

<u>WORKING GROUP 14. Advanced mathematical thinking</u>	2220
Advanced mathematical thinking	2221
<i>Joanna Mamona-Downs</i>	
Lagrange's theorem: What does the theorem mean?	2231
<i>Buma Abramovitz, Miryam Berezina, Abraham Berman, Ludmila Shvartsman</i>	
University students generating examples in real analysis: Where is the definition?	2241
<i>Samuele Antonini, Fulvia Furinghetti, Francesca Morselli, Elena Tosetto</i>	
Is there equality in equation?	2250
<i>Iris Attorps, Timo Tossavainen</i>	
Analysis of the autonomy required from mathematics students in the French lycee	2260
<i>Corine Castela</i>	
Local and global perspectives in problem solving	2270
<i>Martin Downs, Joanna Mamona-Downs</i>	
The application of the abductive system to different kinds of problems	2280
<i>Elisabetta Ferrando</i>	
University students' difficulties with formal proving and attempts to overcome them	2290
<i>Justyna Hawro</i>	
The interplay between syntactic and semantic knowledge in proof production: Mathematicians' perspectives	2300
<i>Paola Iannone, Elena Nardi</i>	
Belief bias and the study of mathematics	2310
<i>Matthew Inglis, Adrian Simpson</i>	
Students' concept development of limits	2320
<i>Kristina Juter</i>	
Habits of mind associated with advanced mathematical thinking and solution spaces of mathematical tasks	2330
<i>Roza Leikin</i>	
Mathematical background and problem solving: How does knowledge influence mental dynamics in game theory problems?	2340
<i>Francesca Martignone</i>	

Student generated examples and the transition to advanced mathematical thinking	2349
<i>Maria Meehan</i>	
Understanding of systems of equations in linear algebra	2359
<i>Maria Trigueros</i>	
Students' choices between informal and formal reasoning in a task concerning differentiability	2369
<i>Antti Viholainen</i>	
Interplay between research and teaching from the perspective of mathematicians	2379
<i>Carl Winsløw, Lene Møller Madsen</i>	
Advancing mathematical thinking: Looking back at one problem	2389
<i>Rina Zazkis, Mark Applebaum</i>	

**SYNOPSIS OF THE ACTIVITIES OF WORKING GROUP 14,
CERME-5, ON THE THEME OF
'ADVANCED MATHEMATICAL THINKING'**

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INTRODUCTION

In 2005 the working group 14 on 'Advanced Mathematical Thinking' (hereafter abbreviated to AMT) had its inaugural meeting in an ERME conference, i.e., CERME-4, and attracted a good response both in terms of the papers received, and the number of participants. I am happy to report that the second meeting, at CERME-5, received an increased number of papers, covering a very varied range of issues. Also it is satisfying that quite a few participants attended both meetings, so we were able to achieve some continuity in building up the group's output, that we hope will be sustained in the future.

The main component of the group's activities was to discuss the nineteen papers that had been submitted and accepted for presentation. Eighteen of these presentations were delivered, and seventeen of the papers were accepted for the post - conference proceedings and hence appear below. During the sessions every author gave a 5-10 minutes talk, which instigated a discussion about or around the paper. How long this was sustained depended completely on the response of the audience. The intention was to use the presentations as prompts eventually to raise more general issues, but in reality some sessions were mostly honed directly to the material presented.

We had arranged the papers in groups according to broad topics; these topics were largely determined by the content of the papers received but we believe they constituted main trends currently taken by research at AMT level in general.

We were somehow ambitious in the number of set themes we treated, which was eight. Also the last two sessions were set apart for an opportunity for the participants to comment on the overall achievements, on the character of the group's title theme, on priorities in research topics and on changes in organization for future meetings. In

order to treat all these themes we decided to place the participants into two groups except for the first and last time-slots (from the seven available).

The rest of the paper first lists the 8 themes taken up, then summarizes the presentations and discussion that took place in each of the 12 sub-sessions, followed by a concluding section.

THE THEMES

- ◆ What is Advanced Mathematical Thinking? This topic has been argued ever since the term was introduced. The main controversy lies in that some researchers claim there is a potential 'continuous' path to lead the cognition from 'school mathematics' to 'university mathematics', whilst others claim that there are unavoidable leaps that require radically different modes of thinking. Further, A.M.T. is a term that educators created for themselves, so it is in place to consider whether its philosophical positions fit with the research identified to this tradition.
- ◆ What is the role of (formal) proof in mathematics? Why do many students have difficulties with proofs for simple consequences of given definitions? Students can 'see' reasons for a result, but might have great difficulty in converting these into an acceptable presentation. How important is it to formulate effective symbolism to achieve proofs?
- ◆ What is the status of the 'problem-solving' perspective in A.M.T.? Should there be more consideration of mathematically based techniques or lines of thought that are more specialized than heuristics, yet have universal potential in terms of application? Have the different strands of problem solving research been developed and integrated to the degree to describe an optimal profile for the management required for being efficient in solving problems?
- ◆ There are many models of general mathematical reasoning extant. Amongst the more recent are the triad of induction/ abduction/ deduction, and the triad procedural/ syntactic/ semantic. In particular, how is mental argumentation at the A.M.T. level accommodated in such models?
- ◆ There are long established and often applied models of conception and 'objectification'. Under what circumstances do these frameworks act in a productive way? How do these models fit in with other words with allied meaning, such as entity and construct?

- ◆ Recently there has been a lot of attention by mathematics educators on students' generation of examples (and non-examples and counter examples). Such activities would seem particularly pertinent to A.M.T. Could it be used in a methodical way in teaching and task design? What would be the advantages / disadvantages?

- ◆ In some universities, general introductory courses are proffered. On the main, one of two directions is taken, either in problem solving or an introduction to the basic fundamentals of mathematics. Mathematics educators are well familiar with the first direction, but there seems less attention to or participation in the latter. How should the community treat the field of basic logical constructions? In particular, how does 'vernacular logic' differs from 'mathematical logic', and how does this difference might effect students' behavior in doing mathematics?

- ◆ The relationship between the mathematics educator and the mathematician, especially in terms of the latter's positions on didactics. Educators in recent years have shown interest in this latter aspect, but conversely it is rare to find studies that try to inform and to convince lecturers about the educators' research output as a channel to change teaching practices.

The issues that have been raised for each theme above did not necessarily reflect the actual presentations and discussion allocated to the theme. This content is described in the next section.

SUMMARIES OF THE SESSIONS

Introductory session, concerning the character of Advanced Mathematical Thinking.

After a brief welcoming address to the participants, three papers were presented and discussed. First to talk was Corine Castela. She compared Chevallard 's anthropological framework concerning the evolution and re-organization of mathematical resources with the tradition of problem solving espoused by Schoenfeld and other researchers. She stated that the former proposes a didactical process that involves six distinct 'moments'. As the mathematics becomes more involved and complex, it becomes infeasible to cover all the moments through direct intercourse with the teacher, so students to succeed have to develop autonomy in developing their own techniques. The second presentation was conducted by Carlo Marchini (the corresponding paper is not published in the proceedings). He described a long-term

project with the purpose to motivate older school students to get involved with abstractly defined mathematical concepts. The mathematical theme taken was 'the finite'. After the students were prompted to realize the weaknesses in informal descriptions including some stated in school textbooks, they were led by their teachers to read and decipher the definitions of 'finite' found in articles on fundamental set theory. In a survey concerning the students' reaction to going through the project, most showed a positive disposition; the students said because of it they were more likely to take on mathematics that at first did not seem to have relevance or usefulness in the 'real world'. The third presentation was by Rina Zazkis. She opted to consider AMT from the 'advanced thinking' angle in a problem solving context. She in particular stressed 'looking back' in a broader context to that the term sometimes is used. She also illustrated what she calls 'outside tools', i.e., applying one's previous knowledge in a situation that does not immediately suggest it, and 'reconstructive generalization' (due to Harel and Tall) where "the existing schema is reconstructed in order to widen the application range".

Session on Proof

Justyna Hawro described a fieldwork that involved tasks including constructing simple proofs that 'should' follow easily from a given definition, generation of examples and analyzing written proofs. From this was extracted an identification of difficulties that students have with proof, and compared these to those of a paper by R. C. Moore with a framework of 'concept image, definition and usage'. She also referred to possible didactical approaches that may alleviate the students' problems. Ludmilla Shvartsman sought to provide students a way of laying a basis for a proof by a process of first giving related propositions and make the students argue whether they are true, and then to induce them to form their own examples and counter-examples. This gave them a sense of how the conditions have to be accommodated in the proof. The mathematical theorem dealt with was Lagrange's Mean Value Theorem.

During the discussion following the presentations several themes arose, including:

- (i) What is the role of generic examples in constructing proofs; would encouraging students to keep these in mind would facilitate their proof production?
- ◆ Do the constructs of semantic and syntactic proof production represent genuine cognitive styles, or mere tendencies of behavior?
- (iii) Do we know enough about the conceptions of proof commonly shared by expert mathematicians?

Session on problem solving

Joanna Mamona-Downs addressed the cognitive effect in taking local and global foci in problem solving. Themes raised here include the need of intertwining the two, the issue of transparency, to distinguish functions from correspondences, and making students aware of general mediums helping the organization of argumentation such as the notion of independence. Roza Leikin talked about the habits of mind that straddle mathematical theories; she illustrated two, reasoning by continuity and symmetry. She also exhorted that students should be encouraged to give more than one solution to any given task as a habit of mind in itself, and as a channel for realizing others. Francesca Martignone treated a case study concerning a task involving Game Theory for which the solver first used episodic exploration based on recursive or temporal thinking, but then made a switch to exploration structured by gained knowledge that allowed a one step resolution of the situation.

The general discourse included the following questions. (1) What is the relationship between Advanced Mathematical Thinking and the notion of Structure, and what is the connection of the latter with 'elegant' or 'smart' solutions? (2) If transparency in a solution is connected to structure, how transparency is recognized? (3) If Advanced Mathematical Thinking is related with giftedness, how? (4) What is the connection between ability to solve a problem and the desire to engage with the task? (5) How do multiple solutions inform AMT? (6) What is the role of the 'didactical contract' in the teaching and learning of AMT?

Session on Models of Mathematical Reasoning

Antti Viholainen talked about students' choices between informal and formal reasoning. He conducted a fieldwork where the tasks involved differentiability. He categorized the students participating as 'formal approachers', 'informal approachers', or students that started informally then switched to the formal. He then indicated how well the students from each category performed.

The ensuing comments from the audience included the following. What is the meaning of 'convincing' for the students? Does the focus on visualization unnecessarily narrow the meaning of 'informal'? Does the formulation of tasks influence the students to take informal or formal approaches? The use of graphing calculators should be avoided or not for considering derivatives of complicated functions from a pedagogical point of view? Is a good understanding of a formal definition implicit in well-grounded visual reasoning?

Elisabetta Ferrando described her own elaborated model on abductive reasoning, a term first introduced by Pierce to accommodate the more creative aspects in the doing of mathematics such as forming hypotheses, intuitions and conjectures. In particular she introduces a term 'abductive statement' that is "a proposition describing a hypothesis built in order to corroborate or to explain a conjecture".

The discussion focused on:

Would it be possible or desirable to teach the identified skills associated with abductive reasoning, including hypothesis generating and justification, in class time?

Students' apparent unwillingness to 'explore'.

Relationships between the ideas of abduction and Schoenfeld's problem - solving heuristics.

The notion of transformational reasoning (due to Harel) represents a case of creative processes associated with abduction.

Can abductive skills be taught explicitly, or must they necessarily arise naturally, at least in the case of certain kinds of problem situations.

Session on models of conceptualization and objectification

Here two papers were presented. Maria Trigueros applied APOS theory to examine students' ability in solving and comprehending the sub-spaces of solutions of systems of linear equations. The emphasis in the study was put on the 'schema' aspect of APOS in relating the many concepts required in this mathematical theme, such as set, function, equality, vector space and geometric interpretations. Her results brought out especially the cognitive importance of the evolution of the notion of variable in its different interpretations. Kristina Juter refers to several well-known models of conceptualization and objectification in her study of the evolution of students' understanding of limits of real functions. In particular she constructed a 'concept map' for this topic based on David Tall's framework of 'three worlds of mathematics', that extend to related notions such as continuity and differentiability. The discussion raised doubts in the wisdom of 'mixing' frameworks; it was suggested that composing too complex schemata can lead to a degree of arbitrariness, and the use of theoretical frameworks are often too self-referential.

Session on student example generation

Two papers were presented and examined student's abilities in generating examples, and the profit to the student in undertaking the relevant activity. Both were set in the context of Real Analysis. Maria Meehan followed the framework of 'boundary examples' due to Watson & Mason, in which a programmed succession of example generation concerning a concept was designed where extra conditions are successively added, and then examples were requested for which all but one of the conditions hold and contravene the remaining one. It is claimed that such programs can lead to enhancement of the students' appreciation of the concept. Also improved skills in verification resulted. Francesca Morselli gave tasks that required students either to construct an example that satisfies a given proposition or to argue that no instances could exist. She was guided by the framework of 'concept image' and 'concept definition', and also how visualization can interfere or be consonant with analytic strategies. In a case study, she illustrates how this type of task can remedy

false preconceptions. The general discussion raised the difficulty how to measure the effect on students from their experiences in example generation.

Session on the fundamentals of Mathematics and logic

Matthew Inglis addressed belief biases in reasoning, i.e., how much people are influenced in judging the status (true, false, undecidable) of syllogisms by the degree that the propositions are realistic or not. For two types of syllogisms (out of four), mathematics undergraduates are shown to perform better than a group of trainee elementary-school teachers. The difference is explained in terms of the 'selective scrutiny' model.

The audience mooted several questions including 1. What is the definition of beliefs employed in the study? 2. How do people make decisions in everyday life? In which situations does logic come in? 3. Is it true that mathematicians on the main master the skill of de-contextualisation in optimal ways? How do mathematicians perform in tests that are not explicitly mathematical?

Iris Attorps talked about mathematical equality within the framework of reification (due to Sfard). She investigated teacher's and student teacher's ability to realize abstract properties that hold for equality, i.e., the symmetric, reflexive and transitivity laws. Also she considered the term 'solving an equation', and taking an equation as a proposition that could either be true or not.

The main points discussed were as follows: 1. What, if any, is the influence of ordinary conceptions of "=" on mathematical conceptions of "="? What are the different meanings of something as simple as "=" in an educational context, and how important is it for the teacher to be aware of them. 2. Why is it necessary for a teacher to understand the properties of equality relations (reflective, symmetric, transitive) formally? 3. More generally, what is the importance of teacher knowledge of a certain type, such as knowing the mathematics, the formal aspects and when it matters, when it is likely to illuminate the pedagogical process?

Session on mathematicians' positions concerning didactical issues

Paola Iannone draws on an extensive project conducted on a group of mathematicians extracting their opinions and experience both from their personal past and from their teaching concerning various aspects of doing mathematics at the tertiary level. This encourages the mathematicians to reflect on the nature of their students' difficulties. In this particular presentation, the protocols selected were intended to illuminate the model of syntactic and semantic knowledge; the main conclusion was that the mathematicians believed that both types of knowledge should concur to produce successful proofs. Carl Winslow adopts the anthropological theory of didactics in order to examine the relationship between mathematicians' research activities and their teaching practices. 'Mathematical organizations' are distinguished for

mathematicians and the students, and are denoted by MO_m and MO_s respectively. 'Didactical organizations' (DO) represent ways of teaching (in a broad sense) MO_s . Three levels of a mathematician's commitment and (perhaps idealized) expectations towards instruction is made: one as a mere duty, one where some of the research techniques can surface in the instruction, and one where the process of teaching can actually enhance the lecturer's own research.

Last two time-slots devoted to reflecting on the group's scope and organization

The last two time-slots available contained no presentations of papers, but were reserved for general discussion.

For the penultimate time slot, the participants were split into two different gatherings. The starting point of the first was to ask the audience to respond to the questions below. (1) Is the perceived discontinuity between secondary and tertiary mathematics due to institutional and pedagogical practices, or is it caused by factors concerning the character of University Mathematics that demand new habits of behavior in reasoning? (2) What ways are there to ease the transition? (3) If AMT is taken as thinking skills needed for Advanced Mathematics, how are they beyond those required at school? (4) What commonalities or differences in mental processes are there in the two levels? The discussion was rather diffused and mostly sidestepped the questions despite of their fundamental significance. It was dominated by the view of some that the research field of AMT has largely changed its main focus from cognitive-based studies starting in the early nineteen eighties, to the tendencies found nowadays based more on societal and effect factors that make the long established work 'obsolete'. Others countered strongly this position on the basis of the existence of different scientific 'paradigms', in the sense of Kuhn, and on much of the actual output of recent educational research. Opinions were often put in a partisan spirit. Some other issues were touched on that were treated in more detail in the final session. The second split-group mostly was on the lines of question 4 above. A discussion was raised concerning the possibility that some tasks accessible to school students might pose the same kinds of problems in their resolution for undergraduates, and so it could be claimed that these tasks might be considered within the scope of AMT.

For the last time slot, all of the participants were together. Each person attending was asked individually to respond to the following two issues. (A) What is the status of the term 'Advanced Mathematical Thinking'? Should the group better split into two different groups according to the two main interpretations of AMT? (B) What organizational changes can be made to enhance the operation of the Group 14 for future meetings?

As far as issue (A) goes, the second question became redundant because the central organizers of CERME decided further groups, beyond the 15 existing ones, could not be accommodated in future conferences. However, it is pertinent to state the

numbers of participants that explicitly expressed opinions whether the two strands (i.e., Advanced thinking, and thinking in Advanced Mathematics) should be treated together or separately. Thanks to the transcript kindly typed by Lara Alcock we have this data; six supported to retain the integrated group, three expressed mixed feelings about this [the others did not offer explicit positions]. Several participants declared that the two interpretations are complementary and that there was no compelling reason not to retain the traditional name 'Advanced Mathematical Thinking' as an umbrella term.

Concerning the management of the sessions, many different opinions were raised, some of which voiced inherent problems. The most evident of these is that because of the number of papers we received, the group was split in two subgroups for most of the sessions. Some participants were disappointed in not being able to attend all the talks. However, were the group kept as a single body it might have affected the flow of the discussion and would have decreased the time available to each paper. Other participants felt that there were too many themes (we had 8 themes adapted from those proposed in the 'Call for Papers' to fit in with the submitted papers). However, AMT is not isolated from other levels of education and there is a vast range of mathematical knowledge at the University level. For these reasons it is difficult to narrow down thematically. Finally there were a few participants who felt that the themes stated in the program were mostly steered towards cognitive factors. On the other hand, also to explicitly incorporate didactical, affective or social factors could end up in hugely broad themes. (In fact to the extent that the papers placed in each theme might not have too much relevance to each other.)

CONCLUSION

On the whole, the participants expressed satisfaction in the interest, breadth and relevance of the material presented and discussed in the sessions. This seemed to belie a divisive aspect in how the term AMT can be interpreted. In the end such discord became largely artificial when it was realized that those papers that did not have an advanced mathematical setting considered tasks that would be both accessible and challenging equally to school students and undergraduates. However, for the sake of clarity of position, it was decided to conduct a post-conference correspondence over the e-mail between the participants in order to propose a suitable delineation of AMT for the purposes of future meetings of the group. The formulation that was eventually decided on is as follows:

"Instead of attempting a characterization of AMT itself, we lay down a working specification of what kinds of educational research would be suitable for consideration by Group 14, CERME. We suggest that the organizing team of Group 14, CERME-6 solicit:

(1) Studies on the thinking processes involved in Advanced Mathematics, incorporating clinical approaches to analyze students' behavior and their difficulties in dealing with abstract concepts, logical reasoning and other cognitive issues commonly associated with tertiary level mathematics.

(2) Theoretical or empirical studies identifying and expounding factors (be they of an epistemological, social, or cognitive character) that influence or install procedures and practices in the learning and teaching of mathematics at the late-secondary and tertiary level. This includes articles dealing with the transition between these two levels."

The issues represented in the papers, presentations and discussion were prolific. It is in place, though, to refer to some general tendencies that were evident. The mathematical backgrounds that were employed were dominated by Calculus/Real Analysis and environments suitable for the general Problem Solving agenda. The need of rigorous statements such that the exact roles of the conditions can be appreciated often surfaced, especially in the context of students' generation of examples. Quite a lot of interest was expressed in the French traditions in the field of Mathematics Education and its relevance to AMT (this interest is happily timely as CERME-6 will take place in France, February 2009).

Joanna Mamona-Downs thanks the members of the organizing team and the invited chairpersons for providing synopses of discussions in the sessions for the formation of this report.

LAGRANGE'S THEOREM: WHAT DOES THE THEOREM MEAN?

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Through teaching calculus in Ort Braude College we developed a method to help the students to better understand the theorems. In this paper we give the example of Lagrange's Theorem.

INTRODUCTION

Teaching mathematics has a twofold purpose: encouraging and supporting knowledge construction by students and developing their creativity by advancing their mathematical thinking. As observed by Dreyfus in Tall (1991), “what most students learn in their mathematical courses is, to carry out a large number of standardized procedures, cast in precisely defined formalisms, for obtaining answers to clearly delimited classes of exercise questions”. However in order to develop advanced mathematical thinking students should also learn and understand theory. As stressed by Mason and Watson (2001), “If they [students] are to make mathematical sense themselves, then they need to be able to assert things for themselves. They need to use technical terms with facility to express their ideas”. Mason and Watson continued by explaining that if students do not understand theory then when they “come to apply a theorem or technique, they often fail to check that the conditions for applying it are satisfied”.

Unfortunately our experience shows that many students have difficulties in learning theory and as a result they are frightened of it. Possible reasons are: the abstractness of the concepts, the formal way subjects are presented and the special language of a theorem or a definition. To try to deal with the problem we developed a system of Self Learning Material (SLM) for three theorems. In this paper we describe it in the case of Lagrange's Mean Value Theorem.

We implemented the designed instructional materials with three groups of about one hundred Engineering students in total. The materials were given as three PowerPoint presentations (according to the theorems) at the site of the course: <http://braude3.ort.org.il>. They were intended for students' self-learning. In addition to the PP presentations we prepared a set of special problems connected with these theorems. Similar problems are described in our paper (2005). These problems were given to the students as an assignment by the system Webassign (see www.webassign.net).

THEORETICAL FRAMEWORK AND LITERATURE REVIEW

Many researchers have written about students problems in understanding and using mathematical theory, particularly theorems, have written that “classroom experience indicates that students do have a lot of trouble with the switch to the Formal Mathematics level” (Leron, 2004), that “The basic knowledge performance and conceptual understanding of the students in mathematics worsen” (Gruenwald and Schott D., 2000). The problem is that students often are taught mathematics as a set of algorithms used in problem solving. Students think “that the theorem can be memorized as a “slogan”, then it can easily be retrieved from memory under the hypnotic effect of a magic incantation. However, using a theorem as a magic incantation may increase the tendency to use it carelessly with no regard to the situation or to the details of it applicability” (Hazzan & Leron, 1966).

Our purpose was to help students to better understand the meaning of a theorem: the conditions and the conclusions. Here we show it for Lagrange’s Theorem. Similarly it can be done for other theorems. We tried to use ideas of Dreyfus & Eizenberg (1990), Vinner (1989), and Zimmermann & Cunningham (1991) on visualization of calculus.

A large part of our SLM is examples. Examples are used in mathematics and in teaching mathematics from the beginning of human history up to present. Hazzan and Zaskis in their early paper (1999) describe the role of examples in teaching and learning mathematics from several perspectives. The significance of examples and their use in mathematical education was reported in several studies (Bills et al., 2006, Zhu & Simon, 1987; Rankl, 1997, Watson and Mason, 2005).

Researchers propose to ask students to construct the requested examples (Gruenwald & Klymchuk, 2002; Hazzan & Zaskis, 1999; Watson & Mason, 2001, 2002, 2005; Selden & Selden, 1998). This is an excellent way of mathematics learning. But, in our opinion, it is a very difficult task for freshmen at their first semester at college, particularly when the students were not presented with learning from examples at school. Our SLM provides students with a large “bank” of different examples; students only need to choose the right examples. Our SLM examples are destined to help to understand why we need this or that specific condition in the theorem and how the conclusions of the theorem depend on the conditions, if the conditions of the theorem are necessary or sufficient. We wanted our students to learn “with examples” (O. Hazzan & R. Zaskis, 1999). We see our “bank” of examples as the first step students learn to construct an example. We argue that following this stage they will be able to start constructing their own examples. Iannone & Nardi (2005) stressed that counterexamples play three roles in learning mathematics: affective, cognitive and epistemological-cum-pedagogical. Understanding the significance of such examples we inserted into SLM tasks to find a counterexample to false statements. Based on these tasks we wanted our students to understand what a counterexample means.

In conclusion we would like to emphasize the main points of our approach:

1. The students were supplied by a bank of examples, comparatively simple, where they could find counterexamples, a task which they usually find to be a difficult one.
2. By using seemingly simple problems the students were solving theoretical problems, finding theoretical conclusions. The students became involved into the research process.
3. Students' learning became motivated by their success.

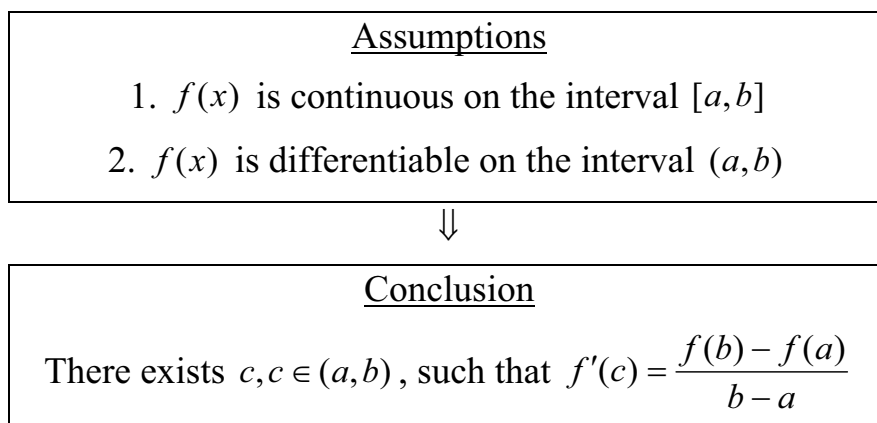
SELF LEARNING MATERIAL

Normally, we give the students a theorem in the form that is clear to specialists. This form is too formal for the students. Our main aim was to turn the students from passive receivers of knowledge into active partners in the learning process. We tried to involve the students in the learning process, step by step, presenting the material in a way that encourages them to take part in formulating, discussing and proving a theorem. Following is a description of the process of learning a theorem:

Revising concepts: The first step is to let the students to revise the concepts, definitions and theorems, that are needed to learn this specific theorem.

Formulation of a conjecture: In the next step we begin to deal with the new material. We present to the students some functions, where only one of them satisfies all the assumptions of the theorem. For each function we ask the students to answer questions concerning the conditions and the conclusion of the theorem. After that we provide several statements, one of them refers to the theorem. We ask the students which statements may be true. For every wrong statement a student can find a counterexample by using the given functions. Now the students are ready to formulate the theorem.

Formulation of the theorem: Here we state the theorem in a schematic form. For example Lagrange's Theorem is stated as



In this way we emphasize what is assumed and what is concluded.

Exploring assumptions and conclusions: In this paper we describe the fourth step: the process of studying the assumptions and the conclusion of a theorem. We tried to

provide the students with exercises and problems where we discuss the following questions: what are the assumptions of a theorem and what are the conclusions, what is the geometrical meaning of a theorem, what happens when one or more of the theorem assumptions are not fulfilled, what assumptions are necessary and which are sufficient. Generally speaking, what does the theorem mean?

Proving the theorem, this is the final step of the process. We intentionally put it at the end because the proof of a theorem distracts the students from its understanding. We give the detailed proof step by step. The students have got this proof to study it. We use different ways to check whether the students really understand the material or whether they learn it by heart.

In the next part of the paper we would like to describe what we present to the students, namely:

- examples explaining the meaning of the conclusion;
- discussion of the geometrical interpretation of the theorem;
- analysis of the assumptions by using carefully designed examples; and
- examples showing that a sufficient assumption is not a necessary one.

EXPLORING ASSUMPTIONS AND CONCLUSIONS

Instructive examples

To help the students to get acquainted with the theorem we offered them several examples. The students were asked to check that the assumptions hold true and determine how many points c satisfy the conclusion.

The examples were:

1. $f_1(x) = x^2 - 2x + 1, x \in [0,3]$
2. $f_2(x) = x^3 - x, x \in [-2,2]$
3. $f_3(x) = x + \sin x, x \in [0,2\pi n]$, where n is a given natural number
4. $f_4(x) = 2x + 5, x \in [0,4]$
5. $f_5(x) = x^2 2^x, x \in [0,1]$

The students were encouraged to study the examples on their own and could see a full solution in the end of the PowerPoint presentation.

The assumptions were satisfied by all five examples. The examples were chosen in order to show that there are several possibilities concerning the number of points c (1 in Examples 1 and 5, 2 in Example 2, n in Example 3, and infinite number in Example 4). Example 5 was chosen to illustrate that the exact value of c can not always be determined even when the existence is known. This example also

demonstrates the concept of the existence theorem: you do not have to compute a point in order to prove its existence.

Geometrical meaning of the theorem

To help the students to understand the geometrical interpretation of Lagrange's theorem we reminded them of the concepts of the secant line, tangent line and parallel straight lines, and asked them to formulate the conclusion of the theorem in geometrical terms. Here too, they could check their work in the end of the PowerPoint presentation, where a dynamical illustration was given in addition to the answer. Before doing it they could use the following questions as a hint: which of the following statements is equal to the conclusion of the theorem? The statements were:

1. c is a local extremum
2. c is a point where the tangent line is parallel line to the secant line connecting $(a, f(a))$ and $(b, f(b))$
3. $f(c) = f(b) - f(a)$

The students could check that the answer is (2), using one of the earlier examples.

What if not?..

To convince the students that both assumptions are important we wrote two false statements:

1. Let $f(x)$ be a function continuous on the interval $[a, b]$. Then there exists c , $c \in (a, b)$, such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.
2. Let $f(x)$ be a function defined on $[a, b]$ and differentiable on (a, b) . Then there exists c , $c \in (a, b)$, such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

After that we gave them two problems:

Problem 1

Consider the following functions:

1. $f_1(x) = \sqrt[3]{x} \sin x$, $x \in \left[\frac{-\pi}{4}, \frac{\pi}{2} \right]$
2. $f_2(x) = |\sin x|$, $x \in [0, 2\pi]$
3. $f_3(x) = [x]$, $x \in [0, 2]$
4. $f_4(x) = \begin{cases} x^2, & x \leq 0 \\ \cos x, & x > 0 \end{cases}$, $x \in [-1, 2]$
5. $f_5(x) = 2x + \sqrt[3]{x^2}$, $x \in [-1, 1]$

Which of them is a counterexample to the first statement?

Problem 2

Consider the function:

$$f(x) = \begin{cases} x^2, & x \leq 0 < 1 \\ a, & x = 1 \end{cases}, x \in [0,1]$$

For what value of the parameter a this function is a counterexample to the second statement?

The answer in the problem 1 is function 5. For this function it is not difficult to check that the conclusion of the statement does not hold, but the assumption is fulfilled. All other functions were chosen in order to teach the students what is a counterexample. The students should understand that a function can not be a counterexample if a conclusion is true. Such functions are $f_1(x)$ and $f_2(x)$. Also a function can not be counterexample to a statement, if it does not satisfy assumptions. Here are the functions 3 and 4. The aim of problem 2 was to show the students how they could build a counterexample. We considered them to understand what if $a \neq 1$. For other a they had to find c and chose those values of a that $c \notin [0,1]$. The answer: $a \leq 0$ or $a \geq 2$.

Sufficient is not necessary...

As our experience shows the students often replace the assumptions by the conclusion and make the inverse statement. In our case they can get the following statement:

If there exists c , $c \in (a,b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$, than function $f(x)$ is continuous on the interval $[a,b]$ and differentiable on (a,b) .

It is not easy to explain to the students that their statement is wrong. The thought, that the conclusion may be right without the assumptions, disagrees with their preconceptions. In order to convince them we asked them to check this statement for functions 2 and 4 (see problem 1). For both functions the assumption holds and the conclusion is wrong: function 2 is not differentiable on $(0,2\pi)$, function 4 is discontinuous on $[0,4]$. After that, we gave them one more counterexample:

$$f(x) = \begin{cases} x^2, & x < 5 \\ 5, & x = 5 \end{cases}, x \in [0,5]$$

This functions is continuous on $(0,5)$ and discontinuous on $[0,5]$, in spite of that

$$f'(0.5) = \frac{f(5) - f(0)}{5 - 0}.$$

At the end the students were asked to build their own counterexamples.

SURVEY, FORUM AND TEST

The results described in this section refer to all three theorems introduced to the students using SLM. In order to examine the effect of implication of the proposed SLM we asked about one hundred students to fill in a 3 questions survey and tested their understanding of the theorems in the final exam.

The 3 questions in the survey were:

Question 1

Did the SLM help you in understanding the theorems and how they are applied?

Question 2

Did the SLM help you in solving problems?

Question 3

Would you like to get similar SLM for other topics of the course?

A summary of the answers is given in table 1. Each number in the table is the number of the students who have chosen that answer (in percents).

Answers	Absolutely yes	Yes	Yes, but not much	No	Absolutely no
Question 1	14%	54%	31%	1%	0%
Question 2	17%	67%	12%	4%	0%
Question 3	63%	29%	6%	2%	0%

Table 1: A summary of the answers

Another source of feedback from the students was an internet forum opened for this proposes on the website of the course, where the students could express widely their opinion on the given material. For example, one student wrote that after he had studied the given material his eyes "opened" and he "saw" these theorems. Another student wrote that he had understood *what a theorem means and how to learn it*.

At the end of the course we decided to compare the students' knowledge of the theorems with their knowledge in another topic - "The definite integral and the basic theorems of integration", that was taught without SLM. For every "differential" problem we inserted into the exam a similar "integral" one:

Problem D-1

Formulate Lagrange's Theorem.

Problem I-1

Formulate The Integral Theorem of The Mean Value.

Problem D-2

Prove Lagrange's Theorem.

Problem I-2

Prove The Integral Theorem of The Mean Value.

Problem D-3

Is the following statement is correct? Explain your answer:

For the function $f(x) = \frac{\sqrt{1+x^4} \cos(\sin x)}{e^{x^2} + 1}$ there exists a point $c, c \in (-\pi, \pi)$, such that $f'(c) = 0$.

Problem I-3

Is the following statement is correct? Explain your answer:

The function $F(x) = \int_0^x [x] dx$ is differentiable on the interval $[0,2]$ and $F'(x) = [x]$.

Problem D-4

Give a counterexample to the following statement:

If a function $f(x)$ is differentiable at a point c and $f'(c) = 0$, then this point is an extremal point of the function.

Problem I-4

Give a counterexample to the following statement:

If a function $f(x)$ is integrable on $[a,b]$, then it is continuous on this interval.

The numbers in the following table are the averages of the grades (a scale 0-100 was used). For example, the average grade in problem D-1 was 74, while the average grade in problem I-1 was only 23.

	1	2	3	4
D	74	57	72	47
I	23	20	27	43

Table 2: Grades

PLANS FOR FUTURE WORK

We consider our paper as the beginning of a wider research on advancement students' mathematical thinking in calculus. We feel that our experience with SLM approach was a success and this encourages us to extend it. In our future curricular design we intend to:

- develop SLM for other theorems of Calculus;

- help students to develop similar material with their own examples; and
- develop means and ways for checking the results.

In our further research we would like to focus, among other issues, on

- learner-generated examples and the changes in students ability to generate various examples and counter-examples
- students' problem-solving strategies
- students progress in their proving activities along the course

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UNIVERSITY STUDENTS GENERATING EXAMPLES IN REAL ANALYSIS: WHERE IS THE DEFINITION?¹

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This paper is part of a wider study, whose purpose is to analyze how university students behave when asked to generate examples in Real Analysis. To this aim, we collected and analyzed written protocols accompanied by individual interviews. In this paper, we focus on the interplay of concept image and concept definition, which is an element revealing interesting aspects of students' way of reasoning.

INTRODUCTION

In problem solving, the role of examples is considered crucial, because they allow to perform exploration and to reach generalization and abstraction (Polya, 1945). On the other hand, many papers show that examples may make students to stick to the explorative phase without feeling the need of generalization, see (Furinghetti & Paola, 1997; Morselli, 2006). Checking on examples may even be the way of proving (Balacheff, 1987; Harel & Sowder, 1998).

Less attention has been paid to the activity of generating examples *per se*, as a special case of problem solving, see (Zaslavsky & Peled, 1996). The generation of examples is a sort of open-ended problem, in which one must decide whether the required example (in our case, a function filling some requirements) exists or not; when the example doesn't exist, it is required to justify why the example does not exist, see (Antonini, 2006). The present paper is set in this second stream of research.

The tasks we consider are set in the field of Real Analysis, that means that our study mainly involves advanced mathematical thinking. It is well known, see (Selden, Mason & Selden, 1994), that at this level difficulties still exist, even for good students.

THEORETICAL BACKGROUND

The complexity of the problem solving process has been studied from different points of view, see (Kilpatrick & Stanic, 1989). In the present paper we draw particular attention to the characters of this process that are more specific to advanced mathematical thinking. As a starting point we take the pattern of the problem solving process, made up of four stages, presented in (Polya, 1945): (i) *getting in touch with the problem*; (ii) *planning*; (iii) *carrying out the plan*; and (iv) *looking back*. According to different situations in which the process is carried out, the focus of the

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analysis of the way a solver copes with this pattern is on different aspects. In the present paper the main point will be that the involved subjects usually have a mathematical culture that allows to deal with the more sophisticated concepts at issue at this level. Thus it will be our concern to analyze how such a mathematical culture intervenes in the process, e.g. whether it is exploited or is a burden.

We take as a reference point for our discussion the duality considered in (Tall & Vinner, 1981) between concept images and concept definition. The term “concept image” describes the total cognitive structure that is associated with the concept; this structure includes all the mental pictures and associated properties and processes. “Concept definition” is defined as a form of words used to specify that concept. We focus on the interplay between these two kinds of concepts as a key issue in shaping the solver's behavior.

Bills et al. (2006) show that generating examples is linked to activities such as visualization, exploration, use of informal language, resorting to prototypes and stereotypes. All these activities happen through the activation of concept images that rely on different representations in different semiotic registers, see (Duval, 1995). We will see that an important point of our study will be the role of visualization, see (Arcavi, 2003; Aspinwall et al., 1997; Dreyfus, 1991), and the duality between analytic and visual strategies, see (Zazkis et al., 1996).

As happens in the process of proving, also in the generation of examples there is an underlying problem, that of what may be a warrant for the solver, see (Rodd, 2000). In some cases visualization, and verbalization may be by themselves mathematical warrants for the solver; in other cases the solver has to look for warrants in more formal arguments. We know from literature that the recourse to definitions, axioms, and theorems is problematic, see (Zaslavsky & Shir, 2005). One problem may be that concept definitions are filtered by concept images and are not correctly applied. In other cases concepts images and definitions are in contrast and generate conflicts. Resorting to concept images or concept definition does not necessarily happen simultaneously, rather there is an intertwining between concept images and concept definition that shapes the process. For example, starting from concept images or from definitions may orient to different modes of reasoning in approaching mathematical situations. At its turn these modes orient to different registers and representations.

METHODOLOGY

In this paper we refer to activities of generating examples in Real Analysis, performed by university students in Mathematics, Physics and Engineering, see (Tosetto, 2006). Totally, we worked with 13 students: 1 student in Physics, 7 students in Mathematics, 5 students in Engineering. They were all attending the last two years of their academic career and they already attended the basic courses in Real Analysis.

To carry out our study we asked them to participate, in a voluntary mode, in an activity which consisted in the following 5 tasks:

- 1) Give an example, if possible, of two functions f and g , having an absolute maximum in $x_0 \in \text{dom}(f) \cap \text{dom}(g)$ and such that the function $(fg)(x) = f(x)g(x)$ has an absolute minimum in x_0 .
- 2) Give an example, if possible, of a twice differentiable function $f : [a, b] \rightarrow \mathbb{R}$, such that f is zero in three different points and its second derivative is positive in the domain.
- 3) Give an example, if possible, of an invertible, continuous function $f : (0, 2) \rightarrow \mathbb{R}$, such that $\lim_{x \rightarrow +\infty} f^{-1}(x) = 1$.
- 4) Give an example, if possible, of two continuous, differentiable functions f and g , such that $f(0) = g(0) = 0$, $f(1) = g(1) = 0$, and the tangent line to the graph of f and the tangent line to the graph of g in the point $(0, 0)$ are perpendicular, as well as the two tangent lines to the graphs in the point $(1, 0)$.
- 5) Give an example, if possible, of an injective function $f : [-1, 1] \rightarrow \mathbb{R}$, such that $f(0) = -1$ and $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow -1} f(x) = 2$.

As you may note, in the tasks 1, 4, 5 it is possible to generate examples, while that is not possible for the tasks 2, 3. The two different kinds of tasks entail different processes and, consequently, different behaviours. All the tasks have an open form, and the students must explore the situation in order to find the required example. When they don't succeed in finding it, they must understand *why* they don't find it, if it depends on their lack of ability in finding it or in the nature of the task, that is to say it is not possible to generate the example. In the latter case, it is necessary to prove this impossibility. It may also happen the converse: the students are not able to find the example for lack of ability but they think this is due to intrinsic reasons and try to prove this impossibility.

The students worked individually on each task. They were asked to comment their solving process to an interviewer, according to the *think aloud* method. In case of difficulty, the interviewer also acted as a prompter, since our aim was to see whether the students were able or not to solve the problem, to see which kind of difficulties they had, to try to help them to overcome these difficulties.

The interviews were audio-taped. During the interview, the students were also allowed to write down notes and sketch drawings. This written data were collected.

In summary, we have at disposal three types of data: transcripts of the interview and written notes by the students, field notes by the interviewer. The data were analyzed according to the method of *grounded analysis*. The gathered data were analyzed through two modes:

- by comparing the outputs of all solvers in the same exercise to identify commonalities and differences

- by going through a single solver's performance to identify interesting features in the solving process.

In this paper, we report some results of our reflections concerning the task 5. The task 5 was given after 4 tasks, 2 of *non existence* and 2 of *existence*. This means that the students had previously encountered both the situations and were fully aware of the possibilities. Due to space restriction, we confine ourselves to the analysis of one solver's performance carried out by going through her pathway and by commenting significant passages of the protocol and the interview.

ANALYSIS OF THE PERFORMANCE

We consider the problem 5:

Give an example, if possible, of an injective function $f : [-1,1] \rightarrow \mathbb{R}$, such that $f(0) = -1$ and $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow -1} f(x) = 2$.

(We remind that in this case it is possible to generate examples).

In the following we analyze the protocol of Letizia, student in Mathematics. We report those excerpts of the interview and of her protocol that evidence important moments of her solving process.

At first, Letizia draws the Cartesian axes and emphasizes the points where the graph passes through. The fact that the limit for x tending to 1 and -1 is 2 is represented emphasizing the points $(-1,2)$ and $(1,2)$. Afterwards she draws a first graph where the point $(-1,0)$ is firstly omitted, after the graph is amended (see Fig. 1). At this point for Letizia the task is filled, that is visualization is a warrant for her. The interviewer suggests to check on the required properties.

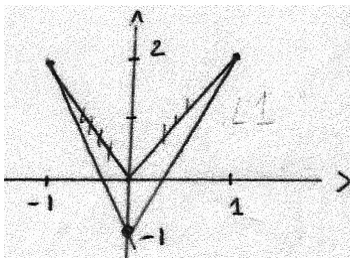


Figure 1

Letizia: Ok. Tell me what's wrong, because for me it's ok.

Interviewer: Try to check whether all the requirements are verified.

Letizia: Well... it is injective, it is well defined, the limits are ok, everything seems ok.

Interviewer: Try to explain me why all the requirements are satisfied.

Letizia: Well, it is a function, it is injective because... it is not injective!

Interviewer: Why did you say it was injective?

Letizia: I was reflecting whether it was a function or maybe, since I constructed my functions with two pieces of straight line and I know that straight lines are injective, then for me the whole function was injective. But this is not injective, because whatever y I take in the image, two values of x correspond to this y .

We note that the strong evoked concept image for injective functions is the straight line. The control of the statement is poor, we may say that she is relying on a prototype present in her concept image (straight lines are injective), as evidenced in the excerpt, without any reference to the concept definition of injective function. Only the intervention of the interviewer causes the shift of her attention to concept definition. This shift is stressed by Letizia's use of verbal symbolic language, that recalls the definition of injective functions ("whatever y I take in the image, two values of x correspond to this y "). We observe that the prompt of the interviewer is not a clear amendment of her mistake, nor an explicit mention of the definition of injective function: it is mainly an invitation to go back to check whether all the requirements are filled.

At this point, Letizia's attention is caught by another issue: the continuity of the function. She realizes that continuity is not required and draws another graph (see figure 2) with a point of discontinuity in O .

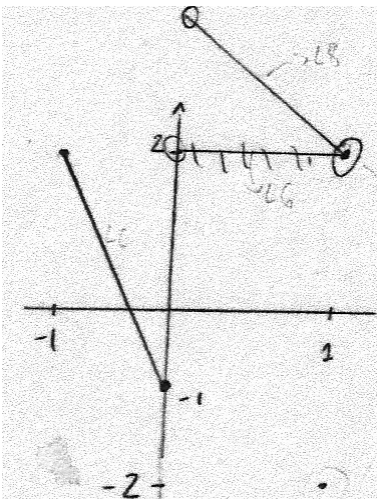


Figure 1

Letizia: But you don't tell me that the function must be continuous. I was thinking of it as continuous, but if it must not be continuous I can also define it in this way. I got it, this is my function, it isn't continuous but you didn't ask me this, so I've done.

Interviewer: Are you sure?

Letizia: No, once again it is not injective! Wait, I just have to take it in this way. This is injective, the other requirements are fulfilled, here we are.

We note incidentally her drawing a first attempt and going back to the text in order to check the graph. This is a kind of working "by trial and error", that allows her to

control and process all the requirement of the task, that is one of the difficulties of such a kind of problems.

The last sentence makes us to reflect on the presence of another concept image, that is based on the prototype of functions as continuous functions only. Furthermore, for Letizia discontinuities are only “inside” the domain of the function. She does not care of the boundary points, as we’ll see in the following.

Interviewer: But I don’t yet agree on the fact that it is injective.

Letizia: Ah, it is true, in $x = 1$ and in $x = -1$ the function has the same values! Then, I take a segment, taking away the point $(1,2)$.

Interviewer: But in $x = 1$, what is the value of the function?

Letizia: I don’t give any value there.

Interviewer: But the function must be defined in $x = 1$.

Letizia: You are right, then the function doesn’t exist because in $x = 1$ the limit must be 2 and this requirement takes away injectivity.

The excerpt shows that for Letizia continuity is a problem, because it interferes with limit. Letizia doesn’t take into account that the value of the limit may be different from the value of the function in the considered point. We may say that her concept image of limit “lives” only for continuous functions.

Another sentence (“I got an idea, I’ll change the function. I’ll take a function with a vertical asymptote in $x = 1$... ah no! the limit has to be a finite value”) confirms this problem with the concept of limit: for Letizia, there are only two possibilities: the value of the limit coincides with the value of the function, otherwise there is an asymptote.

Letizia is in a *cul de sac*: she overcomes the impasse resorting again to reading the text, after the interviewer’s question (“*Why?*”).

Letizia: I think it is not possible because the function must be defined in $x = 1$ and $x = -1$ and in those points the function must have value 2.

Interviewer: Why?

Letizia: It is written here (*she refers to the text of the problem*). No, it is not written that $f(1) = 2$, it is written that the value of the limit must be 2.

Letizia realizes to have a difficulty with limits (indeed, her concept image is not enough to tackle the problem), but she still does not resort to the definition. This stresses that the definition is not part of her concept image of limits: she mainly relies on the intuitive aspects, linked to visualization and common language. Letizia overcomes this cognitive conflict abandoning the graphical register and resorting to the symbolic register. She recalls the example of sequences whose limit behaves in a “strange” (for her) way, such as oscillating functions etc.

Letizia: I define it analitically and not graphically. I could use a sequence that tends to 2. Let's do this, on the negative abscissas I take the one I already drew, and on the positive abscissas... I was thinking of the sequence $-1 + \frac{1}{n}$, when n tends to $+\infty$ it tends to -1 , now I'm considering the negative numbers. I would like to construct a sequence the tends to 2 when n tends to 1 and -1 . With a sequence, I construct a function that tends to 2 in $x=1$ but is not really 2, because the fact that the limit is 2 means that it tends to 2 but is not really 2, as I told before. Hence, the limit does not necessarily have to be the value of the function, but a point to which the value tends. Anyway, I still think that the function doesn't exist for the injectivity, because all the sequences I consider are not good. Now I'm thinking of something oscillating, but it is not good because it is not injective. Then, I'm sure it is not possible to find the function.

The subsequent excerpt shows Letizia's struggle about limits on the boundary points.

Letizia: I was thinking... Can I define my function in $x=1$, by giving any value? No, because if I define $f(1)=3$, then the limit for x tending to 1 of my function is 3.

Interviewer: Why?

Letizia: Maybe, I want the function to be continuous in the intervals where I'm defining it, but it could even be not continuous. If I define $f(1) = -2$, so that it is injective, my problem now is to see what is the value of the limit for x tending to 1 of this function. I don't know what is the value, I mean, looking at the graph I would say that the limit is -2 and not 2.

Only the prompt of the interviewer forces Letizia to actually resort to the definition, as we see in the excerpt:

Interviewer: Try to think of the definition of limit.

Letizia: Ah, but there is a neighborhood with a hole! I mean, I write you the definition of limit (*she writes down the definition, see figure 3*). I must include the point to which the x is tending, then it is ok, the function that I drew is ok, it tends to 2 for x tending to 1. What a nice exercise! Eventually I understand why in the definition of limit it is necessary to exclude the value of the point, I understand the meaning for neighborhood with a hole!

The image shows a handwritten mathematical definition of a limit. It reads: $\lim_{x \rightarrow x_0} f(x) = l$ if and only if $\forall \epsilon > 0 \exists \delta_\epsilon > 0$ t.c. $|f(x) - l| < \epsilon \forall x \in \mathcal{B}(x_0, \delta_\epsilon) \setminus \{x_0\}$. The text is written in black ink on a light-colored background.

Figure 3

The reflection on the definition of limit is the moment when Letizia overcomes her difficulty and succeeds in fulfilling the requirements. She realizes that the perceived conflict between the requirements is not actual, rather is coming from her relying

only on intuition. After this performance we observe two facts: the definition of limit is no more an alien entity, but becomes part of her concept image; this in addition to the fact that succeeding in the task was fostered by the efficient/actual use of the definition.

CONCLUSIONS

The pathway followed by Letizia may be roughly divided in two parts in which her mathematical culture acts according to different modes.

In the first part the main issue is the appropriation and the control of the meaning of the given statement. This has been fostered and made efficient by the use of graphs and the reference to visualization; this seems to confirm the claims about the positive role of visual representations in the learning of mathematics pointed out by many authors, see (Arcavi, 2003) for one. On the other hand Letizia offers a good example of the cognitive boundary of visualization. Visualization may be a hindrance to leaving aside intuition and developing a theoretical thinking relying on definitions, axioms and symbolic mode. As a matter of fact, the second part shows that Letizia is able to reach the conclusion through resorting to the formal aspects of her mathematical culture, such as the definition of limit.

Our results suggest didactical implications concerning the role of definitions. Definitions have to be the end of a path of appropriation of meaning and awareness. Without that, definitions have no future and are not a tool for developing mathematical activities.

Generating examples revealed itself a good way of recovering the meaning of definition through their application and to foster the passage to theoretical thinking.

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IS THERE EQUALITY IN EQUATION?

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In this study we analyse what kind of conceptions secondary school teachers and prospective teachers in mathematics have about equations and how these conceptions are related to the formal definition of the concept of equation. Data was gathered by interviews and questionnaires. The phenomenographic research approach in order to analyse research outcomes was applied in the investigation. The research results indicate that the lack of understanding of the symmetric and reflexive properties of the equality and the existence of different concept definitions of equation can be reasons for teachers' misconceptions concerning equations.

Keywords: conception, conceptual, equation, mathematics teacher, procedural.

INTRODUCTION

According to Sfard (1991) the process of concept formation consists of three sequential stages: 1) *interiorization*: a learner gets acquainted with a concept and performs operations or processes on mathematical objects, 2) *condensation*: a learner has an increasing capability to alternate between different representations of a concept and 3) *reification*: a learner can conceive of the mathematical concept as a complete, “fully-fledged” object. At the stage of reification the new entity is detached from the process which produced it and the concept begins to receive its meaning as a member of certain category.

The first two stages represent the operational aspect of mathematical notation or merely the procedural knowledge of mathematics and the last stage its structural aspect and the conceptual knowledge of mathematics (Using the terms procedural and conceptual knowledge of mathematics, we lean on the definitions given in Haapasalo & Kadjevich 2001). Moreover, the structural conception of a mathematical notation is static whereas the operational conception is dynamic and detailed. To understand the structural aspect of a mathematical concept is difficult for most people because a person must pass the ontological gap between the operational structural stage. Consistently with Tall and Vinner (1981), Sfard distinguishes between the words “concept” and “conception”. The term concept represents the mathematical, formal side of the concept and conception the private side of the concept.

Previous results related to the topic

The equals sign is already encountered at an early age. Especially outside the arithmetic classroom, the equals sign is often used in the meaning ‘it is’ (as in $MATH = DIFFICULT$) or ‘it gives’ (as in $HARD\ WORK = SUCCESS$), and in the classroom, interpreted as a command to perform an arithmetical operation (Sáenz-Ludlow & Walgamuth 1998). The findings of Wagner et al. (1984) show that also many algebra students have an operational interpretation of algebraic expressions, because they try to add ‘= 0’ to the expressions they were asked to simplify.

Over the years, the learning and teaching of (linear) equations have been studied in several surveys, e.g. Sáenz-Ludlow & Walgamuth (1998) and Pirie & Martin (1997). However, these studies seem to concentrate on pupils’ performances on the secondary level and not so much focus on the teachers’ conceptions about this concept. Nevertheless, in her doctoral thesis, Attorps (2006) identified three categories characterizing the conceptions that a group of mathematics teachers ($N=10$) possessed on equations: the teachers apprehended equations as a procedure (focus on the solving of equations), an answer (the equals sign is followed by an answer) or a ‘rewritten’ expression (some kind of object on which algebraic rules are applied). The procedural knowledge of mathematics dominates teachers’ conceptions on equations; all the findings in this investigation pointed out to the same direction: the operational outlook in algebra is fundamental and that the structural approach does not develop immediately (cf. Sfard 1991).

The research questions

In this paper, we continue the analysis of the material collected by Attorps by focusing on the mathematical properties of the equality relation, which is the core component of the concept definition of equation, and considering what blocks for comprehension of concept of equation can be due to the lack of understanding of reflexivity, symmetry and transitivity of the equality. We believe that this will refine our understanding of the concept formation of equation especially on the level of condensation.

In Goodson-Espy (1998), the author completes Sfard’s theory with Cifarelli’s (1988) levels of reflective abstraction (e.g. Piaget 2001) in order to analyse students’ transitions from arithmetic to algebra. This is, indeed, purposeful when students’ problem solving activities are classified and characterized for the making of an assessment of students’ degree of concept formation. Our research method, however, will be based on the phenomenographic research approach (Marton & Booth 1997). This approach seeks to identify how persons in qualitatively different ways understand and experience for example disciplinary concepts.

By the definition of the stage of reification, a participant on this level of understanding of the concept of the equation must acknowledge the reflexive, symmetric and transitive properties of the equality. Therefore, misconceptions and lacking of the understanding of these properties are merely related to the stage of condensation in concept formation of equation. We assume that a participant blinded to the symmetric property of the equality may, for instance, accept $2 = x$ being an equation while stating that $x = 2$ is not an equation but only the solution for the problem represented by, for instance, the former equation. Also, it is possible that the misinterpretation of the reflexive property of the equality may result to that a subject accepts $x = x$ being an equation (because there is something to be solved) but not anymore $0 = 0$ (because there is nothing to be solved). Similarly, the lack of understanding the transitive property of the equality may lead a participant, for example, to think that in $x = (x + 1) - 1 = 2 - 1 = 1$, which is a correct description of the solving of $x + 1 = 2$, there is no equations (because a participant expects an equation always to be of the form $A = B$, where A and B are expressions *not* including the symbol $=$).

Therefore, we ask: how this kind of incomplete understanding of the equality is reflected off the outcome of the interviewed teachers and to what extent teachers' misjudgements in questionnaire can be seen raising from the incomplete understanding of the equality? Another question also arises: on average, does the sense of symmetricity and transitivity develop before the sense of reflexivity in the case equation? This hypothesis we ground by the observation that, on the procedural or operational level, reflexivity property of the equality is merely related to the identical equations of the form $A = A$. Moreover, according to our interpretation, in order to understand that, for example, $x + x$ and $2x$ are only different names for the same object of an algebraic structure, it requires that a subject has already reached the stage of reification of the equality.

Our third research question is related to the existence of a variety of different concept definitions for equation. For example, according to Borowski & Borwein (1989, 194), an equation is a mathematical statement of the following form:

equation, a formula that asserts that two expressions have the same value; it is either an identical equation (usually called an identity), which is true for any values of the variables, or conditional equation, which is only true for certain values of the variables

Judging by the above definition, $0 = 1$ is not an equation because it does not contain any variable for which the assertion would be true identically or conditionally.

However, if we accept – as is done, for example, in Wolfram Mathworld (and also the authors do) – that “an equation is a mathematical expression stating that two

or more quantities are the same as one another”, then $0 = 1$ is an equation. Hence we ask: how the diversity of the definitions for the concept of equation reflects off the data gathered from the participants?

For example, the diversity of concept definitions may be somehow related to the mathematics textbook authors’ different linguistic views to mathematics (e.g. Tossavainen 2005). Namely, similarly as in the case of a natural language we can tolerate false propositions to belong to the language, it is reasonable to assume that a person who recognises the linguistic nature of mathematics also accepts easier the possibility of an equation being false, i.e., equation with truth value ‘false’. Another dimension, where the diversity may unfold, is the context: the equation can be introduced in the arithmetical or the algebraic contexts and, in the latter case, with different views on the concept of the variable and possible amount of them etc. This state of facts, we assume, may encourage learners to form different categories for ‘equation-like’ objects.

METHODOLOGY

The same ten secondary school teachers and 75 student teachers in mathematics that took part in Attorps’s (2006) study also participated in the present study. Five teachers were newly graduated (less than one year’s experience) and five were experienced (between 10 and 32 years’ experience). The data was gathered by interviews and questionnaires. The interviews took place in the schools, where the teachers worked, and were recorded. In the questionnaire 18 examples (e.g. $x^2 - y^2 = (x - y)(x + y)$, $a^b = \sqrt{a\sqrt{a\sqrt{a}}}$, $x = 2$ and $\int f(x)dx = x^2 + C$) and non-examples of equations (e.g. $x^2 - 5x - 10$ and $x + |x - 3| \geq |x - 1| + 2$) were introduced and the participants were asked to answer the question: Which of the following examples do you comprehend as equations? During the subsequent interviews the participants had a possibility to explain and develop their answers. The phenomenographic analysis was then applied to the interview transcripts.

In the interview the participants were also asked to state their own concept definition (actually concept image of that) of equation. These conceptions were divided into four different categories according to on what the participants’ focus was concentrated.

RESULTS

Our first research question has to do with the understanding of reflexive, symmetric, and transitive properties of the equality. The interviews revealed that both secondary school teachers and prospective teachers in mathematics prefer to see the notion of equation as a computational process rather than as a static relation between two quantities. This indicates that only few of them have reified the concept of equation.

Looking at the collected data more closely, teachers incomplete understanding of the *symmetric* property of the equality was easy to detect. Interpretation of the

trivial equation $x=2$ especially points to this direction. For some teachers the statement is only an answer to an equation (for instance to the equation $2=x$). The following quotation illustrates the teachers' conceptions.

Interviewer: How do you comprehend this? [$(x=2)$]

Maria: I apprehend this as an answer. It could be an answer to an equation, but now I became unsure....The value of the unknown factor is already given. That is why it's not an equation.

The conception "The value of the unknown factor is already given" indicates that a teacher may think that the expression on the left-hand side is a process, which is already performed, whereas the expression on the right-hand side must be an answer. One of the teachers in this study gives also an illustrative example of pupils' difficulties concerning understanding of symmetric property of the equality. When asking pupils' conceptions on equals sign, the teacher (Mathias) says that pupils think that "there is to be an answer on the right-hand side. It is difficult for them to understand that $x=5$ is equivalent with $5=x$ ".

Conceptions on the trivial equation $x=2$ in the control group including 75 prospective teachers in mathematics, of which 28 primary school teachers, 34 secondary school teachers and 13 upper secondary school teachers refer to the same conclusion. The trivial equation is apprehended as 'an answer' or as 'an indirect equation'. One student says that it is an answer, because x is already solved. Another student regards it as an indirect equation, meaning both that it is an equation and that it is not. The interpretation 'an answer' indicates the equals sign is used in the meaning 'that is', which is usual in arithmetic.

Table 1 shows the percentage distribution between student teachers' Yes - and No-answers about $x=2$ together with average certainty degree and standard deviation.

Table 1. The percentage distribution between student teachers' Yes - and No answers about $x=2$.

Examples of equations	Percentage (%)	Percentage (%)	Percentage (%)	Average certainty degree 1 ↔ 5	Standard deviation
	No-answers	Yes-answers	missing answers		
$x=2$	55	44	1	3.86	1.28

Over the 50 % of the student teachers do not regard $x=2$ as an equation. The students are relatively sure in their interpretation, since the average certainty degree is almost 4 (The scale from 1=unsure to 5= sure).

The distribution of the Yes- and No-answers to the trivial equation $x = 2$ between the prospective primary, secondary and upper secondary school teachers is shown in Table 2.

Table 2. The student teachers' conceptions about $x = 2$ in percentage in the respective teacher category.

Examples of equations $x = 2$	Prospective primary school teachers	Prospective secondary school teachers	Prospective upper secondary school teachers
Yes-answer	(25%)	(56%)	(54%)
No-answer	(75%)	(44%)	(38%)
Missing-answer	0	0	(\approx 8%)
All	100%	100%	100%

Altogether 75 % of the prospective primary school teachers, and between 38 % and 44 % of the prospective secondary and upper secondary school teachers do not consider the example as an equation. The results indicate that the students apprehend algebraic notions as processes to find out or to do something rather than abstract objects. Process-thinking appears to be frequent for prospective primary school teachers in this study. All in all, seen from this viewpoint, it is quite reasonable why the symmetric property of equality is ignored so commonly.

Also the lack of understanding of the *reflexive* property of the equality was rather easy to notice in this investigation.

Interviewer: How do you comprehend this? [$x^2 - y^2 = (x - y)(x + y)$]

Jenny: You can find here two unknown factors, but if you factorise the left-hand side you receive the right-hand side. Is it always an equation, if there is an equals sign?

The conception – “if you factorise the left-hand side you receive the right-hand side” - indicates that even though a teacher is able to expand the brackets to verify the equivalence he is not capable to identify an equation and to see its structure. The question - “Is it always an equation, if there is an equals sign?” - suggests that it is not easy for the learner to accept that in equation the quantities (expressions) on different sides of the equals sign are the same.

Conceptions in the reference group with prospective teachers indicate that identities like $(x - y)(x + y) = x^2 - y^2$, $\cos^2 \alpha + \sin^2 \alpha = 1$ are apprehended as algebraic rules and formulas (to be applied on ‘proper’ equations) and not as equations. One of the student teachers notes that the statement

$x^2 - y^2 = (x - y)(x + y)$ cannot be regarded as an equation, because the answer will be ‘zero’. The student means that for all values of the variables the left-hand side will be equal to the right-hand side, that is, $0=0$ and there is nothing to be solved.

Another reason for that the rules and formulas are not apprehended as equations may be that the statements like $(x - y)(x + y) = x^2 - y^2$ and $\cos^2 \alpha + \sin^2 \alpha = 1$ according to general conventions, are called with different names like identities (Pythagorean identity, conjugate rule) and therefore cannot be understood as equations in the teachers’ mind.

On the contrast to the previous cases, we were not able to identify from our data the understanding of the *transitive* property of the equality or the lack of it. This is merely due to the way how the questionnaires were constructed for the original purpose, i.e., Attorps (2006). For instance, there were no chains of equations included in the questionnaires. And because the participants were not asked to perform any manipulation on the equations, it was not possible to test how they would judge, for example, a formula of the type $B = C$ in the case where $A = B$ and $A = C$ had been accepted as examples of equations.

Our hypothesis in the second research question was that the sense of symmetricity and transitivity of the equality on average would develop before the sense of reflexivity in the case of equation. Obviously, now we are not able to answer this question thoroughly because our data turned out to be incomplete to reveal how teachers understand the transitive property of the equality. However, some of our findings promote this hypothesis. All in all 40 % of the 75 student teachers misunderstood the beginning premises that two quantities are equivalent in the case of identity $\cos^2 \alpha + \sin^2 \alpha = 1$, and 25 % in the case of $(x - y)(x + y) = x^2 - y^2$, and therefore they classified these statements as rules and not as equations. This result can be compared with the case of $x^2 + (y - 1)^2 = 25$. Only 5% of the student teachers apprehended this statement as non-example of equation. However, because teachers’ sense of symmetric property of the equality also were quite poor in general, it is hard to say whether the teachers’ sense of reflexivity was less developed than the sense of symmetricity or not.

Our third research question is related to the existence of different concept definitions of the concept equation in literature. Different concept definitions, in turn, lead to different interpretations of the concept.

Having asked the secondary school teachers about their conceptions on the concept definition of equation the following four qualitatively distinct categories were found: (1) Equation as a concrete illustration, (2) Equation as a tool to find out unknown, (3) Equation as an equality between two quantities and (4) Equation as a transition to algebraic thinking. The teachers’ conceptions in the three first categories have a process-oriented or procedural view of equations. In

the fourth category the concept is apprehended as a mathematical entity from a general point of view.

In the first category the concrete metaphor of equations is on the focus of teachers' attention. Teachers describe the concept by using concrete illustrations of the concept, e.g. a balance, a swing plank etc. These kinds of illustrations can be found in textbooks both for primary and secondary schools. In the second category the teachers' attention is focused on the act of process of solving equations. The conceptions indicate that the concept recalls strong images of 'doing something', that is, "to find out", "to solve problems" (Maria; Simon). In the third category the structure of equations seems to be on the focus of the teachers' attention rather than the process of solving equations. One of the teachers says "The left-hand side is equal to the right-hand and then you must find an unknown number. I have not before reflected on what the concept of equation means ... $7 + x = 9$, something like this...." (Eric)

The process-oriented metaphor about equations, which is dominant in the first three categories, is replaced by the generalization metaphor in the fourth category. In this category, equations are seen in a universal light as a transition from arithmetical to algebraic thinking. Anna describes: "A type of mathematics where you use letters instead of numbers...When you use equations, you can make certain things universal...". The teachers' conceptions of the definition of equation indicate that teachers define the concept merely in procedural ways by using concrete illustrations and only rarely as a static relation between two magnitudes.

To sum up, our data reveals that for most teachers it is hard to accept the possibility of an equation being false. Especially the representatives of the first two categories seem to be attached to this restricted conception on the equation. This is rather natural: unbalanced expressions or a contradiction relationship can not be illustrated with a balance or a process that, by default, must have got a solution. One of the secondary school teachers, who holds the conception that the statement $e^{x+y} = 1$ is an example of equations, remarks in the interview: "Statement $e = 1$ is wrong for me (because the statement is not true, since $e = 2.718...$) but now you find $x + y$ in the exponent. It is an equation" (Simon). On the other hand, it seems that the representatives of the other two categories have got somewhat more wide-ranging conceptions on the equation concerning false equations.

More than a half of the participants claimed that $V = \frac{4\pi r^3}{3}$ is not an equation. We conclude that this is at least partly due to fact that some participants restrict the concept of equation to such algebraic contexts that only have one variable (which, in addition to that, also is always real-valued, i.e., one dimensional). Our data also indicates that many teachers consider variables in equations merely as unknown numbers. Since this conception is so common, it would be worth

further investigations how it is related to concept definitions of equation presented at modern secondary school textbooks.

DISCUSSION/CONCLUSIONS

Both this study and the modern literature on misconceptions include many examples in which private, intuitive mathematical knowledge leads to erroneous conceptions (e.g. Fischbein 1987, 6; Vinner 1991). For example, if the concept of equation is interpreted only as ‘a balance’ or as ‘an order to do something’, it unavoidably leads to erroneous conclusions. Therefore student teachers should be trained to use definitions as an ultimate criterion in mathematical tasks. This goal can be achieved only if they are given tasks that cannot be solved correctly only by referring to the concept image and thus encouraged to deeply discuss the conflicts between the concept image and the formal definitions (cf. Vinner 1991).

Unfortunately, as our study reveals, both teachers and teacher students hardly even know any versions of the concept definition of equation. Presumably, in modern mathematics textbooks equations are already introduced in the context of arithmetic and later in the context of algebra the concept definition of equation is only rarely discussed. Hence, if teachers’ conceptions on equation are only based on the descriptive use of this concept, it is not surprising that they do not become aware of the structural nature of this concept, e.g. the properties of the equality relation.

However, it seems that, for example, emphasising of the symmetric property of the equality might promote understanding the structural nature of the equation. This would, at least, provide us an auspicious opportunity to discuss teachers misleading division of equations into ‘proper’ equations and their answers. Similarly, teachers’ incorrect grouping of equations into ‘rules’ and ‘proper’ equations might vanish if the reflexive property of the equality was discussed properly in the context of manipulation of equations in teacher training. Generally speaking, discussions and reading about prominent misconceptions can help student teachers both to discover their own misconceptions and to understand pupils’ alternative conceptions (e.g. alternative solution strategies, erroneous conceptions etc.) and learning difficulties in mathematics.

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ANALYSIS OF THE AUTONOMY REQUIRED FROM MATHEMATICS STUDENTS IN THE FRENCH LYCEE

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In France, many previously successful students begin to have difficulties in mathematics in the upper secondary school, especially in the scientific course of study. This paper shows at first some examples of how the problems require an increasing autonomy to use familiar mathematical knowledge. This is interpreted within Chevallard's anthropological framework in terms of evolution and reorganisation of mathematical resources, more exactly of Mathematical Organisations. Secondly, the changes in the mathematical teaching from Collège to Lycée are analysed. It appears that the system leaves up an increasing part of the didactic process up to the student's private work. Hence, the last section quickly presents the way three high-achieving students in Grade 11 prepare for their tests.

The general issue this paper deals with is the following: from one grade to the next one, former successful students begin to cope with important difficulties in mathematics. In France, this first experience of failure regards a significant number of students at two crucial steps of the upper secondary school, Grade 10, which is the first year in the so-called *lycée*, Grade 11 for the students following a scientific course of study [1]. As one cannot consider that teenagers' cognitive abilities regress from one year to another, this steady trend raises two questions:

From one grade to the following, what changes in the mathematics tasks that leads to a student's success or failure? What changes in the way mathematics are taught?

This paper intends to present the way these questions are tackled within the theoretical framework proposed by the Anthropological Theory of Didactics (ATD). This approach of Advanced Mathematical Thinking and Teaching does not come with the predominant trend of studies developed in the English speaking community. Therefore, I consider it interesting to submit this work to the discussion in the Working Group.

WHAT CHANGES IN THE SCHOLAR MATHEMATICAL PROBLEMS?

In France, from primary school through to University first three years, assessment in mathematics relies almost exclusively on problem solving. Therefore to investigate the first question above, we need to study the problems' evolution. Of course problems change because they involve new concepts and theorems. We will not elaborate here on the conceptual difficulties the students may have with this new knowledge and therefore with its use. However relevant this aspect may be, this paper

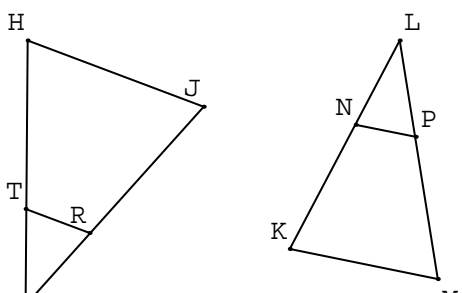
intends to focus on the difficulties regarding familiar knowledge which students previously successfully used.

Analysis of mathematical problems in a given school context: examples

Problems or exercises? This point cannot be eluded. Most papers which deal with problem solving consider that a problem for which a routine or familiar procedure is known is not a problem, only an exercise. This dichotomy is much too rough to be efficient in our study which needs a more progressive scale to differentiate the tasks given to students. Even if a procedure is familiar, the conditions of its use may change a lot from one task to another, thus requiring a variable activity from the student. As following, three examples coming from French textbooks will illustrate this claim and let us see what kind of tools are used to analyze evolutions. The words ‘exercise’ and ‘problem’ will be used without any particular intention in this paper.

The following exercises have in common that they use what in France is referred to as the ‘*Théorème de Thalès*’ [2]. They appear at different moments of the curriculum, from Grade 8 to Grade 10.

Example 1 : Grade 8, Chapter 12 ‘*Triangles et droites parallèles*’.

	<p>Dans les deux cas suivants, calculer la longueur demandée.</p> <p>Figure de gauche : $(TR) \parallel (HJ)$, $HJ = 9$, $TR = 4$, $GJ = 9$, calculer GR.</p> <p>Figure de droite : $(NP) \parallel (KM)$, $LN = 5$, $NK = 7$, $NP = 4$, calculer KM.</p>
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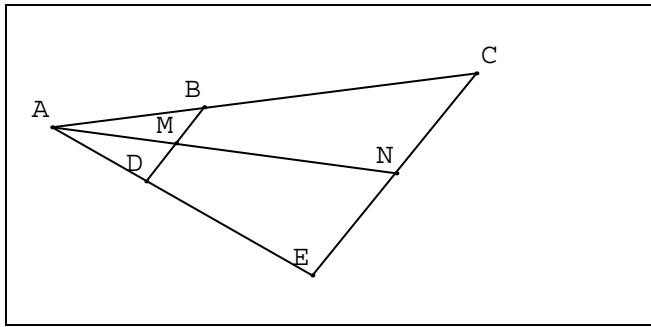
Hatier 4^e, Collection Triangle Mathématiques, 2002, 9 p.195

This exercise appears in the chapter where the *Tales* theorem is taught for the first time. The derived procedure to calculate a missing length is not yet familiar. The students are required to use it in different conditions: variations affect triangles orientation in the sheet, points’ name and given lengths. In particular, the second case introduces the necessity of an intermediary step (calculating LK).

This type of tasks is very frequent all along Grades 8 and 9, so that when they leave ‘*Collège*’, most students recognize on their own that the *Tales* theorem may be relevant from the type of drawing we have above.

Example 2 : Grade 10, Chapter 9 ‘*Configurations du plan*’.

The following exercise appears in a chapter which intends to revise the whole geometric knowledge taught before. Hence, when they face a task, students cannot guess from the chapter context which theorem to use. But, as pointed before, the drawing may be here considered as a good call for the *Tales* theorem.



On a

$$AB = 3 \quad BC = 4,5,$$

$$MN = 3,6 \quad BM = 1,5 \quad AD = 2,5$$

La droite (BD) est parallèle à (CE).

Calculer AE, AM et CN

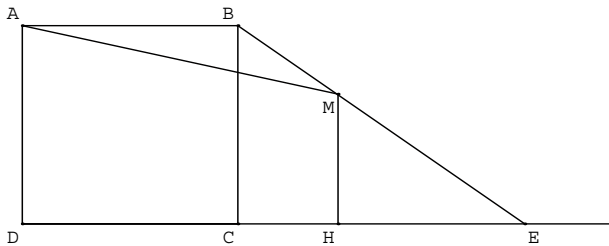
Hachette Seconde, Collection Déclic Maths, 2000, 50 p.247

We shall not consider that the procedure goes on here as a simple routine, especially because of the following analysis. When calculating AM, this procedure leads to the equality $7,5AM = 3(AM+3,6)$. Students have to recognize a linear equation to finish this question. Solving the equation $7,5x = 3(x+3,6)$ is a mere routine in Grade 10. But here, the usual symbol x for the unknown is missing; the general class context refers to geometry and not algebra. Hence students are completely in charge of identifying the type of mathematical question involved and summoning up the relevant knowledge.

Example 3: Grade 10, Chapter 9. '*Fonctions: Généralités*'.

This exercise belongs to a chapter which deals with functions, i.e. a rather new subject for students.

On considère un carré ABCD tel que $AB = 3$. On place le point E sur la demi-droite $[D,C)$ de sorte que $DE = 7$. Soit M un point de $[B,E]$ tel que $EM = x$ et soit H le projeté orthogonal de M sur (DE). Le but de ce problème est d'étudier l'aire du trapèze ADHM.



1. Calculer BE.

2.a) Exprimer les distances MH et EH en fonction de x .

b) En déduire la distance DH en fonction de x .

3. Exprimer l'aire du trapèze ADHM en fonction de x .

Magnard Seconde Collection Abscisse, 2004, 47 p. 292

In this *Calculus* context, the first questions are geometric ones and require calculating some lengths depending on the variable x . Several procedures have been taught relying on *Pythagoras* and *Tales* theorems or using trigonometry. Here, unlike what we found in the second example, there is no strong indication that one of these procedures might be more relevant than the others –the drawing is complex, the *Tales* configuration is not especially visible, the text does not refer to parallel lines or to

some right-angled triangle. Hence the solver needs to summon up the different procedures he knows and check by himself if one or another is efficient. Then he has to adapt it to the x context.

From one grade to the following, what changes in the mathematical problems?

The analyses presented here originate in Robert's works (see for example Robert and Rogalski, 2002). According to her proposals, the difficulty level of an exercise or problem concerning the use of a given procedure is assessed through two questions. At first, is this procedure in some way present in the exercise wording? Secondly, is the procedure efficient in its familiar simplest form or does it need some adaptations? Of course, the analysis must take into account the task context. For instance, we will consider that, in Grade 8, the *Tales* procedure is not used in its common form in the first example (2nd case), while in Grade 10, this exercise would be a routine one.

We can now give at least a partial answer to our initial question. The evolution we met through the three examples analyzed before is paradigmatic of what happens from Grade 9 to Grade 10 and still more from Grade 10 to the scientific course of study in Grade 11 where the rhythm of introduction of new objects becomes higher. Knowledge taught in the previous years is considered familiar. These resources (this word refers to the literature on problem solving, for instance Schoenfeld, 1985) are involved in exercises where they need to be coordinated between them (example 2) and more and more often with completely new objects and procedures (example 3). In this latter case, the charge to perceive the relevancy of some familiar procedure often relies on the student. This responsibility becomes especially demanding when several procedures have been taught for the same type of questions that is more frequent when the curriculum goes on.

In short, Grades 10 and 11 problems require that the students take more and more initiatives of their own using familiar resources; the demand of autonomy as a problem solver increases, probably contributing to the students' new difficulties.

INTERPRETING THIS OUTCOME WITHIN THE ANTHROPOLOGICAL THEORY OF DIDACTICS

The literature (for a review of papers in English, see Carlson and Bloom, 2005) routinely differentiates two dimensions in problem solving, knowing-how and knowing-to-act (Mason and Spencer, 1999). The first one is interested in resources, described as formal and informal knowledge about the content domain (Schoenfeld 1985; Castela, 2000). It emphasizes the ideas of invariance, genericity: every new task has something in common with others and what the solver knows about those may supply him with useful tools. Therefore, the knowledge building process is central, at the individual level as well as the social one. On the contrary, the second one insists on singularity, contingency: a high-achieving solver is someone who is able to tackle with the task 'surprises'. This approach necessarily puts forward the individual and deals with abilities, beliefs, affects, metacognition.

The relative importance granted to each of these dimensions radically differentiates mathematics epistemologies, math education strategies and investigations on math education. One may reduce the resources to the theoretical mathematical knowledge, considering that it expresses the whole relevant genericity of mathematical problems, at least the school ones. This viewpoint is present among French researchers, especially among those who work on primary school (Brousseau, 1986 - the notion of '*glissement métacognitif*'). Within the English-speaking community, the general trend appears to be considering both aspects. Including algorithmic and routine procedures into the field of resources, Schoenfeld (1985) gave evidence that a good problem solver is someone possessing more knowledge, well-connected knowledge and who is moreover able to access to his resources and regulate their use in the solving context. Recently, Carlson and Bloom (2005) have confirmed the importance of well-connected knowledge that appears to influence all phases of the problem-solving process. However, since the end of 80s, most of studies have turned on the knowing-to act in the moment dimension, insisting in particular on metacognitive competencies and more recently on affective variables. The resource dimension appears to be considered as minor.

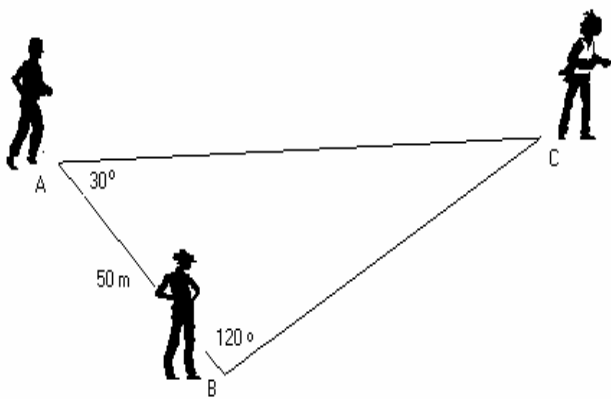
The Anthropological Theory of Didactics (Chevallard, 1999, 2002) chooses a radically different approach of mathematics, focusing on genericity of human practices and social production of knowledge:

La Théorie Anthropologique du Didactique considère que, en dernière instance, toute activité humaine consiste à accomplir une tâche t d'un certain type T , au moyen d'une certaine technique], justifiée par une technologie θ qui permet en même temps de la penser, voire de la produire, et qui à son tour est justifiable par une théorie Θ . (Chevallard, 2002, p.3)

Types of problems, techniques, technologies and theories are the basic elements of the anthropological model of mathematical activity. They are also used to describe the mathematical knowledge that is at the same time a means and a product of the activity.[...]. They form what is called *praxeological organisations* or, in short, *mathematical organisations*. The word 'praxeology' indicates that practice (*praxis*) and the discourse about practice (*logos*) always go together...(Barbé and al., 2005, p.237)

With the *mathematical organisations* (henceforth abbreviated as MO), the ATD proposes a general model of the resources a social group as well as an individual problem solver may build or use while coping with mathematical tasks.

These MOs depend on the institution where the mathematical activity takes place. For the same type of mathematical task, the MO may be different whether this task is tackled with in a mathematics research context or in an engineering one, in a French school or in a Chilean one, in Grade 7 or in Grade 10... Let us consider the following example:



Calculate AC

T: Working out an inaccessible distance

τ : Draw a triangle abc at scale (for instance $ab=5\text{cm}$ and same angles), measure the correspondent length ac and then calculate AC.

θ : Let ABC et $A'B'C'$ be two triangles with two equal angles, $\angle A' = \angle A$ and $\angle B' = \angle B$, then

$$\frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC}$$

This MO is presented in a Chilean Grade 10 text book. It is absolutely impossible to meet this MO in France at the same level because measuring is ruled out in the mathematics course. In France, a possible technique in Grade 10 is to introduce the altitude BH ; in Grade 11, students will use the sinus formula.

In particular, I advance that the technological component is especially dependent on the community of problem solvers. This claim relies on the following analysis. Etymologically a technology is a rational discourse (*logos*) about the technique (*tekhnê*). The technology of a technique is double-faced, including theoretical and pragmatic elements. The first ones, especially theorems, establish the mathematical validity of this technique; the second ones intend to supply the solver with resources to concretely make use of it. Most of what Schoenfeld calls “problem-solving strategies” belong to this pragmatic part of technology. We could use the term ‘*folklore*’ to name this pragmatic part, in the etymological meaning of this English word: it is the science (lore) of the folk [3]. This mathematical *folklore* depends highly on the community of solvers and its experience with the type of tasks. Hence, it may change while the theoretical part is stable.

Let us now illustrate this approach from the examples we have presented before.

Of course, within the limits of this contribution, the following elaboration will be rather rough. In examples 1 and 2, the type of tasks ‘calculating a missing length in a *Tales* configuration’ is at stake with the technique coming from the *Tales* theorem, which is a theoretical element of the technology. In Grade 8, the students’ pragmatic science will probably include some elements regarding the presence of two triangles, whose correspondent sides are associated in the ratios. Such an observation could prevent errors such as considering LN/LK in the first exercise because these two lengths are given. In Example 3, whatever MO relative to the same *Tales* type of tasks has been elaborated, it is not sufficient to cope with the first two questions. A more complex MO would help the student face the demand of initiatives present in the orienting and planning phases of the solving process (Carlson and Bloom, 2005). Dealing with a more general type of tasks ‘calculating a length’, this MO connects several punctual MOs, eventually developing their technology to describe their

efficiency conditions: the *Tales* technique requires knowing that some straight lines are parallel, some points on the same line; the *Pythagoras* theorem and trigonometric techniques need a right-angled triangle.

Regarding the second type of tasks we have met in Example 2, i.e. ‘Linear equations with one unknown’, what is at stake in this exercise is the perception of the type itself. It is characterised by the research of an unknown quantity and not by the presence of the symbol x , which is usually the very point *Collège* students keep in mind.

In short, within the anthropological epistemology of mathematics, the higher degree of autonomy required by Grade 10 and 11 student solving activity is considered as requiring that previous MOs evolve, be completed and reorganised. This approach does not ignore the acting-in-the-moment aspect of problem solving, nor the individual competencies involved. However, it puts forward the knowledge building process in a social education context, that is the didactic learning and teaching problematic.

WHAT CHANGES IN THE WAY MATHEMATICS ARE TAUGHT?

Re-creation of a mathematical organisation in a didactic context

In the Anthropological Theory of Didactics, the process of recreation of a mathematical organisation is modelled by the notion of *process of study* or *didactic process*. This process is organised into six distinct *moments*: the moment of the *first encounter*, the *exploratory* moment, the *technological-theoretical* moment, the *technical* moment, the *institutionalisation* moment, and the *evaluation* moment.

The second moment concerns the *exploration* of the type of tasks T_i and elaboration of a technique τ_i relative to this type of tasks. [...] The third moment of the study consists of the *constitution of the technological-theoretical environment* [...] relative to τ_i . In a general way, this moment is closely interrelated to *each* of the other moments. [...] The fourth moment concerns *the technical work*, which has at the same time to improve the technique making it more powerful and reliable [...] and develop the mastery of its use. (Chevallard, 1999, pp. 250-255, English translation in Barbé and Al., 2005, pp. 238-239).

This model will enable us to describe what changes from *Collège* to Grade 10 and 11.

Collège mathematics teaching: a well developed process of study

From Grade 6 to Grade 9, mathematics teaching goes on very quietly, introducing a limited amount of theoretical objects and correlated MOs. Hence, teachers have time enough to organise the different moments of study. In particular, as we have seen in the first example, they give their students the opportunity to cope with a rich sample of variants of a given type. The common work on the students’ productions is a moment when a collective *folklore* may be elaborated. In short, the teacher creates good conditions for the MO appropriation by the students within the math class. This point clearly appears in Felix’s study on Grade 9 students’ private work (Felix, 2002).

Interviewed on the way they prepare periodic assessment in mathematics, two high-achieving students claim that, regarding exercises, they only read the solutions given by the teacher to check that they have no difficulties, but they are sure they need no more learning. They add that assessing exercises are always similar to the previously studied ones.

Math teaching in Grade 11 scientific course of study: a mere starting off the didactic process

The mathematics syllabus for the scientific course of study introduces a great number of concepts and theorems, each of them controlling several MOs. The teaching rhythm strikingly increases. Consequently, the teacher has not enough time to develop the didactic process for the new MOs. Except for the basic fundamental ones, he hardly begins the second moment, which reduces the opportunity for the class community to elaborate the technological environment, especially its *folklore* component. For instance, when working on the barycentre associativity, the teacher shows that it may be used to prove that three lines are concurrent, that three points are on the same line but he will not vary the exercises involving these techniques. Hence, students lack opportunity to really become aware of the technique subtleties (e.g. how to choose the barycentre that is efficient to prove the concurrence). As for the familiar MOs, it is nearly impossible to go back on them for the evolution and reorganisation process required by the increasing demand of problem solving autonomy. In brief, the teaching system focuses on theoretical knowledge and leaves under the students' responsibility the charge of developing the process of study for the new MOs as well as for the familiar ones.

ABOUT SCIENTIFIC HIGH-ACHIEVING STUDENTS' PRIVATE WORK

From the previous analysis, I draw the following hypothesis: from *Collège* to *Lycée* scientific course of study the changes regarding the mathematical tasks and the teaching conditions require that the students take on more responsibilities as problem solvers as well as mathematics learners. Some previously successful students ignore this new self-teaching charge or fail to face it; therefore they cope with increasing difficulties in mathematics. Others manage to adapt their private work in order to carry on with the process of study initiated by the teacher or to start it again in the case of previously taught MO. In order to investigate on the way they work, I have interviewed three Grade 10 scientific high-achieving students on the following subject: tell me what you have done to prepare yourself for the latest test in mathematics. The salient points I have drawn from this interviews are the following:

They spontaneously put forward the rhythm speeding-up from Grade 10 to Grade 11. Hence, they had to change the way they work at home to prepare for their tests, especially regarding the exercises. At first, they solve again almost every exercise studied with the teacher. While the successful Grade 9 students interviewed by Félix have no doubt on their learning during the class, these students have experienced the necessity to check that they are really able to find the solution. If not, they study the teacher's solution

and try again to solve the exercise. They do not solve new exercises because they would not have any way to control their production validity. But two of them systematically complete this solving written work by a verbal phase in which they describe the solution. In doing so, they begin to decontextualize some generic elements of the solution and to elaborate a personal technology. They are clearly aware that each exercise intends to introduce them to a given type with an associated technique. The third one generally stops working when she can solve every exercise; she is confident of her ability to adapt what she knows to the specificities of the assessing test. However, it may happen that an exercise appears to be especially resistant; in that case, she struggles to draw from the solution elements of the teacher's efficiency. She gives a very convincing sample dealing with the monitoring of the parallelogram relation with vectors. On this occasion, she goes back to a Grade 10 MO.

Thus, these high-achieving students take in charge through their private work a certain development of the *technological-theoretical* moment. In the same study, I interviewed 7 other students with average or weak results; they never refer to this working form, which, in the limit of this clinical study, appears to favour success in mathematics. This confirms the outcomes of a previous investigation regarding university students (Castela, 2004).

PERSPECTIVES

In this paper we propose a double diagnostic to explain the difficulties which former successful French students meet in mathematics in Grade 10 or later, in Grade 11 scientific course of study: 1. the mathematical problems requires the solver to take more and more initiatives; to face this demand, the familiar student resources should evolve; 2. at the same time, the teaching system partially leaves up to the students the charge of re-creating for themselves the Mathematical Organisations at stake in the syllabus. Hence, an important autonomy as a learner appears to be demanded from the students. This generally requires some evolutions of the private work that many students are probably unable to imagine on their own. Therefore I consider it necessary at this point of my investigation to think of experimental proposals to help students to adapt their working style.

NOTES

1. During the last two years in the *lycée* students have to choose a specific course of study. This choice is not totally free. For the science course of study, it highly depends on the student's results in science in Grade 10 (*Seconde*): those who follow this course of study (*Première, Terminale Scientifiques*) were generally rather successful in mathematics.
2. The theorem students learn in grade 8 is the following one: Let d and d' be two straight lines with O in common. A and B are two points on d , A' and B' two points on d' . If the straight lines (AB) and $(A'B')$ are parallel, then $OA/OB = OA'/OB' = AA'/BB'$.
They do not learn that one can directly claim that for instance $OA/AB = OA'/A'B'$.

3. Y. Chevallard referred to this idea of *folklore* during the Baeza Congress on the TAD, attributing its use to some English mathematician. But, I am not sure that he would agree with my proposition to insert a folklore component in the technology.

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LOCAL AND GLOBAL PERSPECTIVES IN PROBLEM SOLVING

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This paper will raise issues concerning the interaction between local and global foci realized in the working mathematical environment. These issues are illustrated by suitably tailored tasks and presented solutions. The discussion is mostly theoretical, but in some cases data from fieldwork is referred to. Predicted difficulties for students in effecting switches in argumentation from local to global perspectives or vice-versa are considered, as well as the consequences on students' general problem-solving ability if they are not overcome. Pedagogical measures are mentioned.

INTRODUCTION

There is some literature by Mathematics educators that claim switches of focus can affect how the original 'make-up' of the task environment is perceived, allowing radically different ways to guide argumentation and solution paths. A particular proponent of this viewpoint is Mason, e.g., Mason (1989). We agree that the ability to bring about new vistas to mathematical situations is central in problem solving of any complexity. However, the literature in the main does not go very far in categorizing switches of focus; further, there is a paucity in considering teaching practices that can enhance students' ability to effect them. One difficulty is that a change in attention sometimes can be made spontaneously; in such cases, how can the researcher analyze the source of the student's line of thought? However, it is rare for the student not to have done some provisional and experimental work, which reveals a new basis of argumentation might be available. In this case, the researcher has a trace that led to the switch of attention that can be analyzed. What for the other students who are not able to effect an essential switch for one reason or another? We would want to give them a firmer experience-base to identify crucial switches of focus. We feel that the topic of executive control is very important in this, in that it encompasses a sense of anticipation beyond exploratory work (see Mamona-Downs & Downs, 2005; see Schoenfeld, 1985, for a discourse on executive control). In this paper, though, we will stress primarily the relationship between thinking in terms of gestalt or integrated mental images, and thinking in more analytical terms. The latter naturally breaks down the structure such that more detailed and dependable argumentation is achieved. In our opinion, these two related themes could be utilized to execute requisite switches in a more deliberate and negotiable fashion compared to

spontaneous 'realizations'. The analysis is required to give information about the global system implied by the setting of the given task; the analysis typically restricts the structure so that simpler argumentation may be made at a different level, and the resultant information is 'lifted' to the global level. The restricted structure has a local character. The 'switch' in how the system is perceived happens either simply by regarding the system as comprising its 'components' together, or by observing constructions that arise in the local examination that become significant on the global level.

The paragraph above motivates the theme of this paper, which is to consider how local and global perspectives can interact in tackling mathematical tasks. By taking selected tasks and solutions, we illustrate several important aspects of this theme, and we conjecture allied students' difficulties and their sources. At times data from fieldwork will be referred to.

AN EXAMPLE OF A LACK OF SENSE OF THE GLOBAL

We consider the following task:

Task 1: M is the real matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

(i) Show that $M^2 - M = 2I$ (ii) Find M^{-1}

Discussion: In an exam that we administered recently, of the students tackling it, about 60 first-year university students studying geology, none used part i. to deduce part ii. . This is despite that 22 students succeeded in the first part. All students attempting the second part employed routines familiar to them, such as calculating $\text{Adj}(M) / \text{Det}(M)$. This procedure is comparatively complex to execute, and the vast majority of students made mistakes, whilst multiplication of both sides of $M^2 - M = 2I$ by M^{-1} is both an easy action to apply and an obvious one to invoke.

What then prevented the 22 students that succeeded in part i. to further apply it to part ii. ? The most conventional way to explain this phenomenon would be to say that students generally prefer to keep to standard methods, and to what they are accustomed to do. However, we doubt that this is the whole 'story'. We conjecture that for most students, a matrix is identical to the array of numbers that represents it, that endows it with an identity of a self-contained, independent, entity. When a related matrix, such as its inverse, is introduced, it is assumed that it is obtained via actions on the array. Matrix algebra (without referring to the arrays) is difficult to conceive because squaring and taking the inverse of matrices are semantic for all matrices simultaneously. (For example, the inverse of a matrix A, if it exists, is the

matrix when multiplied to A is the identity; this property is understood independently from the array). Here M has to be thought of as a particular member of a class of matrices. If students think of M as an isolated entity, their flexibility of thinking about M in respect of other matrices will be impaired. The realization that an entity belongs to a larger class concerns the forming of a global perspective.

EXAMPLES CONTRASTING GLOBALLY AND LOCALLY BASED ARGUMENTATION.

Complicated tasks will tend to need intricate interplay between local and global switches of focus. In simpler tasks, one may judge whether a solving strategy depends largely on a local basis or a global one:

Task 2: For some positive integer n , there are 2^n teams that have qualified to participate in a knock-out competition (i.e., each team plays another team; if a team wins its game it enters into the next 'round', otherwise it does not take part any more. This process continues until there is only one team 'left'). How many games were played in total in the competition?

Discussion: Perhaps the most evident approach would be to consider the number in each round and then sum. Doing this one obtains:

$$2^{n-1} + 2^{n-2} + \dots + 1 = 2^n - 1.$$

Another approach is to realize that there is a one-to-one correspondence between the games and the teams that lose a game. As only one team does not lose a game, it is immediate that the number of games is $2^n - 1$.

We contend that the first approach above has a local perspective in that the argument is based on breaking down the whole structure of the competition into rounds. The information obtained from the different rounds has to be collated in the form of a summation to obtain the number of games overall. The correspondence approach acts on the global level, because all the games are dealt with simultaneously. The two approaches as they are presented above would be roughly the same to apprehend (except the first requires an extra 'step' to simplify the summation). However, it is the second approach where a switch of focus occurs; how the solver naturally first sees the competition is a sequence of rounds, and not in terms of all games together. This means that if you are forming a strategy for answering the task (and not perusing a given exposition), it is much more likely that the first approach will be thought of and taken (and we have data that backs this up, Mamona-Downs & Downs, 2004). However, for other tasks, it is inevitable for a solution path to effect a change of focus, such the next example illustrates.

Task 3: There is a group of islands that are linked by a system of bridges. How can you decide whether you can take a journey that takes you back to where you started for which you have crossed every bridge once and only once.

Discussion: This problem, together with its result, is a contextual form of a basic proposition found in the combinatorial topic of graph theory. Such a journey can be made if and only if at every island an even number of bridges emanate from it. The necessity comes from the fact that if you enter an island, you must leave it on another bridge not used before. The sufficiency is not as immediate, but comes from an argument that if one can complete a circuit that does not include every bridge, one can always construct another circuit that takes up more bridges (e.g., see Bollobas p.14-15).

The task environment, asking for a journey satisfying some conditions, involves all the bridges and islands so it is an issue brought up on the global level. However the criterion addressing the question 'does a journey exist?' very much involves local considerations; you consider any island and the number of bridges emanating from it, and collate the information over all the islands. The switch from the global situation to a single (but representative) island cannot be avoided. This switch might be considered a relatively easy one to make; after all there are only two types of 'entities' involved (bridges, islands), so putting attention on isolating members of one of these (the islands) would not seem out of place. Still, we conjecture that if students encountered this task, the majority would not succeed. (We aim to have collected some data before the congress). Assuming this, how could we make students more likely to catch the shift? One idea would be to treat switches between local and general foci rather like a heuristic, meaning that students have to be made aware of the general idea as being critical in problem solving and be alert to their application by giving them experience via suitable tasks. When the student has difficulties in a task, the teacher can prompt him/her by advising to think 'more locally' or 'more globally' as appropriate. (Our model of the local and the general has some relevance to some of the heuristics mentioned and discussed in Polya, 1973, though these tend to be more explicit in form.)

Some people might find something disconcerting in the criterion that resolves task 3. It seems to dismiss a lot of the information implicit in the global environment, such as if a bridge emanates from a particular island one does not have to consider which other island is linked by the bridge. Because of this, the criterion might be regarded as not respecting the continuity of the completed journey. Because of this, although the argument is simple enough when the advantageous focus is taken, the 'transparency' between the givens and the result is impaired. The topic of transparency is taken up further through considering the next example:

Task 4: Consider the set S of all bijections f whose domain and image is $\{1, 2, \dots, n\}$, where n is even. What is the maximum value of

$$U := \sum_{r=1}^n |f(r) - r|$$

as f ranges over S ? How many bijections in S achieve the maximum?

Discussion: Take the set $N_1 = \{1, 2, \dots, n/2\}$ and set $N_2 = \{n/2+1, n/2+2, \dots, n\}$. Any permutation that maps N_1 onto N_2 (and hence N_2 onto N_1) will maximize U , so the number of bijections maximizing U is $(n/2)!^2$, with the maximal value $2((n/2+1) + (n/2+2) + \dots + n) - 2(1 + 2 + \dots + n/2) = 2(n/2)^2$.

What we want to stress here is on how the kind of bijections that yield the maximum were identified, rather than the numerical results. Our presentation of the solution above does not explain why the sets N_1 and N_2 are introduced. Their appearance must reveal a role for them that is pre-mediated before anything is (formally) written down. This suggests a prior intuitive or 'mental' processing. One way of thinking of the problem is below. U can be interpreted as the 'total displacement'. To maximize this quantity one would promote the rough idea of 'sending' low numbers to high numbers, and vice-versa. It is natural to regard the set of low numbers as the 'half' of numbers of least value, i.e., N_1 , and similarly the set of 'high' numbers as N_2 . The contribution to the total displacement due to a mapping N_1 onto N_2 must be as large as possible (we add up all the 'highest' half of the integers and take away the half with the lowest values); similarly, and independently, for a mapping from N_2 onto N_1 . In this way, the kernel of the solution can be negotiated informally.

The informal argumentation related above would seem to be a good illustration of proof language as described in a paper of Advanced Mathematical Thinking (AMT) Group at CERME 4 (Downs & Mamona-Downs, 2005). The proof language is a channel of interpretation of formal constructs such that the mathematical development can be faithfully guided by the imputing of meaning. In our case, the central interpretation is 'total displacement'.

Rather like task 1, most students will take a 'point-wise' approach to this task, i.e., try to consider particular functions that might seem to give the maximum for U . For example, we have seen quite a few students proposing the bijection $f(i) = n + 1 - i$ for all i , fewer proposing $g(i) = (n/2 + i) \bmod n$. Of course, both are suitable. But the students did not have the language to justify this, and none seemed to appreciate the whole 'space' of solutions. The approach we take is far more global in its perspective, and it allowed us to obtain a much fuller picture. Although local foci are more powerful by enabling analysis, this must be set against the directional help that global oversights can provide.

Indeed, the solution works on the integrated whole; even though there is a partition made into two parts of the domain and image, this action and its consequences remain intimately linked to the global situation and aim. For this reason, we believe that the presented solution has a 'high degree' of transparency; once you have checked up all the details, the basic ideas are easy to synthesize cognitively. (Notice that we do not relate at all transparency with immediacy).

Working on the local level tends to act against transparency, as it tends to make a link, forget about it temporarily, and then to re-establish the link later. Educators often exhort expositions that are transparent because argumentation that lacks transparency can alienate students from mathematics (see for example Hanna & Janke, 1996, in the context of presenting proof). However, as illustrated in task 3, the use of local argumentation is sometimes inevitable, and for the more sophisticated and abstract problems that are found at the AMT level its use is almost ubiquitous. At this level of mathematics, we must make students aware that, in the main, they cannot expect to see through strategies via a constant mathematical framework. They have to see solution paths as consisting of stages for which the 'base' will vary. They have to develop an appreciation of synthetic thinking (see Weber, 2002) and the proof language to obtain a new kind of aesthetic superceding the one offered simply by transparency.

THE ROLE OF CORRESPONDENCE AND FUNCTION

Function, together with set, is the most fundamental concept in mathematics. As we make switches of attention, one would expect some underlying functional backdrop to relate them. In this section, we make a distinction between correspondences and (formal) functions. A correspondence marks the integrated mental recognition of a relation between two sets of entities in the working environment. A function is more analytic, and acts more as an input and output device (see Mamona-Downs & Downs, 2005, section 2.3). A correspondence then has a global aspect, whereas a function in its element-wise action has a local one. We will expand on this theme by considering the following example.

Task 5: Let C be a circle, and suppose that P_1, P_2, \dots, P_n are n points on C . Construct all chords of C connecting two points from P_1, P_2, \dots, P_n . A crossing is a point strictly inside C that is an intersection point of the constructed chords. What is the maximum number of crossings (that is, how many crossings are there with the assumption that only two chords intersect at any crossing)?

Discussion: Call the points P_1, P_2, \dots, P_n circle points. An approach to answer this task is to observe that every crossing is associated naturally with four circle points (the 'ends' of the two chords intersecting at it), and then to argue that any set of four circle points 'generates' one and only one crossing. Then you have a bijection between the crossings and the subsets with four elements of $\{P_1, P_2, \dots, P_n\}$, meaning that the number of crossings equals the choices of picking 4 things out of n , i.e., $n! / (n-4)!4!$. Other approaches exist, but tend to be far more convoluted. For example, one may categorize the cords in such a way that it is expected the cords in each category to have the same number of crossings. Calculate the constant number and multiply by the number of cords in the category, sum and divide the result by four (the procedure counts a cord twice, and a crossing involves two chords).

This task has some similarities to task 2. Both tasks ask about the number of objects of some type in the given system and both have alternative solutions. One solution starts from the global perspective that is broken up and analyzed locally. The other starts from the local entities, highlighted by the statement of the task, whose similarities in property means that assimilation is readily obtained on the global level. For the prior, what is required from the students, apart from switching the focus from the 'givens' to the 'unknowns', is to identify a relationship between two (or more) families of objects either existing in the system or can be constructed from it. However, the realization of the relationship is an instantaneous set-wise comprehension, so is a correspondence. Observing and employing correspondences is hugely important in problem solving as it creates connections that empowers the students' sphere of argumentation. However, for every correspondence there is an underlying function; the two differ only in cognitive terms. The function acts more on an element-to-element basis that checks the presumed implications from the correspondence. For example, once hinting to some gifted undergraduates that there was a relation between the crossings and the subsets of four elements, all the students assumed that the relation was a bijection without explaining it (Mamona-Downs & Downs, 2004). (It is fairly easy to argue that any four circle-points generate one and only one crossing). Translating the correspondence into function constructions such as a formal definition of a bijection is often required for the soundness of the argument. Hence, even though a correspondence lifts the local to the global, analytic aspects occur also.

Usual didactical practices stress function but not correspondence, a fact that we believe severely undermines students' problem solving in general. Unfortunately, some frameworks employed by mathematics educators would seem to encourage this stress. For example, the APOS theory (e.g., Cottrill et al., 1996) leans towards a functional perspective, at the same time claiming a comprehensive aspect.

INDEPENDENCE, INVARIANCE AND FREEDOM OF CHOICE

Task 6: If n is a positive integer and n has an odd number of divisors, prove that n is a square integer.

Discussion: We give a brief solution. We suppose that n has the prime decomposition:

$$n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$$

Then the number of divisors are $(e_1 + 1)(e_2 + 1) \dots (e_r + 1)$ that is odd, so each $(e_i + 1)$ must be odd and e_i even for all i in the set $\{1, 2, \dots, r\}$.

This task may be considered as being a standard result in elementary number theory. However, we can still conjecture the cognitive demands it makes on the student. We assume that the student has met the prime decomposition beforehand.

We shall concentrate here on how the divisors are enumerated from the decomposition. The student has to realize the significance of the decomposition concerning divisors. He/she must recognize that each divisor has the form

$$p_1^{f_1} p_2^{f_2} \dots p_r^{f_r}$$

where $0 \leq f_i \leq e_i$ for each i . This yields a global perspective in the sense of a correspondence discussed in the previous section. To progress, though, a mental action is required where one prime power (w.l.o.g. the first) in the decomposition is allowed to run through its possible values whilst all the other prime powers are kept constant. With this restricted freedom, we have $e_1 + 1$ choices. Now suppose both the first and second prime powers are allowed to vary, the others remaining constant. For each $e_1 + 1$ choices for the first 'component', there are $e_2 + 1$ choices for the second independently. Continuing this line of thought, the number of divisors are given by the multiplication of the $e_i + 1$ as i ranges across 1 to r .

This argument has a much broader context in terms of the number of elements in a Cartesian product. The student is required to realize that if one component takes one value, this does not affect the freedom of choice in value of any other component. This expresses the quintessence of the notion of independence. Even though the multiplication principle behind Cartesian products can be treated to some degree at primary school, from our own experience in teaching undergraduates we have noticed that undergraduates often do not 'catch' independence between components; in particular quite often addition is used rather than multiplication.

The application of the notions of independence, invariance and freedom of choice to Cartesian products constitutes a very particular one; their significance extends far beyond. All point to an analytic and local perspective. Independence suggests two (or more) components of the system that can be worked in isolation. It tends to be related with (standard) decomposition that ultimately provides a new focus on the global level, be it a synthetic one. Invariance suggests a set of transformations that all fix a certain entity within the system. Such a property often leads to more global consequences. Freedom of choice is the sense of what options are open in picking elements in a set. As such, it is broader than the notion of independence. (For example the task, how many permutations on $\{1, 2, 3, 4, 5\}$ are there, requires understanding of freedom of choice but does not involve independence). Freedom of choice is particularly useful in obtaining simplified formats of symbolic processing by invoking the notion "without loss of generality" (w.l.o.g.) with which it is closely related.

This topic has hardly been touched by Mathematics Education. However, these three notions at the AMT level are not only tools that are invoked constantly in the presentation of mathematical theory but they are also tools that students have to master to allow them to organize their own solution strategies. How can students learn the associated problem solving skills? When presenting proofs from a

pedagogical point of view there is a responsibility not only to explain the result and the processes utilized in the proof that sustains the argumentation in general, but also to explain organizational aspects. For instance, students might be given guidance to attain conscious awareness in exploiting invariance properties, to organize an argument on an independence or dependence principle, or to utilize freedom of choice to identify elements that satisfy further conditions.

In this section, we have only given some preliminary thoughts concerning the theme of organizing argumentation. The theme would seem to be a mix of general executive control and the application of specialized techniques that act to identify structure at a local level. Some, perhaps many, of the latter would involve the notions of independence, invariance and freedom of choice.

CODA

For the sector of mathematics education literature that directly examines students' interaction with mathematics, there seems to be separate agendas for concept acquisition and problem solving / proof. Concept acquisition models tend to append indistinct messages about forging connections between concepts, the problem solving agenda is ideal for sealing securely such connections but is weak in explaining the genesis of the concepts involved. Thus educators have tended to display a fractured picture of the entire mathematical enterprise (that does not seem to be acknowledged much). We feel that a structural approach can help in reconciling the two agendas; it suggests a sense of a whole at the same time holding an appreciation how the whole is composed in 'parts'. Concept acquisition can be accommodated in modeling structure, problem solving involves the discovery of new properties that can be deduced from existing structure. In this paper, we have invoked the word 'structure' quite often, without further comment. (The authors propose a framework based on this notion in a recently submitted paper). However, clearly the very basis of structure is on the interplay between local and global perspectives, which is exactly the theme of the paper. So our exposition here contributes to the unity of mathematics education; though we chose to stress problem-solving aspects, we could equally have stressed conceptual formation under the same 'umbrella'.

In this paper we presented some tasks whose suggested solutions illustrated important issues concerning the local and the global. They reflect sundry personal deliberations of the authors. To end the paper, we summarize some of them.

- ◆ Decisive changes of focus in one's working often are precipitated by switching between local and global perspectives.
- ◆ A lack of a sense that an entity belongs to a (global) class restricts how the entity can be operated on.
- ◆ Often the same task can have different solutions where one solution can be characterized more local than the other.

- ◆ Argumentation that retains the same global basis tends to endow it with 'transparency'. However, a 'transparent' solution is not necessarily easier to obtain than one that is more analytic, and indeed transparency cannot be expected in certain circumstances.
- ◆ Pedagogical measures to accustom students in effecting switches between local and global perspectives might follow a heuristic mode, adopting teaching practices set up in the problem-solving agenda.
- ◆ Relationships discovered in a system can be realized as an integrated whole or can be constructed 'point-wise'. For the former, we call the relation a correspondence, for the latter a function. The interplay between correspondences and functions is important for the generation and the reliability in developing new relations.
- ◆ Local perspectives are essential to form mediums to organize lines of argumentation. In particular, we pointed out the role of the related notions of independence, invariance and freedom of choice.

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THE APPLICATION OF THE ABDUCTIVE SYSTEM TO DIFFERENT KINDS OF PROBLEMS

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The purpose of this paper is to show that the tools of the Abductive System can be used for different kinds of problems. Such a study has taken into consideration Peirce's Theory of Abduction, and the result has been the construction of the Abductive System which allows the researcher to analyse a broader spectrum of creative processes; while from a didactical point of view, it could help teachers to be more conscious of what has to be recognized, respected, and enhanced, with respect to a didactic culture of "certainty", which follows pre-established schemes.

INTRODUCTION

Cognitive models of problem solving seldom address the solver's activities such as: the generation of novel hypotheses, intuitions, and conjectures, even though these processes are seen as crucial steps of the mathematician himself (Anderson, 1995; Burton, 1984; Mason, 1995). Most of the problem solving performances is explained in terms of inductive and deductive reasoning, and very little is the attention paid to those novel actions solvers often perform prior to their engagement in the actual justification process, even though autonomous cognitive activity in mathematics learning, and learner's ability to initiate and sustain productive patterns of reasoning in problem solving situation, are issues considered important in the field of research in mathematics. The attempt of this research was to build a cognitive model useful for the analysis and understanding of possible students' mechanisms and difficulties related to the process of conjecturing and approaching to proofs in mathematical analysis. The primary goal was to explore the creative phase of the aforementioned processes (that phase where one looks for or builds the hypotheses aimed at supporting the facts proposed by the problem, or validating the statements). The study started from the consideration of Peirce's definition of Abduction

[...] Abduction is where we find some curious circumstances, which would be explained by the supposition that it was a case of a certain rule, and thereupon adopt the supposition [...] (Peirce 2.624).

The surprising fact C is observed. However if A were true, C would be a matter of course. Hence, there is reason to suspect that A is true (CP. 5.188-189, 7.202)

C is true of the actual world and it is surprising, a kind of state of doubt we are unable to account for by using our available knowledge. C can be simply a novel phenomenon, or may conflict with background knowledge that is anomalous. A is a plausible hypothesis which could explain C . Therefore we consider abduction as any creation hypothesis process aimed at explaining a fact.

Taking into account Peirce's definition of abduction one of the first steps of the research was to give two different problems at two different periods of the semester to a group of students attending freshman year of an engineering degree (more details will be given in the data analysis section).

Problem1: let f be a function continuous from $[0,1]$ onto $[0,1]$. Does this function have fixed points? (Note: c is a fixed point if $f(c) = c$)

Problem2: given f differentiable function in \mathbf{R} , what can you say about the following limit? $\lim_{h \rightarrow 0} (f(x_0+h)-f(x_0-h))/2h$

A first attempt of an a-priori analysis of the aforementioned problems quickly unearthed some difficulties in predicting possible student creative mechanisms according to Peirce's theory of abduction. Peirce's abduction refers to a hypothesis that could explain an observed fact, (which is deemed to be true); on the contrary, problem 1 and 2 present a direct question, which means the solver not only has to find hypotheses justifying a fact, but also has to look for a fact to be justified. More precisely, problem 1 contains a closed-ended question, which means a respondent can select from one or more specific categories to give the answer (in this specific case student can choose between "Yes, the function has a fixed point", or, "No, the function does not have a fixed point"). Problem 2 is an open-response task, which means a performance task¹ where students are required to generate an answer rather than select it from among several possibilities, but where there is a single correct response.

THE ABDUCTIVE SYSTEM

The initial difficulties in the analysis of the problems using only Peirce's definition of abduction, and the new considerations made about tasks requiring not only the construction of a hypothesis but also of the answer, led to the construction of new definitions and tools which have been employed in the analysis of the protocols. I define the *Abductive System* as being a set whose elements are: *facts*, *conjectures*, *statements*, and *actions*: $\mathbf{AS} = \{\text{facts, conjectures, statements, actions}\}$. For *fact* I adopt the definitions of Collins' Dictionary:

- (1) referring to something as a **fact** means to think it is true or correct; (2) **facts** are pieces of information that can be discovered.

For *conjectures* I adopt the definition given by the Webster's dictionary:

Conjectures is an opinion or judgement, formed on defective or presumptive evidence; probable inference; surmise; guess; suspicion.

¹ A performance task is an exercise that is goal directed. The exercise is developed to elicit students' application of a wide range of skills and knowledge to solve a complex problem (NCREL)

The *conjectures* assume a double role of: (1) *Hypothesis*: an idea that is suggested as a possible explanation for a particular situation or condition. (2) *C-Fact (conjectured fact)*: final answer to the problem, or answer to certain steps of the solving process.

Facts and Conjectures are expressed by *statements* that can be *stable* or *unstable*. A *stable statement* is a proposition whose truthfulness and reliability are guaranteed, according to the individual, by the tools used to build or consider the fact or conjecture described by the proposition itself. An *unstable statement* is a proposition whose truthfulness and reliability are not guaranteed, according to the individual, by the tools used to build or consider the conjectures described by the proposition itself. The consequence of this is the search of a hypothesis and/or an argumentation that might validate the aforementioned statement. *Abductive statements* are of special interest for us. An *abductive statement* is a proposition describing a hypothesis built in order to corroborate or to explain a conjecture. The abductive statements, too, may be divided into stable and unstable abductive statements. The former, according to the solver, state hypotheses that do not need further proof; the latter require a proof to be validated.

It is important to clarify that the definitions of *stable* and *unstable statements* are student-centered, namely, the condition of stable and unstable is related to the subject; for example, what can be stable for one student may represent an unstable statement for another student and vice-versa; or the same subject may believe stable a particular statement and this may become unstable later on when his/her structured mathematical knowledge increases (e.g.; he or she learns new mathematical systems; new axioms and theorems). Another situation leading the student to reconsider a statement from stable to unstable is the “didactical contract”; the subject might believe the visual evidence to be sufficient but the intervention of the teacher could underline its insufficiency and therefore the students would find themselves looking for new tools. Furthermore, the statement may transform from unstable to stable inside a process because the subject follows the mathematicians’ path: they start browsing just to look for any idea in order to become sufficiently convinced of the truth of their observation, then they turn to the *formal-theoretical world* in order to give to their idea a character of reliability for all the community (Thurston, 1994).

Behind any statement there is an action. *Actions* are divided into *phenomenic actions* and *abductive actions*. A *phenomenic action* represents the creation, or the “taking into consideration” of a fact or a c-fact: such a process may use any kind of tools; for example, visual analogies evoking already observed facts, a simple guess, or a feeling, “that it could be in that way”; a phenomenic action may be guided, for example, by a didactical contract or by a transformational reasoning (Harel, 1998). An *abductive action* represents the creation, or the “taking into account” a justifying hypothesis or a cause; like the phenomenic actions, they may be conveyed by a process of interiorization (Harel, 1998), by transformational reasoning (ibid) and so on. The abductive actions may look for: 1. *a hypothesis*, to legitimate or justify the

previous met or built conjecture; 2. *a procedure*, to legitimate or justify the previous built conjecture; 3. *tools* to legitimate the adaptation of an already known strategy to a novel situation.

After a broad description, the Abductive System could be schematised in the following way: *conjectures* and *facts* are ‘acts of reasoning’ (Boero et al., 1995) generated by phenomenic or abductive actions, and expresses by ‘act of speech’ (ibid) which are the statements. The adjectives *stable*, *unstable* and *abductive* are not related to the words of the statements but to the acts of reasoning of which they are the expression. Hence, the only tangible thing is the act of speech, but from there we may go back to a judgement concerning the act of reasoning expressed through the adjectives given to the statement. For further details on the Abductive System see Ferrando (2005).

METHODOLOGY

Site and Participants: the participants are freshmen (18 or 19 years old) enrolled in required calculus classes for engineers at the Production Engineering Department of the University of Genova (Italy) during the academic year 2001-2002. The courses cover differentiation and integration of one-variable function as well as differential equations. There are two main reasons for choosing to work with this population: 1) my working experience is with students of this age; 2) the approach to the university frequently revealed a very delicate and difficult issue, since the cultural and didactical reality the students come in contact with is markedly different from their experiences in high school. This gap, in many cases, seems to be critical for the mathematical development of these students. The university approach demands more autonomy in facing mathematical problems; the aim of teaching calculus is not only to provide students with useful tools, but also to prepare them to deal with mathematical concepts and methods in a critical way (understanding the limits of a statement; finding counter-examples, etc.). Students are asked to participate in autonomous work in the creation of hypotheses, conjectures and implement a sense of critique in evaluating their own actions in the problem solving processes; such a request seems to cause several difficulties, suggesting students’ creative abilities has been lost during their scholastic career. At the beginning of the Calculus course the professor introduced me to the students as a Teacher Assistant, working once a week with them in class for a session of three hours, during which the students would solve problems proposed by me, and they would be able to discuss possible problems raised by them. During the week, the students would be able to come to my office for further explanations about topics discussed in class, or about exercises solved autonomously. Clarified my role, the students were asked if someone was interested in taking part in a research project underlying the purpose of the study (as previously explained) and explaining that the participants in the project would be given some tasks to solve, and they would be videotaped. The choice of the participants from the classroom (about one hundred students) was completely left to the students; my only

concern was to have a heterogeneous group from the point of view of both culture and ability, but this could be monitored since I was constantly in contact with them. Finally I got a heterogeneous, but not representative, sample of twenty students.

Data collection: the data (audio-recordings, videos and written texts) was collected through two different exercises given (see introduction), at two different periods of the semester, to the participants in the project. In the problem solving phase the participants were asked to work in pairs (leaving to them the decision about whom to work with); the choice was motivated by the conviction that the necessity of “thinking aloud” to communicate their own ideas gives the opportunity to bring to light guessing processes, creations of conjectures and their confutations, namely those creative processes which in great part remain “inside the mind” of the individual when one works alone, and very often only the final product is communicated to the others (cf. Thurston, 1994; Lakatos, 1976; Harel, 1998). The participants were not asked to produce any particular “structured” solution, my aim being to leave the students completely free to decide their solution process and to autonomously evaluate the acceptability of their solution for the learning community.

ANALYSIS OF THE DATA

The data analysis is based on the analysis of the dialogues (transcribed verbatim from the videotape) with the aim of finding which kinds of reasoning may enhance a creative attitude; and on the analysis of students’ written production in order to look for possible relationships among the various languages (graphic, iconic, and algebraic), and the process of creation of hypotheses, conjectures and facts. The videotape represents a tool for the triangulation of the data, since it gives the opportunity of going over any dialogue students have engaged in during the problem solving process. The theoretical framework is based on the notion of Symbolic Interactionism (Jacob, 1987), whose focus is to understand the processes by which points of views develop, providing models for studying how individuals interpret objects and events. The analysis of the data has been based on Content Analysis (Patton, 1990) that is the process of identifying, codifying and categorizing the primary patterns in the data. The analysis of the protocols was divided into two phases. The first phase showed a comprehensive description of students’ behaviours in tackling the problem; in the second phase the creative processes were detected and interpreted through the elements of the Abductive System.

The following analysis refers only to the second phase (the phase strictly related to the tools of the AS. For the complete analysis see Ferrando 2005)); the excerpt of one protocol is followed by a table divided into two columns where the left column is used to write the excerpts considered relevant to the creative processes (while my own interpretation of the statements are in brackets); the right column has been used to write the interpretation of the excerpts through the tool of the Abductive System;

the vertical arrows, linking one excerpt to another, describe the possible cognitive movement leading from one statement to another one.

TRANSCRIPT OF DANIELE AND BETTA

(For reason of space only the most significant excerpts have been chosen). Problem: Given f differentiable function in \mathbf{R} , what can you say about the following limit? $\lim_{h \rightarrow 0} (f(x_0+h)-f(x_0-h))/2h$.

At the time this exercise was proposed the students have been exposed to the definition of differentiable function given through the limit of the difference quotient. Daniele and Betta are two average-achieving students.

- 1 Daniele: $x_0+h...$
- 2 Betta: $f(x_0)...$

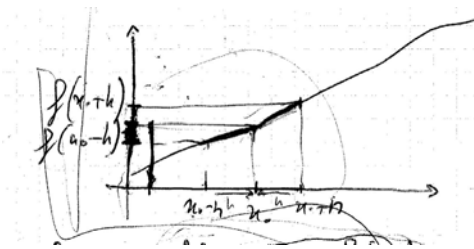


Fig1: Daniele’s graphic interpretation of the difference quotient

- 3 Daniele: in my opinion it is the same thing... when you do the limit of the difference quotient, you do

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \dots \text{this minus this over } h \dots$$

(Note: he signs on the graph the vertical and the horizontal segments)

- 4 Betta: because $f(x_0 + h)...$
- 6 Daniele: minus $f(x_0)...$ is this
- 7 Betta: Ah! OK...ours would be this (see the segments in the figure) over $2h...$ it is the same thing...
- 8 Daniele: therefore...it would be $h \rightarrow 0...$ how much is this?...eh...it will be the slope of the tangent line...
- 9 Betta: namely...the first derivative
- 10 Daniele: in x_0
- 17 Daniele: I mean, we do this...it would be the ratio between this difference | and this one — and in our case it would be the ratio between this difference | and this one —, therefore, $x_0 + h - (x_0 - h)$ that would be $2h...$ and this one that would be $f(x_0 + h) - f(x_0 - h)...$ therefore, the limit for h that goes to zero would be...I mean both go to x_0
- 27 Daniele: eh yes...anyway it is correct...I mean, the difference quotient would be this chord ...namely, it would be the tangent line of this angle, right? The difference quotient...therefore, for h that goes to zero, this...this

chord...shrinks more and more till when it becomes a point and it is the tangent line in that point...in this case it is the same thing

29 Daniele: we should write it down...

30 Betta: how do you write such a thing?

31 Daniele: firstly, if I have an equation and I do the limits of the both parts...it is the same thing...

32 Betta: therefore, if you prove that this is equal to this (namely,

$$\frac{f(x_0 + h) - f(x_0 - h)}{2h} \text{ and } \frac{f(x_0 + h) - f(x_0)}{h}$$

33 Daniele: eh...therefore...yes but I must...it would be...

$$2 \frac{f(x_0 + h) - f(x_0)}{h} = \frac{f(x_0 + h) - f(x_0 - h)}{2h} 2$$

(And they simplify in the following way

$$2 \frac{f(x_0 + h) - f(x_0)}{\cancel{h}} = \frac{f(x_0 + h) - f(x_0 - h)}{\cancel{2h}} \searrow$$

34 Interviewer: but then you have already given for sure that this and this one are equal...

35 Daniele: ehm...yes...

36 Interviewer: I thought you would want to prove that

$$\frac{f(x_0 + h) - f(x_0 - h)}{2h} = \frac{f(x_0 + h) - f(x_0)}{h}$$

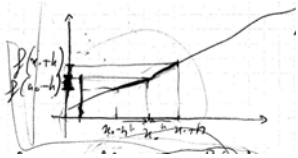
39 Daniele: yes...but you are right! I already thought to be true the equality...then, I looking for...no, no...

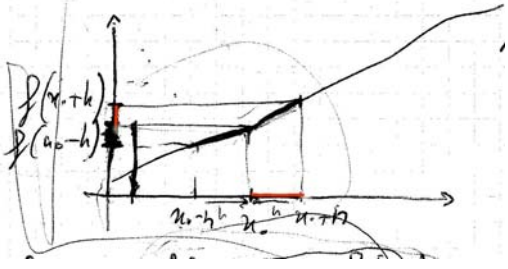
69 Daniele: we did a drawing that misled us

77 Daniele: but now neither the graphic one convinces me anymore...because we used the symmetry respect to f(x₀)...no, no...that one is true

[...]

ANALYSIS THROUGH THE TOOLS OF THE ABDUCTIVE SYSTEM

Excerpt	Interpretation through the tools of the Abductive System
<p>In my opinion it is the same thing...</p>  <p>(Namely, doing</p> $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} \text{ is the same of}$ $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \dots)$ <p style="text-align: right;">Search of a validating hypothesis</p>	<p>CONJECTURE with the role of answer to the problem; therefore, C-FACT.</p> <p>The C-FACT is created by a PHENOMENIC ACTION guided by a feeling, by a visual impact with the graphic representation met for the limit of the standard difference quotient. The statement describing the C-FACT is an UNSTABLE STATEMENT because the visual impact seems to be insufficient to validate the act of reasoning.</p>

 <p>(The two limits use the same tools)</p>	<p>Creation of a HYPOTHESIS through an ABDUCTIVE ACTION guided by the reinterpretation of the frame used for the standard difference quotient: Daniele translates the difference quotient as the ratio between the vertical and horizontal segments (see the figure) and he shifts such interpretation to the present situation. The act of reasoning seems to be expressed by a STABLE STATEMENT since the graphical justification results sufficient for them. Probably such a kind of hypothesis has been also generated by the kind of function sketched by Daniele. The choice of x_0 leads to a sort of symmetry related to $f(x_0)$; namely, $f(x_0+h) - f(x_0)$ and $f(x_0) - f(x_0-h)$ seem to be two segments of equal length.</p>
<p>↓ <i>A new phase starts. I provoke Daniele and Betta with the aim to generate the doubt about the adequacy of their graphical justification</i></p>	
$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} =$ $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$ <p style="text-align: right;">Search of a validating hypothesis</p>	<p>The C-FACT is not changed; and the PHENOMENIC ACTION is always guided by a visual impact. The act of reasoning is expressed by an UNSTABLE STATEMENT.</p>
$\frac{f(x_0 + h) - f(x_0 - h)}{2h} = \frac{f(x_0 + h) - f(x_0)}{h}$ <p>They start with algebraic manipulation to prove the equality. After several attempts they return to a graphic exploration and they find such equality to confirm the parallelism of the two lines. Impossible: they both go through the point $(x_0+h, f(x_0+h))$. The aforementioned hypothesis is refused. They return to the graphic exploration and the c-fact does not change, since the graphic dynamics reinforce their conviction that when x goes to x_0 the line becomes the tangent line. What changes is the approach to prove the c-fact, with a new manipulation of the starting expression</p>	<p>Creation of a HYPOTHESIS through an ABDUCTIVE ACTION probably guided by a fact already acquired, namely if $f(x) = g(x) \quad \forall x \in (x_0 - \delta, x_0 + \delta)$ then $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x)$. the hypothesis is expressed by an UNSTABLE STATEMENT.</p>
<p>The algebraic manipulation brings to the expression</p> $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{2h} + \lim_{h \rightarrow 0} \frac{f(x_0) - f(x_0 - h)}{2h}$ <p>with the construction of a new conjecture.</p> $\lim_{h \rightarrow 0} \frac{f(x_0) - f(x_0 - h)}{h} = f'(x_0)$	<p>This act of reasoning take the connotation of FACT in the sense that they justify it through the graphical interpretation as they did previously with the initial expression and the graphic interpretation this time is enough. A STABLE STATEMENT therefore expresses the fact.</p>

CONCLUSIONS

The Abductive System has been created with the aim of providing some tools, which could identify and describe possible creative processes students implement when they perform conjectures and proofs in Calculus. At the base of such construction there is also the intention to show that the creative processes own some components, and to separate these processes from the belief that it is not possible to talk about it because it is something indefinable and only comparable to a “flash of genius”. The definition of the Abductive System allows the researcher to analyse a broader spectrum of creative processes than those covered by the already given definitions of abduction, and the tools of such system can be employed to study and interpret creative processes in different kinds of open problems: open-response tasks like in this case, and closed-ended question problems (see Ferrando 2005, Ferrando 2006). These findings could be employed in teachers’ training program with the aim of increasing teachers’ awareness about students’ creative ability; to this extent the analysis, through the tools of the Abductive System, of selected protocols would be presented to show how these tools can underline and describe such processes.

From a didactical point of view, it evidences those teaching styles (see Ferrando 2005 for the analysis of a Calculus lesson through the tools of the AS) which can enhance an “abductive atmosphere”, when the teacher does not just deliver the knowledge but he or she creates those conditions where the immediate creation of a fact entails “the necessity” to build or to look for a justifying hypothesis, generating in this way creative mechanisms. Further applications for teachers’ training could consist in discussions and comparisons (supported by videotapes and transcripts) of different teachers’ styles, with the target of evidencing those didactical approaches, which may enhance an “abductive atmosphere”. Therefore, such a research could help teachers to be more conscious about the conditions needed to choose tasks that are suitable to change from a teaching perspective of “certainty”, (based on the teaching of preestablished schemes), to a perspective that enhances creativity, through the choice, in classroom, of target “open problems”. Nevertheless we need to take into consideration, the typology of the sample; which cannot be defined as a random sample, since the students voluntarily offered to participate in the project, and probably were those who positively accepted a didactical contract that encourages an approach promoting the understanding how things work, the making connections among mathematical ideas, creating conjectures and validations of mathematical ideas, rather than a formal deductive approach; anyway, I hypothesize that, since the creative abductive processes do not seem to be an attitude of a particular elite of subjects, what has happened with a particular sample of students may be extended to a larger population of students, if the same previously mentioned conditions are created on the side of the students. The creative abductive attitude met in the students, cannot be considered only an inclination of human nature, but it also probably depends on the scholastic and extra-scholastic experience of the students,

and certain kinds of didactical contract may positively influence such creative processes.

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UNIVERSITY STUDENTS' DIFFICULTIES WITH FORMAL PROVING AND ATTEMPTS TO OVERCOME THEM

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This paper contains the results of diagnostic research on the difficulties that students beginning studies at the tertiary level encounter when reading and doing formal proofs. It also includes suggestions of didactic interventions aimed at overcoming those difficulties. The data for didactic analysis was collected during classes of Introduction to Mathematics that were conducted by the author of this paper. The aim of these classes was to develop the ability to analyse and construct mathematical texts, in particular proofs.

INTRODUCTION

Students with mathematical specialization who begin studies at university experience various difficulties due to the so called “transition point” between the secondary and higher education. This is partly connected with the fact that the move to advanced mathematical thinking requires from an individual the change of cognitive skills and processes. Intuitive reasoning is replaced by formal reasoning where deduction is the primary way for formulating new conclusions from definitions and previously proved theorems. Students must be able to use this way of reasoning to solve problems, which are often new for them and treat on highly abstract ideas.

In order to help students in making this abrupt transition from elementary to higher level, the university I work at organized special classes of “Introduction to Mathematics”. The aim of these classes was preparation to study advanced mathematics through the development of the ability to work on a mathematical text (its understanding, analysis and construction).

Conducting my first classes of “Introduction to Mathematics” in the academic year 2004/2005 I observed how students struggled with problems that required doing and reading proofs. It motivated me to undertake more detailed research in order to answer the following questions:

- 1) What are students' difficulties with making the transition to formal proofs?
- 2) What didactic interventions and instructions could be introduced during classes to help students to overcome the observed difficulties?

In this study I will present results of research from a section devoted to deductive proving of theorems not complex in their logical structure, in which inferences are based largely on definitions.

THEORITICAL BACKGROUND

In the field of Mathematics Education there is much literature discussing the problems of teaching and learning proofs. This fact is justified by the essential role of

deductive reasoning in mathematics and, by the students' poor level in understanding and building mathematical proofs. On the basis of empirical studies different areas of potential difficulties of students were distinguished. Amongst these were: 1) conception of proof (Bell, 1976; Weber, 2003), 2) logic and methods of proof (Siwek, 1974), 3) mathematical language and notation (Weber, 2003), 4) concept understanding (Tall and Vinner, 1981, Weber, 2003).

With respect to concept understanding, the differentiation made by Tall and Vinner (1981) between "concept image" and "concept definition" is generally known. The former refers to the "total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall and Vinner, 1981, p. 152) and it is built up by individual through different kinds of experiences with the concept. The latter refers to a formal definition which determines the meaning of the concept. To these two aspects of understanding of a concept Moore (1994) has added the third one, the concept usage, "which refers to the ways one operates with the concept in doing proofs" (Moore, 1994, p. 252). He has distinguished the following ways of using definition: (1) generating and using examples, (2) applying definitions within proofs, (3) using definitions to structure proof. Encouraging students to analyse provided examples of referents, or produce their own, helps to deepen the understanding of the concept and facilitates the discovery of proof. In the second of the above-mentioned ways, the definition serves to suggest or justify particular steps in a proof, and it also supplies the language to formulate proof. Finally, from the definition we can obtain the structure of a proof.

Concept definition, concept image and concept usage create a scheme which Moore (1994) called concept-understanding scheme. In the terms of this scheme he analysed results of his research that concerned college students' difficulties with doing short deductive proofs. The similarity of the research problem discussed there, to the first of my research questions mentioned above, encouraged me to relate the conclusions of my analysis to the results obtained by Moore, and to draw attention to certain analogies, and differences between them.

METHODOLOGY

The research methodology carried out by me is characterized by Czarnocha and Prabhu (2005) as teaching-research (TR-NYC model). The study which I am presenting here was conducted during the subject classes of "Introduction to Mathematics" in the winter semester of the 2005/2006 academic year. The group used for my research consisted of 20 students from the first year of mathematics. In the process of the research two phases could be distinguished: (1) revealing and distinguishing students' mistakes and difficulties in making the transition to formal proofs, (2) planning and introducing didactic interventions aimed at eliminating the observed difficulties and verification of their effectiveness.

The diagnosis required that classes were organized to include data collection for how particular students coped with the construction of simple proofs or the analysis of

ready texts as proofs. Thus, often before starting the group work on the task, the students individually tried to solve it, and noted down their results. This gave them the possibility of articulating ideas, concepts, and ways of solving the task. Qualitative analysis of the individual works collected let me formulate hypotheses on students' difficulties in the analysis and construction of proofs. I was able to use them in the future planning of didactic activities to overcome these difficulties.

In class individual students' solutions were also used during the group work on the tasks. During this stage, the discussion which accompanied the work proved extremely helpful in revealing other problems and mistakes that the students encountered. Together with my comments and observations they were noted down in the "teacher-researcher's diary", and were later used as research material.

The following are examples of tasks discussed during classes, which I first used as "diagnostic tasks" (they will be referred to in this paper further on):

Task 1: Read the definition and follow the instructions.

Definition: Function $f : D_f \rightarrow R$ is even (odd) iff for every $x \in D_f$ number $-x$ also belongs to the domain of the function f and $f(-x) = f(x)$ ($f(-x) = -f(x)$).

- Give a symbolic notation of the defining condition.
- Give an example of a referent and prove that it falls under the definition.
- Explain when a function is not even (odd).
- Give an example of a non-referent and prove that it does not fulfill the definition.

Task 2: Read the definition and follow the instructions:

Definition: Sequence of numbers (a_n) is geometrical iff $\exists q \neq 0 \forall n \in N : a_{n+1} = a_n q$.

- Give an example of a referent and prove that it falls under the definition.
- Is sequence $(0,0,\dots)$ geometrical? Prove your answer.
- Give an example of a non-referent and prove that it does not fulfill the definition.

Task 3: Analyse the text of the theorem and the proof.

Theorem. Perpendicular bisector of a segment is the set of all points of the plane equally distant from its ends.

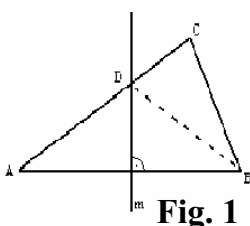


Fig. 1

Proof. Let AB be a segment and m its perpendicular bisector. If C lies on m , then segments AC and BC are symmetrical to m , thus $|AC| = |BC|$. We proved that if a point lies on the perpendicular bisector of a segment it is equally distant from its ends.

We must now show that if a point does not lie on the perpendicular bisector then it is not equally distant from its ends. Let then point C lie outside line m (fig. 1). We connect C with A and B . We presume that C lies on the same side of the line m as B . Then segment AC crosses line m at some point D . Thus $|AD| = |BD|$ gives:

$$|AC| = |AD| + |DC| = |BD| + |DC| > |BC|.$$

Then $|AC| \neq |BC|$, which finishes the proof.

It is worth emphasizing that the concepts and the theorem used in these tasks were known to the students since secondary school; and the way of expressing the definitions, the theorem and its proof came from handbooks [1]. The goal of these formulated tasks was to deepen and/or control the understanding of the different kinds of mathematical texts included, but they also required truth verification or justification of certain statements about the concept, whose definition was given (Task 1 and 2) or that concerned work on the text of written proof (Task 3) [2]. Some of the questions that appeared in the tasks were connected to the activities aimed at overcoming the difficulties with proving which are presented later in this paper.

STUDENTS' DIFFICULTIES WITH PROVING

Analysis of the research material revealed a variety of difficulties that students had with the formulation and understanding of proofs. The source of these difficulties was not only the lack of knowledge but also inadequate cognitive development and fixed false beliefs. In particular, the observed difficulties resulted from: (1) problems with understanding of mathematical concepts, (2) false understanding of the concept of proof, (3) deficiency in logical education, (4) lack of understanding and lack of ability in using mathematical language and notation.

More thorough analysis of a large amount of examples and data helped me to separate and name in detail the students' difficulties. They will be presented in the further part of the paper as conjectures, and will be provided with examples.

The concept understanding

During his research Moore discovered that without an informal understanding of the concept students could not learn and state its definition, which, in consequence, was one of the reasons for students' failure to produce a proof. My observations provided another conjecture on concept definition:

Conjecture 1: Students understood the concept correctly but they made mistakes trying to verbalize its definition.

Such a situation occurred during the work on the theorem quoted in Task 3, which, after reformulation, was as follows: *On a plane, perpendicular bisector of a segment and the set of all points on a plane equally distant from its ends are equal*. Thus, in order to analyse the structure of the proof the revision of the definition of equality of two sets (already familiar to students) was required. The answers received from two students were the following:

Student 1: Two sets are equal if they have the same number of elements.

Student 2: Two sets are equal in the case when if I take the element of the first set it must belong to the second set.

As a result of the discussion about these statements the following examples of sets $A = \{1,2\}$, $B = \{3,4\}$ and $A = \{1,2\}$, $B = \{1,2,3\}$, "equal" in line with the first and second "definitions" were written on the blackboard. Both students reacted to these examples saying: "This is not what I meant". This showed that their answers were

inconsistent with the way they understood the concept, and resulted from an inability to state their definitions precisely.

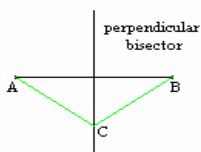
However, students' difficulties resulted not only from the fact that they were not able to formulate a definition correctly. The knowledge of the defining condition also did not guarantee that the students could use it properly to write a proof. In written work on the definition of odd function (cf. Task 1) some of them did not try to formulate the formal explanation that the functions chosen by them fulfilled the definiens, although they knew the definition. This could show that:

Conjecture 2: The students did not know how to use the definition to plan the structure of proof.

This difficulty, connected with the concept usage aspect, is also indicated by Moore.

Moore did not provide a thorough discussion of another issue connected with the aspect of concept understanding, i.e. using the definition to construct or justify subsequent inferences in a proof. During my classes the difficulties in that area usually occurred while analysing written texts of proof, such as the one in Task 3. In the first paragraph of that proof the definition of the perpendicular bisector of a segment was the basis for the formulation of subsequent conclusions. None of the students who analysed the text of the proof individually referred to that definition. For many of them the figure was the basis for justifying the equality of $|AC| = |BC|$.

The following is a piece of one work:



I take segment AB and draw its perpendicular bisector m . If point C lies on that perpendicular then $|AC| = |BC|$. It is easy to construct the perpendicular bisector of a segment, choose a point that lies on it and state that the segments are equal.

The author's comment showed that he or she simultaneously read the subsequent conclusions that were in the text, looked at the figure, and visually checked their veracity without referring to the definition. The fact that other students also formulated similar justifications showed that:

Conjecture 3: The students did not know how, or did not feel the need, to use the definition to evaluate the veracity of subsequent steps in the analysed proof.

At the same time

Conjecture 4: Whilst doing proofs students did not refer to the definition of a concept but to the content of the concept image.

An example of the situations I observed is as the proof of the fact that the function $y = x$ is an odd function (cf. Task 1), some students stated: "its graph is symmetrical in relation to the origin of the coordinate system".

Concept of proof

Whilst Moore analysed issues concerning the students' notion of the purpose of proof in his research, I would like to draw special attention to:

Conjecture 5: The students did not know what constitutes the proof.

The student's answer concerning the proof from Task 3 (already quoted) verified visually the correctness of subsequent conclusions in the text without referring to relations between them, this showed a lack of understanding that "proof is a logical sequence of statements leading from a hypothesis to a conclusion using definitions, previously proved results, and rules of inference" (Moore, 1994, p.263). The following is yet another example illustrating the students' belief that non-deductive arguments constitute the proof:

$$f(x) = x^2 \text{ is even because for } x = 2, -x = -2$$

$$\left. \begin{array}{l} f(x) = 2^2 = 4 \\ f(-x) = (-2)^2 = 4 \end{array} \right\} f(x) = f(-x)$$

The author of these words acted as if they believed that in order to prove that the square function fell under the definiens of even function (cf. Task 1) it was enough to show a single example that fulfilled it. The analysis of written works revealed that the students quite often used the wrong generalization rule: $\alpha(a) \Rightarrow \forall x \in X : \alpha(x)$, where a is a concrete element of set X .

Knowledge of the field of mathematical logic

My observations revealed that deficiencies in logical education are the cause of many mistakes in student reasoning, Moore did not explicitly write about this:

Conjecture 6: The students did not understand the basic concepts of mathematical logic.

The most problematic seemed to be the concept of the quantifier. When in the notation, for example of the definiens, there were more than one quantifier, the situation became even more complex. The following is what I observed during my analysis of students' answers to the question regarding whether the sequence $(0,0,\dots)$ was geometrical (cf. Task 2):

Sequence $(0,0,\dots)$ is not geometrical because condition $\forall n \in N : a_{n+1} = a_n q$ is met for every $q \in R$, so even for $q = 0$, which is contradictory with the definition (as q must be different than zero). Moreover, in the sequence $(0,0,\dots)$ q cannot be different for every term, e.g. $a_2 = 5 \cdot a_1$, $a_3 = 10 \cdot a_2$, $a_4 = 3 \frac{1}{2} \cdot a_3, \dots$ (the existential quantifier is at the beginning of the condition, thus q should be one for all terms).

The student noticed that in the case of sequence $(0,0,\dots)$ the condition $\forall n \in N : a_{n+1} = a_n q$ is fulfilled by every real number q , that, given the student correctly understands the meaning of the existential quantifier, should immediately tell the student that the sequence is geometrical. The second sentence from the quoted work showed yet another difficulty – the student's wrong assumption that if the existential quantifier preceded the general quantifier there only one q common for all the terms of the sequence exists.

Mathematical language and notation

As well as mistakes in the understanding of logical concepts, a lack of ability to make logical analysis of the notation, is another cause of difficulties in the understanding of the defining condition and its usage in the construction of the proof. Here is a part of one work:

$$D_f = R \quad \left. \begin{array}{l} f(x) = x^2 \\ f(-x) = (-x)^2 \end{array} \right\} f(x) = f(-x)$$

Since $x \in D_f$ and $-x \in D_f$ and $f(x) = f(-x)$ then function $f(x) = x^2$ is even.

Without doubt, the student had certain knowledge of the concept of even function (cf. Task 1) as they gave the example of the referent. Justifying this choice they tried to use the definition. However, their “proof” showed that they made out the logical structure of the defining condition incorrectly. From the answer we can conclude how he or she understood the definiens – it is a conjunction of three conditions preceded by a general quantifier: $\forall x : x \in D_f \wedge -x \in D_f \wedge f(-x) = f(x)$. This example confirmed that:

Conjecture 7: The students did not understand the statements formulated in formal language.

The multiplicity of difficulties observed certified that the problems the students encountered were serious. Making the diagnosis enabled me to plan activities directed at overcoming them. Some of these activities I am presenting in the next paragraph.

ACTIVITIES AIMED AT ELIMINATING THE DIFFICULTIES

From the considerations concerning students’ difficulties with proving, stated above, one can notice that there are strong interrelations between them; a difficulty or lack of understanding in one area often led to difficulties in another. This meant their elimination required that I employ interventions in each of the areas discussed. In this paper I am presenting some of them in more detailed way: (1) the creation of situations aimed at developing necessity and the ability of using definitions of the concepts in the proofs, (2) developing the students’ ability of using logical knowledge as the tool to facilitate analysing and constructing mathematical texts, (3) common work on the texts of written proofs.

The conjectures 3 and 4 showed that the students did not realize the role of definitions in deductive reasoning. The identification of the difficulties did not give the answer how to overcome them; it was necessary to reflect on the source of the problem. Some explanations concerning this issue are given by Vinner (1991). He states that although during the problem solving process the concept image and the concept definition “cells” are supposed to be activated, in practice, the second one (even if unavoidable) is not usually referred to. This is because everyday thought habits take over and there is no feeling of the need to consult the formal

definition [3]. Vinner also stresses that students should be trained to use definition as an ultimate criterion in mathematical tasks and gives certain clues how to do that:

This goal can be achieved only if the students are given tasks that cannot be solved correctly by referring only to the concept image (...). Only a failure may convince the student that he or she has to use the concept definition as an ultimate criterion for behaviour (Vinner, 1991, p.80).

In my classes I encouraged situations, in which using the components of concept image (being outside of the scope of the definition) during formulation of the justification, did not yield explicit answers or led to wrong conclusions. With this in mind, the analysis of the so called “special cases” for the given definitions were very useful. For example, in order to solve an “argument” for whether the empty set is convex, we interpreted the definition condition and got the statement: $\forall x, y : x, y \in \emptyset \Rightarrow xy \subset \emptyset$. After referring to the definition of implication and noticing that the antecedent in the defining condition was false for the empty set, it was concluded that the implication was true. This argument was convincing for those who at first answered negatively.

Reading and doing proofs requires certain knowledge of mathematical logic. However, as is stated in conjecture 6, the students knowledge of logical concepts and theorems turned out to be incomplete or misconstrued. This influenced the lack of understanding of the texts formulated in formal language (conjecture 7). Trying to overcome these difficulties, I created situations in which the students were analyzing definitions and theorems in the formal-logical aspect. Siwek (1974) states that activities in which student realizes the logical structure of the texts and transforms them with the usage of the logical knowledge, on the one hand entail that this knowledge becomes more concrete and fixed, on the other hand are necessary to deepen understanding of the content and sense of the texts. During our work on the different definitions and theorems we used the knowledge about the concepts and theorems of logic to: (a) show the logical structure of an expression and notice the relation between informal and formal languages, (b) construct the negations of sentences, (c) write down the sentences and sentence conditions equivalent to data. Writing down a text in formal language enabled us to reveal its logical structure and, after transforming into symbolic notation, often became a good starting point for the construction of a proof. It was also necessary if we wanted both to transform a text on the basis of tautologies to an equivalent form and to construct its negation. By formulating the sentences equivalent to data students could learn that, for example, one definition can be easier to use than another that is equivalent to it. Finally, the construction of the negation was used to look for non-referents or to formulate the justification about falsity of a theorem. In consequence all these activities could have an indirect influence on the development of the skills of using definitions and theorems to plan the structure of a proof or to suggest or justify particular steps in a proof, i.e. in overcoming the difficulties mentioned in conjectures 2 and 3. The

work on mathematical texts also served to develop the ability of the precise and correct expression of thoughts (cf. conjecture 1). I was making efforts to draw the students' attention on the fact that in mathematical statements both the meaning of words and the syntax were important.

The logical knowledge turned out to be a useful tool during another activity aimed at developing students' skills of proving, namely the thorough analysis of the texts of written proofs. Through it students were gaining experiences needed in individual work on mathematical texts but also were accumulating certain clues and patterns to do a proof. Consideration of different examples of written proofs let us construct a certain "plan of activities" consisting of such elements of work on the text which facilitated its understanding. We started usually with the analysis of the theorem which the proof concerned, among others, in the presented above logical-formal aspect. Except "translating" the theorem into formal language and realizing its logical structure this analysis consisted of separating of data conditions and that which follows from the data, understanding the character of mathematical objects which the theorem relates to, and revising adequate definitions. In the process of reading the text of a proof three components could be distinguished. The first one included the analysis of schema of the proof, with reference to the structure of the theorem or proper definition. During this work it was important to expose interrelations between subsequent premises and conclusions rather than their content. The content of subsequent steps in a proof was taken into considerations in the second phase of work. The students tried to understand them and control their correctness through referring to definitions, theorems that the reasoning was based on. As the texts of proofs in handbooks are mostly sketchy instructions how to conduct the reasoning, the students' task was also to fill the gaps in the proof. In the final part of the work we made summaries, in which we tried to realize the guideline of a proof, i.e. the main idea being to find the succession of "links" from the assumptions to the final thesis; but also to reflect on the mistakes and difficulties that appeared during our work and how we tried to overcome them. Throughout these activities the students had the possibility to revise their incorrect understanding of the concept of proof, as mentioned in conjecture 5, particularly their belief about what a proof is and what constitutes it. They could also realize the role and possibilities of usage of definitions in deductive reasoning, with which, as conjectures 2, 3 and 4 states, they had difficulties.

CONCLUSIONS

In this paper I have analysed the different difficulties encountered by students starting their university studies when reading and doing proofs. I have also presented the examples of the activities, constructed and carried out by myself in my classes, aimed at eliminating these difficulties. In conclusion, one can ask the question if my interventions and methods had a positive effect on the performance of my students. Looking for the answer I compared the data obtained from the students at the beginning of the course with the results of the final tests. The results of the analysis

showed the increase of the competence of my students in the area of simple proof construction on the basis of the definition (details will be given in a future paper).

The fact that certain progress took place certifies that the direction of activities used by me was proper. However, they cannot be regarded as sufficient as results achieved by part of the group were still unsatisfactory. Therefore the aim of my further research will be both a more detailed diagnosis of students' difficulties in doing proofs and planning new instructions, didactic interventions or further work on the above-mentioned strategies, in order to increase students' competence in this area.

NOTES

1. In accordance with the main goal of the classes analysing examples of definitions, theorems and proofs we reflected on language and notation of statements, traditional expressions used in mathematics. Sometimes we transformed texts into formal language and completed them.
2. In the case of task 3 before analysing the text of the proof we indicated definitions and theorems which preceded the quoted fragment in the handbook.
3. Vinner draws attention to the fact that when trying to understand a sentence taken from everyday contexts, people usually do not refer to the definitions of the terms in the sentence. This is because most concepts in everyday life are acquired without any involvement of definitions.

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**THE INTERPLAY BETWEEN
SYNTACTIC AND SEMANTIC KNOWLEDGE
IN PROOF PRODUCTION:
MATHEMATICIANS' PERSPECTIVES**

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We draw on a series of themed Focus Group interviews with mathematicians from six universities in the UK (and in which pre-distributed samples of mathematical problems, typical written student responses, observation protocols, interview transcripts and outlines of bibliography were used to trigger an exploration of pedagogical issues) in order to discuss the interplay between syntactic and semantic knowledge in proof production (Weber & Alcock, 2004). In particular we focus on participants' views of how fluency in syntactic knowledge can be seen as a facilitator of mathematical communication and a sine-qua-non of students' enculturation into the sociocultural practices of university mathematics.

Key words: undergraduate mathematics education, mathematicians, syntactic knowledge, semantic knowledge, proof, enculturation, communication, sociocultural practices

INTRODUCTION

In 2004 Weber and Alcock proposed a theoretical framework for understanding the process through which undergraduate students (and mathematicians) engage with proof. Refining and clarifying what is meant by 'formal' and 'intuitive' reasoning (Weber and Alcock, 2004, p210) the authors suggested that proof production can be of two different kinds: *syntactic proof production* and *semantic proof production*. They define *syntactic proof production* as

one which is written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way. [...] A syntactic proof production can be colloquially defined as a proof in which all one does is 'unwrap the definitions' and 'push symbols'. (p210)

and as *semantic proof production*

to be a proof of a statement in which the prover uses instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences that he or she draws. (p210)

In this context *syntactic knowledge* and *semantic knowledge* are the abilities and knowledge required to produce syntactic or semantic proofs (p229). The studies from which this theoretical framework emerged are empirical, data-grounded studies and involved observation of undergraduate students, doctoral students and mathematicians as they worked on proving various mathematical statements (typically in Group Theory or Analysis). The participants were asked to ‘talk aloud’ while writing their proofs and were in some cases interviewed during this process. Amongst the conclusions the authors draw from their studies, is that

The abilities and knowledge required to produce syntactic proofs about a concept appear to be relatively modest. The prover would need to be able to recite the definition of a mathematical concept as well as recall important facts and theorems concerning that concept. The prover would also need to be able to derive valid inferences from the concept’s definition and associated facts. (p229)

while the knowledge required to produce semantic proofs appears to be more complex (p229). The authors conclude that

Hence, writing a proof by syntactic means alone can be a formidable task. However, when writing a proof semantically, one can use instantiations of relevant objects to guide the formal inferences that one draws, just as one could use a map to suggest the directions that they should prescribe. Semantic proof production is therefore likely to lead to correct proofs much more efficiently. (p232)

In this paper we wish to investigate how syntactic and semantic knowledge concur in proof production. The data we draw from illustrate the perspectives of mathematicians as they reflected on proofs produced by their students (as part of written coursework). In what follows we briefly introduce the study they originated in.

THE STUDY

The data we present originate from a study¹ which engaged mathematicians from across the UK as educational co-researchers; in particular, the study engaged university lecturers² of mathematics (more details on the participants to the study can be found in Iannone & Nardi, 2005) in a series of Focused Group Interviews (Wilson, 1997), each focusing on a theme regarding the teaching and learning of mathematics at university level that the literature and our previous work acknowledge as seminal. These themes were:

- Formal Mathematical Reasoning I: students' perceptions of *proof* and its necessity;

¹ Supported by the Learning and Teaching Support Network in the UK.

² In the text we refer to the participants of the study as Lecturers. Meanings of this term differ across different countries. We use it here to denote somebody who is a member of staff in a mathematics department involved in both teaching and research.

- Mathematical objects I: the concept of *limits* across mathematical contexts;
- Mediating mathematical meaning: *symbols and graphs*;
- Mathematical objects II: the concept of *function* across mathematical topics;
- Formal mathematical reasoning II: students' enactment of *proving techniques*;
- A *Meta-cycle*: collaborative generation of research findings in mathematics education.

Discussion of the theme in each interview was initiated by a Dataset that consisted of:

- a short literature review and bibliography;
- samples of student data (e.g.: students' written work, interview transcripts, observation protocols) collected in the course of our previous studies; and,
- a short list of issues to consider. We note that, despite the presence of this list, we gave priority to eliciting participants' own perspectives and kept a minimal role in manipulating the direction the discussions took (Madriz, 2000).

Analysis of the interview transcripts largely followed Data Grounded Theory techniques (Glaser and Strauss, 1967) and resulted in thematically arranged sets of Episodes – see elsewhere (e.g. Iannone & Nardi, 2005) for more details.

The data we present here originate in Episodes from the discussion of the theme *Formal Mathematical Reasoning I: students' perceptions of proof and its necessity*. In these, students' responses to a Year 1 – Semester 1 question that concerned the convergence or divergence of sequences and required the use of the quantified definition of convergence :

The sequence $\{a_n\}_{n \in \mathbb{N}}$ of real numbers converges to a real number L as $n \rightarrow \infty$ if $\forall \varepsilon > 0, \exists N$ in \mathbb{N} such that $n \geq N \Rightarrow |a_n - L| < \varepsilon$

triggered a discussion of what type of knowledge students draw on when engaged with proving the convergence of a sequence.

The students had encountered this definition half way through their first semester. In the Episodes we sample the discussion had revolved around two main issues: why it is necessary to teach and use the quantified statement for the convergence of a sequence; and, how formal and informal understanding of the definition of convergence interact for the production of a correct proof. Issues relating to what difficulties students encounter in internalising and manipulating this statement were also touched upon.

Here after discussing a section of the data, we frame our conclusions in Weber and Alcock's terms and attempt to explore how the two types of knowledge they distinguish (semantic and syntactic) coexist in proof production. In particular we focus on some aspects of the role of syntactic knowledge.

WHY DO WE NEED (AS MATHEMATICIANS AND AS TEACHERS) THE FORMAL QUANTIFIED STATEMENT FOR THE CONVERGENCE OF A SEQUENCE?

The participants agreed on the necessity for the students to learn and understand the quantified statement. Moreover, they all recognised the need for the students to learn how to manipulate the quantifiers correctly and how to write meaningful mathematics sentences (i.e. sentences that comply with formal logical reasoning) using them. Various reasons were offered for this. Below we elaborate three: symbolic language as a compensation for shortage of pictorial/geometrical representation (*loco-visual*); symbolic language as the shared medium of communication amongst mathematicians (communicational); and, symbolic language as a tool for manipulating the logic of mathematical arguments (instrumental).

Lecturer E suggested that quantified statements help defining concepts that are not particularly amenable to pictorial/geometrical representation:

E: You see ... no human can have a ... good intuitive geometrical or pictorial view of what the statement “the sequence does not converge” means, for example. [...] Or say certainly no one can have a geometrical view of the statement “this function is not uniformly continuous”.

Therefore, the symbolic language of quantifiers fulfils the need to express those concepts with as little ambiguity as possible and to compensate for the limited feasibility of pictorial or geometric imagery.

To this reason, Lecturer A adds that symbolic language is the shared language of mathematics:

A: There is a consensus on what things mean because they are used in context in the lectures, in the seminars. [...] I mean, meaning is attached to it [...] otherwise it will be almost impossible to ever write an example sheet again because too much has to go into it.

So, in order that the students begin to partake in the discourse of university mathematics as used by the mathematics community and in order for them to be able to communicate their mathematics to other mathematicians, they need to acquire dexterity in using mathematical language. Furthermore the participants were keen to recognise this as a characteristic mainly of pure mathematics: it is in this discipline mainly that symbols acquire progressively more layers of shared meaning:

E: I think this is a wide problem particularly in pure maths. The process of pure maths as it proceeds, invests more and more meaning in purer and purer symbols. [...] And, you know, you can literally write down a one-line statement that would take ten years to explain.

The third main role for symbolic language is more instrumental: symbolic language is used as a tool for writing proofs and for manipulating formal statements. Lecturer E offers the example of writing the negation of the definition of convergence (i.e. the statement 'the sequence does not converge'):

E: Yes, I guess... the power of symbols is something that they [*the students*] learn to manipulate because... to encourage them [*towards*] more algorithmic things. So for example the negation of a quantified statement I think is much easier as a symbolic definition. Because it is an algorithm: you replace ‘for all’ with ‘there exists’, you replace ‘there exists’ with ‘for all’ and then you get a statement and that is the algorithm.

This is more easily done by using the algorithmic negation, (or *negation by rules*, Dubinsky et. al 1988) using directly De Morgan type laws.

This final suggestion points to the importance, in the participating mathematicians’ perception, of possessing syntactic knowledge. As in Weber and Alcock (2004), this is not meant to be merely procedural dexterity with manipulating strings of symbols, but also ability to construct sentences that follow the laws of formal logic and to unpack and use a definition. In the words of Lecturer E:

E: I mean I try in Analysis to convey the idea, if you like, that by definition the last line of the proof is the definition of something. And that view is not... that approach is not viewed across their homework. [...] And that itself is a difficult idea and is one of the things that first year Analysis should be teaching them. And on the other hand, yes, I mean it is not... of course that is not how we do mathematics and that is not how they have done mathematics so... it is difficult.

The emphasis that Lecturer E places on the need to acquire syntactic knowledge – and to learn how to present proof in ways that are acceptable within the mathematics community – is revealing. While advocating this need he also acknowledges that this very ordered and established way of presenting mathematics does not represent the process by which mathematicians do mathematics. He thus emphasises the role of syntactic knowledge as a tool of mathematical communication (see also Hanna, 2000, p8) and how syntactic knowledge in proof production serves this role.

IS POSSESSING SYNTACTIC KNOWLEDGE ENOUGH? THE ROLE OF SEMANTIC KNOWLEDGE

The mathematicians in our study constantly teamed up syntactic fluency (e.g. with using quantifiers) with what they often referred to as ‘construction of meaning’:

A: This is the definition and that is the meaning, and the meaning I construct is equivalent to the definition.

But how does this ‘construction of meaning’ interact with the (formal) definition and how does this meaning come into being? There is agreement amongst them that this meaning should be constructed with the help of mental images and verbal explanations of the definition, and that those are fundamental parts of being able to work with such statements. Lecturer A refers to his experience as a learner and as a teacher of mathematics:

A: I find it very difficult to work with statements which have quantifiers [...] So the only way for myself in which I can unravel such things is that I have to build up

a mental picture by which I know, ok, ... this is what is going on. [...] So when it comes to convergence I think that the primary notion for the students is asking that no matter what I specify the ε region about the L , from a certain point onward everything fits inside this box. [...] Unless that is the direct connection between images that you have and formalisation I think you are lost. If you just are juggling around ε and δ then it is a completely unworthy process. [...] When it comes to convergence [*this*] is something that is very private: some people work like this and some don't. And I can well imagine that there are students that can work along a string of quantifiers they just do what they are told. You can view this as the recipe, you can do this, you do this and you do this...

So, at least for Lecturer A, fluency in syntactic knowledge must come hand in hand with engagement with the meaning behind the symbols in question, namely an analogous fluency in semantic knowledge. Furthermore he acknowledges the highly personal nature of this enterprise and offers the example of how he himself deals with strings of symbols and with formal logic reasoning: others, he says, may engage in this process quite differently but for him a simultaneous syntactic and semantic engagement is absolutely central.

In other words, for example those of Tall & Vinner's (1981) Concept Image and Concept Definition, from these mathematicians' perspective (and particularly from Lecturer A's quote at the beginning of the paragraph) it appears that the conflict between "meaning" and "definition" (lack of "equivalence") is a crucial source of difficulty in proof production (and mathematical understanding more broadly) for students – as well as for professional mathematicians. Moreover, again framing the above quotes in Tall & Vinner's terms, the interplay between Concept Image and Concept Definition is a highly personal affair, depending on previous mathematical experiences but also, in the case of professional mathematicians, on their specific field of expertise.

INTERPLAY BETWEEN SYNTACTIC AND SEMANTIC KNOWLEDGE IN PROOF PRODUCTION

The participants reflect on the existence of formal definition of a concept and its informal understanding as follows: in the process of 'creating meaning for a concept', as one of them calls it, the need for drawing upon both semantic and syntactic knowledge, often simultaneously, emerges. When only one type of knowledge is used, results are often unsatisfactory. After having discussed students' homework from the first weeks of the Analysis course, where the students were asked to apply the formal definition of convergence of a sequence to find out if the given sequence converged or not, Lecturer D remarks:

D: Again... for example that student of mine who said, you know, why... why does it [*applying the formal definition*] prove convergence? The impression I get is that she would end doing it all, all the side calculations and everything, but she was approaching it because she knew this is what you

are supposed to do to prove convergence, but she didn't really understand why she was doing it, I think.

So, being able to handle and apply, even correctly, the 'formal machinery' (as Lecturer A calls syntactic knowledge) of the convergence of a sequence is not enough to claim understanding of it. From the above it appears that this student has acquired a 'formally operable' definition of convergence of a sequence (Bills and Tall, 1998) in that she is able to apply it correctly to a given situation. But she has not yet created the meaning which grants deeper understanding of the concept of convergence – or more precisely in what ways this string of symbols that her lecturer refers to as the definition of convergence relates to her perception of what convergence is – and which will enable her to apply the same definition in other contexts. With regard to this point Lecturer A responds to Lecturer D as follows:

A: It is the situation between the formal and the informal, I think. I mean... unless the student reaches ever the informal concept I think ... to my mind it should be first very deeply ingrained in the student. And then it should be a justification in order to make sure that this is really doing what it ought to do this formal machinery. And they need to be able to jump from one to the other concept ... and I think this is also how they find the N [*in the definition of convergence see above*]. Because, how do you find the N , how do you pluck it out of the air? You have to have some informal reasoning, some intuition, draw some pictures, do some side calculations and then you say oh, maybe given this ε the N maybe this.

Just learning the formal machinery is of course devoid of meaning. However also relying exclusively on this 'intuition' and 'pictures' is not enough: in fact it can be misleading. What fluency with syntactic knowledge offers here is a shield against such misguidance, a tool of closer scrutiny through which one can establish that conclusions can actually be inferred formally and logically:

Interviewer: Can I just ask a question, I mean, it is very close to [...] what you were wondering about. They [*the students*] say, to me is pretty obvious that one over n , you know, the larger the n is the smaller the one over n is... so it goes to zero. And why do I need to...

A: Why to bother. In fact at that moment you should say ok, this is your informal understanding, you are expecting something that is correct. But then maybe you want to say, well, there is an exotic example ... informally you also draw this conclusion however you are wrong. And so why is that? So that justifies the formal apparatus to sort out what is right and what is wrong.

Therefore, syntactic and semantic knowledge have both to interact while the learner is engaging with proof production (whether the learner is intended here as a student or as a mathematician producing new results in mathematics). Drawing on only one of the two jeopardises both construction of meaning and successful proof production.

DISCUSSION

In sum participants expressed the view that syntactic knowledge

- helps defining and clarifying concepts that escape pictorial representation without ambiguity
- is the “shared language” of mathematics and as such it acquires a socio-cultural dimension: in order for students to enter the mathematical community and communicate their mathematical findings to others they need to become fluent in its language
- can be an effective tool for proof production
- acts as a checking device for intuition and for semantic knowledge.

While semantic knowledge

- guides syntactic knowledge in proof production and it is of great importance when there are parts of proofs that require an act of choice on the part of the prover (in the example of convergence of a sequence, semantic knowledge guides the choice of N in the definition)
- grants deeper understanding of the mathematical concepts considered
- grants flexibility in applying known concepts to new situations.

In addition to the above what also emerges from the data we presented is that resorting to one type of knowledge alone in proof production is limiting, even potentially misleading and ineffective. In fact, our data seem to point to a cyclic process based on drawing on syntactic and semantic knowledge in turn and often simultaneously. Syntactic knowledge is needed both to guarantee unambiguous use of the definition and as a tool that helps manipulate and produce a formal argument. In turn, semantic knowledge is needed to guide the syntactic proof production by drawing on insight into the main properties of the mathematical objects involved.

Semantic knowledge is of great importance, for example, when an act of creativity or choice is involved in proof production. The example the mathematicians referred to is how to find an N for a given epsilon when trying to prove the convergence of a sequence by referring to the formal definition. It may be the case that finding such N can be done in some cases through applying algorithmic procedures (e.g. solving backwards $|a_n - L| < \varepsilon$ given a particular epsilon). However as the mathematicians above reported students were often puzzled about where N came from; we believe that what they were actually reporting there, what is underlying the students' puzzlement is a certain degree of breakdown between syntactic and semantic knowledge.

Furthermore when informal understandings seem to lead to inaccurate deductions syntactic knowledge can re-direct these understandings and shed light on aspects of the argument not necessarily accessible through intuition. Finally, if we consider the

need to produce proofs in the format that can be shared amongst mathematicians, syntactic knowledge functions as a communication tool that serves exactly this purpose.

CONCLUSIONS

In this paper we have discussed how the mathematicians in our study articulated the roles of syntactic and semantic knowledge in proof production, and how they consider their students' acquisition and use both types of knowledge a priority. While discussing the interplay between semantic and syntactic knowledge it appears that, based on their experience as both teachers and learners of mathematics, the participants believe that both types of knowledge need to concur to produce successful proofs and that resorting to only one type of knowledge is not enough. From the data presented above it also emerges that syntactic knowledge has a role within the mathematical community as a tool of communication. In other words, it represents the *genre speech* of this community. Here we use the term *genre speech* in the sense of Bakhtin (1986) and as explored further by Van Oers (2002):

The genre is primarily a social tool of a sign community for organising a discourse in advance and often even unwittingly. It is a style of speaking embodied in a community's cultural inheritance, which is passed to members of that community in the same way as grammar is passed on. (p69)

Therefore syntactic knowledge contributes to mathematics as a social activity by becoming its genre speech, the common language that everyone in the community understands and uses for exchanging ideas and results. We concur with Otte (1990) who emphasises this social role of syntactic knowledge when he writes about proof presentation (referring to how proofs appear in mathematical publications, largely as chains of symbols that convey the logical deductions underlying formal mathematical reasoning):

It is in this way that proofs are both mechanical procedures and social processes. ...although intuition is commonly worshipped in contrast to proof as the highest form of knowing, this attitude is in danger of depriving man of his social nature and thereby of his character as a human subject. (p62)

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BELIEF BIAS AND THE STUDY OF MATHEMATICS

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This short paper compares the influence that so-called belief bias has on responses to the conditional inference task from two groups: first year undergraduate mathematics students and the general well-educated population. It is found that mathematics undergraduates are significantly less influenced by prior beliefs than the general population. The implications of this result are explored within the Selective Scrutiny dual process model. In the final section of the paper possible explanations for where this between-groups difference originates are discussed.

BELIEF BIASES IN REASONING

The effect of belief on reasoning has been the subject of investigation for decades. It was first noted in the context of Aristotelian syllogisms, where participants are asked to decide whether certain arguments are valid or not. Consider, for example, the validity of the following two syllogisms (taken from Evans, 2003):

No police dogs are vicious.

Some highly trained dogs are vicious.

Therefore, some highly trained dogs are not police dogs.

No nutritional things are inexpensive.

Some vitamin tablets are inexpensive.

Therefore, some vitamin tablets are not nutritional.

Both these syllogisms are structurally identical, and both are valid. But across many studies it has been found that valid syllogisms with unbelievable conclusions (such as the second) are accepted by fewer participants than those with believable conclusions (as the first has). Similarly, invalid but believable syllogisms are rejected in fewer numbers than those which are invalid and unbelievable (Evans, Barston & Pollard, 1983).

Earlier researchers (e.g. Inhelder & Piaget, 1958) believed that the processes involved in human reasoning were closely approximated by propositional logic, but effects such as belief bias have cast doubts upon this claim. If propositional logic – with its emphasis on structural validity rather than semantic content – were an accurate model for human reasoning, then the believability of a syllogism's conclusion would not be expected to have any effect on its acceptability. The observation that believability has a large influence on the acceptability of syllogisms poses a major problem for all accounts of reasoning: how can this observation be accounted for? Further, Evans, Barston and Pollard (1983) found an interaction between logic and believability: the

effect of believability on the acceptance or non-acceptance of valid arguments is less than the corresponding effect on invalid arguments. Belief bias cannot, therefore, simply be accounted for through straightforward response bias where believable conclusions are endorsed and unbelievable conclusions rejected.

Although belief bias has mostly been studied in the context of Aristotelian syllogisms, it comes as no surprise that similar effects can be observed on other similar tasks, including standard conditional inference tasks and the Wason selection task (for a review see, for example, Evans, Newstead & Byrne, 1993). The goal of this short paper is to investigate the differing roles that belief bias plays for two populations on such a task. This paper reports on the responses of undergraduate mathematics students, and a control group of well-educated trainee primary school teachers to a conditional inference task. The paper concludes by speculating on what could account for the differences in response detected.

MATERIALS AND METHODS

The material used was a standard conditional inference task. Participants were presented with the following information (for the unrealistic rule):

Here is a statement: *“if an animal is a fish, then it is a mackerel”*. Assume that this statement is true. In each of the following questions one extra piece of information is given. For each question circle one of the following answers: *yes (Y), no (N) or there is not enough information to tell (can't tell) (C)*.

Participants were then given a series of questions which presented modus ponens (MP), modus tollens (MT), denial of the antecedent (DA) and affirmation of the consequent (AC) deductions:

- MP The animal is a fish.
 Is the animal a mackerel? Y/N/C.
- MT The animal is not a mackerel.
 Is the animal a fish? Y/N/C.
- DA The animal is not a fish.
 Is the animal a mackerel? Y/N/C.
- AC The animal is a mackerel.
 Is the animal a fish? Y/N/C.

For the realistic rule, the materials were identical except “mackerel” and “fish” were interchanged. Thus the new rule was “if an animal is a mackerel, then it is a fish”. The normative answers to each part of both versions of the question were “yes”, “no”, “can’t tell” and “can’t tell” respectively.

Our mathematical sample was formed of 183 first year undergraduate students from a high ranking UK university mathematics department. All these students had been highly successful in their school mathematics career, having received top grades in their pre-university examinations. Participants were tested in the first half of the

university's autumn term: consequently the first year mathematicians had had limited experience of university education. They had, however, completed half on an introduction to proof course, which included several lectures on logic and truth tables. In order to have a 'general population' sample, we asked trainee primary school teachers to participate, again from a well regarded UK university; a total of 110 trainee teachers responded to the task. Whilst these trainee teachers came from a wide variety of subject backgrounds, and were not specialising in mathematics, they all had been educated to undergraduate degree level, and thus cannot be said to be truly representative of the population at large. It is reasonable, however, to claim that they are representative of a general, well-educated population.

Participants were tested in groups, they were given approximately five minutes to tackle a questionnaire with two problems, one of which was the conditional inference task described above.¹ Half the sample were given the realistic version and half were given the unrealistic version.

RESULTS

The percentages of each group selecting each response for the two different versions is shown in Table 1.

		Realistic			Unrealistic		
Deduction		Y	N	C	Y	N	C
Experimental Group – Maths Undergraduates							
MP	$P \therefore Q?$	95	4	2	96	1	3
MT	$\neg Q \therefore P?$	2	79	19	1	76	23
DA	$\neg P \therefore Q?$	2	13	86	1	42	57
AC	$Q \therefore P?$	6	4	90	37	0	63
Control Group – Trainee Teachers							
MP	$P \therefore Q?$	98	2	0	94	2	4
MT	$\neg Q \therefore P?$	2	88	10	6	69	25
DA	$\neg P \therefore Q?$	0	19	81	2	81	17
AC	$Q \therefore P?$	16	0	85	81	6	13

Table 1: Percentages of each response to each part of the task from the two groups. Figures for normative answers are shown in bold.

The data showed a strong belief bias effect for the Control Group on the administered task, and, as the literature predicts, the effect was stronger on invalid deductions than for valid deductions. Whereas the MT, DA and AC deductions were accurately classified by 88%, 81% and 85% respectively on the realistic version, equivalent figures for the unrealistic version were 69%, 17% and 13%. The effect, however, was

not as marked for the Experimental Group of mathematics students: 79%, 86% and 90% of this group accurately classified MT, DA and AC on the realistic version, compared to 76%, 57% and 63% on the unrealistic version.

These data were coded into normative and non-normative responses, and analysed using a general linear model with a binomial distribution with three factors: group (maths/general), task type (realistic/unrealistic) and argument type (valid/invalid). The group, task and argument main effects were all significant at the $p < 0.001$ level. The group-task, argument-group and argument-task two-way interactions in the model were all significant at the $p < 0.001$ level. The three-way interaction was not significant ($p = 0.298$). The meaning of these interactions is seen more clearly when these data are shown graphically (see Figure 1).

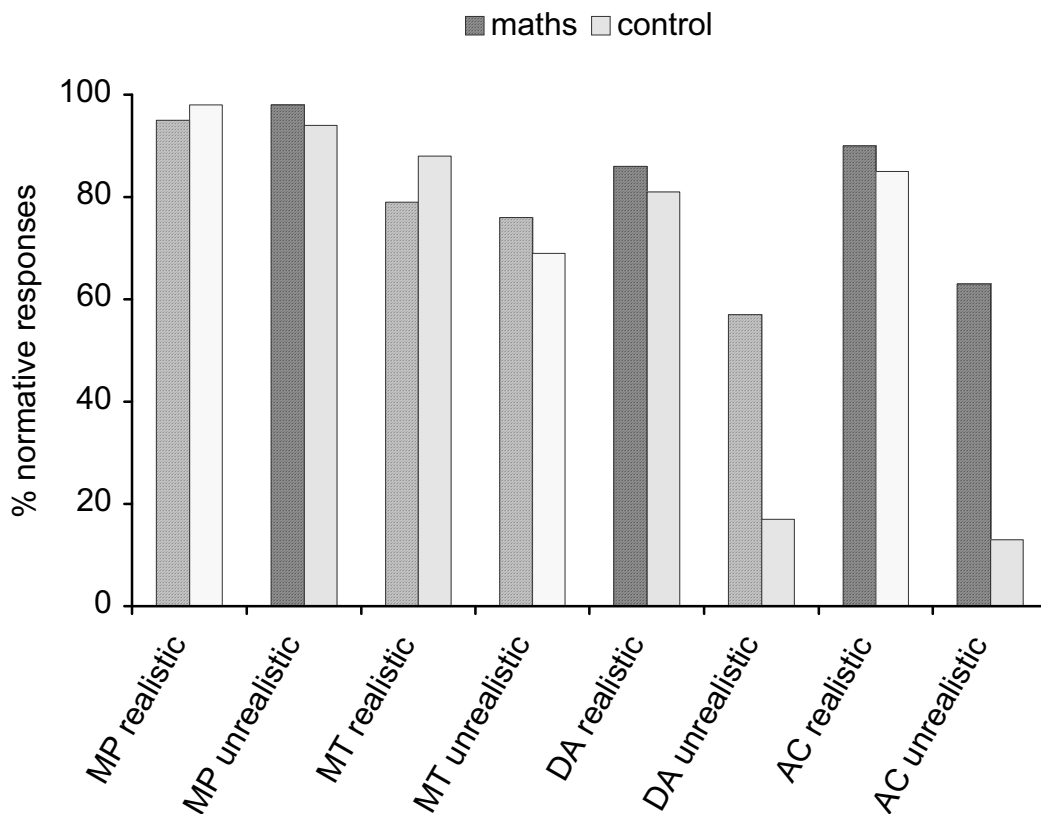


Figure 1: The percentage of normative answers given by the groups to the four arguments, for realistic and unrealistic versions.

Separate analysesⁱⁱ for each of the arguments revealed significant task-group interactions for DA ($p = 0.010$) and AC ($p = 0.006$). The task-group interactions were not significant for either MP or MT.

DISCUSSION

The results from this study show that the first year undergraduate mathematics students were less influenced by the believability of invalid arguments (DA and AC) on the conditional inference task than the general well educated population.

To investigate how this finding can be accounted for, and what its implications are, a brief description of one of the major accounts of belief bias – the so-called *selective scrutiny model* – is required.ⁱⁱⁱ Before the specifics of the selective scrutiny model are discussed, however, the family of reasoning theories it belongs to – those that revolve around the notion of dual processes – are reviewed.

Dual Processes in Reasoning

In recent years, several researchers have brought together several theories that attempt to explain reasoning under one heading: the dual process framework. In essence, the framework puts forward the idea that there are two quite separate parts to the human cognitive system that perform different tasks in different ways (e.g. Evans, 2003; Leron & Hazzan, 2006; Stanovich, 2004).

One part, System 1, operates in a quick and automatic manner, and, very roughly speaking, corresponds with intuitive thought. System 1 is thought to be a large collection of autonomous subsystems, most of which are innate to all humans, but some of which may have been acquired through experience. The subsystems' processes are preconscious in nature and only the results can be actively reflected upon. System 2, in contrast, is slow, sequential and conscious. It is unique to humans and is believed to have evolved relatively recently. It is this part of the brain that allows complex hypothetical thoughts, including abstract logic. System 2 is also involved in expressing the output of System 1, and it has the ability to monitor and

possibly override these intuitive responses, although, as we shall see, this does not always happen.

A good example of how human cognition is shaped by the two parts of the brain comes from examining how chess players decide which moves to make. Consider Figure 2. Black has just played ...Nxa2, what should be White's next move?

Many studies on the psychology of chess have noted that when chess players look at a position they intuitively "see" which moves should be considered (e.g. Hartson & Wason, 1983). For example, in this position White should consider moving the threatened queenside rook, and possibly ought to

investigate going on the attack by placing the knight on b5. Each of these candidate

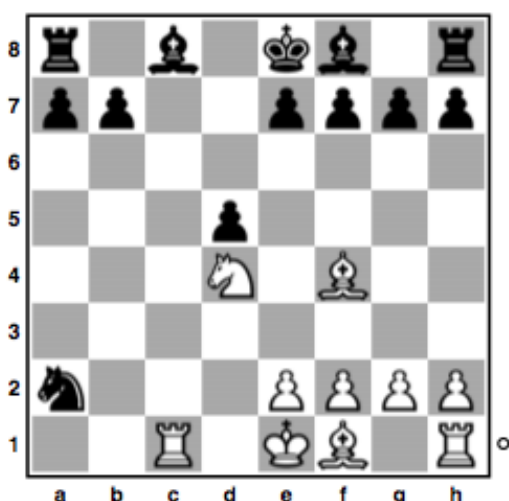


Figure 2: White to play.

moves can then be analysed in detail before finally deciding which to play. It is clear that there are dual processes present in this situation. System 1 heuristically cues which moves appear relevant, and then System 2 performs the slow sequential task of an in depth analysis of the different possibilities. No chess player when faced with this situation would for one moment analyse the consequences of playing their kingside rook to g1. This is not because it is a bad move (although it is), but rather because they simply do not consider it. System 1 does not preconsciously deem it to be relevant, and thus it does not get rejected by System 2. It is not even looked at.

Without such a relevance based System 1, chess (and indeed, life in general) would be impossible. There are simply too many possible moves. If every move had to be analysed by System 2 there would soon be a combinatorial explosion, that would make the processing effort required impractical. However, although System 1 heuristics like this are necessary, as we shall see, they can sometimes be misleading. Note that the general dual process assumption does not attempt to explain *how* System 2 operates, many of the various theories of reasoning can be adapted to fit within a dual process framework (for a full review of dual process theories see Evans, 2003).

The Selective Scrutiny Model: A Dual Process Theory.

First proposed by Evans, Barston and Pollard (1983) the Selective Scrutiny model attempts to account for the belief bias effect on Aristotelian syllogisms by positing, as in the chess example, an initial bias coupled with a formal analytical stage. The initial bias, it is claimed, leads to believable conclusions being accepted with no further analysis, but does not lead to unbelievable conclusions being simply rejected. The Selective Scrutiny model, adapted for the current experimental materials, is shown in Figure 3.

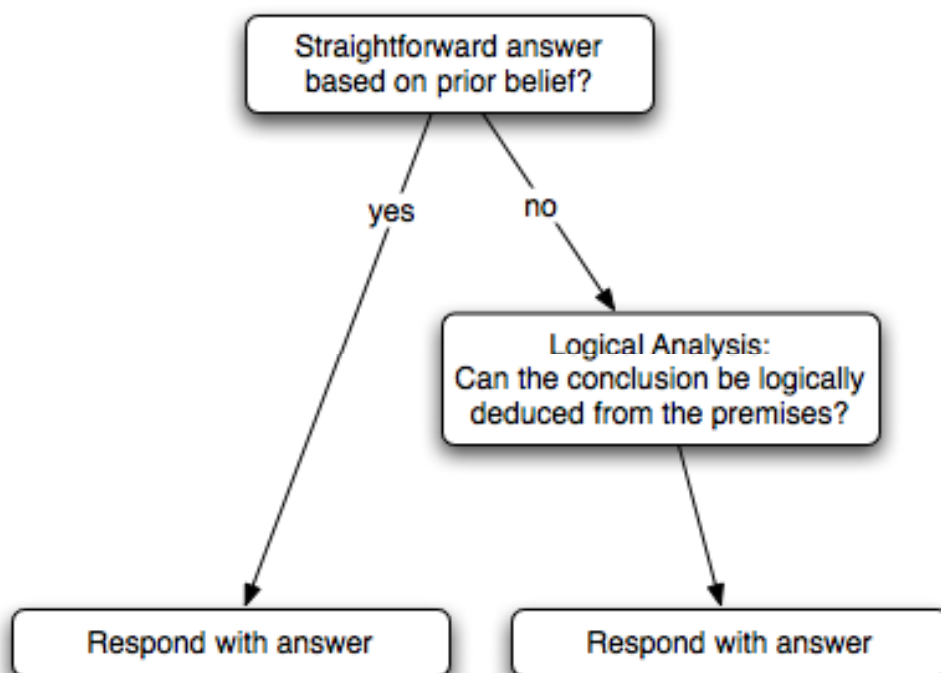


Figure 3: The Selective Scrutiny Model

The model proposes that in questions where there is a straightforward yes or no answer based on prior belief, that answer is given. If, however, there is no such answer, then a logical analysis takes place and the outcome of this process determines the response. The model is neutral as to the question of the exact nature of this logical analysis: any of the mental models (Johnson-Laird & Byrne, 2002), a mental logic theory (e.g. Rips, 1994) or the recent suppositional conditional theory (Evans & Over, 2004) could satisfactorily slot into this role. Evidence for the Selective Scrutiny model comes from various sources in addition to experimental response patterns. Using fMRI techniques, Goel and Dolan (2003) found that different areas of the brain are activated when problems are resolved in favour of belief to when they are resolved in favour of logic. This has been interpreted as being strongly supportive of the Selective Scrutiny account: different parts of the brain appear to be responsible for the belief based response and the logic based response. This account, then, is consonant with the increasingly influential Dual Process framework (e.g. Evans, 2003, Stanovich, 2004).

To further clarify how the Selective Scrutiny model works, consider the unrealistic task. Recall that the rule here was *“if an animal is a fish then it is a mackerel”*. Consider the MT and DA deductions:

The animal is not a mackerel.

Is the animal a fish? (Modal Responses: Control & Maths: N).

The animal is not a fish.

Is the animal a mackerel? (Modal Responses: Control: N; Maths: C).

For the MT deduction there is no obvious response that can be made using prior beliefs. The fact that an animal is not a mackerel does not provide sufficient information on whether or not it is a fish (from prior knowledge and belief). Therefore, the model suggests, the participant conducts a logical analysis to determine whether or not the conclusion (the animal is a fish) follows from the premises. In this case it necessarily does not, and so most participants would be expected to respond “no”. As Table 1 shows, this was indeed the modal response for both mathematics students (76%) and the control group (69%). For the DA deduction, there *is* an obvious response which can be made using prior belief: if an animal is not a fish it is common sense that it can’t be a mackerel. Therefore, no formal analysis takes place and the answer “no” is given. For DA this is indeed the modal response for the control group (81%), but not for the mathematical group whose responses were split between “no” (42%) and the normative answer (“can’t tell”, 57%). A similar response was found for the AC deduction: mathematicians were significantly less influenced by the belief bias effect than the control group.

Accounting for the Group/Task Interactions: Biases and Mathematicians

Why would mathematics students be less affected by belief bias than the general well educated population? In terms of the Selective Scrutiny model it would appear that a

significant number of mathematics students were not influenced by the initial bias which leads to the bifurcation of responses. Instead of immediately accepting believable conclusions without further analysis, apparently some (but not all) mathematics students conducted the analysis anyway.

This explanation – that mathematicians seemed to be significantly less effected by the initial bias than the general population – makes sense in the context of previous research. Inglis and Simpson (2005) found that mathematics students are less affected by the so-called matching bias than the general well educated population on the classic Wason Selection Task. A follow-up experiment indicated, by tracking participants' eye-movements, that whereas the mathematics students' attention was biased in the same manner as the general population, they were better at overruling these biases (Inglis, 2006). Similarly, in the context of the current task it seems that even where a straightforward answer that was based on prior belief did exist, a significant percentage of the mathematics students in the sample conducted a logical analysis nevertheless. This appeared not to be the case for the control group.

The message from the current experiment, coupled with that of previous work that has looked at the reasoning behaviour of mathematicians (Inglis & Simpson, 2004; Inglis, 2006), is that there is a correlation between studying advanced mathematics and the ability to override System 1 reasoning biases. There are several speculative explanations which could account for this finding.

Traditionally one of the proposed benefits of studying mathematics has been the 'thinking skills' it develops. Oakley (1946) put forward this widely held view:

“The study of mathematics cannot be replaced by any other activity that will train and develop man's purely logical faculties to the same level of rationality” (p.19).

Similar arguments have been made by curriculum bodies:

“[Mathematics] graduates are rightly seen as possessing considerable skill in abstract reasoning, logical deduction and problem solving, and for this reason they find employment in a great variety of careers and professions” (QAA, 2002).

One possible explanation, then, is that an advanced education in mathematics develops the skills of abstraction and decontextualisation which are necessary to override initial heuristic biases. However, Inglis (2006) found that there is no correlation between success on the Wason Selection Task and either an undergraduate's length of experience of mathematics, or their attainment at degree level. This finding would seem to sit uncomfortably with the idea that studying mathematics at higher levels develops such skills. Furthermore, in the context of the current experiment it is worth re-emphasising that the data collection took place during the first half of the first term of the experimental group's first year studying at university. It does seem unlikely, then, that the participants' experiences of degree level mathematics could account for the between-groups differences detected. The actual reasoning skills developed (if any) by an advanced mathematics education appears to be an under-researched area.

Stanovich (1999) found that success in reasoning experiments is correlated with measures of intelligence (success on the Wason Selection Task is, for example, correlated with SAT scores). Stanovich has used this result to argue that finding the normatively correct answer in various reasoning tasks – which often requires the participant to override initial biases – is a property associated with intelligent people. Perhaps the undergraduate mathematics students in the experiment reported here are simply more intelligent than the control group. This explanation, whilst appealing in terms of the current belief bias explanation, does not account for the differing range of *mistakes* made by mathematical groups compared with control groups on the Wason Selection Task (Inglis & Simpson, 2004, 2005).

One final possibility is that there is simply a subsection of the population for whom these kinds of skills of decontextualisation are innate, and that these people are disproportionately filtered into studying mathematics at advanced levels. This possibility would, however, run counter to the received wisdom of the value of mathematics (e.g. Oakley, 1946; QAA, 2002).

These possible explanations of the origins of the group/task interactions detected in the current experiment are speculative. Considerably more research would be needed before definitive proposals could be made regarding this issue.

SUMMARY

This short paper has reported experimental data which suggest that undergraduate mathematics students are significantly less affected by belief bias on a version of the conditional inference task than the general well-educated population. The implications of this result have been explored within the Selective Scrutiny account of belief bias, together with speculative explanations as to where this between-groups difference comes from. Given the skills that an education in degree level mathematics is supposed to develop, it seems strange that the research evidence is so limited on this subject. Further work is needed to supply evidence that will be able to distinguish between the various speculative explanations discussed in the final section of this paper.

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ⁱ The questionnaire consisted of two parts, each of roughly equal length. The other part is unrelated to this research, and was reported by Inglis and Simpson (2005).

ⁱⁱ The separate analyses proceeded using a general linear model using a binomial distribution with two factors: task (believable or unbelievable) and group (maths or general).

ⁱⁱⁱ Although the Selective Scrutiny model is one of the major accounts of belief bias, it is not the only one. Mental models theorists account for the effect by postulating a similar kind of bias which happens *after* the major logical analysis (see Evans, Newstead & Byrne, 1993). Space constraints prevent a full discussion of the differences between, and empirical support for, these two theories.

STUDENTS' CONCEPT DEVELOPMENT OF LIMITS

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Students' learning developments of limits were studied in a calculus course. Their actions, such as problem solving and reasoning, were considered traces of their mental representations of concepts and were used to describe the developments during a semester. Several students went through the course with a vague conception of limits which in some cases was wrong. A higher awareness about their mental representations' reliability is required.

INTRODUCTION

The concept of limits is an important and basic notion among others in a calculus course. Students learning limits of functions perceive and treat limits differently. Embracing limits of functions demands certain abstraction skills from the students. There are several cognitively challenging issues to deal with, such as understanding the quantifiers' roles in the formal definition or linking formally expressed theory to everyday problem solving. Students accept different levels of understanding as they have different priorities and abilities. A study on students' conceptual development of limits of functions was conducted at a Swedish university (Juter, 2006a) with the aim to describe students' developments as they learned limits in a basic calculus course. The results imply differences in high achieving and low achieving students' work with limits, but also a lack of differences at some points as will be discussed further on in this article.

A MODEL OF CONCEPT REPRESENTATIONS

Tall (2004) has introduced three worlds of mathematics to distinguish different modes of mathematical thinking, with the purpose to "gain an overview of the full range of mathematical cognitive development" (Tall, 2004, p. 287). The theory of the three worlds emphasizes the construction of mental representations of concepts and has emerged from several theories on concept development, such as Sfard's (1991) work on encapsulation of processes to objects and Piaget's abstraction theories (Tall, 2004). The three worlds are somewhat hierarchical in the sense that there is a development from just perceiving a concept through actions to formal comprehension of the concept. The first world is called the embodied world and here individuals use their physical perceptions of the real world to perform mental experiments to build mental conceptions of mathematical concepts. The mental experiments can be children's categorisations of real-world objects, such as an odd number of items or, later, students' explorations of intuitive perceptions of limits of functions. The second world is called the proceptual world. Here individuals start with procedural actions on mental conceptions from the first world, as counting, which by using symbols become encapsulated as concepts. The symbols represent both processes and

concepts, for example counting and number or addition and sum. The symbols, together with the processes and the concepts, are called procepts (Gray & Tall, 1994) and are used dually as processes and concepts depending on the context. The third world is called the formal world and here properties are expressed with formal definitions as axioms. There is a change from the second world with connections between objects and processes to the formal world with axiomatic theories comprising formal proofs and deductions. Individuals go between the worlds as their needs and experiences change and mental representations of concepts are formed and altered.

Not all mathematical concepts can be regarded as an object and a process, e.g. a circle or an equivalence class that are both pure objects, though in limits this duality is very obvious. Limits can be handled through an explorative approach with tables of function values and graphs from the beginning and later as symbolically expressed entities. Learning limits of functions demands leaping between operational and static perceptions (Cottrill et al., 1996). There is a challenge in understanding the

termination of an infinite procedure as a finite object, such as $\lim_{x \rightarrow \infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = 1$. It is

important to reach all significant stages and be able to change between the different stages. Only then can an individual fully understand the concept if understanding of a mathematical concept is defined as Hiebert and Carpenter did (1992), i.e. to be something an individual has achieved when he or she can handle the concept as part of a mental network. The more connections between the mental representations, the better the individual understands the concept (Dreyfus, 1991; Hiebert & Carpenter, 1992).

In an attempt to create a model for concept development, I have used theories about concept images (Tall & Vinner, 1981; Vinner, 1991) as a complement to the theory of the three worlds. A concept image for a concept is an individual's total cognitive representation for that concept. The concept image comprises all representations from experiences linked to the concept, of which there may be several sets of representations constructed in different contexts that possibly merge as the individual becomes more mathematically mature. Multiple representations of the same concept can co-exist if the individual is unaware of the fact that they represent the same concept. Possible inconsistencies may remain unnoticed if the inconsistent parts are not evoked simultaneously. Concept images are created as individuals go through the developments represented by the three worlds. The model in Figure 1 shows how part of a concept image can be structured as I consider it. The three types of symbols used each represent a concept at the stage of one of Tall's three worlds, as described in the figure. The concepts can be, for instance, geometric series, derivatives of polynomials, definitions of derivatives and limits of functions, theorems, proofs, and examples of topics of related concepts. More links and more

representations of concepts exist around the formal world representations of concepts. There are also parts that are not very well connected to other parts. This situation can occur when individuals use rote learning as they try to cope with mathematics. Students who are unable to encapsulate processes as objects or take the step from procepts to a strictly formalistic exposition can use rote learning as a substitute.

Concept images' magnitudes vary according to the topics of the concept image. A concept image of calculus comprises several sub-levels which can be addressed depending on current circumstances. Parts of concept images can, for that reason, be depicted in terms of *topic areas* as a means to talk about different parts of the concept image as shown in Figure 1. The topic areas are themselves concept images used to divide the larger concept image in different areas of mathematics in components of certain topics, e.g. 'functions' or 'limits'. The sizes of topic areas vary according to what context they appear in, for instance large areas such as 'functions', or smaller areas such as 'polynomial functions'. The classification in topic areas means sub-topic areas at several levels. A component in one topic area can in itself be a topic area. Weierstrass's limit definition belongs to the topic area of 'limit', as do 'limits of rational functions' and the symbols used to express limits. The symbols also belong to the topic areas 'derivatives' and 'continuity'. Topic areas overlap this way as illustrated by the simplistic model in Figure 1.

If a concept is represented in more than one topic area in a concept image and the topic areas the representations belong to are disjointed, then inconsistencies may occur in the way aforementioned. Inconsistencies can appear within a topic area as well, but they are easier to detect due to the relatedness of the topic. The development of concept images never ends and the mental representations generate a dynamical system linked together at various levels.

An example of a topic area, marked by a wider contour line in Figure 1 represents the topic area 'limits'. It comprises a marked oval component representing the limit definition, which is also part of the topic areas 'derivatives', TA2, and 'continuity', TA3. The black rectangular component represents the definition of derivatives. The figure only shows some nodes in each topic area to describe the structures of the complicated relations. There are, in most real cases, more nodes linked in more intricate constellations.

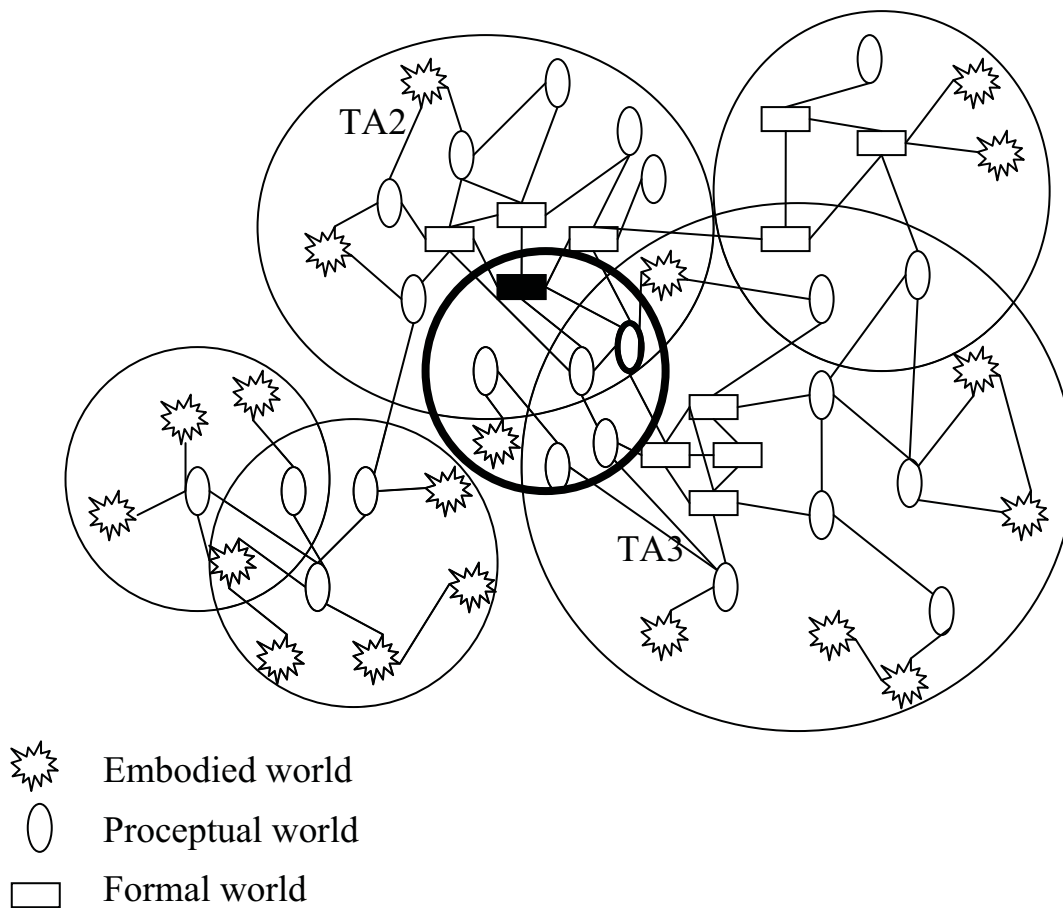


Figure 1. Topic areas (parts of a larger concept image) and components with links in a model of a concept image. The marked part is the topic area ‘limits’. TA2 and TA3 represent the topic areas ‘derivatives’ and ‘continuity’ respectively.

Concept images change on account of outer and inner stimuli, such as discussions, thoughts and problem solving, and a model such as the one in Figure 1 is hence in constant change. It is nevertheless a tool suitable for describing students’ concept developments of limits of functions.

THE EMPIRICAL STUDY

This section describes the sample of students studied and the course they were enrolled in, followed by an outline of methods and instruments used.

The students and the course

There were 112 students participating in the study, of these, 33 were female. The students were aged 19 and up. They were enrolled in a first level university course in mathematics that was divided into two sub-courses. Both of them dealt with calculus and algebra and were given over 20 weeks full time (10 weeks for each course). The students had two lectures (the whole group with one lecturer) and two sessions for task solving (in sub-groups of 30 students with a teacher in each sub-group) three days per week. Each lecture and session lasted 45 minutes. Thus the total teaching time for each course was 90 hours.

The first course had a written exam and the second had a written exam followed by an oral one. The marks awarded were *IG* for not passing, *G* for passing and *VG* for passing with a good margin.

Methods and instruments

Different methods were used to collect different types of data, such as students' solutions to limit tasks and responses to attitudinal queries. The sets of data were collected at different stages in the students' developments to describe traces of changes in their concept images during the semester. The instruments used were designed to take those differences into account. The limit tasks were of increasing difficulty and the attitudinal part was mainly in the beginning of the semester. The students were confronted with tasks at five times during the semester, called *stage A* to *stage E*.

The students got a questionnaire at stage A in the beginning of the semester. It contained easy tasks about limits, such as:

Example 1: $f(x) = \frac{x^2}{x^2 + 1}$. What happens with $f(x)$ if x tends to infinity?

The tasks did not mention limits per se, but were designed as a means to explore if the students could investigate functions with respect to limits. The students were also asked about the situations in which they had met the concept before they started their university studies to reveal if they had related topic areas linked to the topic area 'limits of functions'. Some attitudinal data was also collected.

After limits had been taught in the first course the students received a second questionnaire at stage B, with more limit tasks at different levels of difficulty, for example:

Example 2: a) Decide the limit: $\lim_{x \rightarrow \infty} \frac{x^3 - 2}{x^3 + 1}$. b) Explanation.

c) Can the function $f(x) = \frac{x^3 - 2}{x^3 + 1}$ attain the limit value in 2a? d) Why?

The aim was for the students to reveal their habits of calculating, their abilities to explain what they did, and their attitudes in some areas. The students were asked if they were willing to participate in two individual interviews later that semester. Thirty-eight students agreed to do so; of these, 18 students were selected for two individual interviews each. The selection was done with respect to the students' responses to the questionnaires so that the sample would as much as possible resemble the whole group. The gender composition of the whole group was also considered in the choices.

The first session of interviews was held at stage C in the beginning of the second course. Each interview was about 45 minutes long. The students were asked about definitions of limits, both the formal one from their textbook and their individual

ways to define a limit of a function. The students' topic areas 'limits in theory' and 'limits in problem solving' were investigated with the purpose to discover possible links or inconsistencies. Example 3 was used for that reason:

Example 3: Is it the same thing to say "For every $\delta > 0$ there exists an $\varepsilon > 0$ such that $|f(x) - A| < \varepsilon$ for every x in the domain with $0 < |x - a| < \delta$ " as "For every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - A| < \varepsilon$ for every x in the domain with $0 < |x - a| < \delta$ "? What is the difference if any?

They were also asked to comment on statements very similar to those used by Williams (1991) in a study about students' models of limits. The statements the students commented on are the following (translation from Swedish):

1. A limit value describes how a function moves as x tends to a certain point.
2. A limit value is a number or a point beyond which a function can not attain values.
3. A limit value is a number which y -values of a function can get arbitrarily close to through restrictions on the x -values.
4. A limit value is a number or a point which the function approaches but never reaches.
5. A limit value is an approximation, which can be as accurate as desired.
6. A limit value is decided by inserting numbers closer and closer to a given number until the limit value is reached.

The reason for having these statements was to get to know the students' perceptions about the ability of functions to attain limit values and other characteristics of limits. Connections to language and intuitive representations were discussed to some extent to further describe their concept images. The students also solved limit tasks of various types with the purpose to reveal their perceptions of limits and commented on their own solutions from the questionnaires to clarify their written responses where it was needed.

The students received a third questionnaire at the end of the semester, at stage D. It contained just one task. Two fictional students' discussion about a problem was described. One reasoned incorrectly and the other one objected and proposed an argument to the objection. The students in the study were asked to decide who was correct and why.

A second interview was carried through at stage E after the exams. Each interview lasted for about 20 minutes. Of the 18 students, 15 were interviewed at this point. The remaining three students were unable to participate for various reasons. The students commented on the last questionnaire and, linked to that, the definition was scrutinized again. The task at stage D and Example 3 was brought up again to reveal any changes in perceptions. The quantifiers *for every* and *there exists* in the $\varepsilon - \delta$ definition were discussed thoroughly.

Field notes were taken during the students' task solving sessions and at the lectures when limits were treated to give a sense of how the concept was presented to the students and how the students responded to it. Tasks and results from other parts of the study are described in more detail in other articles (Juter, 2005a-2006c).

THE STUDENTS' CONCEPTUAL DEVELOPMENT

The students' responses to tasks and questions in the questionnaires and interviews have been analysed and categorised. Table 1 shows developments of four of the students with conceptual developments typical for a group of students. The numbers in brackets after each name in the table indicates the number of students in each group. Pseudonym names have been used to retain student anonymity. The digits at stage C indicate the students' preferred choices from the six statements. The bold and larger digits are the students' choices of statements most similar to their own perceptions of limits. The first points at stage E are the students' explanations to what a limit is and the last points are about Example 3 where the students explain the difference between the correct and incorrect definition connected to the task from stage D.

Table 1: Student developments at the five stages A-E

TIME	Tommy (3)	Leo (3)	Mikael (4)	Julia (2)
A	-Links to derivatives -Solves easy tasks well	-Links to nutrition, physics and biology -Solves easy tasks well	-Links to prior studies, problem solving, physics -Unable to solve easy tasks	-Links to prior studies -Solves easy tasks well
B	-Limits are attainable in problem solving -Can not state a definition -Solves routine tasks and explains	-Limits not attainable in problem solving -Can not state a definition -Solves routine tasks and explains	-Limits are not attainable in problem solving -Can not state a definition -Solves tasks and explains well	-Limits are attainable in problem solving -Can state a definition -Solves tasks and explains fairly well
C	-Limits are attainable in problem solving -Limits are not attainable in theory -A limit of a function is how the limit stands with respect to another function, no motion -3, 4 , 5, 6 -Can not state or identify the definition	-Limits are not attainable in problem solving (hesitates) -Limits are not attainable in theory -Thinks of limits in pictures -3, 4 -Can not state or identify the definition -Solves tasks well -Links to derivatives and number	-Limits are attainable in problem solving -Limits are not attainable in theory -Thinks logically rather than explicitly define -2, 3, 4 , 5 -Can not state but can identify the definition after investigation -Problems to solve tasks -Links to prior studies	-Limits are attainable in problem solving -Limits are attainable in theory -It comes closer and closer to A as x comes closer and closer to a -1, 3 , 5 -Can not state but can identify the definition (claims that both def in ex 3 state the same, ϵ and δ come

	<ul style="list-style-type: none"> -Not a real limit if attainable by function -Problems to solve tasks -Links to derivatives and curves -Claims to master the notion of limits 	<p>sequences</p> <ul style="list-style-type: none"> -Claims to master the notion of limits fairly well 	<p>and graphs</p> <ul style="list-style-type: none"> -Claims to master the notion of limits in problem solving but not the definition 	<p>in pairs)</p> <ul style="list-style-type: none"> -Solves tasks well -Links to derivatives, graphs and number sequences -Claims to master the notion of limits
D	-Identifies the error	-Identifies the error, makes other error	-Identifies the error, makes other error	-Identifies the error
E	<ul style="list-style-type: none"> -A limit is a point to tend to but never to reach, motion -Links to integrals -Can not identify the definition -Does not understand the quantifiers' roles 	<ul style="list-style-type: none"> -A function tends to a value as the variable controlling the function changes, it never reaches the limit -No links -Can not identify the definition -Does not understand the quantifiers' roles 	<ul style="list-style-type: none"> -If the x-value tends to a value then the function tends to a value connected to the x-value -Links to Taylor expansions, series and number sequences -Can identify the definition -Can explain the difference between the definitions in ex 3 	<ul style="list-style-type: none"> -If the function goes closer and closer a number as x goes closer and closer to a value -Links to continuity -Can identify the definition after investigation -Can explain the difference between the definitions in ex 3 fairly well

The conceptual development as traces of concept images in Tommy's case shows that when he came to the university, he had a concept image of limits of functions linked to derivatives. Limits were treated with a strong focus on problem solving developing the topic area 'limits in problem solving' but not 'limits in theory', and Tommy thought that functions can attain limit values when he dealt with problem solving but not in theoretical discussions for the duration of the course. Tommy did not grasp the notion of limits, but he was able to solve routine tasks. He was confident about his own abilities to master the notion though he did not reach the third of Tall's worlds (Tall, 2004).

The development in Leo's case differed at some points from Tommy's. Leo thought that limits were never attainable, neither in problem solving nor in theoretical situations. Leo was a better problem solver than Tommy was but he too had theoretical problems.

Mikael had trouble to solve tasks at first, but he showed trace of a concept image of limits in connection to derivatives. He thought that limits are unattainable in problem solving at stage B but changed his mind at stage C. He thought that limits are unattainable in theoretical discussions. He had a reasoning approach to mathematics and managed to investigate mathematical situations to find answers. At the end of the course, he was able to explain the theoretical issues correctly. Mikael showed traces of progression through Tall's three worlds from an exploring attitude to a formal understanding.

Julia understood the notion of limits from the start. She had a concept image from upper secondary school when she came to the university. There were some theoretical difficulties during the semester, but they were overcome at the end.

The quantifiers in the definition caused confusion for all students. There was an opinion among some students that ε and δ in Example 3 at stage C come in pairs and can therefore be placed either way in the example. Mostly high achieving students showed traces of this conception, which can be explained by the fact that low achieving students had not integrated the theory well enough in their concept images to even identify the definition next to a wrong one. The high achieving students did not have this misunderstanding at stage E as they were able to explain the meaning of the quantifiers. The low achieving students did not understand the quantifiers meaning in the definition throughout the course.

The students' problems to connect theory to problem solving became particularly apparent from their difficulties to determine whether limits are attainable for functions or not. Many students interpreted the strict inequalities in the formal definition to say that limits are not attainable. Examples where limits were attainable did not change the low achieving students' beliefs about the definitions' meaning. Some students became frustrated when they saw examples of attainable limits and were asked questions about the definition because they were unable to create a coherent picture of the situation. The students' concept images were divided in disjoint topic areas; 'limits in theory' and 'limits in problem solving'.

Learning limits requires skills from many mathematical areas. Students need to be able to understand formal expositions, perform algebraic manipulations, understand the meanings of quantifiers and absolute values, which students found problematic, and link theory to their every day problem solving. This means that their concept images need to be well developed both in depth and width, i.e. they need abstraction skills and strong links between numerous topic areas. They also need to find inspiration and reasons to go through the hard work to make the knowledge meaningful in their concept images.

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HABITS OF MIND ASSOCIATED WITH ADVANCED MATHEMATICAL THINKING AND SOLUTION SPACES OF MATHEMATICAL TASKS

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ABSTRACT

This paper discusses multiple way problem solving as a habit of mind and claims that it is an effective tool for the development and diagnostics of advanced mathematical thinking. I introduce a notion of solution spaces of a mathematical problem and exemplify the ideas using several problems. The concepts of symmetry and continuity are presented here as important ideas that should be a part of habitual mathematical behavior in the context of advanced mathematical thinking.

INTRODUCTION:

In this paper I consider mathematical creativity as one of the critical characteristics of Advanced Mathematical Thinking (AMT) (Ervynck, 1991). I argue that solving problems in multiple ways is an effective tool for advancing and exploring mathematical thinking and creativity (Leikin & Lev, accepted; Leikin & Levav-Waynberg, in press). Habits of mind are associated with concepts and principles incorporated in one's repertoire so that they are applied successfully in an appropriate problem-solving situation (Cuoco, 1995; and Goldenberg, 1996). Making one step further I suggest that solving problems in different ways as a (meta-mathematical) habit of mind both require and foster advanced AMT. Additionally in this paper I introduce the notion of solution spaces that helps explaining connections solving problems in multiple ways as a habit of mind and the development of AMT. I start with an example of a geometry problem to exemplify main ideas presented in the paper.

PROBLEM 1:

Two circles are inscribed between the sides of an isosceles triangle and of an angle one of whose sides includes the basis of the triangle and the other side passes through its opposite vertex. Find relation between the radiuses of the circles and the altitude of the given triangle (Figure 1).

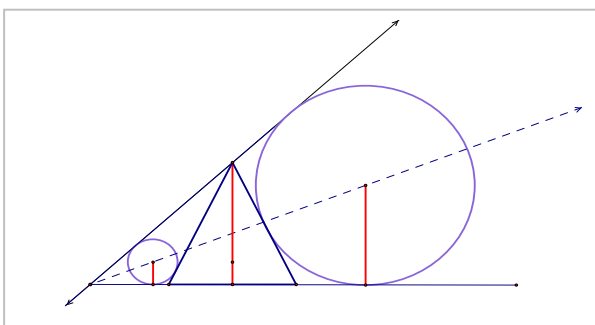


Figure 1: Problem 1

Reasoning by continuity

By continuously changing the figure we obtain three "extreme" situations: Two symmetrical situations in which two sides of the given angle coincide with the two sides of the triangle (Figure 2). In this situation the radius of one of the circles is zero and the other radius equals the altitude of the triangle.

The other "extreme situation" is obtained when the two sides of the angle "become parallel" under dynamic transformation of the given figure (Figure 3). This situation is "in the middle" between the two "extremes" of the first type. Obviously, each of the radii is half of the altitude of the triangle. From here, by symmetry and continuity considerations, the sum of the radii equals the altitude.

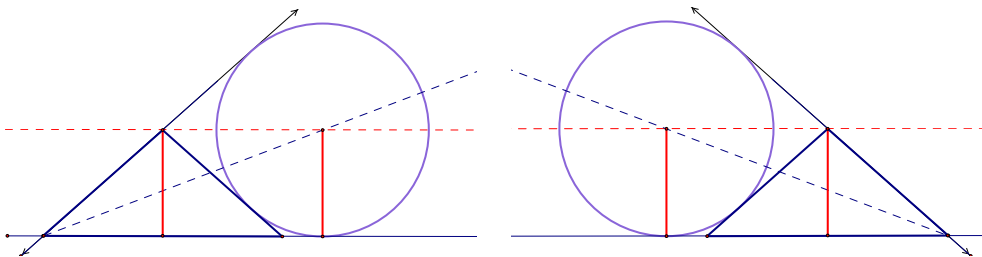


Figure 2: Two symmetrical extreme situations: a lateral side of the triangle is on the side of the angle

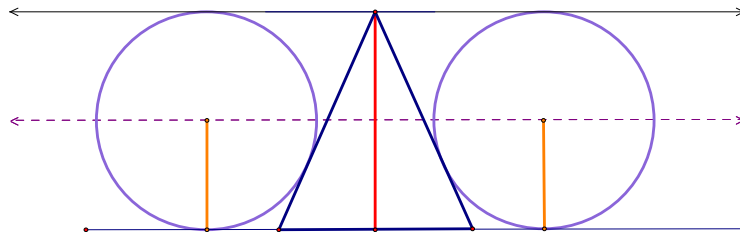


Figure 3: Two parallel lines as an extreme case of the angle

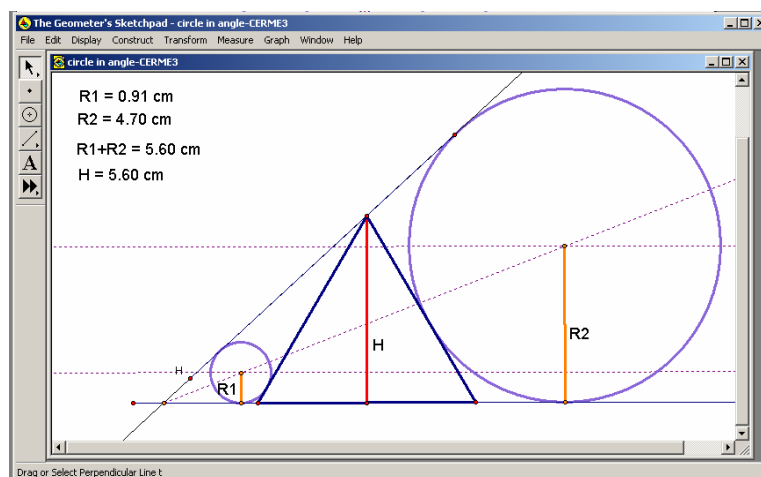


Figure 4: Exploring continuous "change" in DGE: $H=R_1+R_2$

By applying continuity considerations we received the answer to the question. However one may argue: "These considerations do not prove the conjecture". These

considerations performed mentally are analogical to the investigation that one may perform using Dynamic Geometry Environment (DGE, Figure 4). In this case we observe the relation, which continuously does not change: $R_1 + R_2 = H$. Similarly, we "do not believe our eyes" and a formal proof is still needed.

Prove 1 – outline:

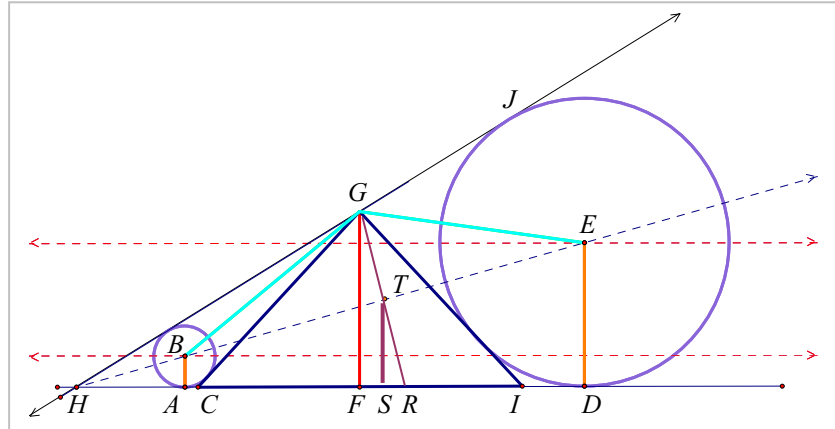


Figure 5: Prove 1

HE , GB and GE are bisectors of the angles GHD , HGC and JGI correspondingly
By calculation of the angles in the given figure, BGE is an isosceles triangle.

Auxiliary constructions:

GR is perpendicular to HE (the angle bisector), T is the intersection of HE and GR , and TS is perpendicular to HD

T is the midpoint of GR (HT is the bisector and the altitude in triangle GHR)

T is the midpoint of BE (GT is an altitude and thus a median in the isosceles triangle BGE).

Thus TS is the midline both in triangle FGR and in the trapezoid $ABED$

Thus the sum of radiuses equals the altitude.

Prove 2 - outline:

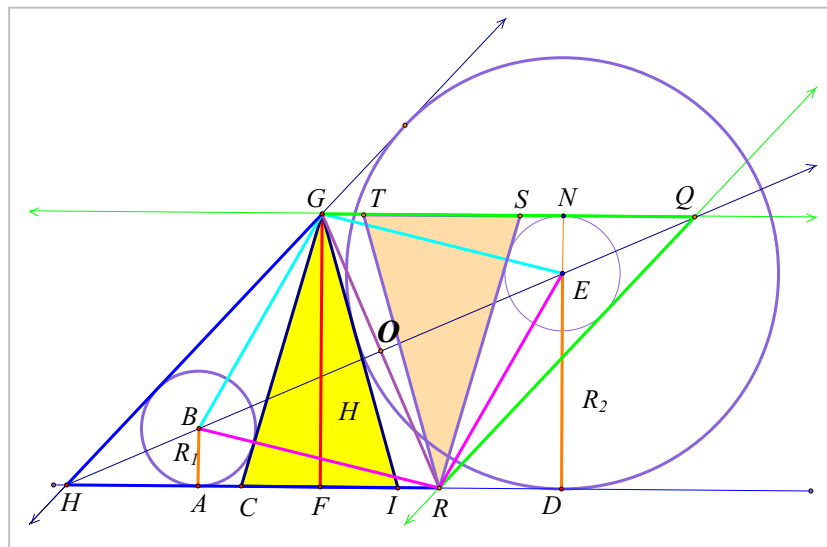


Figure 6: Prove 2

Auxiliary construction: GQ parallel to HA , QR parallel to GH (Figure 6)

By symmetry considerations $HGQR$ is a rhombus and thus GR perpendicular to HQ

By calculation the angles HGB and QGE are equal. Thus points B and E are symmetrical about O . Hence the center of the "small symmetrical" circle E is the center of the "big" given circle. Thus $NE=R_1$ and $R_1+R_2=H$

In 1988 D. Tall pointed out that AMT addresses at least two distinct approaches: one is "advanced forms of mathematical thinking" and another is "thinking related to advanced mathematics". Obviously as the level of mathematics advances it requires advanced forms of mathematical reasoning and the two approaches come together. This paper considers *Habits of Mind* (Goldenberg, 1996) as closely related to AMT. On the one hand habits of mind support developments of AMT, on the other hand AMT includes habits of mind.

One of such habits of mind presented in this paper is applying unifying concepts, such as symmetry and continuity (see Problem 1) when solving problems from different branches of mathematics. The other one is solving (proving) problems in different ways including the use of different representation, different properties of mathematical objects (including different concept definitions and theorems) and the use of mathematical tools and concepts from different mathematical fields (Problem 1).

HABITS OF MIND

From the psychological point of view (Costa, 1991) habits of mind are manifested in humans' ability to behave intellectually when one does not know the answer, e.g., in the situations of dilemmas and uncertainties. These situations usually demand strategic reasoning, insightfulness, perseverance, creativity, and craftsmanship. Employing habits of mind means inclination and ability to choose effective patterns of intellectual behaviour. Psychological characteristics of habits of mind include personal persistence, inclination to choose an effective strategy and ability to apply this strategy to solving the problematic situation. As persistence, creativity and high ability are also characteristics of intellectual giftedness (Renzulli, 2002) habits of mind in mathematical context may be attributed to AMT as advanced forms of mathematical thinking.

Habits of mind considered in mathematical context are guiding mathematical principles for design of mathematical curricula, development of school mathematics culture and individual mathematical reasoning (Cuoco, Goldenberg & Mark, 1997; Cuoco, 1995; and Goldenberg, 1996). Goldenberg (1996) defined habits of mind as

thinking that "one acquires so well, makes so natural, and incorporates so fully into one's repertoire, that they become well mental habits – one not only can draw upon them easily, but one is likely to do so (p. 13)

Some mathematical concepts, principles and ideas may serve as habits of mind. Relevant to this paper – Cuoco (1995) pointed out *reasoning by continuity*, i.e.

thinking about continuously varying systems. Problem 1 exemplifies nicely the application of this principle. The other mathematical concept whose application should be incorporated in school curricula as a habit is Symmetry. Polya (1981) wrote: ‘We expect that any symmetry found in the data and conditions of the problem will be mirrored by the solution’ (p. 161).

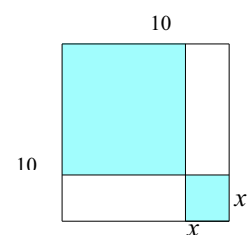
One of the characteristics of the concepts and ideas that may be considered as Habits of Mind within mathematics is their interdisciplinary nature within mathematics, the possibility to incorporate the ideas and concepts throughout the entire school curriculum. In this way continuity that usually is learned formally at the advanced level as connected to the limits, in its common sense may be considered at earlier stages of school mathematical curriculum. For example, why the quadratic equation $x^2+2x+1=0$ has 'two equal' roots while the equation $x+1=0$ only one root? Using continuity consideration teachers can demonstrate and explain to students the distinction. Proving area of a trapezoid using area of parallelogram also may be based on continuous transformation of a geometric figure. When the concept of continuity is commonly used by teachers in the middle grades then in senior high school students (hopefully) can easier understand why a continuous on the segment $[a, b]$ function receive all the values between $f(a)$ and $f(b)$.

The following problems can additionally exemplify application of symmetry and continuity considerations in solving process.

Problem 2: Rhombus $EFGH$ is inscribed in the square $ABCD$. Prove that $EFGH$ is a square (presented in Leikin, in prep).

Problem 3: Given a polynomial $f(x)$ with whole coefficients. It is known that for whole numbers a , b , and c $f(a) = f(b) = f(c) = 2$. Does there exist a whole number d such that $f(d)=3$?

Problem 4: The square with a side equal to 10 meters is divided into four parts. In the two squares (the coloured parts in the figure) blue flowers are planted and in the remaining rectangular parts they planted yellow flowers. For which value of x the area of the blue flowers is maximal? (presented in Leikin & Levav-Waynberg, submitted)



Symmetry as a way of thought was demonstrated as an effective problem-solving tool within the school mathematical curriculum (Leikin, 1997; Leikin 2003; Leikin, Berman & Zaslavsky, 1998, 2000). Solution 2 of Problem 1 is an example of geometric symmetry in solving the problem. Application of geometric symmetry also is an effective solving tool for Problem 2 (rotational symmetry) and Problem 3 (translation). Problem 4 may be solved using symmetry of the parabola (reflection): Since the area (quadratic) function receives equal values on the segment $[0; 10]$ the minimal value of the function will be achieved in the middle of the segment. Ideas of continuity and symmetry are often connected to each other when solving the problem

(e.g., continuity-based solution of Problem 4). Algebraic or logical *symmetry of role* (Polya, 1973; Leikin et al, 2002) used in solving problems usually shorten the solution as, for example, when proving that the ellipse $x^2+xy+y^2=12$ is perpendicular to the line $y=x$ using symmetry. However, when the use of symmetry is not a part of school mathematics culture these solutions are considered as "not mathematical enough", "demonstrating but not proving" or "sophisticated" (Leikin, 2003).

MULTIPLE SOLUTIONS TO A TASK AND AMT

It is commonly accepted by mathematics educators that linking mathematical ideas and deepening understanding of how more than one approach to the same problem can lead to equivalent results are essential elements of the developing of mathematical reasoning (NCTM, 2000; Polya, 1973, Schoenfeld, 1985; Charles & Lester, 1982). The construction of mathematical knowledge is supported by shifting between different representations, comparing different strategies and connecting different concepts and ideas (Fennema & Romberg, 1999). Based on this theoretical perspective solving problems in different ways should become one of the habits of mind incorporated in school mathematical curriculum.

Solving problems in different ways may be attributed to AMT based on Polya's (1973) claim that it characterizes experienced mathematicians since solving problems in different ways requires a great deal of mathematical knowledge. Additionally, Krutetskii (1976) argued that problems with several solutions allow examining flexibility of individual's mathematical thinking by investigating switches from one mental operation to another. Polya (1973) and later Ervynck (1991) stressed that solving problems in different ways characterizes creativity of mathematical thought while some solutions may be more creative (more elegant/short/effective) than others. The discussion of the elegance of a solution acknowledges subjectivity of the view on beauty, simplicity and effectiveness of a solution (Dreyfus & Eisenberg 1986).

SOLUTION SPACES OF MULTIPLE SOLUTION TASKS

In my studies focusing multiple solution tasks I introduce the notion of solution spaces. I suggest applying metaphor used by Watson & Mason (2005) for the description of example generation processes. Same as what Watson and Mason suggest for example spaces, solution spaces are influenced by individual's experiences and memory, as well as by specific requirements of the mathematical tasks for which the solutions should be found. Similarly to the kinds of example spaces introduced by Watson and Mason I consider the following kinds of solution spaces:

Expert solution spaces are spaces of solutions that expert mathematicians can suggest to the problem. I distinguish between two kinds of expert spaces with respect to school mathematics: *Conventional solution space* is generally recommended by the curriculum and displayed in textbooks, into which teacher hopes to induct her students. For example, solution of Problem 4 using calculus is in the conventional

space of this problem. *Unconventional solution spaces* include solutions to problems, which are usually not prescribed by school curriculum. Solutions based on the application of the symmetry and continuity considerations commonly belong to these spaces.

If we want a particular problem-solving approach to become a habit of mind it should belong to the conventional spaces of many problems.

Individual solution spaces are also of two kinds. The distinction is related to the ability of a person to apply it individually. *Personal solution spaces* include solutions that individuals may present on the spot or after some attempt without help of others. These solutions are triggered by a problem and accessible in solver's mind. For example, for most of the schoolchildren in Israel solving minima-maxima problems using calculus relates to these spaces. *Potential solution spaces* include solutions that solvers produce with help of others. The solutions correspond to personal ZPD (Vygotsky, 1978). My experience demonstrates that solutions to Problem 1 presented in this paper exemplify solutions from the potential spaces of the majority of teachers and students. They are "clear and interesting" when guided; however, comparing segments GB and GE when solving the problem is not so trivial. Teachers usually like consideration of the continuous change in this problem as well as in Problem 2 but these considerations are not convincing and their general inclination is to look for a formal proof based on the algebraic manipulations and calculations (e.g. Problem 4, Leikin 2003). Any problem-solving strategy becomes a habit of mind only when it belongs to the personal solution space. However, not each solution from this space is a habit of mind. To be a habit of mind solution should belong to personal solution spaces of many problems from different parts of mathematical curriculum.

Collective solution spaces characterize solutions produced by groups of participants (communities of practice), they determine local conventional spaces and are subsets of expert solution spaces. Collective solution spaces are usually broader than personal solution spaces and within a particular community they are one of the main sources for the development of individual spaces.

SOLUTION SPACES AS AN EFFECTIVE RESEARCH TOOL

By comparing individual solution spaces with expert solution spaces we may evaluate person's mathematical knowledge and creativity (Leikin & Lev, submitted; Leikin & Levav-Waynberg, submitted). Krutetskii (1976), Ervynck (1991), and Silver (1997) connected the concept of creativity in mathematics with multiple-solution tasks. In this context (Silver, 1997, Ervynck, 1991, Leikin, accepted), *flexibility* refers to the number of solutions generated by a solver, *novelty* refers to the conventionality of suggested solutions, and *fluency* refers to the pace of solving procedure and switches between different solutions. Ervynck (1991) stressed that mathematical creativity associated with problem solving in different ways is one of the central characteristics of AMT.

At the same time conventional solution spaces have to include ideas like symmetry and continuity. Probably when one has these habits of mind in the conventional solution spaces he/she will be able to create a solution which will appear unconventional to an expert who introduces the task.

MULTIPLE WAY PROBLEM-SOLVING FOR THE DEVELOPING OF AMT

Based on my the personal experience and the research literature presented earlier I argue that solving problems in different ways may serve as an effective tool for the development of AMT both in the meaning of “advanced forms of mathematical thinking” and “thinking related to advanced mathematics”. The following considerations justify this claim.

As outlined above, solving problems in different ways is an effective tool for construction of *mathematical connections*. When different solution of a problem belong to personal solution spaces one can connect between representations of the mathematical concepts, different mathematical tools and concepts from the same field or different mathematical topics (e.g., NCTM, 2000). In order to develop connected mathematical knowledge Russian educators (Sharygin, 1989; Davydov, 1996) advised *divergence principle* of learning simultaneously several topics. These topics should be connected by unifying mathematical principles, concepts and tools like continuity, symmetry, equivalence; meta-mathematical concepts, like mathematical definitions, theorems and proofs or didactical ideas, e.g., multiple-solution tasks.

One of the basic principles that may be achieved by means of solving problems in different ways is *meeting appropriate difficulties*. Principles of 'developing education' (Davydov, 1996), which integrate Vygotsky's (1978) notion of ZPD (Zone of Proximal Development), and Leontiev's (1983) theory of activity claim that to develop students mathematical reasoning the tasks should not be too easy or too difficult. When solving problems in different ways a teacher has an opportunity to introduce the students to a task for which at least one of the solutions will be situated within personal solution spaces of the students. The additional solutions may belong either to personal spaces of some students whereas for other students they will be included in the potential solution spaces. In this way the students will meet mathematical challenge and expand their personal solution spaces, they will advance the level of their mathematics.

Consequently, combining *individual and cooperative* forms of learning is very important and can support and be supported by implementation of multiple-solution tasks. Learners have to study systematically in individual mode in classroom and at home in order to develop their problem-solving competence. They have to be involved in cooperative learning activities in order to advance in problem solving by supporting each other's ZPD (NCTM, 2000; Davydov, 1996). The role of collective spaces in the cooperative learning is especially important since through interactive process these spaces contribute to the development of individual solution spaces. The

balance between individual and cooperative may provide students with better opportunities for realization of their mathematical potential.

Learning has to be *active*, meaning that learners construct their individual knowledge through *mathematical explorations* and other forms of *doing mathematics* with emphasis on conjecturing that leads to mathematical discussions, proofs and refutations. Reasoning by continuity, symmetry and looking for invariants are integral parts of mathematical exploration. On the one hand, openness of Problem 1 in this paper leads to exploration and applying general considerations to its solution. On the other hand different mathematical constructions and ideas applied at the exploration stage may lead to multiple solutions (proofs) to the tasks.

Finally to allow active participation of the learners in learning process we should vary didactical priorities (Sharygin, 1989): priority of ideas when learning a new topic and performing non-conventional solutions vs. priority of complete answers, rigour and elegant proof when working with known ideas and performing conventional solutions.

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MATHEMATICAL BACKGROUND AND PROBLEM SOLVING: HOW DOES KNOWLEDGE INFLUENCE MENTAL DYNAMICS IN GAME THEORY PROBLEMS?

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The research reported in this paper [1] concerns applied mathematical problem solving processes in the specific field of Game Theory. My research work refers to previous studies that investigated, with different tools and aims, “time exploration” processes. Those studies analysed temporal components of mental processes and put into evidence the crucial role of their “management” during problem solving activity. The main purpose of the research reported in this paper is to construct specific theoretical tools in order to study how “time exploration” processes are influenced by the cultural background during Game Theory problem solving.

INTRODUCTION

In my master thesis (Martignone, 2002) I investigated some aspects of cognitive styles in advanced problem solving activities: in particular I identified similar behaviours for the same person in different problems, and for different people in the same problem. Moreover it came out that, not only the application of learned standard schemes, but also the development of time dynamics played a crucial role during the activity of problem solving [2]. Starting from those considerations, in my Ph. D. thesis I have chosen to closely analyse the interactions and interferences existing between personal cognitive processes (like anticipation processes) and cultural resources (like mathematical models, learned schemes, etc) during the solution of a special type of problems: strategic interaction problems. In particular I’m studying how exploration processes in strategic interaction problems change under the influence of learned mathematical theories, a special attention being paid to the analysis of anticipation processes. In order to do this I selected special types of strategic interaction problems, which could be tackled both with general purpose and with ad hoc mathematical tools and in which there are time components to be handled. The mathematical theory that is suitable to tackle these types of problems is Game Theory. This theory is a mathematical analysis of any social situation in which a rational player (typically a person, but possibly a firm or nation) tries to figure out what the other rational players will do, and choose the best strategy between the possible ones. In strategic interaction problems, the decisions of each player influence the final result of the game: so every player should think, not only about his/her possible moves, but also about what other players should do if he/she wants to construct a successful strategy. For this reason players’ fates are intertwined. I have decided to analyse this type of problems for two main reasons: they are suitable for the development of time exploration processes and they are studied by a particular mathematical theory (Game Theory), though they can also be tackled by general

purpose mathematical tools. There is another distinctive feature that led me to the choice of Game Theory problems: this mathematical domain is enough "localized" in mathematics curricula at every level, then it is possible to distinguish between subjects, in particular mathematicians, who are familiar with it, and subjects who do not know it at all.

THEORETICAL FRAMEWORK

The topic of my research was to study the *development of resolution processes* during game theory problem solving with the aim of drawing up new hypotheses on management of time exploration processes (possibly depending on learned tools). In order to do that, I tried to blend different interpretative tools coming from cognitive psychology and educational studies in Mathematics. In particular I considered Vergnaud's conceptual fields theory (1990), Piaget's theory of schemes (1952), the "frame structures" studied by cognitive psychology (Minsky (1989); Pontecorvo (1994)) and "Mental times" theory (Guala and Boero, 1999 [3]). Starting from this theoretical framework, I have refined "Mental Times theory" focusing my attention to the explorations that are developed during the planning phase of the problem-solving process.

“Exploration time: in open ended tasks requiring the subject to find and concatenate suitable arithmetic operation, plan a geometric construction, build up a proof, and so forth, time projections can be realized from the past onward ('how will he have gone about solving...') or in the future and then towards in the past ('I think up a solution and explore it in order to find the operations to perform, depending on available resources')” (Guala & Boero, 1999; p. 165)

The different components of explorative processes were classified by distinguishing between those that are structured by knowledge (i.e. guided by frame structures, scheme applications and the use of representations) and those that construct/reconstruct "episodic" situations by imagining the temporal development of the events (the term "episodic" comes from Tulving's theory [4]).

As I have already said, the aim of my work is to understand how exploration processes are managed during the resolution of strategic interaction problems. The difficulties concerning planning ahead are well documented and studied in Behavioural Game Theory research: in particular, the statistical data collected by Camerer (Camerer, 2003a, 2003b, Camerer et al. 2003, 2004) show that, during the resolution of strategic interaction problems, anticipation processes are developed only by few subjects, while standard behaviour does not seem to rely upon a long term development of those processes ("limited strategic thinking"). I was interested in Camerer's experiments on a particular game, the *beauty contest game* (BC game): in this game players simultaneously choose a number between zero and one hundred and the player, whose number is closest to $\frac{2}{3}$ of the average of these numbers, wins a fixed prize.

“The mathematical core of game theory is what players think other players will do. In most theories, this reasoning is iterated (A guesses what B will do by guessing what B

will guess A will do, ad infinitum) until mutually consistent responses – an ‘equilibrium’– is reached. For games that are new to players, a more plausible model is that players use a limited number of steps of iterated reasoning. The ‘p-beauty contest (pBC) game’ is a good tool for measuring steps of thinking” (Camerer, 2003a; p.225)

“Rational game theory predicts an equilibrium in which choices are mutually consistent (leading to 0), but in experiments many players assume that others are making random choices, and choose 33, or believe that others are responding to random choices, and choose 22. Models of this cognitive hierarchy organize regularity from many games and show that one or two steps of thinking are typical, although three or four steps are used by analytically skilled undergraduates and game theorists” (Camerer, 2003; p.1674)

Camerer’s results [5] offer a good base of information for pointing out the study of the differences between decisions taken by different subjects, but, in particular, they show that there are special categories of subject that seem to be able to make more step of thinking because of their knowledge.

Subjects	Number	Average step of thinking
Computer scientists	18	3.8
Game theorists	19	3.7
Caltech students	23	3.0
Portfolio managers	24	2.8
Economics Ph.D. class	27	2.3

Table 2 Some data and estimates of τ in Beauty contest Game (Equilibrium=0) (Camerer e al., 2003)

From Camerer’s experiments it seems to be evident that cultural background influence the decisions in the BC game, therefore I decided to analyse in what way knowledge may affect the resolution processes which require a good management of the time components.

METHODOLOGY

The data that I used in my work have different sources: at the beginning of my research I studied Camerer's data that show people's different choices in Game Theory problems and that suggested a good problem for the analysis of anticipation processes (i.e. *beauty contest game*); unfortunately Camerer data present only "photographs" of the final results of decision processes, while I'm more interested in the study of the development of those processes (with the aim of drawing up new hypotheses on the management of mental times). Therefore I needed another type of data that allow me to study the exploration processes. I selected some strategic interaction problems with the aim of analysing analogies and differences in the solving processes of subjects who can use tools coming from mathematical theories, and subjects who don't know those theories. The method used for investigation was that of clinical interviews; subjects were asked to solve the problems and express their thinking process aloud at the same time. The interviews were audio-taped. The analysis is mainly based on the transcripts of the interviews, which were analysed with special attention to verbal tracks in order to detect mental processes [6].

As underlined in the Introduction, the strategic interaction problems seem to offer more possibilities than classical mathematical problems for the investigations on the existence of different anticipation processes and on the relationship between scheme accommodation (related to learned models) and mental exploration of inner times. The chosen problems were: solvable by subjects with different "backgrounds"; rather quick to solve; and easily classifiable in Game Theory (without being well-known examples). I chose different types of non-cooperative games [7], but among them the *beauty contest game* plays a leading role because, as it is seen before, its resolution needs the management of many and complex anticipation processes.

In order to study the influences of "mathematical knowledge" during the resolutions of strategic interaction problems, I interviewed subjects belonging to different categories: mathematicians (inside this category I selected: experts in Game Theory, mathematicians who don't know Game Theory and mathematicians who know a little bit this theory but are not experts) and non mathematicians (inside this category I selected: economists, engineers, medical doctors, etc). In this paper I am going to present the analysis of a very interesting protocol (Sam's case study).

It's important to underline that the aim of my research isn't to collect quantitative data about frequency of particular results, but instead to identify, and study, the processes which produced those results. For this reason my analysis is qualitative and is based on the study of processes that support the generation of the strategies found in the protocols that I collected. Therefore, every protocol is analysed in a double perspective: as bearer of new information about possible exploration processes and as evidence for the existence of recurrent exploration processes.

A CASE STUDY: Sam's protocol

The interview to a 36 years old researcher in Math Education (that in the following will be called "Sam"), dealing with the *beauty contest game*, was fundamental for the development of my research. Sam belongs to the group of mathematicians who don't know Game Theory. The analysis of this protocol shows the strong influence of the subject's "mathematical background" during the management of the exploration processes. As a matter of fact, Sam doesn't face the problem only by trying to figure out what other players will do ("step by step" as theorised by Camerer), but he tackles the situation (described in the problem) using mathematical tools.

In the following I quote the first part of Sam interview. The text proposed was: "There are n players that simultaneously choose a number between zero and one hundred. The player, whose number is closest to $2/3$ of the average number, wins 1000 €. Players do not have any possibility of communicating. Which choice could be the good one?"

Sam: The winner is the one who chooses the number that is nearest to $2/3$ of the average...and there are "n" players..."n"... therefore all of them could also choose the same number because there is not any possibility of agreement...uh...boh...nothing... I have to do a simple thing, I don't see solutions...I try with $n=2$, two players...two players...eh...it's occurring to me that...well, two players are both distant from the average. The first plays a_1 , and the other a_2 , the average is $\frac{a_1 + a_2}{2}$ (see figure A)

The figure shows handwritten mathematical work. At the top, 'n=2' is written inside a hand-drawn circle. Below this, the variables 'a1' and 'a2' are written. Underneath these, the expression for the average is written as $\frac{a_1 + a_2}{2}$.

Figure A

Sam: ...eh... nearest to $2/3$ of average... $2/3$ of average...where is it? Between...that is, it is smaller than average, $2/3$...they are equidistant to the average, $2/3$ of average is lower than average, then wins the one who gave the lowest number. I should make calculations but it seems evident to me. Then we have two numbers, the one who gives the number that is nearest to $2/3$ of the average... $2/3$ of the average is lower than the average, they are

equidistant. [] Here we have a_1 , here a_2 and this is the average ...2/3 of the average stays on this side and then a_1 wins. (See figure B)

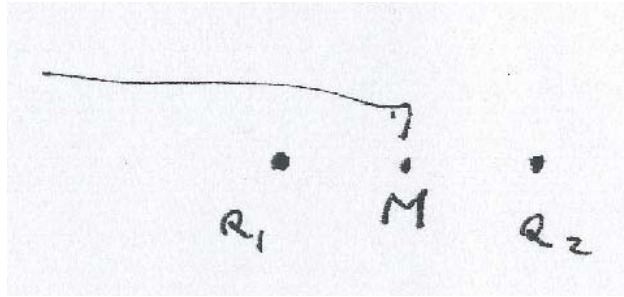


Figure B

Sam: Unless a_1 is equal to a_2 and then both of them win and share out...[...] it's worthwhile to bet on the minimum number: zero. I don't know if it is possible.

Observer: Yes it is. The extremes are included.

Sam: Then, if they are two it is worthwhile to bet on zero.

At first, Sam doesn't manage to analyse the problem without knowing the number of players (probably his exploration processes do not develop because he cannot apply the algorithm for finding the average numerical value); so he decides to fix the value of that variable. He starts to analyse the problem when the players are two (this is a typical attitude of a mathematician: to analyse the simplest case and then try to generalize the achieved results).

The possibility of using the representation of the numerical choices on a "straight line of real numbers" supports Sam exploration processes during his strategy construction (*exploration supported by the representation*): the possible players' choices (the "possible virtual future") become points on a line and the operation on these numbers is transformed into (visual) movements on the interval (I have underlined the words that reveal the superimposition between time exploration processes and movements on the straight line). Moreover his management of anticipation processes, guided by mathematical schemes linked to the concept of "minimum", bring Sam to make a "cognitive jump" from the strategy elaborated for a generic couple of points, to the best choice for all the possible alternatives (*exploration supported by schemes*): the possible explorations of players' choices (temporal movements) are "translated" and "compressed" into only one movement (visual/graphic) on the model. Therefore we can observe that Sam, instead of exploring the interaction problems by simulating players' strategies (*episodic exploration*), constructs a model of the situation: in this way he overcomes the difficulties linked to the management of complex anticipation processes by using schemes and representations coming from his mathematical background (*exploration structured by knowledge*).

PRELIMINARY CONCLUSIONS

Camerer's data show/confirm the existence of different management of anticipation processes (inferred from different choices in strategic interaction problems like the BC game), while Sam's case study lay the foundation for the generation of the following hypothesis: Knowledge and learned schemes allow different exploration processes during problem solving activity (for example the explorations supported by "mathematical" schemes and the explorations supported by graphical representations).

In particular, the analysis of Sam's protocol shows that the mathematical models support and give a structure to the anticipation processes, that otherwise could be hard to carry out ("limited strategic thinking", Simon, 1955).

FURTHER QUESTIONS

Starting from Sam's protocol analysis, I tried to answer these further research questions:

- Are mathematicians' behaviours (influenced by Game Theory or not) different from the behaviours of other subjects with different backgrounds during strategic interaction problem solving?
- In particular, how the mathematical models influence the creation/management of time exploration processes?

In the second part of my Ph. D thesis, in order to test the previous hypotheses and also try to answer these questions, I analysed the analogies and the differences, in terms of exploration processes, among the interviews of different subjects, with different backgrounds, who faced the *beauty contest game* and other Game Theory problems. As a general conclusion, both general purpose and field-specific mathematical schemes and tools seem to be crucial in allowing people to overcome their "limited strategic thinking".

NOTES

1. When the paper was written the research was still in progress: for these reasons it was related only the first part of the research and some preliminary conclusions.
2. "Further explorations of these reasoning patterns [the use of mental imagery to imagine a future situation] in problem solving are needed" (Carlson & Bloom, 2005; p. 69).
3. "The debate about the physical existence of time suggests the possibility that time could also be considered an intellectual construction in order to "treat" (that is, to describe/order/analyse) the flux of external events...we can speak about "Mind times", metaphors that may help in "treating" mental processes, especially those intervening in complex problem solving" (Guala and Boero, 1999, p. 164)

4. According to Tulving (2002), human memory involves two distinct systems: episodic and semantic. Tulving describes episodic memory as the system that allows us to remember personally experienced events, and to travel backwards in time to virtually re-experience those events. By contrast, semantic memory is broadly defined as our 'knowledge of the world'. Tulving's theory made me think about the possibility to improve the analysis of exploration processes distinguishing between the memory activities through (/as a function of) the activated time dynamics. As a matter of fact, the reconstruction of past events requires a generation/management of inner times, which is different from the simple memory activity made to remember a concept.
5. The BC game has been run with many subject pools. The average number chosen is usually 25–40 with a large standard deviation (around 20). This result is robust across countries (Germany, US, Singapore) and ages (high-school students to 70-year-old, well-functioning adults). The lowest averages, 15–20, come from subject pools with unusual analytical skill (Caltech students), training in game theory, and from people entering newspaper contests who are self-selected for their motivation and knowledge" (Camerer, 2003a; P. 226)
6. "[...] verbal reports, elicited with care and interpreted with full understanding of the circumstances under which they were obtained, are valuable and thoroughly reliable source of information about cognitive processes" (Ericsson & Simon, 1980; p.247)
7. In a non-cooperative game the players choose their strategies independently and simultaneously (contemporaneity is to be understood from an informal point of view) and do not have any possibility of communicating (so no binding agreements are possible).

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STUDENT GENERATED EXAMPLES AND THE TRANSITION TO ADVANCED MATHEMATICAL THINKING

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In this paper I describe how example generation exercises were incorporated into an introductory Analysis course at a university in Ireland. Data collected in interviews with 12 students suggests that this pedagogical strategy helped them engage with the formal theory in a way that was novel and yet familiar. A key finding of the study was that in order to ensure the success of such exercises, students must be encouraged to review and validate their initial responses to the examples requests.

INTRODUCTION

As a mathematician I am acutely aware of the role that examples play in mathematical thinking both in understanding, and developing, mathematical theories. As a mathematics lecturer I am also acutely aware that students experience many difficulties in their first encounters with definitions and proofs in advanced mathematics courses at university. In this paper I discuss the implementation and evaluation of the pedagogical strategy of encouraging students to generate examples in an introductory course in Analysis.

THEORETICAL FRAMEWORK AND LITERATURE REVIEW

Examples have a crucial function in guiding and informing mathematical thinking, and in the development of mathematical theories. It follows then that they also have a significant role to play in the teaching and learning of mathematics. The survey paper accompanying the research forum on *Exemplification in Mathematics Education* at PME 30 (Bills et al., 2006) discusses many issues related to the use of examples in mathematics education.

I am particularly interested in the role that examples, and specifically student-generated examples, can play in helping students make the transition to advanced mathematical thinking. Tall (1992) describes the transition from elementary to advanced mathematical thinking as moving from “*describing* to *defining*, from *convincing* to *proving* in a logical manner based on those definitions” (p. 20). Students frequently struggle in making this transition, and a brief glance at a small selection of the research literature will reveal some of the difficulties they encounter (Alcock and Simpson, 2004, 2005; Moore, 1994; Selden and Selden, 1995; Weber, 2001; Weber and Alcock, 2004). To help students overcome some of these difficulties, a number of researchers believe that a useful pedagogical strategy may be to encourage students to generate examples of mathematical concepts (Alcock and Simpson, 2002, 2004; Dahlberg and Housman, 1997; Hazzan and Zazkis, 1997;

Mason, 1998; Watson and Mason, 2001, 2002a, 2002b, 2005; Selden and Selden, 1998).

Dahlberg and Housman (1997) were interested in how a student's "concept image" (Tall and Vinner, 1981) initially develops when presented with a new definition. To this end, 11 undergraduates were presented with the following definition:

A function is called *fine* if it has a root (zero) at each integer.

After studying the definition, each student was asked to generate an example and non-example of a fine function, reformulate the definition, verify if some given functions were fine, and determine the truth of some given conjectures. When initially presented with the concept definition, students either generated examples, reformulated the statement, decomposed and synthesised the statement, or simply tried to memorise it. The authors recommend encouraging students to generate their own examples when presented with a new concept since they found that:

Students who consistently employed example generation had more learning events, were able to encapsulate more examples into their concept image of fine function, and were more able to use these examples than those who primarily used other learning strategies. (p. 295)

In a study of first year university students in a proof-based Analysis course, Alcock and Simpson (2002) identified two criteria necessary for students' success in mathematical reasoning – a good understanding of what objects belong to a given set defined by a formal definition; and, the ability and inclination to use the formal definition in presenting an argument. They suggest that example generation exercises will not only help students get a better idea of what objects belong to a given set, but will help them create a link between the objects and the formal definition.

In their investigation of how learners provide examples, Hazzan and Zazkis (1997) found that being asked to generate examples is a relatively difficult task for students. They recommend giving students this type of task since to construct an example, the learner has to engage with the concept's properties, and not just perform some routine operations that may not require an understanding of the underlying concept.

Watson and Mason (2005) propose that "examples can be perceived or experienced as members of structured spaces" (p. 51) and introduced the term "example space" to describe such a space. They view the extension and exploration of example spaces as being an essential element in learning mathematics:

Learning mathematics consists of exploring, rearranging, and extending example spaces and the relationships between and within them. Through developing familiarity with those spaces, learners can gain fluency and facility in associated techniques and discourse.

Experiencing extensions of your example spaces (if sensitively guided) contributes to flexibility in thinking not just within mathematics but perhaps even more generally, and it empowers the appreciation and adoption of new concepts (p. 6).

In order to promote the extension and exploration of learners' example spaces, a variety of example generation tasks and prompts have been compiled (Mason 1998; Watson and Mason, 2001, 2005). One of these example generation tasks will be further elaborated on in the next section.

That students experience difficulties when they first encounter formal definitions and proofs, will come as no surprise to anyone involved in third level mathematics education. As a university mathematics lecturer, I find research studies that document these cognitive difficulties extremely useful in explaining students' errors, and sensitising me to problems they are experiencing. I am particularly interested in the pedagogical implications of these studies, and since a number of those mentioned above, advocate the use of example generation exercises, I decided to systematically incorporate such exercises into my *Introduction to Analysis* course and evaluate students' experiences of completing them.

RESEARCH CONTEXT AND METHODS

Data Collection

This research was carried out in a second-year, second-semester course called *Introduction to Analysis* at a university in Ireland. This is a formal definition-and-proof based course that covers the main concepts and results in sequences and series of real numbers, such as convergence, completeness, and countability. Students attended 3 1-hour lectures and a 1-hour tutorial each week for 12 weeks. Data was collected primarily via semi-structured interviews with 12 student volunteers at the end of the course. These students were in the second year of a 3-year BSc degree in *Economics and Finance*, and had completed courses in Calculus, Advanced Calculus, Linear Algebra, and Financial Mathematics, where they had encountered some formal definitions of mathematical concepts and proof. All of the students interviewed had taken the Irish state examination – the Leaving Certificate – at the end of their secondary education (17-18 years old). Mathematics is offered at three levels in this examination and all of these students had been awarded honours in the highest-level.

Students were interviewed individually, and interviews, which were tape-recorded, lasted on average 30 minutes. The purpose of the interviews (conducted by research assistant Anthony Cronin) was 2-fold: to have students comment on certain aspects of the Analysis course; and to reflect on their experience of mathematics at university, and compare it to their school experience. Each student was asked the same questions.

64 students attended the course in total. All the students interviewed passed the course, and thus could be deemed to be “successful”. With the exception of one (Liam, who received a second class honour), all received first class honours (70% or

over), placing them in the top 45% of the class. In particular, Anthony, Brendan, and Ciara achieved over 90% in the course, placing them in the top 14%; Donal, Eamonn, and Fergus achieved a score between 80% and 89%, placing them in the top 27%. Being above the 75th percentile, these five students may be considered to be “high-achievers”. The remaining five students interviewed achieved a score between 70% and 79%. As the average score for the course was 64%, the views of the students in this study may be more likely to be representative of those of an “above-average” student.

Each student was asked how he/she found the *Portfolio of Examples* exercise. The responses were analysed for similarities and differences. Categories such as level of difficulty of the exercise, time taken to complete it, and perceived usefulness, emerged from the data. The open-ended nature of the question meant, that while certain students made interesting comments, it proved difficult when analysing the data to interpret how representative these were. A further study, using refined questions based on these responses, would be necessary to do determine this.

Portfolio of Examples

During the *Introduction to Analysis* course, students were asked to keep a *Portfolio of Examples*. Each week, for 6 weeks, they were given a list of exercises of the form “Give an example of...” and were required to generate examples that satisfied the given conditions. The requested examples were based on concepts that had been discussed both formally and informally on the course. Weekly lists of examples-requests were constructed based on the technique of finding “boundary examples” described by Watson and Mason (2001, 2005). This technique involves asking students for an example of Concept A, say. Then asking for an example of Concept A that satisfies Constraint 1. Then asking for an example of Concept A that satisfies Constraint 1 and an additional constraint, Constraint 2, and so on if desired. Once this list of examples has been generated, one then works back up the list and generates an example that satisfies all but the next constraint on the list.

The following are 2 examples of examples-requests taken from the *Introduction to Analysis* course. Further examples are given by Watson and Mason (2001).

Please give an example of a bounded sequence; a bounded, null sequence; a bounded, null sequence that is monotonic; a bounded, null sequence that is monotonic and contains (0) as a subsequence; a bounded, null sequence that is monotonic and does not contain (0) as a subsequence; a bounded, null sequence that is not monotonic; a bounded sequence that is not null; an unbounded sequence.

Please give an example of an infinite set that is bounded above by 10; an infinite set that has least upper bound 10; an infinite set that has maximum 10; an infinite set that has least upper bound 10 but does not have maximum 10; an infinite set that is bounded above by 10 but does not have least upper bound 10; an infinite set that is not bounded above by 10.

Typically the students completed 3-4 exercises of the above type each week for 6 weeks and collected their examples in a portfolio. They were required to hand-up the completed portfolios towards the end of the semester for grading, and the mark received constituted part of their final grade. Due to the relatively large size of the class, examples were not graded on a weekly basis. Students were allowed to work together to generate examples, but if they did so, they could not give the same examples in their portfolios (unless this was impossible). Students were encouraged not to leave blanks in their portfolios – if they could not think of an example, then this meant that they hadn't understood the concept fully and were advised to talk to fellow students, the course tutor, or the lecturer. The lecturer and tutor when approached gave hints to students but tried never to directly give them example.

Initially some students were hesitant that they were completing the exercise “correctly”. Therefore after the first two weeks, they were given the option of handing up their first attempts and receiving informal feedback from the lecturer. This not only reassured the students who chose to avail of it – it enabled the lecturer to give the whole class some feedback and advice on common errors, particularly relating to mathematical notation and representation of mathematical objects.

STATEMENT AND DISCUSSION OF RESULTS

In the interviews conducted at the end of the course, each of our 12 student volunteers was asked for their thoughts on the *Portfolio of Examples* exercise. All the students were positive about the exercise. Jackie felt that a lot of the theory “seemed to be up in the air”, but that the exercise “kind of brought it all together”. Karl said that as a result of the exercise he felt he had a “load of examples” in his head and that the definitions “kind of stuck in your head”.

As Watson and Mason (2005) propose, there was evidence that the exercise had encouraged the formation, exploration, and extension of student's examples spaces. Ciara indicated that it had enabled her to explore concepts more deeply.

Ciara: It's grand kinda thinking that you understand something but when you actually have to come up with stuff and construct stuff, it makes you, it makes you see like, even having a $2n$ instead of an n , it just, it makes you see where like bounded and ... It just, it made them more, it made the stuff in the course more, I don't know... You understood them more after having to do that – definitely.

In his response, Ian describes the extension of an example space:

Ian: You had all the definitions and when, you know, when you do a concrete example you can see what, or you do understand in some ways. It gets you to delve into the definition 'cause, em, you're thinking 'Oh'. Like some of the ones like you'd have to think about for a while, like the examples. And you'd finally find one and you'd realise 'Oh, that's another aspect to the definition'.

The findings also suggest that not only does an exercise like the *Portfolio of Examples* have a role to play in helping students engage with formal mathematics by exploring and extending their example spaces, it does so in a way that mirrors some of the study practices they have engaged in, and been successful with, in secondary school. Consequently, it has a role to play in gently immersing students in formal mathematics and training them in good mathematical thinking practices in a way that is both novel and yet familiar. Before elaborating further on this, the mathematical study practices of our students at secondary school are discussed.

Study practices in mathematics at secondary school

On reflecting on their experiences of mathematics at secondary school, the students in the study generally described it as a subject consisting of a collection of formulae, techniques and procedures that had to be practised and mastered.

Anthony: Then back in school it was just monkey-see, monkey-do type of thing.

Fergus: Like in Leaving Cert. and Junior Cert. it's kind of, you know, it's more a method to do what you do. You apply it. You get, you know, you get your answer. You learn off the method.

This finding is supported by a study on the teaching of mathematics in Irish secondary schools at the Junior Certificate Level (15-16 years old) (Lyons *et al.*, 2003). This study found that the two predominant pedagogical practices in the classrooms observed, were teacher demonstration and student practice of problems.

In another on-going study by the author, 11 first year mathematics students at the same university were asked in interviews how they had studied mathematics at secondary school. All said that they had “practised” problems, or practised problems from previous Leaving Certificate examination papers, or both.

We can conclude that the study practices engaged in by our students for 5-6 years in secondary school largely involved mastering and practising techniques and procedures. One side effect of this practice is that it is very easy for students to assess their own understanding – if you can complete the set of exercises correctly, then you feel that you have understood. Another side effect of this practice is that when a technique is applied correctly and the right answer is obtained, one could say that the student experiences an internal reward – a “yes!” moment - and there is a sense of satisfaction and achievement.

Boaler (2002) suggests that not only does a student's knowledge influence her mathematical capability in approaching a new problem, but also the practices in which she has engaged in learning that knowledge. A student who has engaged in the study practices described above will most likely find it difficult to make sense of the mathematical theory she first encounters at university. Her idea of studying is completing a set of exercises. She feels she has understood if she completes the set correctly, and she obtains satisfaction and a sense of achievement from this.

Portfolio of Examples – a novel exercise

The students in this study had never completed an exercise like this before. They didn't seem daunted by being asked to generate examples – their main source of uncertainty was whether they were presenting their answers correctly. For example, how should one represent a sequence? - As a list of numbers or by giving a formula for the n th term? Should one represent a function algebraically or graphically? Some students took the opportunity to “show-off” with their examples, while others were creative with the presentation style of their portfolios.

Portfolio of Examples – a familiar exercise

There were features of the exercise that appealed to the students and perhaps this was because of their similarity to the study practices they had engaged in at secondary school. After studying and performing well in mathematics at university for 2 years, Donal made the decision not to take any more mathematics courses in his final year because “This theory is gone beyond me like”. His problem seemed to be that despite studying mathematics for 2 years he hadn't developed good practices for making sense of formal mathematics. For him the examples exercise was “perfect”, because it actually enabled him to study the course material in a meaningful way.

Donal: You'd go through your notes like, 'cause you had to go through your notes as well, so at the same time you were kind of studying, but at the same time you were kind of doing an exercise so, you know, you're writing. And the one thing you're writing, you're writing down numbers, which is to me the key like. I have to know numbers. [*Gently bangs on table*].

Donal differentiated between studying and doing an exercise. At secondary school studying equated to doing exercises, but in university studying involved making sense of the notes – something that he had difficulty doing. He enjoyed the fact that the example generation exercises gave him the chance to work things out while he studied, and allowed him to play around with concrete examples.

Donal: I like to know that there's numbers there and I can work things into it like, you know what I mean, play around with things.

What Donal is describing is good mathematical practice – a mathematician would most likely behave in this way when trying to make sense of a new concept.

By completing an exercise that involves playing around with examples and “actually doing things”, Donal is engaged in a familiar practice, yet is making sense of the theory in a way that he hasn't before. He felt the exercise enabled him to “put a kind of mental picture” in his head. He indicates that he has begun to change his study habits when he remarks that he has started to make notes while he studies Analysis. He finds this rather strange because before it was “just sums and stuff”. It is unclear whether the example generation exercises are responsible for this.

Anthony, who came first in the *Introduction to Analysis* course, said that despite his best efforts in some of his previous university mathematics courses, he was sometimes left with a feeling of not quite understanding some of the theory.

Anthony: You might kind of having an idea and go ‘Oh that looks like it make sense’ but not quite getting down to what it meant.

For Anthony the *Portfolio of Examples* exercise, gave him the opportunity to test his understanding and give him a sense of achievement that he understood the concepts.

Anthony: Like you kind of knew that you understood it then because you got, you got through them all and you’d be like ‘Oh, that’s grand, that’s grand’.

Portfolio of Examples – more than an exercise

Anthony’s last comment requires further discussion. Some contradictory comments emerged with regard to the amount of time it took to complete the exercise, and how difficult the exercise was perceived to be. One student claimed you could complete each weekly set of examples in at most 20 minutes, whereas another student said she had spent 3-4 hours on one of the sheets. Similarly there were discrepancies between how easy or difficult the exercise was perceived to be. This could be explained by student ability, if it hadn’t been for the fact that some of the high-achievers said they had spent a lot of time on it.

A key lesson must be learned here. Anthony and Brendan both commented on the fact that after completing the weekly example generation exercises, they often reviewed the work, either by themselves or with a friend. Both commented about they often found what they had initially written was incorrect.

Anthony: You might go and see one of your mates and he’d be like ‘What’s this one?’ or whatever. And then you’d have to think about it and eventually like it helped you kind of clear ... clear up what was going on.

Brendan: And the amount of times you found yourself wrong is surprisingly large like. You actually go through it and you go ‘Oh, hold on a minute, no, that’s continuous. Oh no’. You go back and you solve it again. So it’s kind of good.

Validation of the work is a key aspect of this exercise, especially when there isn’t usually one correct answer for the examples requests. From their comments it was clear that the process of reviewing their examples was often responsible for extending their example spaces. As Bill et al. (2006) remark: “A succession of examples does not add up to an experience of succession”.

CONCLUSIONS

As a lecturer I found the *Portfolio of Examples* exercise to be a useful pedagogical tool. It encouraged the students engage with the formal theory in a novel, yet familiar way, and provided them with an example of good mathematical practice. It also enabled me to detect students’ difficulties (Watson and Mason, 2005).

A key finding of this study is that it is extremely important that students validate their examples. In order to encourage them to do this, it may be useful in future to have students correct each other's work in small groups. This could help detect errors in their thinking, and encourage debate of the concepts.

What one would really like to ascertain is whether or not example-generation exercises like those described in this paper help students learn analysis more effectively in a measurable way. This question cannot be answered by the interview data and is beyond the scope of this study. However it is a key question for future research. A detailed analysis of students' portfolios may be useful in this respect. It may also be useful to analyse students' responses throughout the semester to various tasks. For example, proof production tasks or tasks requiring the student to decide if statements are true or false, may be particularly useful.

Not only do we want students to have the ability to generate examples, we also would like them to have the inclination to do so (Selden and Selden, 1998). It is not clear that they are more inclined to generate examples after completing such an exercise and maybe one can't expect this after just one semester. Perhaps if students are encouraged to engage in example generation in all their advanced mathematics courses, then they may come to naturally adopt it as mathematical best practice.

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UNDERSTANDING OF SYSTEMS OF EQUATIONS IN LINEAR ALGEBRA ¹

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Linear Algebra make use of the concepts involved in the solution of linear systems of equations; that is why this topic is important in such a course. In this study six students who were taking a course based on APOS theory were interviewed at the beginning of the course and at the end of the course to study the viability of a proposed genetic decomposition, their difficulties, their reasoning pattern and the evolution of their schema (as defined in APOS theory). Results show that an important prerequisite for the students to profit from the opportunities the course offers to deepen their understanding of linear systems of equations depends strongly on the development of their schema for variable, and that a course based on APOS theory helps students in the evolution of their systems of equations schema.

Key words: linear algebra, equation, systems, APOS, schema.

I. INTRODUCTION

Linear systems of equations is a topic that students have encountered in their mathematics classes since secondary school. During linear algebra courses students have the opportunity to generalize students' knowledge from systems of two equations to systems involving more than two equations and to be introduced students to the use of matrix techniques in the solution process.

Knowledge about linear systems of equations is used and needed throughout the whole course of linear algebra in the solution of different kinds of problems and in the understanding of most of the concepts related to this subject.

It is known that even advanced students have difficulties understanding the concept of solution to a system of equations (Cutz, 2005) and that they also face difficulties when deciding the number of solutions of a given system, when representing those solutions geometrically, when interpreting graphs related to systems of linear equations and when trying to pass from one representation to another (Ramírez, et al., 2005).

The identification of the nature of these difficulties and their association with the way with which students construct the concept of solution of a linear system of equations is of great importance in the developing and implementing good instructional strategies. It is also important to study if a course in linear algebra helps students in their understanding of this kind of systems and how their use along a linear algebra course helps students to build the necessary relations between different concepts

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related to linear systems. APOS (Action-Process-Object-Schema) Theory provides a research tool that has been successfully used in other areas of mathematics such as abstract algebra and calculus, for similar purposes.

This study focuses on students who have taken a course based on APOS theory, that did not include the topic of systems of linear equations explicitly (except for a brief geometric treatment). We center our attention on the evolution of students' learning of the concept of systems of linear equations throughout the course and the conceptual needs that this evolution entails.

RESEARCH QUESTIONS

This study is part of a larger project on linear systems of equations based on APOS Theory. The research questions addressed in this particular study are:

- Is the preliminary genetic decomposition viable for this concept?
- What are the difficulties found by students when they are learning systems of linear equations?
- What are the reasoning patterns followed by students when they work with problems related to linear equations?
- Given that systems of equations is a recurrent theme during the whole course, are there changes in students' notions of solution of a system of equations from the beginning to the end of the course? If not, what are the conceptual needs of the students to be able to understand the concepts associated with the solution of linear systems?

THEORETICAL FRAMEWORK

In order to study in depth the nature of students' understanding and to identify sources of difficulties in students with respect to the evolution of solution to a linear system of equations concept, we designed an empirical research study based on APOS Theory (Action, Process, Object, Schema) (Asiala, et al., 1996; Weller, et al., 2003; Dubinsky and McDonald, 2001) which followed the steps that we describe here:

We developed a possible genetic decomposition of solution to a linear system of equations and its possible evolution, which consists of the constructions the researcher thinks that students should make in order to construct the concept and those coordinations between them that underlie the evolution of the defined schema, and we analyzed activities that students can perform to make the necessary mental constructions. The notion of schema has been used in several studies (Mc Donald et al., 2001; Baker et al., 2001; Trigueros and Oktaç, 2005), and its definition in APOS Theory is as follows:

A schema for a certain piece of mathematics is an individual's collection of actions, processes, objects and other schema which are linked consciously or unconsciously in

a coherent framework in the individual's mind and may be brought to bear upon a problem situation involving that area of mathematics. An important function and defining characteristic of the coherence is its use in deciding what is in the scope of the schema and what is not (Asiala et al., 1996).

According to this theoretical analysis, the schemas an individual must bring to the study of systems of equations are set, function, equality and vector space. This means that understanding of systems of linear equations in the context of Linear Algebra requires that the individual should have constructed coordinations between the actions, processes, objects and other schema that are considered in the construction of each of them. Equation and function objects are coordinated into a function that verifies if a given tuple is a solution of a given equation. This process is encapsulated so that it becomes possible to consider the set of all possible solutions for a given equation.

The equation, set and solution schemas are coordinated to construct a process that takes the intersection of the solution sets of two or more equations in a system. This process is then encapsulated so that it becomes possible to compare two systems in terms of their solution sets, to study their properties and to interpret the systems geometrically when possible.

Schemas for equality and equations are coordinated to construct a process that transforms an equation into an equivalent one. This process is coordinated with the schema for systems to construct a process to find an equivalent system of equations, and a process to determine the solution set from this equivalent form. This process of finding the solution set of a system of equations is encapsulated into an object, and then it is possible to study its properties and to relate it to its geometric interpretation.

Also, the process of constructing an augmented matrix and considering it as the representation of a system is encapsulated so that it becomes possible to perform row operations on the matrix to be able to find the solution set from the reduced form and to compare solution sets associated with different augmented matrices.

METHODOLOGY

A description of the methodology of this part of the study follows:

I. Observation of an introductory Linear Algebra course during one semester at an undergraduate institution in Mexico, in which engineering majors were being taught using the pedagogical component of APOS framework. This pedagogy is based on the ACE (Computer Activities – Class Discussion – Exercises) cycle and the CAS Maple was used to handle the computer constructions (for a detailed description of this cycle, see (Asiala et al., 1996). The textbook chosen was one written by RUMEC (Research in Undergraduate Mathematics Education Community) (Weller et al., 2002).

II. Two interviews with 6 students from the same course about the notions of linear systems were conducted. The first interview consisted of 13 questions on systems of linear equations; it was done after the first exam of the course (three weeks had passed since the beginning), the second interview consisted of 11 questions where students had to consider systems of equations in the context of other linear algebra concepts. It was performed at the end of the course. Two of the students were identified as belonging to the top level according to their grades on the first exam, two of them as performing at the medium level, and the remaining two were classified as being poor, but otherwise they were chosen randomly within these three subgroups.

In this study we present the analysis of two questions, one from the first interview (numbered 8 in the interview), and one related to the same concepts from the second interview (numbered 1 in the instrument) together with our *a priori* analysis of them and related student performance. We chose these questions because they are representative of the general results that are being obtained in the project.

Question 8, first instrument. Consider the system of equations $\begin{cases} x + 2y + 3z = 3 \\ 2x - y - 4z = 1 \end{cases}$. How many solutions does the system have? Find the solution set. If we graphed the solution set, what would be its geometric representation?

A priori analysis of Question 8: This question considers a typical system of equations with an infinite number of solutions. Students need to coordinate their schema for solution with that of set and interpret geometrically the solution set. If students have a weak algebraic background they will show difficulties in separating the object *solution* from the process of finding the solution. They will probably find it difficult to think about a line in \mathbb{R}^3 . Students who have the notion of solution as an object or schema may find it difficult to represent it geometrically.

We are interested in observing whether these difficulties are found in students at this level, and the strategies students follow in their responses. We would also like to identify those schema that they are able to coordinate.

Question 1, second instrument. Let $H_1 = \{(x, y, z) : 2x - y - 4z = 0\}$ and $H_2 = \{(x, y, z) : x + 2y + 3z = 0\}$ be two subspaces of \mathbb{R}^3 . Find their intersection. What does this intersection represent geometrically? Is the intersection a subspace of \mathbb{R}^3 ? Justify your answers.

A priori analysis of Question 1. In this question we tried to see if students can associate the problem of finding the intersection of the given spaces with a system of equations and solve it to find the intersection. We also expect students to be able to interpret geometrically subspaces of \mathbb{R}^3 . We expect that the students who have a very weak algebraic background will find this question difficult, since the notions of solution, function, variable and set are considered a prerequisite in the understanding of new more abstract concepts. We also expect that some weak students will profit

from the opportunities offered by the course to reflect on the notions of solution, function and set, to construct them and to use them in the construction of new linear algebra concepts. In the case of students who have a good understanding of the prerequisite concepts, we expect to see an evolution of the coordination of schemas needed to understand the concept of solution to a linear system, and to be able to apply it in new settings.

SOME RESULTS

In this section we summarize students' performance related to the two interview questions. We found that we can divide the students in two groups according to their responses to the question 8 in the first interview; those who show a good background from elementary algebra courses, and those who do not show such background.

Three students were classified in the first group. Their answers showed that they could interpret variables in the expression as unknowns to be found, and later as related variables (Trigueros and Ursini, 2003), and that they had an understanding of the meaning of *solution to an equation* as an object; two of them showed the interiorization of actions into a process that generalized this understanding to the notion of solution to a system of equations, and showed the possibility to coordinate solutions to different equations into an understanding of solution set associated with it. Examples of responses follow:

J: "The system has an infinite number of solutions given by these expressions (shows solution obtained: $y = \frac{5-10z}{5}$, con $z = t$, $y = \frac{5-10t}{5}$ $x = 3 - 2\left(\frac{5-10t}{5}\right) - 3t$). The solution set... well it would be like this $\{(x, y, z) | x = \dots, y = \dots, z = t\}$ and then x and y are given by these expressions and $z=t$."

We note that this student did not collate his results as a set, opposed to student M:

M: (after working on a procedure to solve the system and committing an error in a sign, she writes: $S = \{(x, y, z) \in \mathbb{R}^3 | x = -1 + t, y = 1 - 2t, z = t, t \in \mathbb{R}\}$)

These students could coordinate the solution and set schemas to interpret the system as such and could perform the process of reduction of the system to find its solution. They could coordinate the given equations to their geometric representation and could also coordinate the solution set to its geometrical representation. The responses from the other student in this group show he had not fully interiorized the actions needed to construct the process that permits to generalize the *solution to an equation* concept to that of *solution to a system of equations*, but he was able, with some

difficulty, to coordinate the equations with their geometric interpretation and perform actions on the equations to solve the system and find a possible solution. Here we present part of this student's work:

W: "With the augmented matrix... this is the solution

$$\begin{aligned} y &= 1 - 2z \\ x + 2y + 3z &= 3 \\ x + (2 - 4z) + 3z &= 3 \\ x &= 1 + z \end{aligned}$$

and then... $\begin{matrix} 1 - 2z + 2 + 2z + 3z = 3 \\ 3z = 0 \end{matrix}$ (here we observe that he committed an error in calculation), and this should be a line, because up here I can give any value to z , but in this second set of equations there should be something wrong, but, ...I don't know.... If I do it again... This are two planes and the intersection is a line given by $y = 1 - 2t, x = 1 + t \dots$ the solution set is $\{(x, y) | x = 1 + t, y = 1 - 2t \quad \forall t \in \mathbb{R}\}$ "

We also observe that W presents his final solution as a set in \mathbb{R}^2 , which shows that the coordination between the process of solution of the system is not fully coordinated with the geometric representation of the solution.

The other three students did not understand the meaning of solution, or solution set, had difficulties differentiating the different uses of variable (unknown, general number and related variables) in the expressions employed (Trigueros and Ursini, 2003), and always tried to find memorized answers to the questions. They did not show any evidence to have constructed the prerequisite schemas mentioned above, nor to have interiorized the actions needed in the solution of the system or to have coordinated those actions with the geometrical representation of equations. As there was no meaning attached to their answers they showed plenty of difficulties with the solution of the system of equations and with the geometrical interpretation of the equations. Examples of responses follow:

S: " (After doing some algebraic procedure in which she has errors of a conceptual nature in algebra). The solution is this. The solution set... I think is $x = 3 - 2y - 3z$.

$$\begin{aligned} x &= 3 - 2y - 3z \\ y &= \frac{-5 + 10z}{-5} \\ -5x &= -5 + 25z \\ \frac{-5x + 5}{25} &= z \end{aligned}$$

... I think it is..., the... geometric interpretation? A line, I think, but... well it really can be other things... but has many solutions... I don't know I would have to write the matrix... but it is not squared...I really don't remember that"

L: “ I don’t remember, I think these are planes, but I really don’t know, something in \mathbb{R}^3 ...a point, or a cylinder or something else... the solution set?... if one cuts the other there should be one solution.”

Results from the question of the second interview and their final exams show that the two students who demonstrated they had constructed a schema for the notion of solution of a system of equations, recognized the system in the question and could solve it, and interpret the solution with no problem; considering all the work analyzed, and not only the questions reported here, they also demonstrated an evolution in their notion of solution set. They were able to relate the object *solution to a system of equations* to that of *vector space* to be able to show when the intersection of two subspaces was or was not a subspace of \mathbb{R}^3 .

The student who didn’t show he had encapsulated the object *solution to a system*, had more difficulties solving this question. He was able to recognize the question as related to a system of equations and to give the geometric interpretation of each of the given subspaces, showing that he had coordinated the object equation and the object solution set between them and the relationship between equations and their geometrical representation. He could solve the system but had difficulties to interpret the solution geometrically, that is, he did not show evidence of coordination between the object solution set in an analytical representation and in a geometrical representation. He also was not able to relate the solution set of the system to the concept of vector subspace.

W: “The solution set is a line, I think , not sure... let me see , yes it is a line... and I know it is a subspace because these two are subspaces and their intersection goes through the origin so it is a subspace.”

From the group of students who had a weak understanding of elementary algebra only one showed some evolution in her systems of equations schema. She was able to identify the geometric representation of the given subspaces and to consider their intersection as a line because the planes are not parallel, and to associate the solution set with a subspace, although she had problems remembering which were the necessary conditions she had to verify for a subset of a vector space to be a subspace.

L: These are planes. They intersect in a line because the normal, the vectors are not parallel, and there are only two planes... the solution set, the line is a subspace because it is contained in one subspace and goes through the origin.”

The other two students still showed many problems with the notion of variable. Although they both used in their explanations “new words” related to the concepts introduced in the course, they clearly demonstrated they had not associated any

meaning to those concepts and could not even find the solution to the system of equations. They are probably at an action level regarding the different components of the schema, and apparently unable to coordinate any of them.

M: “I have to subtract, no... (after some manipulations) this is the solution

$$\begin{pmatrix} 2x - y - 4z = x + 2y + 3z \\ 2x - x - y - 2y - 4z - 3z \\ x - 3y - 7z \end{pmatrix}.”$$

We note these three students thought they had to add or subtract the two given equations and that would be enough to represent their intersection.

CONCLUSIONS

Results of this study clearly show that the construction of a schema for variable that includes the interpretation and differentiation between the different uses of variable: unknown, general number and variables in functional relationship, and the understanding of solution of an equation as an object constitute necessary conditions for students to make the constructions that are needed in order to construct the systems of equations schema. The lack of those previous notions in the interviewed students seems to interfere severely with their possibility to make the constructions needed to understand new abstract concepts. The worlds related to mathematics that students use when they describe their work when solving problems, as those presented to them in this study, can become more sophisticated as they progress in their mathematics courses, they may include those new terms that are introduced in a linear algebra course, but they are unable to interpret the meaning of their words, probably because they have not made the necessary constructions to understand them as processes or objects.

One student showed some evolution of her schema for equations and for that of solution. This is evidence that some conceptual evolution is possible when students follow a pedagogy where they have to do actions on mathematical objects, but the evolution was insufficient for the student to fully understand the previous concepts.

Students who showed a good understanding of elementary algebra, in particular of the different uses of variable, showed an evolution in the constructions related to the systems of equations schema. Two of them were able to use appropriately their schema in the solution of a problem posed in a different context within Linear Algebra, the other showed an evolution in his system schema since he could identify the problem and interpret it geometrically, although this evolution was not enough for him to encapsulate the notion of solution to a system of equations. The analysis of these two questions has still to be complemented with the results from the analysis of the other questions from the interview to be able to conclude about the evolution of students' schema for systems of equations.

We expected to find that following a Linear Algebra course that made use of the ACE cycle, and that used activities designed taking APOS Theory as a basis together with the use of finite fields and computer activities, would enrich these students' mathematical understanding. We expected that the activities in the course would allow them to reflect on their constructions while solving problems, but that only happened partially. In fact, at least two students profited from the course in the sense expressed above, and there is some evidence of evolution in the schema of other two students. We hope that detailed analysis of the whole course and the students' answers to the other questions in the interview will shed more light into students' difficulties and their possible conceptual evolution. We also hope that a more complete analysis will allow us to make didactical suggestions to improve the design of the genetic decomposition of this concept. At the present moment, we can only affirm that a good understanding of elementary algebra seems to be really important for the students to learn concepts related to systems of equations and linear algebra.

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STUDENTS' CHOICES BETWEEN INFORMAL AND FORMAL REASONING IN A TASK CONCERNING DIFFERENTIABILITY

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Mathematical reasoning often includes both informal and formal elements. In this paper we study subject-teacher students' choices between informal and formal methods in a problem solving situation. Most of the students chose an informal approach for a primary method, even if results of a written test suggested that many of them would have had a potential ability for a formal approach, too.

Key words: advanced mathematical thinking, argumentation, definition, derivative, informal and formal, mathematical reasoning, problem solving, visualisation.

INTRODUCTION

Formal mathematical results can often be argued either formally or informally: They can be proved by using definitions, axioms, previously proven theorems and formal language, or they can be justified by using visual or other more concrete interpretations. A formal proof is usually demanded for the final form of the argument, but more informal explanations can in several cases help an individual to understand, even on a more conceptual level, why a claim is really true (Raman, 2002 & 2003). Informal arguments also have an essential role in thinking and problem solving processes of professional mathematicians (Raman, 2002; Stylianou, 2002).

In this paper we investigate university students' use of informal and formal reasoning in the case of the concepts of derivative and differentiability. The main goal is to study the students' preferences between informal and formal methods in a problem solving situation. The students were Finnish subject-teacher students in mathematics and they were mainly at the final phase of their studies.

WHAT IS FORMAL AND WHAT IS INFORMAL IN MATHEMATICS?

Nowadays the mathematical knowledge as a whole is considered as a very abstract and rigorous system. At least in principle, the formal definitions and formal arguments have the conclusive role in determining what finally is true and what false in mathematics. So, the truth value of a mathematical proposition is –in principle– fully independent of individual preferences or interpretations, social conventions, negotiations and subjective conceptions (Goldin, 2003). However, in practice, mathematical proofs, made by professional mathematicians, are not fully formal, but the social factors have an important role in the acceptance of a proof (Hanna, 1991; Dreyfus, 2000). This means that it is more essential to ask whether the proof convinces the mathematical community than whether it is fully formally correct.

Mathematically correct formal proofs are usually not very illuminative. Especially, for students having little experience in formal mathematics, it can be difficult to understand why the claim is true alone on the basis of a formal proof, and thus more explanatory arguments are needed. Raman (2002, 2003) defines that a private argument is an argument engendering understanding and that a public argument is an argument having sufficient rigour for a particular mathematical community. A private argument can be based on visualisation or on other informal aspects, whereas a public argument is usually based on definitions, axioms and previously proven theorems. Rodd (2000) presents a corresponding division: He divides arguments in warrants and justifications. According to his terminology, a warrant is an explicit mathematical argument, which is needed to convince a person of the truth of a mathematical proposition, to convince that the proposition is part of the common mathematical knowledge. By a justification Rodd means a personal reasoning which is needed for securing personal beliefs in the truth of the proposition. He also states that a formal proof is often not a sufficient justification for students that a claim is true. On the other hand, according to Rodd, a mathematical warrant can also include other factors besides elements of a formal proof. He shows that, in some situations, mathematical warrants can be visual. Arcavi (2003), furthermore, proposes that visualisation “can be an analytical process itself which concludes with a solution which is general and formal”. According to Dreyfus (1994), the results of cognitive science have shown that visual reasoning has powerful potential in the learning of mathematics and visualisation has a very essential role in the epistemology of mathematics. Dreyfus proposes that visual reasoning “should be developed into a fully acceptable and accepted manner of reasoning, including proving mathematical theorems”. However, he says that “better judgement of visual arguments should be developed”.

Therefore, it is not possible –nor appropriate– to define absolute criteria for one to judge which factors of mathematical thinking, expressions and outputs are formal and which are informal. Rather, it can be thought that the level of formality varies from fully informal to fully formal. In this paper, the word formal is used to refer to factors concerning the formal axiomatic system of mathematics (definitions, axioms, theorems and so on), and the word informal refers to all other factors connected to mathematics and mathematical thinking. The informal factors are often connected to visualisation, but they also can be based on other representations. For instance, Goldin’s (1998) classification of representations brings well out the variety of different aspects in mathematical thinking. Formal arguments (proofs) are often very strict, but the strictness and foundations vary among the informal arguments. Harel’s and Sowder’s (1998) classification of proof schemes reveals this variety.

RESEARCH QUESTIONS AND METHODOLOGY

The research questions of the study can be stated as follows:

1. To what extent do the students use informal methods and to what extent do they use formal methods when solving a problem concerning differentiability?

2. How are these choices related to students' potential abilities to consider the subject matter informally and formally?

The study presented in this paper is part of a larger study, whose goal was to study the relationship between informal and formal reasoning in the case of the derivative. The data of the whole study consists of a written test and interviews of some participants of the test. Altogether 146 subject teacher-students in mathematics took part in the written test. They had all passed at least 20 Finnish credits (about 30-35 ECTS credits) in mathematics, and, therefore, they were either in the middle or at the final phase of their studies.¹ In total, 21 participants of the written test were interviewed a couple of days after the test. The interviewees differed from each other with regard to their performance on the test, with regard to the amount of passed studies in mathematics and with regard to their success in studies in mathematics. The questions in the test and in the interviews considered continuity, the derivative and differentiability. The formal definitions of these concepts were given to students both in the test and in the interviews. The interviews were videotaped so that the video camera was focused on the paper.

The choices between informal and formal methods are studied by analysing the interviewees' solution processes in the case of the following problem:

$$\text{Let } f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = \begin{cases} x^4 \cos(1/x^3), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Is the function f differentiable?

Special attention was paid to the differentiability at the point $x=0$. Due to the lack of time, this task was not considered in the interviews of two students. In addition, in the case of one student, this task was considered only at a superficial level. Thus, the analysis is based on the interviews of 18 students.

The analysis of the solution processes especially focused on the distinction between informal and formal methods. In this case the method was classified as formal if it was based on an explicit use of the definition. Using the type of the method as a criterion, students' performances were divided into episodes so that an episode was considered to end if a method was changed from informal to formal or conversely. Every episode ended either without a solution or with having a proposed solution. In the latter case, students' convincedness of the proposed solution varied. If an episode finished without solution or if an achieved solution did not convince the student,

¹ In Finland the minimum requirement to achieve a qualification to teach mathematics in secondary school was to pass at least 35 Finnish credits mathematics at the university level if a student was not majoring in mathematics. The students majoring in mathematics had to pass about 65-75 Finnish credits depending on the university. Nowadays, ECTS credits are used in Finland.

he/she usually changed the method. An analysis was carried out on the basis of the video recordings.

The estimates about students' potential abilities to consider the subject matter (in this case the derivative and differentiability) informally and formally are based on their answers to the following tasks of the test:

1. a) How would you explain, by using visual interpretations, why the derivative of a constant function is everywhere zero?
b) Prove the same by using the definition of the derivative.
2. Claim: Let $f : R \rightarrow R$ be a differentiable function and a given point $x_0 \in R$ be fixed. Then

$$\lim_{h \rightarrow 0} \frac{f(x_0) - f(x_0 - 3h)}{3h} = f'(x_0).$$

- a) How would you argue this claim visually by using a diagram?
b) Prove the claim formally by using the definition.

In tasks 1a and 2a students were asked to argue a claim visually. However, some of the students correctly used other informal representations than visual ones in their answers. In tasks 1b and 2b, the students were asked to prove the same claims formally. The answers for each of these four questions were graded by giving each of them 0-2 points. A general principle in the grading was that two points were given if the answer was correct and well reasoned, and one point was given if the main idea was correct but the answer included some deficiencies or mistakes. The tasks 1a and 1b were meant to be simple and the tasks 2a and 2b more complicated to solve, and according to the results, this turned out to be the case. The points from questions 1a and 2a were added together, resulting in a variable which indicates a potential ability to argue claims informally. Respectively, the points from the questions 1b and 2b produced a variable which indicates a potential ability to prove claims formally. The values of these variables vary then between 0-4 and only integer values are possible.

It is quite justified to claim that a student who has received high points from the informal tasks has a potential ability to consider the derivative informally. His/her performance on the informal tasks shows that he/she knows at least one viable informal interpretation of the derivative and that he/she knows a principle how to build informal arguments. Respectively, high points from the formal tasks shows that the student knows the main principles of formal proving and that he/she knows how to use the definition of the derivative in proving. On the other hand, it is quite unreliable to make conclusions in the cases where a student received low points from the tasks: The low points do not necessarily mean that the student is unfamiliar with the above-mentioned issues.

The questionnaire also included questions concerning students' backgrounds. Because it was not possible to get data from student registers in the case of some

students, the necessary data concerning the total amount of passed studies in mathematics and the students' success in these studies was asked in the questionnaire. Thereby, these were the students' own estimates and thus not fully reliable. The success in studies in mathematics was estimated by asking the students to describe it verbally, and then the researcher classified these written descriptions into three classes: poor, average or good success.

DESCRIPTIONS OF SOLUTIONS

It turned out that the interviewees could be divided into four groups on the basis of the episode analysis of their solution processes. The division was made as follows: Class I consists of students whose solutions include only a formal episode, and Class II, for one, consists of students whose solutions do not include any formal episode at all but only an informal one. The solutions in the cases of the students in Class III began with an informal episode, and it was followed by a formal episode. Any other combinations of episodes did not come out. In the case of six students the foundation of the performance was totally incorrect, and thus these performances were not comparable with the others. These six students constitute Class IV, which was eliminated from the follow-up analysis. The interviewees' performances in each class are briefly described below. The classification as a whole is presented in Table 1.

Class I: Formal approachers

(3 students: Lydia, Keith and Sean)

The foundation in the performances by Lydia, Keith and Sean was the formal definition of derivative. All of them performed the following kind of calculation:

$$\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h^4 \cos(1/h^3) - 0}{h} = \lim_{h \rightarrow 0} h^3 \cos(1/h^3).$$

After that they concluded that because the values of the cosine were bounded to the interval from -1 to 1, the limit existed and it was zero. Thus they concluded that the function f was differentiable at the point $x=0$. None of these three students met serious problems during their performances and they all used less than seven minutes for this task. No informal interpretation seemed to have any explicit role in their reasoning. In addition, they did not show any sign to doubt their reasoning, apart from Keith, who was a little unsure about his calculations.

Class II: Informal approachers

(3 students: William, Scott and Victor)

It was common to all students in this class that they did not use the formal definition of the derivative at all when they considered the task of the interview. All of them sketched the graph of the function f , and this graph had an essential role in their

reasoning. William's and Victor's reasoning was mostly based on visualisation, and Scott used computations in addition to it.

Scott used almost 20 minutes for considering the symmetry and the continuity of the function f . Finally, he claimed that the function was not differentiable and he argued this by the frequent oscillation near zero.

Victor's criteria for the differentiability were continuity and the fact that there was no "very tight bend" in the graph. For him the sketched graph revealed that the limit of the function at zero was zero and that the function was thus continuous. In order to find out if the latter criterion was fulfilled, Victor tried to study the shape of the graph at zero. Finally, he said that he believed that the function was differentiable, but he did not present any proper argument for this.

William sketched the graph of the function, and on the basis of it he understood that the function f oscillated between the functions x^4 and $-x^4$, and thus the function was continuous. He also saw that the tangent line drawn to the graph at the point $(0,0)$ was a horizontal line coinciding with the x -axis. He argued this by studying a secant line going through the points $(0,0)$ and $(h, f(h))$. William reasoned that the secant line was in its steepest position when it touched the graph of the function x^4 , and therefore, when h went to zero, the secant line pivoted to a horizontal position. William's conclusion was that the function f was differentiable and the derivative was zero at the point $x=0$. William in fact did not seem to have any problems during his performance and he did not show any signs of hesitance of his performance. The whole process took less than eight minutes. After that, the interviewer asked William also to prove the claim formally. William managed to do this without any difficulties

Class III: Students who first tried an informal approach but later changed to a formal approach

(6 students: Robert, Nellie, Regina, Harry, Eric and Kathleen)

The students in Class III first tried to study the problem by using an informal approach, but, in this way, they did not manage to achieve a convincing solution. Finally, they switched to a formal method.

Regina, Nellie, Harry and Kathleen tried to approach the problem visually on the basis of the graph of the function. Regina sketched the graph and on the basis of it reasoned that at zero the function oscillated too much to be differentiable. Also Harry managed to sketch the graph, but he did not make any conclusion on the basis of it. Nellie, as a criterion for the differentiability, considered that the graph of the function did not include jumps or corners. However, she was not able to make any conclusions, because she did not manage to sketch the graph. Also Kathleen had problems in sketching the graph. Eric and Robert did not draw any graph, but their informal methods were more computational. On the basis of the expression, Eric tried to find out how the function behaved near zero. He was especially interested in the

continuity at this point. Robert tried to study the behaviour of the function by calculating its values at separate points.

All the six students used several minutes for the informal studies described above, but after that they began without any hints by the interviewer to calculate the limit of the difference quotient, according to the definition. Regina and Harry managed to carry out the calculation correctly like the students of Class I. Nellie had problems in calculating the limit for the expression $h^3 \cos(1/h^3)$ at zero and Robert reasoned the value of this limit erroneously: He remembered that the limit of $\sin(x)/x$ is one when x approaches zero, and he believed that a similar result is valid for cosine, too. He applied this, and, in addition, he made some errors in calculation. Kathleen met several problems in calculating the limit of the difference quotient. Eric calculated it correctly, but his argument for the differentiability was erroneous: He explained that the reason for the differentiability at zero was that the limit of the difference quotient at zero was equal with the value of $f(0)$.

Class IV: Students whose performance was based on an incorrect foundation

(6 students: Leonard, Henry, Matt, Ralf, Sabrina and Neil)

The students in this class studied the differentiability by criteria which were clearly in conflict with the formal theory. Ralf, Leonard and Matt thought that continuity was a sufficient condition for the differentiability. Sabrina's method was to find out if the derivative of the expression $x^4 \cos(1/x^3)$ had a limit at zero. Henry in a serious way confused the concepts related to the continuity with the concepts related to the differentiability, and Neil's performance was quite confused, too.

ANALYSIS OF THE RESULTS

In Table 1, the results concerning the students' performances in informal and formal solving attempts in the task of the interview are presented. The marking "passed" means that the student by this type of method received a solution which convinced him/her and, in addition, no part of the solution contradicted with the formal theory. Otherwise, the marking "failed" is used. Table 1 also includes achieved points from informal and formal tasks in the written test, the amount of passed studies in mathematics in university and estimates of the average success in these studies.

Among these 12 students, there appeared to be a strong tendency for informal reasoning. The task of the interview proved to be more difficult to solve informally than formally, but in spite of that only three students started by a formal method, while nine students preferred an informal method. The difficulty of the informal method compared with the formal method came out as follows: Of the nine students who attempted an informal approach (the students of Classes II and III), only one (William) managed to present a proper informal solution. Instead, five students (Lydia, Keith, Sean, Regina and Harry) managed in their attempt on a formal

solution, and only four students (Nellie, Robert, Eric and Kathleen) failed in it. In addition, the presented formal solutions for this task were relatively short.

		Interview		Written test		Background	
Student	Class	Informal method	Formal method	Informal points	Formal points	Studies in mathematics (in Finnish credits)	Estimated success
Lydia	I	–	passed	4	4	60	good
Keith	I	–	passed	3	4	119	–
Sean	I	–	passed	4	3	130	good
William	II	passed	passed	4	4	40	average
Scott	II	failed	–	2	4	100	average
Victor	II	failed	–	3	1	35	average
Robert	III	failed	failed	0	3	60	average
Nellie	III	failed	failed	2	2	52	average
Regina	III	failed	passed	2	4	59	good
Harry	III	failed	passed	3	4	38	good
Eric	III	failed	failed	2	4	55	average
Kathleen	III	failed	failed	1	0	40	poor

Table 1: Profiles of the interviewed students.

The tendency for informal reasoning is, furthermore, stressed by the following observation: On the basis of the results of the written test, many of the students who preferred an informal approach, would have had a potential ability for a formal approach. In fact, five of these students received four points from the formal tasks of the test. Among these five, William, Regina and Harry showed that they were able to solve the task of the interview formally, and Eric was the only one of them who failed in the formal solution of the task of the interview.

In Class I, the success in the written test had been good both in the informal and in the formal tasks. This indicates that these students had a strong potential ability to consider the concept of the derivative both informally and formally. The success of the students of Classes II and III on average in the informal tasks (average value 2.33) was not so good as that of the students of Class I (average value 3.67). Yet, the students of Classes II and III chose an informal approach for their first method, while the students of Class I chose a formal approach. Thus, the test results seemed not to have a determinant influence on the choice between the informal and the formal approaches. Instead, a general feature concerning the background knowledge about the students presented in Table 1 is that the students of Class I had studied a lot of mathematics (Keith and Sean very much) and their success in these studies had been good, whereas the students of Classes II and III had studied less mathematics or their success in these studies had been less good.

CONCLUSION

Some previous studies have shown that, in connection with basic analysis, students have difficulties in consulting formal definitions (Cornu, 1991; Juter, 2006) and often have a tendency to avoid using them (Pinto, 1998; Vinner, 1991). The results of this paper support the latter one of these claims: The students' choices of methods indicate that in the task of the interview nine out of twelve students preferred informal methods to the use of the formal definition. In addition, the results of the written test reveal that many of the students who chose an informal approach for their primary method would have had a strong potential ability for a formal approach as well.

Vinner's (1989) results concerning the choice between visual and formal/symbolic methods are in several ways contrary to the results presented in this paper. Among college students he found a tendency to avoid visual methods and visual arguments. Tasks in his written test were more easier to solve visually than formally, but in spite of that a significant part of the students chose a formal approach, whereas in my study the formal solution of the task could be assumed to be more straightforward compared with any informal solution, but most of the students preferred an informal approach. Conceptions about the form of an acceptable solution might be one notable factor influencing this distinction between the results: Maybe the informal solutions were considered more acceptable in an interview than in a written test.²

Because the students of Class I had studied a lot of mathematics with a good success, it is probable that they had more experience in working with the definitions, and their self-confidence in their abilities to use definitions was better than that of the students in the other classes. According to the data used, this seemed to be a more crucial factor influencing the choice between informal and formal approaches than the potential ability to consider the subject matter informally and formally. However, a substantially larger number of students would be needed to study quantitatively such dependencies as these.

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2 In the questionnaire used in Vinner's study it was told that both visual and algebraic solutions are acceptable.

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INTERPLAY BETWEEN RESEARCH AND TEACHING FROM THE PERSPECTIVE OF MATHEMATICIANS

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In this paper, we examine the relation between teaching and research on mathematics in universities. We suggest that this relation can be fruitfully examined from the perspective of mathematicians' praxeologies (organisations of didactical and mathematical practice). We illustrate the approach with data from an interview study involving five top-level mathematicians.

INTRODUCTION

One of the specificities of higher education is that scientific research is “close by” in several ways: through teachers who are also researchers; as a future career option for students; and because the contents and methods taught can be conceived as “closer” to actual research than pre-university education. The last point is particularly strong in the case of mathematics, where the school subject is essentially based on mathematical knowledge developed before 1900. The distance between “academic mathematics” and “school mathematics” is manifest and the study of their relation is a classical theme in the didactics of mathematics (e.g. Chevallard, 1985; Brousseau, 1986, 33ff). On the other hand, research activity is often used as a kind of ideal for pupils' exploration of school mathematics, both in the mathematics education literature (we return to this in the next section) and in official curricula.

On this background, it is strange that university mathematics courses tend to be rather “didactic”, in the common sense. The learning of students in such courses seems to be very different from learning through research. Burton (2004, p. 198) concludes from a large-scale interview study with British mathematicians' that

the gap between mathematicians' views of mathematical knowing and that encountered by learners is monstrous. It could be said to be an indicator as to why so much teaching in mathematics fails...

What are the reasons for this gap? How do the mathematicians articulate the relations between research and teaching practice? If they also see a “gap”, how do they explain it: is it simply due to norms and choices, in the sense that they do not see it as an ideal that research activity and teaching are closely related? Or does it mainly arise from constraints and necessities, for instance of an institutional nature? In this paper, we look more closely at these questions. Above all, we try – supported by a case study of five mathematicians' views – to state them more precisely. We begin by putting them in into a wider context, in part linked to higher education in general.

MOTIVATION AND BACKGROUND.

The meaning of the term “university” is one that varies considerably, both historically and geographically. In particular, three very different university models have developed within Europe over the past centuries (cf. Mora, 2001):

- the “Humboldtian” national universities of Northern Europe, democratically governed and serving the ideal of *Einheit von Lehre und Forschung* (unity of teaching and scientific research),
- the “Napoléonic” university of the Mediterranean region, a prestigious institution controlled by the State and entrusted above all to form its élite of civil servants,
- the “Oxbridge” university, with independent and private colleges, which is more similar to the Medieval university in terms of aims and structures.

In Europe, these models still occur as implicitly assumed bases for lay discussions on university politics. But in the last decades, the functioning and functions of universities have been rapidly changing, and there is a tendency of *convergence* towards universities as corporation-like “service providers of the knowledge age” (*ibid.*, 108). To these new knowledge corporations, both education and scientific research are crucial “products” which are being marketed, sold and frequently evaluated. We have several international rankings of universities, based primarily on “measures” of their performance on research and education. In this global competition, top academics may soon be “traded” at the price level of soccer stars...

On this background, it is not surprising that the *relation* between teaching and research gets renewed attention: what will be left of the *Einheit von Lehre und Forschung* in mass universities that compete for funding and students, and where teaching and research compete for the time of its employees? Should they (continue to) co-exist in the same institutions and be delivered by the same people? If so, why?

From the perspectives of the university models mentioned above, these questions may seem almost sacrilegious. The classical models simply assume a general and intrinsic symbiosis, valid in any discipline. In the so-called *higher education* research literature, this hypothesis of a general “nexus” between teaching and research (TR-nexus, for short) has been scrutinised in hundreds of papers. The term “nexus” means a semantic “connection” of phenomena that influence each other, in positive or negative ways. For instance, Neumann (1992) found “a strong belief in a symbiotic nexus between teaching and research” among senior academic administrators in Australia, and identified three levels: (1) the *tangible* nexus, relating to the communication in teaching of the newest knowledge; (2) an *intangible* nexus, referring to effects on the working modes of teachers and students resulting from the fact that the teachers are also researchers; (3) a *global* nexus, where the connection is situated at the institutional rather than at the individual level. A large number of studies have attempted to find evidence for positive or negative correlations of type (3), rather than just beliefs. In a seminal paper, Hattie and Marsh (1996) examined a total of 58 studies from

which they extracted a total of 498 correlation coefficients between measures of quality of research and teaching in institutions of education and research. The weighted average of these coefficients turned out to be within the total variation; after excluding outliers it was a mere .05 (p. 525). Hattie and Marsh concluded that “the relationship between teaching and research is zero, and it would be more useful to investigate ways to increase the relationship” (p. 533). In fact, ten years earlier, Elton (1986) stated that questions about a possible connection between teaching and research “when put on a departmental or institutional scale cannot conceivably be answered through simplistic quantitative methods” (p. 300). After examining several classical arguments, he suggests that

it is necessary to distinguish between three activities - teaching, scholarship and research. It is then likely that at this [*the individual*, auth.] level teaching and research can fertilize each other, but only through the mediation of scholarship (p. 303).

Here, *scholarship* can be seen either as a “common factor” that could fertilise both teaching and research (the position of Elton), or as an overarching concept for “scholarship of discovery”, “scholarship of teaching” etc. (cf. Steen, 2000, 334). So we are led to consider the TR-nexus at the *individual level* ((1) and (2) above) which is where scholarship is enacted. Also we must do so with *disciplinary specificity*, if we are to surpass superficial ideas of scholarship. Then the fruitful question is not whether teaching and research support each other *automatically*, but *what* such a mutual support is, and how it can be furthered. In short we must explore the *nature of and conditions for* a positive TR-nexus in university mathematics scholarship.

A significant part of the tertiary mathematics education literature can be said to point in that direction, along the following three lines:

- studies of innovative *teaching* which aims at student activities that are implicitly or explicitly “research like” (e.g. Grenier and Godot, 2004; Legrand, 2001; Mahavier, 1999; Steen, 2000)
- studies of the nature of “advanced” mathematical *thinking and learning* among students (e.g. Tall, 1991; Rasmussen et al., 2005)
- studies of the learning perspective in mathematicians’ research, or *scholarship of discovery* (e.g. Burton, 2004; Misfeldt, 2006).

However in virtually all existing studies, the main object of research is either *teaching and student learning*, or *mathematicians’ research*, rather than the full interaction of the three. In the next section we propose a theoretical framework that takes into account these crucial elements, and we define the nexus in terms of them. Then, as a first illustration of how this model can be used, we consider the TR-nexus from the perspective of five mathematicians.

AN ANTHROPOLOGICAL APPROACH TO THE TR-NEXUS

As noticed by Rasmussen et al. (2005), mathematical scholarship is an *activity* that involves not just cognitive but also communicative acts. To do this we use the anthropological theory of didactics (we use it freely here and refer readers unfamiliar with it to Chevallard, 1999 or Barbé et al., 2005, sec. 2). Winsløw (2006) used this theory of didactics to propose a framework in which tertiary didactics of mathematics may be considered the study of the interplay between two types of human activity, namely:

- *mathematical organisations* (MO) where tasks (enacted by students, teachers and researchers) are mathematical in the usual sense, associated with mathematical techniques, technology and theories;
- *didactical organisations* (DO) where the tasks (enacted by teachers) are to induce students into enacting a MO, using didactical techniques, which may, in principle, be described and justified using didactical technology and theory.

The didactical transposition is mediated by the DO and can be located in the transition from the regional mathematical organisation MO_m of the professional mathematician (where tasks are in part research problems), to the local mathematical organisation MO_s to be enacted by students. In this sense, MO_s can be thought of as a delimitation (sometimes a drastic reduction) and adaptation to learners of a smaller section of MO_m , while the DO is the praxis through which MO_s is constructed and “delivered” (in a sense, by *devolution* in the sense of Brousseau, 1986) to students:

$$(MO_m - DO) \leftrightarrow MO_s$$

Here, MO_m and DO are the two families of praxeologies enacted by the mathematician, in research and teaching. The nexus is in the interplay between these two, potentially at all four praxeological levels:

- between *tasks* of teaching (DO) and *tasks* of research (MO_m) in the material sense that they are distributed in the working time of the mathematician. This corresponds to a *minimal* nexus which arises by material necessity in the activity of the mathematician.
- between DO *techniques*, e.g. for constructing MO_s -tasks and deliver explanations and presentations (of MO_s -technology and theory), and MO_m *techniques*. We call this the *implicit* nexus (close to the *intangible* nexus mentioned above).
- between DO *technology and theory* and the specificities of the knowledge block (*technology, theory*) of MO_m that could contribute to explain and justify didactical choices, including those involved in preparing MO_s . We call this the *explicit* nexus (it seems close to the *tangible* nexus mentioned above).

The minimal nexus is always present under the assumption that the mathematician does both teaching and research, which is of course a necessary assumption to study the TR-nexus at the individual level. One might have the minimal nexus without

much retroaction from students' activity to that of the teacher, i.e. in a setting that could be described as $(MO_m - DO) \rightarrow MO_s$. In this setting, the mathematician does his two "jobs" independently and sometimes even without much consideration of what the students do. Indeed, DO techniques could be simply inherited from his own experience as a student, without being affected by his subsequent experience as researcher. However, the implicit and explicit nexus clearly require a certain retroaction from the students' praxis of MO_s on the activity of the teacher. It could be limited, for instance by institutional constraints, to minor adaptations of DO techniques. At the other extreme, it could lead to genuine interaction between MO_m and MO_s e.g. when students get involved directly in current research of the mathematician. But most of the time – prior to this last situation, which is likely to be rare, at least in undergraduate teaching – it would be a dangerous illusion to imagine the absence of a DO, i.e. genuine collaborative learning of professors and students.

WHAT MATHEMATICIANS SAY

The two organisations MO_m and DO are enacted and to a large extent controlled by mathematicians, and their interaction even more so. The former is difficult to "observe" directly. Indeed existing studies of what corresponds to MO_m , such as those by Burton (2004) and Misfeldt (2006), are based on mathematicians assertions about it. But to our knowledge it is new to analyse such data with the praxeological model outlined above, and with the explicit intention to investigate the nexus $MO_m - DO$.

Context and Design of study

Our data come from interviews with five mathematicians (3 full professors, 2 associate professors) at the University of Copenhagen. It is the biggest and oldest university in Denmark – and the only one to figure in the top 100 of international ratings. It is close to the "Humboldtian" tradition: professors' time is, at least in theory, equally divided between research and teaching. As in most European universities of similar status, research is highly valued and the main factor in hiring and promotion.

The interviews were done individually, based on an interview guide, and lasted for about an hour. The interview guide aimed at having respondents explain (1) their current research, (2) their current teaching, with an emphasis on the bachelor level, and (3) the relations they saw between (1) and (2). Questions were adapted to the development of the interview, with the goal of identifying main characteristics of the MO_m and DO involved, as well as their interplay as it could affect students' work (MO_s); for instance, we specifically asked about techniques involved, and their interplay. Our main interest lies in part (3) of the interviews, but the two first parts (on MO_m and DO separately) were indeed useful to enable concrete references and examples. Interview transcripts (in Danish) can be made available through the authors.

Our purpose here is to illustrate some main points from our preliminary analysis of the data. The interviews with mathematicians form part of a larger study of the TR

nexus, which is currently in process; it involves also interviews with professors of geography.

Some instances of interplay

It may not be surprising that all five mathematicians cite the supervision of more advanced students – particularly during master thesis work – as an instance of teaching which can draw significantly on their experience and current work as researchers, at least indirectly. Although this seems to be relatively rare, three of them have profited in their own research from having master thesis students work with sub-problems coming from their own research projects, and two have even done joint research with excellent master students (in some cases published by the student alone). Four of them think that this is also possible, at least in principle, at the bachelor level, but they did not have actual experience with this.

Even in a first year course, students may, according to one respondent, get an “experience which is somewhat like that of a researcher”:

We try to do that. (...) We give them relatively open tasks. (...) That is, where it is not just an exercise, with a question and a specific point in the text book to refer to and a unique answer. (...) For instance (...) we ask them to compute π by using the formula for a function, like arcus tangens or something, which gives π in some point, and then use the Taylor series at a point where they know it [*the value of the function*]. That I would say is a completely standard exercise. They also learn to estimate the error. But then we can go on and ask, can you with certainty find the first 100 decimals in π , using this method. Or when can they be certain that they found the first 100 decimals. (...) And that we give them as an open task which we don't even ourselves completely, well of course we could, but we don't even consider beforehand if we know precisely what the perfect solution would be, because we are not after the perfect solution, we are after *them* thinking about what are the problems involved in this task. [*explains the central difficulty of evaluating the error term*] (...) just they explore this *problematique* we feel they have achieved a lot. And that we think reminds us of our own research, as for the processes (...) that type of exercises we give a lot. (...) We take a standard exercise, and open it up a bit (...) to see how far can you go on this type of task. (...) they need to get the experience that here they have to explore a domain by themselves.

Notice that a general idea (in fact, DO-technique) for constructing student tasks is explained here: you expand on a standard type of task from MO_s , to obtain a task with some of the overall characteristics of a task from MO_m (as explained). The techniques used by students come, of course, from MO_s ; in some cases the same task would not have the “research flavour” if attacked by more advanced techniques. And in this way, asking such “twisted” MO_s -tasks can be in itself a mathematical challenge:

That in a way you could call a kind of research. (...) One thing is to prove a theorem with all available mathematical methods. But to ask yourself, can you prove this theorem by just using the following methods, it is in itself a mathematical question. (...) [*similarly*] we ask ourselves, can we solve this *without* using this or that result.

Another respondent reports on asking a problem (about matrices) in a first year course, to which he didn't know the answer; after a lot of activity among students and teachers, it was completely solved by a student. This, however, seems to be an exceptional case (and an optional exercise for enthusiastic students).

A main component in the DOs of all five mathematicians is *lecturing*, at the bachelor level often to large audiences of students. One respondent cites the preparation of lectures as an activity which is both supported by and inspiring for his research:

My teaching couldn't at all be like it is without my research. (...) When you teach, you learn the material... it's the teacher who learns something from a course. (...) My whole approach to mathematics is shaped by research, how I feel mathematics should be understood. But also the other way: you work through things you thought you knew when you prepare a lecture, and then you discover you didn't know them, you find new things...

Notice that here the strong implicit nexus (DO-MO_m) is considered very fruitful for the respondent, but it is open how (if) it affects the work space (MO_s) of students.

Another respondent says that his access to advanced MO_m knowledge blocks helps him to select topics and perspectives to present, in a way which is relevant in the students' long way towards modern research: "I can tell them things which are not in the text books". This amounts to a explicit nexus with a strong direction *from* MO_m *to* DO. But to him, there is no way undergraduate teaching could contribute to his work with MO_m, given that even at master level, MO_s may not come close to it. Also, he seems to refuse the idea of an implicit nexus (at least for undergraduate DOs).

Research-like – what does it mean?

Overall, the *explicit* nexus seems to occur in two ways: (a) in the teachers' work on DO (presentation and exercise construction) where he makes certain choices in order to let MO_s approach – possibly from a long distance – certain topics and methods from MO_m; (b) in the supervision of advanced students' projects, where MO_s involves directly elements from MO_m (at the level of technology and theory). These forms of nexus are recognised by all five interviewees. However, (b) seems to be exceptional. About (a) one can say that it does not in itself mean that MO_s becomes research like, in the sense that students' actual activity involves tasks and techniques similar to that of a researcher. But then, after selection of appropriate topics, could one leave the DO to non-researchers? Here, a certain doubt is expressed:

Interviewer: it almost sounds like one could let university researchers devise the "menu" and then let other serve it?

Mathematician: I don't believe so... I don't think others can explain it as well as I can. (...) In principle, I could say, I write up the lecture notes and then one could hire a slave worker to deliver them, but that I think would be a very weird policy to have.

The *implicit* nexus occurs in several ways, which are (naturally) harder to categorise. However, one can distinguish two major areas: (a) construction of MO_s tasks, from

project problems to exam exercises; (b) organisation of the students' work with MO_s -theory (including modes of presentation). In both cases, the nexus is at the level of techniques, and therefore it is typically more difficult to articulate. As one mathematician puts it,

I think it's fun to make such exercises, one can use a lot of time on it. (...) I just let my imagination run freely, and then I cut down to something the students can manage.

The interplay between DO - and MO_m -techniques may not always be conscious:

But, eh, there was one of the students who said to me, after an exam on a course where I had made the exercises, then he said, and I had not thought about that before, "but it's really research to make these exercises", and, it really is, for one has to come up with something they can't have seen anywhere else.

However, to some of the respondents, MO_m -techniques are explicitly mobilised in constructing MO_s -tasks as well as in organising students' work with MO_s -theory:

I think I use my research a lot in my teaching. (...) I like the way, sometimes, when I choose exercises, to elucidate problems I have myself encountered in my research. It often happens, when you work with [*respondent's specialty*] that you end up with some problem on finite-dimensional matrices.

A very difficult task is when people ask you for literature about something you know well, but you have come so far away from books. It's much more fun... The essence you rarely find in a book. So I often give people a book where they read up to a point, and then the rest I give as exercises (...) 'cause often I can't find a book which is suitable.

Continuity or rupture?

A final question posed to respondents used a mathematical metaphor: in your work, do you regard teaching and research as separate parts or is there rather a kind of continuum? Despite the relative vagueness of the question, the previous discussion of their research and teaching seems to have given sufficient basis for the respondents to reply with certainty (4 opted for continuity, 1 for difference). One qualified continuity as follows (the others being essentially on the same track):

There is clearly a continuum. The point where the two things collide is that what my research is about here and now is not often what I try to make the students learn. There is a time conflict. (...) My own research is the funniest, right, because it's me playing, while in the other, I try to make the students play.

The option for a more principal difference is also accompanied by a reservation:

It is to a certain degree dependent on what area of mathematics we talk about. Certain areas are far more problem focused, in sense that the problems are given. (...) Everything has to be motivated by them. (...) But then there are other areas where it looks different, and where people may look differently at questions like the one you ask.

One of the respondents who argue for continuity also mentions that the answer could vary according to the amount of theoretical requirements which are required for even understanding the basic questions in a mathematical specialty. Although more evidence is clearly needed to assess this variation, we doubt that the specialty of the respondents alone can be said to account for the difference in points of view on this issue.

Institutional obstacles to continuity

We have already mentioned the institutional and didactical reasons to look for ways to strengthen the TR-nexus. It is therefore interesting to note that the participants mention several *obstacles* for this, even if many of them would like research to be more present in students' activities. Time constraints and students' lack of interest and knowledge are mentioned by some (but not all). An obstacle which seems important to all respondents is the requirements of the study programme. As one puts it,

In an ideal world, I would like to see [*long silence*] that the students learned the mathematical method. (...) Of course they need some concrete ideas, definitions, to work from, but I would like that going from them... there is a bare field, a beginning to a path, and then they have to pave it. (...) The barrier is that there is a concrete syllabus, they have to learn that and that, and the other method is time demanding. (...) In physics, they require, they have to learn this particular theorem. (...) I think we should put more weight on the working methods, rather than learning mathematics in a mechanical way.

This experience of a pressure to “cover a large amount of topics”, resulting in poor practice blocks of MO_s, is of course not new. But we think that the need to strengthen the bonds between research and teaching is an important reason – and potential help – to consider it anew, and as a didactic problem. In this light, it is interesting to note that four out of five mathematicians interviewed could in fact imagine that research problems, or problems close to research, could be taken up already in the context of the project work at the end of the B.Sc.-studies.

CONCLUSIONS

The transition from “school like” introductory courses to more “research like” activities (cf. Grenier et al., 2004) is complex and difficult in university mathematics education. The five mathematicians' points of view seem to indicate that even in those introductory courses, a research point of view may in several ways help to improve the quality of students' activities and learning, and that it may be possible to strengthen the bonds between mathematical organisations of students and teachers by promoting those forms of DO that mobilise professors *as researchers*. Giving more institutional legitimacy to such DOs may not be simple, though. Some students and professors may not want them, for different reasons perhaps. And the individual professor has to respect a syllabus which is often rigid and demanding. Nevertheless, there seems to be considerable potential among mathematicians to enact such DOs.

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ADVANCING MATHEMATICAL THINKING: LOOKING BACK AT ONE PROBLEM

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In this paper we present and analyze a particular classroom interaction with a group of secondary mathematics teachers that involves “looking back” on a particular problem. We argue that a guided looking back via a chain of horizontal and vertical mathematizing is a pedagogical practice that contributes to advancing mathematical thinking of students. We further recognize a few traits that can be considered as identifiers of advanced mathematical thinking at various levels of sophistication.

PRELUDE

On a hot day in Prague, during a relaxed stroll following PME 30 activities, a colleague offered the following problem.

There are 40 objects in a store, each object has a different weight, and these weights range from 1 to 40 pounds (in whole numbers). Using a balance scale, and weighing one object at a time with the available weights, what is the minimum number of weights that is sufficient to measure the weight of each one of the 40 objects? What are these weights?

We strongly recommend that the reader stops here and spends some time pursuing the problem, as the following discussion will present a solution and spoil the pleasure of discovery.

REFOCUSING THE QUESTION

In his review of “Advanced Mathematical Thinking” edited by David Tall (1991), Pat Thompson wrote: “If in the distant future an archaeologist were to build an image of mathematics education based on artifacts from the educational research community, she might conclude that, as late as 1990, mathematics had not progressed past proportional reasoning” (p. 279). Indeed, this edited collection was a landmark in the short history of advanced mathematical thinking (AMT) as an area of research in mathematics education. It also intensified the conversation on what constitutes AMT and how it can be identified and supported.

It is 15 years later and the conversation on what is advanced mathematical thinking continues. Tall (1991) characterised AMT as transition “from describing to defining, from convincing to proving in a logical manner based on definitions” (p. 20). Tall

also suggested that advanced mathematical thinking must begin in early elementary school and should not await till after high school. On the other hand, Tall (1995) claimed that “the full range of creative advanced mathematical thinking is mainly the province of professional mathematicians and their students” (p.71, quoted in Selden & Selden, 2005, p.3). These two observations appear not in accord with each other, unless we draw a distinction between AMT and creative AMT. These two perspectives highlight one source of disagreement in the attempts to define AMT, specifically, whether the adjective “advanced” describes thinking or mathematics, that is, whether AMT means thinking in advanced mathematics or advanced thinking in mathematics. Harel and Sowder articulate this tension by relocating the hyphen in considering “advanced-mathematical thinking” (i.e., thinking in advanced mathematics) versus “advanced mathematical-thinking” (i.e., mathematical thinking of an advanced nature) and take a stand for the latter.

This difference in perspectives on what constitutes AMT resulted in shifting the focus, or at least the description of the research area, to tertiary mathematics (Selden & Selden, 2005). However, this tension is not exclusive to AMT. In fact, there is no agreement on whether “mathematical thinking” is restricted to thinking in the subject matter of mathematics or applying certain ways of thinking (e.g. logical, rigorous) in different, not necessarily mathematical, situations. Some equate mathematical thinking with “thinking like a mathematician”, dwelling into a non-mathematical circularity in definition.

Edwards, Dubinsky and McDonald (2005) suggest that advanced mathematical thinking involves rigorous deductive reasoning and does not rely on sensory perception. They argue that AMT involves abstract concepts and deductive proofs and consider it as a transition that most often occurs during students’ undergraduate experience. On the other hand, Harel and Sowder (2005) specifically acknowledge that they focus on advanced thinking in mathematics rather than on thinking in advanced mathematics, and suggest that AMT involves overcoming epistemological obstacle. They further suggest that “person’s ways of thinking involve at least three interrelated categories: beliefs, problem-solving approaches, and proof schemes” (p. 31).

Considering these suggestions or definition attempts, we believe that the heart of the disagreement is in treating advanced/not advanced mathematical thinking as dichotomy vs. continuum. In the former, presented with an excerpt of mathematical thinking one should be able to classify it as “advanced or “not advanced”. In the latter, presented with two excerpts of mathematical thinking addressing the same situation, one should attempt to say which one is “more advanced” than the other. However, as in incomplete relation of order, some comparisons could be impossible.

Mathematicians, when unable to solve a general problem, often formulate a specific sub-problem or a related problem that appears solvable. We shall follow this approach in mathematics education (and this may be our example of AMT in

mathematics education, that is, of “thinking like a mathematician”). In fact, Rasmussen, Zandieh, King and Teppo (2005) have done just that, focusing on “advancing mathematical activity” rather than addressing the question of “What is advanced mathematical thinking”. In a similar fashion, the question that we address in this paper is “What identifies AMT in a problem solving situation?” And in order to avoid the expectation of comprehensive characteristic, we rephrase further: “What are some of the identifiers of AMT in a problem solving situation?”

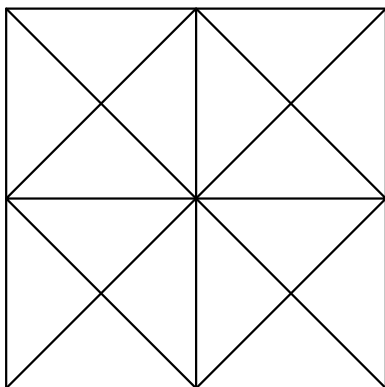
Despite the disagreement on the definition, there seems to be a consensus that teaching practices contribute to the development of advanced mathematical thinking (Selden & Selden, 2005). With this in mind we ask, “What practices are advancing mathematical thinking?”

ONE EXAMPLE AND ONE FEATURE OF AMT

We follow Harel and Sowder (2005) in the assumption that “advanced” is a characteristic of mathematical thinking rather than of a mathematical content. Further, we take the relativistic approach and view the progress from elementary to advanced as a continuum rather than dichotomy. This view is problematic as it makes the distinction dependent on an individual learner’s knowledge and experience. In a research on problem solving, there appears to be an agreement that what presents a problem for one learner can be just an exercise, or “not a problem at all”, to another. Similarly, what could be seen as AMT for one learner or in one situation may not be advanced for another. Edwards et. al. (2005) exemplify what can be not-advanced thinking or practice when dealing with advanced mathematics. We are compelled to provide a counterpart, that is, an example of what we consider AMT in not-so-advanced mathematics.

Consider for example the following task, given to students in Grade 3 class:

How many triangles can be found in the following drawing?



After about ten minutes of meticulous counting, the teacher invited students’ suggestions. Several volunteers contributed their answers, which ranged from 13 to

42. Then the teacher turned to the class, asking, “Should we all try to find 42 or is there someone who counted more?” To her surprise, one student objected the count.

Dave: 42 is a wrong answer.

Teacher: What is your count?

Dave: I am not sure yet.

Teacher: So why do you suggest that 42 is incorrect?

Dave: Because it should be 4 times something because there are 4 corners the same

Dave, in our view, exhibited advanced mathematical thinking, though he didn't have the terminology to express it. Translating his idea to conventional mathematical language is to claim that the figure has a (rotational) symmetry of order 4 and therefore the number of triangles should be a multiple of 4. (The correct count is 44).

Dave brought the tools of symmetry and divisibility to the problem of counting and geometry. Since these tools are not embedded in the problem itself, we consider them as *outside tools*. We suggest that the use of outside tools in a problem solving situation is a feature that identifies what can be considered advanced mathematical thinking at any level. In fact, significant progress in mathematics as a discipline is made by introducing tools from other, seemingly unrelated, areas of the discipline. Consider the classical example the ancient mystery of doubling the cube or trisecting an angle. The problems are situated in geometry but they were solved by using extensions of fields, that is, introducing tools from Abstract Algebra. Or the renown Fermat's Last Theorem. The problem is situated in elementary Number Theory but the solution involved elliptic curves and ideas from ring theory.

A STORY OF ONE PROBLEM

Let us return to the problem in the prelude to this paper. It was presented as part of the homework assignment to a group of inservice secondary school teachers. In what follows we describe a particular classroom interaction that took place after the assignment had been completed.

The solution that all the students agreed upon was that 4 weights are sufficient and these weights are 1,3,9,27. Having solved the problem, one may consider the task as completed. However, in Polya's tradition, having reached a solution does not complete the problem solving process, as there is the need for “looking back”. Looking back – phase #4 in Polya's framework, that follows understanding the problem, planning and carrying out the plan – is often misinterpreted in mathematics classrooms to mean “checking the solution”. In fact, checking is only one small part of looking back, the phase that may also include considering various ways of reaching the solution, connecting to other problems, extending and generalizing. We

demonstrate how a guided looking back may contribute to the advancing of mathematical thinking.

Step 1: Mathematizing

In sharing the solution there was no initial agreement on the language students used to describe their solutions. To stimulate a more productive communication an invitation to mathematize was presented by the instructor: *The fact that 4 weights are sufficient in case of 40 objects, how can the idea be expressed mathematically?*

The consensus reached by the students after a debate stated that every natural number from 1 to 40 can be expressed by adding and subtracting numbers from the set $\{1, 3, 9, 27\}$ or, alternatively, that every natural number from 1 to 40 can be expressed by adding and subtracting powers of 3, up to 3^3 .

Step 2: Sharing methods

Adopting a common language opened the gate for sharing the solution processes. Mike's approach can be considered as a lucky leap of faith. Darlene and Gaby present a systematic reasoning, with different degrees of sophistication.

Mike: My initial guess was that this can be done with powers of 2, that is, 1,2,4,8,16,32. It gave 6 weights and felt pretty good. But then someone said it can be done with less than 5. So trying powers of 3 was the next logical step.

Darlene: To reach the minimum number of weights needed, I thought of avoiding duplication. 1 has to be there. Now if you chose 2, then we can measure 1 in two different ways, putting 1 against the object, or putting 1 with the object and 3 against the object and 1. So I didn't take 2. If I take 3 there is no duplication. Now, with 1 and 3 we can measure 1, 2, 3 and 4. Adding 5 will create a duplication, like $2=3-1$ and also $2=5-3$. So we do not take 5. 9 is the next one that does not introduce redundancy. So we have 1,3 and 9. And then the next thing to try is 27, to fit the pattern.

Gaby: We start with 1 and 3. So the largest one you can measure is 4. So the smallest one you cannot measure is 5. As such the next weight to consider is 9, which is $4+5$. With 1, 3 and 9 the largest we can measure is 13 ($=1+3+9$). So the smallest one you cannot measure is 14. Then add 14 and get 27. Now the largest we can measure is 40 ($1+3+9+27$). I have also noticed something interesting: Adding to the sum of powers of 3 the next natural number also gives you a power of 3.

Step aside

Gaby's last observation created an opportunity to pursue a different problem. Rather than confirming the student's observation, it was put in doubt: *Do you really believe this is the general case?*

The need to convince the instructor may be similar to Mason's (Mason, Burton & Stacey, 1982) stage of "convincing the enemy", that follows "convincing yourself" and "convincing a friend". While "friends" could have been satisfied with checking a few more examples, convincing the enemy required rigorous formulation of the conjecture, as $2 \cdot \sum_{i=0}^n 3^i + 1 = 3^{n+1}$ and a proof by mathematical induction.

Step 3: Exhaustive checking and extension

While Gaby's method of finding weights was considered resourceful by her classmates, it still left the open question of representing *all* the numbers between 1 and 40 using powers of 3. Most students reported that they confirmed the existence of such representation in all the cases in order to be satisfied with their solution.

Having verified the desired representation, students felt that the problem was exhausted. However, the instructor was not satisfied. In order to continue the conversation, but avoid explicit hint, the following was offered: *I have a strong desire to keep exploring. Do you share this desire?*

The following suggestion was offered:

Mike: I think if we add another weight to the set, 81 or 3^4 , we can measure up to 121. I checked a few. I know this is not enough. We need a proof.

Step 4: Generalization and Proof

The invitation to formulate what exactly has to be proved resulted in generalised conjecture, namely, that any natural number can be expressed by adding and subtracting powers of 3. However, after about 20 minutes of proof attempts, either in pairs or individually, no significant progress was reported.

There is a thin line between supporting students who are stuck and giving out the solution, or the key idea. Walking this thin line the following question was posed: *Where in your mathematical experiences have you encountered powers of 3?*

After some additional probing the idea of "base 3" representation of numbers surfaced. It was taken for granted that every number can be represented in base 3. As such, the expanded notation of this representation involves "place values" that are powers of 3 and the digits in use are 0, 1 and 2.

Students' proof attempts focused on the switch between the "additive" representation in base 3 to the combination of addition and subtraction. The existence of such switch has been assured by the following:

If there are only 1 and 0 in a base-three representation, the desired representation with "powers of 3" is given. If at some place value there is a "2" the following manipulation is required, and may be implemented several times:

$$2 \times 3^k = (3-1) 3^k = 3^{k+1} - 3^k$$

This assured the existence of the desired representation for any natural number and was further reconfirmed empirically.

Step 5: Further generalization

The following recap was offered: To represent numbers with powers of 2 we need addition only. To represent numbers with powers of 3, we used addition and subtraction. What about other powers?

Extension of investigation led to the idea of “balanced” base representation, that is, any odd base representation can be converted to a “balanced” one, using numbers $0, \pm 1, \pm 2, \dots, \pm n$ for base $2n+1$. Specifically, the “multipliers” for powers of 5 in balanced representation are $-2, -1, 0, 1$ and 2 . A method of switching from the standard base representation to a “balanced” base representation was offered.

Step 6: Leaving the door open

What about even bases?

This question was posed by students, rather than the instructor. Unfortunately (if there are unfortunate events in mathematics), balanced representation is not possible for even bases; clearly, as the number of digits in use is not odd, a “balance” around zero cannot be achieved. However, such a conclusion may close the doors for any further conjecturing. To keep the door open we invite a different investigation: If balanced representation is not possible, what *is* possible?

LOOKING BACK AS ADVANCING MATHEMATICAL THINKING

According to Freudenthal (1968) mathematics can be best learned by doing, and mathematizing is the goal of mathematics education. Freudenthal distinguishes between ‘horizontal’ and ‘vertical’ mathematizing. Horizontal mathematizing involves translating the problem from non-mathematical field (some context related to “real world” situation) to the field of mathematics. Vertical mathematizing involves manipulating and moving within the mathematical system itself. In other words, “to mathematize horizontally means to go from the world of life to the world of symbols, to mathematize vertically means to move within the world of symbols” (Van Den Heuvel-Panhuizen, 2003). Rasmussen et. al. (2005) broaden the definition of horizontal mathematization and suggest that it involves “formulating a problem situation in such a way that it is amenable to further mathematical analysis” (p. 54). Step 1 described above fits the original definition of horizontal mathematization, while steps 4,5 and 6 are in accord with the amended definition. That is, initially the problem is translated from “real-life-like” situation to a mathematical situation, and then it is amended and reformulated to allow further mathematical analysis. This further mathematical analysis is the vertical mathematization.

We can consider the chain of horizontal mathematizing as follows, labeling the transition in reformulating the problem as T_i ($i=1,2,3,4$):

How can the weight of 40 objects be determined with minimum number of weights? $=T_1=>$ Can natural numbers from 1 to 40 be expressed by adding and subtracting powers of 3? $=T_2=>$ Can *any* natural number be expressed by adding and subtracting powers of 3? $=T_3=>$ Using powers of other numbers, is there an analogical representation? $=T_4=>$ Having described “balanced” representation for odd bases, what about even bases?

While transitions T_2 , T_3 and T_4 all involve generalization, these generalizations appear to be of different character. Harel and Tall (1991) distinguish between three different kinds of generalization: (1) expansive, where the applicability range of an existing schema is expanded, without reconstructing the schema; (2) reconstructive, where the existing schema is reconstructed in order to widen the applicability range and (3) disjunctive, where a new schema is constructed when moving to a new context. In this sense T_2 is an example of expansive generalization, where the scope of applicability of existing representation is extended from the interval $[1,40]$ to all natural numbers. Attending to T_3 resulted in reconstructive generalization, where previously available representation of numbers with powers of 3 became an example of balanced base representation. T_4 may result in disjunctive generalization if a separate case for even numbers is found, or, alternatively, it may turn to be another reconstructive generalization, if representation is found to include powers of both even and odd numbers.

We note here that the “key” proof in the above looking back on the problem relied on representing numbers in different bases, what could be considered an “outside tool” for the given problem. We claimed earlier that the use of outside tools is one of the identifiers of AMT. Furthermore, Harel and Tall (1991) emphasize the value of reconstructive generalization, as it is cognitively more demanding and more powerful than the other types. In relation to our discussion in this paper, we suggest that producing (that is, formulating and proving) reconstructive generalization is an identifier of AMT.

CONCLUSION

We believe that “profound” (vs. superficial) looking back at a problem provides opportunities for “advancing” mathematical thinking. This is done through series of mathematizing, where horizontal mathematization reformulates the problem in a way that invites further mathematical analysis and vertical mathematization advances the solution, preparing the background for further horizontal mathematization. This chain of mathematizing is advancing mathematical thinking of learners who are involved in looking back on a problem. Within this chain we suggested two features that identify AMT: the use of outside tools and reconstructive generalization.

Teacher’s role in advancing mathematical thinking of students in looking back on a problem has not been our main focus in this discussion, though this role is crucial initially in progressing through the chain of horizontal/vertical mathematizing. To begin with, the teacher is the one who pushes and reformulates the problem to allow

for further mathematical investigation. It is also the teacher who “keeps the door open” as an invitation to continue the exploration. At a more advanced stage, similarly to Mason’s et. al. (1982) suggestion of developing an “internal enemy” when proving an argument, learners develop an “internal teacher”, that is, become able to invite themselves for further mathematical activity via horizontal mathematization. We consider this internal ability as yet another identifier of AMT.

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