in descending order of significant importance, were the most significant factors for predicting performance in geometry tasks. In the case of 8<sup>th</sup> graders mental rotation appeared to be a more significant predictor of students' geometry performance than image manipulation.

### Crosstabs Analyses Results

The Extended Logistic Model of Rasch (Rasch, 1980) was used in order to create interval scales presenting item difficulties and students' performance on the geometry test and on the spatial abilities test. Using the information retrieved from this analysis, students were assigned to five levels of performance concerning geometry tasks and to five levels of performance concerning spatial tasks. Crosstabs tables of geometry performance level by spatial ability level were obtained for each age group in order (a) to trace the spatial ability level of students who have achieved high performance in the geometry test, and (b) to examine the performance level achieved in the geometry test by high spatial ability students. The crosstabs results are summarized in Table 2 and Table 3.

The results presented in Table 2 indicate that students who evidenced high performance in the geometry test (geometry performance level 4 and 5) belong to the high spatial ability groups (spatial ability level 4 and level 5) as formed based on the responses in the spatial ability battery used. This is the general observation for all three age groups, except from the case of 4<sup>th</sup> graders, where a percentage of 41.1% of the students whose performance in the geometry test ranks them in level 4 have been assigned to level 3 as far as spatial ability is concerned.

	Geometry performance L4*			Geometry performance L5*		
	4 <sup>th</sup> graders	6 <sup>th</sup> graders	8 <sup>th</sup> graders	4 <sup>th</sup> graders	6 <sup>th</sup> graders	8 <sup>th</sup> graders
Spatial ability L3*	41.1 %	15.6 %	13.2 %	0 %	2.4 %	5.1 %
Spatial ability L4*	26.0 %	21.3 %	30.9 %	12.5 %	9.5 %	18.6 %
Spatial ability L5*	23.3 %	58.2 %	54.6 %	87.5 %	88.1 %	76.3 %

(\*L=level)

**Table 2: Crosstabs of high geometry performance level by spatial ability level**

The majority of primary school students (grade 4 and grade 6) who were assigned in spatial ability level 4 and level 5 reached a performance in the geometry test which ranked them in geometry level 3 (except from 6<sup>th</sup> graders coming from the highest level of spatial ability group, who reached level 4 in geometry performance). In the case of 8<sup>th</sup> graders the majority of high spatial ability students reached level 4 in geometry performance, while there is also a percentage of students remaining to scores that assigned them to geometry level 3.

	Spatial ability L4			Spatial ability L5		
	4 <sup>th</sup> graders	6 <sup>th</sup> graders	8 <sup>th</sup> graders	4 <sup>th</sup> graders	6 <sup>th</sup> graders	8 <sup>th</sup> graders
Geometry L3	63.5 %	50.0 %	33.7 %	50.0 %	35.7 %	24.9 %
Geometry L4	30.2 %	41.9 %	51.1 %	34.0 %	42.3 %	48.0 %
Geometry L5	1.6 %	6.5 %	12.0 %	14.0 %	22.0 %	26.0 %

(\*L=level)

**Table 3: Crosstabs of high spatial ability level by geometry performance level**

To sum up, crosstabs analyses indicated that the majority of the students who attained high scores in the geometry test are also included in the high spatial ability groups. The reverse, though, is not valid, since students with high spatial ability have not necessarily attained the best scores in the geometry test.

### Similarity and Implicative Analysis Results

For the purposes of this paper we refer to the similarity diagrams and the implicative graphs produced by CHIC when conducting similarity and implicative analysis of the data (Bodin, Couturier, & Gras, 2000). The similarity diagram allows for the arrangement of tasks into groups according to their homogeneity and the implicative graph contains implicative relations, which indicate whether success to a specific task implies success to another task related to the former one.

The objective of conducting Gras's similarity analysis was to examine whether students of the three different age groups confronted spatial ability tasks and geometry tasks involving geometrical figures in a similar or in a different way. For this purpose, three similarity diagrams were produced for each age group from the application of implicative analysis. The first similarity diagram refers to geometry tasks involving 2D figures and spatial ability tasks, the second diagram refers to geometry tasks involving 3D figures and spatial ability tasks, and the last one refers to tasks involving nets of geometrical solids and spatial ability tasks. Due to space limitations here we do not present the diagrams, but we sum up the observations that arise from them.

In the case of all three age group students, the majority of spatial ability tasks formed separate clusters from the clusters involving 2D figures. The same picture was revealed in the similarity diagram presenting spatial ability tasks and geometry items involving 3D figures. So, the most important finding is that students generally dealt with spatial ability tasks in different ways than with tasks involving 2D or 3D geometrical figures. The way students have confronted spatial ability tasks in relation to the geometry tasks involving nets of solids differentiated in relation with their age. More specifically, in the case of 4<sup>th</sup> graders, tasks involving nets of solids were confronted in a totally different way than the spatial ability items. The young students could not see any similarity in the underlying geometrical concepts. But in the case of 6<sup>th</sup> graders and more clearly in the case of 8<sup>th</sup> graders the involvement of spatial ability tasks in the same clusters with net-representations items provided evidence that the older students realize to a bigger extent that the same cognitive processes underlie spatial abilities and manipulating net-representations of three-dimensional figures.

The objective of conducting Gras's implicative analysis was the investigation of the presence of any implicative relations between spatial ability and geometry tasks. As mentioned above, the implicative graphs produced contain implicative relations, which indicate whether success on a specific task implies success on another task related to the former one. Three implicative graphs were produced, one investigating relations between tasks involving 2D figures and spatial items, one investigating relations between tasks involving 3D figures and spatial items, and the third one investigating relations between tasks involving nets of solids and spatial ability tasks.

In the case of spatial ability tasks and geometry tasks involving 2D figures as well as in the case of spatial ability tasks and geometry tasks involving 3D figures, the first and the most important observation was that the implicative relations observed are "intra-categorical", that they concern the same category of tasks. No implications between the two different categories of tasks were observed. The picture changed when we examined spatial ability tasks and geometry items involving nets of 3D geometric figures. Though in the case of the younger students (4<sup>th</sup> graders), no implicative relations were observed between nets items and spatial ability tasks, this

was not the case for the older students. Apart from intra-categorical relations, the analysis revealed implicative relations between tasks from the two different categories: nets items leading to spatial ability tasks and spatial ability tasks leading to tasks referring to nets. In the older students' minds, we might think, there are not only intra-categorical relations, but it seems that successful performance on geometry tasks including nets implies success on spatial ability tasks and vice versa.

## DISCUSSION

In this paper we have tried to extend the research on geometry and spatial ability. Specifically, we investigated how three different components of spatial ability, as proposed by Demetriou and Kyriakides (2006), namely image manipulation, mental rotation and coordination of perspectives are related to primary (grade 4 and grade 6) and secondary (grade 8) students' geometry performance in tasks involving 2D figures, 3D figures, or nets of geometrical solids.

The results of this study indicate that students' performance in the spatial abilities test was the most significant predictor of their geometry performance. Examination of the three different subcomponents of spatial ability we have measured, revealed that image manipulation and mental rotation are predicting factors of both primary and secondary students' geometry performance. Additionally, coordination of perspectives is a predictor of geometry performance only in the case of primary school students.

On the other hand, though, the crosstabs analyses revealed that only part of the students who were included in the high spatial ability groups have reached the higher levels in the geometry test. This finding seems to be contradicting the previous findings concerning the predictive role of spatial ability factors as far as geometry performance is concerned. But, one should keep in mind that the spatial abilities aforementioned are not the only predictors of students' geometry performance. These factors explain only a part of the variation of geometry performance. In subsequent research other cognitive as well as metacognitive factors can be investigated in this direction.

The similarity and implicative analyses conducted in this study provides evidence that students of all three age groups generally confronted spatial abilities tasks and geometry tasks involving two- or three-dimensional figures in a different way. Consequently, no implicative relations were evident between tasks of these categories. On the contrary, in the case of tasks involving nets of geometrical solids, only the students in grade 4 considered these tasks totally different from spatial abilities tasks. Younger students did not recognize any similarities between those two categories of tasks, while the older students in the study confronted a number of tasks involving nets similarly to spatial abilities tasks. This implies that the older students can realize that the same cognitive processes underlie spatial abilities such as image manipulation and mental rotation on one hand, and manipulating net-representations

of 3D geometrical figures. This finding is in line with Potari's and Spiliotopoulou's proposition that the whole process of developing solids and handling their net-representations requires the student not only to "see" the objects and recognize their elements, but also to mentally "combine the latter in a transformed position and probably take into consideration the reverse process" (Potari & Spiliotopoulou, 2001, p. 41).

Our findings make us consider once again the idea that systematic training in spatial abilities should be a principal aim of geometric teaching. They also raise questions that need to be examined further on the role of certain spatial abilities in different geometrical tasks.

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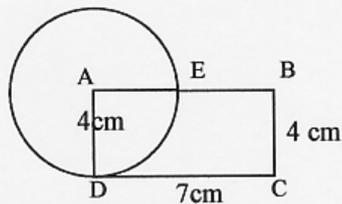
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**APPENDIX : Examples of tasks used**

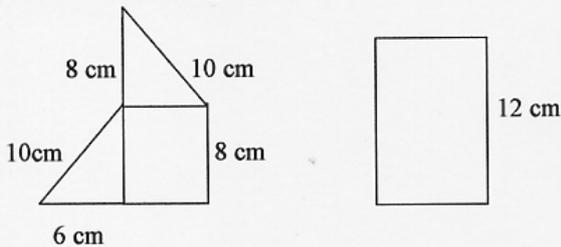
On the figure sketched freehand here (the real lengths are written in cm), are represented a rectangle ABCD and a circle with center A, passing through D. Find the length of segment EB.



**(Problem 2D: Circle and Rectangle)**

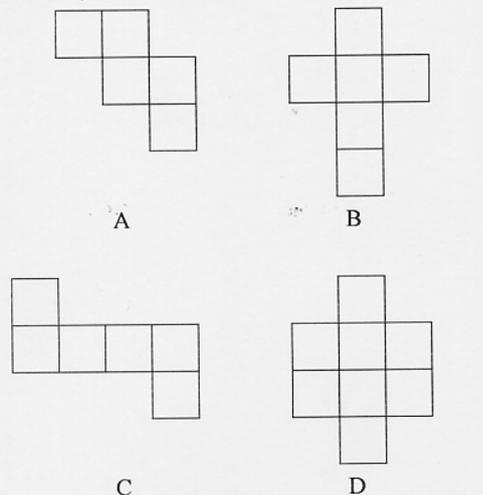
**(Problem 2D: Equal area figures)**

Figure A and Figure B have the same area. Find the width of Figure B.



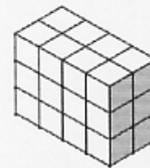
**(Recognition of the net of cube)**

Which of the following represent(s) a net of cube? Circle your answer(s).



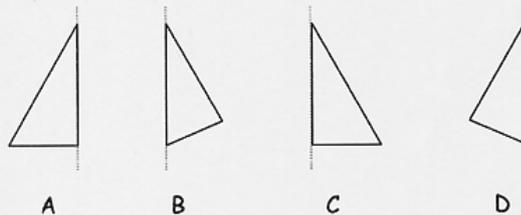
**(Task 3D)**

How many small cubes do we need to construct the following solid?



**(Spatial ability: image manipulation)**

What will the piece of paper look like if it is folded around the axis?



## **SPATIAL ABILITY AS A PREDICTOR OF STUDENTS' PERFORMANCE IN GEOMETRY**

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*The aim of this study is twofold: First, to investigate whether certain abilities compose 5<sup>th</sup> and 6<sup>th</sup> grade students' spatial ability and second to examine the relation between students' spatial ability and their performance in geometry. To this end, a model for spatial ability was formulated, and validated. The major constructs incorporated in this framework were spatial visualization (Vz) and spatial relations (SR). It was also hypothesized that tasks derived from the ETS kit of Factor-Referenced Cognitive Tests, constituted markers of the two constructs. A confirmatory factor analysis demonstrated that Vz and SR predict fairly well students' spatial ability. Path analysis revealed that spatial ability constitutes a strong predictor of students' performance in geometry.*

### **INTRODUCTION**

The development of a general spatial ability is an important factor associated with geometric understanding (Bishop, 1980). Spatial ability implies the generation, retention, retrieval, and transformation of visuo-spatial information (Colom, Contreras, Botella, & Santacreu, 2001) and is used in problem solving activities which particularly require the processing of visuo-spatial information. The ability to represent and process spatial information is important for many common activities, such as finding our way to and from places in the environment, moving furniture, packing a suitcase, and catching a ball (Hegarty & Waller, 2005). The National Council of Teachers of Mathematics (2000) emphasizes the importance of spatial abilities in mathematics education and stresses the significance of developing students' spatial skills throughout the geometry curriculum. It recommends that 2D and 3D spatial visualization and reasoning should be core skills that all students develop.

Although, there is much literature on the measurement of spatial abilities (Lohman, 1988), there is limited information on the relation between clearly defined spatial ability factors and measures of geometry ability. The relation of spatial and geometry abilities will enable educators to develop appropriate programs for the teaching of geometry. The aim of the present study is twofold. First, to investigate whether certain spatial abilities compose 5<sup>th</sup> and 6<sup>th</sup> grade students' general spatial ability and second to examine the relation between students' spatial ability and their performance in geometry.

## **THEORETICAL CONSIDERATIONS**

### **Definition of Spatial Ability**

Although various definitions have been used to describe spatial ability by psychologists and mathematics education researchers, there is a lack of a unique operational definition. Thus, the term spatial ability is identified as spatial cognition, spatial intelligence, spatial reasoning and spatial sense (Lohman, 1988), while Linn and Petersen (1985) define spatial ability as the mental process used to perceive, store, recall, create, edit, and communicate spatial images. Most other researchers define the concept of spatial ability through the use of factors resulted from analytic studies. For example, Lohman (1988), based on the results of a meta-analysis, proposed a three-factor-model including the spatial visualization (Vz), the spatial orientation (SO), and the spatial relations (SR) abilities. Vz is the ability to comprehend imaginary movements in a three-dimensional space or the ability to manipulate objects in imagination. The manipulation could be in a holistic, as well as piece-by-piece fashion and the movements must be imagined. Vz is often differentiated from SO by the mental processes and stimuli involved (McGee, 1979). SO is a measure of one's ability to remain unconfused by the changes in the orientation of visual stimuli. SO requires the mental rotation of the object as a whole whereas Vz requires the movement of parts of the object. Tests that measure spatial orientation require the subject to imagine how a shape would appear from a different perspective and then to make a judgement from that imagined perspective. SR is defined by the speed in manipulating simple visual patterns such as mental rotations and describes the ability to mentally rotate a spatial object fast and correctly (Carroll, 1993). Some researchers assert that the differing element between SO and SR is that in SO situations the body orientation of the observer is an essential part of the problem (McGee, 1979; Carroll, 1983).

### **Mathematics Education and Spatial Ability**

From almost the earliest days of intelligence testing, spatial ability has been considered to be closely related to academic achievement, particularly to success in mathematics. In addition to general intelligence, mathematical reasoning is typically thought to require abilities associated with visual imagery, as well as the ability to perceive, number, and space configurations (Hegarty & Waller, 2005). There is a substantial literature in which relations between factors of spatial ability, such as visualization, mental imagery, and mathematical performance have been investigated (e.g. Bishop, 1980; Presmeg, 1992). Though there are some differences in the literature, the importance of spatial ability to the development of mathematical thinking is supported by many researchers (Bishop, 1980; Tartre, 1990; Gutiérrez, 1996). Connor and Serbin (1985) found that the skills of spatial orientation and spatial visualization contribute meaningfully to predicting mathematics achievement. In a meta-analysis that included 75 studies, Friedman (1995) found that correlations

between spatial and mathematical ability generally ranged between 0,3 and 0,45. Although moderate in size, these correlations suggest a substantial relationship between spatial and mathematical abilities. However, research has shown that spatial ability correlates more highly with ability in geometry than with algebra (Bishop, 1980). Tartre (1990) found that spatial orientation skill was involved in specific ways in the solving of geometry problems, while Saads and Davis (1997) showed the importance of both spatial ability and language use in the on-going development of geometric thought. Tso and Liang (2002) showed that a significant correlation existed between students' spatial abilities and van Hiele levels of thinking in geometry. They suggested that spatial abilities are important cognitive factors in learning geometry and that incorporating spatial visualization and manipulation into learning activity could improve geometric learning.

### **THE PROPOSED MODEL AND THE PURPOSE OF THE STUDY**

Notwithstanding the extent of research into students' spatial ability, recent research has not investigated systematically the relation of spatial ability with students' performance in geometry. Accordingly, the literature does not provide the kind of coherent picture of how students' spatial ability affects their geometry thinking that is desirable for current approaches in geometry instruction. In this paper, we propose a model, which may enable 5<sup>th</sup> and 6<sup>th</sup> grade students' spatial ability to be described across two dimensions. As it is highlighted in Figure 1, we speculate that students' spatial ability is not a unitary construct, but it is defined by two spatial factors, spatial visualization (Vz) and spatial relations (SR). Although, many other spatial factors have been identified, in the proposed model we incorporate only Vz and SR which are the most prominent ones (Colom, et al., 2001). Based on a synthesis of the literature, it is also hypothesized that tasks derived from the Form-Board Test, the Paper-Folding Test, and the Surface- Development Test of the ETS kit (Ekstrom et al., 1976) constitute markers of the Vz factor, and tasks derived from the Card Rotations Test and the Cube Comparisons Test are SR markers respectively.

In order to capture the nature of students' performance in geometry, our model (see Figure 1) incorporates forms of geometry situations according to Van Hiele's theory (Van Hiele, 1986; Burger & Shaughnessy, 1986). Fifth and 6<sup>th</sup> grade students are expected to reach maximum the 3<sup>rd</sup> Van Hiele level, so in the model we only incorporate visualization, analysis and abstraction situations. Students at Level 1 (Visualization) recognize figures by appearance alone, often by comparing them to a known prototype. Students at Level 2 (Analysis) analyse figures in terms of their parts and the relationships between these parts, establish the properties of a class of figures empirically, and use properties to solve problems. At Level 3 (Abstraction) students perceive relationships between properties and figures.

The purposes of the study were: (a) to examine whether 5<sup>th</sup> and 6<sup>th</sup> grade students' spatial ability is composed by distinct spatial abilities (Vz and SR) and (b) to examine the relation between students' spatial ability and their performance in geometry.

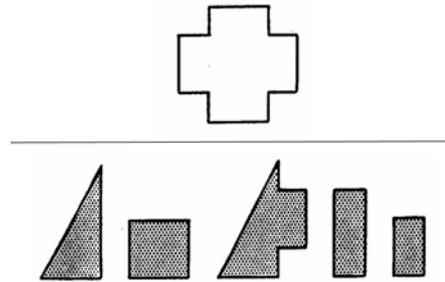
## METHOD

### Participants and Instruments

Data reported in this paper were collected through tests administered to 187 5<sup>th</sup> and 6<sup>th</sup> grade students in four urban schools in Cyprus (in Cyprus 6<sup>th</sup> grade is the last year of primary school). In the study two tests were used: the spatial ability test and the geometry test. The tests were administered during the students' mathematics course. Students were asked to answer each test in 40 minutes. The administration of the geometry test took place one week after the administration of the spatial ability test.

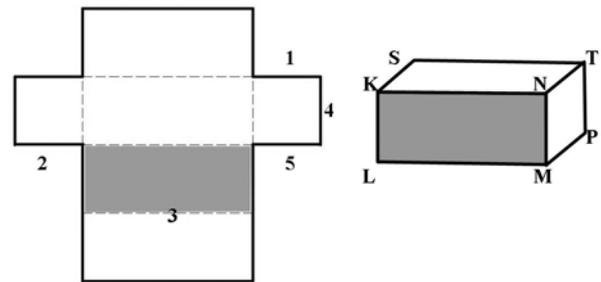
#### Form-board (3 tasks):

Indicate which of the pieces, when fitted together, would form the outline.



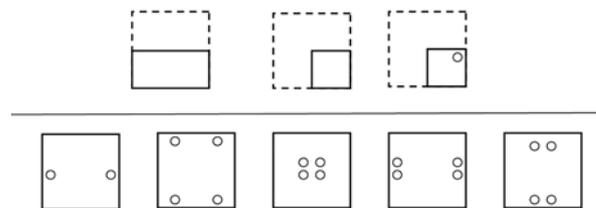
#### Surface-development (4 tasks):

The diagram shows how a piece of paper might be cut and folded as to make the solid form. Dotted lines show where the paper is folded. Indicate which lettered edges in the drawing correspond to numbered edges or dotted lines in the diagram (Segment "3" corresponds to the edge "LM").



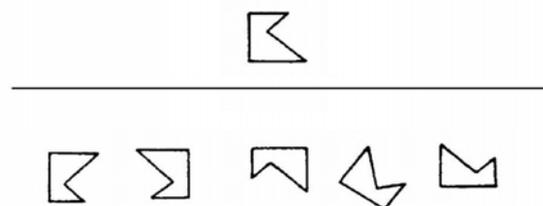
#### Paper-folding (4 tasks):

The final drawing of the folded paper shows where a hole is punched in it. Select which drawing shows how the punched sheet would appear when fully reopened (the paper cannot be rotated while folded or unfolded).



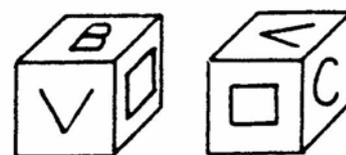
#### Card-rotation (4 tasks):

Under the card there are five other drawings of the same card, one of them merely rotated. Indicate the rotated card.



#### Cube-comparison (4 tasks):

Assuming no cube can have two faces alike, indicate whether these drawings can be of the same cube or not.



**Table 1: Examples of Spatial Ability Tasks**

The spatial ability test contained 19 tasks which were adopted from the Form-Board, the Paper-Folding, the Surface-Development, the Card-Rotation and the Cube-Comparison tests, which are included in the ETS kit (Ekstrom et al., 1976). These tests are considered strong markers of the corresponding factors, as described above, by several researchers (Lohman, 1998; Colom, et al., 2001). All tasks were modified in order to become suitable for 5<sup>th</sup> and 6<sup>th</sup> grade since the original ones refer to students in grades greater than 8<sup>th</sup>. Examples of the tasks are presented in Table 1.

The geometry test contained 10 tasks. Three of them were visual tasks which presented several drawings and students were asked to recognize specific shapes. The four analytic tasks required students to investigate the properties of figures and to calculate the area of figures. Finally, to solve the four abstract tasks, students were asked to perceive relationships between properties and between figures. Due to space limitations, examples of the geometry tasks are presented in Table 2.

	Circle the parallelograms:
Visualization task	
Analytic task	<p>Draw in the following grid a non right-angle triangle. The vertices of the triangle should lie on the bullets.</p>
Abstract task	<p>(a) Write down two differences and two similarities between a rectangle and a parallelogram.</p> <p>(b) Could you describe to a friend of yours how you can transform a parallelogram first to a rectangle and then to a square.</p>

**Table 2: Examples of Geometry Tasks**

## Data Analysis

The goal of the analysis was to estimate the relative strength of the proposed model. Because we proposed a theoretically driven model about the components of spatial ability, our first interest was in the assessment of fit of the hypothesized a priori model to the data. The assessment of the proposed model was based on confirmatory factor analysis (CFA). One of the most widely used structural equation modelling computer programs, MPLUS, was used to test for model fitting (Muthen & Muthen, 2004) and three fit indices were computed: The chi-square to its degrees of freedom ratio ( $\chi^2/df$ ), the comparative fit index (CFI), and the root mean-square error of approximation (RMSEA). The observed values for  $\chi^2/df$  should be less than 2, the values for CFI should be higher than .9, and the RMSEA values should be lower than .08 to support model fit (Marcoulides & Schumacker, 1996).

## RESULTS

In this section, we refer to the results of the analysis, establishing the validity of the latent factors and the viability of the structure of the hypothesized latent factors. In this study, we posited an a-priori structure of the proposed model and tested the ability of a solution based on this structure to fit the data and then conducted a path analysis to examine the relation between students' spatial ability and their performance in geometry.

### The Distinct Nature of Spatial Ability

To examine the first aim of the study, we proposed and examined the validity of a model consisting of two second-order latent factors (Vz and SR) which should have been able to model the performance of students on the tasks addressed in the spatial ability test and compose the third-order latent factor spatial ability. As it is highlighted in Figure 1, the Vz factor was composed by three first-order latent factors and the SR factor was composed by two first-order latent factors, respectively. The proposed model incorporated also a second-order geometry performance latent factor composed by three first-order latent factors: the visual, the analytic and the abstraction factors. CFA showed that each of the tasks employed in the present study loaded adequately (i.e., they were statistically significant) on each factor, as shown in Figure 1. It also showed that the observed and theoretical driven factor structures matched for the data set of the present study and determined the 'goodness of fit' of the factor model ( $\underline{CFI}=.92$ ,  $\underline{\chi^2}=125.81$ ,  $\underline{df}=90$ ,  $\underline{\chi^2/df}=1.39$ ,  $\underline{RMSEA}=.04$ ).

The r-squares (shown in parentheses in Figure 1) also illustrated that modest to large amounts of variance are accounted for all tasks corresponding to each spatial factor and suggested that ability in form-board, paper-folding and cube-comparison tasks explained the shared variance of their corresponding tasks much better than did ability in surface-development and card-rotation tasks. The structure of the proposed model also addressed the differential predictions of the ability in the tasks used for Vz and SR. Considering the effects among the two spatial ability factors revealed that

ability in the paper-folding and form-board tasks were the primary source explaining students' Vz ( $r^2=.95$  and  $r^2=.91$ , respectively), while ability in cube-comparison tasks was the primary source explaining SR ( $r^2=.93$ ). However, the structure of the proposed model revealed that Vz and SR have almost the same prediction validity on spatial ability ( $r^2=.99$  and  $r^2=.97$ , respectively). The abstract and the analytic factors were the primary source explaining geometry reasoning ( $r^2=.99$  and  $r^2=.99$ , respectively) while the visual factor had a moderate effect ( $r^2=.89$ ).

### **The Relation between Spatial Ability and Geometry Performance**

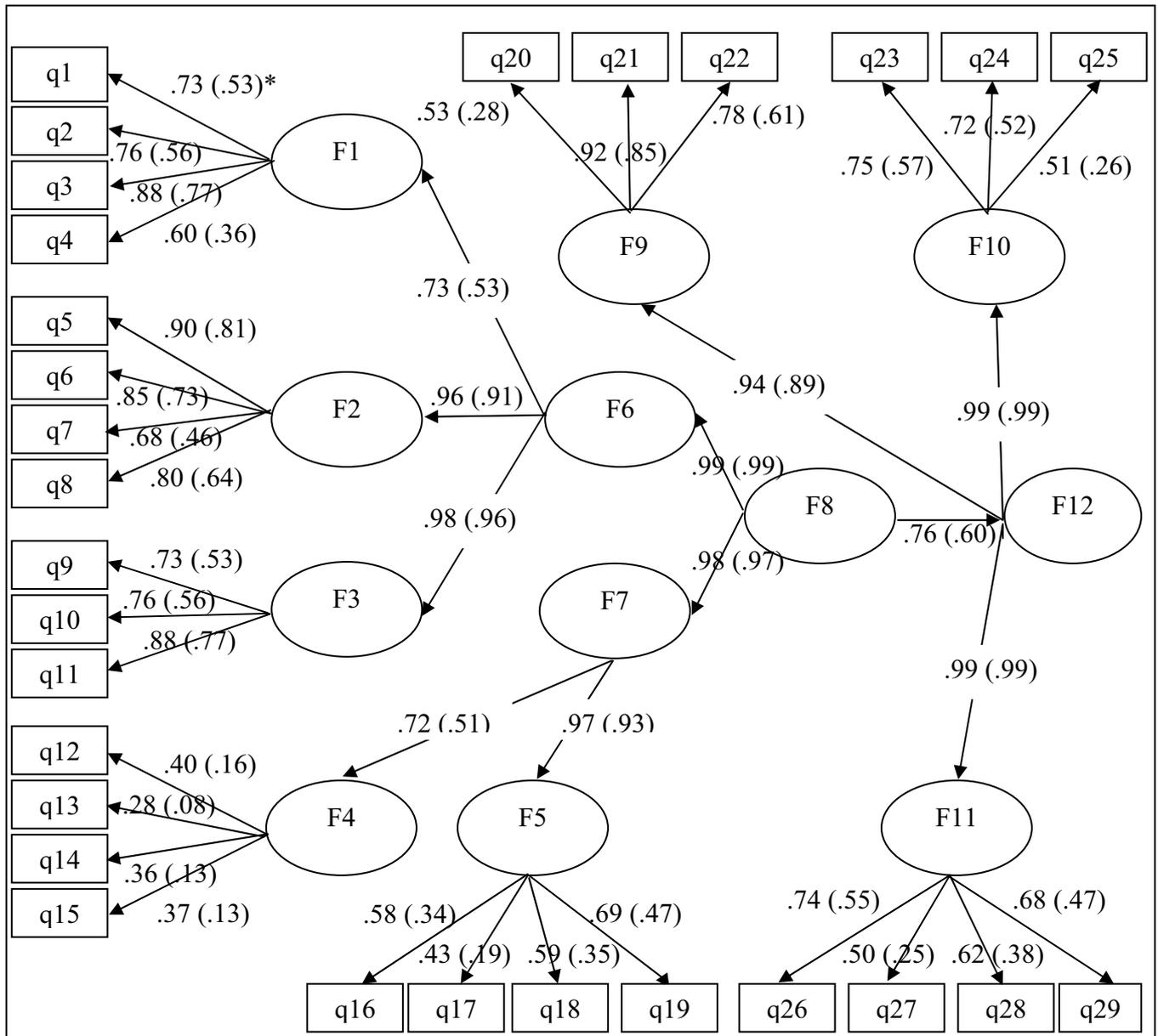
Path analysis was used to investigate the relation between students' spatial ability and their performance in geometry. Thus, we tested the validity of a model where the second-order latent variable geometry performance is regressed on the third-order latent factor spatial ability; assuming a causal effect between spatial ability and geometry performance (see Figure 1). The model fitted the data, and fitting indices were adequate to provide evidence that supported the relation implied in it ( $CFI=.92$ ,  $\chi^2=125.81$ ,  $df=90$ ,  $\chi^2/df=1.39$ ,  $RMSEA=.04$ ). These results gave strong evidence to the assumption that spatial ability is a predictor of geometry performance. The regression coefficient of spatial ability on geometry performance was extremely high ( $r=0.76$ ,  $z=4.37$ ,  $p<0.05$ ).

## **DISCUSSION**

Despite the fact that spatial ability has become widely recognized as a basic skill and, as such, has been incorporated in many tests of general aptitude or intelligence, its status as a predictor of geometry thinking has never been clearly established. There is also a need for a theoretical framework that outlines the space of different cognitive abilities associated with representing and processing spatial information. It was argued in this study that few models exist to help researchers and educators explain how spatial ability actually affects geometry thinking. Hence, the goal of this study was to articulate and empirically test a theoretical model to help educators build new understandings about the structure of spatial ability. The model integrated two prominent factors of spatial ability, spatial visualization (Vz) and spatial relations (SR) (Lohman, 1988; Carroll, 1993) and extended the literature in a way that validated a model examining the relation of spatial ability and geometry performance based on empirical, quantitative data.

The model proved to be consistent with the data leading to the conclusion that Vz and SR mediate 5<sup>th</sup> and 6<sup>th</sup> grade students' spatial ability. Specifically, it was found that Vz and SR contribute to students' spatial ability, with the two factors being the same important, showing that spatial ability is not a single, undifferentiated construct, but instead is composed of separate abilities, such as spatial visualization and spatial relations. These distinctions have evolved over decades of ongoing research. The validation of a path analysis model revealed a causal effect between spatial ability and geometry performance, indicating that spatial ability constitutes a strong predictor of students' performance in geometry. This particular finding suggests that

an improvement of students' spatial ability may result to an improvement of their geometry performance. The underlying assumption is that students improve at solving geometry tasks with practice in spatial ability tasks. These findings show also that spatial ability is important in various geometrical situations



**Figure 1: The Structure of the Proposed Model**

Note: F1=Surface Development test, F2=Form board test, F3=Paper Folding test, F4=Cube Comparison Test, F5=Card Rotation test, F6=Spatial Visualization, F7=Spatial Relations, F8=Spatial Ability, F9=Visual, F10=Analytic, F11=Abstraction and F12=Geometry Performance, q1-q19 refer to the spatial ability tasks and q20-q30 refer to the geometry ones.

\* The first number indicates factor loading and the number in parenthesis indicates the corresponding  $r^2$ .

The model used in this study offers teachers and researchers a means to examine the complexity and sophistication of spatial ability. From the perspective of teachers, the model may be used in order to include in their instruction activities that contribute to the development of Vz and SR and consequently to the development of their geometry performance. From the prospective of researchers, it is likely that the model could be useful as a prototype for further analyses of the structure of spatial ability and its relation with geometry performance. The model can also be linked with recent research on people's cognitive style and how they represent problems in mathematics. Kozhevnikon, Hegarty and Mayer (2002) showed that visualizers with high spatial ability are more successful in problem solving because they are more likely to construct diagrams or schematic spatial representations of the spatial relations between objects described in a problem. As a result, the model could be used to further examine how students with different levels of thinking in spatial ability and cognitive style can grasp the spatial relations in pure geometrical situations.

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## COMPUTER GEOMETRY AS MEDIATOR OF MATHEMATICAL CONCEPTS[1]

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*This work takes cue from a “paradox”, based on the destabilization of the concepts of equiextension and equidecomposability, which occurs when specific computer software for the game Tangram is used. We presented this paradox to the students of various ages, with the aim to investigate their behaviours in consequence of it. We examine the strategies used and the concepts, geometrical or not, employed to justify the paradox.*

### INTRODUCTION

The literature on real or apparent paradoxes in mathematics is very rich. Particularly well-known is “Riddles in Mathematics” (Northrop, 1944), which classifies them into groups such as arithmetical, geometric, algebraic, logic, etc. But, nowadays a new type of paradox appears, that concerns graphical visualisation on a computer screen.

In particular, there are different paradoxes about Tangram (Pellegrino, 1986); for instance, that of Dudeney is based on ingenious dispositions of all seven pieces, which misleads the visual perception.

Here we use the word ‘paradox’ with the etymological meaning: a thing that contradicts a common opinion. We can also give this definition:

“something which at first sight seems false, but which is in reality true, or something which seems true, but which is false, or something which is simply contradictory” (Northrop, 1944)

Yet in our case, the conflict arises between some images on the computer screen and some concepts of Euclidean geometry. The paradox don’t is inside a theory, it suggested in the computer world, but it refers to geometrical world. We could denominate it “*xenodox*” (from Greek word “ξένος” that means “foreign”), with reference to it. In other words, our use of word paradox can be inappropriate, but we choose it for simplicity.

This work doesn’t concern the computer geometry. It uses particular and unsuitable Tangram software, that presents an apparent paradox and creates a cognitive conflict. The aim is to investigate the student’s behaviour in front of this paradox, especially to observe their use of geometrical intuition and knowledge, when they search to justify an apparent contradictory geometrical situation. We want to observe also how much the ordinary geometry is consolidating in the minds of the students. Another aim is to study the possible misconceptions that this kind of activity can do to emerge.

In this research, a particular paradox provides a basis for consideration of some mathematical concepts: area, equiextension, congruence of shapes, geometrical transformation (translation, rotation, glide reflection), invariance for congruence. We

think that this riddle would be a good starting point to study ‘discrete – continuous’ conflict, concept of irrational number. We propose to work afterwards about this.

At the beginning we studied the paradox from a mathematical point of view, after we presented it to students between the ages of 10 and 18 [3]. Finally we analysed the different types of answers they gave.

## **THEORETICAL FRAMEWORK**

Our first question is whether it is opportune to submit a paradox to the students. From the constructivist point of view, the answer is certainly yes, in that the paradox is an instrument for creating a stage of disequilibrium; this causes a cognitive conflict, which in turn generates or facilitates acquisition of new knowledge (Henry, 1991). In our work, the conflict is a result of the contrast between previous geometrical knowledge and what appears on the computer screen. In other words, a conflict occurs between what the mind thinks and what the eyes see. For the conflict to be overcome we need investigate it and to deepen some mathematical and informatics knowledge.

Another possible answer to the previous question is: to submit a paradox to the students it is an occasion to study their argumentations, their thinking and also possible mathematical misconceptions.

In didactical research in mathematics is well-known the role of misconceptions as a source of errors. The misconceptions have an influence on the learning and, of consequence, they are an useful instrument in order to prevent learning difficulties. The presentation of ‘strange’ situations, that can be in conflict with the opinion of the students, often provokes to emerge of misconceptions. The awareness of misconceptions influence is documented in research in didactics of mathematics (Schoenfeld, 1985; Maurer, 1987).

This research is based on computer’s drawings, that are objects of Geometry I or “Natural Geometry” in which “the source of validation is the sensitive” (Houdement, Kusniak, 2003). “To produce new knowledge in this paradigm, all the methods are allowed: evidence, real or virtual experience and of course reasoning. The backward and forward motion between the model and the real is permanent and enables to prove the assertions: the most important thing is to convince.” (Houdement, 2007). In secondary school the drawings are only one starting point for more in-depth study, but also in primary school it is important the transition from intuition to the argumentations.

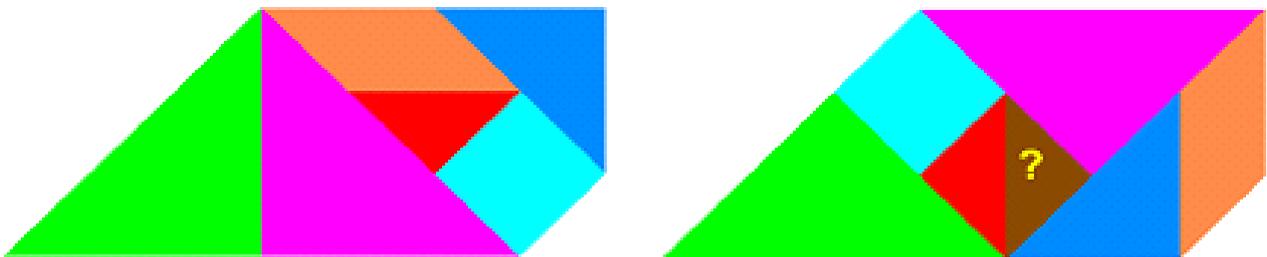
For some time the role of informatics instruments in teaching–learning of mathematics has been under discussion in didactic of mathematics. The studies have focussed particularly on computer mediation for comprehension of mathematical concepts, and is investigating new opportunities for interaction between knowledge and learner (Bottino & Chiappini, 2002). Often the computer is considered as a tool for presenting concepts in an innovative and stimulating way. But learners are warned

of the limits of computers and software. For example, in geometry, it is emphasised that drawing with Cabri-Géomètre software allows the user to formulate conjectures, but not to prove them.

In this paper, the computer is taken to be a cultural artefact (Saxe, 1991) which mediates geometrical concepts (equiextension, conservation of quantity, isometry, etc.) and informatics concepts (computer geometry, approximation).

### THE PARADOX

The Chinese game Tangram has been a candidate for graphics programmes, since operational systems started to elaborate sophisticated graphics. TANGRAM software, designed and manufactured by the Dutch professor Mark Overmars, allows the user to construct shapes at different levels of difficulty. With a click of the mouse you can shift or rotate or overturn each piece to construct a shape. When the shape is formed correctly, there is ‘applause’ from the computer, and another shape to build appears on the screen. The idea of this work born casually: I used this software with the aim to drawing shapes of Tangram and I made some attempts of constructions of shapes. One of these, an irregular pentagon, was very interesting because it was possible to construct it on the screen indifferently with 6 or 7 pieces of Tangram, but an applause arrive only in the second case. The following images, realised with software above-mentioned, show the “strange” situation described. What explanation for this apparent ‘deception’?

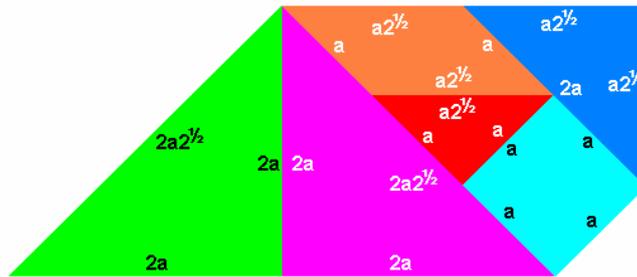


**Fig. 1: the paradox**

At the beginning, I studied this problem for a personal curiosity. I used Paint Brush to verify that the two pentagonal shapes in Fig.1 can be perfectly superimposed on one another. Seemingly they can be formed in a different way with a different number of pieces and it would seem to mean they have different areas! I decided to investigate this in deep and, after my personal study, I found a possibility to use it as the starting point of a didactical research.

We label  $a$  the cathetus of the smaller triangles and we base the other measurements on  $a$  (Fig. 2). There it appears to be something strange in the central part, in which a

segment has a length of  $2a\sqrt{2}$  as the hypotenuse of an isosceles triangle with cathetus  $2a$ , while its length becomes instead  $3a$  as the “sum” of three segments of length  $a$ .



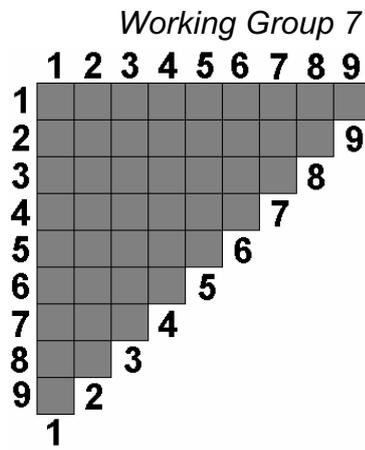
**Fig. 2: algebraic explanation**

Using this algebraic notation we have also that the areas of the two shapes of Fig. 1 are respectively  $15a^2/2$  and  $8a^2$ . In order to find why the paradox arises, we can take a two dimensional concrete tangram and we can attempt to reproduce the two shapes. We can observe that it can be constructed only with seven pieces. The shape on the screen made up of six pieces appeared to have no concrete existence. Why is this?

Evidently the software does not conserve the extension of some of the pieces. If we observe the Fig. 1, we note that the squares are oriented in the same way in both shapes and that their dimensions do not change. Other pieces of the tangram, the parallelogram, the ‘medium’ triangle and the small triangle are unvaried as they are rotated of  $90^\circ$  in the passage from the first to the second shape of Fig. 1. But the ‘large’ triangles ‘shrink’ in the rotation. In this software when a shape is rotated of odd multiples of  $45^\circ$ , a similar shape is obtained, but it is not congruent to the initial shape.

It would be interesting and significant to investigate about the reasons for this, but it lies outside our aims. Yet it is important to furnish an explanation to the students, after the paradox’s presentation. It is sometimes said that the computer is ‘wrong,’ but a more accurate observation is that computer graphics are not ‘Euclidean’. We choose to re-propose a geometrical explanation based on pixels since, en effect, the focus of our work was into geometry. We found this possible “trivial” explanation: that succeed because computer graphics use the pixel. We are conscious that is it possible to do others justifications, more appropriate and pertinent, but we want remain in geometry world and we prefer to take this problem as a starting point for speaks about discrete geometry.

The computer screen use a discrete geometry based on pixels, a “graphical atom”; every picture on the screen is composed of pixels. A geometrical shape is realised by putting pixels together in an opportune way. Of consequence, a diagonal line that forms an angle of  $45^\circ$  with the horizontal is formed by the same number of pixels composing each of its orthogonal projections (Fig 3). In this strange triangle, the cathetus and the hypotenuse are all made up of the same number of squares.



**Fig. 3: a “pixel-triangle”**

So when the software a piece of the Tangram is rotated by an odd multiple of  $45^\circ$  it is transformed into a shape that is similar but not congruent.

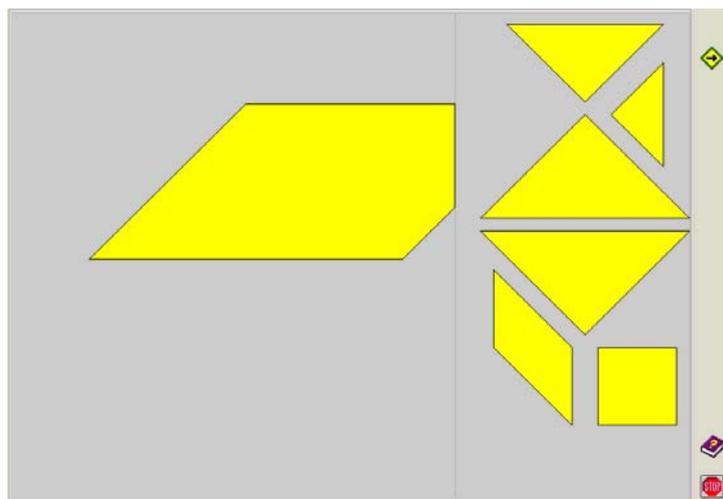
The computer graphics is discrete and not continuous and not allows the representation of irrational. In other words, we meet again

“...the idea of the world as a finite set of atom-points probably held by the early Pythagoreans” (Speranza, 1997).

On the screen, length and area are measured in the same unit, the pixel. Now it is clear why it is possible to put the pieces of tangram side by side virtually and obtain the result shown in Fig 1. The measurements of some sides of the tangram pieces are expressed as a function of  $\sqrt{2}$ , but this cannot be shown correctly on the screen.

## RESEARCH METHODOLOGY

We presented the paradox to learners at different stages of schooling, in different ways according to their age.



**Fig. 4: the task on the screen**

In primary school, we involve two classes of children 10-11 years old (45 pupils). They worked in small groups (4-5 pupils in each group), with the sequent modalities. At the beginning the activity is proposed with the computer, under teacher guidance. The pupils were asked to construct the following pentagonal shape with six pieces,

using TANGRAM software previously mentioned (Fig. 4). If it was necessary, the teacher suggested some help. The task was: “Cover the pentagonal shape in the screen with these six pieces of Tangram, that compare in the right side.” They were around the computer, they used in turns the mouse, all of them observe the screen and suggest how to put the pieces on the screen.

The learners were next asked to rebuild the same shape on the screen using the seven pieces of Tangram. The ‘strange’ situation was emphasised by the teacher. Afterwards the children were asked to think about it and to work all together on the desk. We provide them with a sheet of paper in which there were copies of Fig 1. They were also asked to attempt to explain the ‘paradox’. Another sheet of paper for each group was used for writing a report of observations and of discussion results. Finally we proposed to rebuild the pentagon using concrete two-dimensional pieces of Tangram. The aim of this final activity was to do a final answer to the question. In this case, the children conclude that “the computer made a mistake”. The teacher prefers to say: “The computer makes what it can do!” and he delays later an in-depth explanation.

In the secondary school the experimentation involved one class of 25 students 12-13 years old and two classes of 42 students 17-18 years old. The work was at the beginning all together in the computer classroom and after individual: the students were asked to look the projection of the computer screen. We make both constructions, with 6 and 7 pieces, and we ask to comment on what they saw. After each student was asked to investigate about the copies of two shapes of Fig. 1 onto paper, to make comments in writing and to provide some explanations of the paradox.

Later we analyse the protocols of each primary school groups, the papers wrote from secondary school students and also the films recorded during the activities.

## **RESEARCH RESULTS AND CONCLUSIONS**

This research allows us to make only a qualitative description of learner’s behaviour. It is however richly significant in analysing learning of some concepts and in investigating learner’s conceptions.

On the whole, the mains interesting results came from Primary School pupils. First of all, we must explain that we worked in two particular classes: the teachers are also researchers that usually collaborate with us. Always they, in their didactical activity, ask to the pupils to do a justification of their answers to the questions. Those classes were already familiar with formulae of area and perimeter and they had already worked on “equi-composed” shapes.

The younger learners carried out the activity with enthusiasm and showed great familiarity with the computer, perhaps even blind faith. Only one girl, after her group had built the shape with six pieces, declared that it was useless to continue and that the next exercise was impossible. The others believed what they saw on the screen, at

least until the teacher pointed out the paradox. In the second part of activity, when they worked in groups on a sheet of paper, with reproductions of two constructions, they used different approaches to the problem. At the beginning, we have two different kind of explanation:

- *explanation based on 'big' triangles*: the children perceive that the problem concerns the bigger triangles of Tangram, using observation and visual intuition (7 groups). To test this hypothesis, they choose different methods: measure with scale the sides of big triangles, measure and calculation of areas with comparison, use of scissors and superposition.

- *explanation based on equi-estension*: “In the first shape a piece must have the same area equal that two shape in the second” (1 group). This argumentation is based on geometrical idea that congruent shapes have the same area, but the observation shows that in the second there is a piece more.

The others 2 groups attempt to explain, without success. In particular, in one of these a pupil refuses the question: the paradox creates unease and it stops the resolution.

The most interesting conjectures arise from the first explanation: how justify the role of the big triangles? Some groups (4) found the cause in a difference of occupied space, that depends to the mutual position of big triangles:

“In the first shape the big triangles are near to one another and they take up a lot of space. In the second shape the big triangles are distant and they take up less space”.

So they doubt the conservation of quantity. The pupils think that the positions of triangles (near or far) change the occupied space. They correctly found the reason of the paradox, but they justify their argumentation using a misconception: two shapes occupy different space if they are near or far. This phenomenon is documented in researches with lower aged pupils (Montis, 2003), but here re-emerge. It was a surprising answer for us; we suppose that the conservation of quantity has not been completely acquired. It may be that the paradox leads the learners into error, but further investigation about this, which we are unable to report here, confirmed this hypothesis. Only one group attempts a check with calculation of areas: they calculate the area of the small triangle more in second shape; they calculate also the difference between the areas of big triangles present in the first and the second shape, finally they compare these areas, but with failure.

Also the invariance for isometry seems in doubt. In two groups the problem it is found in the different orientations of big triangles. In effect, they distinguish a triangle that “stand up” from another that “lie down”, because they are perceptively different.

“If we put the triangles ‘in height’, they occupy more space. If we put ‘in largeness’, they occupy less space”.

This is another misconception: the space occupied by a shape change with the its ‘position’. Others activities confirmed that also the shape can be change with position

(Vighi, 2003). A possible explanation of it is that frequently in Italy the focus of Geometry is the measure of perimeters and areas. A sort time is dedicated to “division in equal parts” of shapes.

The learners also showed the ‘conflict between perimeter and area’ (Marchetti, Medici, Vighi & Zaccomer, 2005): some pupils wrote that the perimeter of the shapes is the same but the internal areas may vary.

We observe two main differences for learners aged 12-13: the missed recourse at perception and eyes, the systematic use of the scale, measure and formulas. They first measured the shapes sides they had copied onto paper and made long laborious calculations of the areas. They supplied empirical verification that the problem is absurd and that equi-extension does not take place, but they could not supply any explanation. They conclude: “The computer makes a mistake”. We attempt to throw again the problem, without success. We don’t have significant results in this class: the pupils renounce to think and make only a lot of calculation.

Learners aged 17-18 gave more significant replies. In the first part of the task, some observed that the problem lies in “the arrangement of the pieces”. Others ascribed it to “automatic adjustment” made by the computer or to a problem in segment’s contact (overlapping? Few millimetres incorporated in separation lines?). Others simply noted that it is impossible for one of these constructions to exist (oneness of solution). Others were worried that “the computer does it, but it can’t be true”. Others made a dangerous distinction between area and “occupied space”: the misconception above reported reappears. In the second part of the task they mainly used the numeric and algebraic register, similar to ours used in Fig. 2. Some of them used geometry and broke the small triangles as units of measurement, without however reaching any conclusions. One student made recourse to a Euclidean geometrical theorem and emphasised that it was not verified by the computer screen. Others suspected the existence of properties of areas which they are as yet unaware of. Some students speak about “a problem of proportion” connected with “shapes similar but not identical”, but nobody locates the problem in particular rotations. The transformation geometry seems not interiorised.

Only one student identified the possible cause of the problem in the presence of irrational  $\sqrt{2}$  and of the ratios between measurements of sides. His comments prompted us to re-propose the “crisis of the incommensurable” and discuss the concept of number, particularly irrational number, with the class.

This paradox was chosen as a focus of research as a vehicle for significant thought processes, as well as instrument to reflection on geometrical concepts. The hands-on verification on paper led the younger learners to the conclusion that “you can’t place trust in computer”. For the older learners it was a source of enquiry and finding out. From the teaching point of view, the opposition between the Euclidean and virtual reality is constructive. In this case we choose to use it for supplies an important opening for the introduction of ineluctable and necessary irrational number. This does

not mean that the mathematical approach is preferable, rather it emphasises the relativity of concepts also in mathematics. Nowadays there is insistence on the history and epistemology of mathematics in teaching. Our experiment shows a good way of continuing the work: introducing learners to a historical learning process through the modern instrument of the computer. Starting from a point as ‘grain of sand’ to a geometrical entity with no dimensions, from discrete to continuous, from irrational number perceived with many decimal digits but not infinite, from the limited to the unlimited, from the finite to the infinite. In other words, we share the following sentence:

“New technologies offer occasions to allow to learner a complete *mathematical experience* and, in this meaning, they offer the possibility to realize a *reasonable didactics of mathematics*” (Paola, 2001).

## NOTES

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2. [paola.vighi@unipr.it](mailto:paola.vighi@unipr.it)
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