Proceedings of the Nineteenth ERME Topic Conference

Mosvold Reidar, Fauskanger Janne, Ferretti Federica, Vondrová Naďa

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ETC 2026
CONFERENCE

Proceedings of the Nineteenth ERME Topic Conference

Connecting the Learning of Mathematics Teaching to Practice

Charles University, Faculty of Education
Prague (Czech Republic)
4–7 February 2026

Edited by

Reidar Mosvold
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Introduction to the proceedings from ETC19

In the first eight CERMEs, a single working group explored the education of mathematics teachers. Subsequently, the group split into three: *Mathematics Teacher Education and Professional Development* (TWG18); *Mathematics Teaching and Teacher Practice(s)* (TWG19); and *Mathematics Teacher Knowledge, Beliefs and Identity* (TWG20). This ERME topic conference brings these groups together again to consider what it takes to learn teaching.

As the professional education of teachers has done little to change conventional mathematics teaching, scholars have argued for greater attention to the learning of practice, learning to "do" mathematics teaching. The topic conference and the papers in these pre-conference proceedings focus on what it takes to get the learning of practice at the forefront, to infuse regard for teaching as a practice (in and outside the classroom) throughout the system of professional education (pre-service and in-service), in goals, design, activities, and assessments.

Research-based contributions related to the following three questions are included:

1. What foundational things do mathematics teachers need to learn to do as they learn to teach?
2. What is involved in learning to do mathematics teaching?
3. How can we design and carry out professional education for the learning of practice?

At the conference, we will share our growing understanding and raise cross-study questions, warranting claims in relation to teaching and the learning of practice. Considering recent developments in the three TWGs, this topic conference and all its papers encourage attention to issues of equity (in mathematics teaching and in the recruitment and education of teachers) and the design of activities and tasks for learning teaching in ways that impact practice. In addition, for studies carried out in the context of one's own practice, explicit attention is given to how multiple roles are ethically managed and how analytic clarity is maintained in design, methods, and claims.

Typically, each working group at CERME addresses a distinct topic. We could have organized the papers for this ERME topic conference as distinguished among TWG18, 19, and 20, but our goal was specifically to work across this separation. Therefore, we categorised papers into four domains foundational to teacher development. (We use the word in bold as shorthand for the domain.)

- Consideration of **curriculum** of teacher education (mathematical and pedagogical knowledge and skill) and the central endeavour of extending this curriculum to prospective and practicing teachers.
- Organisation and **enactment** of design, interaction, and discourse of teacher education.
- Understanding, relating to, and responding to **teachers** as people and learners.
- Attending to broader social, cultural, and political **issues** that matter for teacher education, including imperatives of social justice.

Research on teacher development typically attends to all of these domains, yet any individual study will likely foreground one as it manages others. Identifying and categorising our work in relation to these domains helps us put diverse studies into productive conversation with each other. These domains are adapted from ones TWG19 has used productively in recent years. Whereas those domains are on the level of mathematics teaching (mathematics, enactment, students, and issues), the domains we apply in this conference are bumped up a level, from teaching to teacher learning and development.

For this ERME topic conference, we have organised people into four working groups, with the aim of having all four groups mostly analogous. No submissions focused on the **issues** domain, so we will use only three of the domains. Each working group has a set of papers foregrounding curriculum, another set foregrounding enactment, and another foregrounding teachers. Instead of organising the proceedings by working group, we have decided to organise it by domain, to display work in each domain across the full conference.

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CURRICULUM

Towards a framework for building on students' work in whole-class discussions

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Building productively on students' work in mathematical discussions is challenging work, yet a practice pre-service teachers (PSTs) need to learn during teacher education. In this study, we propose a framework for how such building on students' work can play out, based on PSTs' imagined discussions on a proving task. The framework consists of five categories, with an increasing level of complexity. We discuss the potential of the framework as a tool in practice-based teacher education.

Keywords: Mathematical discussions, students' work, reasoning and proving, practice-based teacher education.

Introduction

Researchers agree that building on students' thinking is fundamental for effective and equitable mathematics education (Shaughnessy et al., 2019; Smith & Stein, 1998). Thus, teachers should be able to recognise students' thinking, ensure it is shared with other students or the whole class, and facilitate further mathematical activity building on students' contributions. Following a practice-based approach to teacher education (Ball & Cohen, 1999), pre-service teachers (PSTs) should get experience with such practices in teacher education. However, effectively using students' responses to support mathematical learning is a key challenge in teaching (Stein et al., 2008). Research indicates that PSTs can easily master the eliciting and sharing of students' strategies in whole-class discussions, but that they do need support in how to use these strategies in further instruction (Amador et al., 2022). Moreover, teachers often listen to students' mathematical thinking with the aim of correcting or assessing answers (Hoover & Ball, 2025), a disposition that we see as counterproductive when orchestrating mathematical discussions. Hence, we believe it is useful to pay special attention to *the practice of building on students' work in mathematical discussions* in mathematics teacher education.

This paper aims to make a first attempt at developing a framework in the form of a typology for how students' work can be used in mathematical discussions. In the study, we draw upon data from PSTs' lesson plays (Arnesen & Valenta, 2025) as examples of mathematical discussions. We intend the framework to be a functional tool in practice-based teacher education, which will be discussed in the final part of the paper. It also holds potential as an analytical research framework.

Theoretical background

We see mathematics as a discourse on mathematical objects, where the exploration of properties of and relations between mathematical objects is central (Sfard, 2008). To agentively participate in such discourse, students need to be able "to take responsibility, to be active and to take ownership of the learning process" (Alrø & Skovsmose, 2004, p. 40); hence, shared authority between teacher and students is a vital component of mathematics teaching. In a classroom of shared authority, the teacher recognises, credits, and invites peers to evaluate students' mathematical contributions (Arnesen & Rø, 2022). Alrø and Skovsmose (2004) proposed the idea of students and teachers engaged together

in co-inquiry, where the participants are 'equal' even if their roles (and competencies) are different: "In a dialogue, there should be no use of power or force, nor persuasion of the other, and no winning" (p. 41). We believe orchestrating discussions so that students' contributions are valued—by which we first and foremost mean *used*—is necessary for shared authority or co-inquiry to take place. Naturally, also mistaken or incomplete arguments provided by students can be used as a collective learning opportunity in a meaningful way (Zhuang & Conner, 2024) and thus valued.

As stated in the introduction, using students' work in discussions is challenging for teachers and PSTs. In our study, the discussions focused on proving; that is, they involved a type of mathematical reasoning where the teacher's expertise may far exceed the students'. Still, it is possible for the teacher to build interaction so that the eventual proof arises from co-inquiry (Lampert, 1990). Here, we study discussions aimed at constructing a *generic example*, which is an argument based on one or more examples, pointing to some underlying mathematical structures that make the argument generalisable to all possible examples (Mason & Pimm, 1984). Thus, it differs from an empirical argument, which is also based on examples: While an empirical argument is based on checking *that* the claim holds for some examples and concluding generality based on that, an argument by generic example put forward some generalisable feature of the example which is used to explain *why* the claim holds in the given example, and then also in general. An empirical argument is mathematically invalid, but an argument by generic example is usually considered a valid proof in school mathematics (Rø & Arnesen, 2020).

Methodology

Participants, context, and data collection

The study is based on data from Norwegian teacher education for grades 1–7. PSTs in this program take a mandatory mathematics education course during their first two years, corresponding to one full term/semester. In their third year, they have the option to take an additional equally sized course in mathematics education. This more advanced course forms the context of the study. The second author of the paper was one of the teachers of the course in the year the data was collected. The PSTs had no additional mathematics courses in the program, nor did they hold a previous degree in mathematics.

Reasoning and proving were among the topics in the course. To support PSTs' understanding of different types of arguments, and also to provide opportunities for them to imagine and, in a way, enact a whole-class discussion concerning a task on proving, the second author designed an assignment in the form of a lesson play (see Arnesen & Valenta, 2025). The assignment is presented in Figure 1.

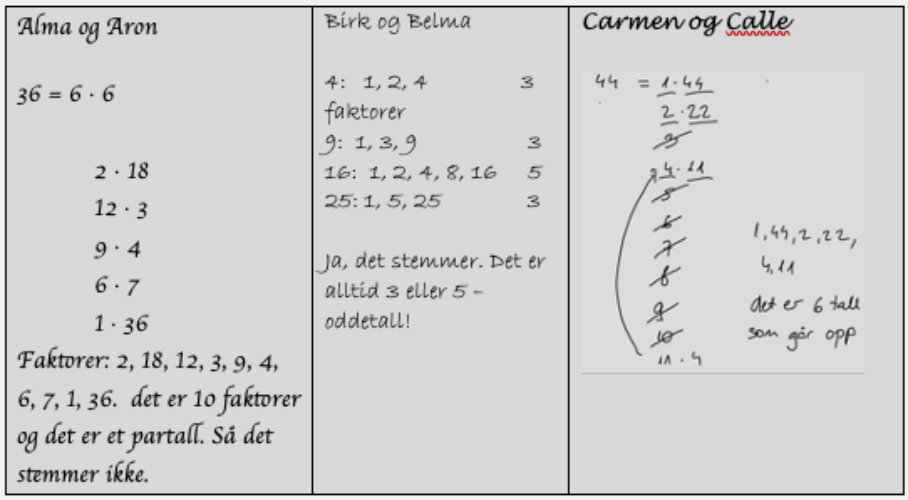
The three student pairs' responses were designed to inspire PSTs in how it is possible to proceed in part a of the assignment (and all presented solutions using similar ideas), and all have potential that can be developed in a whole-class discussion to strengthen the argument and support students' learning about proving. These potentials were characterised a priori as follows:

In a 7th grade class, students have been working on prime numbers. In the class, it is emphasised that prime numbers have only two factors, and the students are given the task of investigating how many factors other numbers have. At one point, Thomas, one of the students, suggests that all square numbers have an odd number of factors. The teacher finds this to be an interesting conjecture and asks the class to investigate whether it is true—and if so, why.

a. Prove the conjecture that all square numbers have an odd number of factors by using a generic example. The proof should be adapted to 7th grade students.

b. The students in the class have worked in groups on this task and below are three examples of their written work. The class will now engage in a whole-class discussion, where the teacher's goal is to construct a proof using a generic example together with the students, and by building on the three responses.

Your task is: Imagine an ideal whole-class discussion where the teacher and the students build a proof you suggest in a. Write the discussion explicitly—include exactly what the teacher says, what the student(s) say, and what is written on the board. The conversation should be 1.5 to 2.5 pages long.
Note: You are to write a fully detailed imagined conversation in the form of a transcript, not a description of a conversation.



Alma and Aron: (..) There are 10 factors, and that is an even number. So it is not correct.

Birk and Belma: (…) Yes, it is right. It is always 3 or 5 factors—odd numbers!

Carmen and Calle: There are 6 numbers it is divisible by

Figure 1: The task given to PSTs. Part b requires them to write what we call a lesson play

- Alma and Aron's (pair A's) response has the potential to show how the number of factors for a number can be investigated. There is a computational error, which makes the conclusion wrong. However, 36 would be a counterexample if it had an even number of factors, and the conclusion that the conjecture was not true would be right in that case; that is something that can be promoted in the discussion to support students' understanding of mathematical proving.
- Birk and Belma (pair B) check several examples. Their argument is empirical, but a potential for the discussion is that the investigation suggests that the conjecture can be true. Further, their work gives opportunity to point out the limitation of empirical arguments. It is not clear how they found the factors, and a discussion about that would make the definition of a factor explicit and allow for comparing the approaches.

- Carmen and Calle (pair C) are not checking a square number, indicating that they do not know what it is. Thus, a potential is that it gives an opportunity to express the definition of square numbers explicitly in the discussion; it can be used to make it clear where the "odd factor" is coming from. Also, 44 can be used as an example of non-square numbers; the students' list of products giving 44 shows factors that always come in pairs. Another potential is the systematic enumeration of the product giving the number, which is needed when finding a number of factors in examples and would strengthen pairs A and B's approaches.

The PSTs worked in small groups on the assignment, and the data we analyse in this study are the 17 lesson plays of whole-class discussions that were submitted.

Method of analysis

In the first round of coding, we were guided by the analytical question: What characterises PSTs' use of students' work in the lesson plays? In particular, we built on the a priori analysis of the students' work, looking for how the mathematical ideas, strategies, or mistakes inherent in the work were taken up in the discussions. This phase was carried out individually, with several meetings in which we discussed our findings and gradually agreed on a list of codes derived from the data, which is presented in Table 1. The codes regard two central aspects of how student work could be taken up in the discussion. We did not examine other characteristics of mathematical discussions, such as the types of questions the teacher used or how students were more generally included in the discussion, as these were outside the scope of the study.

Table 1: Final list of codes for analysing how students' work was used in discussions

Eliciting students' work **at the beginning**	Building the final argument
Elicits and connects students' work, with a focus on structure/ideas/strategies	Explicitly combines structure/ideas/strategies from several students' work, including mistakes
Elicits one or more students' work, with a focus on structure/ideas/strategies	Explicitly combines correct structure/ideas/strategies from several students' work
Elicits one or more students' work, with a focus on conclusions	Explicitly builds on structure/ideas/strategies from one student's work
Does not elicit students' work	Implicitly extends students' work
	Argument detached from students' work
	No valid argument

For the second aspect, "Building the final argument", we distinguished between explicit and implicit building on students' work. *Explicit* means that the argument builds on ideas or structure in the students' work, and this is clearly credited in the discussion. *Implicit* means that the teacher seems to proceed from the students' work without crediting the students, and it is typically only the numbers/example that are used, not the ideas or strategies. *Detached* means that the discussion makes a 'leap' where the ideas, examples, and strategies brought into the final argument are not clearly connected to the students' work at all, although the strategies or examples might coincide (they are,

e.g., brought in by a new student instead of taken from the given student work). Almost all discussions ended with a valid argument, although sometimes with a few clarifying details missing.

During earlier phases of the analysis, we had also looked for how the PSTs took up in their lesson plays the potentials related to the mistakes and shortcomings of the students' work. We found, however, that these were treated in similar ways across the lesson plays. In almost all the discussions, the calculation mistake of group A was discovered and corrected, and usually it was pointed out that the conjecture was correct for the number 36. The error made by group C was usually also discovered and corrected, and it was usually used as a reminder of the definition of square numbers. Also, the shortcomings of empirical argumentation were often a component of the discussions, following the eliciting of students' work. However, since these characteristics differed little, they did not contribute to the typology described below; instead, they are common to all the types.

In the final round of analysis, we aimed to identify a typology of how student work is used in whole-class discussions. When constructing types from a data set with variables (two aspects of Table 1) taking multiple values (the codes), the types do not need to be constructed based on frequently observed cases, or even based on real, identified cases; they can also be constructed as hypothetical combinations of codes (George & Bennett, 2005). Aiming to construct a complete and usable typology, we included types that we felt were 'typical' and often observed, but also types that we only observed one or two cases of. Some types are a mix of several observed cases, so that all observed cases fit with one type. The "Building the final argument" variable had the most influence on the resulting typology.

Findings

The process of analysis resulted in five types, each more complex than the previous when it comes to how the potentials of the students' work is utilised. We present each type with a short example.

T1: Present and correct (some or all) students' work before constructing a detached argument

The 'detached argument' appeared quite often in our data material. In these discussions, the teacher allowed two or three groups to present their work. Usually, the teacher's focus was to elicit answers and conclusions, but we also found some cases where the teacher spent time on students' strategies. Then, the teacher kind of 'cleared the scene' and built the argument without proceeding from the presented students' work. This final argument could be reminiscent of the students' work (after all, there is only a limited number of available strategies). Still, there was no connection made with the students' work. Thus, there was potential inherent in the students' work that was not made use of.

One example of this type is found in the work by Group 8. Here, pairs A and B (but not C) had presented their conclusions, i.e., the number of factors for their chosen examples, and the mistake made by pair A had been corrected. Yet, nothing was written on the blackboard so far. The teacher prompted: "Ok. Does it mean it always has to be like this, since it was so for the five square numbers we have checked so far?" The class discussed the impossibility of checking all numbers, and the teacher asked about what to do. A new student, Truls, said: "There is a free space when we multiply the numbers. I can show you, can I write on the board?" Truls wrote the first five square numbers, together with a complete list of factorisations for each number. He then gave a valid argument, which

was approved by the teacher and the class. Although this argument is somewhat reminiscent of the students' work (in particular, pair C), Truls' argument was never connected to group C's work. Thus, it appeared detached from the first part of the discussion.

T2: Present and correct (some or all) students' work, and implicitly build on some of them

This type was also very common in the data material, although there were some differences in how the discussions played out. Most of the discussions began with letting all pairs present. The argument that followed implicitly built on students' work, but without explicit reference, again not taking up much of the potential. Notably, two of the discussions identified as this type did not include pair C's solution at all. One of them was Group 11, which we use to exemplify this type.

Here, the teacher asked for a square number, and Belma suggested 25, which is $5 \cdot 5$. Then, the teacher asked pair B which number they used. The teacher wrote 36 on the board, with $6 \cdot 6$, and asked the class which other multiplications that yield 36. The students gave suggestions, including Aron's $6 \cdot 7$ which was removed from the list. Then, the class compared the two numbers and realised that both included a number multiplied by itself. The argument provided was incomplete (that the factors apart from the square root will always come in pairs was only briefly mentioned), but what is more important here is that the strategies or ideas inherent in the student's work were not made explicit or elicited. Thus, although the argument came as a continuation of discussing pairs A and B's solutions, the final argument did not explicitly build upon their ideas.

T3: Present and correct (some or all) students' work, and explicitly build on correct strategies/ideas

Here, the teacher presented the students' work and explicitly used some of the correct strategies or ideas in one of them to proceed with the argument, thus utilising more of the potential of the students' work. One example is given by Group 5. The teacher had elicited all three pairs' conclusions. Then, the teacher asked the students to "go back and have a closer look at Aron and Alma's solutions". One student detected the mistake of including $6 \cdot 7$, and after establishing that the conjecture was indeed true for their case as well, another student noted the pairwise structure in pair A's work. Following a further prompt from the teacher, the final argument was built. Thus, the structure of pair A's work was explicitly used to construct the final argument.

T4: Present and correct (some or all) students' work, and explicitly build an argument by combining ideas, possibly also including mistakes

In this type, the teacher combined more than one student pair's work. In group 7's discussion, the teacher had written down all multiplications that yielded 36 and 4 on the board, following the discussion of pair A *and* pair B. Then, the students looked for what was common in the two lists and proceeded from there. Another example, where the mistake in pair C's solution was used productively, is group 14's discussion. Here, the teacher highlighted that for 44, which is not a square number, the number of factors was even, whereas for 36 (from pair A), it was odd: "Well, do you have any thoughts about that?" Carmen suggested that you cannot get 44 by multiplying two equal numbers. This idea was further discussed by the class and turned into a final argument.

T5: Continuously and explicitly build on students' work from the beginning

This type differed from the rest in that it did not first present solutions and then proceed. Instead, the teacher continuously combined students' work, without the 'break' between presentation and argumentation. Interestingly, this type was very uncommon in the data material. Group 6's discussion is used to exemplify. Here, the teacher connected pair A's and pair B's use of examples, before listing (and correcting) the factors of 36 (pair A) using the systematic enumeration of pair C. The students looked for patterns, and they used the structure of writing the factors from pair B's examples. From this, they identified the multiplication with the same factor twice and built the argument from there.

Discussion

The proposed framework can be used as a tool in teacher education as a support for PSTs when learning the practice of building on students' work in mathematical discussions, which is known to be challenging (Amador et al., 2022; Stein et al., 2008). It can be used to make awareness of the different approaches available, during planning, and—maybe most fruitfully—during post-enactment reflection. For example, in the situation where data was collected, we can imagine a final part of the assignment where PSTs together reflect upon how students' work was used in their discussions, before refining their discussions accordingly. The framework has a hierarchical structure, where the complexity increases with the potential that is taken up from the students' work. For example, the PSTs involved in our setup could notice that many did not utilise pair C's work at all, and thus 'lost' much of the potential available from the students' work. Or they would recognise their discussion as a 'detached' or 'implicit' type and reflect upon the shortcomings of such discussions. In particular, the framework can help strengthen PSTs' awareness that eliciting students' thinking can reveal much more than 'right' or 'wrong' answers, and that recognising other potentials is an important step towards a more equitable or co-inquiry teaching (Alrø & Skovsmose, 2004; Hoover & Ball, 2025).

We acknowledge that had we used different lesson plays, lesson plays from another school level, or transcripts of real-time teaching, we could have gotten different frequencies of types. Still, we believe that the framework could be used to categorise other approximations to (or even real) mathematical discussions. In our study, the PSTs aimed to construct a proof with their fictive students. We think that it is likely that other mathematical discussions based on mathematical reasoning, such as discussions following problem-solving activities, can also be analysed using the framework. However, more empirical studies are needed to refine and possibly extend the framework.

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Studying curriculum designers' decisions and their rationale in defining goals and activities for learning and teaching statistics

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This study examines the design of a school mathematics curriculum in Greece, focusing on statistics, through the lens of the implementation chain and activity theory. By analyzing interviews with curriculum designers, this research explores designers' decisions about content, learning trajectories (LTs), teaching approaches and teacher support systems by examining how resources influenced these decisions and what challenges arose in the process. Findings illustrate how the interplay of research, teaching and educational policy, seen from the lens of Activity Theory (AT), shapes curriculum design, offering new insights into decision-making processes and the teaching and learning of statistics.

Keywords: Mathematics curriculum, decision making, teaching mathematics, statistics.

Introduction

Teaching and learning mathematics are complex processes. One of the most important factors that impacts mathematics teaching is the directions and perspectives about teaching and learning that the curriculum offers. In the latest years, while world faces excessive changes in almost every field of life, many countries either make amendments on the existing educational program or even drastically change their whole teaching philosophy by designing new curricula. Although research has shown that the way teachers use and enact the curriculum resources in their classrooms differs, it also highlights that curriculum resources impact how mathematics teaching is actualised and developed (Rezat 2024; Rezat et al., 2021). Studying both the process of designing and implementing a new mathematics curriculum can contribute to understanding the distance between designing and implementing and how this can be bridged. In the study, part of which is reported in this paper, the process of designing and implementing a new mathematics curriculum in secondary education in Greece is investigated through the lens of the implementation chain (Koichu et al., 2022). This reflects the three interconnected levels: the Intended Curriculum, the Implemented Curriculum, and the Attained Curriculum, distinguished by Robitaille (1980). Research on teacher design includes also what Pepin et al. (2019) refer to as the macro/national level when mathematics teachers work in national design teams for renewing the national curriculum. In the study reported in this paper the focus is on the design of a new mathematics curriculum with a national group consisting of teachers and researchers. In particular, we study the designers' decisions and their rationale while they develop the potential learning outcomes and the recommended teaching approaches of a new mathematics curriculum for primary and secondary education in the area of statistics. Research on the designing process in the Greek context has focused on the design of the mathematics curriculum emphasizing the interaction of the activities of research, teaching and educational policy (Potari et al. 2019). This process has also been investigated from the designing activity and positioning of three leading coordinators aiming to integrate two reform curricula (Sakonidis et al., 2022). Here, we focus on the

decisions and the rationale of a group of designers in the area of statistics. Statistics has been neglected in mathematics education in Greece, so decisions about introducing it to the mathematics curriculum are of particular importance. Therefore, we pose the following research questions: a) what are the curriculum designers' decisions, b) on what resources do the designers draw on making these decisions, and c) what are the challenges the designers face in the decision making process?

Theoretical framework

The term 'intended curriculum' mirrors the official plan or vision and represents what students should learn, in theory. It is created by the curriculum designers, education authorities or ministries of education and can be found in curriculum documents, textbooks, syllabi and policy guidelines. The 'implemented curriculum' is the enacted version. It embodies what teachers actually teach in the classroom, a reality influenced by teachers' interpretation, resources, time constraints and teaching style. As a result, it is often narrower or presents differences from the intended curriculum. The 'attained' or 'realized curriculum' is the achieved learning. It expresses what students learn, retain and can apply, depending on their prior knowledge and their engagement with the learning process. Each level plays an important role in a curriculum reform process by bridging policy and practice while identifying gaps between the levels. Thus, the implementation chain provides a conceptual map for designing and monitoring a curriculum reform in mathematics. By paying attention to each level, educational systems can create reforms that are both visionary and realistically impactful in classroom and as a result, later, in society (Robitaille, 2006). Thus, the implementation chain shows how curriculum reform moves from design to interpretation and finally to teaching activity.

Engeström (1987) expanding on Vygotsky's (1987) ideas, introduced a framework for examining activity systems. Within this model, six elements are identified: the subject or agent (which may refer to an individual or a community), the object (which is the focus of the subject's activity), the tools or resources that are employed to shape the object in alignment with a particular goal, and the communities, the rules and the division of labor; with the three latter embodying the social and historical dimension of the activity. This framework emphasizes that an activity system is oriented towards transforming the object to achieve a desired outcome. Based on the Activity Theory (AT) (Engeström, 2001), the mathematics curriculum designing process is conceptualized as the interaction of three interrelated systems: mathematics education research, mathematics teaching and educational policy. The interrelations both within and between these activity systems become apparent through the negotiations among the members of the new curriculum's design group. Collective decisions made in this group influenced the curriculum directions and perspectives. Research on decision making through AT perspectives (Stouraitis, 2017; Potari & Stouraitis, 2020) has pointed several social aspects of decision making. Activity is collective and the object is socially negotiated and determined. Thus, decision making has social, historical, moral and systemic aspects (Engeström, 2001b). Nevertheless, individual subjects have their own histories and positions that form the way they individually carry out the socially formulated activity. Thus, these histories and positions formulate their contribution to the collective decision making (Stouraitis, 2017).

Concerning the teaching and learning of statistics in the new curriculum, the designers consider the important role of statistics in mathematics education. Teaching and learning of statistics eagers

students' ability to understand, analyze and interpret data in an increasingly information/data-driven world. Statistical skills such as data literacy, graph interpretation, understanding variability, probability reasoning and the ability to use statistics-related digital tools for data analysis is not just a part of Potential Learning Outcomes (PLOs) but will become an essential skill for informed decision making in society, science, business and technology. In GAISE II (2020) statistical literacy is seen as the ability to understand and interpret statistical language and representations to critically evaluate data-driven claims and to make informed and ethical decisions based on data in diverse real-world problems and context. These perspectives about statistics teaching and learning in schools have oriented the design of curriculum in several countries and in the new mathematics curriculum in Greece.

Methods

The study aims to investigate the implementation chain of the new Greek mathematics curriculum, ensuring a comprehensive understanding of the designing process's rationale and its impact on teaching. The first phase involved a detailed review of the revised curriculum documents, including the Potential Learning Outcomes (PLOs) and instructional guidelines such as the Curriculum's Philosophy Document. Data were also generated by semi-structured interviews with four out of six members of the stochastic mathematics designer group to gather insights on the designing process. Two members of the designing group were unable to attend the interview meetings due to personal reasons. The other four were: two secondary education mathematics teachers with research background; one in statistics and the other in mathematics education, and two academic professors; one in the area of statistics and the other in statistics education. In this paper we focus on teachers' considerations about the decisions made in the designing process. Teachers play a vital role in the implementation chain as enactors of the curriculum. Thus, we analyse the interviews of the two secondary education teachers (D1 & D2).

The interviews contained open questions about the designers' role in the designing process, the designing of PLOs, the aims and designing criteria of the new curriculum and the teacher support system for the implementation phase. Data from interviews were coded by grounded theory approaches. Firstly, codes were assigned to each decision. Then, the decisions were grouped into four main categories as discussed in the results section. In each category we identified the related resources and challenges that arose. The resources and challenges were related to the activity systems of teaching, research and educational policy to understand the rationale of the designers' decisions.

Results

During the designing of the new curriculum, the members of the stochastic mathematics designer group were called to make a series of critical decisions concerning the mathematics curriculum. These decisions involved: a) selecting specific content areas and key ideas in relation to statistics, b) creating Learning Trajectories in order to formulate the Potential Learning Outcomes (PLOs), c) clarifying the preferable teaching approaches in the teaching guide, and d) proposing a teacher-support system for the implementation of the new curriculum. Each decision they made was influenced by research resources, prior curriculum resources, knowledge about statistics learning and teaching and teaching experience. Use of these resources often raised challenges in the decision-making process.

Selecting specific content areas and key ideas

In the designing process of the new curriculum, designers of the group working on the statistics carefully selected content areas and key ideas based on a set of well-defined criteria. A central consideration was curricular relevance, ensuring that the chosen concepts of probability and statistics (e.g. variability, statistical inference) align with both national and international standards and students' developmental stages; D1 states:

> we mainly used the GAISE II for documenting what each PLO means. We considered what happens in other countries with corresponding learning outcomes, at what level the students are, and in what order it is recommended that the concepts are introduced. We searched for model projects that would first help us understand and then explain the meaning of each learning outcome. In other words, research resources helped us justify our choices or modify them if we couldn't support them with proper documentation.

Research resources influenced the Potential Learning Outcomes as designers tried to align the content with other international projects and curriculums; D2 says: "we drew information from previous curricula, European curricula, the curriculum of Cyprus, European platforms and international platforms such as TIMSS, as well as from the program 'Educate'.".

Designers also prioritized real world knowledge applicability by selecting ideas that foster connections between stochastic thinking and every day decision making such as interpreting data in media; D2 stresses:

> for us, it was a necessity to include stochastic mathematics in the curriculum, because they are connected with decision-making, design science and data science, which has now become very important. Today, as an individual, you need to manage data, make decisions and deal with uncertainty in all aspects of your everyday life.

In the decision-making process challenges arose from balancing international frameworks with local needs as designers tried to align requirements based on research resources with the reality of the Greek educational system. They faced the challenge of justifying every choice with proper documentation, while ensuring that concepts matched students' developmental stages and could be effectively applied. These decisions and challenges are manifestations of the interplay between the research activity system and the teaching activity system.

Creating Learning Trajectories (LTs) in order to formulate the potential pearning putcomes

Another key criterion of the decision-making process was not only choosing the cognitive content but also determining its structure throughout grades, meaning that topics are introduced gradually and build on prior knowledge, while preparing learners for more abstract reasoning in later grades. Specifically, curriculum designers placed strong emphasis on learning trajectories, which map how students' understanding of statistics and probability develops over time. To them, LTs provide a developmentally coherent pathway, ensuring that statistical concepts such as distribution, variability and inference are built on early experiences with data representation, chance experiments and informal reasoning; quoting D2: "we observed in the literature, particularly in the American context, that in middle school the box plot is described differently than in high school and we respected that",

D1 adds:

> we decided that students in the first year of secondary school should mainly work with quantitative data. In the second year of middle school, they would focus primarily on time-related data. During that year, we will also introduce variability through a single box plot. Later, in the second year of high school, students should work deeper in that concept with two box plots this time and, in fact, be able to compare them.

By carefully sequencing content, designers aim to help students move from intuitive interpretations of randomness to more formalized statistical thinking, avoiding gaps and cognitive difficulties while moving from grade to grade; D1 says: "ultimately, we are not interested in the students just being able to make comparisons; what matters to us is that they understand the concept of variability. We want them to be able to use the comparison as a starting point to discuss the variability of a sample.".

It can be concluded that when selecting these trajectories, designers applied several key criteria that come from both research and teaching experience such as: to ensure that concepts are introduced in appropriate stages with coherence and progression; to guarantee smooth transitions between grade levels and topics; to integrate conceptual and procedural knowledge; to balance exploration and interpretation with formal reasoning; to embed the stochastic ideas in real-world context to promote statistical literacy. Conclusively, they aimed to the creation of well-designed LTs that provide a mental scaffold, trying to support both deep understanding and long-term knowledge retention.

Sequencing content raised challenges as it requires balancing research principles with classroom reality while avoiding gaps or overlaps. Maintaining coherence across grades while integrating both conceptual understanding and procedural skills also proved challenging, especially when trying to embed real-world applications to promote statistical literacy. These challenges make apparent the interplay between all three activity systems: policy, research and teaching.

Clarifying the suggested teaching approaches in the teaching guide

When developing the curriculum's supportive documents such as curriculum's philosophy and teaching guide, designers decided to clarify the suggested teaching approaches by drawing on both cognitive and sociocultural criteria for stochastic mathematics. From a cognitive perspective, approaches were selected to foster progressive development of students' reasoning about randomness, variability and data, emphasizing learning strategies such as simulations on familiar environments, dynamic visualizations and opportunities to move from informal to formal statistical reasoning. A key aspect in each of these processes was the students' ability to interpret rather than to use algorithmic approaches. D1 stresses:

> in the new curriculum, within each PLO lies a mathematical practice. Asking the students to just create a chart is fine, that's not an insignificant PLO, but it may have existed in previous curricula as well; it's not that innovative. What is important now is how the students will use the chart and how they will compare it with another representation. I believe that the emphasis on the interpretation is a leading feature of the new curriculum.

These approaches are valued because of their alignment with the development of conceptual understanding. Additionally, designers decided to stress the sociocultural aspects that apply to

teaching and learning, such as equity and inclusivity, that should be taken into account, as they guide the process of selecting content that is engaging, culturally relevant and adaptable to diverse learning. D2 states:

> the new curriculum slightly changes the way of thinking. Statistics is an intriguing subject area that does not require particularly advanced prior knowledge. In other words, even if a student has not been very strong in mathematics, the field of stochastic mathematics offers more entry points for all students. This is something that is generally supported by all colleagues who have taught stochastic mathematics in secondary education. They have observed that even the most disengaged students become involved.

D1 adds:

> it is essential for students to develop the ability to rely on sound reasoning. In the PLOs for stochastic mathematics, the requirement to manage uncertainty reflects a sociocultural practice. Moreover, this skill can be extended and applied to everyday life, highlighting its relevance beyond the classroom context.

In developing the curriculum's supportive documents, challenges arose from balancing cognitive and sociocultural goals, demonstrating the interplay of the research and teaching activity systems. Designers had to ensure that students will be able to interpret data deeply while creating content that is inclusive and relevant for diverse learners.

Proposing a teacher-support system for the implementation of the new curriculum

Curriculum designers were called to make deliberate decisions about the type and characteristics of the teacher support system of the new curriculum regarding its implementation phase. For them, the success of the implementation depends on the teacher-support system that will be created by the policy makers to communicate the rationale of the curriculum properly. They concluded that the support system should be structured to ensure that teachers can effectively translate curriculum goals into classroom practice, by including professional development programs that strengthen teachers' understandings of key statistical concepts and pedagogical strategies, as well as teaching resources such as model lesson plans, assessment tasks and digital tools for data exploration. D1 argues that: "a dynamic teacher's guide should be established. After two to three years of curriculum implementation, the examples and resources should be reviewed, updated, and enhanced to ensure they remain relevant.". Designers also considered the importance of ongoing collaboration and mentoring, encouraging professional learning communities where teachers can share practices, troubleshoot challenges and adapt materials to familiar concepts and contexts. As D1 points out: "ideally, a professional learning community should be established. When the curriculum is implemented simultaneously across all grades, it is essential that teachers receive personalized support, tailored to the specific grade level they are responsible for teaching.". Designers also stress the need for fostering equitable access to students by providing culturally relevant tasks, inclusive teaching approaches and resources for diverse classrooms. By embedding these layers of support, the gap between the intended curriculum and the enacted curriculum can be bridged, increasing the likelihood of sustained and meaningful change in stochastic mathematics education.

In designing the teacher-support system, challenges arose from ensuring teachers could effectively translate curriculum goals into practice. The designers' decisions should be research-based while balancing the demands of policy makers and teachers' professional development to bridge the gap between intended and enacted curriculum. All three activity systems are apparent in these challenges.

Concluding remarks

The design of the new mathematics curriculum in Greece relating to statistics illustrates a comprehensive approach to curriculum design which involves careful and research-informed decisions across multiple aspects, including content selection, learning trajectories, teaching approaches and implementation support. Designers selected the introduction of big stochastic ideas in probability and statistics, such as variability and statistical inference while ensuring alignment with national and international standards, students' developmental stages, real-world applicability and data literacy. They structured content through learning trajectories, sequencing topics to build from intuitive understandings in early grades to formal statistical reasoning in later grades, integrating conceptual and procedural knowledge in authentic contexts. Proposed teaching approaches were clarified in the supportive documents, emphasizing interpretation, reasoning and the progression from informal to formal statistical thinking. They also considered equity, inclusivity and engagement to make stochastic mathematics accessible to all students. Finally, designers proposed a robust support system for implementation, including professional development programs, model classroom activities and professional learning communities to ensure the successful transition into the new educational reality. Decisions were influenced by research resources, prior curriculum resources, knowledge about statistics learning and teaching and teaching experience. Decisions often raised challenges that make apparent the interplay between the three activity systems: research, teaching and policy. These findings align with prior research findings (Sakonidis et al., 2022) which emphasize the interrelation of these activity systems, when making curriculum-oriented decisions. Building upon the work of Potari & Sakonidis et al. (2019) who analyzed curriculum design through the lens of AT, this study highlights how these interrelations affect the design of stochastic mathematics educational core.

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A purpose-turn, not a practice-re-turn

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The ETC conference focuses on teacher education for the learning of practice. In this paper, we discuss the challenges of a practice-re-turn in teacher education and argue for a purpose-turn – a stronger focus on conceptual discourses in which to critically engage both content and practice. We illustrate the importance of a focus on purpose in the narrower sense of mathematics teaching, which highlights its relevance to teacher education. Next, we draw on studies within our wider projects to illustrate that the educational purposes and functions of practice are often not critically engaged with student teachers. We argue that teacher education should have a stronger focus on critical engagement with the educational purposes of teaching, and that research on teacher education should have a stronger focus on critical engagement with the purposes of teacher education.

Keywords: Teacher education, theory-practice, teacher professionalism, teleology.

Introduction: The practice-turn

The idea of practice-based teacher education is not new. Work has been done on mapping what (good) teachers do and systematising this (see Zeichner, 2012). The main idea is that identifying the core practices in the work of teaching would inform what takes place in teacher education. The research work is to identify the practices that are most strongly linked with learners' learning, so that these practices can be taught in teacher education. Research also works to identify the teacher education pedagogies that may serve this purpose – here, the pedagogies of representing, decomposing and approximating core teaching practices (Grossman et al., 2009) have received substantial attention.

However, Grossman et al. (2009) carefully pointed out that "the professional reasoning underlying the practitioners' actions may be invisible" to the learners seeing the practice represented. We read this to suggest that the practice is characterised as much by what the teacher does, as by why they do it. In a narrow sense, this suggests that teacher education should provide access to the reasoning behind teacher actions. A natural consequence of this is to recognise the need for a discourse in which to engage with the reasoning and its grounds. In a broader sense, this opens professional reasoning to considerations of the broader implications of teaching, of what is taught to whom and with what effects. What is the learners' learning that teaching practices must lead to? The intended learning – the purpose – informs the professional reasoning and subsequent judgement of teachers, which together with their actions constitute teachers' practice.

If teacher education has turned from a focus on practice to a focus on knowledge and now back again, there is reason to be cautious about this re-turn. One critique of the idea of identifying and listing core practices to be taken up in teacher education is that breaking teaching into smaller practices to be learned loses track of the bigger purpose of making content comprehensible to (possibly disinterested) learners (Cirade & Crumière, 2019; Kennedy, 2016). Cirade and Crumière suggest a stronger focus on what it takes to occupy the position of teacher in the institution of the school. This recognition of the context of teaching draws attention to the disciplinary, social, economic, political and cultural issues that teachers need to understand to engage in judgements about the educational desirability of

their teaching. The professional judgement is not limited to judgement about the next action in the classroom; it includes "judgement about the educational desirability of [their] actions and arrangements" (Biesta, 2015, p. 10). These are the questions we gather here under the umbrella term of educational *purpose*, the teleological dimension of both school and teacher education.

In reflecting on this re-turn to a focus on practice, the broader context of educational systems under political influence should not be ignored. There is political pressure to emphasise "what works" (Skourdoumbis & Rowe, 2025), and, as Zeichner warned back in 2012, reduce the length of teacher education and take away anything in the programmes that do not have immediate use value (see also Bylund & Player-Koro, 2024). The title of a recent report on the need to reform Swedish teacher education says it all: "Content knowledge and teacher skills" (SOU 2024:81). Unlike some other countries, Sweden has no expert committee for teacher education, and in this situation, the political governing of teacher education becomes sensitive to ideological turns.

Within the research and teacher education discourses, arguments for the continued relevance of propositional, theoretical and contextual knowledge to future teachers' professional judgement prevail. And few teacher educators would disagree that to teach the practice of teaching is to teach the professional reasoning that informs teacher choice. But this is not always reflected in practice. Swedish primary teacher education appears to emphasise "knowledge, skills and competencies that are dependent upon context and that are geared more toward *knowing how* instruction should be carried out than toward *knowing why* education practice is the way it is" (Sjöberg, 2019, p. 130).

There is reason to be cautious about the effect on our teacher education practices when the political discourses of effectivity and skill and the academic discourses of focus on what works converge. We therefore set out to interrogate the question: how is attention to the educational purpose engaged in teacher education? We view the teacher as a professional, where teachers are trusted to make informed decisions based on the purpose they have in mind, though naturally within some constraints. Such a position emphasises knowledge of learners, content and the educational system, and requires reasoning and discretion on the part of prospective teachers (PSTs) (see also Rusznyak & Österling, 2024). The required knowledge can be a combination of contextual and decontextualised knowledge, where for the sake of brevity we refer to the latter as *theory*. The research question thus is, what opportunities to engage with justifications for educational choices and what invitations into discourses where broader critical engagement can unfold are provided in teacher education?

Below, we first provide some examples from existing data on the difficulties arising from a teaching practice that does not align with the purpose of the lesson. Then, we share results relevant to the question above from studies on teacher education conducted in our research teams.

The challenges of practice non-aligned with purpose

We are concerned with the extent to which teacher education could better prepare teachers to critically engage the educational justifications of what and how they teach. To interrogate this, we studied the take-up by students from mathematics teacher education in Sweden. In this longitudinal project (TRACE), we followed 15 PSTs from their final semester of practicum teaching through their first two or three years of teaching. We returned each semester for lesson observations and interviews.

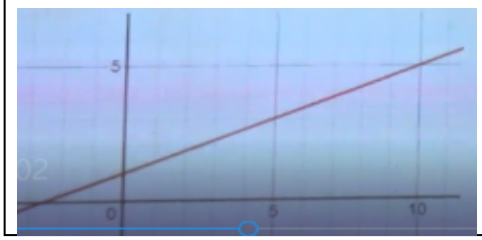
Erik (pseudonym) was a teacher in the TRACE project. In an interview about his teacher education, he expressed his main takeaway to be that "when I do a lesson, (…) I can do it in a lot of different

ways". He saw his task as being to make all learners understand and wanted ways to make teaching effective. He referenced "planning according to Brousseau" and joint problem-solving.

Having been a secondary school teacher for 18 months, Erik is engaging learners on a task of determining the equation of a linear function, the graph of which is given. Before the task is introduced, Erik presents the general method, making statements that indicate that he is inspired by joint problem-solving; he instructs the learners that it is a group task and that they will later show their thinking. After a learner has presented a solution at the board, he asks for different approaches and comments. Yet the task (see Figure 1) leaves little room for problem-solving or discussing different approaches; it directs learners to follow a given sequence of the work and a given procedure.

GROUP TASK

Hint: the k-value, choose Δx so Δy is easily determined from the graph (a whole number).



m =

k = *Δy/Δx* =

negative/positive k?

y =

Figure 1: Task from Erik's lesson

Erik's teaching practice in this case therefore corresponds to the purpose of mastering a previously demonstrated procedure. The task and guidance during the group work leaves little room for activity corresponding to the purpose of joint development of a procedure with its related need for mathematical substantiations and mathematical communication. Both purposes, although contrasting, are highly relevant. Erik wants his learners to work mathematically, but he also needs them to learn the procedures they must know according to the curriculum. Hence, contextual constraints make it harder to align teaching practice with just one purpose (see also Christiansen, 2024; Christiansen et al., 2023), and the educationally desirable is balanced against what is practically possible.

In another case, Tamra tried to engage her learners in an exploration of prime numbers (Christiansen, 2024). She started with strong overtures for learner agency by inviting them to choose their own approach to determining whether a given number is prime. In addition, she asked them to substantiate their answers. However, shortly into the group work, she modelled a procedure for investigating whether a number is prime, which she subsequently used to check learners' proposed answers while the learners listened. Thus, by marking one procedure as correct, the overtures for learner agency and call for learners' substantiations vanished. Interviews with Tamra suggested that she had expected learners to engage their prior learning about divisibility rules, and when learners did not come up with the expected procedure, she introduced a stronger framing. She clearly stated that she had wanted to use a problem-solving pedagogy, but also that her desired outcome for the lesson was about mastery of a procedure. The many contextual and institutional constraints – such as the content-pacing requirements, the constraints of a 45-minute lesson, and the need to manage a classroom of 13-year-olds and their expectations of mathematics learning – encourage teaching that focuses mathematical procedures, making purposes of problem-solving discussions difficult to attain.

In these examples, Erik and Tamra make an effort to include problem-solving teaching, as they were taught, but they also have to make decisions in a complex classroom setting, navigating contextual and curricular demands. From 71 recorded lessons in the TRACE project, Willert (2023) found only six occasions when the new teachers successfully engaged learners in exploratory discussions. Whereas theory from their teacher education did not provide enough guidance in this situation, without this theory, they might not have aimed for teaching beyond providing correct procedures.

Could teacher education better prepare teachers to consider the educational desirability of what they have to teach and to align their practice with desired purposes? Or do PSTs currently have rich opportunities to engage with educational purposes and develop conceptual discourse in which to critically engage such issues? Existing studies from our research teams provide an insight.

The engagement with theory and purpose in mathematics teacher education

Several theories and frameworks have drawn explicit attention to considerations of purpose. We mention here three well-elaborated theories/frameworks specific to mathematics (education). The *competencies framework* which inspired PISA (Niss & Højgaard, 2019) works with a broad description of mathematics that includes both applications/modelling and reasoning. It invites teachers to consider which competencies are addressed in their teaching. As this has become part of the curriculum in Sweden, such considerations have been drawn into teacher education. The *anthropological theory of the didactic* stresses justifying discourses in mathematical and didactic praxeologies alike, and operationalisations thereof assist in understanding the practices of teacher education within conditions and constraints (e.g., Asami-Johansson et al., 2020). By formulating a theory about the nature of mathematics as a discourse, and learning as increased ability to partake in this discourse, the theory of *commognition* formulates a purpose for teaching; this is further nuanced by additions to the theory that help inform the choice of pedagogy (see Nachlieli & Elbaum-Cohen, 2021). These theories draw attention to the mathematics taught and link this to teaching approaches. In that sense, they all serve as a reminder to view teachers' practice as encompassing not just actions but professional reasoning and judgement linked to educational purposes.

However, from several studies conducted by our teams and outlined below, we must conclude that PSTs often have little support for using theory to justify purposes. The studies span a range of methodological approaches – systematic reviews of existing literature, text analyses of teacher education resources, longitudinal case studies of novice teachers, and case studies of teacher education. The analyses were theory-informed deductive analyses.

For instance, to get a sense of what is privileged in teacher education, we studied what was foregrounded in the assessment of PSTs during practicum (Christiansen et al., 2019). We did so by analysing the observation protocols used at a range of teacher education institutions in different countries. We found that the teleological dimension, reflected in reference to goals in the protocols, was rarely present. What was surprising to us was that the importance of formulating a goal or purpose as a basis for instructional decisions was generally not stressed in the assessment of PSTs.

The impact of the policy level on teacher education is illustrated by a study on the written tasks connected to practicum in a Swedish teacher education programme. The policy turned practicum into separate courses, from previously being integrated into mathematics education courses. After this change, written tasks no longer required students to link theory and practice (Österling, 2022).

A case study of the course literature in one Swedish teacher education programme for Grade 4–6 teachers explored the relationships between theoretical and practical ideas in the texts (Pansell & Jatko Kraft, 2021). Often, theory was used without any links to practice, or methods were proposed without justification in theory. Furthermore, the students were invited into a local practice-based conversation but not to take a critical stance towards it, which would have included a critical engagement with the foregrounded educational purposes. A comparative case study foregrounds cultural differences in what PSTs in India and Sweden, respectively, see as purposeful content (Raval et al., 2025). Without critically engaging such cultural differences, teacher education is at risk of uncritically reproducing practices. These studies suggest that our own practices could do more to support PSTs in utilising theory to engage critically with practice.

Review studies on teacher education point in the same direction, however, with some notable exceptions. In a review of research on mathematics teacher education for primary school teachers with a focus on oral communication in the classroom, a range of core practices were identified, including, amongst others, classroom leadership, noticing, preparation for teaching, and contextual activities (Christiansen et al., 2024). The teacher education pedagogies used were categorised according to the categories of Grossman et al. (2009) as representation, decomposition or approximation. It was further noted for each paper whether PSTs were taught a discourse in which to engage questions of why to use the core practice in question. Of the 40 papers included in the review, 31 included some element of legitimising discourse and this was more frequent in cases where PSTs engaged in approximations of practice (Christiansen et al., 2024).

Other reviews found that justifications for practice were rarely made explicit. A review of empirical studies of instructional technology integration in mathematics education found that PSTs engaged in lesson planning for use of IT or rehearsed teaching using IT but that the purposes of the didactic practices they were taught were not presented to them, or were not clearly linked to practice (Mensah et al., 2024). The situation appeared better when it came to representations, reflected in research, of the practice of lesson planning to mathematics PSTs. Of 25 studies, 17 represented the *process* of lesson planning – e.g., listening to an experienced teacher going about planning, or planning together with mentor teachers or university teacher educators (Jones et al., under review). Such representations are likely to include engagements with decisions informing the planning and with the knowledge base for the decisions. However, only one paper explicitly addressed the formulation of lesson goals.

Lesson planning was also the focus of a study on how planning is addressed in the learning outcomes of 18 Swedish primary teacher education programmes, with a focus on mathematics (Jones, 2025). Eleven of these programmes required the PSTs to provide critical justifications for their planning, whereas 16 of them stressed teaching according to the national curriculum. A similar study of learning outcomes in 37 courses for prospective Grade 4–6 mathematics teachers, across 13 Swedish higher education institutions, showed that PSTs were frequently expected to explain existing "works" (theories and research) or to apply these, or to unpack the way the mathematical content had been didactically transposed. Only in 18 outcomes were students expected to explicitly justify or discuss their analyses or teaching choices (Christiansen, under review). A case study of mathematics teacher education practices in Ghana also found that the purposes of the teaching practices that were demonstrated to PSTs were not engaged (Mensah, forthcoming).

In summary, these studies suggest that the engagement with purpose, both narrowly and broadly, is unevenly foregrounded in the teaching of mathematics PSTs. We found it quite widespread in the reviewed literature on teaching PSTs about classroom communication but generally absent in the literature on the inclusion of IT. The studies on practical assessment protocols, course learning outcomes and course literature also give an uneven picture.

Discussion

In this paper, we argue that it is important for teacher education to foreground the purposes of teaching practices, in two respects. (1) Purposes understood narrowly as unambiguous learning objectives must be foregrounded because PSTs must be prepared to make judgements, grounded in a professional knowledge base, that link the learning outcomes of the lesson and the teaching practices. For mathematics in particular, this involves discerning how different purposes of problem-solving or procedural learning require different teaching approaches, and reflecting on the consequences of choices made to opportunities for mathematics learning. As our two cases illustrated, there is a risk of working with partially contradictory purposes in a lesson, leading to practices that do not align well with either of the intended purposes. (2) Purposes understood more broadly as the educational desirability of the content and teaching under the conditions and constraints of the institutional, political, social, economic and cultural context must be foregrounded in teacher education so that PSTs can critically engage with the roles they come to play in society.

Most researchers advocating a practice-turn in teacher education would acknowledge the importance of professional judgement based on research and theory. However, there is also a political discourse surrounding teacher education. In the worst case, teaching is seen as the execution of particular skills or techniques and teachers are viewed as policy implementers (Rusznyak & Österling, 2024). Our concern is that the desire for a shorter and cheaper teacher education is a strong motive for emphasising skills and techniques at the cost of engaging the teleological dimension where the teacher is expected to be able to discern nuanced purposes behind teaching practices.

In teacher education, a stronger focus on the purposes of teaching – including critical reflections on the disciplinary content, the socio-political context, the nature of the curriculum – would position professional justifications as the cornerstone of practices. Strengthening the discourse around the purposes of mathematics teacher education would be a welcome distancing from the policy discourses of skills and effectiveness. We invite colleagues to take part in a purpose-turn for teacher education.

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Early algebra in the encounter between blind and non-blind students

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This article aims to investigate aspects of the development of algebraic thinking with blind and non-blind students that can inform mathematics teaching practices regarding the inclusion in the early years of elementary school. Using on a qualitative approach, data were collected in a fourth-form and a fifth-form class from two public schools in Brazil. The data were collected over three months and recorded through in field diary, and through audio and video recordings. In conclusion, the findings emphasize the importance of dialogue, gestures and the creation of a mental image of a recursive sequence in algebraic generalization. The importance of audio description of what is recorded on the board when collectively sharing strategies is also highlighted. This aspect helps to keep visually impaired students engaged in dialogue.

Keywords: Visual impairment, inclusion, algebraic thinking, mathematics education.

Introduction

The development of algebraic thinking in the early years of elementary school involves work with generalized arithmetic, functional thinking, and modeling, with its core comprising the generalization and expression of generalization through various languages, such as natural language and pictorial representation (Kaput et al., 2008). In this sense, the focus is not on algebra content, anticipating it in the early years in the same way as it was taught in the final years of elementary school, but rather on a way of thinking algebraically based on three principles: "It builds heavily on background contexts of problems. It only gradually introduces formal notation. And, it is tightly interwoven with the following topics from early mathematics curriculum" (Carraher et al., 2008, p. 236). Nacarato and Custódio (2018), Moretti and Radford (2021), and Filha et al. (2022) have explored patterns in the early years of elementary school.

Nacarato and Custódio (2018) highlight that, when exploring patterns with children, it is essential to address vocabulary related to algebraic thinking, such as sequence and regularity. When working with pre-established repetitive sequences, the authors emphasize the need to repeat the pattern at least two and a half times to allow children to identify it. Playing, using the body and movement, musical resources, manipulative materials, recording (oral, written, pictorial and photographic), and problematizing are fundamental aspects in the development of algebraic thinking. Moretti and Radford (2021) recognise the importance of corporeality in the teaching and learning of early algebra. According to the authors, students' cognitive processes are constituted by tactile, perceptive, and synesthetic sensory modalities, among others. During tasks that focus on developing algebraic thinking through the pattern exploration, students mobilize various semiotic resources, such as written and spoken language, gestures, and actions (Moretti & Radford, 2021).

Filha et al. (2022) investigated the development of algebraic thinking in the early years of elementary school with blind students. The research participants attended regular school in one shift. In the extracurricular shift, they attended a rehabilitation and support center for students with visual impairments. The authors' interventions were developed during the extracurricular shift (outside the regular school hours) and consisted of developing a didactic sequence. The results provide evidence that the students understood the concepts of equality and inequality. Furthermore, the authors emphasize the importance of children's literature, playfulness, manipulative materials, and sensory books in fostering the development of algebraic thinking in the early years. Although several studies have focused on the development of algebraic thinking in early years students, none is explicitly related to how this development occurs among blind and non-blind students within the same class at that level. Thus, the objective of this article is to investigate aspects of the development of algebraic thinking with blind and non-blind students that can inform mathematics teaching practices regarding the inclusion in the early years of elementary school.

Theoretical framework

The theoretical framework of this paper is organized around a critical and dialogical epistemology based on Paulo Freire's critical pedagogy, Skovsmose's critical mathematics education, and Radford's semiotic and embodied theory of algebraic thinking. Freire (1972) conceives humanization as a process that encompasses aspects such as respect for all participants in the educational process, the development of democratic attitudes, and commitment to dialogical and equitable participation (Freire, 1972). Alrø and Skovsmose (2004) also contribute to reflection on the construction of dialogical and humanizing interactions in mathematics classes, underscoring that the qualities of communication influence the qualities of learning that occur in the classroom. They also highlight the importance of student engagement during the investigative process. According to Skovmose's (2019) research in mathematics education, curriculum, and classroom teaching practices encompass different groups of students, including those with disabilities. The author conceives of inclusive education as an encounter with difference, opening up possibilities for reflection on the encounter between blind and sighted students in mathematics classrooms.

Concerning early algebra, the paper draws on Radford's theory (2010, 2018), which defines three types of algebraic thinking: factual, contextual, and symbolic. In factual algebraic thinking, "indeterminacy remains unnamed; generality rests on actions performed on numbers; actions are made up here of words, gestures, and perceptual activity" (Radford, 2010, p.56). In contextual algebraic thinking, "the general objects are named through an embodied and situated description of them (e.g., "the next figure", "the top row", etc.)" (Radford, 2010, p.56). In symbolic algebraic thinking, indeterminacy is expressed through alphanumeric symbols. Radford (2006), also point out that pattern generalization involves (a) identifying a common characteristic in some elements of the sequence, which is called commonality; (b) generalizing this commonality to all elements of the sequence; (c) using this commonality to determine any term of the sequence.

Methodology

Focused on a qualitative approach, this research has as its data production context two classes in the early years of two public elementary schools in the city of Belém/PA, Brazil, where blind and non-

blind students are enrolled. The initial criteria for choosing the two classes participating in the research were to have blind and non-blind students together in the same class, as well as the availability of the class teachers to discuss and develop tasks that involved the development of algebraic thinking in partnership with the researcher. The fifth-grade class consists of 26 students, aged between 9 to 12 years old. Among them is Pedro, a boy with adventitious blindness. The fourth-grade class consists of 29 students, aged between 8 and 10. One of them is Maria, a girl with congenital blindness. The classes were followed for three months. The data were collected from audio and video recordings of mathematics classes, field diaries, photographs, records of tasks carried out in class, and all manipulative materials used by students. During data analysis, the videos and audios were revisited multiple times to identify critical events (Powell, Francis, & Maher, 2004). These events were transcribed and analyzed. The tasks developed were designed by the researchers, who are sighted, and subsequently discussed and validated by the community of blind teachers and researchers. This dialogue helped to ensure the accessibility of tasks. During the data production phase at school, these tasks were discussed with the teachers and developed in partnership between the teacher and the researcher. This article analyzes the tasks developed for fourth- and fifth-grade elementary school students, focusing on exploring patterns in repetitive and recursive sequences.

Exploring patterns in a repetitive sequence

During the first pattern exploration tasks developed in the fourth-grade class, the importance of working with ordinal numbers and the concept of odd and even using manipulative materials was noted. Each pair of students received a cup with several small cubes. The teacher asked them to take a small cube from the cup. Meanwhile, she wrote the number 1 (one) on the board, stuck a piece of tape underneath it, and attached the cube to the tape. The teacher and the students discussed whether it was possible to form a pair with just one cube. The students responded that it was not possible, because 1 (one) is an odd number. Afterwards, the students put the cube back in the cup, and then the teacher asked them to take two cubes, then three. Then, everyone, including Maia, took four cubes from the cup and arranged them on the table, forming pairs. Then, the teacher asked:

Teacher: Maia, is this odd or even?
Maia: Even.
Teacher: Why?
Maia: Because there is none left.

The dialogue around the concepts of odd and even continued until number 12. While going through this process, the teacher spoke out loud about what she was doing and what was being written on the board. Then, the repetitive sequence exploration activity began.

“An egg carton,” said a student, laughing as he saw the teacher taking out some egg cartons and distributing them to the class. Each pair received an egg carton with 30 cavities arranged in six rows and five columns. First, the students touched the box. In view of this, aspects of the box arrangement were collectively discussed. In this context, the following aspects were listed: left, right, top, bottom, the difference between rows and columns in the egg carton, and the nomenclature for ordinal numbers. Afterwards, they were asked to place the boxes on the desk, and the teacher filled them with the balls from the first six positions (Figure 1).

Figure 1: Fourth-grade students with the material to complete the task (Source: Researchers' archive)

Description: A fourth-grade classroom. The students are sitting in pairs at their desks and have received egg cartons filled with small white balls.

While large Styrofoam balls were placed in the first, third, and fifth cavities, and the second, fourth, and sixth cavities received small ones. The pairs also received a cup with balls of both sizes so that they could continue the sequence. They were also handed a paper sheet with the task questions: 1) What is the third element of the sequence? And the fifth? 2) What is the seventh element of the sequence? Justify your answer. 3) What is the eighth element of the sequence? How do you know it? 4) Is there anything in common between the elements present in the second, fourth, sixth, eighth, and tenth positions? Explain it. 5) Is there anything in common between the elements present in the second, fourth, sixth, eighth, and tenth positions? Explain it. 6) Which ball is in the fortieth position? 7) How can we determine any element of the sequence? 8) Choose a letter to represent each of the balls in the sequence. 9) Considering the letters you chose, which one will be in the twelfth position? How did you arrive at that conclusion?

Student Maia reached her fingertips to feel the balls. She passed her hands over the balls from left to right, and wanted to know whether they were eggs. The teacher replied that they were Styrofoam balls. Maia noticed that the balls shared some characteristics: some were large, while others were small. Maia identified the commonality (Radford, 2006). Then, her partner analyzed and identified the commonality by observing the balls, and agreed. To identify the missing elements in the sequence that started in the seventh position, the pair of students filled the empty cavities with the balls from the cup. The balls present in the cup would allow them to fill the cavities up to the tenth position. The pair managed to continue solving the tasks up to the sixth question, realizing, for example, that the balls in the second, fourth, and sixth positions were even numbers. The duo identified the commonality and continued the sequence. The sixth question constituted a challenge for the pair. The teacher read the question aloud to the students, and Maia said she did not understand the word "fortieth". Thus, the students could solve the questions that used ordinal numbers to refer to some positions, but had difficulty with the term "fortieth".

The teacher discussed the meaning of the word "fortieth" with the pair and provided examples using smaller ordinal numbers. Hence, the students could answer questions similar to the sixth one when formulated with smaller ordinal numbers, with a manipulative representation in the box. However, the fortieth position remained a challenge, underscoring the need to work on ordinal numbers with the entire class. Maia's gestures allowed her to notice that the arrangement of the large and small balls was repeating itself: large and small. Maia focused on the size of the balls and identified the commonality in the sequence. Thus, the blind student understood the commonality coordinating the perception of her body, with the spatial displacement of her hands and arms from left to right, with

the perception of the balls and their sizes, with positions, words, and with the spatial arrangement of the box. This highlights the importance of the body in algebraic generalization (Moretti & Radford, 2021) in the encounter between blind and sighted students. Furthermore, it was possible to see that limiting the number of balls to which the students had access favored algebraic generalization, as they continued the sequence up to the tenth position using the balls and continued it without them. The challenging moments of the task reinforce the need to delve deeper into ordinal numbers.

Manipulating patterns in a recursive sequence

Each student received a sheet of paper containing the questions, a set of popsicle sticks, and five white Styrofoam cups. In the first cup, there were three popsicle sticks; in the second, six; and in the third, nine. The fourth and fifth cups were empty. The paper sheet contained the following questions: 1) How many popsicle sticks are there in the third cup in the sequence? Explain how you came to this conclusion. 2) How many popsicle sticks will be placed in the fourth cup in the sequence? And in the eighth? Justify your answer. 3) Maria Júlia, a morning-shift student, said that the cup in the ninth position contains 21 popsicle sticks. Do you agree with her? Why? 4) Pedro, a morning-shift student, said that the cup in the 11th position contains 23 popsicle sticks. Do you agree with him? Why? 5) Write a message or record an audio message to another student explaining how to find the number of popsicle sticks in the cup in the 20th position. 6) Write a message or record an audio to another student explaining how to find the number of popsicle sticks in any cup.

The teacher and the students, read the questions aloud, after which the students completed the task individually. They then gathered in groups of three or four to discuss their answers, and the groups collectively shared their findings. When working individually on the task, Pedro felt the popsicle sticks and counted the ones in the first, second, and third cups, without removing them (Figure 2). Using this strategy, he answered questions that referred to the cups that were already filled with popsicle sticks.

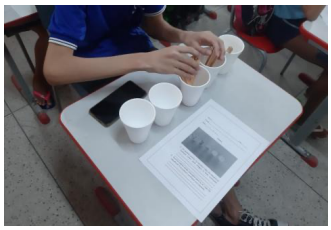

Figure 2: Counting the popsicle sticks in each of the cups (Source: Researchers' archive)

Description: A student sitting on the chair placed in front of his desk. On the table, five Styrofoam cups are arranged in sequence horizontally. Starting from left to right, in the first cup, there are three popsicle sticks; in the second, six; and in the third, nine. The fourth and fifth glasses are empty. The student is touching the sticks in the second and third cups.

The action of manipulating the popsicle sticks with his hands was essential for the student to realize that each cup he counted had an increase of three sticks, which led him to identify the commonality. Although the student had popsicle sticks available, he did not use them to fill the fourth or fifth cup, nor to count the number of popsicle sticks in an inexistent cup. After identifying the commonality, the student abandoned the strategy of counting the sticks; he began counting in threes on his fingers and, on other occasions, performed the calculation mentally. Pedro's action provided evidence that he identified the commonality present in the sequence and created a mental image of it, meaning it

was not necessary to continue using the manipulative material to identify the next elements in the sequence. Pedro generalized the commonality for the missing elements of the sequence, thereby meeting the second characteristic defined by Radford (2006) as essential to pattern exploration.

For the question, "5) Write a message or record an audio to another student explaining how to find the number of popsicle sticks in the 20th position", the student chose to create a written record. The student used a clipboard for the embossed manuscript. Pedro's answer, contains the following text: "To complete the count of the popsicle sticks, you have to count by 3s, twenty times." After completing the task individually, the students gathered as a group and discussed their findings. Then, they shared the group's response with the whole class (Figure 3).

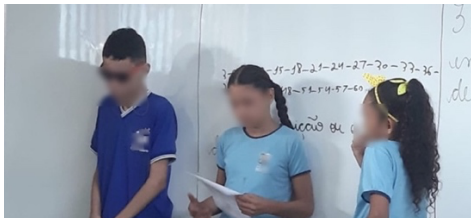

Figure 3: Group of students sharing their resolution strategies (Source: Researchers' archive)

Description: Three students share their strategies for solving the task with the class.

During the collective discussion in which Question 5 was addressed, a group of students, comprising four boys and one girl, went to the board and shared two ways of solving the question. In the first one, they just had to add threes at a time, a strategy that they represented on the board as 3-6-9-12-15-18-21-24-27-30-33-36-39-42-45-48-51-54-57-60. In the second, it was necessary to multiply 3 by 20, a strategy represented on the board by the multiplication algorithm. At that moment, the teacher asked them to explain out loud what they had written on the board. She then asked the students what each of the numbers they recorded referred to. The students explained that number 3 referred to the growth of the sequence, and the number 20 referred to the position or number of cups. The teacher then wrote this information on the board, directly next to the numbers to which each one referred, and explained aloud what she was writing.

During the collective discussion in which Question 6 was addressed: "Write a message or record an audio to another student explaining how to find the number of popsicle sticks in any cup", there was a group that said "count in threes" and another that said "if the first cup has three popsicle sticks and the second has six, just count in threes". A group of three girls comprising a student group recorded an audio response to the question, which was shared with the class at the time of the presentation: "We count in threes, until we reach a specific position with a specific result." This passage shows evidence of the emergence of contextual algebraic thinking, in which students express generality using natural language. The teacher asked the students to replay the audio. In light of this, she wrote the answer sentence on the board to analyze with the class and underlined the terms "specific position" and "specific result". This passage shows evidence of the emergence of contextual algebraic thinking (Radford, 2018) in which students express generality using natural language. Another group responded that it was multiplying by three. The teacher then asked what should be multiplied by three. Some students answered that it was the position the cup occupies; others answered that it was the number of popsicle sticks. The teacher and the students tested both possibilities and concluded that

the correct solution was to multiply the position occupied by the cup by 3. One of the students suggested that the position of the cups be represented by the Flamengo football team, using either its logo or its initial letter. Then, it was collectively agreed that the letter "f" would be used, and the teacher wrote on the board "3xf = number of popsicle sticks." While writing on the board, the teacher read aloud what she was writing. Factual, contextual, and symbolic algebraic thinking (Radford, 2018) emerged during task development; It is noteworthy that symbolic algebraic thinking, in which indeterminacy is expressed through alphanumeric symbols, emerged only in collective dialogue between the teacher and the students.

This moment of sharing strategies is essential for students to perceive other ways of approaching tasks and identify different strategies, expanding their repertoire in relation to these and mathematical vocabulary. Therefore, it became clear that all students must remain engaged in dialogue at these times. In general, many students and teachers conceive their explanations and organize their arguments, prioritizing the visual aspect. Such an explanation tends to exclude and disengage the visually blind student, as they do not perceive themselves as the intended recipient of such discourse. For the collective discussion to be interesting for the blind student, the teacher must deliver an audio description of the images recorded on the board and encourage all groups to produce an audio description of what they write on the board. This aspect becomes essential for the blind student to remain engaged in the task during the moment of collective sharing and systematization. According to Alrø and Skovsmose (2004, p. 236), "thinking aloud refers to the process of making mathematical reasoning and reflections public. Thinking aloud helps to verbalise ideas that otherwise would not have come to the surface of communication and inquiry." Thinking aloud in meetings between blind and non-blind students involves the audio-describing of strategies recorded on the board.

Considerations

The results of this study provide evidence that an inclusive teaching and learning environment, which contributes to humanization in mathematics classes, involves considering the characteristics of people with disabilities when developing tasks that all students in the class will perform. Furthermore, it suggests that blind students understood commonality (Radford, 2006) by coordinating their body perception, with spatial displacement of their hands and arms from left to right, with tactile perception of the sequence elements, with words, and with the spatial arrangement of the sequence elements. It is noteworthy that generalizing commonality across all elements of the sequence involved different strategies: the first involved coordinating gestures and words with manipulable material to complete the sequence. The second consisted of creating a mental image of the sequence and coordinating it with finger-counting and mental-calculation strategies, thus continuing the sequence without the need for manipulatives. The findings also emphasize the importance of the dialogue students established with one another and with the teacher in algebraic generalization. The students shared different strategies for approaching the task. At the same time, the teacher asked questions that enabled them to think aloud, i.e., to explain aspects of their strategies that were not initially explicit and that were essential for algebraic generalization. The importance of audio description of what is recorded on the board when collectively sharing strategies is also highlighted. This aspect helps to keep visually impaired students engaged in dialogue.

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From theories about playful learning to classroom practice: Directions for future research

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In this theoretical paper, we draw on existing research to explore how playful learning, specifically guided play, can be integrated into Norwegian lower primary school mathematics education. Our examination of theory and research highlights multifaceted obstacles that hinder full realisation: teachers' uncertainty regarding their evolving role, the lack of practical implementation guidance, curriculum ambiguities, and policy pressures towards standardised testing. Explicit guidance for implementing efficient, curriculum-intended playful learning is unavailable to Norwegian teachers. Countering perceived practical constraints, we provide directions for future research. These include the need for a framework detailing practical examples of playful learning and mathematisation for educators and policymakers, advocating for future research-practice partnerships, and embedding student-centred teaching like guided play in pre-service and in-service teacher education.

Keywords: Playful learning, lower primary school, guided play, educational policy, curriculum implementation.

Introduction

The ERME Topic Conference on Connecting the Learning of Mathematics Teaching to Practice initially prompted a major question: how to effectively establish respect for teaching as a practice within both pre-service and in-service professional education systems. This raises critical questions concerning the focus, domains, and practices school mathematics should encompass, and how teachers should be prepared. We explore these questions and provide directions for future research in the context of lower primary school, considering the value of playful learning.

Mathematics education research documents the great potential playful learning holds as a pedagogical approach through which students can engage in mathematical problem solving. Nesbitt et al. (2023) proposed a connection between active, playful learning and the six 21st-century skills. Blinkoff et al. (2023), reviewing existing research on the science of learning and interest development, identified the students' interest as a key motivational factor that may mediate the relationship between active playful learning, and the development of the six Cs needed for success in school and life: creative innovation, content expertise, collaboration, communication, confidence and critical thinking (Golinkoff & Hirsh-Pasek, 2016; Hirsh-Pasek et al., 2020). This is particularly apparent in playful situations with adult guidance, that is, what is termed guided play (Skene et al., 2022). Thus, we aim to answer the following question: *Drawing on research literature, which directions for future research can be provided to bridge theory on playful learning and teacher practice in lower primary schools?*

Literature

Guided play and the learning of mathematics

A significant theoretical contribution is the conceptualisation of play, not as a singular activity, but rather as a spectrum of experiences (e.g., Zosh et al., 2018). This provides a nuanced understanding of 'play', ranging from entirely child-directed 'free play' to adult-directed didactic instruction, with guided play occupying the intermediate position.

Weisberg et al. (2013; 2016) adopt an integrated approach to guided play, acknowledging the benefits of both child-led exploration and intentional instruction. The approach is characterised by a delicate balance: the child retains significant autonomy and control, while the adult functions more as a 'coach' or a 'guide' than an 'instructor' or a 'director' (e.g., Weisberg et al., 2013; Zosh et al., 2018). The adult provides purposeful scaffolding and subtle guidance towards specific learning objectives, which aligns with the school's mandate for learning.

Guided play is frequently highlighted as an optimal learning experience for the learning of mathematics (Fisher et al., 2012; Hopkins et al., 2019; Weisberg et al., 2016). The effectiveness of guided play stems from its adherence to core principles of effective learning: it is active, engaged, meaningful, and socially interactive (Zosh et al., 2018), and interest is identified as a key mediator in this process (Blinkoff et al., 2023). Guided play integrates elements of free play and instruction and is highlighted in the research literature for its effectiveness in developing early mathematics skills (e.g., Blinkoff et al., 2023; Toub et al., 2016). Children engaged in guided play demonstrate significant academic benefits, notably achieving greater positive effect on spatial and mathematics vocabulary (Skene et al., 2022). For instance, the seven-year-olds in Flaten's (2025b) study are afforded valuable learning opportunities to actualise arithmetic knowledge, including the cultural tools of parity and place value, refining their justifications toward a structural and general level of reasoning and practicing how to sole mathematical problems in a stepwise manner. Thus, the dichotomy between play and learning is not constructive (Björklund et al., 2018; Hopkins et al., 2019; Pyle et al., 2017).

Taking a cultural historical activity theory perspective, van Oers (2014) argues that rather than treating play as a tool for delivering fixed curricular outcomes, the focus should shift to seeing mathematics as emerging from children's engagement with their environment, through the "mathematisation of children's actions and utterances by more knowledgeable others" (van Oers, 2014, p. 114). This theoretical stance advocates for learning environments that encourage creative problem-solving, exploration and children's degrees of freedom, rather than strict adherence to pre-determined procedures. According to van Oers (1996, 2014), mathematising is fundamentally a form of playful mathematics and direct instruction can be meaningfully integrated into playful mathematics activities, without sacrificing the students' engagement. Thus, play should not be seen as separate from academic goals, but as a context where children can appropriate mathematical knowledge and lifelong skills when supported by the teacher and peers as more knowledgeable others.

The preceding discussion underpins the emphasis placed on guided play as a pedagogically appropriate playful approach to learn mathematics in school, which is an approach aligned with

objectives of primary education and the school's mandate for learning. However, even if policymakers and mathematics education researchers can agree to move beyond the false dichotomy between play and learning (e.g., Björklund et al., 2018), there are still several significant and often interlinked challenges to realising playful learning in school mathematics.

Obstacles for implementing playful learning in mathematics in lower primary school

Teachers and their contextual factors are the most significant factor for implementing playful learning. Teachers play a fundamental role across the entire teaching process, from planning to evaluation. Adopting a playful learning approach in school mathematics requires shifting the teacher's role from a traditional lecturer to an active dialogue collaborator. Research shows teachers often report uncertainty regarding their role and how to effectively integrate mathematics into playful learning situations (e.g., Bubikova-Moan et al., 2019; Jay & Knaus, 2018). Without training and understanding of mathematics and child development, teachers may fear losing control or failing to meet academic standards when moving away from traditional methods.

Research provides several recommendations regarding the teacher's role in playful situations. For instance, teachers should be responsive to children's ideas and interests, situate known mathematical concepts, and subtly challenge concept meanings to extend experiences without disrupting play (Björklund et al., 2018; Fisher et al., 2012; Pramling Samuelsson & Björklund, 2023; Wickstrom & Pyle, 2024). A recurring theme is that teachers need to behave as a "coach" or a "guide", asking open-ended questions rather than showing ways to solve mathematical problems. However, examples of how to do so in a playful context in school mathematics are lacking (Parker & Thomsen, 2019) and teachers need experience in student-centred teaching (Bubikova-Moan et al., 2019).

A lack of student-centred teaching experiences in Scandinavian early childhood education presents a paradox, given the region's social-pedagogical tradition and holistic view on children's learning (OECD, 2006). This tradition has influenced the Norwegian Framework Plan and school curriculum (Ministry of Education and Research, 2017; 2020). The Framework Plan (for kindergartens, ages 1–5) prioritises lifelong learning, supporting engagement in key mathematical areas (numbers, spaces, shapes) through play and exploration, rather than explicit school preparation (Ministry of Education and Research, 2017). The entry of six-year-olds into school via Reform 97 (R97) aimed to facilitate a smoother transition from early childhood play-based learning (Kirke-, undervisnings- og forskningsdepartementet, 1996). The current 2020 mathematics curriculum also emphasises play as an essential teaching approach for achieving competence goals in the first four school years (ages 6–9). Nonetheless, while play features in three competence goals beyond Year 2 (Ministry of Education and Research, 2020), the curriculum offers no directions on its implementation.

Consequently, a gap exists between the official, enacted, and experienced curriculum. Bjørnestad and colleagues (2024), evaluating the 'six-year-reform' (R97-2020 curriculum) for the Norwegian Directorate for Education and Training, provided insights into 531 responding first-grade teachers' experience and practice. Furthermore, half of these first-year teachers reported that national diagnostic assessments in mathematics influence subject progression (Bjørnestad et al., 2022). External policy mandates and curriculum delivery pressures are consistently cited as major obstacles to implementing a play-based approach (Bubikova-Moan et al., 2019; Jay & Knaus, 2018; Lillejord

et al., 2018). This increased focus on learning outcomes and standardised testing often result in what is referred to as the "narrowing of the curriculum" and reduced time for play. Despite the Norwegian primary school context being highly influenced by a social-pedagogical tradition that emphasises holistic learning and play in the curricula, further discussion is needed regarding the consequences of this increased focus on learning outcomes and testing.

Bjørnestad et al.'s (2024) final report drew on three surveys (to school authorities, leaders and Year 1 teachers), classroom observations, and interviews with students, teachers, and school leaders. The researchers concluded that small, cramped classrooms and a lack of suitable play materials constitute practical constraints for teachers wanting to adopt a more playful approach. Bjørnestad et al. (2024) reported that first-grade teachers "did not explicitly emphasise play when discussing mathematics" (p. 166, own translation). Although there is a consensus within the Norwegian educational system regarding the value of play, teachers must individually operationalise the playful approach in their mathematics classroom and navigate their way through obstacles by themselves. Since the curriculum lacks implementation specifics and research is limited, teachers are left to figure it out. This confirms a gap between research and practice. Educators and policymakers need examples of what playful learning in school mathematics might look like and how to implement it.

Teachers' attitudes form a related obstacle: while policymakers and mathematics education researchers move beyond the dichotomy of play and learning, teachers may still hold this view. Bjørnestad et al. (2024) reported that Norwegian first-grader teachers did not emphasise play in mathematics. Furthermore, a meta-synthesis by Bubikova-Moan et al. (2019) of 62 studies (ages 0–6, 24 context) reported that teachers struggle to see how playful approaches lead to academic learning. This causes uncertainty about when and how to intervene to support specific learning objectives. Due to this lack of clarity and concern over balancing student interest with academic goals, guided play implementation in primary school is uncommon (Lillejord et al., 2018; Parker & Thomsen, 2019).

Discussion

We argue that the practical constraints experienced by teachers in the study by Bjørnestad et al. (2024) does not need to be major obstacles. While the size of Norwegian classrooms is difficult to resolve quickly, Flaten's (2025c) experiences suggest minor adjustments may facilitate implementing playful learning in mathematics teaching. Flaten's studies (2025a, 2025b) exemplify playful activities that are fully realisable within the seemingly constrained classroom context. The participating teachers reported relatively little effort was needed to capture students' interest (Flaten, 2025c) which is a key aspect of play-based learning (Blinkoff et al., 2023). Furthermore, neither classroom size, setup, nor equipment were seen as major obstacles (Flaten, 2025c).

The second obstacle, the individualised operationalisation of playful learning within a context focused on testing and explicit learning goals, is more challenging. There is a need for an explicit focus on playful learning in primary school mathematics curricula, and a framework where researchers and educators provide examples of what playful learning looks like and how lower primary teachers can mathematise students' play. Such a framework is necessary for teachers interested in playful approaches, those who see the freedom of free play conflicting with the school's learning mandate, and those who favour direct instruction. Broadly, policymakers and parents, who

may impose external pressure and advocate for more traditional approaches, also need to see examples of how playful learning contributes to students' mathematical learning.

However, providing concrete interaction examples might risk undermining the responsiveness expected of teachers in guided play. We need research into teachers' strategies of implementing and guiding students in playful mathematics, and how these strategies influence the students' freedom within various playful mathematics activities. Still, the research of Flaten (2025) suggests that the teachers constructively drew on their written instructions. The observations made by Flaten (2025c) imply these resources supported the teachers in the initial phases, without constraining their development in applying playful approaches.

Thus, we encourage further research where researchers work closely with practitioners to bridge the gap between research and practice. In recent decades, there has been a shift towards evidence-based practices in education, and research-practice partnerships have emerged as a promising approach to bridge this divide: to meet practitioners' needs and support their use of research (Coburn et al., 2013; Sjölund et al., 2023).

Building on Wickstrom and Pyle's (2024) arguments within the kindergarten context, we recommend further exploration of how guided play, incorporating elements of direct instruction where appropriate, can be strategically integrated within the mathematics curriculum in lower primary school. Research is encouraged to study how various ways of engaging with students in guided play affect their freedom to direct the activity. Furthermore, mathematics education researchers must provide examples of how teachers' implementation may incorporate various strategies. These examples must show how strategies may involve teacher-led elements and instruction, following the arguments provided by van Oers (2014) and Björklund et al. (2018). Providing this nuanced picture of what guided play is supports teachers in adopting a playful approach and may help limit the gap between theory and practice versus how it may be perceived by policymakers and educators.

Our recommendations also extend to mathematics teacher education. Drawing on the role of play in the Norwegian curriculum and research documenting its potential in school mathematics, pre-service and in-service teachers may learn how to implement a guided play approach by engaging with real students when implementing such a student-centred teaching. Playful learning needs to become part of pre- and in-service teacher training. This calls for mathematics education research which is classroom-based and bridges the gap between theory and practice. There is a need to provide practical insights into successful implementation of playful learning in school mathematics. Such research, across various areas of the mathematics curriculum may prompt school leaders to enable teacher collaboration in planning and implementing playful mathematics activities.

In conclusion, we suggest that by pursuing these research directions, which involve developing a framework and systematising knowledge about lower primary students' learning of mathematics through playful learning, the mathematics education research community can better support teachers to realise the intentions in the National guidelines and achieve successful implementation.

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A task for teacher education: Learning for the practice of teaching fraction division

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This study describes a task for teacher education aimed at promoting learning for teaching practice in fraction division within an online continuing education context. The task was grounded in the Mathematics Teacher's Specialized Knowledge (MTSK) model and in interpretative knowledge, articulating teachers' specialized knowledge with the ability to understand, analyze, and respond to students' reasoning. The proposal fostered the revisiting of specialized mathematical knowledge, pedagogical decision-making, justification of choices, and the discussion of teaching strategies, thereby creating a space for critical analysis of professional practice. The results indicate an expansion of participants' specialized knowledge and a recognition of different approaches to teaching, while also revealing difficulties and uncertainties concerning classroom implementation.

Keywords: Fraction division, teacher education, learning for practice, specialized knowledge.

Introduction

Studies indicate that fraction division is one of the most challenging mathematical topics for both students and teachers. Many students struggle to grasp its underlying concepts, while teachers often rely on mechanical algorithms with little conceptual grounding (Adu-Gyamfi et al., 2019). This positions fraction division as a fertile ground for reflecting on the relationship between teaching practice and professional learning.

From this perspective, we designed and developed a teacher education proposal centered on a Task for Teacher Education focused on the teaching of fraction division, aiming to foster learning for practice. The design of this proposal was grounded in the Mathematics Teacher's Specialized Knowledge (MTSK) model (Carrillo-Yañez et al., 2018), the concept of interpretative knowledge (Di Martino, 2019), and recent studies on the teaching of fraction division, considering both its challenges and teaching practice as a privileged space for the development of professional knowledge (Stromholt et al., 2024). The guiding research question is: In what ways can a professional development task centered on fraction division foster the learning of teaching practice by teachers? The aim is to present, analyze, and discuss how a professional task can promote the learning of teaching practice related to this topic, while simultaneously supporting the development of the specialized and interpretative knowledge required for its teaching. The study thus seeks to understand the process by which teachers learn to teach more effectively through reflected practice, and how this experience contributes to their ongoing professional development.

Theoretical framework

Fraction division constitutes a central topic in Mathematics Education, as it involves distinct meanings (partition and measurement). Its understanding implies not only applying the "invert and

multiply" rule but also interpreting the underlying ideas and adjusting teaching actions according to students' different forms of reasoning.

The concept of *Pedagogical Content Knowledge* (PCK), introduced by Shulman (1986), is particularly relevant in this context, as teachers need to articulate meanings, select appropriate representations, and adopt strategies that foster understanding. Recent studies reinforce the importance of teacher education proposals that integrate multiple registers of representation and support pedagogical decision-making, thereby broadening learning opportunities and promoting more reflective teaching practices (Tossavainen, 2025).

Within this scenario, models such as MTSK (Carrillo-Yañez et al., 2018) specify the particularities of mathematical teacher knowledge, encompassing both mathematical knowledge and PCK. This perspective seeks to create opportunities for prospective teachers to learn to teach both *from* practice and *about* practice (Ball & Forzani, 2009). Along these lines, the work of Wood and Horne (2022) highlights the importance of professional tasks that bring teachers closer to real practice and develop the competence to understand, interpret, and respond to students' thinking. This competence is related to *Interpretative Knowledge* (Di Martino et al., 2019), defined as teachers' ability to make sense of students' reasoning by interpreting their responses — correct, incorrect, or unexpected — in order to respond pedagogically rather than evaluatively. In light of these discussions, this study presents and analyzes a teacher education proposal structured around a Task for Teacher Education (TTE) aimed at fostering learning for teaching practice in the context of fraction division.

Context and method

This study reports on the results of a teacher education proposal implemented through a Task for Teacher Education (TTE) on fraction division, involving Brazilian elementary school teachers (ages 7 to 14). The group consisted of two experienced mathematics teachers (Bruno and Ana), two prospective teachers in the final stage of their Mathematics degree (Dina and Carlos), and two pedagogues with teaching experience (Célia and Eva). The professional development proposal was conducted online over a period of three days, with sessions lasting two hours each day. Data was collected through an online questionnaire sent to participants, observations during the Google Meet sessions, teachers 'submitted work via email, as well as audio, video, and chat recordings. The TTE, discussed collectively, aimed to develop specialized and interpretative knowledge of fraction division and to foster connections between theory and teaching practice. The MTSK model proved effective both for identifying elements of teaching and for guiding the design of TTEs.

Structured in two parts (Figure 1), the task addressed possible needs concerning teachers' specialized knowledge and encouraged the articulation between theory and practice in a collaborative environment. This setting enabled experiences like those offered to students, thereby promoting conceptual understanding, deeper mathematical–pedagogical reflection, and the learning of practice. Specifically, Part I focused on broadening teachers' knowledge of estimation strategies, meanings of division, and problem posing, whereas Part II aimed to develop specialized and interpretative knowledge related to representations, algorithm validation, resources, and students' difficulties.

Part I – Questions for Teachers
Task: Operating with Fractions
(*You should always explain your reasoning by describing the process you used to answer the question. You may do so using diagrams, words, calculations, drawings...*)
Consider the following expressions below:

a) 5:2
b) $\frac{2}{5} : 4$
c) $7 : \frac{1}{2}$
d) $\frac{12}{15} : \frac{3}{5}$

i) Without resorting to the algorithm, indicate what you consider to be the approximate or exact value of each expression. Explain your reasoning.
ii) Solve each of the above expressions, indicating the exact value and presenting the process you used to find the answer.
iii) Formulate a problem for each of the above expressions, ensuring that its solution requires the specific expression.

1) Consider the task above:
a) Solve the task (from the section above) yourself.
b) Represent each of the expressions (a, b, c, and d) in at least two different ways (you may do so using diagrams, words, calculations, drawings...). Justify your answers.
c) Indicate a set of possible specific difficulties that students might encounter when solving this task (specify the grade level of the students).
d) If you had to teach fraction division to a seventh-grade class, how would you approach it? Would you use any resources? Which ones? Why (with what purpose)?

Part II
2) Teacher Gabriela implemented the same task with her seventh-grade classes. Some student work differed from what she had anticipated. Below are examples of these responses. Three students solved item (d) as follows:
Luís: $\frac{12}{15} : \frac{3}{5} = \frac{12}{15} \cdot \frac{5}{3} = \frac{4}{3}$

Rafael: $\frac{12}{15} : \frac{3}{5} = \frac{12 : 3}{15 : 5} = \frac{4}{3}$

Mônica: $\frac{12}{15} : \frac{3}{5} = \frac{12}{15} : \frac{9}{15} = \frac{12}{9} = \frac{4}{3}$

a) For each of the students' responses, indicate whether you evaluate it as mathematically correct (appropriate) or not, justifying the mathematical reasoning evidenced.
b) Provide constructive feedback to the students; that is, make sense of their solutions in such a way that they may later contribute to the development of their mathematical knowledge.
c) Which of the above approaches would you select to teach your students? Justify your choice.

3) Another seventh-grade student, when solving the task, asked:
Gabriel: *Teacher, why do we have to multiply when solving a fraction division? That is, why do we invert the divisor and multiply in order to obtain the result?*
a) What would you answer Gabriel? Justify your response.
b) Which representation could we use to give meaning to Gabriel's question?

Figure 1: Task for Teacher Education (TTE) - Fraction Division

The analysis was structured around seven categories, developed based on the specific objectives of each task question and linked to the MTSK theoretical framework and Interpretative Knowledge, in order to support the investigation of the research question. These categories encompass key dimensions of teaching practice: representations, algorithm validation, use of resources, students' difficulties, estimation strategies, interpretative knowledge, and problem posing.

Teachers' written productions and plenary discussions were considered, analyzing the types of solutions, justifications, and debates. The analysis involved identifying evidence in each response, grouping it according to common knowledge, and examining its relation to the selected categories, ensuring alignment with the goals of professional development and research.

Results

The results are presented by highlighting the potentialities, challenges, and limitations of the professional development task, based on seven a priori categories, which were established in alignment with the objectives of the task questions and with the theoretical framework on fraction division. These categories — *estimation as a professional practice; anticipating students' possible difficulties; selection of resources for learning; validation of strategies; representation as a mediator of practice; interpretative knowledge; and problem posing as a learning strategy* — reflect the central themes addressed in the task. This analytical organization provides an integrated understanding of how the task supported both the learning of teaching practice and the development of specialized and interpretative knowledge.

Regarding *estimation as a mathematical practice*, the study shows that the task fostered discussions on this process, highlighting its complexity, the challenges involved, and the specialized knowledge required. The central role attributed to the algorithm in solutions, justifications, and teaching practices revealed participants' difficulties in mobilizing alternative strategies such as estimation. As Carlos stated: "For me, it is difficult to estimate; I immediately think of the algorithm in my head"; and Célia added: "I immediately think of the least common multiple (LCM) or convert to decimals; that is how I learned, and I do not know another way to explain or do it." The difficulty with estimation seems to be related both to conceptions of mathematics and to prior formative experiences, indicating that early and algorithm-centered teaching restricts the development of strategies such as mental calculation and estimation. On the other hand, the task enabled collective discussions on mental calculation, number sense, and properties of division. In this context, teacher Ana highlighted: "I thought like this for the operation ($\frac{12}{15} \div \frac{3}{5}$): I know that $\frac{12}{15}$ is less than 1 and greater than $\frac{3}{5}$, so the answer will be greater than 1. I made an estimate." This strategy shows how the teacher used benchmarks to anticipate the interval of the result, resorting to estimation without relying on the algorithm.

In the task focused on *anticipating students' possible difficulties*, it was possible to foster discussions and reflections on the obstacles that might emerge in learning fraction division, both conceptual and procedural. This discussion proved significant, as the in-service teachers mostly pointed out procedural difficulties. In contrast, the prospective teachers, lacking classroom experience, found it more difficult to anticipate potential obstacles, explicitly citing their limited teaching practice. As Carlos noted: "I do not know what difficulties students might have with the task because I have not yet taught." Ana added: "One difficulty would be with fractions themselves, because students usually do well with whole numbers, but fractions are more difficult for them." However, the ability to anticipate students' difficulties constitutes part of the specialized knowledge required for teaching, underscoring the need for professional development that fosters the growth of this kind of knowledge.

Regarding the *selection of resources for learning* in the teaching of fraction division, the task highlighted the centrality of the "invert and multiply" algorithm, justified by teachers both through

the emphasis given in textbooks and through their own school experiences. Manipulative resources were also mentioned, such as the Tangram and the fraction strip, as well as the use of software, although not always directly related to fraction division. Bruno (in-service teacher) noted: "I usually use the Tangram in my classes because students enjoy it." Ana (in-service teacher) said: "I usually use IM [invert and multiply], which is in the textbooks and is how I learned." Célia (pedagogue) explained: "I use the fraction strip, but not for teaching division." Eva (pedagogue) added: "I use mathematics software." In discussing these choices, teachers reflected on the importance of using resources with mathematical intentionality, recognizing the need to understand their potential and limitations and to select representations appropriate for each topic. In this process, it became clear that a well-founded choice of resources is not merely a didactic aid, but rather an essential element to foster students' conceptual understanding and the development of teaching practice.

The task encouraged teachers to reflect on justifying algorithms and validating students' solutions, especially unusual ones. Ana noted: "[...] now I understand why the algorithm works and what to do with it, because I always used IM but had never stopped to think about what it meant." Bruno added: "Many of my students ask me why this is necessary when dividing, and I always responded that it was simply the rule." In the interpretative knowledge task, teachers analyzed three student solutions, validated them, and provided feedback, encountering uncommon procedures such as Divide Numerators and Denominators (DND) and Equalize Denominators (ED). Bruno observed: "Rafael's solution (DND) is not something that appears in textbooks; I have never seen it, and I do not know if it always works." Eva commented: "I think Mônica's solution (ED) is not incorrect, but it is not necessary to find equivalent fractions to perform fraction division; we usually do this in addition and subtraction." Dina added: "Now, seeing Monica's solution (Task 2), I can think of a pictorial representation to try to justify the procedure" (Figure 2).

These discussions expanded teachers' repertoires beyond the standard algorithm, incorporating strategies based on equivalences and pictorial representations. Mônica's production stimulated further reflection, leading teachers to reconsider Rafael's solution and recognize the feasibility of using equivalent fractions even when numbers are not multiples, opening space for a new understanding of these solutions.

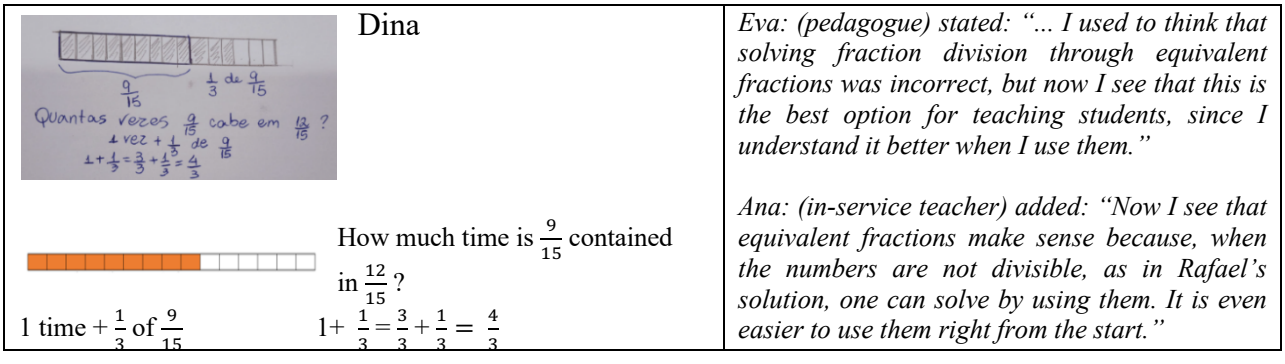


Figure 2: Teacher Dina's representation and teachers' discussion

The task on *representation as a mediator of practice*—that is, producing pictorial representations of fraction division—proved powerful for fostering understanding of this operation and for broadening knowledge about reference units, part–whole relationships, meanings of division, and types of

representation. An example was the "wall problem," designed by teacher Carlos ($\frac{2}{5} \div 4$): *"Two-fifths of a wall remain to be painted. Four painters were hired, and they must each paint the same portion of the wall. How much will each painter paint?"* From this situation, it was possible to discuss different forms of representation and their articulation with mathematical language.

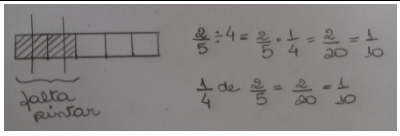
 Instructor: *"Where is the representation of $\frac{2}{20}$?"* Dina (prospective teacher): *"The $\frac{2}{20}$ was simplified; we found $\frac{1}{10}$."* Instructor: *"Yes, but does the answer you found correspond to your pictorial representation?"* Teachers: *(they were not able to explain why or how to do it).* Instructor: *"Where is the pictorial representation of $\frac{2}{20}$?"* Dina (prospective teacher): *"I am trying to relate it to the area of the chocolate."*	Ana (in-service teacher): *"For me, this is a sharing problem and discrete."* Instructor: *"When would it be continuous for you?"* Ana (in-service teacher): *"When the result is not exact?"* Instructor: *"And isn't this result inexact?"* Bruno (in-service teacher): *"We usually deal with discrete and continuous only in Statistics."* Dina (prospective teacher): *"This was something new for me—the meanings of measurement and sharing, and the representation, this issue of discrete and continuous."*

Figure 3: Teachers' discussion

The discussion Figure 3, for example, highlighted the importance of articulating mathematical language and representations as an essential condition for an effective understanding of procedures. It also emphasized the relevance of reflecting on the diversity of possible representations for the same problem as exemplified in Figure 4, with different registers for the operation 5 ÷ 2.

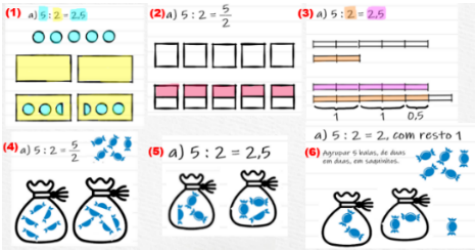


Figure 4: Representations of the operation 5 ÷ 2

The teachers stated that knowing these different registers of representation can contribute to learning, as they support the construction of concepts. In this way, teachers realized that teaching and learning Mathematics implies recognizing that cognitive activity requires not only natural language or images, but the articulation of multiple registers of representation and expression. In this context, the debate on the reference unit was highlighted, based on the representation developed by Carlos (Figure 5).

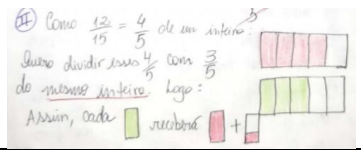
Carlos: 	Like $\frac{12}{15} = \frac{4}{5}$ of an integer. I want to divide $\frac{4}{5}$ with $\frac{3}{5}$. Therefore: Thus, each green part will receive one and a third of the red part.

Figure 5: Teacher Carlos's representation of the operation $\frac{12}{15} \div \frac{3}{5}$

This reflection highlighted the importance of understanding the flexibility of the reference unit in partitive representation: while the dividend and the quotient must maintain the same unit in relation to the whole, the divisor can assume a different one (such as $\frac{3}{5}$). In the measurement meaning, however, the dividend, divisor, and remainder must share the same unit; thus, the quotient $\frac{4}{3}$ would take $\frac{3}{5}$ as its reference, rather than $\frac{4}{5}$. Another key point was recognizing the challenge of understanding and producing pictorial representations with fractions other than unit fractions. The discussions showed that it is not enough to represent cases with numerator 1 and then generalize, as there are difficulties in carrying out the geometric procedures of fraction division.

Problem posing proved to be a powerful resource for knowledge construction. Through posing problems, teachers revisited the concepts of partition and measurement, as well as the correspondence between the required operation, the statement produced, the solution, and its pictorial representation in an appropriate context. They also reflected on the importance of problems posing as a school mathematics activity. As Bruno (in-service teacher) stated: "I did not think it would be so difficult to create a problem for fraction division. We usually do not think about posing problems, only about solving them. Now, with my difficulty, I see how important it is." The task also made it possible to discuss problems related to the measurement meaning and to identify inadequate contexts for the operation.

Conclusion

This study contributed to understanding the process of learning to teach fraction division, highlighting the relevance of tasks that elicit pedagogical decisions and justifications. These include the choice of didactic resources, the selection of numbers for examples, the validation of algorithms, the use of estimation strategies, problem posing, and the use of pictorial representations. The teacher education proposal enabled teachers to engage in teaching practice in an authentic situation with structured support. The task also fostered collective discussions that broadened participants' specialized knowledge, leading them to report significant advances in their understanding of the topic and a willingness to apply such knowledge in the classroom. Despite recognizing different approaches to teaching, the teachers showed difficulties and uncertainties, especially due to their concern about not being able to respond to unexpected student questions on the topic. These results underscore the need for professional development experiences that promote the articulation of differ formative task representations and support well-founded pedagogical choices (Tossavainen, 2025).

In addition, the study sheds light on what is important to consider when designing similar tasks — namely, the inclusion of opportunities for decision-making, justification, and reflection on the teaching process.We argue that the design of mathematical tasks focused on learning for practice constitutes a promising avenue for professional development, as it makes evident the articulation between content and teaching practice. The findings of this study have important implications for teacher education, showing that practice-centered tasks requiring pedagogical decisions and justifications can develop specialized and interpretative knowledge. They also highlight the need for continuous contextualized professional development with collective discussions, enabling teachers to reflect on strategies, understand students' reasoning, and support well-founded pedagogical choices.

Our findings are consistent with research in different contexts (Adu-Gyamfi et al., 2019; Lee, 2017), expecting that the ideas presented here may benefit teacher educators in different countries, contributing to a stronger preparation of teachers for teaching practice related to fraction division.

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Exploring the subject-specificity of practice-based mathematics teacher education: The case of examining student work

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We explore practice-based mathematics teacher education through the lens of instructional exchanges, focusing on how participants in mathematics teacher education examine student work in approximations of problem-based lessons. We propose three mathematically-specific categories of perception—responsiveness, serviceability, and normativity—that can be the learning content at stake in those activities. These categories exemplify distinctively mathematical teaching competencies that mathematics teacher educators may cultivate.

Keywords: Mathematics instruction, practical rationality, noticing, algebra, geometry.

Introduction

In this conceptual paper, we explore implications of research on mathematics teaching for practice-based mathematics teacher education (PBMTE; Ball & Forzani, 2009). Research on mathematics teaching points to the instructional exchanges between student mathematical work and mathematical content at stake in instruction that teachers manage (Herbst & Chazan, 2011). Applying them to PBMTE, we illustrate what it might mean for PBMTE to emphasize its subject-specific character. We do that by focusing our argument on teaching teachers how to examine students' mathematical work in problem-based lessons. As PBMTE curricula cannot address all mathematical content teachers will teach, we advance that PBMTE should foreground mathematically specific categories of perception (Herbst et al., 2023) in interpreting student work on content-related tasks. We call for research into how mathematics teacher educators (MTE) cultivate attention to these categories as their clients (prospective or practicing teachers) learn to respond to students' mathematical work.

The PBMTE field has identified core practices warranting cultivation by clients, including mathematical tasks of teaching (MToT) listed by Ball et al. (2008). Among these is interpreting students' work on problems, often enacted by having clients examine written work or videos (Sherin & Han, 2004; Walkoe, 2015). While mathematical topics are often explicit as instructional content, the learning goals for clients are frequently described generically—including the cognitive skills of noticing: attending, interpreting, and deciding how to respond—rendering the mathematical opportunity to learn less visible. This generic framing offers little guidance for MTE to select student work, understating the need for a distinctively mathematical gaze in the design of PBMTE. Building on work on practical rationality of mathematics teaching, we argue that MTE need to consider mathematical categories of perception when organizing PBMTE tasks involving examination of student work, and researchers should inspect client's socialization into these categories.

Theoretical framework

The practical rationality framework articulates the teacher's perspective managing an economy of meanings, involving exchange of students' work on tasks for fulfillment of instructional goals—both manifestations of mathematics requiring simultaneous attention (Herbst & Chazan, 2011). When

MTE engage clients in examining student work, those engagements may also have exchange value in the form of knowledge for teaching. In line with Chazan and Herbst (2023), we use the notion of instructional exchanges to frame the content of PBMTE from the perspective of MTE: When clients examine student work on a mathematics problem, how does mathematics play a role in identifying the exchange value of that activity? Herbst and Chazan (2023) proposed that PBMTE entails a second-order instructional exchange represented in Figure 1. We further unpack this diagram to identify the specific forms of knowledge at stake when clients examine student work on a problem posed within a particular lesson.

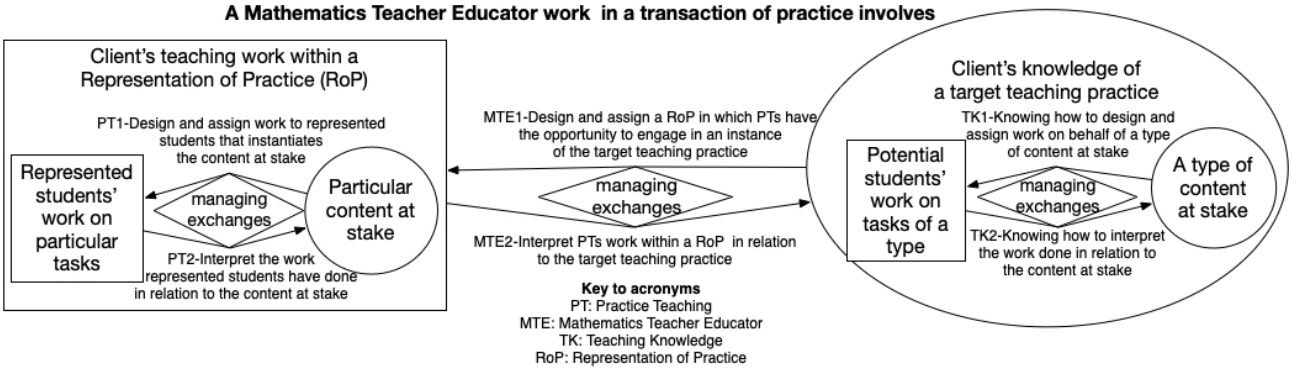

Figure 1: Instructional exchanges in PBMTE (adapted from Herbst & Chazan, 2023, p. 5)

PBMTE engages clients in MToT within reduced complexity settings—approximations of practice (Grossman et al., 2009). These may use social resources (peers, stand-in children, adults, AI agents) and material or technological resources substituting for real classrooms, including videos, cases, dialogues, lesson plans, storyboards, student work samples, or software. We term these artifacts Representations of Practice (RoP; Herbst, 2018), emphasizing their semiotic role: They evoke real classroom work while only partially resembling classroom artifacts or participants. These resources also evoke mathematical content from school curricula. Clients' actions with RoP represent teaching by pointing to classroom teaching, drawing from a semiotic infrastructure that can account for the PBMTE curriculum. We apply this perspective to a specific MToT: examining student work.

Though this is a conceptual contribution, experiences observing Story*Circles* (Herbst & Milewski, 2018), where clients anticipate lesson development by creating storyboards, help us ground our argument. Story*Circles* is a virtual form of Lesson Study where clients collaboratively design lessons (Herbst & Milewski, 2018). We recently observed groups of secondary teachers from Bulgaria, India, Philippines, South Africa, and the United States engaged in this collaborative process. U.S. teacher groups composed storyboards that represented problem-based lessons in algebra or geometry, which were then annotated by other groups before being revised by the original group. In composing their lessons, groups used problems given by designers and aimed at knowledge specified by designers, yet clients retained freedom to frame those problems and to decide on whose students' work to focus discussion on. Each lesson included moments in which the teacher had to consider student work for whole-class discussion. Examples from Story*Circles* assist our interpretation of Figure 1 and our illustration of this conceptual contribution. On the left side of Figure 1, clients are shown engaging in perusing and annotating written student papers from instances of problem-based lessons such as

The Midpoint of Hypotenuse[1] (MH) or *The Line through two Points*[2] (TP). Clients' examination of student work in Story*Circles* simulates moments when teachers observe students before deciding whose work to highlight in a classroom discussion. As clients annotate digital copies, they rehearse the examination of student work and have opportunities to learn knowledge tied to the MToT of examining student work in problem-based lessons. What is the nature of this knowledge? As noted above, clients' examination of student work can be interpreted by MTE as evidence of cognitive skills associated with noticing student thinking. From the perspective of MTE, however, one might ask whether examining student work on MH affords the same or different opportunities to learn the mathematics of examining student work as examining student work on TP.

Our research on teaching practice within the Practical Rationality framework offers resources for PBMTE, particularly for addressing whether examining students' work on different problems affords similar learning opportunities. In studying instruction in problem-based lessons, we have found teachers employ different categories of perception when examining student work that enable them to carry out MToT and manage instructional exchanges (Herbst et al., 2023). In this prior research, we documented three categories of perception: responsiveness, serviceability, and normativity. In this paper we propose these categories can be useful to identify the knowledge at stake in PBMTE activities that approximate the examination of students' work. Whereas responsiveness, serviceability, and normativity are comparable to correctness in describing what teachers might see in student work, they foreground different ways student work can be seen in relation to lessons and their problems and target knowledge. Responsiveness describes perceiving student work as responding to the problem posed. Serviceability captures perceiving student work as a contribution to building the target knowledge. Normativity refers to perceiving student work as abiding by the norms of the instructional situation used to frame the problem. These categories function as lenses shaping teacher perception of student work.

The argument

A PBMTE activity that approximates the examination of student work on a problem-based lesson creates opportunities to learn how such work can be perceived regarding responsiveness,

[1] This geometry lesson, taught before students learn about circumcenter, starts with a swimmers' problem (Three swimmers are going to jump into a rectangular pool and race toward a buoy. One of them will do so from a corner of the pool, a second from a side of the pool adjacent to that corner, and the third from the other adjacent side. Suppose the swimmers are in position and you have to position the buoy. Where should you position the buoy to make the race fair?) and ends with stating the theorem that the midpoint of the hypotenuse of any right triangle is equidistant from the vertices.

[2] This algebra lesson starts with a temperature conversion problem (This winter, your aunt from Canada is visiting you and she is only familiar with degrees Celsius as that is how local weather is reported there. In Celsius water boils at 100 degrees and freezes at 0. She asked you to set the room to 20 degrees Celsius, but your thermostat only works in Fahrenheit. How can you make sure that you are setting the thermostat to the exact temperature that your aunt wants?) and ends with the presentation of the equation of a line that passes through two given points.

serviceability, and normativity. Beyond problem-specific strategies and generic noticing skills, clients' engagement affords mathematical learning opportunities organized by those categories.

Responsiveness: Does the student work produce the kind of answer requested?

Clients need to learn to perceive whether student work is responsive to problems—does it produce requested answers, even if incorrect? Work may display combinations: correct and responsive, responsive but incorrect, unresponsive and incorrect, or unresponsive but correct. Some combinations are theoretically predictable and mathematically important. Students may generate correct work that's unresponsive when solving different assigned problems. In the MH problem, clients anticipated students might place the prize first and swimmers afterward. This might yield mathematically correct work, but the existence of a solution depends on the prize's location relative to pool dimensions. [3]

This distinction illustrates mathematically significant aspects of responsiveness extending beyond MH. At stake is the relationship between mathematical features of a problem and the mathematical neighborhood of the target knowledge. Mathematical objects have properties not shared by all class members. For any rectangle, interior points exist that cannot be midpoints of segments with endpoints on adjacent sides. Thus, placing the buoy first may prevent students from finding answers. The general issue is that more mathematics is available for interpreting problem settings than the mathematics which is instrumental to pose the problem, which students can use (deliberately or not) to correctly make the problem unsolvable.

Students' lack of responsiveness—choosing to solve a different problem—is a predictable perturbation to problem-based lessons. While not necessarily undermining lesson viability, teachers must manage these perturbations. The possible perturbations and the mathematical analysis of contingencies relating differently chosen problems and target knowledge at stake constitute a subject-specific item of knowledge at stake in PBMTE tasks of examining students' work.

Serviceability: Does the student work help move toward the target content at stake?

Clients need to learn to perceive whether student work is serviceable for the content targeted in the lesson—is the student's work on the problem serviceable for bringing up the knowledge at stake? Particularly interesting eventualities arise when student work is responsive yet not serviceable. We consider two cases that represent different ways in which lack of serviceability can show up.

The first case is contextualized in the MH lesson. Students might invoke the circumcenter theorem (the point of intersection of the perpendicular bisectors of a triangle is equidistant from the vertices) to say it should be the location of the prize. This knowledge could be available to students who have read ahead in the book, allowing them to solve the problem as an application of known knowledge rather than as a bid to construct that knowledge. It takes for granted that the three perpendicular bisectors will intersect at a point, and because the theorem applies to any triangle, its use waives the need to ponder what difference it makes (if any) that the triangle at hand is a right triangle.

[3] If students draw the rectangle first and then place a point inside it, this point will be a fair target for all three swimmers if and only if it is located in the quadrant of the rectangle with a vertex which coincides with the swimmer in the corner.

A second case comes from the use of the TP lesson. Just as some students might read ahead in the book, other students might ask the internet or text their older siblings and obtain the rule of thumb to subtract 30 (or 32) and divide by 2 (or multiply by 5/9). Students could still translate that information into an equation and solve it algebraically, hence calculating the temperature to which to set the thermostat. As in the first case, their answer might be responsive to the problem, yet the lack of any explicit calculation of the slope of the line does not help evince the considerations needed to model the equation of a line given two of its points.

While both sources (reading ahead in the book or getting assistance from outside) might produce correct answers and be responsive to the problem, a teacher should perceive their lack of serviceability and note the mathematical qualities that make both cases different. In the first case, the circumcenter theorem offers a more general solution to the problem (say, one that would work even if the pool were not rectangular, and that would work for translations of the problem to other contexts). This is a common feature of mathematical knowledge: True knowledge items are usually subsumed under more general true knowledge items and the midpoint of the hypotenuse being a case of circumcenter is an example of that. Indeed, the search for generalizations is an engine in the creation of mathematical knowledge, so as students look back on their knowledge, they would ordinarily see more recent knowledge as generalizing some more limited knowledge they might have had earlier. And yet, the manner in which this knowledge is encountered matters, particularly for a teacher who has posed a problem in order to usher knowledge that is not yet present, as the case of the MH lesson illustrates. The public emergence of a solution based on the circumcenter could derail the whole lesson because the timelessness of validity in the mathematical knowledge of the discipline and the chronogenesis of knowledge in the classroom are in conflict (Chevallard, 1991): Whereas discovering the midpoint of the hypothenuse theorem can motivate the circumcenter theorem later, as soon as the circumcenter theorem is adopted, the former becomes a forgettable corollary.

In a Story*Circle* on the MH lesson, teachers were shown Delta's work, which used knowledge of circumcenter. However, none of the composing group participants talked about how it disrupted the chronogenesis. They instead commented on the lack of a referent circle or radii in the diagram Delta used and on his weird vocabulary (with one participant avowedly not knowing about circumcenter). The possibility that students will access the timeless body of knowledge and alter the chronogenesis of instruction is quite real, not the least because teaching through problems is not common and students are more used to a didactical contract in which problems call for the application of more general known knowledge rather than for conjecturing new knowledge. Even though the circumcenter solution may jeopardize the design of the lesson, it does so in ways that are mathematically defensible: As the point being sought is indeed the circumcenter, the students' capacity to reduce the pool problem to a problem about circumcenter of a triangle is a mathematical habit of mind worth developing. If the students did not know the proof for the case of the circumcenter, this could be done, and the fact that the circumcenter necessarily lies on the hypotenuse of the right triangle may still be proven. It is thus essential for clients to learn not only to recognize this solution's lack of serviceability for the lesson but also its virtues for the course of studies, and to use that recognition to figure out what to say in response.

The second case of lack of serviceability is different in its relationship to the study of mathematics writ large. Its characteristic is the use of a rule to perform the temperature conversion. Of note here is that obtaining the desired temperature seems to be independent of understanding how the two scales covary–for example, the students don't seem to need to know where the 9 and the 5 come from in order to say that 68ºF will work. In other words, the sort of responsiveness that one gets from the application of this rule solves the problem without leaving much of a trace for why one needed to learn any mathematics to solve it. The general goal of knowing how to create the equation of a line through two points remains elusive, even though, of course, the teacher could ask "where do the 9 and the 5 come from" or "would that change if instead of Celsius degrees she was using another temperature scale, where water boils at B degrees and freezes at A degrees"? In other words, the teacher would need to pose a different problem rather than build on the student's contribution. In a Story*Circle* working on TP, teachers were shown Gamma's work, who said, "So from physical science, we learned to convert one to the other, you just need to divide by 5, multiply by 9 and then add 32." A participant suggested taking advantage of Gamma's work, "So, not that the math work is wrong. But how can we put it into an equation?.... How can we make that fit a linear equation?" But another participant said that [Gamma's work] "wouldn't got me doing math right now." And a third participant tried to mediate, asking Gamma, "How would you find that conversion factor?"

The lack of serviceability in these two cases is different. In the first case, lack of serviceability is lethal to the lesson though it relies on a mathematically productive disposition that a mathematics teacher should recognize. The second case's lack of serviceability is grounded instead in the temperature context of the problem and in a disposition toward solving problems by any means necessary, which should at the very least catch a teacher's attention. Though this lack of serviceability is not lethal to the lesson, knowing how to respond to work like Gamma's requires a teacher to be firmly aware that solving problems by any means necessary is not why one uses problems to teach mathematics. Along these lines, we could note that examining student work in the context of these two lessons supports considerations of different mathematical ways in which students' work can responsive but not serviceable, and to develop mathematically informed ways of responding to those contributions. This consideration distinguishes among responsive and correct but not serviceable solutions to problems. It distinguishes solutions which are related to the mathematics at stake (generalizing the mathematics to subsume the knowledge at stake) from solutions which are related to the context of the problem but elude the knowledge at stake. This distinction is generative, as other problems chosen to teach something might be examined in terms of whether students could get correct answers without engaging in the mathematical work envisioned, and whether they might achieve so either by the application of more general mathematical knowledge or by the application of knowledge from the real world setting used to pose the problem.

Normativity: Does the student work abide by the norms of the instructional situation?

Clients need to learn to perceive whether student work is normative according to the instructional situation (or type of task) used to frame the problem. To scaffold students' work on a novel problem, a teacher may frame it using known instructional situations the class is familiar with. A teacher could frame the MH problem as a construction, for example, by suggesting students make a rectangle, place swimmers, and consider prize placement methods. Using an instructional situation to frame a problem

can reasonably move a problem-based lesson forward. But such framing does more than scaffold work; it constrains teacher perception of student work, setting expectations that may alienate ideas. Construction norms might become expectations teachers and students hold toward each other in MH if the problem is framed as a construction.

Thus, the third example of what can be at stake when clients engage in the examination of student work on a problem has to do with discerning differences among students' work in relation to the instructional situations that could have been used to frame the problem posed and the norms of those situations that student work could breach. Clients must consider how they perceive student work that includes serviceable features and yet flouts norms of the instructional situation. As clients interpret students' work on the MH problem that flout construction norms, what is at stake extends beyond specific work samples to norms flouted and analysis of the expectations create by the construction framing. The knowledge at stake involves analyzing what expectations are set by the instructional situation used for framing and what expectation breaches could be anticipated.

Discussion: Researching the mathematical opportunities to learn of PBMTE tasks

We have illustrated how the categories of perception responsiveness, serviceability, and normativity add subject-specific dimensions to clients' learning opportunities when examining student work for problem-based lessons. Teachers need to interpret all student work types, and these perception categories support anticipation of what they need to be prepared to respond to. Thus, the right-hand side of Figure 1 could include responsiveness, serviceability, and normativity as part of the mathematical knowledge at stake when students practice examining students' work.

Other equally subject-specific categories of perception might be available when examining work produced in other types of lessons. Noticing cognitive skills may still apply, yet what to attend to and interpretation methods might vary according to other mathematical considerations. The structure of instructional exchanges may permit researchers to elicit these from teachers similarly to how responsiveness, serviceability, and normativity were elicited (Herbst et al., 2023). Other MToT beyond interpreting student work are available to organize PBMTE activities. Beyond the mathematical topics providing practice context for MToT, MTE need to consider what approximation aspects make learning opportunities mathematical.

Research on mathematics teaching attending to mathematical demands of teaching work can inform PBMTE by providing bases for describing the mathematical opportunities to learn. Meanwhile, much PBMTE occurs without these opportunities to learn being explicitly articulated as mathematical, yet mathematics education researchers could help document what is being made available to learn. Mathematics education researchers can assist PBMTE practitioners by observing practice, interviewing and surveying participants, to discern reproducible competencies clients derive from approximations of practice and how mathematics participates in delineating such learning opportunities.

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Exploring foundational practices for teaching mathematics: The case of fractions

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Fractions remain a persistent challenge in school mathematics, yet their conceptual complexity also makes them a productive site for developing foundational teaching practices. This study draws on a classroom episode to examine how essential teaching moves, including explaining concepts in multiple ways, eliciting and interpreting student reasoning, addressing misconceptions, supporting transferable skills, and fostering engagement, are enacted in practice. The analysis illustrates how these moves can be observed in classroom interactions and how they contribute to both conceptual understanding of fractions and the development of skills that extend to other areas of mathematics. The findings further suggest broader implications for teacher education and professional development, particularly the importance of creating classroom environments where student ideas are surfaced, valued, and used as resources for learning.

Keywords: Fractions, teacher professional development, teaching practice, mathematics education.

Introduction

Fractions have long been recognised as one of the most challenging areas of school mathematics. Davis et al. (1993, p. 63) described "the teaching and learning of fractions" as "not only very hard" but, in the broader scheme, "a dismal failure." More recent studies confirm that fractions remain difficult for students (Schulz, 2023). Li (2025) conducted a comparative study of British and Taiwanese students' procedural proficiency in fraction addition and found that more than 30% of British students relied on the incorrect strategy of adding numerators and denominators (for example, 1/3+2/5=3/8), compared with fewer than 4.5% of Taiwanese students. This pattern reflects the "whole number bias" which has been identified as a major source of difficulty in students' understanding of fractions (Namkung et al., 2018).

Fractions sit at the intersection of whole number arithmetic and rational number reasoning, bridging primary mathematics, where arithmetic is largely confined to whole numbers, and secondary mathematics, where learners must work flexibly with rational numbers, ratios, and proportional reasoning. Their complexity draws together multiple mathematical ideas, including ratio, proportion, division, equivalence, decimals, and percentages (Tsai & Li, 2017), and requires teachers to make connections across real-world contexts, manipulatives, diagrams, spoken language, and symbolic notation (Lesh, 1981). These features make fractions a particularly productive site for teachers to develop foundational practices such as explaining concepts in different ways, eliciting and responding to student thinking, facilitating discussion, and balancing mathematical accuracy with pedagogical responsiveness. Accordingly, this study examines teaching episodes to identify critical moments where these foundational teaching moves are practised. The aim is to provide insights into how

teaching fractions can support the development of essential skills that are transferable across mathematics teaching broadly.

Why focus on fractions?

Fractions are widely recognised as one of the most difficult topics in school mathematics, largely due to the intricate relationships among their conceptual meanings, representational forms, and fundamental arithmetic operations (Tsai & Li, 2017). Research indicates that the way fractions are taught in schools often results in a limited understanding among students. A key criticism is the overemphasis on the part-whole construct. Traditional instruction frequently presents fractions as parts of a whole, for example interpreting 3/5 as three out of five parts, but does not encourage students to understand fractions as a quotient, such as 3 divided by 5 (Kieren, 1993). This narrow focus can hinder the development of a conceptual understanding of what a fraction truly represents.

Tsai and Li (2017) identified five essential elements in the development of students' understanding of fractions, which include: (1) the five constructs of a fractions: the part-whole, measure, quotient, operator, and ratio constructs of fractions; (2) the concept of equivalent fractions; (3) the procedural fluency for and conceptual understanding of fraction operations; (4) the relationship between fractions, decimals, and percentages; and (5) the transition between different forms of representation involving fractions. However, as Clements and Del Campo (1990, p. 188) argued, "for the majority of school children, fraction knowledge is something that is crammed into their heads in school mathematics." Essential fraction concepts are often missing.

Recent research has examined effective and practical ways of teaching fractions that promote deep conceptual understanding. For example, Björkhammer et al. (2023) examined whether a whole-class instructional sequence centred on number-line representations would enhance students' fraction knowledge. While much attention has been given to curriculum content and instructional materials, comparatively less focus has been placed on the development of teaching practice itself. There remains a need to support teachers in refining how they teach fractions. Addressing this gap calls for an examination of the foundational teaching moves that underpin effective fraction instruction.

Foundational teaching moves: Connecting theory and classroom practice

The construct of foundational teaching moves in this study refers to the core instructional actions that enable teachers to engage students with mathematical ideas through representation, questioning, and eliciting reasoning. This construct is informed by perspectives from practice-based teacher education (Grossman et al. 2009), which emphasise identifying and decomposing the essential elements of teaching practice that can be learned and refined. It also resonates with the concept of teacher noticing, as these moves often arise from teachers' in-the-moment interpretation of students' thinking and classroom interactions. Mason (2011, p. 35) describes teacher noticing as a "collection of practices designed to sensitize oneself." Foundational teaching moves can also be understood as core instructional actions that reflect the norms, tools, and discourses of mathematics classrooms. In this respect, Goodwin's (1994) concept of professional vision clarifies how teachers' perception and interpretation of classroom events are shaped by the practices of their professional community. Together, these perspectives provide a theoretically grounded lens for analysing teaching moves,

including selecting and connecting multiple representations, prompting students to explain their thinking, and responding to misconceptions in ways that advance conceptual understanding.

In the field of fractions, research reports that students' difficulties may arise from the ways teachers mediate representations through manipulatives and visual models, which provide opportunities for teachers to enact foundational teaching moves. For example, Nickson (1998) found that children could identify one quarter of a pizza but struggled when the same fraction was represented as a rectangle, highlighting the need to emphasise invariant relationships. Situations in which students encounter different representations of fractions provide natural opportunities to implement foundational teaching moves. Careful teacher guidance is therefore essential to prevent superficial learning when using concrete materials.

Aims and research questions

This study aims to analyse teaching episodes to identify critical moments in which foundational teaching moves are enacted. The goal is to provide insights into how instruction in fractions can support the development of essential skills that are transferable across mathematics teaching more broadly. Specifically, the study addresses the following two research questions:

1. How are foundational teaching moves, such as explaining concepts in multiple ways, eliciting and interpreting student thinking, and responding to misconceptions, used during fraction lessons?
2. In what ways do these teaching practices support the development of essential skills that are transferable across other areas of mathematics teaching?

Method

The data reported in this paper were gathered from a larger study investigating teaching and learning during a one-year intervention aimed at developing the fraction proficiency of a class of 35 fifth-grade students' (ages 10–11). The class teacher, Ms Zhu (a pseudonym), voluntarily participated in the study as she was motivated to refine her foundational teaching practices to better support students in developing a meaningful understanding of fractions. At the time of the study, Ms Zhu had over 10 years of teaching experience. The participating school, a government-funded institution, is located in a suburban area of central-west Taiwan, and the class was selected because the school, teacher, and students typically reflected fifth-grade classrooms in the region, with a standard class size, a range of student abilities, and teaching aligned with the national curriculum. The teacher's willingness to collaborate and implement the intervention as intended further made this class suitable for an in-depth exploration of teaching practices and student learning.

Drawing on Tsai and Li's (2017) fraction proficiency framework and a review of the Taiwanese curriculum, the author designed 15 tasks in consultation with Ms Zhu prior to implementation. Following the problem-based learning model proposed by Li and Stylianides (2018), the intervention adopted a three-phase procedure for each task: (i) launching the task by the teacher, (ii) exploring the task through small-group discussion with the teacher circulating among groups, and (iii) closing the task with a whole-class discussion facilitated by the teacher. This structured approach provides

opportunities for students to engage with fraction concepts collaboratively while allowing the teacher to enact foundational teaching moves.

Classroom observations were collected via audio and video recordings and transcribed. Analysis followed Creswell's (2012) inductive coding process, with initial codes generated through repeated review of transcripts focusing on teacher actions and student engagement. Codes were refined by removing redundancies and grouping similar codes into broader categories. Reliability was supported by systematically checking codes against the transcripts at multiple stages of analysis, ensuring that categories remained grounded in the data and consistently reflected classroom practice. The resulting categories were summarised into key themes capturing teacher practice, student reasoning, and interaction patterns. The inductive approach allowed patterns to emerge directly from classroom practice, providing insights into how foundational teaching moves support fraction learning.

Results and discussion

To illustrate how the teacher used foundational teaching moves, this section presents one teaching episode in which Ms Zhu and her class discussed a task adapted from a TIMSS-1999 eighth-grade mathematics item (Figure 1). This episode was selected because it highlights how Ms Zhu engaged students with a mathematically demanding task while explaining concepts in multiple ways, eliciting and interpreting student thinking, and responding to misconceptions. Analysing this interaction provides a lens for examining how such practices supported the development of fraction understanding and transferable skills across mathematics teaching.

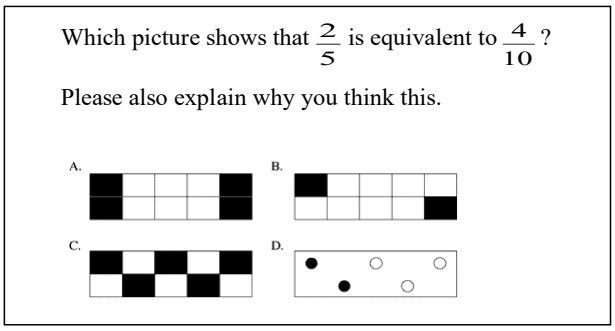
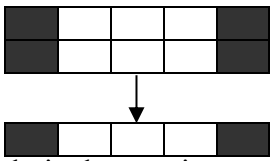


Figure 1: Equivalent fractions

An excerpt of the whole-class discussion is provided below to show Ms Zhu's teaching in action:

Mark: Look, if we wipe off the half of it, like this…



Mark: Look, it changes into two-fifths, so they are equivalent fractions.
Teacher: [Looking at Mark's drawing] Who wants to comment on it?
[The class was silent]
Teacher: Mark wipes off half of it. That also changes the size of the diagram, so the units of these two diagrams are different. [She points to Mark's drawing and the original diagram.] Look, they are different in size. Remember in previous lessons, we talked

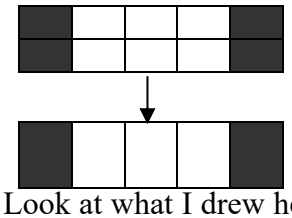
about the importance of the equality of the unit in fractions. So, I don't think Mark can just make the diagram smaller.

[Teacher pauses for a couple of seconds]

Teacher: Anyone else want to come forward and share your idea?

[Several students raise their hands. Ms Zhu picks Liana, who goes to the board.]

Liana: I didn't change the diagram; I just took off the middle line.



Liana: Look at what I drew here. These two diagrams are still the same size. The top one is divided into 10 parts with 4 shaded, so that's four-tenths. The bottom one is divided into 5 parts with 2 shaded, so that's two-fifths.

Liana: These two diagrams are actually the same, which means two-fifths and four-tenths are equivalent fractions.

This excerpt illustrates five key teaching practices that directly address the research questions and will be described below.

1. Explaining concepts in multiple ways: Ms Zhu responds to Mark by emphasising that altering the size of the diagram changes the unit, which invalidates the equivalence he suggested. She not only corrects the misconception but also connects the moment to earlier lessons on the equality of units. In doing so, she demonstrates how revisiting and reframing prior knowledge in different representations can deepen students' conceptual understanding. This aligns with research showing that teachers' flexible use of multiple representations enhances students' conceptual learning (Wilkie & Roche, 2023).

2. Eliciting and interpreting student thinking: Instead of immediately correcting Mark, Ms Zhu pauses and invites comments from the class. The silence is allowed space, and she resists the temptation to fill it herself. As Rodríguez et al. (2025) note, this approach signals to students that their ideas matter, and it gives them time to reflect before contributing. By interpreting and building on student contributions, the classroom shifts from teacher-centred explanation to collective sense-making.

3. Responding to misconceptions: Mark's suggestion provides a natural opportunity to address a common misunderstanding of fractions, namely the belief that equivalence can be achieved simply by "wiping off" or resizing. Ms Zhu identifies the flaw in his reasoning and scaffolds his understanding by explicitly emphasising the importance of maintaining equal units. This practice aligns with research showing that misconceptions can be used productively as learning opportunities rather than dismissed as errors (Copur-Gencturk, 2021).

4. Supporting transferable skills: The discussion extends beyond the specific task by reinforcing fundamental ideas of fraction equivalence and the equality of units. These are not isolated skills; they underpin proportional reasoning, ratio, and algebraic thinking, making them transferable across many areas of mathematics teaching and learning. As Tossavainen and Helenius (2024) found, many student teachers understood only the part-whole and quotient interpretations of fractions. This highlights the importance of grounding classroom discussion in multiple fraction constructs,

including equivalence and unit equality, as it supports students in developing meaningful and transferable fraction knowledge.

5. Fostering student engagement: By inviting Liana to present her alternative approach, Ms Zhu emphasises student reasoning as central to the learning process. Liana's explanation, supported by clear representations, demonstrates how peers can clarify and extend each other's thinking. This interaction models for the class how to reason mathematically, compare solutions, and justify claims, reflecting practices that lie at the heart of effective teaching and learning (Grossman et., 2009).

Limitations, implications, and directions for further research

Several limitations should be acknowledged. Although the selected episode provides an insight into how foundational teaching moves were enacted, it cannot fully capture the breadth of the teacher's practice across different mathematical topics and contexts. The qualitative nature of the data also means that the findings are contextually situated and constrained by the methods employed, which limits their generalisability. Furthermore, although careful attention was given to multimodal aspects of the data through video and audio recordings, including gestures and facial expressions, certain subtleties of classroom interaction may not have been fully represented in transcription.

Despite these limitations, the study offers several implications. First, it demonstrates how foundational teaching moves, such as explaining concepts in multiple ways, eliciting and interpreting student thinking, and responding to misconceptions, can be observed and analysed in ordinary classroom interactions. This highlights the value of using teaching episodes as a reflective tool in teacher education. By examining concrete examples of classroom practice, teachers can develop a nuanced understanding of how their instructional choices influence student reasoning and engagement (Sherin & van Es, 2009).

Second, in line with Hiebert and Grouws (2007), the findings of this study also suggest that explicit attention to these teaching moves can support the development of skills that are transferable across multiple areas of mathematics. For example, emphasising the equality of units and the principles of fraction equivalence not only reinforces conceptual understanding in fractions but also encourages active student engagement, as learners are prompted to reason, discuss, and justify their thinking. This process provides a foundation for proportional reasoning, ratios, and algebraic thinking and highlights the broader pedagogical value of fostering deep mathematical understanding through interactive and participatory learning.

Third, the study has implications for the design of professional development and curriculum planning. Providing opportunities for teachers to observe, discuss, and analyse teaching episodes may help them recognise effective strategies for scaffolding student thinking, promoting dialogue, and addressing misconceptions in real time. In addition, the analysis highlights the importance of creating classroom environments where student reasoning is made visible and valued, allowing teachers to use student ideas as a resource for further learning (Kazemi & Hubbard, 2008).

Finally, these implications provide guidance for researchers studying teaching practices in classroom contexts. The study illustrates the potential of using classroom episodes as units of inquiry to examine how instructional strategies are enacted and how they contribute to the development of transferable

teaching practices. As van Es and Sherin (2008) suggested, this approach can inform both theory and practice, bridging the gap between observed teaching practice and pedagogical frameworks.

Future research could examine teaching episodes across classrooms, grade levels, and mathematical topics to identify patterns in foundational teaching moves, including differences related to teacher experience, instructional approaches, or student characteristics. It could also explore how classroom interactions influence student reasoning, engagement, and concept transfer, offering a holistic view of how teaching strategies shape learning.

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Implementing a four-step lesson structure to teach problem solving

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Problem solving is a key goal in mathematics education. Yet, teaching it remains challenging. This study investigates the potential of a four-step lesson structure to support pre-service teachers (PSTs) in teaching mathematical problem solving in primary education. Grounded in the frameworks of Pólya, Takahashi, and Stein, a four-step lesson structure was incorporated in a second-year course of a bachelor's teacher education program. Twenty-seven PSTs implemented this structure in Grades 4-6 and reported on it through lesson reports and video recordings. Findings showed success in the first three steps: problem exploration, monitoring students' work and whole class discussion. However, the fourth step – making mathematical connections and looking back – was more difficult. The four-phase structure thus shows promise but highlights areas needing further attention in teacher education.

Keywords: Problem solving, teacher education, teaching practice.

Introduction

Problem solving has become a central goal in mathematics education worldwide. Yet, research consistently shows that teaching problem solving is challenging for teachers (Hourigan & Leavy, 2022; Lit et al., 2025; Schoenfeld, 2022; Verschaffel et al., 2020). Unlike procedural tasks, problem solving requires students to engage in exploration and independent thinking. To foster these skills, teachers must present rich and meaningful problems that invite students to search for their own approaches. Crucially, teachers need to resist the urge to intervene too early, allowing students time and space to grapple with the problem independently. Once students have had the opportunity to engage with the problem, teachers must then facilitate a whole-class discussion that brings together diverse student strategies and ideas. Literature consistently shows that orchestrating such discussions – especially in ways that connect and build on students' mathematical thinking – is a demanding and complex aspect of teaching (Dunning, 2022; Gil et al., 2019; Stein et al., 2008).

This complexity underscores the need to better prepare teachers for teaching mathematical problem solving. In response, our teacher training college in the Netherlands introduced a course on problem solving in the second year of the bachelor's program. Alongside this curricular innovation, we investigated how pre-service teachers (PSTs) develop the skills to teach problem solving in primary education. This study focuses on the potential of a four-step lesson structure designed to support PSTs in planning and delivering problem-solving lessons. The four steps are: (1) understanding the problem, (2) solving the problem, (3) whole-class discussion, and (4) connecting and looking back.

The research question guiding this study is:

> How do preservice teachers (PSTs) implement the four-step lesson structure to teach problem solving in primary education?

Theoretical framework

There is substantial evidence that students can learn to solve non-routine problems: complex tasks for which students do not have a readily available solution strategy (Schoenfeld, 2022; Takahashi, 2021; Verschaffel et al., 2020). However, traditional instruction-first lesson structures fall short in fostering problem-solving skills. Problem solving cannot simply be taught through direct instruction; rather it must be experienced by productive struggle. To support this, a task-first approach is recommended, which is commonly organized into a three-phase structure: launch, explore, and discuss (Hourigan & Leavy, 2022; Russo & Hopkins, 2017; Stein et al., 2008). This structure aligns with principles of inquiry-based learning, which have been shown to enhance mathematical understanding and foster positive attitudes – especially when teachers are trained in its implementation (Alreshidi & Lally, 2024). Despite its benefits, task-first instruction remains less prevalent in mathematics education than traditional instruction-first models. Teachers often perceive instruction-first lessons as more efficient and better suited for students needing support, while task-first lessons are seen as more engaging and conducive to developing persistence and mathematical creativity (Russo & Hopkins, 2017).

In task-first lessons, the classroom discussion following student exploration is crucial. To support teachers in orchestrating these discussions, Stein et al. (2008) proposed five practices – anticipating, monitoring, selecting, sequencing, and connecting – which are widely adopted in teacher education (Ayalon & Rubel, 2022; Dunning, 2022; Gil et al., 2019). Among these, purposeful selection of student work is essential for fostering meaningful mathematical discourse; without it, key mathematical ideas may remain unexplored, and the shared strategies may fail to engage students effectively. Ayalon & Rubel (2022) observed that pre-service teachers tended to select students' work that demonstrated a variety of approaches increasing in complexity but placed less emphasis on mathematical connections. The last practice, making mathematical connections, is frequently reported as challenging for pre-service teachers (Cumhur, 2022; Gil et al., 2019).

To address the challenges of teaching mathematical problem solving, we developed a four-step lesson structure that synthesizes insights from three foundational sources: Polya's (1945) four steps in problem solving, the Teaching Through Problem Solving (TTP) lesson structure widely used in Japan (Takahashi, 2021), and the five practices for orchestrating mathematical discussions proposed by Stein et al. (2008). This integrated framework aims to provide PSTs with a practical and theoretically grounded lesson structure for designing and implementing problem-solving lessons in primary education. The resulting four-step structure consists of:

- Step 1 Understanding the Problem. This phase corresponds to Polya's first step and involves introducing the problem and ensuring students comprehend its context and constraints.
- Step 2 Solving the Problem. Students engage in solving the problem. Following Takahashi's (2021) adaptation, this step merges Polya's original phases of devising and carrying out a plan. Simultaneously, the teacher monitors student work to gather insights that inform the selection and sequencing of student solutions for the upcoming discussion. These teaching moves are aligned with Stein et al.'s (2008) practices for preparing productive mathematical discourse.

- Step 3 Whole-Class Discussion. In this lesson step, selected students present their approaches while the teacher facilitates a whole-class discussion to deepen understanding and explore diverse strategies. Takahashi (2021) identifies this step as central to the TTP approach, often occupying a significant portion of the lesson time.
- Step 4 Connecting and Looking Back. This final step integrates Polya's "looking back", Takahashi's "look back and summarize", and Stein et al.'s practice of "connecting." The teacher explicitly links student approaches to key mathematical concepts and encourages reflection on the problem-solving process.

Methods

At HU University of Applied Sciences Utrecht, the Netherlands, pre-service teachers can pursue a bachelor's degree in primary teacher education. In the second year of the program, a dedicated course on mathematical problem solving is offered, targeting PSTs who are teaching children aged 8-12 years in weekly teaching practice. During this course, PSTs introduced three non-routine problems to their students. Problem solving is a relatively new component in the Dutch primary school curriculum, and most PSTs entered the course with little to no prior experience in teaching or engaging with problem-solving tasks themselves. Moreover, many were placed in schools where problem solving was rarely, if ever, addressed in regular instruction.

Data collection was aligned with the standard assessment procedures of the course. For each PST, the dataset included: a written report detailing the three problem-solving lessons taught, including lesson plans, an individual reflective note, and a video recording of one of the lessons. In July 2024, 56 PSTs submitted their course reports and were invited to provide consent for the use of their materials in this study. Of these, 19 declined consent and 10 submitted incomplete reports, resulting in a final sample of 27 participants.

Lesson reports and videos were independently coded by the first and second author, focusing on the implementation of each lesson step. Coding continued until full agreement was reached. Categories were refined iteratively throughout the process.

- Step 1: Understanding the Problem was coded based on whether the PST engaged students in identifying key characteristics of the problem or encouraged them to restate the problem in their own words – indicating a meaningful understanding. If the PST focused primarily on surface-level elements such as relevant terms, numbers, or the question itself, this was coded as a more superficial understanding.
- Step 2: Monitoring was primarily coded from the written report, as video recordings often lacked visibility or audibility during student work phases.
- Step 3: Coding focused on the rationale provided in the report for selecting and sequencing student work. These decisions were then cross-checked with the video for implementation fidelity.
- Step 4: Connecting and looking back was coded from the video, with attention to meaningful connections made by the PST – such as highlighting multiple representations of the same idea,

explaining why a particular approach worked, and reflecting on the problem-solving process, heuristics, or learning outcomes.

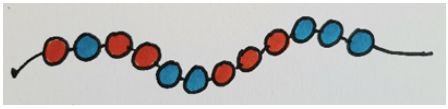
Overall, the integration of lesson reports and video recordings strengthened the validity of the analysis and provided a comprehensive view of PSTs' implementation of the four-step lesson structure.

Results

The lesson reports and videos of 27 PSTs provide insights into how they implemented our four-step lesson structure. The results are presented step by step.

Step 1: Understanding the problem

The majority of the PSTs (20 out of 27) supported their students in identifying relevant words and numerical information within the problem and in articulating the question being posed. Among these, five PSTs demonstrated a deeper level of sense-making by posing open-ended questions that encouraged students to explore key characteristics of the problem. For example, one PST introduced the bead sequence problem (Figure 1) in a meaningful way:



Someone creates the following bead sequence: 1 red, 1 blue, 2 red, 2 blue, 3 red, 3 blue, and so on.
What colour is bead number 100?

Figure 1: Bead sequence

PST:	What are key words and numbers?
Student:	One, one, two, two, three, three.
Student:	Red, blue.
PST:	What can you expect to come next? What is the colour of the bead that comes after this?
Student:	Four, four.
PST:	Four red and four blue beads. What exactly is the question?
Student:	What colour is bead number 100?
PST:	What kinds of answers could those be?
Student:	Red or blue.
PST:	So, every time there is one more bead here, and one more here.

The remaining seven PSTs introduced the problem by reading it aloud and instructing students to note down the important information. These PSTs did not engage in questioning and transitioned students to individual work within one to two minutes, completing this lesson step with minimal interaction.

Step 2: Students solve the problem, while PSTs monitor their approaches

In their lesson reports, all but one PST indicated that their students engaged productively with the problem, and that only a few students required their help to get started. PSTs circulated through the classroom, observing students' progress. This allowed them to gain insight into the strategies employed by most students. However, some PSTs noted that it was difficult to fully grasp students' approaches when students did not write down their thinking processes.

One PST reported difficulty in initiating student engagement with the task. In her class, many students left their seats to seek help. The video recording shows that this PST struggled to maintain classroom control, which may have hindered the effectiveness of this lesson phase.

Step 3: Whole class discussion

Nearly all PSTs (24 out of 27) invited students to present their approaches to the class, actively supporting them to ensure clarity and peer understanding. A majority of 16 PSTs invited two or three students to present their approaches to the class. Another seven PSTs allowed four or five students to present, while one PST – exceptionally – gave presentation turns to eight students, responding to the students' enthusiasm and willingness to share. In contrast, three PSTs began this phase with a teacher-led demonstration of the approach they considered most effective.

The criteria for selecting students to present their approaches were described in the PSTs' lesson reports. Interestingly, student selection was predominantly based on classroom observations of students' work. Only four out of 85 presentations were given to volunteers (see Table 1), while this might well have been the usual practice by PSTs to give turns to students.

Table 1: Selection criteria for student contributions

Selection criterion	1st student	2nd student	3rd student	4th student	total
Variation	6	14	20	4	44
Accessibility	11	5	1	0	17
Reward	2	0	0	3	5
Interesting for problem solving	2	2	0	0	4
Volunteer	0	2	1	1	4
Misconception	2	1	0	0	3
Most used	1	0	1	0	2
No reason	0	1	0	0	1
Other	3	2	0	0	5
Totals	27	27	23	8	85

The most frequently cited selection criterion was to ensure variation in approaches, which accounted for 44 of the 85 selections. This suggests that PSTs aimed to expose the class to a range of strategies, thereby showing that multiple strategies can be applied to non-routine problems. The second most common criterion was accessibility, mentioned in 17 cases. Notably, accessibility was most often used to justify the selection of the first presenter, where it even surpassed variation as the primary consideration. This indicates a strategic choice to begin the discussion with an approach that would be comprehensible to all students.

Less frequently, five students were selected as a form of reward for their effort, indicating a deliberate attempt by PSTs to motivate and affirm individual student contributions. Additionally, PSTs selected students based on approaches they deemed valuable for learning problem-solving (four cases), or due to the presence of interesting misconceptions (three cases).

Step 4: Connecting and looking back

Following student presentations, 15 PSTs implemented both components of the fourth lesson phase: making connections between different strategies and encouraging students to reflect on the problem-solving process. One PST focused solely on making connections, while two others engaged only in retrospective reflection without drawing connections between approaches. Notably, nine PSTs did not implement this lesson step at all. In their lesson reports, many PSTs evaluated this last step to be challenging, citing time constraints, declining student engagement, and a lack of suitable questions to stimulate students' reflection.

The example below illustrates a typical classroom discussion during step 4, based on the bead sequence problem (Figure 1):

PST:	If we look at these two approaches, what similarities do you see?
Student:	The one on the left involves more writing, the one on the right more drawing.
PST:	Yes, that's a difference, but what is the same?
Student:	That each time a red and a blue are added.
PST:	Yes, the numbers with red and blue, and here the letters. Can you name an advantage of this approach?
Student:	It's clear.
PST:	Yes, clear. Who else can name an advantage of drawing?
Student:	(inaudible)
PST:	And who can name an advantage of this strategy, the chart?
Student:	Faster.
Student:	Even clearer.
PST:	What two types of approaches do we see? I just mentioned them.
Student:	Calculating and drawing.
PST:	Yes, this is drawing, and what do you call this? No one? A table or a chart.
PST:	And what did you think of this math problem, was it OK?
Students:	Yes!

In this example, the PST facilitated mathematical connections by prompting students to compare solution strategies and identify similarities and differences. Most PSTs asked similar questions, that proved difficult to answer for younger students. Regarding the looking-back component, this PST referred to heuristics – though framed simply as "approaches." Three other PSTs explicitly addressed the use of heuristics, while an additional three discussed the overall problem-solving process. Another three PSTs reflected on the lesson goal with their students. Eight PSTs concluded the reflection phase by asking students to evaluate the perceived difficulty of the problem.

Conclusion and discussion

This study examined how 27 PSTs implemented a four-step lesson structure while teaching mathematical problem-solving lessons in primary education. Despite limited prior experience and placement in schools where problem solving was not emphasized, most PSTs successfully introduced appropriate non-routine problems and actively engaged their students. Nearly all PSTs selected

student work to facilitate classroom discussion, often aiming to showcase a variety of approaches to the problem, consistent with findings from Dunning (2022) and Ayalon & Rubel (2022). Only three out of 27 PSTs began the discussion by demonstrating their own approach, to ensure full understanding of all their students. However, the final step – connecting mathematical ideas and looking back on the problem-solving process – proved challenging. Nine PSTs omitted this step entirely, and many others reported difficulties in its implementation. These challenges align with previous research (Cumhur, 2022; Gil et al., 2019).

This study contributes to the growing body of knowledge on preparing teachers to teach problem solving. The four-step lesson structure shows promise as a pedagogical framework: it encouraged PSTs to move away from a traditional instruction-first lesson structure and adopt a task-first lesson structure, focusing on students' mathematical thinking and promoting productive classroom discourse. Several factors may have supported this shift, including the provision of a curated set of non-routine problems and targeted guidance in lesson preparation, particularly in anticipating students' approaches, a strategy shown to be effective in prior departmental research (Kool, 2022).

One limitation of this study is the small sample size of 27 PSTs. Of the 56 PSTs invited, 19 declined consent for the use of their materials. While they were not required to provide reasons, it is presumed that the presence of children in the video recordings influenced their decision. In addition, a similar number of PSTs did not submit their work on time. As a result, the findings may primarily reflect the practices of the most engaged or high-performing PSTs in our program.

Despite these limitations, this study offers valuable insights into preparing teachers for teaching mathematical problem solving. It underscores the need for additional support in developing PSTs' ability to connect mathematical ideas and look back on the problem-solving process. Our next design research cycle will focus on the impact of PSTs' noticing of students' mathematical thinking. Future research should examine the replicability and sustainability of these findings, including long-term impact on teaching practice. Comparative studies involving in-service teachers could further illuminate how general teaching experience shapes the implementation of problem-solving instruction.

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From reading about modes of reasoning to classifying reasoning and proving tasks: How pre-service teachers select tasks and apply theoretical concepts

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Mathematics textbooks do not usually include units dedicated to reasoning and proving (R&P) and it is up to teachers to prepare such units for their classes. Thus, the selection of suitable R&P tasks and recognising different modes of reasoning in their solutions is paramount. This study examines how pre-service mathematics teachers select R&P tasks and connect academic readings with their practical application. Results indicate that pre-service teachers associate R&P tasks mainly with their classroom roles, difficulty or external factors. They struggle to identify correct modes due to misunderstanding terms like "model" or believing reasoning is intrinsic to tasks. Reading about modes of reasoning might be a useful initial activity, but a more comprehensive approach is necessary for improving future teachers' perceptions and understanding of different types of reasoning.

Keywords: Pre-service teachers, reasoning, proof, reading, task selection.

Introduction

The Czech National Curriculum for primary schools (ages 6–14) states that teachers should develop pupils' argumentation competencies and critical reasoning skills. Consequently, reasoning and proving (R&P) should be incorporated into teaching. R&P can facilitate learning and communicate to pupils how new mathematical knowledge is agreed upon (de Villiers, 1990). However, textbooks usually devote no explicit thematic unit to R&P; it is therefore up to teachers to include and select appropriate tasks and to decide what forms of justification are appropriate and expected from pupils.

Not only do pupils face difficulties with R&P (e.g., Buchbinder & Zaslavsky, 2009; Fischbein, 1982), but teachers also often struggle with R&P (e.g., Knuth, 2002a). Thus, pre-service mathematics teachers (PMTs) must be prepared for such teaching during their university studies. In our course, we aimed to introduce PMTs to the many approaches to R&P available in school mathematics—distinguishing these from formal proof—and to develop their understanding of the roles that R&P might have. To develop R&P in their lessons, PMTs must be able to recognise the potential of tasks for such a development. Thus, among other tasks, PMTs were asked to select a task they consider R&P-related, explain why they see it as such, and—based on reading a theoretical paper—categorise the reasoning involved. Further, to enrich PMTs' *pedagogical content knowledge for teaching proof*, it was considered important to introduce PMTs to pupils' *proof schemes,* as this could enhance their ability to interpret pupils' thinking and understand their conceptions and misconceptions related to R&P (Lesseig, 2016). The article about *modes of reasoning* was chosen, however, as it was more accessible for PMTs inexperienced in reading scientific papers, due to its clear language, manageable length, and illustrative examples. In this study, we investigate which tasks PMTs identify as R&P-related, why they select them, and how they apply the modes-of-reasoning framework in practice to get an insight into their understanding of R&P and its facilitation in teaching.

Theoretical background

Teachers are expected to develop R&P skills in their pupils because R&P is an essential part of mathematics. They therefore need a good understanding of R&P as teachers communicate "whether consciously or not, ideas about the ways in which 'truth' is determined in a field" (Shulman, 1987, p. 9). Among the knowledge they require is an awareness of the many roles R&P can play (e.g., explanatory), which are useful in mathematics and its teaching (de Villiers, 1990).

School mathematics, however, differs from the discipline of mathematics: "it has different warrants for truth, different forms of reasoning" (Watson, 2008, p. 1). This opens the door to forms of justification that mathematicians might not accept. Different authors distinguish levels of reasoning or proof based on rigour, formalism, or the purpose it serves (e.g., Balacheff, 1988; Hanna, 1990). Others emphasise the social function of proof to convince. Harel and Sowder (1998) describe what is convincing through the notion of persons' *proof schemes.* In addition to *deductive proof schemes,* they identify *external conviction proof schemes* (e.g., appeal to authority) and *experimental proof schemes* (e.g., empirical testing). Building on *proof schemes* and a textbook analysis, Stacey and Vincent (2009) proposed a classification of solved R&P tasks called *modes of reasoning*. These modes align with the *proof schemes: external conviction scheme* corresponds to *appeal to authority,* the e*mpirical proof scheme* includes *concordance of a rule with a model* and *experimental demonstration,* and the *deductive proof scheme* includes *deduction using a model*, *a specific case,* and *a general case.* They also note an additional mode that does not fit any *proof scheme* class: *qualitative analogy.*

There is a pressing need to prepare pre-service teachers for R&P-related teaching, as both pupils and teachers often struggle with R&P. The literature documents many difficulties, such as tendencies to reason empirically where deductive proof is expected, or not understanding the generality of proof (e.g., Buchbinder & Zaslavsky, 2007; Fischbein, 1982).

Though issues with R&P are well documented, relatively few interventions focus on how to overcome them (Stylianides et al., 2017), particularly regarding the development of pre-service teachers' *pedagogical content knowledge* (Shulman, 1987; see also Cihan & Akkoç, 2019). This study focuses on the first part of a longer intervention aimed at developing PMTs' *subject* and *pedagogical content knowledge for teaching proof*, specifically *specialised content knowledge* and *knowledge of content and students* (Lesseig, 2016), which includes knowledge of pupils' *proof schemes.* The intervention goal was to prepare PMTs for R&P lesson planning. It also targeted PMTs' beliefs about the nature and role of R&P, which tend to be rather limited (Knuth, 2002a, 2002b). As an initial activity to develop this knowledge, we selected academic article reading since such reading can shape teachers' beliefs and practices (Foster & Inglis, 2019).

To develop R&P in their lessons, a pre-requisite for PMTs is that they can recognise the potential of the task to develop R&P. Thus, such a selection might be seen as "a core practice" for teachers (especially when there is no thematic unit devoted to R&P in textbooks) possibly influencing the teaching quality. However, not much is known about teachers' task selection process (Bardy et al., 2024).

In our study, the PMTs were asked to choose a R&P task and provide explanation for their decision. The study is guided by two questions: What tasks do PMTs select as R&P tasks, and why? How do they utilise in practice the theoretical framework of modes of reasoning that they read about?

Methods

The participants were PMTs in their second year of a two-year master's program at the Faculty of Education, Charles University. Because the master's program is open to students from other faculties, the cohort varied in previous experience studying and teaching mathematics. Most had already completed the compulsory mathematics courses in their university studies, finished two mathematics education courses, and undergone practical training at a lower secondary school. Many had some teaching experience as unqualified teachers (mean 2.4 years, median 2.0 years).

Data were collected in the winter semester 2022–2023. Nineteen of the twenty PMTs submitted their work (referred to here as PMT1, PMT3– PMT20). PMTs were assigned homework to select a task related to R&P appropriate for eighth grade (13-year-olds) and to classify it using *modes of reasoning* based on the article by Silverman and Even (2016), which they had read. PMTs were also asked to explain why they chose the task, why they considered it R&P-related, and to submit their work electronically. The aim was to gain insight into their understanding of types of reasoning, the basis on which they select R&P tasks, and whether they can apply theoretical notions to mathematics tasks.

The area of mathematics to which each task belongs was coded inductively (Saldaña, 2013). An inductive–deductive approach was used to code why a given task was selected and considered R&P: some codes were derived from the literature and from analysis of interviews in which PMTs shared their conceptions of R&P (see Michal, 2023). Next, connections among codes were examined and grouped into categories describing why PMTs chose a task (e.g., *external aspects of the task*, *serving a specific purpose*, *demands on pupils*, *other reasons*).

The PMTs' assignment of modes of reasoning to their tasks was assessed from two perspectives. First, we considered whether the assigned mode would be judged appropriate by experts. If not, we examined whether the assignment nevertheless aligned with an *external-conviction*, *empirical*, or *deductive proof scheme* (Harel & Sowder, 1998). This evaluation was not possible in all cases because some PMTs provided only tasks and reasons without solving the tasks.

As an illustration, consider PMT5. Her task concerned a pictorial proof of Pythagoras' theorem. The rationale for selection was coded as *deducing/discovering* within *the task purpose* category: "This task specifically guides pupils to discover the properties of Pythagoras' theorem independently." She indicated the solution by instructing pupils to "cut […] the template" and "describe the properties of the squares above the sides." As no further reasoning was involved, the approach was coded as an *experimental demonstration*, which agreed with the *mode* she assigned: "This task corresponds to the […] experimental demonstration […] Pupils cut out individual parts and, by rearranging them, discover [the statement]." (PMT5)

Results

The PMTs chose tasks from different fields of mathematics: a geometry-related task (seven out of ten concerned perimeter, area, or volume), algebraic identities or divisibility (three), diagonals in

a convex n-gon (one), functional thinking (two). Three PMTs chose non-standard tasks requiring insight, including a logical puzzle.

Justification for considering the task to be R&P related

The reasons the PMTs gave for classifying a task as R&P varied. Some stemmed from the form of the task or their experience (*external aspects*), while others related to the solution *process* or its *purpose*. The task was also considered R&P, for example, based on the *demands it places on pupils*.

External aspects: Two PMTs identified the task as R&P based on its keywords ("verify" or "prove"). Some PMTs chose tasks based on their prior teaching. In one case (PMT19), a negative pupil experience—unexplained statements—prompted a desire to know why it was valid.

Task purpose: The PMTs often considered an R&P task to be either one in which pupils discover mathematical properties on their own (see PMT5 above), or one that shows the origin of statements: "[...] it is important that pupils can justify where the formula came from and that it did not just fall from the sky." (PMT3); "[Pupils] are presented with a formula for calculating the [length of a] diagonal, but they have to show where it comes from and that it is correct." (PMT8)

PMT12 mentioned the process of *generalisation*: "[In] the task, it is necessary to justify the volume of the cylinder in a specific case and generalise [...]" In connection with inductive reasoning, rather than a *generic case* (Stacey & Vincent, 2009), PMT15 also commented on generalisation. Seven PMTs mentioned *deducing* or *discovering.* In some cases, however, this was not a criterion by which they decided whether the task was R&P, but merely a mention of a feature considered important in the task. Two PMTs (PMT7 and PMT13) used *reasoning* as a synonym for *deduction,* and in both cases, deducing included both the creation of the hypothesis and its justification.

Other PMTs considered it helpful that the task connected knowledge from different areas (e.g., algebra and geometry, PMT13). It was chosen by PMT8 to show pupils the need to prove.

Demands on pupils: Four PMTs viewed the task to be "developing R&P" when it required "outside the box" thinking, and pupils had to find the solution independently: "[The] task definitely develops pupils' reasoning skills – it cannot be plugged into a pre-learned formula." (PMT9)

Assigning the appropriate mode of reasoning

Table 1 displays the modes PMTs assigned to their tasks and those expected based on the solutions provided. If the solution was not explicitly stated but could be inferred, it is marked with an asterisk.

Several PMTs offered multiple modes, even attributing these to possible pupils' approaches. However, seven PMTs did not indicate how they would solve the task. In such cases, assigning the correct *mode of reasoning* was impossible. In six of those instances (all but PMT3), it was evident that the *mode of reasoning* was assigned solely on the assignment itself, not on the reasoning: "I believe this task could be *experimental demonstration* […]" (PMT18) *Mode* assignments based on the wording of the task itself were additionally identified in PMT1 and PMT7, where, however, an intended solution, or at least a hint, could be inferred from the task wording or additional provided explanations. Both PMTs assigned the *mode* incorrectly.

Table 1: PMTs' assignment of modes of reasoning to selected tasks

Student	Modes of reasoning		Student	Modes of reasoning	
	Assigned	Expected		Assigned	Expected
PMT1	**Concordance of a Rule With a Model**	**Experimental Demonstration***	PMT12	*Deduction Using a Model*	*Not solved*
PMT3	*Deduction Using a General Case*	*Not solved*	PMT13	Deduction Using a General Case	Deduction Using a General Case*
PMT4	Deduction Using a General Case	Deduction Using a General Case*	PMT14	**Deduction Using a Specific Case**	**Deduction Using a General Case***
PMT5	Experimental Demonstration	Experimental Demonstration	PMT15	Experimental Demonstration	Experimental Demonstration
PMT6	Experimental Demonstration	Experimental Demonstration		**Deduction Using a Specific Case**	**Experimental Demonstration**
	Deduction Using a General Case	Deduction Using a General Case	PMT16	Experimental Demonstration	Experimental Demonstration
	Deduction Using a Specific Case	**Experimental Demonstration**		**Qualitative Analogy**	**Appeal to Authority**
PMT7	**Deduction Using a Model**	**Deduction Using a General/Specific Case***		**Deduction Using a Model**	**Experimental Demonstration***
PMT8	Appeal to Authority	Appeal to Authority	PMT17	Deduction Using a Specific Case	Deduction Using a General/Specific Case*
	Deduction Using a Model	*Not solved*	PMT18	Experimental Demonstration	*Not solved*
	Deduction Using a General Case	Deduction Using a General Case*		*Deduction Using a General Case*	*Not solved*
PMT9	*Deduction Using a Model*	*Not solved*	PMT19	*Deduction Using a General Case*	*Not solved*
PMT10	*Deduction Using a Model*	*Not solved*	PMT20	*Did not answer*	
PMT11	*Deduction Using a Model*	*Not solved*			

One of three modes PMT8 suggested was also marked as *unsolved.* She claimed: "[...] pupils will choose *deduction using a model*, where they sketch a cube with edge a [...]" Though her answer is unclear, it seems that she considered the sketch to be the *model*. However, the sketch did not serve as R&P, but "only" had an illustrative role, supporting algebraic reasoning (which PMT8 correctly assigned to *deduction using a general case*). Misunderstanding of the *model* occurred in other cases where an illustration or an object was present in the assignment (e.g., PMT1). Four PMTs (PMT9, PMT10, PMT11, and PMT12) expressed this explicitly ("In this case, the model is probably the board on which the individual tokens are placed. [So it is a] *deduction using a model*." PMT10) or implicitly.

Eleven PMTs mentioned, described, or hinted at a solution for at least one of the assigned modes (17 in total). Ten can be considered correctly matched solution-mode pairs, authored by eight PMTs. Cases where no agreement was found between the proposed solution and the assigned *mode of reasoning* are marked in bold in Table 1.

Next, data was considered from a more general classification perspective using the relation between *modes of reasoning* and the broadest categories of *proof schemes*. After assigning *empirical*, *deductive,* or *external schemes* to corresponding modes, 14 out of 17 cases were well assigned.

Discussion

PMTs selected R&P mainly from geometry, which is unsurprising because the notion of theorem in the Czech elementary and lower-secondary curriculum typically appears with geometrical statements. Similar patterns are reported elsewhere (e.g., Furinghetti & Morselli, 2009). The association may also reflect a desire to leverage the visual nature of geometry, as many people prefer pictorial support when proving (Wu, 1996). At the same time, Wu (1996) warns that confining proof to geometry risks conveying a distorted view of mathematics.

Beyond some superficial criteria, PMTs tended to characterize an R&P task as one in which pupils discover, deduce, or see where statements come from. It aligns with roles of proof (de Villiers, 1990) such as *explanation* or *discovery*; *verification* also appeared in PMTs' accounts. The importance of sharing and cooperation, mentioned less often, relates to *communication*. Hints of *systematisation* emerged when PMTs discussed interconnectedness. Overall, these selection criteria can be considered appropriate in light of classroom-suitable roles of proof (de Villiers, 1990; Hanna, 1990).

When a solution was hinted at, PMTs mostly assigned modes that experts would accept; where not, the assignments still often fit an *external*, *empirical*, or *deductive proof scheme* (Harel & Sowder, 1998), suggesting they made good sense of Silverman and Even (2016). In some cases, however, PMTs did not provide a solution yet still assigned a mode, which may indicate they treated the mode as an intrinsic property of the task. While some tasks do cue a method (e.g., "verify by measurement" → *experimental*), this is not generally the case, and such a view could be limiting in teaching.

A common error concerned the term *model*: *concordance with a model*, *deduction using a model*, *experimental demonstration*, and *deduction using a specific case* all involve specific cases (Stacey & Vincent, 2009), likely causing confusion among modes. Relatedly, reasoning meant as a *specific case* may be interpreted as an *experimental demonstration*, or vice versa (Mason & Pimm, 1984). This highlights the didactic value of classifying R&P by how a pupil might understand it—through *proof schemes*—rather than treating modes as fixed task attributes. Indeed, several PMTs presented a possible pupil's solutions, supporting the prominence of *proof schemes* (Harel & Sowder, 1998) alongside, or even instead of, *modes of reasoning* (Stacey & Vincent, 2009) in the course.

Conclusions

The PMTs in our study mainly selected R&P tasks from geometry. They considered whether a task was R&P-based primarily on the purpose of its inclusion in teaching (many of these were related to R&P roles), but also on its perceived non-standard nature or complexity, and occasionally on external aspects. Other factors, such as the PMTs' experience, influenced the decision to include a task. This suggests the need to communicate how R&P can be included in non-geometric topics of primary school and raise awareness of the different R&P roles to all PMTs.

The PMTs had some problems assigning *modes of reasoning*. This was partly caused by assigning modes based solely on the wording of the task, but also to their lack of understanding of some modes.

There were tasks where the PMTs listed multiple possible modes, and some even stated that different pupils may approach the reasoning differently. It should be mentioned that the article (Silverman & Even, 2016) does not contains examples of *concordance of a rule with a model* and describes other *modes of reasoning* very briefly. Other examples, for which the article was included in the intervention, can be considered useful for PMTs. Yet, the concept on the *proof schemes* seems a more viable option to include in the course in the future. Nevertheless, *modes of reasoning* can serve as a brief introduction to demonstrate broad approaches to R&P.

The research had many limitations. The sample size was small and consisted of PMTs at one institution. We also do not know how much effort they put into trying to understand the assigned reading and doing the homework. Some of the answers were ambiguous, and additional interviews would have been beneficial to lower the subjectivity of qualitative analysis.

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Tasks of teaching in the context of building thinking classrooms

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This study identifies tasks of teaching that teachers might encounter in the process of selecting problems when building thinking classrooms. Thematic analysis of transcripts from teachers' group discussions indicates plausible categories of tasks of teaching that are discussed. Although the identified categories can be described as common tasks of teaching, our analysis shows that teachers experience some of them as particularly challenging in the context of building thinking classrooms—thus highlighting the potential of this approach to studying tasks of teaching.

Keywords: Mathematical problems, building thinking classrooms, tasks of teaching.

Introduction

Numerous efforts have been made to reform mathematics teaching and develop more progressive teaching practices. One such effort is Building Thinking Classrooms (BTC). This approach involves a set of practices that aim to stimulate students' thinking and engagement (Liljedahl, 2021), for instance by having students work on mathematical problems in visibly random groups. Instead of sitting at their desks, the students collaboratively work on problems as they stand around vertical boards, and the BTC approach thus aims to alter traditional norms and routines of mathematics classrooms. Although BTC has been adopted by teachers in many countries, few research efforts have explored what is entailed in teaching according to BTC. Our study aims at contributing to this.

To study the entailments of teaching according to BTC, we draw on Ball and Forzani (2009), who consider the "work of teaching" as "the core tasks that teachers must execute to help pupils learn" (p. 497). Tasks of teaching break down the complex work of teaching into smaller components that are recognized as meaningful within the professional community (Hoover et al., 2014). Based on this, we seek to identify and consider tasks of teaching that might plausibly be involved in the work of teaching according to BTC. Here, we focus on a specific slice of the work of teaching that teachers are typically faced with in their initial implementation efforts: selecting mathematical problems that are appropriate for use in BTC. We ask: What tasks of teaching might teachers encounter in the process of selecting problems for thinking classrooms? In considering this research question, we wonder if these tasks of teaching are specific to the context of BTC or not. We analyze data from *Partners in Practice*, which is an ongoing research and development project that aims at creating a collaborative environment for exploring and developing more progressive teaching practices.

Theoretical foundation

Two sets of concepts form the theoretical foundation for our study. First is the concept of tasks of teaching, which is related to a conceptualization of teaching as "work." Second is the concept of problem in the context of Liljedahl's (2021) framework for developing "thinking classrooms."

Tasks of teaching

The *work of teaching* involves core tasks that teachers must carry out to support student learning. Ball and Bass (2003) describe the identification of these tasks as a form of "job analysis"—an effort to define what needs to be accomplished in teaching, independent of the specific methods used to achieve it. Similarly, Ball and Forzani (2009) consider *tasks of teaching* to be the elements that constitute the work of teaching mathematics—the most foundational activities teachers must perform to help students learn—and they argue that careful analysis of core tasks of teaching is important. When teachers learn to carry out their job, they do this by trying to apply more general principles to specific challenges that arise in the work. As they learn how to carry out the specific tasks of teaching, they also develop skills that can be applied more generally in teaching (Ball & Forzani, 2009).

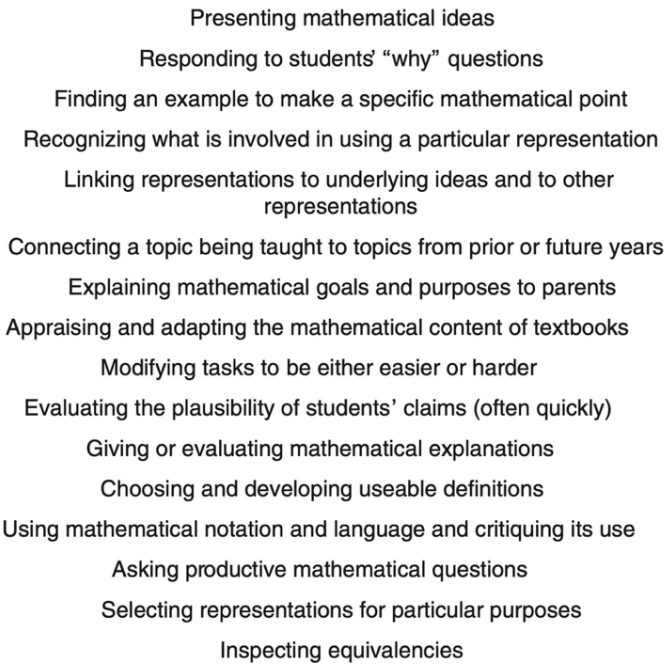

Figure 1: Ball et al.'s (2008, p. 400) list of mathematical tasks of teaching

Figure 1 displays a list of common tasks of teaching, which provide a foundation for the practice-based theory of mathematical knowledge for teaching (Ball et al., 2008). Mathematics teachers must regularly deal with these tasks of teaching in their work; solving them requires mathematical knowledge for teaching. Such core tasks of teaching are described as universal (Hoover et al., 2014), and we wonder if an approach like BTC, which specifically aims at breaking with many common patterns of teaching, might involve tasks of teaching that are specific to this context. In our effort to conduct a "job analysis" of a slice of BTC work, we hope to identify some specific tasks of teaching, which might contribute to better understanding of what is involved in the work of teaching according to Liljedahl's (2021) principles and practices, and we anticipate that scrutiny of these specific tasks of teaching might also lead to insights that can apply more generally to research on teaching.

Mathematical problems in a thinking classroom

Liljedahl (2021) developed the BTC framework based on two core assumptions: (1) students engage in limited cognitive activity during typical mathematics lessons; (2) the structure and organization of classrooms are rooted in models of education from the industrial era, which are no longer effective for fostering deep learning. He finds the lack of student thinking particularly problematic, since meaningful learning depends on active cognitive engagement. The second assumption points to the presence of entrenched institutional structures in today's classrooms that may hinder rather than support learning. Yackel and Cobb (1996) identified various norms that shape mathematics instruction, many of which are resistant to change. The BTC approach is designed to challenge and dismantle such norms to promote student thinking.

Like others, Liljedahl (2021) argues that classroom practices have remained largely static despite multiple reform efforts. Students typically remain seated, work individually with textbook tasks, while teachers stand in front of a board and teach by telling. This reinforces a passive view of learning, according to Liljedahl. BTC seeks to disrupt this pattern. Drawing on observations and development work, Liljedahl outlines 14 key practices that define the BTC approach.

The nature of problems is important in a thinking classroom, and the first core practice in BTC focuses on the types of problems used to stimulate student thinking (Liljedahl & Girox, 2024). Mathematical problems play a central role in mathematics education—not only in teaching itself (e.g., Hiebert et al., 2003), but also in research aiming at understanding classroom practices and student thinking (e.g., Tekkumru-Kisa et al., 2020), as well as in professional development (e.g., Boston & Smith, 2011; Johnson et al., 2016). Problem-solving tasks (*problems*), inviting students to explore, collaborate, and discuss, are important in reform-oriented mathematics teaching and particularly when building thinking classrooms (Liljedahl, 2021). Liljedahl (2021) argues that these problems must possess certain essential qualities—most importantly, they should actively engage students in mathematical reasoning. The goal is to foster a culture of mathematical thinking within the classroom.

Rather than beginning with standard curriculum tasks, Liljedahl often introduces non-curricular tasks (*problems*) that are designed to be engaging and thought-provoking. These problems help students develop the habit of thinking deeply and collaboratively. Once students are comfortable with this mode of engagement, the teacher gradually transitions back to more traditional, curriculum-aligned tasks. This approach contrasts with conventional teaching methods, where students spend time mastering foundational skills before attempting problem solving. In BTC, problem solving is the starting point, not the end goal. In BTC development initiatives, it is thus important to explore what tasks of teaching teachers might encounter in the process of selecting problems.

Methods

In *Partners in Practice*, twelve lower secondary mathematics teachers and three teacher educators partnered to explore teaching with problems. Regular sessions were held to explore key concepts in a problem-solving approach to mathematics teaching, offered hands-on opportunities to solve mathematical problems, and encouraged teachers to engage their students in similar problem-solving activities. Throughout the sessions, discussions covered a range of topics, including how problem

solving is defined in the recent Norwegian national curriculum and the types of tasks featured in the textbooks currently in use. Activities involved emphasis on introducing rich problems and facilitating discussions where students are positioned as active problem solvers. The collaboration ran for 12 two-hour sessions over a period of 18 months. In this paper, we focus on data from session 9, which introduced a period where the group started exploring BTC explicitly. Before the session, the teachers had started reading Liljedahl's (2021) book, and the session involved an activity where the teachers got first-hand experience of a "thinking classroom" as students working on a non-curricular problem. We analyze transcripts from this session, where teachers discussed selection of problems for BTC.

Our analytic approach was inspired by thematic analysis (Braun & Clarke, 2022), and we followed these steps: 1) familiarization with data by splitting up the transcripts into meaningful episodes (n=89) and writing memos, 2) initial coding of episodes based on the memos, and 3) further developing codes and sorting into themes. As an example, in one of the group discussions, Nelly mentioned that students' errors become "so visible" in BTC, since everyone can see what other groups have written on their boards. Some of the other teachers in the group then jumped in to make another point about students' activity level—this was identified as a separate episode—before Nelly returned to the issue of visibility of errors in what was identified as episode 49 in the transcripts from the first session:

Nelly: It is so visible, like "what have they done?"
Olga: They are very concerned about that already in Grade 8.

This episode consisted of two utterances; episodes could contain one or more utterances, and they were distinguished by shifts in meaning. The initial memo of this episode reads:

> Nelly here continues her reflections about the fact that students' errors become so visible in that everyone can see what is written on the boards. This appears to be a challenge that is particularly prevalent in BTC.

The initial coding of the episode was "How to relate to student errors in BTC" (step 2), and this was further developed into the following category of tasks of teaching: "BTC-related challenges." We created this special category of tasks of teaching to encompass multiple tasks of teaching that appeared specific to the context of BTC. In this episode, the more specific task of teaching can be described as *attending to student errors that become visible on the vertical boards*, but we decided to categorize it under a broader category of BTC-related challenges, when the reference to BTC was explicit ("visible" refers to vertical boards). In the third analytic step, we agreed that this task of teaching mainly attended to the students, so we assigned it the theme "Students."

Results and discussion

Table 1 presents frequencies of the categories of tasks of teaching that we identified from the episodes analyzed, as well as themes that were attended to. From the total of 89 episodes identified, we have selected three episodes as representative examples to illustrate the findings. The example episodes illustrate the three most frequently occurring categories of tasks of teaching identified.

Table 1: Overview of categories of tasks of teaching versus simplified themes

Tasks of teaching	Content	Organization	Students	Teacher	NA	*Total*
Select and evaluate problems	22	5	9	1	0	*37*
BTC-related challenges	4	6	2	2	0	*14*
Facilitate student learning	3	6	4	0	0	*13*
NA (No task of teaching)	2	4	1	3	1	*11*
Organize student collaboration	0	4	1	0	0	*5*
Handle challenges and needs	0	1	3	1	0	*5*
Present the problem	1	3	0	0	0	*4*
Total	*32*	*29*	*20*	*7*	*1*	*89*

The category of tasks of teaching that appeared most frequently in the teachers' discussions was that of selecting and evaluating problems. Related to this task, the teachers often discussed the content of the tasks. "Content" in our analysis, includes the types of tasks to use when using a BTC approach (non-curriculum tasks and curriculum tasks) as well as the mathematical content of the tasks. The first example is representative of the 22 episodes coded as "Select and evaluate problems/Content":

Jon: Our tasks should probably be a combination of the thin-slicing tasks and some that are a little more logical. We need to have some that are more independent, so that you don't always feel like you're just working on the same thing, just that you're now standing at the boards. We need to have a change-up there.
Bea: I don't think we should give up problem solving just to have thin slicing.
Jon: No, but we need to have both. Because if we're always going to do math standing up and not push it forward a little, I'm going to lose a lot of lessons. I need to have a little bit of both. I don't need to feel like I must have three lessons where we're just sitting and working since we're so far behind now.

In the process of selecting problems for BTC, and in this example, the teachers' discussion shifts slightly from the previous episodes' focus on progression to different types of mathematical tasks, which they name thin-slicing tasks, logical tasks, and problem-solving tasks. The concept "thin-slicing task" is used by the teachers as parallel with curriculum tasks as defined by Liljedahl (2021). In this episode, Jon addresses the challenge of working with more "standalone" problem-solving tasks, or what Liljedahl (2021) calls non-curriculum tasks, on the whiteboards versus the progression in the subject. Jon underlines that BTC is more than "standing at the boards" working on traditional tasks, and that different types of tasks are needed. It seems that the teachers here view working in BTC with exploration of problems on the whiteboards as something they do alongside regular mathematics instruction, where they use curriculum tasks, or what they name thin-slicing tasks. The teachers want to include both problems that stimulate student thinking (Liljedahl & Girox, 2024) and curriculum tasks. Problems inviting students to explore, collaborate, and discuss are important in

BTC (Liljedahl, 2021). Liljedahl often introduces problems that are designed to be engaging and thought-provoking and to help students develop the habit of thinking deeply and collaboratively before more traditional, curriculum-aligned tasks. The teachers, however, seem to value using the two types of tasks in parallel. This episode also illustrates how researchers can gain valuable insights into participating teachers' classroom practices and tasks of teaching (Ball & Bass, 2003; Ball & Forzani, 2009)—crucial for professional development (cf. Boston & Smith, 2011; Johnson et al., 2016)—through their discussions of instructional tasks (cf. Tekkumru-Kisa et al., 2020).

The second example illustrates challenges that the teachers seemed to relate directly to BTC. We coded this episode as "BTC-related challenge/Organization", and we selected this episode since "Organization" was the most prominent theme within this category:

Jon:	We've had some good tasks in the first two weeks that they've used on boards. We're always a little inexperienced with that consolidation [in BTC].
Bea:	Is it the summaries?
Jon:	The summaries, yes. It's a bit like that.
Bea:	I haven't read this.
Jon:	Difficult to summarize for both those with very basic skills and those who have come further.
Bea:	If you take everything, it takes too long.
TE:	That's a very good point. It's demanding. Here's something we must help each other with. Look at the summary phase.

It is important to notice that Jon is an experienced teacher who has used problem solving with his students for decades. Jon refers to "good tasks," but he does not elaborate on what it means. Earlier in the discussion, Jon said that he prefers a combination of problem-solving tasks (non-curriculum tasks) and curriculum tasks, and that both are important in BTC. The teachers have worked on mathematical problems with their students on vertical boards for two weeks at this point, and based on this experience, Jon expresses that the consolidation phase of a BTC lesson is challenging. Jon surely has experience with the general practice of summing up a problem-solving discussion. Yet, he finds it challenging to carry out the consolidation in BTC, particularly when some of his students have "very basic skills" and other students "have come further." Although Jon does not explicitly state what he finds challenging about consolidation, he must already be aware of the complex work of organizing and purposefully sequencing student strategies or solutions, emphasizing the importance of giving students the opportunity to express their thinking and orient themselves toward each other's thinking (Liljedahl, 2021). When Bea expresses that she has not read about the consolidation phase, it confirms that these teachers are in an early phase of learning BTC. The teacher educator (TE) confirms that the BTC-related challenge of facilitating the consolidation phase is demanding, and it is something that would be addressed in later sessions.

The third example illustrates the category of facilitating student learning (Table 1). The teachers discussed practical issues related to the organization of introducing BTC with their students. We coded this episode "Facilitate student learning/Organization," and it illustrates how the teachers in their discussion are concerned with their own experience from the same session. The teachers do not explicitly mention the students in this episode, but just before this episode, the teachers discussed students' negative attitudes and focused on dividing students into groups.

Oda: I feel the frustration myself just when we were sitting here, because I'm like Lea who must think for myself.
Kai: Yeah.
Oda: I need to use trial and error, then it will come.
Oda: I'm not fast with those things. Or I can be fast in certain areas.
Kai: Right.

Based on previous experiences in the same session, where they had worked on a problem of fractions on vertical boards in the role of students, Oda, Lea, and Kai seem to feel the frustration of not having the opportunity to think individually before collectively working on the problem on the boards. Oda says that she needs to try it out on her own first—using trial and error—before discussing the problem with others. Based on their first-hand experience with BTC, the teachers seem to be aware of the challenge of facilitating individual students' learning, suggesting that individual thinking time can be something to take into consideration when organizing and facilitating student learning in a BTC lesson. This differs from Liljedahl's (2021) recommendation to establish a common ground for students' thinking when introducing a problem orally before working on the boards. The first-hand experience from these teachers highlights the challenge of facilitating student involvement in BTC. It also illustrates that breaking down the complex work of teaching into smaller components (Hoover et al., 2014) to facilitate helping students learn (Ball & Forzani, 2009) is an important issue.

Conclusion

In conclusion, we want to highlight two observations from our study. First, although some of the identified tasks of teaching seemed to be specific to the context of BTC—and were coded as "BTC-related challenges"—we suggest that they can still be considered common tasks of teaching (cf. Hoover et al., 2014). For instance, even though the teachers debated the use of curriculum tasks versus non-curriculum tasks in the context of BTC, selecting appropriate problems or activities is a common task of teaching mathematics—beyond this context. Similarly, although Jon considered the consolidation phase to be particularly challenging in BTC, summarizing a problem-solving discussion and consolidating students' learning, is also a common and challenging task of teaching. In other words, although teachers experienced something as challenging in this particular context, the task of teaching—when considered in a certain grain size—is still a common task of teaching that is not necessarily specific to BTC.

Second, our approach to studying tasks of teaching has highlighted some valuable nuances. Whereas Ball and Bass (2003) identified tasks of teaching from careful analysis of classroom videos, we identified tasks of teaching based on analysis of teachers' discussions in a professional development context. This approach enabled us to discover how an experienced teacher like Jon perceived the seemingly common task of consolidating students' thinking as challenging in the context of BTC. In other words, common tasks of teaching might be perceived by teachers as specific to certain contexts, and this is important to attend to in professional development. The language of "tasks of teaching" might help teachers connect new practices to familiar challenges they are regularly faced with in the work of teaching mathematics. In addition, the context in which we have explored tasks of teaching involved providing teachers with first-hand experiences, and it was such an experience that prompted Oda to identify the challenge in the third example.

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The interplay between student teachers' content knowledge and teaching practices: Insights from the ERME community

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In Norway and internationally, reforms for teacher education programmes have changed from talking about practice to enacting practice. A main tenet of this turn is that student teachers learn about content and students in and from practice. However, in our work with student teachers enacting mathematical discussions in role-plays of teaching, we have encountered challenges related to the learning of content knowledge and the challenges it implies for the student teachers' development of teaching practices. To improve our teacher education programme, we need to better understand the interplay between content knowledge and practices in teacher education. In a metasynthesis, we analysed papers from TWG18–20 from CERME13 and -14 for how the ERME community addresses this interplay. The results indicate three interplays: the impact of content knowledge on teaching practices, the theoretical framings of these interplays, and reflection for addressing the interplay.

Keywords: Mathematics teacher education, content knowledge, practice-based teacher education, professional development, metasynthesis.

Introduction

The turn towards practice-based teacher education has led to increased efforts to support teachers' learning of practices that are fruitful for students' learning (Charalambous & Delaney, 2020). Although a solid knowledge of mathematics and teaching is a prerequisite for learning the practices of teaching (Ball & Cohen, 1999), the development of mathematical knowledge for teaching has not been a primary focus of research on practice-based teacher education. Instead, the assumption has been that student teachers (that is, pre-service teachers in teacher education or in-service teachers in professional development) learn in and from practice (Ball & Cohen, 1999). For example, in planning a mathematical discussion, the student teacher needs to break the mathematical goal into a sequence of subgoals and consider which moves and representations can support students' work towards those subgoals. Inspired by the practice-based approach to teacher education, we – the authors and our colleagues in a research project – designed several lessons for a mathematics teacher education course. Although student teachers' opportunities to learn to lead discussions were documented, our analyses revealed that the complexity of the mathematical content sometimes hindered their development of teaching practices.

The goal of the designed lessons was for student teachers to learn to promote students' reasoning and proving in mathematical discussions. Reasoning and proving is known to be challenging to teach (Stylianides & Stylianides, 2017). Despite our best efforts to provide student teachers with opportunities for learning mathematical content through investigating and enacting practice, it seemed somewhat arbitrary whether the opportunities were grasped or not (Valenta & Enge, 2025). Moreover, challenges in the student teachers' mathematical knowledge were occasionally observed to hinder the planning and enactment of the discussions (Arnesen et al., 2025). Our challenges in designing lessons that support student teachers' development of knowledge and practice encouraged

us to review the papers of the working groups on teacher education, practice, and knowledge (TWG18–20), to learn how our colleagues in the ERME community address similar challenges. Through a metasynthesis of the papers, we answer the research question: *How can the interplay between student teachers' mathematical knowledge and their teaching practices be described, based on the results of the research reports of TWG18–20 from CERME13 and CERME14?*

Theoretical background

Two major foci in teacher education research are the practice-based approach and the knowledge needed for teaching. In this section, we present the main ideas of influential contributions to teacher education research on these foci. First, the aim of centring teacher education around the practices of teaching is to support student teachers' development of teaching practices. Teaching can be seen as a collection of practices, which are routines performed habitually and constantly (Lampert, 2010). These practices include those performed inside and outside the classroom, such as interpreting and responding to students' thinking, leading classroom discussions, planning instruction, and choosing and adapting tasks. By conceptualising teaching as a collection of practices, the assumption is that the repetition of action is a necessary component of learning to teach. Through approximations of practice, student teachers can rehearse practices in contexts less complex than those of authentic teaching. However, the ability to perform routines also depends on the teacher's skills. Through representations of practice, student teachers can develop their noticing skills (van Es, 2010) by learning to attend to and interpret students' thinking.

Second, a necessary component of learning to teach mathematics is to develop mathematical knowledge for teaching. Ball et al. (2008) described mathematical knowledge for teaching as comprising subject-matter knowledge and pedagogical content knowledge. Similarly, Carrillo-Yañez et al. (2018) refer to mathematical knowledge and pedagogical content knowledge. In the framework of Ball et al. (2008), subject-matter knowledge consists of common content knowledge (the mathematics students learn), horizon knowledge (understanding how mathematical topics are related across grades), and specialised content knowledge. The latter refers to the mathematical knowledge required for teaching, which extends beyond what students typically learn: it involves the knowledge to unpack mathematics in more detail than people who are not teachers need. For example, knowing when representing subtraction as a difference is more useful than representing subtraction as taking something away, given the mathematical content at stake. Carrillo-Yañez et al. (2018) described content knowledge differently, consisting of knowledge of the topics, the connections, and the ways of proceeding in mathematics. The second domain of the frameworks by Ball et al. (2008) and Carrillo-Yañez et al. (2018), pedagogical content knowledge, involves knowledge of learning (including understanding typical misconceptions and predicting students' thinking) and teaching (e.g., knowing how to sequence instruction and recognising which students' contributions to build upon). Lastly, it involves knowledge of curriculum and standards.

Methods

To provide a coherent description of the interplay between student teachers' mathematical knowledge and teaching practices based on the research of the ERME community, we conducted a metasynthesis of the research reports of CERME13 and CERME14. A metasynthesis includes an investigation of

the qualitative findings of a selection of research reports (Sandelowski & Barosso, 2007). A metasynthesis differs from a literature review in that the researcher obtains new results, based on the empirical results of a selection of papers, rather than summarising trends in the papers. Since research on approaches to teacher education typically addresses knowledge or practices, novel interpretations are needed to investigate the interplay among these components.

A qualitative metasynthesis consists of several steps. First, the studies need to be retrieved. We retrieved all the research papers from the three thematic working groups that are part of the original working group exploring mathematics teacher education (TWG18–20) from the last two CERMEs. The reports from these two CERMEs represent advancements in the research on supporting student teachers' development of practices and knowledge, written after we implemented the lessons that made us aware of the crucial role of mathematical knowledge for developing teaching practices. We included and excluded papers based on topical parameters of inclusion (Sandelowski & Barosso, 2007). The topical parameter was that the paper considered student teachers' (in teacher education or professional development) learning of practices or content, which we identified based on the papers' methods sections. We categorised papers concerning the learning of practices by the methods section's reference to work on representations and approximation of practice. Papers were considered to concern learning of content if work on mathematics or the unpacking of mathematics was referred to, paralleling (or explicitly using) the content knowledge part of Ball et al.'s (2008) framework or the mathematical knowledge part of Carrillo-Yañes et al.'s (2018) framework. Moreover, we only considered papers reporting empirical, qualitative studies, a feature of metasyntheses (Sandelowski & Barosso, 2007). The second step was appraisal of the studies. In this step, we read the retrieved papers using a reading guide that addressed our research interests. Hence, we made notes of the context of the study, the theoretical framing, and the findings. In this step, we excluded papers that did not discuss their work on learning practices or content in light of content knowledge or how content knowledge influences practices. Table 1 shows the included papers.

In the third step, we interpreted the findings of the selected reports, using practices and content knowledge as imported concepts (Sandelowski & Barosso, 2007). Hence, we analysed the findings to describe the student teachers' content knowledge and practices. After analysing the student teachers' knowledge and practices, we analysed the interplay between these concepts. This last analysis yielded three interplays: specialised content knowledge is positively correlated with student teachers' practices, reflection as a means of strengthening the link between knowledge and practice, and theoretical approaches addressing the interplay between knowledge and practice.

A weakness of qualitative metasyntheses is that the researchers do not have access to the data underlying the reports they analyse. Hence, interpretations of results need to be conducted carefully. In the research reports, the interplays between content knowledge and practices were the subject of analysis or a subject of the authors' discussion. However, neither of the reports framed the interplay in terms of the theoretical constructs of content knowledge and practices, and the identified interplays presented in the results are products of our analysis. In our analysis process, we both read all the reports and critically reviewed each other's interpretations before agreeing on a set of interplays between content knowledge and practice.

Table 1: Distribution of included papers, with author names

	TWG18	TWG19	TWG20
CERME-14	Rotem & Eraky Ødegaard Kuntze et al.	Öcal et al.	Demirci & Akkoç Martin et al. Silva & Ribeiro Semanišinová & Slabý
CERME-13	Graf et al. Kongshavn Serrazina et al. Wöller	Tunc-Pekkan & Kayıtmaz	Climent et al. Semanišinová & Sajka

Findings and discussion of research

Our metasynthesis identified three ways to describe the interplay between content knowledge and teaching practice in the research reports of TWG18–20 at CERME13 and CERME14. In all three descriptions, we found that specialised content knowledge played a crucial role in the development of the student teachers' practices.

How specialised content knowledge correlates with student teachers' practices

The metasynthesis revealed two direct relationships between content knowledge and practices in the data material, neither of which was very surprising. The first direct relationship, and the most common, was that student teachers who themselves struggled with a particular mathematical content also struggled to perform teaching practices that built on that content. For example, practices could involve making sense of students' mathematical thinking, posing meaningful questions, or choosing tasks. These practices were hindered by poor common content knowledge and specialised content knowledge. In Silva and Ribeiro's (2025) study, some of the student teachers did not recognise students' errors because they made the same mistakes. The authors concluded that "the content of teachers' [mathematical knowledge] impacts the type and level of interpretative practice and knowledge" (p. 7). Martin et al.'s (2025) study demonstrated that student teachers' task selection practice was hindered due to inconsistencies between tasks, contexts and representations, as well as solutions. Since knowing when different representations are useful is a part of specialised content knowledge, our interpretation of the report was that poor specialised content knowledge hindered task selection. Accordingly, Graf's (2023) study demonstrated that a student teacher's poor specialised content knowledge made her unsuccessful in supporting students in solving the task. The teacher was able to solve the task herself, but did not demonstrate flexibility in different solution methods and guided the student only according to her own solution strategy.

In Graf's (2023) study, the second direct relationship was also identified. The relationship was that good content knowledge led to the successful enactment of a particular practice. Graf (2023) demonstrated that the teacher with more robust specialized knowledge for teaching asked meaningful and supportive questions to her student. The teacher demonstrated robust specialised content

knowledge by "focusing on understanding the arithmetical operation …[and] knowing fitting ways to guide the child in this understanding process" (p. 3208).

Reflection as a means of strengthening the link between knowledge and practice

The second identified interplay was a method for strengthening the link between knowledge and practice in teacher education: reflection. Three of the reports demonstrated that post-practice reflection can be a context for addressing or even developing the relationship between mathematical knowledge and teaching practices.

In Kongshavn's (2023) study, student teachers reflected on an enactment of a role-play of a lesson where the mathematical content was proof and proving. The student teachers pointed to teaching episodes in which they wished they had been able to support the students properly by knowing more ways of proving. Interestingly, when interviewed before their lesson, the PSTs did not worry about content-specific matters but focused only on (more general) pedagogical aspects of teaching. Thus, Kongshavn's study indicates that work on practices can strengthen PSTs' awareness of the need for specialised content knowledge.

Also, Serrazina et al. (2023) connected content knowledge and practice through reflection in cycles of planning, enacting, and reflecting. The reflections, carried out in small groups, included discussions about the processes of reasoning their students had been engaged in. During the reflections, it became evident to the researchers that the student teachers held misconceptions about the different reasoning processes (an aspect of specialised content knowledge), and that "these episodes show the relevance of discussing the processes of reasoning with peers and experts in articulation with classroom practice" (p. 3298).

Wöller (2023) also addressed reflection and claimed that "reflective competence seems to have a 'mediating character' between basic knowledge and action-related competence" (p. 3496). In this report, specialised content knowledge came to the foreground, particularly when the student teachers discussed the theoretical foundations of open-ended tasks they implemented in the classroom.

Theoretical approaches addressing the interplay between knowledge and practice

Most of the reports we analysed were situated in the 'Shulman tradition', referring to his fundamental work or work building on it (e.g., Ball et al. (2008) and Carrillo-Yañez et al.'s (2018) frameworks). Many reports were based (also) on other ideas or open inductive coding. However, these reports did not use a theoretical framework for addressing practice. Four reports used a theoretical framework for analysing the interplay between knowledge and practice in their results. The frameworks were situated in a knowledge perspective, a practice perspective, and a noticing perspective.

First, the Mathematics Teacher's Specialised Knowledge (MTSK) framework (Carrillo-Yañez et al., 2018) was used in two reports to analyse how misconceptions affected practices. The MTSK framework builds on Ball et al.'s (2008) framework but focuses specifically on mathematical knowledge, using a more fine-grained substructure. Mathematical knowledge in the MTSK framework includes subcategories for *Knowledge of Topics* (KoT), *Knowledge of the Structure of Mathematics* (KST), and *Knowledge of Practices in Mathematics* (KPM). Semanišinová and Sajka (2023) found that the student teachers' lack of knowledge about topics, beyond what would be

described as common content knowledge by Ball et al. (2008), affected their practices: “The gap in their KoT affected their understanding of the students’ solution (KFLM) and the feedback they gave to the student (KMT)” (Semanišinová and Sajka, 2023, pp. 3898).

Ødegaard (2025) analysed the interplay between content knowledge and practices from the practice perspective. Here, content knowledge was conceptualised within the ‘complexity’ of an approximation to practice. Within these frames, the author found that, with high complexity and little support provided, the student teachers’ practice of leading mathematical discussions towards the proof of a general conjecture was hindered. We interpreted that the mathematical support missing in the approximations concerned specialised content knowledge, as the student teachers needed support in evaluating the representations they used and unpacking the concepts of fraction division.

Another identified theoretical framework for connecting content knowledge and practice is the concept of noticing. Kuntze and colleagues (2025) analysed PSTs’ noticing skills when working with a vignette showing students and a teacher working on algebraic generalising. Noticing has been defined as attending to, interpreting, and making teaching decisions (e.g., van Es, 2010). The PSTs interpreted the students’ response well. Still, they were unable to provide satisfactory feedback to the students, instead agreeing with the vignette teacher (whose response was intentionally designed to be incomplete or insufficient). Kuntze et al. (2025) claimed that there is a need to include a ‘noticing focus’ on the mathematical properties of the classroom situation. By “comparing and relating different mathematical approaches of the participants in the classroom interaction” (p. 7), which we analysed as focusing more on specialised content knowledge, student teachers can be supported in learning to respond to students’ thinking.

Final remarks

Our question about how the interplay between knowledge and practice has been treated in the last two CERMEs is answered through the three interplays we identified in our metasynthesis. Based on the results, we provide implications for teacher education and suggest areas for further research.

Our results showed that student teachers’ specialised content knowledge stood out as crucial for enacting meaningful practices. Particularly, poor specialised content knowledge is a hindrance to the development of teaching practices, according to our initial conjecture. Following the practice-based approach, we recommend that future approaches to teacher education incorporate methods for developing specialised content knowledge through practice. Moreover, our results highlight reflection as a promising approach. Through reflection, student teachers can increase their specialised content knowledge, which can positively impact their practices, and the divide between knowledge- and practice-based pedagogies in teacher education can be bridged. To develop approaches that address the interplay between specialised content knowledge and practice through reflection, design-based research is needed.

Moreover, in design-based research, theories that can capture all aspects of the interplay are needed. The metasynthesis revealed that several theoretical constructs (Mathematics Teacher’s Specialised Knowledge, noticing, complexity) have been used to uncover what hinders student teachers’ practice. However, they do not provide directions for pedagogies in teacher education. For example, we argue

that pedagogical knowledge for teaching, as exemplified by the KFLM and KMT categories of the MTSK framework, does not wholly capture the practice-based part of the relationship. However, it is undoubtedly important for – and strengthened through – practice. Similar problems arise when conceptualising the interplay from a practice perspective. Complexity, for example, did not provide a fine-grained lens to study content knowledge. Networking of theories that capture several dimensions of knowledge and practice can be fruitful. Moreover, we believe that theories for competence development through reflection can provide a helpful lens.

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Potentials and limitations of computer-based assessment in mathematics teachers' pedagogical practice: A systematic review

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This systematic review article analyzes the potentials and limitations of computer-based assessments from the perspective of mathematics teachers' pedagogical practice. Following PRISMA 2020 guidelines, English-language articles from the Scopus and Web of Science databases (2015-2025) were analyzed. The results show that computer-based assessments support and mediate teachers' pedagogical content knowledge and formative assessment skills. It enables the collection of large amounts of data and provides instant, automated feedback, saving time and offering personalized learning paths. However, computer-based assessments also present limitations, such as teachers' difficulties in data interpretation, technical issues, and software gaps in certain topics. In conclusion, while computer-based assessments have the potential to transform the teacher's role into a data-driven guide, this requires overcoming current technical and training-related challenges.

Keywords: Computer-based assessment, mathematics education, pedagogical content knowledge, formative assessment.

Introduction

The educational landscape is undergoing significant changes due to the integration of technology, which is fundamentally transforming how we teach and learn (Haj-Yahya & Olsher, 2020). In recent years, digital technologies have revolutionized education, providing both teachers and students with new and effective learning and assessment tools (Piaw, 2012; Zawacki-Richter et al., 2020). The use of digital assessment, especially formative assessment platforms, has shown that teachers can better observe students' thought processes and respond to their difficulties with the content being taught (Haj-Yahya & Olsher, 2020). Computer-based assessment CBA can reduce the influence of human subjectivity and provide immediate feedback (Redecker & Punie, 2017). In the context of mathematics education, CBA creates the opportunity to assess factual and procedural knowledge for a large number of students (Hoogland & Tout, 2018). Although CBA is known for its effectiveness in measuring learning outcomes, its impact on teachers' daily practice has been less studied (Haj-Yahya & Olsher, 2020).

Compared to paper-based tests, teachers spend significantly less time on assessment and analysis of results, which allows them to focus on better planning of their teaching (Öz & Özturan, 2018). Based on the data obtained, teachers can make better informed decisions, adapt teaching methods, and provide students with more targeted support (Sailer et al., 2021). For teachers, this offers an opportunity for self-reflection and continuous professional development. However, the limitations of CBA include the difficulty of assessing open-ended responses, problem solving tasks, and higher-level thinking questions, as well as various technical constraints (Seifert & Lindmeier, 2024). Many teachers feel they lack the digital competence and support needed to adopt new systems.

The theoretical foundation of this study is Lee Shulman's (1987) concept of pedagogical content knowledge (PCK). Our analysis is primarily guided by the PCK components related to diagnosing student understanding and selecting appropriate teaching strategies. This theory emphasizes that effective teaching requires both strong subject-matter expertise and an understanding of how students learn a given topic, including the typical difficulties they encounter, such as common misconceptions and appropriate instructional strategies. Although other frameworks exist, such as Mathematical Knowledge for Teaching and TPACK, this review adopts Shulman's original PCK model because the primary focus is on how teachers interpret and respond to assessment data. This function lies at the core of PCK, while technological competence emphasized in TPACK and the depth of domain-specific knowledge emphasized in MKT are not the central concerns here. Together with Black & Wiliam's (1998) theory of formative assessment, which examines how evidence is generated, interpreted, and used to improve learning, PCK provides a lens for understanding how teachers can use computer-based assessment data to inform their instructional decisions and better guide the learning process (Maskos et al., 2025).

While alternative theoretical frameworks, such as Mathematical Knowledge for Teaching (MKT) by Ball et al. (2008) or Technological Pedagogical Content Knowledge (TPACK) by Mishra & Koehler (2006), exist, Shulman's (1987) original PCK model was chosen for this review. This is because the primary focus of the analysis is on how teachers interpret assessment data to understand student misconceptions and adjust instructional strategies—core functions of PCK. These components are essential for analyzing the pedagogical utility of CBA, distinguishing the focus from purely technological competence (TPACK) or the depth of disciplinary mathematical knowledge (MKT).

Although previous studies have addressed the potential and limitations of CBA, the results are scattered across different fields. There is no comprehensive synthesis that consolidates and systematizes these results to understand the main opportunities and challenges in the context of mathematics education. This systematic review is particularly relevant to the ETC2026 conference theme, specifically addressing the question "What is involved in learning mathematics teaching?" With the integration of digital technologies, the nature of learning to teach is evolving. Our review demonstrates that computer-based assessment (CBA) transforms the role of the mathematics teacher, moving it toward data-driven guidance. CBA supports the learning of teaching by enabling them to collect large amounts of data, identify student misconceptions and knowledge gaps (the diagnostic component of PCK), and subsequently rapidly adapt instructional strategies. Thus, this article provides a systematized synthesis of how technology mediates the practical application of PCK and outlines the new skills required for learning mathematics teaching in the digital age.

Method

The aim of this systematic review is to analyze and synthesize existing academic literature on the potential and limitations of CBA in mathematics teachers' pedagogical practice. The review was conducted according to the PRISMA (Preferred Reporting Items for Systematic Reviews and Meta-Analyses) 2020 guidelines (Page et al., 2021), which ensures the transparency and reproducibility of the research.

A systematic search was conducted in the academic databases Scopus and Web of Science (WoS). These databases cover a wide range of fields in education and social sciences. The search used Boolean operators AND and OR and a combination of keywords. The following complete search string was applied to the Title, Abstract, and Keywords field (TITLE-ABS-KEY in Scopus; TS= in WoS. The search terms Scopus Search String: TITLE-ABS-KEY(("computer-based assessment" OR "digital assessment" OR "online assessment" OR "e-assessment") AND (teacher OR teachers OR "teacher practice" OR "pedagogical practice" OR "teacher professional development" OR teaching) AND (mathematics OR maths OR "mathematics education" OR math) AND ("pedagogical content knowledge" OR PCK OR "formative assessment")). The search terms WoS Search String TS=("computer-based assessment" OR "digital assessment" OR "online assessment" OR "e-assessment") AND TS=(teacher OR teachers OR "teacher practice" OR "pedagogical practice" OR "teacher professional development" OR teaching) AND TS=(mathematics OR maths OR "mathematics education" OR math) AND TS=("pedagogical content knowledge" OR PCK OR "formative assessment"). The search was limited to peer-reviewed articles published in English between 2015 and 2025. The search was conducted on September 15, 2025.

The article selection process took place in two stages. First, the titles and abstracts were reviewed, and then the full texts of the selected articles were evaluated. Only peer-reviewed articles focusing on the use of CBA by mathematics teachers and its impact on pedagogical practice and professional development were included. Studies that focused solely on technical aspects, were non-educational, or only concerned student performance were excluded. Articles that dealt with subjects other than mathematics were also excluded.

The initial search yielded 78 articles, with 23 from Scopus and 55 from WoS. The lower number of results from Scopus and the low number of duplicates (2) suggests that the search strings might not have been fully equivalent across databases or that the specific index selections and document type filters led to this distribution. The higher result count from WoS is likely due to its broader inclusion of conference proceedings and specific index collections (SCI-E, SSCI), whereas Scopus indexing may have been more restrictive regarding document type. The very low number of duplicates suggests the search strings effectively retrieved complementary, rather than overlapping, sets of relevant studies, thus ensuring the comprehensiveness of the review. Despite this, the search retrieved a sufficient and relevant set of articles for the qualitative analysis. Duplicates were removed (2). A total of 76 articles remained. As a first screening process, abstracts were read to check whether they reflected CBA, the role of the teacher, and whether the article focused on the teaching of mathematics. After this check, 23 articles remained. During the collection of articles, a table was created to note the article's title, author(s), year of publication, what teacher practices the article focuses on, its potentials, and its limitations. During the review of the articles, studies that were not empirical and that did not directly focus on or whose focus was not on the (implicit or explicit) PCK of teachers were removed. The term 'PCK' was used broadly as an inclusion criterion to encompass teacher knowledge concerning student difficulties, curriculum content knowledge, and instructional strategies, which are the core aspects examined in relation to CBA data usage. This broad interpretation ensures that studies relevant to PCK's practical application, regardless of whether they explicitly used Shulman's (1987) terminology, were considered. A total of 7 articles remained.

This was followed by an in-depth analysis of the full texts of the articles. During the in-depth analysis, the table was supplemented with information about the study's methodology (study design, sample type and size, data collection method, country where the study was conducted). The table also collected information about the CBA, what their potentials and limitations were, and what relationships were found between CBA and teachers' PCK or formative assessment. In addition, the main findings and results of the study were highlighted.

Results and discussion

The analysis of CBA shows that while it has significant potential for improving teachers' pedagogical knowledge and instructional work, there are also significant limitations associated with it. Based on the analyzed articles, the main potentials and limitations of CBA can be divided into content-related and technical aspects.

Joshi et al., 2023, Confrey & Shah, 2021, Annuš & Kmeť, 2024, and Faber et al., 2017 highlight how CBA allows teachers to collect and analyze large volumes of data on student knowledge to make more informed pedagogical decisions to change the learning process. CBA offers both students and teachers quick and detailed automated feedback on knowledge and progress (Annuš & Kmeť, 2024; Confrey et al., 2018; Faber et al., 2017; Joshi et al., 2023). CBA provides students with automatic and immediate feedback, which supports both self-assessment and error analysis (Annuš & Kmeť 2024; Faber et al., 2017; Joshi et al., 2023). This helps students better understand their errors and shortcomings (Annuš & Kmeť, 2024). Digital tools and automated assessment systems increase students' motivation and interest in learning (Annuš & Kmeť, 2024; Faber et al., 2017; Hommik & Hõim, 2015). The data helps teachers identify students' knowledge gaps and misconceptions.

Most of the analyzed articles (Confrey et al., 2018; Confrey & Shah, 2021; Faber et al., 2017; Joshi et al., 2023) point to flexibility, that is, the rapid ability to adapt and modify teaching content according to student needs. They also emphasize the potential of CBA to provide students with customized learning paths that match their individual needs and learning pace (Annuš & Kmeť, 2024; Confrey et al., 2018). These are important aspects for remedial students, for acquiring in-depth knowledge, and for high-achieving students, to whom CBA can offer additional tasks (Faber et al., 2017). It can be said that CBA allows for faster adaptation and modification of curriculum content according to student needs, which is particularly important in times of change such as pandemics. Joshi et al. (2023) and Faber et al. (2017) emphasize the ease of use and accessibility of digital tools, which promotes their widespread adoption. In addition, various CBA environments provide visualization and simulation tools that help students better understand complex concepts (Hommik & Hõim, 2015). Thanks to automated grading, teachers spend less time on assessment and feedback, allowing them to focus more on helping students. Also, the use of CBA is more environmentally friendly, as it generates less paper waste (Alderton & Pratt, 2021).

Although CBA provides a wealth of data, teachers may not have the skills to interpret it in depth or link it to learning theories (Confrey et al., 2018; Confrey & Shah, 2021). Discussions revealed that data analysis and interpretation pose challenges for teachers. Hommik & Hõim (2015) highlight that teachers need additional training in data interpretation, which would help them better apply the data in practical teaching. Alderton & Pratt (2021) point out the problem that algorithms can create

situations in data interpretation where the teacher does not understand how the software arrives at certain answers, thereby limiting their pedagogical autonomy. Technical problems, such as platform stability and a lack of devices, can hinder the effective implementation of CBA (Hommik & Hõim, 2015). Consequently, teachers may feel resistant to new technology or lack the skills to integrate it into their daily practice (Confrey et al., 2018; Faber et al., 2017). In addition, it is pointed out that frequent technical errors reduced the motivation of both teachers and students (Annuš & Kmeť, 2024; Hommik & Hõim, 2015). Some software lacked content or sufficient assessment systems for certain topics, which limited its use (Annuš & Kmeť, 2024; Hommik & Hõim, 2015).

The analyzed studies have applied diverse methodological approaches, combining both quantitative and qualitative methods. Quantitative studies (e.g., Faber et al., 2017; Joshi et al., 2023) relied on large-scale questionnaires and experimental designs, while qualitative studies (e.g., Alderton & Pratt, 2021; Confrey et al., 2018; Confrey & Shah, 2021; Hommik & Hõim, 2015) used interviews and video analyses. Participants included both teachers and students, with sample sizes varying significantly. For example, the study by Faber et al. (2017) included 1808 students, which is a large and statistically significant sample, while the sample of Joshi et al. (2023) was skewed in terms of gender balance, which reduces generalizability. The study by Faber et al. (2017) used a larger sample size, while Hommik & Hõim (2015) relied on a sufficiently small sample for an in-depth case study. The scarcity of qualitative studies focusing specifically on the teacher's interpretative role underscores the challenge in fully articulating the complex, nuanced relationship between CBA data and the application of PCK in real classroom scenarios.

To illustrate this implicit connection to PCK, studies showed that CBA data directly enables core PCK functions. For example, the detailed feedback on student responses, such as the type of errors students make most often (Confrey & Shah, 2021; Faber et al., 2017), allows teachers to move beyond general knowledge to specifically diagnose common misconceptions in a given topic. This diagnostic function—understanding student difficulties—is a primary component of PCK. Furthermore, CBA tools that provide individual learning paths, such as "Learn with M.E." (Annuš & Kmeť, 2024), enable teachers to immediately select and adapt appropriate instructional strategies (another PCK component) to address those gaps, thereby strengthening their practical application of PCK. Thus, even without explicit reference, the system's output facilitates the core function of PCK: interpreting student evidence to inform instructional decisions. Tools like "Learn with M.E." provide teachers with information that allows them to create customized learning paths that support students' individual development (Annuš & Kmeť, 2024). At the same time, a study by Alderton & Pratt (2021) shows that although the software is marketed as a PCK-supporting solution (e.g., gap identification), it can actually limit a teacher's professional responsibility and promote a mechanical approach to learning (Alderton & Pratt, 2021). This may limit the effective application or growth of deeper PCK.

CBA is an ideal tool to support formative assessment, as the systems provide continuous feedback on student performance, which is the most important part of formative assessment (Annuš & Kmeť, 2024; Confrey et al., 2018; Hommik & Hõim, 2015; Joshi et al., 2023). Unlike traditional assessment methods, CBA provides detailed and fast feedback, which allows teachers to adjust the learning process in real time and plan the next lesson according to the data. The data helps teachers identify student problems as soon as they arise and provide timely support. In addition, studies show that the

use of digital tools and quick feedback increase students' motivation, which in turn supports the learning process.

Conclusion

The analysis shows that CBA is a crucial tool for supporting and mediating teachers' PCK by providing them with accurate data on students' knowledge and learning paths. Because the software collects and analyzes student data, teachers gain a unique overview of students' knowledge gaps and misconceptions, which directly supports the diagnostic component of PCK. This detailed information allows them to adapt their teaching and focus on specific topics (i.e., selecting appropriate instructional strategies). This data-driven approach helps teachers better understand what and how students think, which mediates PCK's practical application (Confrey et al., 2018; Confrey & Shah, 2021; Faber et al., 2017). However, it is important to note that the effective use of this data depends heavily on the teacher's ability to interpret it in-depth, which remains a significant limitation (Confrey et al., 2018; Confrey & Shah, 2021). This data-driven approach helps teachers better understand what and how students think, which mediates PCK's practical application. Although the availability of data is a major potential, it is crucial to ensure that teachers are adequately trained to interpret and apply this data.

CBA effectively supports the formative assessment process by providing fast, automated, and detailed feedback to teachers and students. Based on the data, customized learning paths and personalized learning materials can be created to support each student's individual development (Confrey et al., 2018; Hommik & Hõim, 2015). This is particularly valuable in mathematics education, where students' levels and needs are often very different. However, it must be considered that the teacher must remain the leader of the process.

Despite its potential, CBA is associated with significant challenges. Technical errors and a lack of content can diminish motivation and effectiveness (Annuš & Kmeť, 2024; Hommik & Hõim, 2015). In addition, the software, especially when linked to high-stakes assessment systems, can promote a mechanical and fragmented approach to mathematics, which is contrary to richer, deeper learning (Alderton & Pratt, 2021).

CBA has enormous potential to transform the role of the mathematics teacher into a data-driven guide. For this potential to be fully realized, current limitations need to be overcome, improving the technical quality of the software, developing its content, and training teachers in the use of these environments (Alderton & Pratt, 2021). To address the recurring limitation of data interpretation, future research must prioritize investigating the efficacy of professional development programs aimed at enhancing teachers' assessment literacy and data-driven decision-making skills. Future research should focus on long-term, in-depth studies that examine how CBA supports the use and refinement of PCK and leads to lasting changes in teaching practices. While large quantitative studies are important, more qualitative case studies are needed to understand the social and pedagogical aspects of CBA implementation.

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Teaching through representation: Fostering student autonomy through bar models in mathematics learning

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Teaching mathematics through visual representation requires teachers to integrate visual tools with pedagogical responsiveness. Such representations engage learners' reasoning and promote autonomy in mathematical sense-making. This study explores the challenges and potential of using bar models through an individual case study, examining how teachers employ representations to scaffold a learner's reasoning and support independent problem solving. Using qualitative discourse analysis of teacher–student interactions, it investigates how teachers adapt scaffolding and feedback in response to the learner's representational choices. Findings show how questioning, feedback, and representational mediation shape the learner's conceptual engagement and ownership of learning. The study advances understanding of how representational teaching influences individual learning trajectories and informs professional development that promotes learner autonomy in mathematics.

Keywords: Visual representation, bar models, scaffolding, analogical reasoning, student autonomy.

Introduction

Visual representations are central to mathematical understanding, enabling learners to interpret relationships, structure problems, and develop reasoning. Among these, bar models (Kho et al., 2009), provide a versatile means of representing key mathematical relationships. However, the effective use of representations in mathematics instruction does not unfold spontaneously; it requires deliberate pedagogical choices. The teacher's role extends beyond selecting appropriate visual tools to scaffolding their use, responding to individual learner thinking, and mediating meaning through interaction. This study investigates how a teacher develops the capacity to teach through representation in a one-on-one instructional setting, focusing on the use of bar models supported by virtual manipulatives. It examines how representations are introduced, negotiated, and refined in interaction with a single learner, and how the teacher uses comparing correct and flawed solutions to reveal misconceptions and guide reasoning. Using qualitative discourse analysis of teacher-student interactions, the study explores how representational scaffolding, questioning strategies, and real-time feedback shape the learner's conceptual engagement and autonomy in problem solving. The findings highlight the pedagogical potential of individualised representational teaching to foster mathematical understanding and inform teacher professional learning that promotes learner independence.

Theoretical framework

This study is grounded in a theoretical framework that integrates constructivist, sociocultural, and cognitive perspectives to explain how visual representations mediate mathematical understanding and autonomy. Central to this framework is the Model Method (Kho et al., 2009), a representational approach to problem solving that enables learners to translate mathematical relationships into visual-symbolic forms (Nesher et al., 1982; Kintsch & Greeno, 1985). The method emphasises the use of

bar models to represent part-whole, comparison, and unitary concepts (Fong, 1999), allowing learners to visualise and reason about quantitative relationships, supporting reasoning about how parts combine to form a whole, how one quantity compares to another, and how values scale proportionally based on a unit. The representational foundations of the Model Method align closely with Bruner's (1966) constructivism, which views learning as an active process of meaning-making through progressive engagement with representations. Bruner's distinction between enactive, iconic, and symbolic modes of representation provides a basis for understanding how learners move from concrete manipulation to abstract reasoning. In this context, bar models exemplify how learners externalise relationships among quantities and transition between visual and symbolic forms of mathematical thought.

From a sociocultural perspective, Vygotsky's (1978) theory emphasises that learning is mediated through tools and interaction. Representations such as bar models act as mediators that structure thought and demonstrate how meaning-bearing tools contribute to cognitive development. In an individual instructional setting, the teacher's scaffolding and discourse serve as the social means through which the learner internalises strategies for reasoning independently. Within this process, teachers can use such representations to guide learners' progress within their Zone of Proximal Development (ZPD). Vygotsky viewed learning as a fundamentally social process, with the ZPD representing the space where learning occurs through guided interaction. It marks the gap between what a learner can accomplish independently and what can be achieved with appropriate support, providing an optimal context for learning through challenge and collaboration with a more knowledgeable other. Through such mediated interaction, knowledge is first co-constructed on the social plane (interpersonal) and subsequently internalised as individual understanding (intrapersonal) (Vygotsky, 1978).

Comparison, as a form of cognitive support, plays a critical role in promoting conceptual understanding (Richland et al., 2004). By juxtaposing correct and flawed solutions, teachers enable students to discern key mathematical relationships, notice underlying structural similarities and differences, and refine their conceptual frameworks. Insights from cognitive psychology further illuminate this process, highlighting how analogical reasoning supports the development of flexible and transferable understanding. Research on analogical reasoning demonstrates that recognising relational similarities between mathematical representations facilitates problem-solving and knowledge transfer.

Side-by-side comparisons of example solutions enhance this process by drawing attention to both similarities and contrasts (Namy & Gentner, 2002). According to Gentner's (1983) structure-mapping theory, effective comparison requires explicitly defining the elements being aligned so that learners can map correspondences between valid and flawed solution strategies. Through this explicit guidance, teachers help students identify deep structural relationships that might otherwise remain unnoticed, thereby supporting more flexible and integrated conceptual learning.

Within this framework, learner autonomy in mathematics is conceptualised not as working independently in isolation, but as developing the capacity to regulate one's own mathematical thinking, make informed representational choices, and engage critically with problems and reasoning.

Autonomy emerges through scaffolded participation and reflective engagement, as learners assume greater responsibility for selecting and interpreting visual tools and strategies. In the individualised instructional context examined in this study, autonomy is viewed as a socially mediated and cognitively grounded outcome of representational teaching, an interplay between teacher guidance and the learner's growing control over meaning-making processes.

Research questions

The research questions outlined below guide the exploration of how representational teaching practices are developed and their impact on students' engagement, conceptual growth, and emerging autonomy as mathematical learners:

RQ1 What kinds of teacher–student interactions emerge when a teacher uses bar models for addressing misconceptions? RQ2 How can teachers structure scaffolding to support students' transition from guided to autonomous use of bar models for error correction in word problems?

Methods

This study adopts a design-based single case study approach to investigate how representational teaching practices unfold within a structured instructional design. By analysing moment-to-moment teacher–student interactions, the study explores how representational tools and dialogue mediate conceptual understanding and the development of learner autonomy. The approach integrates design experimentation with interpretive analysis to connect theoretical insights with pedagogical application.

The study focuses on an individual case conducted in Singapore, examining how representational teaching practices using bar models support student engagement and the development of conceptual understanding and autonomy. The one-on-one instructional session involved a Primary 3 (Grade 3) student who had prior experience using bar models to represent the four basic operations but had not yet received formal instruction on fractions. This context provided an opportunity to explore how the teacher's representational scaffolding and discourse evolved to support meaning-making and strategy development as the student engaged with new fraction concepts.

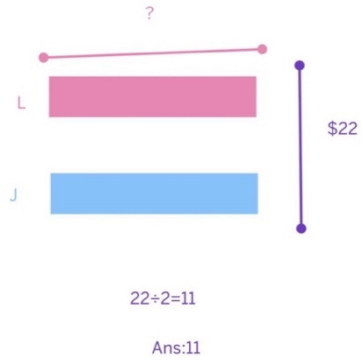
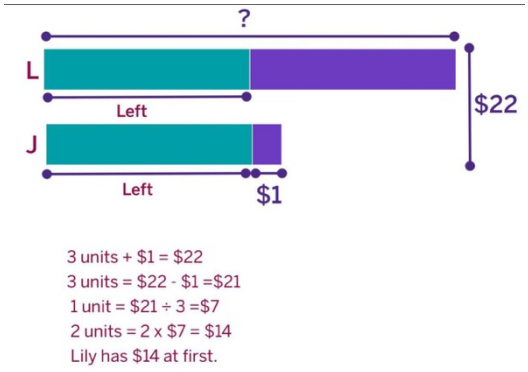
The instructional sequence followed a two-phase design. Phase One introduced the student to the digital bar model tool through guided exploration, followed by a short diagnostic quiz to assess initial understanding. Phase Two centred on comparison-based reasoning, in which screenshots of the student's incorrect solutions from the quiz were paired with correct solutions modelled by the teacher-researcher. These side-by-side representations (Table 1) functioned as methodological prompts to elicit reflective dialogue and capture how representational comparison mediated conceptual understanding and emerging autonomy.

Data sources and analytical approach

Data for the individual case study comprised 50 minutes of video recordings of teacher–student exchanges during the instructional sequence and 45 minutes of interactive app data capturing the student's manipulation of digital bar models. These recordings captured the moment-to-moment interactional dynamics through which the teacher and student co-constructed understanding, including verbal reasoning, gestures, and references to the digital bar model displayed on the mobile

device. The digital bar model artefacts generated during these exchanges were treated as embedded products of interaction, providing visual evidence of the student's evolving representations and conceptual thinking within the dialogue. Analysis was conducted using an interpretive qualitative approach grounded in the study's theoretical framework. The video data were reviewed iteratively to identify episodes of representational mediation, moments in which the teacher's discourse of representations served as mediating tools for meaning making. These episodes were analysed in relation to scaffolding within the zone of proximal development, tracing how teacher prompts and comparison-based representations supported shifts in the student's understanding. Evidence of metacognitive regulation and emerging autonomy was identified through indicators such as self-explanation, representational decision-making, and verbalised strategy adjustments. This analytic process enabled a fine-grained interpretation of how representational dialogue and guided comparison mediated conceptual growth and supported the development of learner autonomy.

Table 1: Comparison of valid and incorrect bar model solutions (Individual)

<table>
<tr><td colspan="2">Lily and James have $22 altogether. If Lily spends $\frac{1}{2}$ of her money and James spends $1, then they will have the same amount left. How much money does Lily have at first?</td></tr>
<tr><td>Kai's Solution (LHS)</td><td>Suggested Solution (RHS)</td></tr>
<tr><td> 
</td><td> 
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</table>

Results

This section presents the findings of the study in relation to the two research questions. Analysis of the teacher–student interaction revealed how representational work with the bar model supported conceptual reconstruction and the student's progression toward autonomy. Four interactional themes (RQ1) and a five-stage scaffolding trajectory (RQ2) emerged and are examined with reference to the relevant turns in the transcript excerpts.

RQ1: Four themes were identified within the interactional processes supporting conceptual reconstruction:

1. Representational dialogue (Table 2). The teacher and student negotiate the meaning of a representation. The dialogue treats the diagram as a tool for reasoning, and meaning emerges through discussion, making the representational assumptions visible.

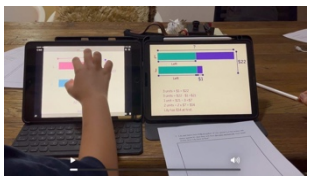
2. Representational mediation (Table 3). Teacher questioning mediated the student's meaning-making by foregrounding the diagram's key visual features (halves and unit structures) and guiding

the student to interpret how these features encode the equivalence of Jame's and Lily's remaining amounts. Turn 38 marks the first point in the episode where the student initiates clarification rather than awaiting direction. By questioning the structure of the bar model, he signals awareness of a gap in his understanding and begins to appropriate the representational language previously modelled by the teacher.

3. Guided comparison (Table 4) supported the student in distinguishing structural differences between the "before" and "after" diagrams.

4. Misconception surfacing (Table 5). Teacher probing revealed the student's misinterpretation of the bar model's story-state structure.

Table 2: Transcript excerpt showing evidence of representational dialogue

Turn	Speaker	Talk	Moment
1	Teacher	The question asks, Lily and James have $22 altogether. You put two parts here because it says, 'the same amount left'...explain to me what your diagram is trying to say.	 Image 1
3	Teacher	Why are Lily and James "the same"?	
4	Student	Because they both have $11 (see Image 1)	

RQ2: The analysis of the scaffolding trajectory and autonomy development revealed a progression across five stages of support:

1. High-level representational framing. At this initial stage, the teacher supports the student in orienting to the bar-model representation by drawing attention to how the diagram depicts the quantities and relationships in the problem (Table 2: Turn 1). By linking these visual structures to linguistic cues in the task ("same amount left"), the teacher establishes how the bar model encodes the problem context. This orientation enables the student to interpret the diagram as a meaningful representation in the scenario.

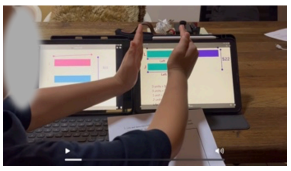
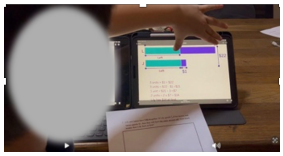
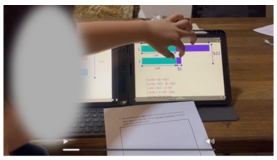
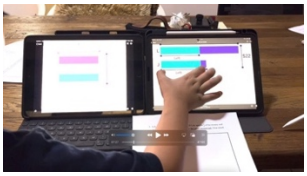
2. Prompting and strategic questioning. In this phase, the teacher uses carefully timed prompts to shift greater interpretive responsibility to the student. For instance, the question *"Do you think there are missing parts?"* (Table 5: Turn 15) invites the student to attend to the representation and make their own evaluative judgments. The subsequent prompt, *"Which part of the diagram tells you that?"* (Table 3: Turn 23), requires the student to justify their thinking by referring directly to features of the diagram. Through these questions, the teacher deliberately hands over more of the interpretive work, encouraging the student to draw their own conclusions from the representation.

3. Comparison-based scaffolding. At this stage, the teacher supports the student in comparing the "before" and "after" (Table 4) representations to clarify the quantitative relationships in the problem. The teacher reinforces this distinction and guides the student back to the purpose of the comparison, leading the student to identify what needs to be found (Turn 32).

4. Metacognitive regulation. This stage is marked by the student beginning to monitor and articulate his own reasoning. This shift is first signalled by the student's return-questioning in Turn 38 (Table 3), where he asks, "How big is one unit?", showing that he is independently examining the diagram and testing his interpretation. This metacognitive stance becomes more explicit in the later turns, showing the student articulating inferential steps of his reasoning:

42 Student: One unit is $7…
43 Teacher: How do you get 7?
44 Student: Because 3 units plus $1 is $22... so 21 is 3 units divide by 3 to get 1 unit... then 2 units is $14.

Table 3: Transcript excerpt showing evidence of representational mediation

Turn	Speaker	Talk	Moment
22	Student	When James spent $1, he had the same amount of money as Lily	
23	Teacher	Which part of the diagram tells you that?	Image 2
24	Student	(Non-verbal) (see Image 2). Lily spent ½ of her money (see Image 3). Then James spent $1. Then they have same amount left (see Image 4).	 Image 3  Image 4
38	Student	How big is one unit? Which one is one unit? Is this one unit?	
39	Teacher	What do you think? Because you finished off with 2 equal units, right?	Image 5
40	Student	So, this is one unit (Image 5).	

5. Emerging autonomy. By this stage, the student demonstrates increasing independence in using the bar model to derive and justify numerical relationships. Across these turns, the student shows only minimal dependence on the teacher's prompts:

47 Teacher: If I ask you...how much more money does Lily have than James at first?
48 Student: Lily has $6 more than James at first.

49 Teacher: How do you get $6?

50 Student: $6 plus the $1 becomes one unit and that's how much Lily has more than James at first.
51 Teacher: So, Lily has how much at first?

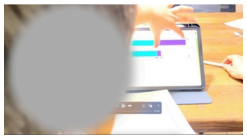
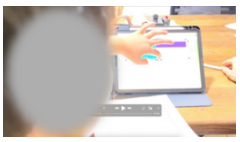
52 Student: $14
53 Teacher: And James?
54 Student: $8
55 Teacher: And then how much more money does Lily have?
56 Student: $6

He uses the diagram autonomously to interpret the representation and complete the reasoning.

Table 4: Transcript excerpt showing evidence of guided comparison

Turn	Speaker	Talk
29	Teacher	Can you tell me where is the 'before' and where is the 'after'?
30	Student	This is 'after'... (points to the right-hand diagram showing the final amount) and this is 'before' (points to the left-hand diagram showing the initial amount)
31	Teacher	So, this (LHS) is 'after'...at the end of spending all that. money. What's the question asking?
32	Student	How much Lily has 'before'.

Table 5: Transcript excerpt showing evidence of the teacher surfacing misconceptions

Turn	Speaker	Talk	Moment
13	Teacher	Ok. Same amount left is after they have spent the money, right? So, do you think there are missing parts? The whole thing is $22. Do you think there are missing parts?	
14	Student	Yes.	
15	Teacher	What are the missing parts? You can look at this diagram and then tell me what the missing parts are.	Image 6
16	Student	This... (see Image 6). ... and this (see Image 7).	
			Image 7

Discussion

This study examined how interactional scaffolding with bar models supported a student's representational understanding and developing autonomy. The findings show that representational challenges were not only procedural but deeply structural. The student initially attended to surface features of the visual model, such as equal bar lengths, rather than the relational meaning encoded in the part–whole and before–after structure. The teacher's prompts helped the student move beyond this surface-level interpretation by explicitly distinguishing total amounts, amounts spent, and amounts remaining, thus supporting the relational coordination that is central to effective mathematical modelling. This progression can be understood through Vygotsky's Zone of Proximal Development (ZPD). The teacher's dialogic scaffolding mediated the student's engagement with the bar model, enabling reasoning beyond his independent capability. From a sociocultural perspective, the bar model functioned as a cultural tool through which the student could externalise and manipulate quantitative relationships. As he appropriated this tool, perceptual noticing gave way to structural interpretation, a shift consistent with Bruner's account of scaffolding and the enactive–iconic role of representations in supporting conceptual organisation. The interaction showed a shift from other-

regulation toward emerging self-regulation. Early modelling and guided prompting helped establish a shared representational frame, and comparison-based discussion supported the student's developing structural noticing. A particularly significant moment occurred when the student initiated a return-question in Turn 38, signalling a shift in agency within the interaction. This student-generated query reflected both recognition of conceptual uncertainty and an emerging capacity to steer his own sense-making. Such moments, along with his appropriation of the teacher's representational language ("units"), exemplify representational mediation and mark movement toward greater autonomy in his reasoning. These findings highlight two implications for instruction. First, learners may require explicit support to shift from perceptual to structural interpretation of visual models. Second, student-initiated return-questioning may serve as an important indicator of readiness for reduced support, providing teachers with a diagnostic cue for when to fade scaffolding and promote autonomy.

Conclusion

Future research should investigate how teachers can be prepared to notice and systematically prompt structural interpretation of visual models and recognise moments of student-initiated inquiry as opportunities to reduce support. Such work would deepen our understanding of how autonomy develops through interaction and how instructional design can more deliberately cultivate it.

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Exploring the professional mathematics teacher's knowledge in school and university practices

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Mathematics teachers' knowledge has been a focus of research in mathematics education for over forty years. This perspective has been complemented in the last decade by a growing interest in professional competencies, such as lesson planning competence. This article, therefore, addresses the knowledge mobilized by two teachers as they reflect on lesson plans developed within a formal education program. To do so, we use the Mathematics Teacher Specialized Knowledge (MTSK) model, which demonstrates that these teachers use their knowledge of mathematical topics and certain elements of mathematics teaching and learning in an interrelated manner. Simultaneously, the research process itself makes them aware of the need to reflect on their own lesson plans.

Keywords: Mathematics Teacher's Specialised Knowledge (MTSK), teacher practice, mathematics content knowledge.

Introduction

In the field of education, the most renowned and influential work on teachers' professional knowledge is that of Shulman (1986). In mathematics education, Ball et al. (2008) have made significant contributions to the study of this area. Their research presents a comprehensive model that illustrates the complexity of the knowledge required by teachers in their daily professional lives, both in and outside the classroom. From a different perspective, Carrillo-Yañez et al. (2018) have proposed shifting the focus from the knowledge teachers need to the mathematical knowledge they use in teaching mathematics. This has been further detailed through the conceptualization and categorization of an analytical model that considers the specialized, understand from an intrinsic perspective (Scheiner et al, 2019), components of mathematics teachers' knowledge, referred to as Mathematics Teacher's Specialised Knowledge (MTSK) (Carrillo-Yáñez et al., 2018).

On the other hand, there is a growing interest in understanding how mathematics teachers mobilize and develop various professional competencies, particularly the competency associated with lesson planning (Blömeke et al., 2008).

The mobilization of this competency requires specific elements of professional knowledge, particularly the teacher's professional experience. Such experience is essential for effective lesson planning and for the daily practices that occur within the classroom. Kaiser and König (2025) observe that "…explicit references to the teaching profession have been rare in the last two decades." (p. 1073). Consequently, this study addresses the identified gap by clarifying, conceptualizing, and delineating the aspects that in-service teachers value and prioritize when designing their lesson plans. In doing so, it contributes originality and novelty to the scholarly examination of this construct.

This study examines the characterization of the knowledge mobilized by two in-service teachers when planning lessons within the context of a continuous professional development course, where the history of mathematics serves as a cross-cutting resource. Based on the above, we pose the following research questions:

- What aspects do in-service mathematics teachers consider and prioritize when designing their planned classroom activities?
- What specific characteristics are evident in the aspects teachers take into account when reflecting about their lesson plans?

To address these questions, we will use the MTSK model to characterize the focal points of attention for both teachers.

Theoretical framework

According to Carrillo-Yañez et al. (2018, p. 240) the Mathematics Teacher's Specialised Knowledge (MTSK) model "features a reconfiguration of mathematical knowledge, a reinterpretation of pedagogical content knowledge and a new way of conceptualising the notion of specialisation". Building on Shulman's (1986) and Ball et al. (2008) work, the model is structured in three domains: Mathematical Knowledge (MK), Pedagogical Content Knowledge (PCK) and Beliefs, that will not be addressed in this study, since the training program didn't address them. Figure 1, illustrates the subdomains within each domain.

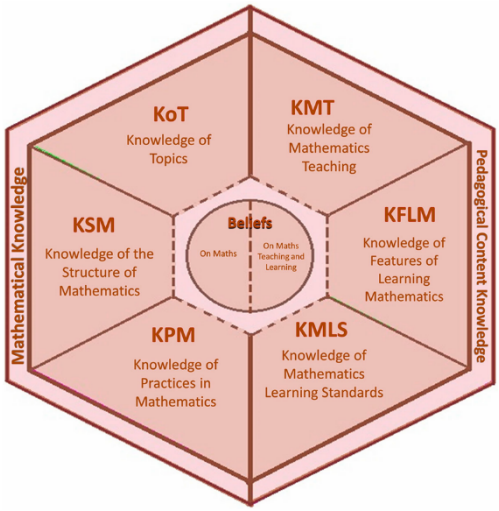

Figure 1: MTSK model proposed by Carrillo-Yañez et al. (2018)

The domain of Mathematical Knowledge is organized in three subdomains: First, Knowledge of Topics (KoT) that describes what and in what way the mathematics teacher knows the topics they teach; it implies thoroughgoing knowledge of mathematical content (e.g. concepts, procedures, facts, rules and theorems) and their meanings. Second, Knowledge of the Structure of Mathematics (KSM), that entails teacher's knowledge of connections between mathematical items and are based on (i) temporal considerations, which respond to (non-curricular, but mathematics-related) questions of sequencing, and (ii) considerations of demarcation of mathematical objects, which produce inter-conceptual connections. Finally, Knowledge of Practices in Mathematics (KPM) focuses specifically on means of production and mathematical functioning. The term practice is understanding as any

mathematical activity carried out systematically and which conforms to a logical basis from which rules can be extracted.

In the other hand, Pedagogical Content Knowledge is comprised of three subdomains: Knowledge of Mathematics Teaching (KMT), encompassing theoretical and practical knowledge (both personal and institutional) specific to mathematics teaching and awareness of the potential of activities, strategies and techniques for teaching specific mathematical content, along with any potential limitations and obstacles which might arise. Second, Knowledge of Features of Learning Mathematics (KFLM), contemplating knowledge associated with features inherent to learning mathematics (e.g. usual mistakes, strengths, emotional aspects of learning mathematics). The main sources tend to be their own experience built up over time along with research results in Mathematics Education. The third subdomain, Knowledge of Mathematics Learning Standards (KMLS) encompasses the knowledge of the mathematical contents to be taught at any particular level, and the learning standards (any instrument designed to measure students' level of ability in understanding, constructing, and using mathematics, and which can be applied at any specific stage of schooling). Its main sources are the curriculum specifications, non-official curriculum documents, and research literature.

Methods

This is an exploratory study, with some features of instrumental case studies (Bassey, 1999). This work occurs during a course about the history of mathematics and their implication for mathematics teaching for in-service mathematics teachers. This course belongs to a wider instruction process, linked to a PhD programme in mathematics education (for in-service mathematics teachers) at the Antonio Nariño University in Colombia. This course is one of four in a series in which in-service teachers are expected to develop a deeper understanding of the history, philosophy, and epistemology of mathematics, as well as notions linked to mathematical thinking. One of the authors of the current proposal was the teacher educator who led the course.

There were twenty-five in-service mathematics teachers of different levels (from elementary to university), and diverse knowledge backgrounds (engineering, mathematics teaching, actuarial science, and other subjects). All of them had postgraduate qualifications in mathematics teaching or related subjects and are considered experts in mathematics teaching by their colleagues. Data were collected using two instruments. All participants were asked to design a five session planning for real classrooms. For this particular study, from the 25 teachers, two were selected to be interviewed, in order to deepen in the focus of their considerations when planning. The interviewer used a semi-structured interview to invite teachers to reflect about the foundations of their lesson plan.

The following instructions were given to the teachers. Participants wrote their proposals digitally and sent to the educator.

Using the sources contained [in the book] The History of Mathematics: A Reader*, edited by John Fauvel and Jeremy Gray, designing a teaching proposal for one of the levels (elementary, grades six through eight, grades nine through eleven, undergraduate) of at least five one-hour lectures, and fully develop the design of two such lectures addressing one of the following topics.*

1. *Number systems (select one from the following: whole numbers, integer numbers, fractional numbers, rational numbers, irrational numbers, or complex numbers) and the history of their development. (…)*
7. *Infinite sums and their development over the centuries (from sums of geometric progressions and Archimedes' method of exhaustion to Newton and calculus). (…)*

Three criteria were taken into account when selecting participants for the interview. First, the proposals had to fall within topics 1 and 7. Second, the educational level considered for the design of the proposals should be comparable (school and university). Third, participating teachers had to accept the invitation to be interviewed to find out more about the development of the proposals. Based on that, the productions of two in-service mathematics teachers (Katy and Mary) were selected.

The semi-structured interview aimed to gain a deeper understanding of the motivations behind Katy and Mary's decisions when designing their proposal. To this end, three initial questions were posed, and the one-hour interview was conducted virtually. The session was video-recorded and fully transcribed. Three open questions were asked, followed by some contingent follow-up questions that allowed to deepen into the knowledge they used to design their respective plannings. The questions were:

1. *When developing your lesson plan, what key aspects did you consider in its preparation, in addition to the reference book? In other words, what elements other than the use of original historical sources did you draw upon to propose the activities contained therein?*
2. *To what extent do you find the proposed plan applicable in your professional context? What limitations, potentialities, scope, and difficulties can you foresee in implementing your sequence?*
3. *What are your opinions on the applicability of your colleague's plan in the classroom? What can it be of use to inform your considerations when planning your lesson?*

Data analysis was carried out in two stages. In the first stage, the Katy and Mary's proposals were read in full to identify common and uncommon elements. In the second stage, for the video recording and its transcription analysis, the thematic method of qualitative analysis (Nowell, et al., 2017) was employed. Thus, it was carried out following the iterative and reflexive requirements based on the following phases presented by Squires (2015).

In Phase 1, the researchers engaged in an initial immersion in the data. In Phase 2, they proceeded to generate preliminary codes by systematically cataloging the information line by line—specifically those segments pertinent to the research topic—under either the MK or PCK categories. For this process, Atlas.ti software was employed. These initial codes were informed by the data itself, as salient themes began to emerge.

In Phase 3, the researchers identified potential themes related to the central research question within each dataset (both MK and PCK). These themes were subsequently compared with specific categories or subdivisions of MK and PCK, resulting in theme–category pairs. Finally, the researchers refined the initial codes by integrating them into broader themes and subdivisions, thereby ensuring that the resulting correspondences were coherent and well substantiated.

In Phase 4, a review of the theme–category pairs process was carried out, looking for coherent patterns within each group of superordinate and subthemes. In Phase 5, researchers generate a map or visual representation, determining the essence of each theme–category pairs, and ensuring that the data extracts are coherent and aligned. Finally, in Phase 6, the theme–category pairs are described based on the evidence that supports each of them.

In order to minimize potential bias in the interpretation of the results by the researcher-instructor, the coding process for identifying themes and constructing theme–category pairs was independently conducted by each of the authors. Subsequently, the results were reconciled through contrast and comparison, yielding a 90% agreement across the individual analyses.

Findings

Through the first two phases of the analysis, 55 codes related to the aspects Katy and Mary value when designing their teaching activities were obtained from the interview transcripts. In the third phase, twelve groups of related codes were identified as potential themes for answering the posed questions. Due to the diversity of the codes, each group was studied. Three groups were highlighted as the most comprehensive in terms of the number of codes or citations they incorporate. These were then contrasted with the knowledge domains of the MSTK model to determine whether they were sufficient or required adjustment. We will present now the MTSK elements of knowledge that let us understand the knowledge used by teachers, while answering each question.

What aspects do in-service mathematics teachers consider when designing their planned activities for the classroom?

Mathematical knowledge is the most important element Katy and Mery use while designing lesson plans. Although the selection of content was somehow limited by the way the task was presented, there was still some flexibility in the specifics of the content. For instance, Katy chose the integer number system over the other available systems. It should be noted that, in this regard, the KoT elements predominate over the KSM and KPM ones. Regarding the KSM, both refer to temporal consideration. Addressing integers in seventh grade is appropriate for Katy's students since they have previously studied the natural number system. Mary points out that "…they (learners) have many difficulties in (…conceive) the difference between a sequence and a series (…and) to be able to take the sequences into the infinite (…) series.".

Katy and Mary emphasize the difficulties students experience with this topic in their justification of their planning (KFLM). These difficulties put the focus on the content and seem to be justifying the exploration of alternative teaching methods (KMT). Katy presents a KFLM argument when she states that she designed her sequence based on the integer system because "…from my experience I see many difficulties (…) especially with the negative numbers.". On the other hand, Mary found difficulties in students "…to calculate the result of a series (…) or (…) the pattern (…) of the (…) sequences" as a recurrent element present every time she has taught the course. This highlights the close relationship between the KoT and the KMT, one of the bases for understanding the need for mathematics teachers to have topic-specific mathematical and pedagogical content knowledge.

The second element is consulting information sources for lesson design, which has a place in the KMLS. According to Katy and Mary, these sources can be of two types: those that provide information about the specific mathematical knowledge to be taught, such as school textbooks and history of mathematics books, and those that provide information about curricular aspects and normative orientations of the learning process, such as national curriculum guidelines, competency standards, and school curricula.

Finally, they place high value on a given structure of ten main elements of lesson structure, assuming it as a theory guiding planning design- therefore fully aligned with KMT: the learning objective; methods of disseminating mathematical knowledge; form of content presentation (e.g., problems, examples, and exercises); age of students; materials to be used; possibilities for managing practice; number of students in a classroom; time available for each session; and organization of individual and group work, including its strengths and limitations.

What specific characteristics are evident in the aspects that teachers consider when designing their practice plan?

Some particularities revealed in the interview with the teachers are important to highlight. Both teachers use electronic resources to develop their lesson plans, which are closely related to KMT. However, each teacher assigns a different role to these resources due to the students' age. For Katy, it is a way to familiarize students with mathematical software, considering they are approximately 12 years old. Mary, on the other hand, uses a robust model developed by another author to help her students visualize a phenomenon. She also takes into account that her students are older than Katy's.

Another aspect of KMT becomes evident when Katy suggests that students may formulate mathematical problems, arguing that this activates the transfer of learned knowledge "…the student always (should transfer) within a real context if it is possible or (study) the application of the topic address.". On the other hand, Mary doesn't consider it a possibility. When asked if there was a reason for it, the teacher couldn't give justification, just claiming that she considered it not possible.

For both teachers, the organization of classroom work depends on the nature of the topic covered. For example, if the topic is related to the history of mathematics, they convey by proposing that it can be worked on in groups or in plenary sessions to promote collective discussion among peers. However, if the topic is 'entirely' mathematical, they highly value students addressing the problem individually before considering collective work. This is probably based on their beliefs about mathematics and its learning as a solitary activity.

Finally, Katy and Mary claim that this kind of reflection is not routinary for them, as well as they seem to not be aware of evaluating their own lesson plans. However, when asked about the strengths and limitations of their proposals, they realize the necessity of this process. It is only after answering questions related to these aspects that they become aware of other elements of PCK worth considering, such as students' prior knowledge, specific expectations, and needs.

Katy and Mary were assigned to evaluate each other's proposals. This allowed them to offer a modest opinion on the design of the sequence. They mostly focus on the changes they would make, which are based almost exclusively on possibilities for managing classroom activities (related to KMT). For

Katy, it would be necessary to offer students a session to address preconceptions (KFLM) required for the activities proposed by Mary. Mary, on the other hand, always incorporated individual work, even if the nature of the activity was unrelated to mathematical objects.

Final considerations

The relevance of professional knowledge in teaching practice is widely acknowledged (Shulman, 1986, Ball et al., 2008; Carrillo-Yáñez et al., 2018), both in mathematics education research and in many (if not most) teacher training programs. However, it is notable that research on in-service teachers, and even more so on expert teachers, remains relatively scarce.

In this study, we examine how two expert teachers mobilize different elements of their specialized knowledge when reflecting on their respective lesson plans. We observe how elements of *Knowledge of Topics* intersect with *Knowledge of Mathematics Teaching* and *Features of Learning Mathematics*, forming the foundation for their classroom planning activities. These latter two subdomains also serve as significant sources for their justifications, as both teachers focus on structuring their instruction around elements that allow them to teach mathematics with theoretical support, while also addressing the challenges faced by their students.

In future developments of this study, we aim to delve deeper not only into the knowledge they use, but also into how they become increasingly aware, through reflection, of the need to construct new knowledge. For instance, when considering the emphasis placed on student errors in shaping the direction of classroom activities, it appears that the tendency is to avoid such errors rather than to employ them as a foundation for fostering metacognitive processes in students. Leveraging errors in this way would contribute to broadening the dimensions of knowledge proposed in the model.

In this regard, future investigations should take into account both the temporal dimension and the type of experience of the participants under analysis. Drawing on the research perspectives advanced by Kaiser and König (2025), studies that approach this issue through mixed and qualitative paradigms are expected to contribute significantly to broadening our understanding of the proposed knowledge model.

Furthermore, the approach we take here is based on the assumption that in-service teachers can serve as an important source of information for identifying the characteristics and needs of the knowledge used by educators. This knowledge should be taken into account in initial teacher training programs, in order to lay the foundation for a solid knowledge base that ensures high-quality mathematics instruction. The contrast between the two educational levels in which teachers operate can yield valuable insights that inform the specificities of initial teacher training, in accordance with the distinctive approaches of curricular programs offered across diverse contexts.

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Initiating and establishing mathematical practices of determining and transforming numbers as a foundational skill in fostering mathematics teaching

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Teaching can be understood as a bundle of practices in which teachers and students jointly engage. This paper focuses on practices of determining and transforming numbers, which are central for developing number understanding – a central goal of primary mathematics education. Based on video analyses of primary mathematics lessons in Germany, fostering situations are examined using interpretative interaction analysis. The study reconstructs how practices establish between teacher and children in the classroom as part of teaching-integrated support. The findings underline the importance of such practices as foundational elements of mathematics teaching and point to implications for teacher education, especially in preparing teachers to notice and support mathematical practices.

Keywords: Mathematical practices, discourse, professional development, number understanding.

Introduction

An understanding of numbers and operations is considered fundamental to mathematical learning and is therefore one of the so-called basic skills that all children must acquire. At the same time, it is well known that not all children develop a fundamental understanding of numbers and operations. As part of the German nationwide professional development initiative QuaMath "Developing quality in mathematics classrooms and teacher professional development" (Prediger & Selter, 2024), we are developing a PD to support primary school teachers in diagnosing and fostering children to develop an understanding of numbers and operations. The accompanying research focuses on the classroom practices related to children's understanding, as well as on the process of initiating and establishing these practices in learning situations.

Theoretical framework

The conceptual understanding of natural numbers and their operations is a key objective of primary school mathematics teaching and a critical point in the learning process. Understanding of numbers includes that children count systematically, be aware of the relation between quantity and number, understand and connect different representations of numbers and interpret numbers in relation to other numbers (Sayer et al., 2016). A lack of understanding of numbers can encourage counting-based calculation and lead to arithmetic being treated as mere manipulation of digits according to rules, without any understanding. Therefore, it is important for teachers to identify children's understanding of numbers and encourage practices that foster the awareness of mathematical structures and enhance understanding.

Following Steinbring (2005) conceptual understanding at different individual levels develops through interactive negotiation of a shared thematic meaning. He emphasises that mathematical signs and

symbols do not carry meaning on their own but must be interpreted and negotiated by the children through interaction. Specifically, for example, dot fields (Figure 2) in the context of developing children's understanding of numbers must be interpreted and negotiated in terms of their structure. Therefore, math lesson and especially fostering situations should encourage children to become aware of the structure of mathematical signs and symbols and the relation of numbers.

To examine how exactly such dot fields are interpreted and negotiated in the classroom and what types of counting are introduced, required and discussed in the classroom, or how children describe relationships between numbers and explain changes, the corresponding classroom mathematical practices must be considered. Mathematical practices can be understood as a recurring pattern of how children and teachers act and speak about content. Following Cobb and colleagues (2001, p. 121), "a practice is seen to be an emergent phenomenon rather than an already-established way of reasoning and communicating into which students are to be inducted". Practices describe students reasoning from a social perspective while their understanding can be reconstructed from a psychological perspective (Cobb et al., 2001). Reconstructed practices and the understanding of children that becomes apparent in them can become the subject of professional development processes for teachers, by reflecting on them in detail.

To conceptualise this link between classroom practices and teacher learning, we draw on the Three-Tetrahedron Model for content-related professional development (Prediger, Roesken-Winter & Leuders, 2019). The model emphasises that PD content must be specified with respect to research on the classroom level – for example, by taking into account students' typical learning pathways, common misconceptions, and the interactional patterns that shape their opportunities to learn. In line with this perspective, we use the model's nesting strategy by grounding elements of professional development in the mathematical structures, representational practices and challenges that emerge in the classroom. Importantly, this also includes the relations between teacher, students, material and mathematical content, as these relational dynamics shape how mathematical practices are enacted and can therefore be taken up productively in PD settings. Thus, the reconstructed practices form both an analytic lens and a resource for teachers' reflection and growth.

This paper focusses on the classroom mathematical practices regarding the determination and transformation of numbers and the corresponding interpretations with a focus on children experiencing mathematical difficulties and provides an outlook on how these can be addressed in the context of further PD. In doing so, we highlight that not only students' interpretations but also the relations between teacher, children, material and mathematical content shape how such practices emerge, stabilize, or remain fragile. These relations can be made explicit and productively taken up within professional development.

Design

This study is situated within the German nationwide professional development initiative QuaMath (Prediger & Selter, 2024), which develops professional education formats for mathematics teachers. Within this project, we designed a practice-based professional development for primary school teachers aiming to strengthen their skills in diagnosing and fostering children's understanding of numbers and operations. After attending input sessions ("Modules") on various

aspects of diagnosis and support, teachers are provided with task materials and written implementation guidance and invited to implement them in their classrooms during the distance phase ("Classroom trial") (Figure 1). They documented their experiences and diagnostic observations on a reflection sheet, which was discussed in the subsequent module. Each module included 30 minutes for preparing the trial and, at the beginning of the next session, 30 minutes for joint reflection.

Figure 1: Design of the professional development programme

Building on this program, the present study focuses on the classroom level. In the first iteration of the PD programme, the classroom trials were video-recorded in a total of eight primary school classes (grades 1–4). In this way, data were collected that give insight into how fostering situations unfold as part of classroom-integrated support, when teachers and children work on tasks designed to support understanding of numbers. The video material allows for the reconstruction of what happens when teachers implement the materials in practice, making it possible both to reflect on the design of the materials and to analyse how children's numerical understanding develops in interaction. Furthermore, selected scenes can be transcribed and fed back into subsequent PD cycles as critical incidents, enabling teachers to reflect on their own practices. In this sense, the empirical findings from the first cycle inform the iterative refinement of both the material and the PD programme. Our main research question is: *What mathematical practices related to quantifying and mentally transforming numbers can be reconstructed in the interaction, and how is children's understanding of numbers stimulated in this process?*

The analysis followed several methodological steps: (1) The video data consisting of several hours of recorded classroom interaction, were reviewed and episodes relevant to the research question were selected. In line with the methodological approach, only those scenes were selected in which interpretive hypotheses could arise and which offered sufficient analytical depth, i.e., so-called "critical episodes" (2) These episodes were transcribed and (3) on this basis, interpretative interaction analysis was applied to reconstruct, turn by turn, what and how practices are initiated and established in fostering interactions and how children's understanding becomes visible in their uptake of the practices. (4) Finally, the practices were compared with other analyses of other scenes and examined in a generalised way (Krummheuer, 2002).

Analysis of an episode

Result of the analyses is the reconstruction of mathematical practices related to determining and transforming numbers as well as their relation to children's developing understanding. In order to understand both the initial emergence of these practices and their subsequent stabilisation in interaction, several episodes from a lesson were selected in which the children are asked to determine and transform quantities. As mental transformations are not directly visible, the reconstruction draws on indirect interactional indicators such as response timing, observable strategy shifts, children's verbal contributions and gesture-based references to structure. In the following, one of these episodes

is presented as a critical episode, both to illustrate the methodological procedure and to exemplify initial findings. The scene and its subsequent development are then presented in an abridged form. It is important to note that the analysis does not aim to demonstrate fully developed understanding, but to show how the enacted practices offer or limit opportunities for stimulating number understanding.

Learning environment "Changing quantities"

The overall aim of the task is to foster children's number understanding, with a particular focus on two aspects: determining quantities through structured subitizing and mentally transforming these quantities. The task supports children in recognizing and then automatising number representations based on the five-structure (0, 25, 50, 75, 100). In addition, the learners are encouraged to mentally transform these quantities by adding or removing 10 or 20 counters. Rather than manipulating concrete materials, it is the mental imagination of these operations that raises awareness of structures, which is particularly important for children who rely on counting strategies (Häsel-Weide et al., 2025).

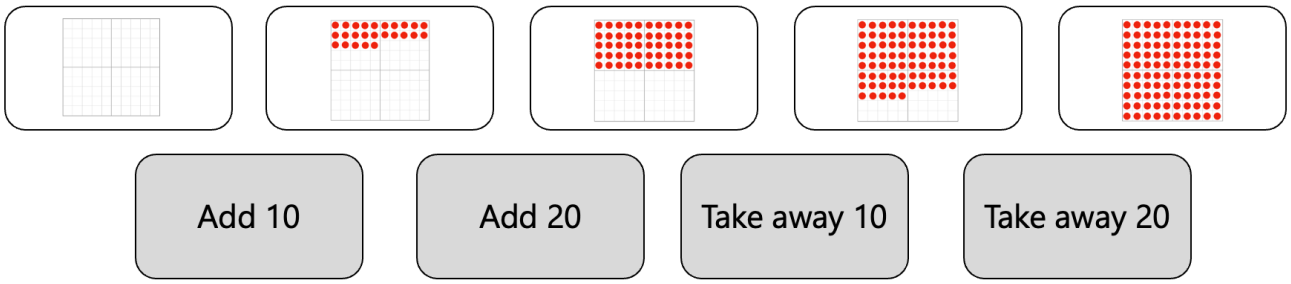

Figure 2: Overview of the dot cards and action cards[1]

To put this aim into practice, the children work in pairs with a set of dot cards and action cards (Figure 2). The cards are placed face down. One dot card (white back) and one action card (grey back) are turned over. The children take turns in determining the quantity, mentally transforming, and stating the new number. If appropriate, the task can be noted on an optional worksheet. Afterwards, the cards are turned face down again (remaining in play), and a new pair of cards is drawn to repeat the process with another pair of cards.

Exemplary Episode: Initiation of the teacher-initiated practices

The following episode is from the beginning of a second-grade lesson after the number range has been extended to 100 according to the German curriculum. The episode focuses on the two children Robin (aged 8, 3rd year of schooling) and Pia (aged 7, 2nd year of schooling) who work together on a task. After each child has solved one pair of cards, the teacher joins the children's table and then the subsequent scene begins.

[1] In the German language, terms were deliberately chosen that are not identical to 'add' and 'subtract' but reformulate the actions.

Robin:	Now it's my turn. I take this one (turns over the dot card 75) and this one (turns over the action card 'Add 20').
Teacher:	How many is it then?
Robin:	One Hundred.
Teacher:	No (shakes head).
Robin:	No.
Teacher:	Count again. How many are there now (points at the dot card)?
Robin:	Wait. One, two, three, four, five (.) seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen (looks at the points on the dot card). (…) I can't. It's difficult (places both cards in front of the teacher).
Teacher:	Okay. Let's see what this number here is called (points to the dot field). # (points one after the other to the first dot in each row while Robin counts).
Robin:	#Ten, twenty, thirty, forty, fifty, sixty, seventy, eighty.
Teacher:	No. Seventy and then five more (runs finger along the five dots in the eighth row).
Robin:	five- (…) Eighty-five.
Teacher:	…five-?
Robin:	Seventy.
Teacher:	Exactly. So now add twenty, Robin (points to the action card). Imagine two more rows of ten were added (runs finger twice above the first row of dots).
Pia:	Ahh. I know.
Teacher:	So we don't have seventy-five anymore, but rather?
Robin:	Uhm (glances at Pia).
Pia:	Ninety-five (whispers).
Robin:	Ninety-five.
Teacher:	Yes, correct (nods). Great. Exactly.

In the episode, Robin uncovers a dot card with 75 dots and an action card "add 20," which she initially reads as "add 10." The teacher asks for the result directly and does not comment or correct the mistake. Robin responds with "one hundred," and is then instructed to "count again". She begins to count dots individually and finally signals that the task is too difficult. The teacher initiates counting in tens by pointing to the first dot of each row. Robin follows this strategy immediately but counts to eighty, treating a row of five dots as a complete ten. The teacher corrects this by decomposing the number into "seventy and five," which stimulates decomposition of a number into tens and ones, and highlights the distinction between tens and ones through different gestures. Although Robin repeats parts of this decomposition, her answers remain uncertain and partly inconsistent. It is not clear, whether Robins number understanding is stimulated by this practice. Regarding determining numbers, we see several practices in this episode (Figure 3): Counting in ones while simultaneously tapping on the dots, counting in tens while simultaneously tapping on the rows in combination with stroking the incompletely filled row. After the process of determining the quantity, the teacher asks for mental transforming. Thereby, he also seems to encourage cardinal number understanding by mentally adding a set of two rows of dots. However, this transformation is indicated but by imagining additional rows above it, which does not allow for structured subitizing as the structure of the material is not utilised. Pia immediately says that she knows the answer, while Robin remains silent. She then whispers the result to Robin, who repeats it. In this way, the task is completed collaboratively. With regard to Robin and Pia's understanding of numbers, it can be reconstructed that although Robin is able to count in tens, she does not yet determine the quantity independently, and her interpretation of the dot field as tens and ones combined with counting in tens and teacher-guided decomposition seems only partially consolidated. It remains unclear whether Robin would be able to implement the change independently. Pia seems confident in this regard.

Further course of the scene

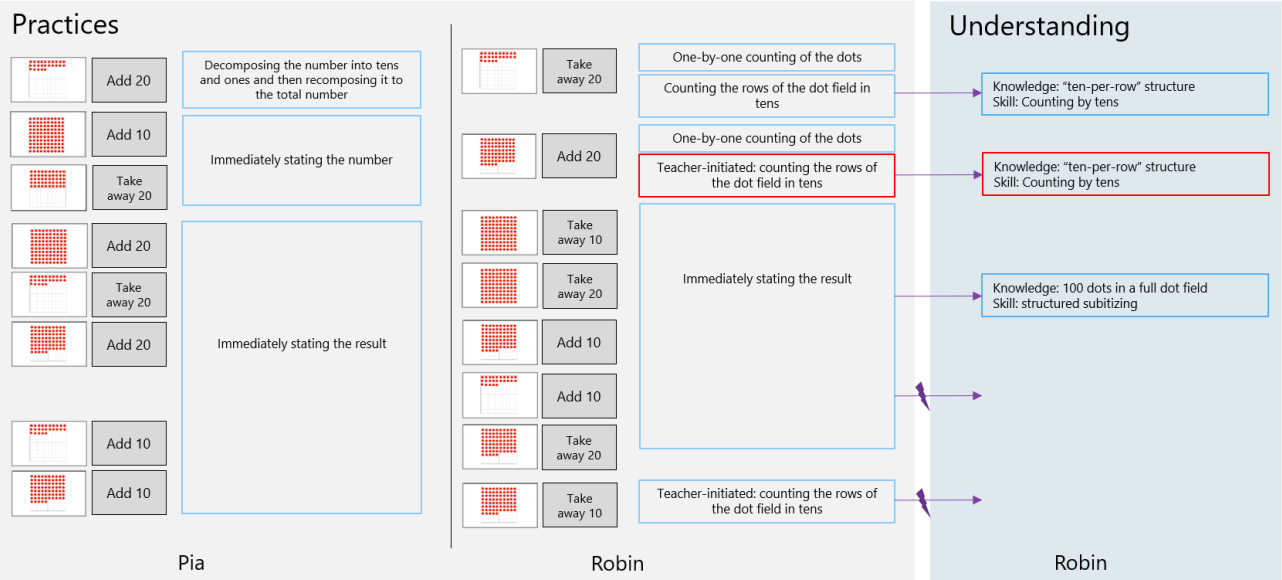

Figure 3: Mathematical Practices and relating understanding

After the episode, the children continue working without the teacher. At first, Robin and Pia take turns. Pia quickly develops a routine: she initially states the number on the dot card and soon after directly gives the result. After a while, she skips the first step and immediately states the result (Figure 3). This practice seems to combine structured subitizing with mental transformation. Robin states the results, too. But, except when the *100* card appears, she consistently gives incorrect results, which Pia then corrects. It seems that she guesses based on previously revealed cards and her earlier answers. She continues instead of the mistakes and does not change the strategy to determining first and transforming afterwards. After a while the teacher then returns to the table and asks Robin to turn over two cards. When asked to state the number of dots, Robin answers incorrectly ("ninety-five–eighty"), which may indicate uncertainty in number determination or an attempt to give the result directly and continue to imitate Pia's practice.

Teacher:	Now, look very carefully. Come here, let's look at this again. Look here (*takes a pen*). Ten (*moves the pen along the first row of the dot field*), (*moves the pen along the further rows of the dot field*).
Robin:	Twenty, thirty, forty, fifty, sixty, seventy. Fifty-seven.
Teacher:	Noo.
Robin:	Uh seventy-five.

The teacher initiates a counting-based practice by modelling counting in tens through pen movements along each row. Robin follows this modelling ("twenty, thirty, … seventy") but ends with an inverted result ("fifty-seven"). After the teacher's correction, she adjusts to "seventy-five". The teacher's focus on guiding Robin through counting procedures and producing the correct result does not invite a joint negotiation of the dot field's internal structure or support Robin in developing a more structurally informed approach.

Reconstructed practices and relating understanding

Looking at the mathematical practices of quantifying and transforming numbers the teacher initiates practices such as counting in tens and number decomposition to support cardinal number understanding, and structural features of the material are made explicit through gestures and prompts. As the work continues, Robin does not sustain the teacher-initiated practice of counting in tens, instead she adopts Pia's practice of immediately stating the result, effectively attempting a combination of structured subitizing and mental transformation. This practice of giving the result immediately, however, reduces opportunities for jointly articulating quantity and structure, thereby limiting the negotiation of the dot field's arrangement beyond the basic insight that each row contains ten dots. As a consequence, Robin cannot draw on the material's structural affordances independently, which restricts the stimulation of her number understanding. The teacher-initiated practices thus does not take hold, possibly because of various reasons: the practice is not consistently necessary across all dots cards; performance differences between the two children could create social differences (Robin "needs" to count, whereas Pia immediately states the result); does not lead Robin independently to the correct result. Over the course of the interaction, Robin continues to struggle with the number 75, while she determines 100 correctly in a glance. The later visualisation of adding two rows above the dot field supports producing a result, stated by Pia, but does not invite further structural insight within the field, limiting progression toward structured subitizing and mental transformation of 75. With regard to transforming of quantities, the analyse shows that Pia confidently carries out the transformation and immediately states the result, though it remains unclear how she envisions the underlying process. By contrast, Robin does not carry out the mental transformation herself; instead, she draws on Pia's previously stated results from similar card combinations as a heuristic to produce an answer, thus employing an expedient but non-conceptual strategy.

Concluding discussion

The learning environment affords multiple entry points for children with different prior knowledge and invites peer interaction. It has clear potential to build number understanding, support collaboration, and establish classroom mathematical practices (e.g. structured subitizing, mental transformation). However, analysis of the actual implementation shows that potentially productive practices such as performing number determination and transformation one after the other, or counting in tens, decomposition from teacher modelling, only partially applied. Their establishment appears to depend on the learner's current insight, the goal orientation (conceptually oriented instruction vs. producing the correct answer) and the social dynamic in the classroom. Similar to the study by Häsel-Weide & Nührenbörger (2025), discourse oscillates between rule-guided procedure and understanding-oriented exploration, and participation shifts between one-to-one support focussing on the student with apparent difficulties and shared engagement. This foregrounds a design challenge: to align interactional norms with conceptual aims and progressively shift from teacher-supported participation to student-initiated use of practices, thereby consolidating number understanding.

Beyond the classroom level, the results point to important implications for teacher education and professional development aligning with the nesting strategy of the Three-Tetrahedron Model

(Prediger, Roesken-Winter & Leuders, 2019). First, analytically rich, critical episodes such as the one presented here can serve as concrete and authentic material for professional reflection, enabling teachers to discuss their instructional decisions and explore alternative actions. Second, the findings underline that well-designed materials do not automatically lead to the intended mathematical practices in classrooms. For PD facilitators, this implies that potential difficulties in the implementation of such materials need to be anticipated and explicitly addressed within the professional development setting, rather than assuming that the tasks will "work by themselves".

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ENACTMENT

The 5P framework: A practice-based model for learning to teach mathematics

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Teacher education faces the challenge of preparing preservice teachers not only to understand mathematics but also to do the work of mathematics teaching. This paper introduces the 5P framework—present, perform, perceive, participate, polish—a cyclical, practice-oriented model that structures preservice primary school teachers' opportunities to enact, observe, collaborate, and reflect on teaching. The framework was implemented with forty preservice primary teachers who engaged in repeated cycles of lesson design, microteaching, peer observation, workshops, and reflection. Data from lesson plans, observations, and reflective journals show that the 5P framework supported teachers in anticipating misconceptions, employing multiple representations, and developing reflective practice. The model highlights how professional education can be designed to foster learning, teaching as practice, with implications for supporting equitable learning opportunities through representations and collaborative redesign.

Keywords: Mathematics teacher education, preservice teachers, reflection, practice-based learning.

Introduction

Teacher education worldwide continues to grapple with a persistent gap between theory and practice, a challenge explicitly highlighted in international policy frameworks such as UNESCO's Education 2030 Framework for Action and the OECD's Teaching and Learning International Survey (TALIS). These frameworks emphasise the critical importance of well-prepared teachers who can develop not only through knowing *about* teaching but through learning to *do* teaching. This study focuses on preservice teachers who require organized opportunities to cultivate pedagogical skills through guided practice rather than solely through theoretical frameworks.

To address this gap, several innovative approaches have emerged. Co-teaching partnerships between teacher educators and practising teachers have shown promise in helping pre-service teachers bridge theory and practice through collaborative mathematical discourse (Downton et al., 2018). Similarly, video-based assignments, or "video papers," have been introduced as a curriculum component to support pre-service teachers in connecting theoretical knowledge with classroom practice (Krumsvik & Smith, 2009). Research in Indonesia's teacher education programs has also highlighted effective strategies in teaching processes, assessment, and student feedback to improve the preparation of prospective mathematics teachers (Jupri & Herman, 2017).

This challenge is particularly pronounced in mathematics education, especially for preservice primary school teachers, where pre-service teachers must master both subject matter and the pedagogical skills

necessary to convey concepts effectively and diagnose students' misconceptions. Fractions provide a clear example of this complexity. They are foundational to a wide range of mathematical topics, including ratios, proportions, and algebraic reasoning, yet they are consistently reported as one of the most challenging concepts for learners to grasp (Lamon, 2007).

Related to practice-based approaches to teacher education (Jao et al., 2018; Silver, 2009), Santagata, 2016), this study introduces the 5P framework—Present, Perform, Perceive, Participate, and Polish—as a coherent practice-based model designed to help preservice teachers learn mathematics teaching through iterative cycles of action and reflection. Unlike instructional models such as 5E (Bybee, 2015), which primarily focus on supporting student learning, within the 5P framework, preservice teachers do not encounter learning models merely as theoretical constructs; rather, they come to understand these models through active engagement in the work of teaching. By designing lesson plans, enacting microteaching simulations, observing peers, and mentoring others, they experience how pedagogical theories operate in practice. This iterative engagement enables them to internalise instructional models more deeply and more accurately than through theoretical study alone. The emphasis of the study remains on how preservice teachers learn the professional work of mathematics teaching through practice. Guided by this approach, the study addresses the following research question: How does the 5P framework support preservice teachers in learning mathematics teaching as a practice?

Theoretical framework

Teaching involves specialised professional knowledge distinct from both pure mathematics and general pedagogy (Ball et al., 2008). Schön (2017) highlights the importance of reflection-in-action and reflection-on-action, while Kolb (1984) conceptualises learning as a cycle of experience, reflection, conceptualisation, and experimentation. Applied to teacher education, these insights emphasise that teachers learn best when they engage in cycles of enactment and reflection, rather than only absorbing theory. This study adopts these notions as a foundation, positioning preservice teachers' learning as a practice-rich, cyclical process. Several frameworks have sought to structure practice-based learning. The lesson analysis framework supports video-based reflection by focusing preservice teachers on how instructional choices affect learning (Santagata & Angelici, 2010). The three-minute rehearsal cycle uses pedagogies of practice to help preservice teachers rehearse questioning and eliciting student thinking (Arbaugh et al., 2019). Castro (2015) demonstrates that curriculum interventions can shape preservice teachers' use of materials, while Hobden (1999) shows that reflection on beliefs can shift conceptions of mathematics teaching. While these frameworks advance understanding of practice-based teacher education, they often focus on general pedagogical skills rather than subject-specific challenges in mathematics and do not always integrate structured observation, collaborative redesign, or equity considerations.

Ball et al.'s (2008) model of Mathematical Knowledge for Teaching (MKT) is used to conceptualise how preservice teachers learn to anticipate misconceptions, choose representations, and diagnose student thinking. In this study, MKT is operationalised through preservice teachers' ability to: (a) identify common fraction misconceptions, (b) select and justify mathematical representations, and (c) design explanations that support conceptual understanding. This model offers a theoretical

framework for understanding the changes observed across the 5P cycles. The construct of teacher noticing, particularly attending to and interpreting students' mathematical thinking (Jacobs et al., 2010), frames how preservice teachers learn through the Perceive stage. During structured observation, preservice teachers are expected not only to describe peer teaching but also to analyse the thinking students might display, which is essential for developing diagnostic competence.

The 5P framework is grounded in practice-based teacher education literature (Grossman et al., 2009; Jao et al., 2018) which emphasises repeated cycles of approximating practice. In this study, the 5P framework operationalises these cycles by engaging preservice teachers in designing lessons, enacting microteaching, observing peers, collaboratively analysing teaching decisions, and refining lesson plans. The 5P framework is positioned as a model that helps preservice teachers understand pedagogical theory through action.

Methods

This study was conducted in a mathematics methods course for preservice primary teachers at a private university in Yogyakarta, Indonesia. A total of 40 preservice primary teachers (PSTs) participated in the study, all enrolled in the same compulsory fifth-semester course focusing on practice-based learning. The PSTs were organised into ten groups of four, and each group engaged in all three cycles of the 5P framework within the university classroom context. All participants provided informed consent. Each 5P cycle lasted approximately three weeks, giving a total implementation period of nine weeks. Cycles were not conducted within a single lesson; instead, each stage (Present, Perform, Perceive, Participate, Polish) unfolded over multiple class meetings, allowing PSTs time to plan lessons, rehearse teaching, observe peers, and revise materials. Roles rotated systematically across cycles so that all PSTs experienced all three key practice stances, teacher (and mentor), observer, and audience (student), by the end of the study. In Cycle 1, selected groups acted as teacher (and mentor), observer, and audience (student). In Cycles 2 and 3, these roles rotated, ensuring equity of experience and balanced exposure to teaching practice.

The 5P cycle was implemented across three cycles. Each cycle included:

1) *Present***.** PSTs collaboratively prepared a lesson and explicitly identified misconceptions, drawing on literature and classroom observation notes collected earlier in the course. Their presentation to peers required them to justify pedagogical decisions and anticipated student thinking.
2) *Perform***.** The same PSTs conducted a 20–25 minute microteaching simulation in the methods classroom. Before teaching, they met with the instructor for a planning conference. Only one PST taught at a time, but all group members contributed to planning.
3) *Perceive***.** Assigned observers used a structured rubric aligned with MKT constructs (Ball et al., 2008) and teacher noticing dimensions (Sherin & van Es, 2009). Observers focused on student thinking, representation use, and instructional clarity, rather than teacher behaviour alone.
4) *Participate.* Mentors (teacher simulators from previous sessions) lead workshops, analyse observations, discuss misconceptions, and guide peers in creating lesson plans for new objectives. They also guide the other groups in creating creative lesson plans for other learning objectives based on microteaching experiences. Preservice teachers develop skills in collaborative problem-solving, peer critique, and promoting equitable learning experiences.

5) *Polish***.** Groups produced a revised, final lesson plan incorporating feedback from observation and workshops. PSTs also wrote a reflection journal connecting their revisions to relevant theoretical concepts (MKT, teacher noticing, reflection-in-action).

This cycle will iterate three times until every group of students has turned a presenter-teacher-mentor or peer-reviewed. Data collection included: (1) lesson plans: examined for use of representations, anticipation of misconceptions, and alignment with learning objectives, (2) teaching observations: scored descriptively (frequencies and percentages) across cycles to examine growth trends, (3) reflective journals: analysed thematically to identify evolving understanding of MKT, teacher noticing, and reflective practice. Mathematical Knowledge for Teaching (MKT) was operationalised through: (i) identification of misconceptions, (2) accuracy and clarity of mathematical explanations, (3) choice and justification of representations. Teacher noticing was operationalised as documented evidence in journals and rubrics of: (1) attending to student thinking, (2) interpreting student strategies, (3) proposing instructional adjustments. The data were analysed by mixed methods. Descriptive statistics (percentages, frequency counts) were used on observation rubric data and lesson-plan codes to track changes across cycles (e.g., percentage of PSTs addressing misconceptions). A thematic coding approach was used to analyse journals and workshop transcripts, supported by theory-driven categories (reflection-in-action, misconception awareness, representation choice).

Result and discussion

Development of Mathematical Knowledge for Teaching

Analysis of lesson plans, observation rubrics, and reflective journals indicated a steady development of Preservice Teachers' (PSTs') Mathematical Knowledge for Teaching (MKT). MKT was defined following Ball, Thames, and Phelps (2008) as PSTs' ability to: (1) anticipate misconceptions, (2) select and justify appropriate representations, and (3) provide mathematically precise explanations.

Table 1: Percentage of participants using misconception cases in discussions

Cycle 1	Cycle 2	Cycle 3
30% using misconception cases	50% using misconception cases	70% using misconception cases

In the first cycle, only 30% of participants incorporated strategies to address fraction misconceptions. For instance, one participant demonstrated a common error by stating, "To compare 1/3 and 1/10, just compare the denominators: since 10 > 3, then 1/10 > 1/3." This response reflects a typical misunderstanding in which students assume that larger denominators always produce larger fractions, without considering the fraction's overall value. As the cycles progressed, preservice teachers increasingly recognised the importance of addressing these misconceptions. By the final cycle, 70% of participants explicitly integrated strategies to confront such errors in their lesson plans. One example of a refined instructional approach involved the use of a number line: "Use a number line from 0 to 1 to show that 1/3 is at 0.33 while 1/10 is at 0.1, so 1/3 is larger. Discuss why denominators cannot be compared directly without considering the whole." This shift demonstrates not only an awareness of students' potential errors but also the ability to design teaching interventions that directly target conceptual misunderstandings. This pattern demonstrates an emerging capacity for

specialised content knowledge. Growth in MKT aligns with Kolb's (1984) experiential learning cycle, where PSTs in the Perform stage applied conceptual knowledge, while the Perceive and Polish stages enabled reflective abstraction and refinement. Schön's (1983) reflection-in-action is also visible: PSTs adjusted explanations in response to student thinking observed during microteaching.

Representations as tools for equity

Participants increasingly incorporated multiple representations in their teaching practices over the course of the cycles. During the early cycles, instruction was heavily dominated by symbolic approaches. However, by the final cycle, 70% of lesson plans included a variety of concrete and visual representations, such as fraction strips, traditional food-sharing scenarios, or number lines, allowing students to engage with fractions in more intuitive and meaningful ways.

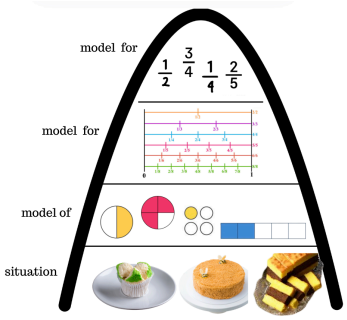

Figure 1: Part of S3's polished lesson plan with Indonesian realistic mathematics education

Figure 1 illustrates an example of a polished lesson plan created by a preservice teacher using principles of Indonesian Realistic Mathematics Education (RME). It demonstrates how structured representations can bridge concrete experiences with formal mathematical symbols. These multiple-representation tasks provided equitable access to learning for students with diverse competencies, particularly benefiting those with weaker symbolic skills. This development reflects the representational dimension of MKT (Ball et al., 2008), as PSTs deliberately selected representations to support diverse learners. The emphasis on representational diversity shows an increasing awareness of equity, ensuring that all students have accessible entry points to learning.

Growth in teacher noticing through structured observation

The Perceive stage played a key role in sharpening preservice teachers' ability to notice important classroom interactions and student thinking. Early in the cycles, participants' observation journals often focused on teacher actions rather than on learners' understanding. For example, one participant reflected after observing a peer: "I realised my peer skipped explaining why 2/4 = 1/2, and then I remembered I often skip that step too." This insight illustrates the beginning of a reflective process, where preservice teachers became aware not only of their peers' instructional choices but also of their own habitual practices. By the third cycle, the content of observation notes had shifted substantially. Participants increasingly documented students' cognitive responses and struggles rather than merely recording teacher behaviours. In Cycle 1, an observation note read: "Teacher writes = 2/4." By Cycle 3, similar notes demonstrated a deeper focus on learners: "Students looked confused when the teacher simplified 2/4. A visual would help." This progression indicates a growing ability to interpret classroom interactions through the lens of student thinking and to anticipate instructional adjustments

that could support understanding. This shift reflects the noticing framework described by Sherin and van Es (2010), involving (1) attending to student thinking and (2) interpreting what student actions reveal about understanding. The Perceive stage, supported by structured rubrics aligned with MKT indicators, was instrumental in this development. The 5P cycle's design allowed PSTs to experience Schön's (Schön & DeSanctis, 1986) reflection-in-action: they observed misconceptions emerge in Perform, recognized them in Perceive, and addressed them in Polish. This iterative noticing–reflection process contributed to more responsive and intentional teaching decisions in later cycles.

Collaborative refinement through the Participate stage

Workshops conducted during the Participate stage provided structured opportunities for collaboration and co-construction of teaching materials among preservice teachers. In one session, participants worked together to redesign a fraction comparison task to make it more accessible for learners. One preservice teacher noted, "If students cannot read the word problems, they can still compare the fraction pictures. This way, everyone can participate." This adjustment demonstrates a deliberate effort to create equitable learning experiences. In another group, participants developed a fraction card game aimed at supporting pupils with limited procedural fluency, allowing these learners to engage with fractions through hands-on practice rather than relying solely on symbolic computation. These collaborative redesigns align with Hobden's (Hobden, 1999) argument that shared reflection can reshape beliefs about mathematics teaching. The redesign discussions also reflect Kolb's (Kolb, 1984) "reflective observation" phase, where PSTs critically examined teaching outcomes and considered alternatives.

Reflection, identity formation, and the Polish stage

The Polish stage emphasized the role of reflection as central to ongoing professional growth. By the final cycle, lesson plans exhibited notable improvements, including clearer sequencing of activities, explicit anticipation of common misconceptions, and the integration of multiple representations such as fraction strips, number lines, and real-life contexts. These refinements indicate that preservice teachers were not only attending to content but also strategically designing lessons to support diverse learners' understanding. One participant, S34, reflected on this development: "At first, I thought teaching fractions was just showing rules. Now I see that students need to experience fractions in different ways. I feel like a real mathematics teacher when I can design such lessons." This statement highlights the transformative impact of reflective practice, showing how engagement with observation, collaboration, and iterative lesson design can shift teachers' perspectives on teaching and learning. The progression observed during the Polish stage aligns with Schön's ((Schön, 1983) model of the reflective practitioner, where reflection-in-action and reflection-on-action support continuous improvement. It also resonates with Kolb (Kolb, 1984) experiential learning cycle, illustrating how teachers move from concrete experience through observation and reflection to the revision of practice.

Conclusion

This study explored how the 5P framework—Present, Perform, Perceive, Participate, and Polish, supports preservice primary teachers in learning to teach fractions. Across three cycles, the preservice

teachers showed clear development in anticipating student misconceptions, using multiple representations, noticing student thinking, and refining their lesson plans. These improvements reflect growth in Mathematical Knowledge for Teaching (MKT) and in their ability to make instructional decisions based on student understanding. The findings suggest that the 5P framework helps bridge the theory–practice gap by giving preservice teachers repeated and structured opportunities to enact, observe, discuss, and revise teaching. Through rotating roles as teacher, observer, and mentor, they learned to connect theoretical ideas such as reflection, noticing, and representational thinking to their actual teaching practice. Future research could examine how the framework applies to other mathematical topics or how it influences teaching once preservice teachers begin working in schools. Overall, the study highlights the value of structured practice-based experiences in helping future teachers develop more confident, thoughtful, and equitable approaches to teaching mathematics.

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Using simulations to study prospective teachers' building on student thinking

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Teaching that builds on student thinking actively engages with student sensemaking and respects the sense that students make of mathematical situations, both of which crucial for engaging all students in the learning of mathematics. In the context of a 15-year project to design and use teaching simulations focused on eliciting and interpreting student thinking, we present a methodological approach using simulations and post simulation interviews to provide insight into the ways that prospective teachers build on student thinking. The study demonstrates that simulations can prompt prospective teachers to engage in eliciting and interpreting relevant to building on student thinking and reveal important aspects of their capabilities.

Keywords: Teacher education-preservice, teacher educator, teaching simulations.

Introduction

A prospective teacher examines the student work shown in Figure 1. The teacher asks the student some questions about the work to gain a clearer sense of the process the student used to compare the fractions; how the student understands equivalent fractions and why the process of generating them works; and how the student understands comparing fractions with the same denominator.

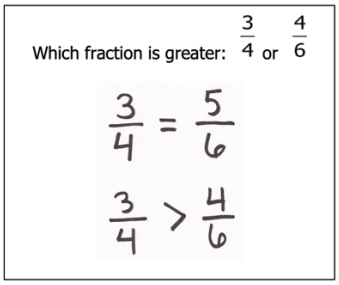

Figure 1: Student work sample

This work of asking questions to learn about student thinking and making sense of what students know and can do is a routine part of classroom teaching. It is core to the profession and provides a foundation for other parts of a teacher's work. It is often referred to as "eliciting and interpreting of student thinking." Prospective teachers (PTs) need robust opportunities to develop the content knowledge, practical skill, and principled discernment required to do this work. Simulations are a promising approach to supporting PTs in learning to teach (Mikeska et al., 2021). They provide PTs with access to teaching situations that might otherwise be limited in the field. They can be structured to support work on aspects of teaching practice, while suspending others that might interfere with or unnecessarily complicate teaching. They can position those needing to learn about PTs' teaching close to the work and create favorable conditions for documenting that teaching would be difficult to match in the field. These attributes also make simulations useful for studying the practices of PTs.

Researchers have been using simulations as opportunities to observe and analyze the components and qualities of PTs' teaching (Cohen et al., 2020; Dieker et al., 2014).

In this paper, we explore the promise of simulations as a research tool for supporting insight into PTs development of the kinds of knowledge and skills needed for teaching. In particular we are interested in teaching oriented to, and building on, students' mathematical sensemaking. We argue that simulations provide opportunities for researchers to see the turn-by-turn teaching interactions that are foundational to this kind of teaching. In this context, we focus on the following research questions:

1) Is it possible for a simulation to reveal aspects of PTs' ability to build on student sensemaking?

2) What could variation in building on student sensemaking sound like and look like when PTs *elicit* student thinking and, later, share their *interpretations* of student thinking?

Situating the use of teaching simulations

Mathematics teaching aims to support each student in developing a deep understanding of mathematics, agency in using mathematics in a wide array of contexts, and a positive belief in themselves as "doers of math" (Aguirre et al., 2013). For teachers, this means inviting, listening to, and valuing what students say and the mathematical resources they bring to learning. It is well documented that teaching rooted in students' sensemaking provides rich opportunities for learning (Carpenter et. al., 1989). This research illustrates many paths that students may use to solve problems and also that it is productive to teach in ways prioritize the uptake of student thinking. By "teaching that builds on student thinking," we mean teaching that engages with student sensemaking and respects the sense that students make of mathematical situations. Teaching in this way requires finding out about students' thinking through practices such as eliciting and interpreting student thinking (Jacobs et al, 2010). For the field to learn about how PTs engage in such practices, it is crucial to explore ways of studying it.

Simulations of teaching can provide researchers with insights into the ways in which PTs build on student thinking through their engagement in eliciting and interpreting student thinking. The choice to use simulations, in contrast with approaches that rely on school-based contexts, allows researchers to standardize the mathematical content and the nature of student thinking that will be encountered and that all PTs will have the opportunity to elicit and interpret in response to that thinking. To generate the context for practice, we developed a "decomposition" that breaks complex professional work of eliciting and interpreting student thinking into pieces that can be learned and studied (Grossman et al., 2009). Our decomposition includes components of student thinking teachers may ask about and interpret (mathematical processes and understandings) and how PTs engage with what a student says and does (see Shaughnessy & Boerst, 2018). Building on student thinking goes beyond asking students about their ideas, to also ask follow-up questions that are purposefully connect (i.e., do not dismiss or propose an alternative to) with students' ways of using or talking about mathematical processes and practices (for a more thorough unpacking of this idea, see Shaughnessy et. al., 2025). This decomposition connects with the practice of noticing (Jacobs et al., 2010), including a focus on attending to student strategies as the basis for interpretation and formulating next steps. We also share the sense that attending to student thinking can be challenging. When students

use unfamiliar approaches, teachers may opt to redirect or take over the thinking which shows disrespect for student thinking that can be detrimental to learning (Jacobs et al, 2014). In this paper, we focus on "building on student sensemaking," which is comprised of two parts, namely "attending to student thinking" and "respecting the student's mathematical thinking."

This decomposition is the foundation for building a simulation that approximates practice (Grossman et al., 2009). The approximation strategically replicates some, but not all, features of teaching situations that influence the ways in which teaching practices are implemented. In our simulation, PTs initiate their work by analyzing the written work of a student, generating questions to ask the student, taking up what the student says to explore the student's process and understanding more deeply. A simulation grounded in written work provides PTs with the opportunity to elicit and interpret how the student is approaching the problem (process) and what they understand about the mathematics of the situation (understanding). The student work shown in Figure 1 provides some information about the student's process for comparing two fractions. It also illustrates a mathematical error that may relate to steps in the student's process and/or aspects of the student's understanding of equivalence. It could also be a mistake that the student will notice and correct. Importantly, this error does not preclude that the student may understand other elements of the process, such as understanding the use of common denominators to compare fractions. Simulations enable researchers to see and hear how the PT takes up what the student is saying and doing, the extent to which the PT probes for understanding when errors are present, and how the PT characterizes the student's thinking. Further, by talking with the PT after the simulation, the researcher can learn how the PT has interpreted the student's process and understanding, how the PT characterizes the student's approach, and, more generally, the PT's characterization of the student and their thinking.

Methods

There are four parts to our approach. First, a PT examined a piece of student work, such as the one shown in Figure 1, and prepared to learn about the student's process and understanding. In the second part, the PT had five minutes to interact with the simulated student to elicit the student's thinking around this sample of student work. In this case, the student had used the "common denominator" method to solve the problem. Specifically, the student decided to convert the ¾ to a form they believed was equivalent and generated 5/6. The student did this because "fractions are easier to compare when they have something in common." The student believes that 3/4 = 5/6 because "adding a two to the top and the bottom" aligns with the idea that equivalent fractions are created by "doing the same thing" to the numerator and denominator. While not shown in the work, the student then compared 5/6 and 4/6 and determined that 5/6 is greater by comparing their numerators. The student understands that when the denominators of two fractions are the same, the pieces in both fractions are the same size. This allows the student to determine which fraction is greater by knowing how many of the same-sized pieces are in each fraction. We train the simulated student (a TE taking on the role of a student using a well-defined set of response rules) to use this process, to reason in particular ways about the mathematics, and to use specific phrases when responding to commonly asked questions. Third, the researcher interviewed the PT to learn about their interpretations of the student's thinking (e.g., the student's process and understanding) and their understanding of the

mathematics underlying the student's process (see Shaughnessy & Boerst, 2018 for more detail). Fourth, the researcher engaged the PT in a feedback conversation about the performance.

We video recorded this teaching simulation cycle and collected all written artifacts. The simulation and post simulation interview were scored in real time as the activities unfold using a web-based performance checklist keyed to the performance components identified in our decomposition of eliciting and interpreting student thinking. In the simulation, the PT asked questions to uncover the student's process and understandings of the process they used. This provided a window for the TE to observe the ways in which the PT "takes up student thinking" and "respects the student/student's thinking." In the post simulation interview, the TE asked questions to learn how the PT interpreted information gathered about the student's process and understanding. This provided an opportunity for the TE to listen for how the PT characterized the student as a knower and doer of math, as well as how the PT characterized the student's process as generative, interesting, misguided, "the right/wrong way," etc. In the interview, the PT spoke with another adult about the student, instead of speaking directly with the student. These different contexts had the potential to bring out different facets of the PT's insights into the student's thinking and their characterizations of that thinking.

For this paper, we analyzed data collected in 2025 from an administration of the common denominator simulation with 20 PTs at a teacher education site participating in a larger research project focused on the validity and reliability of the simulation. While the performances were appraised in the moment to support the provision of formative feedback, in this paper, we focus on analyses that we did with the collected video records. We first applied a detailed observational tool to each performance (see Shaughnessy et. al., 2025). An app we developed converted the observational tool marks into a rubric score ranging from 1 to 4 for each performance area (with increased scores indicating greater proficiency). In this analysis, we focus on rubric scores for "attending to student thinking" and "respecting the student/student's thinking" to understand how the components of building on student sensemaking were represented in the PTs' performances. Table 1 includes brief characterizations of proficiency levels in the rubrics. After analyzing the rubric results, we qualitatively analyzed how PTs engaged in the simulation and in the post simulation interview to better understand the ways in which PTs demonstrated building on student sensemaking.

Table 1: Characterizing levels of proficiency by performance areas

Level	***Attending to Student Thinking (simulation)***	***Respecting the Student/Students Mathematics (simulation and post simulation interview)***
1	Does not take up what the student says or writes.	Does not consistently demonstrate respect for student's mathematical thinking
2	Takes up what the student writes	Demonstrates respect for student's mathematical thinking when talking with others about the student
3	Takes up what the student says.	Demonstrates respect for student's mathematical thinking when interacting with the student
4	Takes up what the student says and writes.	Consistently demonstrates respect for student's math thinking when interacting with the student and interacting with others about the student

Results

We focus on whether it is possible to use a simulation to study how PTs build on student thinking. We first examine "attending to student thinking," which appraises whether the PT took up and used what the student said or wrote during the conversation as the PT interacted with the student. Next, we examine "respecting the student/student's mathematics," which appraises how the PT valued the student's process and understanding as something to be built upon.

Performance area: Attending to student thinking

"Attending to student thinking" is assessed through examining the simulation performance, focusing on whether the PT takes up what the student says and/or writes. There is no component of the post-simulation interview that corresponds with this performance area. Eighteen of the 20 PTs (90%) earned a level four score on the rubric. This means that these 18 PTs took up what the student said and wrote. Qualitative analyses revealed many instances in which PTs asked specific questions about the student's initial work sample. There were also a few times when PTs asked the student to write and talk about that writing. Elena asked the student to solve a follow up problem:

Elena:	So if this were to be a different problem... Such as, let's see, 4/5 or 6/7. How would you solve it? Which one would be greater?
Student:	Well, since they don't have anything in common, I would choose one of these to make something in common. Did you want me to write it?
Elena:	If you can.
Student:	Okay. So then I would change 4/5 into something that has sevenths. So that means...
Elena:	Okay. So how would you know that 6/7 is greater than 5/7?

These 18 PTs also demonstrated uptake of what the student said to them during the simulation. Qualitative analyses of these performances showed these PTs probed what the student said and/or incorporated the vocabulary the student used. For example, Marcela revoiced what the student said and encouraged the student to "teach" them the method being used:

Student:	I decided that I needed to change three fourths into five sixths.
Marcela:	Cool. 3/4 to 5/6. I like learning things. Please teach me.
Student:	So, the way that I did that was I added two to the numerator and I added two to the denominator to make my equal fraction.

The remaining two PTs (10%) earned a level two score on the rubric. Qualitative analyses of these PTs' interactions with the student revealed a pattern of a PT asking a question and the student responding. The PT then either briefly acknowledged the answer by saying "okay" or "good" or asked a question that was not about what the student just said.

Performance area: Respecting the student/student's thinking

Respecting the student and the student's thinking is studied by examining the simulation with a focus on the nature of a PT's response to the student's process and/or understanding. It can be further studied in the post-simulation interview by focusing on the PT's interpretation of the student as a learner of math and the degree of respect they have for the student's mathematical process and understandings. Thirteen of 20 PTs (65%) earned a four on the rubric by consistently demonstrating respect for the student's thinking in the simulation and in the post-simulation interview. During the simulation, these PTs actively engaged with the student's process from the outset of the interaction

and probed the student's understanding of their process. For example, from the second question in the simulation, this PT (Daniela) elicited and probed understanding in respectful ways even though it produced fractions that were not equivalent:

Student: Well, first, I decided to change three fourths into five sixths.
Daniela: I see that you changed it to 5/6. Can you tell me why you decided to change that?
Student: Well, I wanted to have something in common with the other fraction. So I started with these two fractions. And so I wanted to have something in common with the four sixths. So that's why I wanted to have six.
Daniela: Okay. So, you said you wanted it to be similar. Do you think—no, no. I'm so sorry. Um Oh, my goodness. I pressed on my questions too. So, you changed it to be similar. Do you think changing it will affect your answer?
Student: Well, I mean, I made sure that this fraction was equal to this one. So as long as it's equal and it has something in common with this other one, then I can go ahead and do my comparing. In the end, I can come back to the fractions I had to start with.
Daniela: Can you explain what you mean by equal? How are 3/4 and 5/6 the same number?

Acknowledging that the student used an unsound method to generate equivalent fractions is not a sign of disrespect. In the interview, several PTs conveyed a sense that while the process was incorrect, what the student thought was reasonable on some level. For example, one PT (Sabrina) stated:

> The student knew that a way to compare fractions is to get the same numerator or denominator. In this case, they chose the denominator. The only thing is like about the adding, the student just said that they added because it's easiest. So, I don't think they got that part of the mathematical process, but they definitely understood that a part of compare fractions is to get the same denominator.

Many PTs acknowledged that the student generated fractions that were not equivalent and also named strengths in the student's understanding. For example, Larissa stated, "he understood to more easily compare fractions, it's good to find like a common denominator, but he did it wrong because he used addition." Another PT (Maribel) stated, "They understood that it's easier to solve fractions with the same denominator. They also understood which fraction is greater and what the numerator means."

Three of the remaining PTs (15%) earning a three on the rubric demonstrated consistent respect for the student's thinking in the simulation. However, this group demonstrated forms of disrespect at some point during the post-simulation interview. In response to a question about what the student understood, all PTs in this subgroup assumed the student didn't understand mathematics either because the student "used the wrong way" to determine equivalence and/or because the student was not using a "real method." For instance, PT (Ryan) stated, "So there is really no math involved here. I feel like they just kind of guessed. They didn't do any of the math involved in it. They didn't get the LCD or anything like that."

The final four PTs participating in the simulation (20%) earned a two on the rubric. They demonstrated consistent respect for the student's thinking during the post-simulation interview and demonstrated forms of disrespect during the simulation interaction with the student. For instance, in the simulation, after asking only one question, the PT (Diego) questioned the accuracy of the student's work and proposed an alternative approach quickly thereafter:

Diego: Is that correct? How did you come up with that answer?
Student: Well, I added two to the numerator and two to the denominator to get five sixths.
Diego: Oh, okay. And if we were to draw this out in an array, would we be able to see that they are equivalent fractions?

In another case, the PT (Naila) persisted in asking the student to use a method even after the student explained it was not used:

Naila:	What would happen if we represented them with something like using an area model representing pieces of pizza?
Student:	When you draw pictures when the fractions are kind of close to each other, I'm not that great at drawing. So usually it's hard to tell which one's larger, so that's why I use the numbers.
Naila:	Okay. Yeah, that's totally okay. And then if we ever did use the area model to demonstrate which fraction is bigger, would it be the same or different amount?

Instead of inquiring about the student's process or understanding, these examples illustrate a form of disrespect by almost immediately pressing the student to adopt about someone else's approach.

Discussion

Simulations present a tool that researchers can use to analyze PTs' skill in "building on student thinking" through their work to elicit and interpret student thinking. Through examination of PTs' engagement in a simulation, we showed that it is possible for researchers to use observational checklists and associated rubrics to discern degrees of proficiency within PTs' simulation performance and responses in post-simulation interviews relevant to two component practices of building on student thinking, namely attending to student thinking and respecting the student/student's thinking. The simulation could elicit and capture affirming examples as well as examples when PTs did not attend to or show respect for student thinking. Some aspects of these practices were shown during interaction with the simulated student. Other aspects did not surface until the post simulation conversations between the PT and TE. Having the two contexts expanded our opportunities to hear how PTs were characterizing the student and their mathematics. Since it is crucial for teachers to be able to engage respectfully and professionally when interacting with students (Jacobs & Empson, 2016) and characterizing a student's thinking after an interaction (Chappuis & Stiggins, 2017), it is useful to have simulations designed to provide insight into both.

Further, we argue that the ability to discern these differences in teaching and provide granular examples of them is a service to the field as both are crucial for those teaching teachers to be able to see and articulate what it could look like and sounds like to take up and respect student thinking when engaging in eliciting and interpreting. These are the basis for observing, providing feedback, and reflecting on teaching (Grossman et al., 2009). This also provides a resource for PT reflection on their eliciting and interpreting, as well as plan for subsequent action.

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Training strategies for teaching how to teach in preservice teacher education

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With the goal of exploring how to teach teaching in early years and primary preservice teacher education, we carried out a systematic literature review of teacher education tasks aimed at developing pedagogical content knowledge. The search extracted 26 tasks, categorised according to the strategy employed to promote knowledge among preservice teachers. The results show that a variety of approaches are used, in which the preservice teachers plan and implement lessons, analyse classroom situations and tasks, which might be real or hypothetical, conduct their own research based on their practice, imagine classroom scenarios, and solve mathematical problems. In future studies, we aim to make connections between training strategies and other features, and to explore the relationship between characteristics of tasks and the knowledge they develop.

Keywords: Pedagogical content knowledge, training tasks, educational strategies, preservice teacher education.

Introduction

Pedagogical content knowledge (henceforth PCK) is considered one of the distinguishing features of teachers' professional knowledge, an essential element of their work brought into play whenever they engage in the act of teaching (Depaepe et al., 2013). It has been shown to have a direct effect on the quality of teaching (Baumert et al., 2010). Since the term was coined by Shulman (some 40 years ago), it has been the focus of many studies into mathematics education (Grigaliūnienė et al., 2025).

In the course of defining the scope of PCK, Shulman also made use of the term pedagogical reasoning (Shulman, 1987), which focuses on teachers' decision-making and is concerned with professional knowledge in use. Research into the pedagogical reasoning of experienced teachers when they put their PCK into practice in comparison with newly qualified teachers has drawn attention to the need to provide opportunities in teacher education for trainees to apply their PCK to the process of pedagogical reasoning (Pilous et al., 2023). This emphasis on professional knowledge in use can be understood as a situated vision of PCK, one in which its dynamic nature is foregrounded.

The training that preservice teachers (PSTs) receive is largely defined by the tasks they are given (Grevholm et al., 2009). Placing the focus of this study on teacher-education tasks thus enables us to explore teacher education and how this is conceptualised from the perspective of research into the subject. Drawing on definitions of a mathematical task (Mason & Johnston-Wilder, 2006; Watson et al., 2013), we consider a training task to be a structured situation which PSTs are invited to explore with the aim of focussing their attention on a mathematical or didactical mathematic idea or process. In this paper we consider the format taken by training tasks in mathematics teacher education (Grevholm et al., 2009). In this respect, the ICMI Study 15 Conference Proceedings include a range of successful training tasks for primary mathematics teacher education (Clarke et al., 2009), which are representative of categories of tasks and strategies oriented towards research. Among the strategies can be found the use of Lesson Study, portfolios for mathematics teaching, video-based

analysis of authentic lesson excerpts and students solving mathematics problems, hypothetical classroom situations, problem-solving tasks and use of children's literature.

Given that, learning to teach is closely connected to development of trainees' PCK and their familiarisation with professional practices, the research question of this study is: What are the key training strategies that feature in tasks aimed at developing preservice teachers' PCK (regarding mathematics teaching) in teacher education? We conducted a systematic review of academic papers on teacher education tasks at early years and primary levels in the Scopus and Web of Science (WoS) databases, targeting strategies which the research had identified as contributory to the development of PCK. The contribution of this study is thus an overview of how PSTs learn to teach.

Description of methodology

The systematic literature review was carried out in adherence with the *PRISMA 2020* protocol (*Preferred Reporting Items for Systematic reviews and Meta-Analyses*, Page et al., 2021). The eligibility criteria for inclusion were: (a) peer-reviewed journal papers; (b) empirical research; (c) teacher education tasks focusing on development of teacher knowledge; (d) teacher education tasks implemented in early years or primary preservice teacher education; (e) studies published between 2005-2025; and (f) papers published in English, Spanish or Portuguese.

The keyword search expression used were: *(math* AND knowledge AND (task* OR assignment*) AND ("future teacher" OR "preservice teacher" OR "prospective teacher" OR "teacher education"))*. These terms were established using the *UNESCO* thesaurus and *Eric* so as to ensure accuracy. The search fields were title, keywords and abstract. In total, 141 (WoS) and 445 (Scopus) papers were retrieved. The search was subsequently refined for both databases with respect to language (English, Spanish and Portuguese), area of knowledge (Social Sciences and Mathematics in Scopus, and Education & Educational Research & Mathematics in WoS), and year of publication (between 2005 and 2025), retrieving 91 (Scopus) and 347 (WoS) papers. The search was done on 14th March 2025. After rejecting duplicated publications 353 papers remained. In the initial reading phase, skimming title and abstract, 146 were included and 207 excluded (because they did not meet the following eligibility criteria: journal paper (2), empirical research (9), knowledge development (77), and initial teacher training (119)). A second reading phase of full title saw a further 69 articles excluded. In the final phase, the number of papers was whittled down to 23 papers on the basis of including PCK in their consideration of training tasks, while the remaining 54 did not. The final selection also included 3 studies recommended by experts. These 3 articles met the eligibility criteria and enrich the evidence by broadening both the participant profiles and the conceptual framework. Specifically, they provide insights into tasks implemented with early childhood education pre-service teachers ([5] –see Table 1) and into different teacher knowledge models ([25], [17]). The total number of papers included was 26 (Table 2), 19 written in English, 5 in Spanish and 2 in Portuguese.

Results

All teacher-training tasks extracted from the 26 papers included were classified following inventory in Clarke et al. (2009) – marked in Table 1 in italic – with addition of further classifications of our own. In total, we identified seven different types of tasks (Table 1).

Table 1: Classification of teacher-training tasks by format. (If the training task uses more than one strategy, it has been included more than once, associated with each corresponding strategy)

Category of teacher-training task	Sub-category	Papers	n
Design, implementation and reflection	*Lesson Study*	[1] (i) [1]; [2] (i); [3] (i)	3
	Task design and classroom resources	[4] (v); [5] (iv); [6] (i); [7] (ii); [8] (ii); [9] (v); [10] (ii)	7
	Lesson implementation	[6] (i); [11] (i)	2
	Portfolios		0
Analysis of clasroom situations	In-person lesson observation	[20] (ii)	1
	Hypotetical situations	[15]; [16] (iii); [17] (ii); [18] (iii); [19] (iii)	5
	Video-based analysis	[7] (ii); [12] (iii); [13] (ii); [14] (iv); [15]	5
Analysis of tasks and classroom resources	Analysis of classroom tasks	[15]; [16] (iii); [21]; [22] (ii); [23]	5
Imagining of hypothetical situations	Drawing tasks	[24] (iv)	1
	Imagining of possible student responses	[22] (ii)	1
Research into practice		[25] (ii)	1
Analysis of pupils' work		[21]; [22] (ii)	2
Solving mathematical tasks		[4] (v); [9] (v); [12] (iii); [15]; [21]; [23]; [26] (iii)	7

[1] For reasons of space, we have indicated with numbers and in abbreviated form the references corresponding to the articles obtained in the systematic search. In addition, we report on the size of the samples, grouped by ranges. Thus, (i) represents between 1 and 10 PTs, (ii) between 11 and 35, (iii) between 36 and 71, (iv) between 72 and 105, and (vi) between 106 and 168. These groups have been adapted to the sizes reported in the studies. If there is no indication, it is because the sample size is not specified.

Table 2: Articles obtained in the systematic search

[1] Cardoso, L., Da Ponte, J. P., & Quaresma, M. (2023). https://doi.org/10.1108/IJLLS-02-2022-0027

[2] Duarte, N., Mendes, J. P., & Pinto, H. G. (2024). https://doi.org/10.35763/aiem26.5428

[3] Valenzuela-Molina, M., Ramos-Rodríguez, E., & Flores, P. (2021). https://doi.org/10.17648/acta.scientiae.5650

[4]Alonso-Castaño, M., Alonso, P., Mellone, M., & Rodríguez-Muñiz, L. J. (2021). https://doi.org/10.3390/math9243300

[5] Alsina, A., Pincheira, N., & Delgado-Rebolledo, R. (2023). https://doi.org/10.47553/rifop.v98i37.2.99241

[6] Castro, W. F., Durango-Urrego, J. H., & Pino-Fan, L. R. (2021). https://doi.org/10.29333/ejmste/11139

[7] Figueiredo, M. P., Gomes, H., & Rodrigues, C. (2018). https://doi.org/10.1080/1350293X.2018.1487164

[8] Nason, R., Chalmers, C., & Yeh, A. (2012). https://10.1007/s10857-012-9209-0

[9] Montes, M., Chico, J., Martín-Díaz, J. P., & Badillo, E. (2024). https://doi.org/10.1016/j.jmathb.2024.101132

[10] Putra, Z. H., Dahnilsyah & Aljarrah, A. (2021). http://dx.doi.org/10.22342/jme.12.3.14143.563-580

[11] Venkat, H., & Mathews, C. D. (2024). https://doi.org/10.4102/sajce.v14i1.1509

[12] Climent, N., Contreras, L. C., Montes, M., & Ribeiro, M. (2024). https://doi.org/10.1007/s10857-023-09590-7

[13] da Ponte, J. P., & Branco, N. (2013). http://doi.org/10.1590/S0104-40602013000400010

[14] Montes, M. Á., Climent, N., & Contreras, L. C. (2022). https://doi.org/10.17811/rifie.51.1.2022.27-36

[15] Pascual-Martín, M. I., Climent, N., Codes, M., Martín-Díaz, J. P., & Contreras, L. C. (2023). https://doi.org/10.47553/rifop.v98i37.2.99221

[16] Kang, R., & Liu, D. (2018). https://doi.org/10.1007/s10763-016-9760-8

[17] Putra, Z. H. (2020). https://doi.org/10.18690/rei.13.4.365-394.2020

[18] Son, J. W. (2013). https://doi.org/10.1007/s10649-013-9475-5

[19] Son, J. W. & Sinclair, N. (2010). https://doi.org/10.1111/j.1949-8594.2009.00005.x

[20] Livy, S., & Downton, A. (2018). https://doi.org/10.1016/j.jmathb.2017.11.004

[21] Ortiz, J. J., Batanero, C. & Contreras, J. M. (2012). https://www.relime.org/index.php/relime/article/view/243

[22] Stephens, A. C. (2006). https://doi.org/10.1007/s10857-006-9000-1

[23] Stylianides, G. J., & Stylianides, A. J. (2024). https://doi.org/10.1080/10986065.2023.2172617

[24] Lin, Y. C. (2022). https://doi.org/10.1080/1359866X.2021.1880546

[25] Colaco, S., & Branco, N. (2016). http://dx.doi.org/10.6018/reifop.19.2.254851

[26] Rosli, R., Goldsby, D., Onwuegbuzie, A. J., Capraro, M. M., Capraro, R. M., & Gonzalez, E. G. Y. (2020). https://doi.org/10.22342/jme.11.1.9482.59-76

The first category includes tasks involving the process of lesson planning, implementation and analysis, or recognisable parts of this process. This category (with 42,3% of articles containing tasks associated with this type of strategy) comprises four sub-categories: lesson study, task design and classroom resources, lesson implementation, and teaching portfolios (Table 1).

Three of the 26 papers describe the use of lesson study with PSTs. The approach involves setting learning goals, identifying areas of potential difficulty for students, selecting or creating tasks, managing the implementation phase, and reflecting on student learning and the effectiveness of the task [1]. In other instances (seven papers), training strategy focused solely on planning stage. Certain

tasks of this type require the lesson plan to respond to a classroom situation presented in a video excerpt, in terms of focusing on students' needs regarding an item of mathematical content [7].

The second category, with 38,5% of articles containing tasks associated with this type of strategy, consists of tasks that place an emphasis on immersing PSTs in classroom situations as critical observers. In this instance, three forms of observation can be distinguished: those in which PSTs attend actual lessons [20]; those using video-recordings of real lessons; and those which consider hypothetical situations created by the researchers (Table 1). In video-based analysis strategies, PSTs may be required to observe both students' learning and teachers' performance ([12], [13], [14], [15]), or solely student discourse and responses to content matter, as well as preservice hypothetical reactions to a given situation [7]. Several variations on pure lesson observation can be noted. These have in common the presentation of a hypothetical situation given by researchers, but vary in the constitution of their subsequent elements. It can range from presenting a hypothetical classroom context, learning goals and classroom tasks (Figure 1), to including interventions on the part of the pupils. These situations are used as a springboard for PSTs to describe how the students might demonstrate their understanding (Figure 1), interpret the students' reasoning and describe how they would respond as teachers, choose a classroom task to fulfil a particular learning goal, provide an appropriate solution to the classroom task, and evaluate the learning potential of a task.

> *You have just completed a series of lessons with your fifth-grade mathematics class with the following learning goals: "The pupils will understand how to add fractions and will understand the concepts underlying this operation". You want to give the students a problem to assess whether you have achieved this learning goal. You are considering the following problems. (Your students are familiar with all these materials).*
>
> *Problem 1: Solve 1/4 + 3/8 using your fraction pieces.*
> *Problem 2: Solve 1/4 + 3/8 drawing a diagram on graph paper.*
> *Problem 3: Solve 1/4 + 3/8 using the common denominator algorithm.*
> *Problem 4: Solve 1/4 + 3/8 using pennies, where 8 pennies equal 1.*
>
> *Sue is a pupil in your class. Imagine that Sue has achieved the learning goal. Explain how Sue would answer each of the four problems: What (exactly) would Sue have to do and say to convince you that she understands the concepts underlying the addition of fractions?*

Figure 1: Extract from the training task in Kang & Liu (2018, p. 131)

The third category, with 19,2% of articles containing tasks associated with this type of strategy, consists of training tasks focused on the analysis, as opposed to the design of lesson activities and resources. It includes a sub-category, comprising approaches based solely on the analysis of classroom resources, but none of the papers in the review were found to include such an approach. Some papers describe approaches in which PSTs are given a mathematical problem to solve, and are then asked to identify its goals and the mathematical processes involved [21], or to suggest what learning goals the teacher that set it might have [22].

The fourth category, with a representation of 7,7% in the 26 articles, of special interest for its capacity to promote PSTs' pedagogical imagination (Skovsmose et al., 2023) and also not included in Clarke et al. (2009), consists of tasks in which it is the PSTs who are required to create hypothetical

situations. In one such task the PSTs were asked to draw a picture of a classroom environment in which they are teaching, and the students are learning relevant content [24] (Figure 2). Another asks the PSTs imagine hypothetical student answers to given mathematical tasks [22].

> *Assume that you are teaching mathematical content (e.g., addition, fractions, ratios, etc.) to your students. Draw a classroom environment in which you are teaching, and students are learning relevant content. … What is the teacher doing in the picture? What are the students doing in the picture? Which types of resources (mathematics tasks, activities, tools) are* [sic] *the teacher using in the picture?*

Figure 2: Extract from the training task in Lin (2022, p. 495)

In one of the papers included in the review, the authors describe a teacher education approach that requires PSTs to undertake research on an aspect of their teaching practice [25]. We consider this approach as a separate category, as it brings to the fore the formulation of research questions about one's own practice, and the concomitant process of data collection and analysis, as distinct from the process of reflecting on one's practice characteristic of the first category above.

The sixth category comprises teacher education tasks based on the consideration of students' responses (whether early years or primary education) ([21], [22]). It has a representation of 7,7% in the 26 articles. In this type of task PSTs are faced with hypothetical student answers to a given mathematical task, and are required to summarise the students' reasoning, or explain thinking underlying any potential errors they might make. We distinguish this category from the fourth, that of hypothetical situations, as rather than presenting a typical classroom scenario, the training tasks here consider solely the possible student answers to a classroom activity. As such, the tasks in this category are perhaps less impactful as those which attempt to immerse the PSTs in actual lesson contexts, such as the first and second categories.

The final category is composed of tasks requiring PSTs to solve mathematical problems (representing 26,9% of all papers). It is worth noting that, of all the papers retrieved in the systematic search, only one included this approach as the only strategy under consideration [26]. In some of the remaining papers, the task commenced with finding the solution to a problem but then followed this up with analysis of the pedagogical implications underlying it and/or potential student answers. In other papers, the solution occupied the latter stages of the task and was preceded by a planning phase.

Seeking possible relationships with the above types of tasks, we analysed the PCK components that each paper aims to develop, following those identified in Depaepe et al. (2013). Thus, considering the results of this review, tasks involving *imagining hypothetical situations*, followed by tasks involving *analysis of classroom situations* and *analysis of tasks and resources*, place the greatest emphasis on developing knowledge about students' (mis)conceptions and difficulties. This second type of task, *analysing classroom situations*, is even more important in developing knowledge about instructional strategies and representations, while the first type (*design, implementation and reflection*) focuses on knowledge about mathematical tasks. In the above, we have considered *solving mathematical tasks* separately because they coincide in placing the same emphasis on knowledge about students and about teaching (strategies and tasks). The same applies to the only *research into*

practice task. Only three types of tasks address knowledge about curriculum and media, with little coincidence in this focus among tasks of the same type.

On the other hand, from the size of the groups of preservice teachers tackling the formative tasks (Table 1), we can extract strategies that seem viable for large groups (more than 70 students), such as *task design and classroom resources*, *video-based analysis*, *drawing tasks*, and *solving mathematical tasks*, each strategy being of a different type. In contrast, the strategies of *lesson study* and *lesson implementation* are only reported for groups of up to 10 PSTs.

Considering the continents in which the teacher-training task were implemented, 12 of the 26 articles report on training tasks implemented in Europe (Portugal and Spain), 6 papers refers to training tasks implemented in Asia-Oceania, 5 papers in North America (all of them in United States), 3 papers in South America and only one paper in Africa (South Africa).

Final reflections

These seven types of teacher education task aim to develop PCK in different ways. Among them are approaches which situate PSTs in the context of the teacher's working life, requiring them to plan and implement lessons, and to reflect on their performance. Other approaches promote learning how to teach through the analysis of classroom tasks, resources, pupils' answers, and discussion of classroom scenarios. The strategy in which PSTs take on the role of researchers goes beyond analysis and reflection and requires them to gather evidence from their own practice as the basis of their enquiry. Other approaches use both actual and imagined situations. Or it might be that the PSTs themselves provide the hypothetical scenario. One strategy gives PSTs the role of problem-solver to develop their mathematical awareness and is utilised mainly in combination with other strategies.

In all instances, the aim is to develop PCK by means of reflecting and taking decisions. The approaches thus provide opportunities to learn PCK through the use of pedagogical reasoning.

With respect to approaches included in Clarke et al. (2009), four new categories have been added: analysis of tasks and classroom resources; imagining of hypothetical situations; research into practice; and analysis of pupils' work. Of particular interest among the tasks involving imagined situations, in our view, are those in which PSTs imagine their future teaching selves. In contrast to teacher education tasks based on existing situations, which essentially promote reflection, such activities stimulate the practice of imagining what has yet to come into existence. This extension of PSTs' reflection on the past and present into future brings an important added dimension (Chapman, 2008).

The types of tasks, the relationships found between types and elements of PCK, as well as strategies related to the number of PSTs the task is aimed at, can serve as guidelines for teacher trainers when designing or adapting training tasks. Many other questions remain open, such as the possible connections between different training strategies and knowledge models conceptualising PCK.

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Meta-didactical praxeologies in a teacher education pathway: The dynamics of sharing between teachers and researchers

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The present study describes a teacher education pathway with 16 Italian in-service primary and middle school teachers. The pathway, designed within a research–action collaboration, centred on mathematical activities related to sun shadows, integrating Outdoor Education and digital technologies. Drawing on the framework of Meta-Didactical Transposition, we analyse moments in which teachers and didacticians shared reflections within communities of inquiry. Using Qualitative Content Analysis of audio and video recordings, we investigate how meta-didactical praxeologies are shared during the design and implementation of classroom activities. The key dynamics include the potential transformation of praxeologies through shared reflection, and the emergence of divergent positions between didacticians and teachers during this sharing process.

Keywords: Teacher education, meta-didactical transposition, shared praxeologies.

Introduction

The concept of *communities of inquiry* (Jaworski, 2006) refers to spaces where questioning one's own perspectives and opening up to new ones become opportunities for professional growth. In this contribution, we present a year-long teacher education pathway, focusing on moments of shared reflection between teachers and researchers in mathematics education. We refer to these researchers as didacticians (Jaworski, 2006). They work directly with the teachers to foster richer opportunities for students' mathematical learning. Such dynamics foster the development of new practices for both groups, who collaborate *as learners* (Wagner, 1997). Through discussions, teachers are encouraged to reflect critically on their teaching practices and on the mathematical theories that underpin them; didacticians, in turn, have the opportunity to examine the critical aspects of the proposed pathway, maintaining a continuous dialogue with the concrete teaching experience contributed by teachers, who observe students' responses to the jointly developed activities on a daily basis.

We argue that this collaboration provides the foundation for a teacher education pathway that is both meaningful for teachers and able to evolve over time. Central to the design is the creation of conditions that foster reflective processes, sustained through dialogue with colleagues and mathematics education didacticians.

Theoretical background

To examine these reflective processes that take place within a community of inquiry, we draw on the Meta-Didactic Transposition (MDT) model (Aldon et al., 2013; Arzarello et al., 2014). The model is rooted in the Anthropological Theory of Didactics (ATD), which conceptualises knowledge and practice through the notion of praxeology (Chevallard, 1999; Chevallard & Bosch, 2020). A praxeology, developed within specific institutional contexts, is structured around two fundamental

dimensions (García et al., 2006): *praxis*, or "know-how", which includes tasks and techniques, and *logos*, the discourses and theories that justify and support those techniques.

The ATD describes mathematical and didactical praxeologies, but in the context of teacher education, meta-didactical praxeologies also play a significant role. These emerge from teachers' and researchers' reflections on teaching activities, and it is the sharing of such reflections that triggers the generation of new praxeologies for both groups.

According to the MDT framework, a teacher education program should aim to develop teachers' existing praxeologies towards new ones, which consist of a blending of the two initial praxeologies that reasearchers and teachers have at the start of the teacher education program (Figure 1). The result of this interaction of teachers with the community of researchers it's called shared praxeology (Arzarello et al., 2014). In this paper, the community of mathematics education researchers is represented by the didacticians, who are themselves researchers in this field.

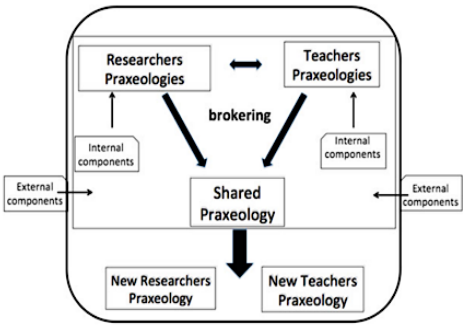

Figure 1: The Meta-didactical Transposition Model (Arzarello et al., 2014)

Method

The teacher education pathway (a process that comprises both teacher education activities and teaching experiments) involved 16 volunteer primary and middle school teachers (grades 1 to 8) from the same Italian school. This convenience sample, together with the *didacticians* from the University of Eastern Piedmont, forms a research–action group that has been collaborating for about ten years.

Building on the theoretical framework outlined in the previous section, the present study analyses the data that emerged during the pathway for in-service teachers in order to address the following research question: *Through what dynamics are meta-didactical praxeologies shared within a community of inquiry during a teacher education pathway?*

To investigate such question, we adopted a qualitative approach, namely *Qualitative Content Analysis* (Mayring, 2015). The analysis was carried out deductively on audio and video recordings of meetings with teachers, with the aim of identifying the didactical and meta-didactical praxeologies at play. In addition, the MDT framework was employed to describe the choices made by the *didacticians* from a teacher education design perspective.

Design and analysis of the teacher education pathway

In previous years, this group of teachers and didacticians has been exploring the topic of mathematical argumentation in different contexts. In 2024, the didacticians proposed a pathway to the group of

teachers that opened up new directions and topics compared to prior years, focusing on mathematical activities related to the shadows of the sun. The idea was to embed a mathematics education programme within a specific field of experience, fully exploiting its potential through outdoor activities carried out in direct contact with the phenomenon itself. The present work details the proposed pathway and explores some outcomes of the year 2024/2025 of shared work between didacticians and teachers.

The didacticians therefore proposed an initial phase centred on three main topics: (1) the field of experience of sun shadows, from both a mathematical and didactical perspective; (2) Outdoor Education (Priest, 1986), as a teaching practice for designing and conducting meaningful learning experiences outside the classroom; (3) digital technologies, regarded as contemporary tools that can support the exploration of the field of experience both outdoors and in the classroom.

In the following section, we outline praxeologies enacted by the didacticians in defining the pathway and in designing a set of tasks for teachers, with the aim of guiding the teachers along a coherent path towards the acquisition of knowledge that, it is hoped, will prove useful for their future teaching design and implementation. Subsequently, and for reasons of space, we present excerpts from the fourth and fifth tasks only, illustrating meta-didactical praxeologies that emerged through the interaction between a didactician and several teachers.

First task: The sun and shadows

The first task proposed to the teachers was to study descriptive materials on mathematical activities involving the shadows of the sun, re-examined through the lens of Outdoor Education (OE). These materials, produced and progressively refined over the years by a research group led by Boero and composed of both researchers and teachers (Boero et al., 1995), include not only teaching sequences designed for different school levels and centred on "doing mathematics" with sun shadows, but also pedagogical reflections intended to support teachers in implementing such activities in their classrooms. For this task, the teachers were organised into heterogeneous groups, bringing together both primary and middle school teachers.

Second task: Digital technologies

The second task proposed to the teachers was designed to introduce, through a practical activity, the theoretical framework of instrumental genesis (Rabardel, 1995). The teachers were asked to explore *Sunlitt*, an application closely related to the field of experience of sun shadows, though not originally developed for educational purposes. Their analysis of the app was supported by a table prepared by the didacticians, intended to guide their work step by step. The table first invited teachers to describe the application as an artefact and to identify possible utilization schemes (Rabardel, 1995). For this task as well, the teachers continued working in heterogeneous groups.

Third task: Designing new activities

The introduction of digital tools, such as the *Sunlitt* app, aimed to prompt teachers to rethink sun-shadow activities in today's educational context and to open discussion on technologies, previously absent in this group. The guiding question was: how might these activities change if supported by tools usable both indoors and outdoors?

Teachers were then asked to design a short mathematics–science pathway, centred on activities involving the shadows of the sun and integrating *Sunlitt*, using a template provided by the didacticians and working in heterogenous group.

Fourth task: Designing teaching experiments

The first part of the teacher education pathway, which was more theoretical in nature and aimed at introducing theories and methods from the world of research, has come to an end. However, as we have seen, it was not limited to lectures, but also included workshop activities, which were made more relevant to teachers' daily work thanks to the use of boundary objects that accompanied the proposals.

We now describe the more operational phase, in which teachers are called upon to put into practice the theories they have encountered at the suggestion of the didacticians. They must design teaching programmes for their own classes, to be integrated into the reference curriculum.

In the first phase, the teachers worked together, with a vertical and shared approach to planning. The aim was to identify a common core for all, so as to avoid excessive dispersion among the designs, especially in view of the final sharing moment. This approach also allowed teachers to support each other during the implementation phase. Once the common core had been defined, the group was then divided into primary and lower secondary subgroups, thus forming communities of inquiry that were more homogeneous in terms of context, curriculum and teaching objectives.

Fifth task: Implementation

Once the designs had been finalised, the teachers implemented them in their classrooms. Even before the start of the implementation phase, however, they felt the need to request and organise meetings with the didacticians.

These meetings provided an opportunity for in-depth discussion of teaching practices and of the most effective ways to present the activities to students. They also fostered a collaborative process of continuous redesign of the activities, carried out both in response to students' feedback and in consideration of external factors—such as weather conditions—that inevitably influence activities involving shadows and sunlight.

At the end of the implementation phase, the teachers met again to share their reflections, both on the concrete experience of putting the designs into practice and on the theories introduced during the initial phase of the pathway.

The tasks described above, which together constitute the teacher education pathway, encompassed different cycles of sharing meta-didactical praxeologies. In the next section, we focus specifically on describing the meta-didactical praxeologies teachers and didacticians shared between the fourth and fifth tasks.

Analysis and discussion

During the meetings organised to redesign activities during the implementation phase, the focus was primarily on praxis, particularly on how to propose tasks and on the problem-solving techniques expected from students or towards which they should be guided. Less attention was devoted to logos,

partly because these relatively short meetings were mainly intended for sharing reports of classroom activities and for redesigning them in light of students' responses.

Below we present an excerpt from didactician R., who discusses with teachers some possible teaching difficulties related to a subsequent activity. In this activity, teachers were asked to take measurements with the gnomon[1] in their own class, then exchange the collected data across several classes and compare up to three gnomons of different heights. The potential difficulty lies in the fact that the heights of the three gnomons (11 cm, 12.5 cm, and 15 cm, respectively) do not easily support the identification of proportional relationships between the gnomons' heights and the lengths of their shadows; rather, they risk producing an excess of data that may be difficult for students to interpret.

R.: If it (the gnomon) were 10 and 15, it would objectively be easier to relate to 10 and 15. So, in my opinion, they will have to think about it for a moment. If one had been double the other, like 10 and 20, then it would have been much more immediate, they would have seen it almost immediately. Because we are used to double, half, a quarter... there it is much more difficult. Then, of course, you are there with them to help them, and you can shift their attention to "what is invariant" but always passing through the shadow triangle. [...] What changes? If one is always the same height, the other piece changes.

The didactician reflects on alternative ways in which the task of collecting data with gnomons could have been presented to students, with the aim of facilitating the recognition of proportional relationships between the heights of the gnomons and the lengths of their shadows. The choice to work with gnomons of different heights is acknowledged as educationally valuable, but also potentially problematic, since the selected measurements were not specifically designed to support the explicit development of proportional reasoning in students.

The ongoing meetings had essentially practical purposes, providing opportunities for teachers and didacticians to engage in dialogue with the aim of ensuring the successful implementation of the activities. At the end of the experience, we met again to share the reflections that had emerged both during the classroom implementation and throughout the ongoing meetings between teachers and didacticians. Here a statement by a teacher (B.).

B.: Because when the children saw these figures, they weren't intuitive for them, and the problem of the length of the gnomon had already been raised at the beginning, because if they had been double or triple ... the figures would probably have been simpler. So, when faced with this table, they were a little lost, so I had to guide them a lot. So, what did I do? I calculated the ratio between the shadow and the gnomon for the short, medium and long [...]. Then we calculated the proportions. We did it on this Excel sheet.

The excerpt from B. refers back to the discussion on teaching practices conducted during the ongoing redesign phase with teachers. It seems that the teacher was prompted by the R.'s shared reflections on *praxis* as a possible alternative for addressing the challenges arising from the random selection of

[1] A gnomon is a nail or a stylus whose shadow (or the tip of the shadow) is used to indicate the hours on a sundial. In our work, we used this instrument to reflect on its shadow and observe how it changes over the course of a day and across different days.

gnomon heights. This was confirmed by her observation of the difficulties encountered by her students, which led her to revise her own technique for tackling the proposed task with the class.

Having discussed that, for a given time of day, the ratio between the gnomon's height and the shadow's length remains constant, teachers assigned to students a final task: to determine the shadow length of a 27.5 cm gnomon using measurements taken at 13:10 (1:10 p.m). During the final meeting after implementation, significant exchanges emerged between the didacticians and the teachers. R., for example, discussed a protocol shared by the teachers (Figure 2) and offered a reflection on the meaning of ratio in relation to proportional thinking, which students generally find difficult to grasp and internalise. This difficulty often arises because students remain anchored to a mechanical approach, focused on setting up a proportion, without truly understanding it as the equality between two constant ratios.

R.: The big problem is that they haven't linked it to what you want to teach as a proportion, which is different. [...] They don't see an equality of ratios. They don't feel it as such because the procedure they use to solve it is not a ratio. Think about it. They can understand it as meaning when they see a triangle, but when they hear the numbers, what do they do? They multiply the product of the extremes by the product of the means. They don't feel that this is a constant ratio. They feel something else. […] Look at this (pointing to Figure 2). Here it is quite clear that they knew what they wanted to do, they have an idea. But the problem is that since they have to apply this blessed proportion, they apply it with numbers even without a logical criterion behind it, and so 13:10, which is a time for them, suddenly becomes a proportion because the symbol is the same.

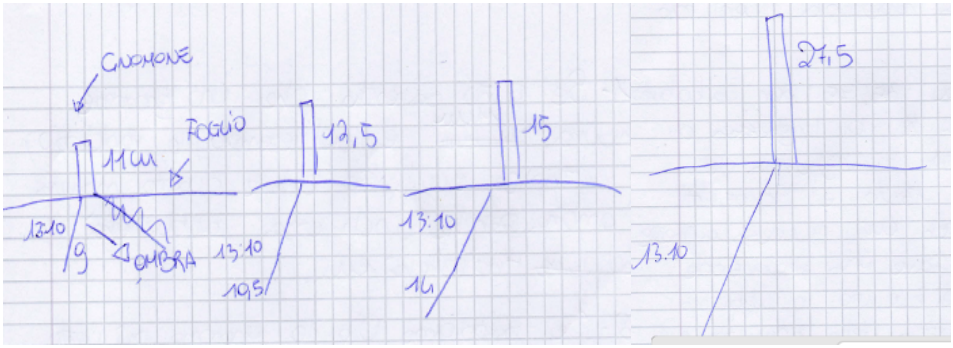

Figure 2: A representation made by a student of four different gnomons and their shadows at the same time (13:10) of the same day

The reflection by R. relates both to *praxis*—the ways in which proportional thinking is typically taught in schools—and to *logos*, as it stresses that, from a mathematical perspective, proportional thinking is not merely the mechanical application of a procedure. Rather, it entails recognising a linear relationship between a system of two variables, which enables one to draw conclusions about a situation or phenomenon that can be characterised by a constant ratio (Karplus et al., 1983, p. 219).

A teacher (P.) responds to R.'s reflection by offering a personal perspective that contrasts with the didactician's position.

P.: At least as far as I am concerned, but for everyone, the definition of proportion is the equality of two ratios. So we give it that way, we explain it that way, and we start doing everything that way. Then, when in a class of 22, three understand this concept, I start to say, OK, but mechanically, that is to that as that is to the other.

> [...] you have to put corresponding things in proportion, and they more or less understand this, but if you explain it to them simply as a ratio between numbers that remains constant, in my experience, only three or four people really understand it. So, if the aim is to make them really understand what proportion is, I understand why we should focus on that. If the aim is to apply proportions to practical situations, even for those who are not particularly gifted, in my opinion, at an educational level, we say that one is to the other as the other is to the other.

This teacher's viewpoint, grounded in everyday teaching practice, is more concrete and focused on the practical management of exercises that require proportional reasoning. P. points out that, in his experience, presenting proportion as the equality of two ratios is understood by only a few students, whereas a more mechanical approach—such as "this is to this as that is to that"—proves more accessible and enables a larger number of students to apply proportions in practical contexts. Such exchange highlights divergent meta-didactical praxeologies: on the one hand, the didactician's theoretical stance, and on the other, the teacher's pragmatic approach, which remain apart and struggle to converge on common ground.

Conclusions

In relation to our research question, we have identified that some of the praxeologies shared by the teachers were didactical (Garcia et al., 2006), both in the discussion about designing and implementation. These were primarily aimed at specifying the choice of activities to propose to students and the ways in which to conduct them (*praxis*). In the final feedback meetings, after the actual implementation of their designs, the teachers shared with didacticians some reflections on their didactical praxeologies, which we identify as meta-didactical praxeologies (Arzarello et al., 2014), showing increased attention to mathematical content, in particular proportional thinking, which the field of experience of sun shadows was able to foster. This led to a deeper reflection on *logos*, which, however, was never detached from *praxis*.

The dynamics of sharing meta-didactical praxeologies that characterised this process are, in our view, particularly significant, as they illustrate both the potential transformation of pre-existing didactical praxeologies—as in the case of teacher B.—and the emergence of divergent positions between didacticians and teachers, as in the case of teacher P. While not exhaustive, the analysis provides valuable insights. It will be important to extend the study with a broader investigation that encompasses all the moments of the pathway in which praxeologies come to the fore. Such an extension would not only enable a deeper research perspective, but also offer indications, from a teacher education design standpoint, of which elements proved effective and which aspects require revision in order to foster more productive sharing and deeper reflection.

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A transfer process centered model for effective professional development programs

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Research in mathematics education clearly indicates that using digital learning environments helps to develop a conceptual understanding. However, schools have not exploited these findings so far. Teachers state that there is a no visible agreement on the tenets of technology-enhanced teaching (TET) and note a lack of respective professional development programs and teaching materials. The key issue is to translate research findings into teaching practice. The TET professional development program (TPDP) MaTeGnu for upper secondary mathematics is conceptualized to facilitate the crucial transfer process from TPDP to teaching. In this paper we present the underlying transfer process model for TPDP and the results of two evaluation studies of the TPDP MaTeGnu drawing on two impact levels in the transfer process model, teaching practice and student learning. The findings provide significant empirical evidence on the effectiveness of the MaTeGnu concept.

Keywords: Professional growth, professional development program, evaluation, conceptual understanding, PDP transfer process.

TPDP as driver for the advancement of teaching

Teachers can act as intermediaries for the implementation of research findings in teaching practice. The initial and secondary phases of teacher education form the basis, while in the tertiary phase professional development programs provides ongoing support for teachers throughout their careers, helping them to implement subject-specific teaching concepts. However, for PDPs to be effective, the design must incorporate elements that genuinely reflect teachers' teaching practices. In their systematic review on PDPs with a special focus on TPDPs, Lipowsky and Rzejak (2019) identified ten essential characteristics for effective PDPs, among them integration of input, testing, feedback, and reflection phases as well as promotion of cooperation among teachers. These characteristics are incorporated in an offer-and-use model for PDPs that identifies five categories of influencing factors and success metrics of PDPs. These are the quality and quantity of learning opportunities (offer), the characteristics of the teacher educators, the perception and utilization of learning opportunities by participants(use), the characteristics of participants and the school context. All these factors influence the transfer process, which in turn predominantly moderates the effects of PDP, that are described by four levels of success, leaning on Kirkpatrick's four levels of training evaluation (Kirkpatrick & Kirkpatrick, 2016). Although Lipowsky and Rzejak (2015, 2019) put the transfer process in the center of their offer-and-use-model, they do not elaborate on this pivotal element. In our proposed *TPDP transfer process model* (see Figure 1), we aim to answer the important question of how this transfer process should be designed to implement particularly effective TPDPs. It consists of four interrelated components, the *prerequisites*, the *offer*, the *transfer process*, and the *four levels of impact*. The teachers individual transfer process forms the focal point of the model. Teachers encounter new concepts and knowledge in the TPDP. Integrating these elements into one's individual teaching

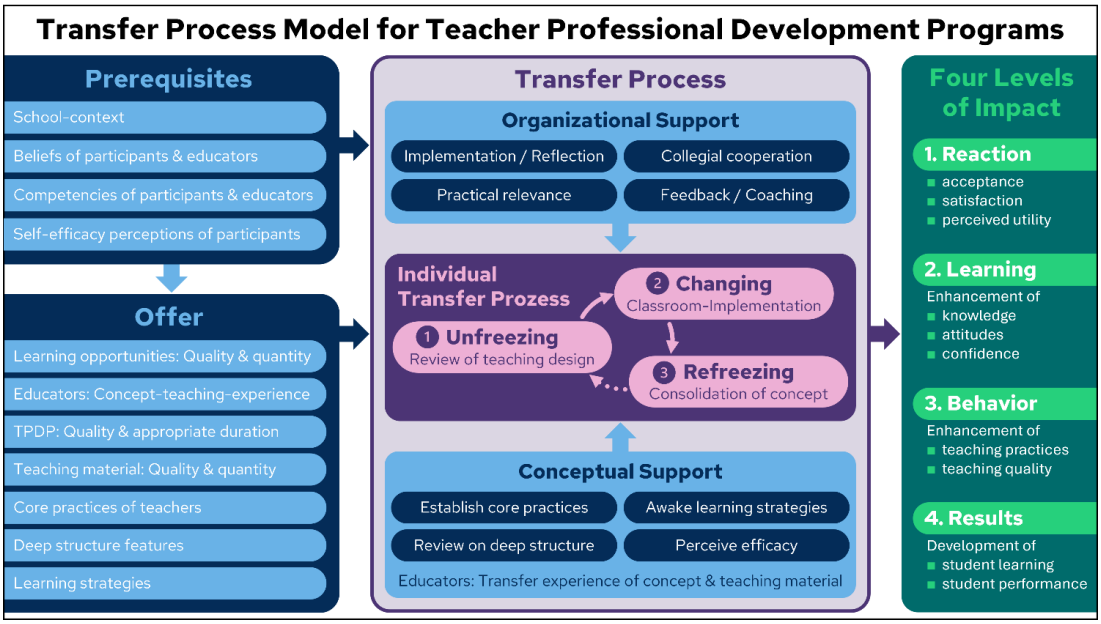

Figure 1: TPDP transfer process model (Digel and Roth, 2025)

practice is an *individual process*, that Fütterer et al. (2024) describe in three stages: *unfreezing*, *changing* and *freezing*. At first, teachers carefully review their teaching design, focusing on specific learning goals (*unfreezing*). Next, teachers adapt their teaching and review the classroom implementation iteratively (*changing*). As soon as they are convinced with the results, they consolidate their concept (*refreezing*). This complex endeavour demands for *organizational* and *conceptual support* that needs to be provided by the TPDP. Impact in turn can be identified on four levels: the participants perception of utility and relevance (*reaction*), their acquisition and enhancement of knowledge, skills and attitude (*learning*), the enhancement of teaching practices and quality (*behaviour*) and finally on the most remote level the advancements in learning and performance of students through adapted teaching practice (*results*). The *TPDP transfer process model* can only be outlined briefly here. For a more detailed discussion see Digel and Roth (2025).

Implementation of the TPDP transfer process model in MaTeGnu

MaTeGnu seeks to remediate current issues in upper mathematics education, namely significant deficits in mathematics skills (Rolfes et al., 2021), missing conceptual understanding and a focus on formalism in teaching. As conceptual understanding is rooted in the development of basic mental models (Greefrath et al., 2023), which in turn have proven to be beneficial in technology-enhanced teaching for learning mathematical concepts (Digel & Roth, 2022), MaTeGnu as a TPDP aims to exploit the potential of technology for instructional quality to support students' conceptual learning with a focus on basic mental models. The objective of MaTeGnu is to provide systematic support for teachers in upper secondary mathematics teaching, incorporating the topics derivatives, integrals, analytic geometry, matrix calculus, statistics, and exponential functions. To ensure a sufficient focus on subject-specific content and student learning processes and an appropriate relationship between the duration of the programme and the desired goals (Lipowsky & Rzejak, 2019), MaTeGnu TPDP covers the entire upper secondary school cycle, spanning three years (*organizational offer: appropriate duration*). As part of the *organizational offer* and *support* MaTeGnu literally accompanies the participating teachers with their upper secondary mathematics course through the

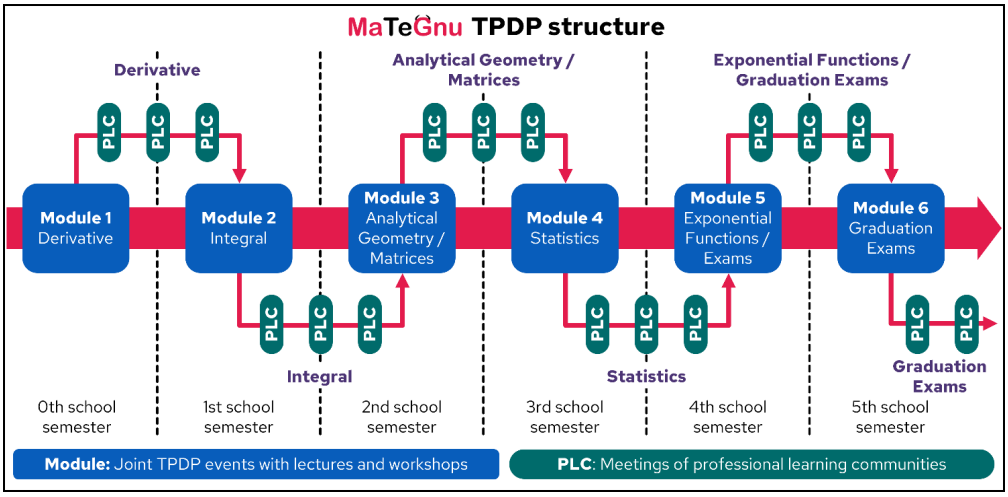

Figure 2: MaTeGnu TPDP structure (Digel and Roth, 2025)

complete upper secondary cycle (*organizational support: implementation*, *practical relevance)* through Module events and professional learning communities (see Figure 2), which enables professional cooperation, a key influential factor for PDP and student learning (Lomos et al., 2011).

The joint full-day TPDP events with lectures and workshops (*quality & quantity of learning opportunities*) address the teaching of the following time span until the next event with subject-specific (technological) pedagogical content knowledge and core practices, e.g. on basic mental models, digital learning environments and tool use (*offer: core practices of teachers, deep structure features, learning strategies*) and dedicated teaching material[1] (*offer: quality & quantity of teaching material*). The heart of the MaTeGnu transfer process form the PLCs led by a MaTeGnu teacher educator tandem, where continuous *conceptual* and *organizational support* is provided. MaTeGnu teachers seek and receive support for and exchange ideas about the implementation of the concept and teaching material (*coaching; collegial cooperation*), discuss (*feedback*) and reflect collectively on their teaching experiences and their transfer process (*implementation / reflection*), identify best practice and analyse student interaction (*perceive efficacy*) and receive individual support from MaTeGnu teacher educators (*feedback / coaching*). The MaTeGnu teacher educators themselves participated in the complete three-year MaTeGnu TPDP cycle, to gain extensive experience in the MaTeGnu teaching concept and material. Thus, the teaching material's practicality is assessed in different teaching contexts and refined as well as elaborated based on the individual transfer process experiences of the MaTeGnu teacher educators. (*Educators: concept-teaching-experience*).

A central objective of conceptual support is to develop and consolidate *core practices* for teaching with technology for understanding. These core practices target conceptual understanding with basic mental models and technology-enhanced teaching (e.g. foster self-guided *learning strategies* in digital learning environments; coach reflected digital tool use and documentation in problem solving and modelling), but also constructive alignment of teaching and assessment (Biggs, 1996) regarding both technology and conceptual understanding. Implementing this principle results in higher learning outcomes, higher perceived competence and less feeling of pressure (Leber et al., 2016).

[1] The teaching material is OER and the first module (derivatives) can be accessed under https://mategnu.de/m/l1e

Evaluation of the MaTeGnu concept

The evaluation of MaTeGnu TPDP also validates the underlying TPDP transfer process model and hence incorporates all four levels of training impact outlined above. In this article we present selected results on conceptual understanding in the content area of derivatives (module 1) from a first study, located on level four (results) and from a second study, located on level three (behavior).

The first study assesses student learning, precisely the conceptual understanding of derivatives. This fourth level of training impact, which is the furthest remote from the training itself, but at the same time mostly core objective of PDPs is seldom evaluated in studies concerning the effectiveness of training (Stahnke et al. 2025). To evaluate the actual effects of the TPDP on the students' learning, the study compared students in mathematics classes participating in MaTeGnu to those in regular mathematics classes regarding their conceptual understanding in the topic of derivatives.

> RQ1: Does the understanding of the concept of derivatives of students in MaTeGnu mathematics classes differ from that of students in regular mathematics classes?

The second study assesses the enhancement of teaching practice, more specifically integrating the crucial aspect of constructive alignment, it assesses the mathematics class exams as indicator for an aligned teaching and assessment practice based on conceptual understanding. Exam tasks for the topic derivatives of participating teachers before the start of the program are compared with those deployed during the TPDP implementation in the MaTeGnu mathematics class.

> RQ2: Are there differences in the assessment practice based on conceptual understanding of the participating teachers before and during MaTeGnu TPDP in the topic of differentiation?

Method

To address RQ1, a test on conceptual understanding of derivatives (CUD) based on Klinger (2018) is utilized together with a test on basic mental models of derivatives (BMMD) with items from Greefrath et al. (2023). The test is scheduled for 45 minutes and evaluates the learning outcomes of students after a series of lessons on derivatives. The combined test (CUD, BMMD) is carried out in a multi-matrix design, with a total of 58 items, distributed across six booklets with 14 anchor items. Due to limited space this paper presents the results on conceptual understanding (CUD), further results can be found in Digel and Roth (2025). The open answer items in the CUD test are coded with the solution criteria provided by Klinger's code manual (2018) and transferred to a dichotomous rating using predefined criteria thresholds. The dichotomous data from CUD is analyzed using IRT in a joint dataset of EG and CG. The person ability scores from the Rasch model are used for further analyses on EG and CG and subscales of the test. Cronbach's alpha is calculated for reliability, and a Welch-Test is performed to determine differences between experimental (EG) and control group (CG). The students from the *18* MaTeGnu mathematics classes of the MaTeGnu teacher educators form the experimental group (N_{EG}=*151*, *76* female, *75* male, *0* diverse; age *M=17.0*). Data was collected from June to December 2024. Prior to the start of the MaTeGnu, *45* mathematics classes from the schools of the teacher educators also participated in the CUD (control group. No MaTeGnu TPDP treatment took place before and during teaching of derivatives in this group (N_{CG}=*571*, *276* female, *294* male, *1* diverse, age *M=17.1*). Data was collected from June to November of 2023.

For RQ2 qualitative content analysis is used to analyze exam tasks. The assessment practice based on conceptual understanding is operationalized in the coding manual in two dimensions. The dimension activities examines which operators according to the operator list of the central final exams (IQB, 2019) are used, at which level of cognitive activities the task is located according to the hierarchy of cognitive activities (Barzel & Ebers, 2020) and if the task evokes merely automated procedures or demands for custom processing indicated by the structure of task types (Eilers & Bergmann, 2010). The dimension of conceptual understanding examines which basic mental models of derivatives (Greefrath et al., 2023) and concept-oriented skills (Hußmann & Prediger, 2010) the tasks address. The data presented here also refers to the evaluation study in the MaTeGnu teacher educator sample (*N=18*). Data was collected in Spring 2023 and the first half of 2024. To enhance reliability, the analysis is carried out as a mixed task pool pre and post ($N_{Pre} = 56$ and $N_{Post} = 54$). Intercoder reliability is determined from ten percent of double-coded tasks.

Results

Study 1 on conceptual understanding of students (level 4): The total CUD data shows good reliability in the Rasch model ($rel_{WLE} = 0.80$) and mostly acceptable reliability in the divided groups and subscales. Table 1 lists Cronbach's alpha values for reliability, mean values and standard deviations for each subscale in the control and experimental group and the results of the Welch-tests between both groups. In the MaTeGnu classes (EG) students show significantly higher conceptual understanding than students in the control group with a small to medium effect. The difference between EG and CG is highest in the subscale (SIT), formed by the contextualized tasks, with an almost medium effect. In the subscale for graphical derivations the scores in EG are slightly lower than in CG, but not significant with a discussable reliability of this subscale in the experimental group.

Table 1: Cronbach's alpha reliability, mean value, standard deviation and effect size of the CUD and its subscales in experimental (EG) and control group (CG)

	TOTAL		**SIT**		**GF**		**GD**	
	CG	EG	CG	EG	CG	EG	CG	EG
α_C	0.57	0.60	0.70	0.80	0.46	0.55	0.65	0.50
M	5.03	6.06	1.54	2.27	1.66	2.07	1.84	1.69
SD	8.20	9.01	2.46	2.73	1.83	2.50	2.42	1.93
Welch-Test	**t(225)=3.78 p<0.001**		**t(225)=4.92 p<0.001**		**t(210)=2.93 p<0.01**		t(257)=-1.08 p=0.140	
d	**0.346**		**0.451**		**0.268**		-0.099	

CUD Subscale SIT: situational context; GF: formal-symbolic with graph; GD: graphical derivation

α_C: Cronbach's alpha reliability; M: mean value; SD: standard deviation; d: Cohen's d effect size

Study 2 on assessment practice based on conceptual understanding (level 3): The double coding using the coding manual shows good intercoder reliability (*ICC = 0.74*). In the activities dimension (see Figure 3) at the first measurement point (PRE) approximately half of the tasks address the *lowest level* (level 1) of cognitive activation in the *procedural* scale. In addition to the expected number of *basic* structure types, there are also *reasoning* as well as *difficult determination* and reversal types in

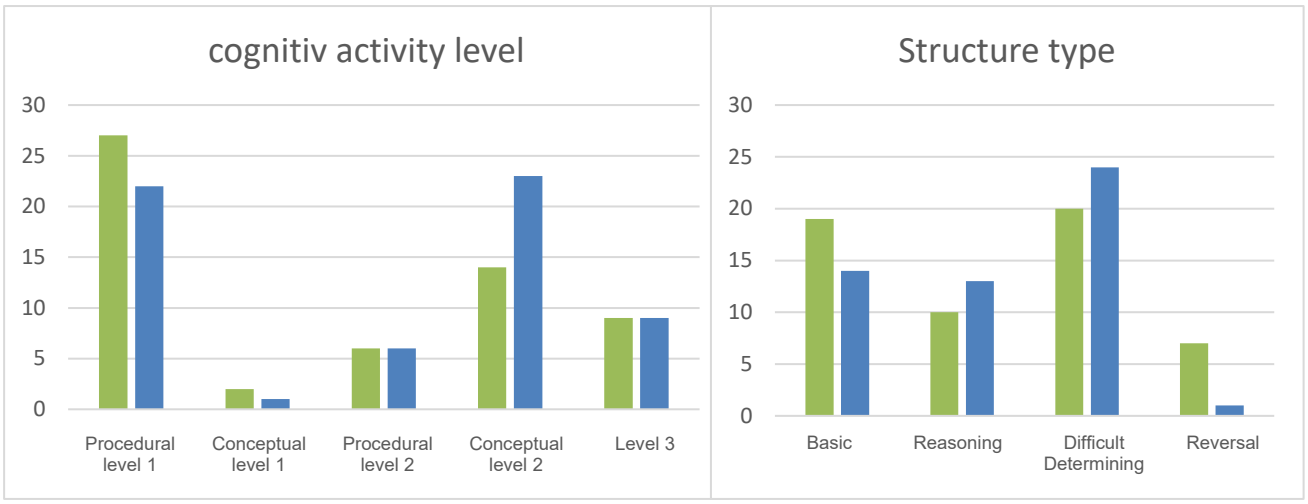

Figure 3: Level of cognitive activities in and types of tasks in CG and EG

the PRE tasks. The POST tasks show a shift from *procedural level 1* to *conceptual level 2* as well as a shift from *basic* structure type to *reasoning* and *difficult determination* types.

In the dimension of conceptual understanding (see Figure 4) there is a focus on basic mental model *tangent slope* in PRE (11 tasks), which decreases in POST (9 tasks). The basic mental model *local rate of change* is already present in six PRE tasks but still increases to eight POST tasks. The basic mental models *local linear approximation* and *amplification factor* are not present in PRE tasks, but the former is present in three POST tasks.

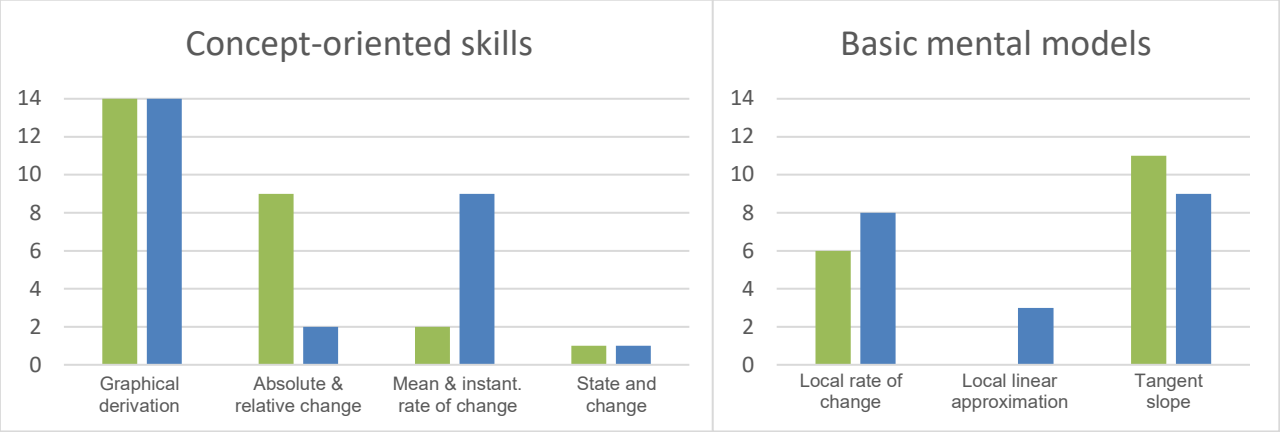

Figure 4: Tasks addressing basic mental models and concept-oriented skills in CG and EG

As for concept-oriented skills the total number of tasks addressing these skills does not change from PRE to POST. Tasks demanding *graphical derivation* are the most frequent and those requiring *distinction between state and change* are least frequent, both at the same level PRE and POST. The distribution of tasks addressing *absolute and relative change* and those addressing *mean and instantaneous rate of change* are inverted PRE to POST.

Discussion

For RQ1 we can conclude that MaTeGnu students show significantly higher conceptual understanding and score most different from others in contextualized tasks, where conceptual

knowledge must be applied. As for RQ2, a broader range of basic mental models is addressed in the MaTeGnu classes' exam tasks and more demanding concept-oriented skills (mean and instant rate of change) are required. In the activity dimension higher levels of cognitive activities are reached, and tasks shift from the procedural to the conceptual domain. The number of basic structure tasks decreases and makes room for more difficult determination and reasoning tasks. Regarding the MaTeGnu TPDP concept, the results at the most distant level of TPDP impact and the enhancement of conceptual teaching together with the attainment of constructive alignment underline that the focus on the transfer process in conjunction with *conceptual and organizational support* is effective. Improved student learning through TET with a focus on basic mental models indicates that the TPDP model aspects *deep structure features, learning strategies, quality & quantity of teaching material* (*conceptual offer and support)* seem to have promoted the transfer process. The enhancement of teaching practice (evident in the results of the second study) can likely be attributed to the practical TPDP model aspects *core practices, implementation & reflection, cooperation* and *relevance*. The evaluation cycle for the first cohort (currently in progress) is expected to provide further insights into the effects of *concept-teaching-experience, feedback & coaching* on the transfer process. Finally, the MaTeGnu implementation of the *TPDP transfer process model* uses technology as a conduit to foster conceptual understanding. Nevertheless, a technology-centered evaluation on the impact levels two to four is also subject to current research.

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Cultivating coherence in teacher education

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This article examines coherence in teacher education through a model called Campus Day, in which all students return to the university campus for one day each week during the seven-week practicum. The campus days are used for subject-matter courses and for reflection on how campus coursework and practicum are connected, aiming to strengthen the structural coherence in teacher education. We also use core practices as a tool to strengthen the conceptual coherence across campus courses and practicum. This is an ongoing project, and early results include the joint development by campus and practicum teachers of two local core practices, as well as students' reflections on how these core practices relate to their practicum.

Keywords: coherence, teacher education, core practices.

Literature

Teacher education has been consistently criticized for a lack of coherence (Hammerness, 2006). This mainly concerns the gap between university campus coursework and school practicum experiences. This divide between theory and practice is called the "two worlds pitfall" (Feiman-Nemser and Buchmann, 1985), and even after forty years, this problem remains. The tension between theory and practice discourages pre-service teachers from engaging in innovation, which helps maintain the status quo in schools. Improving coherence between theory and practice in teacher education has important implications for pre-service teachers' learning and, ultimately, student learning.

Hammerness (2006) suggests two aspects of coherence. The first is structural coherence, which involves the sequencing of courses and practicum, as well as what occurs simultaneously. The second is conceptual coherence, which describes the extent to which there is a shared vision of teaching and learning within a teacher education program. Arguably, structural coherence is a means of creating opportunities for collaboration to develop conceptually coherent programs, while the conceptual coherence aspect makes a difference to the content and quality of pre-service teachers' learning.

A strong coherence can be seen as a destination, but Richmond et al. (2019) argue for a more dynamic view of coherence, which is an ongoing process rather than a fixed outcome. This means that coherence can be viewed as a continuous negotiation and adaptation to meet the diverse needs of all stakeholders involved in teacher education, potentially also as a more democratic approach to building coherence (Floden et al., 2021; Philip et al., 2022).

To improve coherence in teacher education, international trends have focused on bridging the gap between academic knowledge and practical experience in various ways (Canrinus et al., 2019; Darling-Hammond, 2017; Hermansen, 2020). One trend is the global shift toward school-based teacher education models, such as partnerships, enhanced school experiences, and apprenticeships. Another trend that Norway has adopted since 2017 is establishing master's-based teacher education programs that combine academic learning with practice-oriented approaches (Advisory Panel for

Teacher Education, 2020; Smith, 2015). UiT, the Arctic University of Norway, piloted integrated, research-based, five-year master's programs for primary and middle school teachers (grades 1-7 and 5-10). This pilot laid the foundation for Norway's implementation of five-year master's programs across all teacher education courses, starting in 2017. Both the pilot and the Norwegian reform build on Finland's goal to create inquiry-oriented teachers (Toom et al., 2010), aiming to improve teacher quality through strong subject knowledge and research skills (Advisory Panel for Teacher Education, 2020; Jakhelln et al., 2019).

Generally, Norwegian programs excel at integrating subject matter with education (didactics) and preparing teachers with research competence (Munthe et al., 2024; Antonsen et al., 2024; Sæther et al., 2024; Drageset et al., 2025). Despite these advances, Norwegian teacher education programs still face criticism for lacking coherence between the practicum, campus, and research approaches (Hermansen, 2020). This issue is also evident internationally, where the gap between theory and practice remains despite a widespread understanding of the need for coherence in teacher education (Canrinus, 2024; Darling-Hammond, 2017). Therefore, innovation and research on the structures, curricula, and pedagogies of teacher education are essential to increase its relevance (Toom & Husu, 2024). While it is clear that coherence is vital, there is little understanding of why it is not being addressed more explicitly in the Norwegian context.

In recent years, teacher educators have introduced *core practices* programs (Grossman et al., 2009), also known as *high-leverage teaching practices* (Ball et al., 2009), as a practice-based method for creating more coherent teacher education. This approach examines what teachers do and characterizes teaching practice as a set of specific practices. By breaking down teaching practice this way, it becomes easier to focus on, investigate, and rehearse smaller, more manageable parts (Grossman, Compton, et al., 2009). The goal is not to create an exhaustive list of core practices, but rather to offer a way of thinking about learning to become a teacher and organizing pre-service teachers' learning more effectively. The idea of core practices builds on traditional methods used by teacher educators during practicums: highlighting specific teacher tasks, such as presenting, assessing, or establishing routines, and encouraging students to rehearse them. By establishing shared core practices across campus courses and practicums, we can develop a common language and set of learning goals, moving toward a shared vision.

The core practices approach has also faced criticism. One common critique is that core practices can oversimplify the complexity of teaching, reducing it to a set of prescriptive techniques and overlooking the complex, context-dependent nature of effective teaching (Philip et al., 2019; Zeichner, 2012). Another criticism is that core practices fail to incorporate work on social justice in teaching (Philip et al., 2019). This criticism is countered by arguments that pre-service teachers can learn to teach responsively within different contexts (Baldinger & Munson, 2020; Lee & Yeo, 2022; Munson et al., 2021) and can develop both routines and reasoning (Campbell & Baldinger, 2022).

The idea of core practices is easily linked to the mathematics education research literature. One example is the first high-leverage teaching practice recommended by Teaching Works (TeachingWorks, n.d.): Learning to lead a group discussion. The goal of such a discussion is to use each other's ideas as resources to build knowledge. This is achieved by sharing ideas, explaining and

arguing for them, and listening to others to discuss those ideas. There is extensive literature on how to lead discussions in the classroom and on how students contribute (e.g., Erath et al., 2021). Such a discussion also connects to themes like social and sociomathematical norms (Yackel & Cobb, 1996) and positioning theory (Drageset & Ell, 2024). This demonstrates that core practices can be rehearsed during the practicum and also be integrated into campus mathematics education courses.

Method

This article describes an ongoing project at UiT The Arctic University of Tromsø called Focus on Practice. The project employs two models, *Campus Day* and *Micro Practicum*, to enhance coherence. In Norway, practicum periods during teacher education typically last three to four weeks in schools each semester. During these weeks, students have no contact with the campus, and during campus periods, they do not engage with the practicum. Our model seeks to connect the campus and practicum in both directions. Campus Day serves as a link between the practicum and campus by holding sessions on campus every Wednesday during the practicum. Campus Day includes campus courses related to students' subjects, where experiences from practicum are discussed in context, along with reflection sessions for students across different classes focusing on cross-disciplinary topics, ethics, and dilemmas. The practicum teachers join pre-service teachers on selected campus days. Micro Practicum occurs within courses, where pre-service teachers visit their practicum classrooms for a day or a few hours, two to four times each semester, applying what they have learned on campus (e.g., teaching, observing, assessing).

On Campus Days, we use core practices as a tool for improving conceptual coherence. To develop core practices that were locally grounded, we drew inspiration from Teaching Works (TeachingWorks, n.d.) to create our own. First, the project leadership drafted two core practices, which were then further developed and refined through collaboration between campus teachers and practicum teachers. The goal was to develop an operationalization of these core practices that was meaningful for use and rehearsal in both the practicum and campus courses across various subjects. These were outlined in lists, where each bullet point could be rehearsed as smaller parts of the core practices (see figures 1 and 2).

FORMATIVE ASSESSMENT: CAPTURING AND UNDERSTANDING STUNDENTS' THINKING

Gain access to student thinking by asking questions and assigning tasks, encouraging students to elaborate on their thoughts and explain their reasoning.

Position students as sensible thinkers with valuable ideas.

Be careful with telling, correcting, or adjusting.

Attempt to understand the students thinking, ideas, logic, and reasoning, and consider whether there could be alternative interpretations of the student's thinking.

Use the collected information as a basis to give formative assessment.

Figure 1: Formative assessment: Capturing and understanding students' thinking

CLASSROOM CONVERSATION

Take the role as facilitator, invite students into the discussion, give them time and space to develop ideas (thinking time, turn and talk) and formulate them.

Consciously use talk moves, such as genuine questions, elaborating, clarifying, revoicing, making connections, request justification, challenge.

Involve as many students as possible into the conversation.

Assure that the conversation is respectful and tolerant to other perspectives.

Facilitate for the students to practice conversation skills, such as sharing thoughts, elaborate on and justify ideas, active listening, interpreting others' statements, expressing agreement or disagreement.

Figure 2: Classroom conversation

Focus on Practice works with all students in the first two years of the five-year master's program for teachers at grades 5-10. This report builds on the work done in the second year of the program, involving a cohort of 14 students, with an emphasis on the Campus Day model. Our research question is: *How does the Campus Day model, through the use of core practices, improve coherence between campus courses and practicum?*

The data was collected during reflection sessions held on campus days. In total, each of the 14 students submitted three written reflections. They were asked about what they felt capable of, what was challenging, how they understood and applied core practices, and to provide open feedback.

The analysis was conducted using a reflective thematic analysis (Braun & Clarke, 2006). First, we transferred all the written reflections into NVivo. Then, we developed thematic categories to describe different types of reflection. The main categories were linked to questions, while we developed sub-categories to describe the variations within each category. The sub-categories were continuously revised to determine whether any utterances should be moved or if any sub-categories should be split or merged. Additionally, we consistently updated the sub-category names to ensure they accurately reflected the utterances within. In this article, we will report on the categories of Capability, Challenging, and Understanding of core practices. Since this article reports on an ongoing project, more data will be collected and analyzed, including oral group reflections. The results presented here are preliminary and capture what is clear at this early stage.

Early findings

Based on our data, we have two types of early results from our informants, who are second-year students in a five-year teacher education program. One consists of pre-service teachers' reflections on their abilities and the challenges they face, and the other includes their reflections on the usefulness of the two core practices.

Most of the participating pre-service teachers tell us that they are “capable of creating a good relationship with students.” This is a central topic in their pedagogy course, and most practicum teachers also emphasize it. Additionally, one student states that “I manage to communicate

knowledge in an understandable way to students, and also attempt to explain multiple times and simplify information to make it understandable." Many have similar statements where they express confidence in their ability to instruct or communicate subject matter.

At the same time, teaching the subject matter is also the most common challenge, for example, "I find modeling on the blackboard difficult; it's something we don't talk about or get training in at campus." The other main challenge is related to unwanted behavior.

> A: Correcting unwanted behavior in students with special needs can be challenging.
>
> B: I find it difficult to manage disruptive students. Every student needs a different response and consequence for their actions.

There are also several other capabilities and challenges mentioned by some, such as planning, leading a classroom, thinking on your feet, and self-efficacy.

The reflections on the use of the two core practices were helpful for gaining insight into how pre-service teachers applied or thought about them in relation to the practicum. Here are a few reflections on *formative assessment: capturing and understanding students' thinking*.

> C: Don't rush ahead; instead, take a bit more time to encourage the students to explain themselves. Ask questions like: What have you done? What were you thinking? How can we approach this task?
>
> D: Capture and understand students' ways of thinking. I feel like I've become more aware of this when students offer suggestions in class, especially in encouraging them to elaborate on their responses.
>
> E: Be brave and dare to try to understand student responses without imposing too many preconceived notions about how you think beforehand.
>
> F: Received tips on how to elicit students' strategies when they think and work. Understand that patience is an essential approach to delve deeper into exactly what the students are thinking.

As we can see, the use of core practices has influenced their approach to teaching, with an emphasis on probing (C, D, F), being patient (C, F), and using the time to discuss and gain insight into students' thinking (C). This suggests that core practices can be a tool for influencing pre-service teachers' learning, and that core practices can connect the campus (where they are introduced) more closely to the practicum (where they are rehearsed). Also, campus days ensure that they can rehearse and reflect multiple times over a short period.

Some reflections also showed a lack of understanding, by saying things such as "I understand core practices as the things a teacher does that are outside of ordinary teaching" and "I think that as a teacher, you are both an educator, conflict mediator, psychologist, nurse, conversation partner, caregiver, and all these are core practices." While these utterances were fewer, it is still interesting to see how the idea of core practices is misunderstood. The common misunderstanding was that core practices relate to tasks outside of teaching, as these two examples show.

Discussion and conclusion

This article explores the work with Campus Day as part of the larger project Focus on Practice. Campus days can be seen as facilitating what Hammerness (2006) refers to as structural coherence. By having students visit campus once a week, we create structural links between the campus and the practicum. With one campus day each week, preservice teachers can shift between theoretical reflection and rehearsals multiple times within a short period. Moreover, using core practices enhances conceptual coherence by creating shared practices emphasized both on campus and in the practicum. Our two core practices also help develop a shared language, which is a step toward a shared vision. In this way, core practices also address the two worlds pitfall (Feiman-Nemser & Buchmann, 1985) by integrating theory, such as positioning theory (Drageset & Ell, 2024) and classroom discourse (Erath et al., 2021), with rehearsals in practice.

While most students seemed to find valuable ways of incorporating core practices into their practicum, some misunderstood and confused core practices with non-teaching roles. This information is vital for the further development of our project, as it provides insight into the challenging nature of core practices and helps us develop more precise operational definitions that align with Grossman et al.'s (2009) idea of deconstructing teaching.

This project aims to address the persistent critique of teacher education (Hermansen, 2020) by developing methods for working dynamically with coherence in teacher education, in line with recommendations from Richmond et al. (2019), both across disciplines and between the campus and practicum. We need to develop core practices that are beneficial across subjects and practicum, and continue to dynamically develop these through stakeholder collaboration as society and the work of teaching evolve. Although much changes over time, coherence remains crucial to preparing pre-service teachers for the work of teaching. By doing this, we seek not to see our program as either academic or practice-oriented, but as both. The main achievements so far are our creation of structures for coherence through micro practice and campus days, and our joint development of a tool to work with conceptual coherence (core practices). Our data are still limited, but the early findings are promising, as they indicate some influence on pre-service teachers' practicum experiences.

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The *Aesthetigram* as a methodological and didactic tool to explore pre-service teachers' relationship with mathematics

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This article presents the aesthetigram, a novel methodological tool used, in our particular context, to foster pre-service elementary teachers' reflection on their relationship with mathematics during initial teacher education. Grounded in a phenomenological perspective, the aesthetigram invites participants to engage with a mathematics-oriented artistic stimulus to articulate and map their personal reactions to that stimulus. Through this process, pre-service teachers (PST) are given an opportunity to recognize and reflect on their emotions and attitudes towards mathematics that shape their learning and future teaching practices. The article details the adaptation of the aesthetigram for use in a mathematics content course, illustrates its operationalization with student examples, and discusses its dual role as a didactic and research instrument.

Keywords: Pre-service teacher, mathematics, methodology, relationship with mathematics.

Introduction

Reflections on the training of future primary school teachers in mathematics often involve questioning what is necessary and what they require to effectively teach mathematics. As Boyd et al. (2014) highlight, expectations regarding the competencies and knowledge that teachers must develop in mathematics are substantial. Furthermore, teachers' beliefs about mathematics and their confidence in their own mathematical skills and knowledge also warrant careful examination. Indeed, many researchers and teacher educators seek to contribute to a better understanding of the conditions and factors that foster the development of confidence among teachers in teaching mathematics (Bates et al., 2013; Boyd et al., 2014).

Future teachers carry with them a relationship to mathematics shaped by their past school experiences. While some have developed a positive relationship with mathematics over the years, others maintain a rather negative disposition (Bjerke, 2019; Boyd et al., 2014). This issue is particularly salient for prospective primary school teachers, who often exhibit high levels of mathematics anxiety linked to negative school experiences and feelings of incompetence (Bates et al., 2013; Bjerke, 2019; Boyd et al., 2014). Whether characterized by confidence or anxiety, this relationship with mathematics inevitably influences not only their own learning but also the way they will teach mathematics to their students (Aslan, 2013; Ebbelind, 2019).

Considering the above, it appears essential to invite PSTs to reflect on their own relationship with mathematics, fostering awareness of the beliefs and emotions that accompany it. This introspective work constitutes a key step towards developing a professional stance and transmitting to their future students a more positive relationship with mathematics. Such an approach could enable students to approach the discipline with less apprehension and, consequently, contribute—at least partially—to breaking the cycle of deeply ingrained negative beliefs about mathematics (Lake & Kelly, 2014).

Objective of the article

Given the aforementioned observations, we sought to propose an intervention for PST that promotes reflection related to their relationship with mathematics (didactic aim) and, simultaneously, to document and study this reflection (methodological aim). This article is methodological in nature, and its objective is to present this intervention, called the aesthetigram.

Below, we will discuss the context in which the intervention was introduced to PST. We will also outline its theoretical foundations and genesis, as well as the adaptations we proposed to tailor it to the specific context of reflecting on the mathematical relationship of PSTs. Finally, we will provide examples of students' productions using this intervention to better understand its operationalization and its didactic and methodological potential.

Research context

The intervention presented here was implemented within a mathematics course for new pre-service primary school teachers at the University of Montreal. This course is a "content course" that aims to revisit primary-level mathematics content. The course has a dual objective: to lead future teachers to a deep conceptual understanding of primary-level mathematical concepts and to encourage reflection on their relationship with mathematics. Thus, the course addresses not only the skills of PSTs but also their relationship with mathematics, which inevitably shapes their attitudes (Kalder & Lesik, 2011; Novak & Tassell, 2017).

The first objective is justified by the fact that these PST often possess a superficial understanding of mathematical concepts, closer to procedural than conceptual knowledge (Özpinar & Arslan, 2022). To foster genuine conceptual understanding, we employ an approach inspired by the "Thinking Classroom" approach (Liljedahl, 2021). This approach, based on discovery through problem-solving, encourages students to work collaboratively to solve problems that, for most students, qualify as "true problems" (Liljedahl, 2021). Students are required to document their problem-solving processes, which the teacher then uses to compare and discuss various methods employed. In this context, the teacher's role is to provoke students' reasoning to lead them to independently find means of validation. Importantly, there is no traditional lecturing; rather, the focus is on highlighting certain reasoning errors, their causes, and consequences, and on meaningful problem-solving strategies that promote conceptual understanding. This pedagogical method is demanding, unsettling, and for most students, quite different from their previous schooling experiences.

Consequently, the course, both by its objectives and its approach, can induce anxiety in many students. This stress and anxiety toward mathematics, often felt by future teachers, is rarely addressed directly during their university training, although it is clearly observable in classroom contexts. It is from this widespread and recognized mathematics anxiety among PST (Boyd et al., 2014; Novak & Tassell, 2017) that our second objective emerged: to provide students an opportunity to reflect on their relationship with mathematics, whether positive or negative. The tool we adopted for this purpose is the creation of an aesthetigram, which we present below.

Theoretical foundations

Within the context of teacher education, some authors argue that "becoming a teacher starts the day children enter school for the first time and encounter different educational models" (Jong, 2016 in Ebbelind, 2019, p. 3250). It therefore seems desirable to draw upon these experiences and lived realities to better understand the relationship that PST have with mathematics and, the teaching of mathematics. The tool named aesthetigram offers an interesting avenue to facilitate such reflection.

Indeed, White and Lemieux (2017) created this tool with the intent to foster engagement and deepen critical reflection and analysis of an artwork in an arts education course, as well as to document these processes. Its development and use are grounded in a phenomenological approach:

> Phenomenology, therefore, is the study of the human consciousness. Put another way, phenomenology is the scientific study of the essential structures of experience […] those referents can take many forms – cultural norms, personal beliefs, memories, et cetera- layers of influences upon any given experience […] it is concerned with fundamental relationships between an experiencing subject (a person) and objects of that experience – with how objects are experienced, rather than why. (White & Lemieux, 2017, p. 3)

Even though it was developed within arts or literature, we propose that this tool is also appropriate in the context of a mathematics class. Indeed, this perspective on the objects explored with the aesthetigram allows future teachers not only to draw upon their past experiences but also to project themselves into the future to better understand themselves and, above all, to become aware of their relationship with mathematics. The tool also pursues two aims: didactic and methodological (Lemieux & Dufour, 2024). Didactically, we aimed to facilitate metacognitive reflection among PST regarding their relationship with mathematics. From a scientific standpoint, we sought to document this reflective process for research analysis.

For this study, we based our approach on a conceptualization of the relationship to knowledge that aligns closely with the phenomenological perspective described above. Both Beaucher (2010) and Charlot (1997) emphasize the subjective, emotional and dynamic nature of this relationship. More precisely, Charlot (1997) detaches knowledge from being merely an object and conceives it more as a process or activity, describing the relationship to knowledge as "the organized set of relationships that a subject maintains with everything related to 'learning' and knowledge" (translated from French; Charlot, 1997, p. 94). In his theory, Charlot (1997) proposes three dimensions to describe and observe the relationship to knowledge: the epistemic, identity and social dimensions. Thus, the aesthetigram concretizes these relationships both for the participant and for the researcher seeking access to them.

Description of the aesthetigram construction process

More concretely, the aesthetigram is a mapping of individual and personal reactions to an (aesthetic) artwork. It is therefore not a "conceptual" map in the traditional sense, since what is related are feelings and reactions, not concepts *per se*. The detailed steps of the aesthetigram construction process and their corresponding explanations are presented in Table 1 below.

In its original context, the artwork itself was the object of study. In our context, the artwork serves as a trigger for metacognitive reflection on an object present within the artwork—but which can also be

abstracted from it—the relationship to mathematics. Hence, the purpose of using an artwork is not to analyse it *per se* but to inspire reflection. Indeed, as the process unfolds, one can observe a progressive detachment from the artwork towards a more personal proposition relying on the artwork without making it the central object. To provoke this reflection, we selected a children's literature work which theme directly concerns the relationship with mathematics: a chapter from the youth novel *The Number Devil: A Mathematical Adventure* by Hans Magnus Enzensberger (1997[1].

Table 1: Steps of the aesthetigram construction and its operationalization in our research context (inspired by Lemieux & Dufour, 2024; White & Lemieux, 2017)

<table>
<tr><th>Steps</th><th>Explanations</th></tr>
<tr><td>Contact with a work of art</td><td>The artwork is used here to provoke reflection. The artwork selected is a chapter from The Number Devil: A Mathematical Adventure.</td></tr>
<tr><td>Chronologically note their reactions</td><td rowspan="2">Participants note very personal or more technical/operational reactions. All reactions are valid. We suggested to note between 8 to 12 reactions through their reading of the chapter. They could note them manually or digitally, with intensity rated from i to iv (low to very high). For example, a participant notes (among others):
• Pierre doesn't like the demon but still wished to dream just to see him. (i)
• Demon's reactions are exaggerated and very humiliating for Pierre (children's book). (iv)
• Reading the story makes one think about the toxic behaviour of a parent towards their child. (iv)</td></tr>
<tr><td>Note the intensity of each reaction</td></tr>
<tr><td>Categorize reactions</td><td>Using a guide proposing various categories of reactions, participants identify the category for each reaction. Multiple categories are possible depending on interpretation, providing a new level of reflection based on reactions rather than the artwork directly.
For example, reaction 1 was categorized as Judgement (I), meaning "a spontaneous and initial reaction."
Reactions 2 and 3 categorized as Thought, which is described as "when your reading leads you to question yourself about one or more of the following aspects". This category then splits into 7 sub-categories among which (I) the affective nature of the artwork, the feelings it inspires, (II) the specific content, like a character's action, or (VII) in relation with your own social context. Both reactions 2 and 3 from our example have been placed in Thought's sub-category VII.</td></tr>
<tr><td>Transfer reactions, intensity, and category onto sticky notes</td><td>The aim here is to deconstruct the chronological order to reorganize reactions differently. At this stage, reactions form a non-ordered set without precise organization.
Judgment (I)
Pierre doesn't like the demon, but still whished to dream just to see him.(i)
Thought (VII)
Demon's reactions are exaggerated and very humiliating for Pierre (children's book). (iv)
Thought (VII)
Reading the story makes one think about the toxic behavior of a parent towards their child. (iv)</td></tr>
<tr><td>Map the notes</td><td>This is the construction phase of the aesthetigram itself. Reflection reaches a new level as participants organize their reactions according to their own criteria. Examples above led to the aesthetigram in Figure 4, which we will discuss below.</td></tr>
</table>

[1] We used the French version translated by Jean-Louis Schlegel (1998).

Write a justification accompanying the aesthetigram	A commentary accompanies the aesthetigram to explain and fully understand the participants' reflection processes; without this, it is impossible to move from a private stance (phenomenological mapping) to a public stance allowing external readers to understand the lived experience (Lemieux & Dufour, 2024, p. 5). In our context, instructions were adapted for students, who could submit their justification as written or video productions.

From a methodological standpoint, the analysis focuses more on the justification component of the datas than on the aesthetigram itself. In line with the theoretical framework adopted for this study, we identify through coding, for example, distinct units of meaning based on the three dimensions proposed by Charlot (1997): epistemic, identity and social. This leads to two complementary perspectives on the datas: a narrowed one leading to an individual-level view on each student, and a broader thematic analysis structured around the three dimensions.

Examples of aesthetigrams and of their accompanying justification

Since 2022, we have been employing this tool in our course. Within the framework of our research on students' relationship to mathematics, some student's productions have been analyzed as described before. As this article focuses primarily on the methodological aspect, such analyses are not presented here. However, we deemed it pertinent to provide examples of aesthetigrams along with their justifications to illustrate the types of outputs that this tool can generate.

The aesthetigram by Dominique (Figure 1) has not been translated, as our attention here is directed more towards its form and the justification it offers (Figure 2).

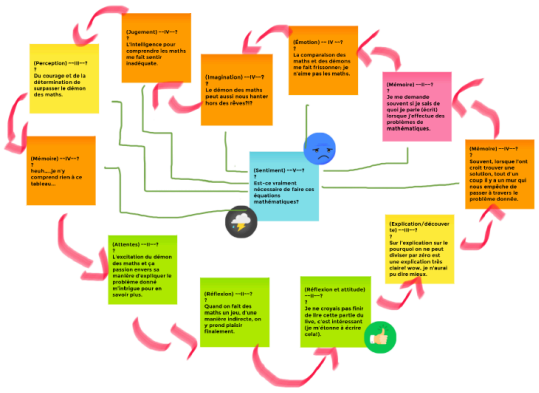

Figure 1: Dominique's aesthetigram

[…] My original idea was to make a roller coaster with my aesthetigram. […] I started placing sticky notes in the shape of roller coaster hills […] roller coasters because sometimes I get math, sometimes I don't, sometimes I like it, sometimes I don't. I like it when I understand, I don't like it when I totally lose my mind. Also […] the amount of effort I put in to finally "get it." […] But unconsciously, I started making a circle. A circle that kind of looks like a heart. […] So yeah, it's really a vicious circle! […]

In the end, I decided to keep it kind of heart-shaped. Like a round heart! A vicious circle made with love because it's not mean, even if, like in the excerpt, it still says "the number devil" […]

Still, by making an aesthetigram, I realized that I see it as a vicious circle, but other people doing the same thing—if I looked at what they did—they all have different ways of seeing math. […] It really makes you realize that I only see it as a vicious circle, but on the other hand, that can be broken because there isn't just one way to see math […]

Figure 2: Extract of Dominique' justification transcript

Dominique's justification illustrates how the student can distance herself from the artwork in order to engage in a deeply personal reflection. In this excerpt, one can observe her perception of mathematics as well as her feelings when engaging with the subject. It is apparent that this reflection allows her, on the one hand, to become aware of her apprehensions towards mathematics and, on the other hand, to explore elements of hope related to more positive aspects of the discipline. Furthermore, this work, and likely the course context, enable her to recognize that this perspective is not unique and can be subject to change. This excerpt constitutes a pertinent example of the phenomenological approach underpinning the tool. Indeed, Dominique reports on a lived experience in relation to an object—here, mathematics (how).

In the following excerpt, it is also possible to perceive her awareness of her future role as a teacher and of how her perception of mathematics may influence that of her students (Figure 3).

> I push myself to succeed [in math] because I know—I'm aware—that I'm going to teach it anyway. I'm going to teach it to my students, and when I do, I don't want to do it with disrespect. I want to do it with some kind of passion. They must be able to like it, and whether I like it or not, well, that's just how it is!

Figure 3: Extract for Dominique' justification transcript

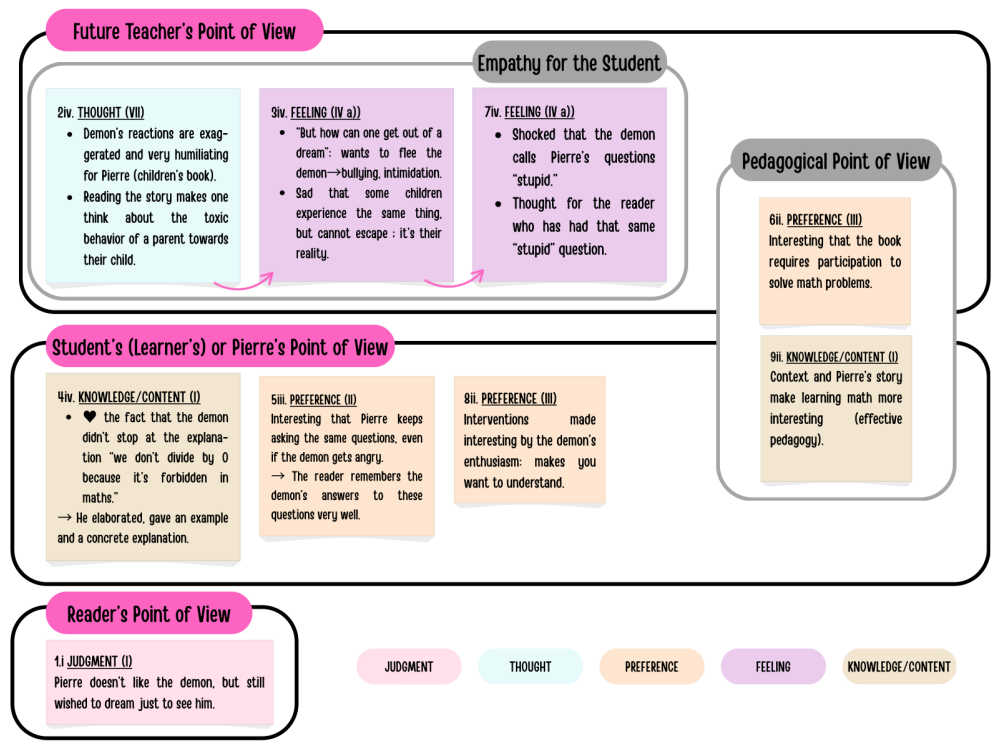

Figure 4: Charlie's aesthetigram, translated and slightly modified for formatting purposes, while faithfully preserving its original content

Moreover, certain aesthetigrams and their accompanying justifications reveal that some students approach their reflection through the various roles they hold in relation to mathematics. Charlie's aesthetigram provides an example of this approach (Figure 4). Indeed, the organization of her responses is based on the different stances she adopts that of a future teacher, in which the participant adds subcategories such as "empathy for the student" and "pedagogical perspective", that of a student (learner), and that of a reader. In this aesthetigram, it is evident that Charlie positions herself primarily

in an “observational” mode: she expresses empathy for the character, whom she identifies as one of her future students. Consequently, she assumes the stance of a prospective teacher to observe the character’s reactions and reflections, aiming to understand them—this emerges as the central element in the justification of her aesthetigram.

Although both examples invoke, at some point, the stance of a future teacher, the role of the artwork in the unfolding of their reflection differs. Dominique more closely identifies with the character, whereas Charlie associates this character with a future student in her class. This distinction influences the way they approach their own relationship to mathematics, yet the reflections they develop remain equally relevant and rich for analytical purposes.

Conclusion

In this paper, we aimed to present a specific tool, the aesthetigram. We also discussed certain modifications made to adapt this tool to encourage prospective teachers to reflect on their relationship with mathematics, while preserving useful records for analysing such reflections. Although the device presented here may appear primarily didactic, it also serves as a methodological instrument that provides access to PSTs’ relationship to mathematics and their ongoing reflection on this relationship, which is often in flux throughout their teacher training. The intervention does, however, present some limitations. It may feel somewhat heavy or destabilizing for students, as they are required to rely on their own feelings to construct the aesthetigram, often without clear guidance on what is considered sufficient. From a research standpoint, the analysis is quite dependent on the classroom context at the time of implementation. Nevertheless, through this work, we hope to open a window onto a deeper understanding of the evolution of the relationship with mathematics during the initial training of prospective mathematics teachers.

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Direction of educational change in professional development: Who holds the mandate?

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This paper explores the ethical challenge of the mandate for educational change within professional development. Drawing on a long-term educational design research project focused on problem solving and problem posing, we examine how collaboration between researchers and teachers evolves across four phases. Initially, researchers led the design and implementation. Over time, teachers became active co-creators, contributing to planning, data collection, and analysis. This shift from research on teachers to research with teachers enabled a shared mandate for change, fostering sustainable professional development. Our findings suggest that change is more likely to occur and sustain when it originates from within the teachers themselves, supported by collaborative structures. The study highlights how educational design research can bridge ethical concerns and promote teacher agency in relation to change.

Keywords: Educational design research, professional development, change, problem solving, problem posing.

Introduction

Changing educational systems and their practices through professional development presents specific challenges. One such challenge, among others, concerns the ethics of deciding who holds the mandate to determine the direction of change (Ebbelind et al., 2023; Ebbelind & Helliwell, 2024). That is, who has the authority to decide whether to change one's practice, or even who has the right to suggest that teachers change their practice. These questions have been referenced in past Cerme TWG18 topic group reports (Fausganger et al., 2022; Friesen et al., 2023). In Bolzano (2022), for example, the TWG18 report concludes that ethical considerations regarding teacher change have become an important topic of discussion, and the related question of how we can learn more about the roles of different participants in PD has arisen (Fausganger et al., 2022). In Budapest (2023), it is noted that the "TWG18 discussions did not tend to focus on research ethics in the more traditional sense, regarding technical requirements such as data collection procedures, consent, confidentiality; rather, the focus centred on the ethical and moral dimensions of the researcher's participation" (Fausganger et al., 2022, p. 3054). It is concluded that such issues require further exploration in relation to professional development (Fausganger et al., 2022) and thus form the rationale for this paper.

Professional development refers to the ongoing process through which individuals enhance their skills, knowledge, and competencies related to their careers or professions. According to Liljedahl (2014), research on the professional development of mathematics teachers can be categorised into studies focused on knowledge and/or beliefs, the effectiveness of professional development, or the methods used in professional development. Through this diversity, findings from research on

professional development have informed its effects on teachers' practices, its effectiveness and specific aspects of professional development (Sowder, 2007). Overviews demonstrate that collaborations, the combination of tools, and reflections on shared experiences, where groups of teachers work together over time, have a positive impact on classroom practice (Avalos, 2011; Sowder, 2007). In general, regardless of direction, professional development involves some form of change, quite often changes in classroom practice with the aim to improve students' learning (Avalos, 2011). In regard to changes in classroom practice, a question arises who holds the mandate to decide on the direction of this change (Mason, 2003; Pimm, 1993).

> Their change is not our business; how, when and if they change is surely their concern alone. […] If I as a teacher educator can only feel successful if the teachers I work with change (and in ways I want them to), I am setting up both myself and the teachers I am working with quite dramatically. I believe it is dangerous to lose sight of how difficult personal change can be - and we should not talk lightly or glibly about it, let alone expect or demand it. (Pimm, 1993, p. 31)

Mason (2003) raises ethical concerns when external forces try to make teachers "adopt practices that are deemed valuable or even necessary" (p. 282). This ethical concern will be explored in this paper within the context of a mathematics educational design research study conducted in Swedish preschool class. The focus will be on how the ethical concern of who decides the direction of change can be understood in relation to various types of collaboration between researchers and teachers. We specifically focus on whether and how different forms of collaboration between researchers and teachers in educational design research interact with the ethical concern inherent in the mandate to make decisions about directions of change.

Educational design research

Educational design research is an approach that combines basic research with applied research to enhance the impact of research on practice (Anderson & Shattuck, 2012). Jacobsen and McKenney (2023) recently identified three orientations within educational design research: research for, on, and through interventions. They conclude that a key concern in educational design research is ensuring that research methods appropriately align with the research questions, theoretical frameworks, and contexts. This presents a challenge because the iterative cyclic process of designing and testing interventions occurs within an inherently complex educational environment. Each design cycle involves preparing for teaching, implementing the teaching, and finally, conducting a retrospective analysis of the teaching and learning. Most often, the design of new educational materials is part of this cyclic process. These educational materials include, for example, specific tasks, worksheets, digital tools, or extensive professional development programs (Bakker, 2018).

Realistic settings are important in educational design research because solving instructional problems requires a detailed understanding of the contexts in which they occur. Gaining this understanding needs collaboration between teachers and researchers, though the extent of this collaboration can vary (McKenney & Reeves, 2018). One way to differentiate the roles of teachers and researchers in design research is by examining their specific tasks and their joint work (Cai et al., 2018). For example, who defines the instructional problem, identifies the learning goals, applies available research, and analyses the data. According to Cai (2019), educational design research involves building productive

and sustainable partnerships between researchers and teachers, as well as exploring the traditional professional roles of both (Cai, 2019). To conclude, teachers play a vital role in collaboration due to their familiarity with their students and the educational context (Cai et al., 2018). Regardless of the level of collaboration, many studies emphasise the importance of professional development in relation to educational design research (e.g., Jacobsen & Mckenney, 2023).

The context of the study

In Sweden, six-year-olds attend preschool class, which is part of the formal education system, aiming to provide a smooth transition between preschool and school. Although the preschool class was introduced in 1998, a national curriculum for preschool was only implemented in 2017. Problem solving and problem posing were, however, already emphasised in both preschool and primary school curricula prior to 2017. Nonetheless, there were few studies on problem solving in early mathematics, and even fewer on problem posing (Palmér & van Bommel, 2020; van Bommel & Palmér, 2021). The educational design research study presented here was initiated in 2014, and the initial aim of the educational design research was to explore the possibilities and limitations of teaching problem solving and problem posing to young students at the start of school. The study shows positive outcomes regarding students' mathematics learning (e.g., Palmér & van Bommel, 2020; van Bommel & Palmér, 2021), as well as their views and feelings about problem solving and problem posing (Ebbelind et al., 2024).

During the educational design research study, the collaboration between the researchers and preschool class teachers has evolved (Ebbelind et al., 2023; da Ponte et al., 2023). The development of these collaborations can roughly be divided into four phases. In the first phase, the teaching was planned and carried out by the researchers, who also collected and analysed the empirical data. The teachers were present in the classrooms as observers. In the second phase, the results from phase 1 were further developed in collaboration with teachers from one municipality. The teaching was still planned by the researchers but implemented by the teachers, who also collected the empirical data later analysed by the researchers. In the third phase, some teachers from phase 2 were occasionally asked to implement new problem-solving tasks and to collect students' documentation from these lessons. This phase also included the introduction of problem posing into the study. The teachers wrote reflections on the lessons, and at times, they were asked to conduct interviews with their students, during which students' reflections on the lessons were discussed. In the fourth and ongoing phase five preschool class teachers from the second and third phases continued to collaborate with the researchers. The planning of teaching is done collaboratively between the teachers and the researchers, and it is carried out by the teachers who also collected the empirical data, as in phase 3. The analysis is partially conducted in collaboration between the researchers and the teachers, where the teachers now act as both producers and consumers (prosumers) of ongoing research (da Ponte et al., 2023). External funding offered the preschool class teachers time in their employment allocated for the study.

The background to the development of the four phases, as well as their impact on research quality, has previously been examined by Palmér and van Bommel (2021) and in another related study, Ebbelind et al. (2023) focused explicitly on the participation of two teachers, Diana and Nicolina and

how the project might support or hinder change from the teachers' perspectives. For Diana, participation provides a safe space to be the teacher she aspires to be. Meanwhile, for Nicolina, the project serves as a pathway to becoming, offering alternative career possibilities. Both teachers feel a need and envision different ways of being, believing that change is possible through their involvement. Consequently, the project can support change if they choose to pursue it.

Method

First, in this section, the participants and ethical considerations are presented. Following this is a presentation of the empirical material and analysis for this paper.

Participants and ethical considerations

In the educational design research study on problem solving and problem posing, the unit of analysis was the teaching and learning of mathematics in preschool classes, while the methodological research questions in this paper arose in parallel. As such, the educational design research study is in this paper viewed as a collective instrumental case (Stake, 1995) on which a methodological meta-analysis is conducted. The case being collective indicates it involves several teachers; the case being instrumental signifies an aim of developing a general understanding beyond the case itself.

Some preschool teachers (hereafter teachers) participated in phases 1, 2, 3, and 4, while others only took part in phase 2. Regardless of the phase, all participating teachers received information about the study and provided written consent to take part. Since students documentation were collected in all phases, and students interviewed in phases 3 and 4, permission for their participation was also required. The guardians of the students were informed about the study and consented to their child's involvement. The students were briefed about the study by their teachers. All data was stored in accordance with standard data management guidelines, meaning it was kept securely, scanned, and stored on a password-protected server. Therefore, all ethical regulations for research in Sweden (Swedish Research Council, 2024) were adhered to.

Empirical material and methodological meta-analysis

To answer our research question, we conducted a methodological meta-analysis on empirical data from all phases of the study, collected between 2013 and 2024. This data includes researcher notes (phase 1), written reflections from teachers (phase 2), interviews with teachers (phases 3 and 4), and notes from joint meetings between researchers and teachers (phases 2, 3, and 4). One of the researchers conducted the individual interviews with the teachers in phases 3 and 4. These interviews lasted approximately one hour and were transcribed.

The analysis of these empirical data focused on the shifting distribution of mandate and decision-making across the four phases of the study. First, we organised all data chronologically by phase, focusing on the development of collaboration over time (Palmér & van Bommel, 2021). Then, using a qualitative thematic analysis, we identified patterns related to (a) perceived mandates for change, (b) experienced needs for change, and (c) enacted decision-making concerning change. This analysis proceeded in three steps. First, we examined how teachers and researchers were positioned in each phase with respect to responsibility for generating and implementing change. Then we examined

mandate shifts over time, mapping moments where changes were negotiated. Finally, we interpreted how shifts in mandate interacted with the collaboration in each phase.

Results

In this paper, we ask if, and how, different forms of collaboration between researchers and teachers in educational design research interact with the ethical concern inherent in the mandate to make decisions about directions of change.

In phase 1, when the teachers were observers, they were positioned in a way that might oppose change. Even if they regarded the observed mathematics teaching as desirable, they did not gain any experience of teaching problem solving nor receive support to feel a personal need for change. However, in this phase, the researchers may have experienced change, as the teaching of mathematics in preschool class was the primary goal of the study. Furthermore, this phase was essential for progression, as the researchers found the results on the level of the students to be positive; consequently, the changed teaching carried out by the researchers led to phase 2.

In phase 2, the researchers instructed the teachers to implement the problem-solving lessons developed in phase 1, which can be seen as an external force encouraging teachers to adopt valuable or necessary practices regarding mathematics. In phases 3 and 4, however, as the teachers became involved in all aspects of the educational design research, it became evident that they started using their classrooms to explore students' learning and understanding of problem solving, as well as theories of learning. When participating in all parts of the educational design research, the teachers also obtained a mandate to make decisions about the directions of change concerning their mathematics lessons. In phases 3 and 4, as they sought to gain a deeper understanding of the mathematical content in the tasks and how to apply it to monitor the problem-solving processes, the direction of change emerged from the teachers. In phase 3, conditions for sustainable change were established, and in phase 4, this became a reality. During these phases, both researchers and teachers raised educational concerns where joint professional experiences and reflections guided the decision-making process, leading to a shared mandate on the direction of change.

During phases 3 and 4, there is a shift in focus from researching (on) teachers to researching with teachers. Teachers have various options for different positions. Participation in the project provides them with opportunities to adopt practices they find valuable or necessary. The research process enables participating teachers to recognise a need, envision a different way of being, and believe that it is achievable. Feeling part of the group appears to be essential when discussing whether the project supports potential change.

Based on the above, the developed collaboration between researchers and teachers through the four phases involved a movement from researchers having the sole mandate for the desired change to a joint mandate for decision-making about the direction of change. Furthermore, collaboration in

phases 3 and 4 suggests that change is more likely to occur because it originates from within the teachers, as they are part of the decision-making process about whether and what to change.

Discussion

The ethical challenge of making decisions about the directions of change is emphasised by Mason (2003) and Pimm (1993). Change is known to be difficult (Avalos, 2011; Pimm, 1993) and rarely occurs unless it originates from within the teacher (Ebbelind & Helliwell, 2024), as it is their concern whether they change. When external forces try to make teachers adopt practices they see as valuable or even necessary, teachers often respond with counterarguments about their practices that oppose change (Mason, 2003). By connecting this ethical challenge of change to the four phases in a case of educational design research on problem solving and problem posing in preschool classes, we gained a deeper understanding, as it became clear that the ethical challenge is addressed differently in each of the four phases of the study. However, these findings are situated within the Swedish preschool class context. Future research should examine whether similar mandate shifts occur in other institutional contexts, where teacher autonomy may differ significantly.

In this paper, the ethical challenge of deciding on directions of change has been addressed. One of the main aims in professional development is to transform teaching into something different from what it was before the development (Liljedahl, 2014). Researchers have attempted to provoke change as they see it as necessary, but this has often fallen short because implementing teacher changes is difficult (Ebbelind et al., 2023; Mason, 2003). In the educational design research study presented here, in phase 3, conditions for sustainable change were established, and in phase 4, this became a reality. However, the changes in phases 3 and 4 may not have been possible without the groundwork laid in phases 1 and 2.

It is also important to acknowledge that researchers themselves underwent changes during the project. Their initial role as designers shifted toward facilitators and co-creators, requiring flexibility and openness to teachers' perspectives. This reciprocal change underscores the collaborative nature of educational design research (Bakker, 2018). This shift in position raises important ethical considerations about power dynamics in research partnerships. By adopting a co-creator stance, researchers challenged hierarchical structures and aligned with participatory principles.

Therefore, to summarise, if professional development genuinely aims to innovate and improve classroom practice, it aligns with educational design research, which involves developing theories and new forms of instruction (Anderson & Shattuck, 2012). It also provides practical and theoretical answers to questions relevant to research and educational practice (McKenney & Reeves, 2018). A collaboration between teachers and researchers, especially in line with phases 3 and 4, appears to be beneficial. Such collaborations are not only advantageous for educational design research but also support professional development in mathematics. This collaboration helps reduce perceived gaps between researchers and teachers, as well as the ethical challenge of making mandates about the directions of change.

Future professional development initiatives may prioritise shared decision-making frameworks to avoid enforcing change. Researchers should develop processes that allow teachers to express their needs and co-create solutions, ensuring ethical integrity and sustainable impact.

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Becoming a mathematics teacher with dual AI roles student and mentor simulations in professional identity development

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This study investigates how dual-role AI chatbots, enacting both student and mentor roles, shape preservice teachers' professional identity development. Sixteen participants interacted with AI-students simulating common misconceptions and received AI-mentor feedback designed to guide pedagogical reasoning. Findings suggest that chatbot interactions fostered PSTs' self-awareness of noticing, while also offering a safe environment to rehearse responses to unexpected student ideas. The AI-student was often perceived as realistic, though concerns were raised about polished answers. The AI-mentor was valued for its guidance on questioning strategies, identification of strengths and weaknesses, and motivational support, yet some participants found feedback insufficiently specific. These results highlight the potential of dual-role AI simulations to support professional identity development while underscoring the need for more authentic student modelling and specific feedback.

Keywords: Mathematics teacher education, professional identity, generative artificial intelligence.

Introduction

Teacher noticing is central to teacher competence, as it shapes how teachers attend to and interpret classroom events (van Es & Sherin, 2002). The development of professional identity in mathematics teacher education is closely connected to the situation-specific skills involved in teacher noticing. Professional identity is described as a dynamic and continuously reconstructed sense of self, shaped by personal experience, social interaction, and contextual demands (Rushton et al., 2023). Professional identity mediates between teachers' dispositions and their in-the-moment performance in classrooms. This identity formation process requires structured opportunities to approximate practice in safe and supported environments. Grossman et al. (2009) emphasise that teacher education should provide reduced-complexity enactments, scaffolding, and cycles of reflection to support the gradual assumption of professional roles. Syntheses of mathematics education research position attending, interpreting, and responding as a central bridge from knowledge to practice and document persistent challenges (Santagata et al., 2021) for preservice teachers (PSTs). Empirical studies continue to link noticing expertise to instructional quality, showing that higher levels of noticing are associated with richer whole-class discussions centered on student strategies (Jacobs & Empson, 2025). Prior research has shown that PSTs often rely on procedural reasoning and experience difficulties in interpreting students' mathematical thinking, which in turn limits their pedagogical decision-making (Parodi et al., 2023).

Artificial intelligence chatbots have recently emerged as flexible partners for PSTs to rehearse responsive teaching. Research using virtual students reports gains in noticing expertise and identifies distinct noticing pathways linked to the quality of subsequent teaching moves, suggesting that chatbot environments can approximate open, real-time pedagogical decision-making (e.g., Bastian et al.,

2025; Galiç et al., 2025; Son, 2025). Beyond teacher noticing, scenario-based work with chatbots demonstrates potential for surfacing epistemic and communicative dimensions of questioning, again highlighting both the strengths and blind spots of PSTs' response decisions (Galiç et al., 2025). Studies show that AI-based simulations can foster responsive teaching and questioning while simultaneously raising concerns about authenticity and scalability (Lee et al., 2024; Zhang et al., 2024; Zhuang & Zhang, 2025).

Beyond mastering mathematical knowledge and pedagogical strategies, PSTs need to see themselves as future professionals who can guide learning, reflect on their practice, and position themselves within a teaching community (Grossman et al., 2009). Recent studies highlight that AI chatbots can create interactive spaces for PSTs to rehearse teaching moves, confront student misconceptions, and receive feedback that fosters reflection (Lee et al., 2024; Zhang et al., 2024; Zhuang & Zhang, 2025). Building on this understanding, AI can be positioned both as a simulated student and as a mentor to support professional identity development through teacher noticing. As a student, enabling teachers to see the consequences of their instructional decisions and practice with their professional knowledge in action; and as a mentor, providing structured opportunities for reflection and adaptation as teachers negotiate and refine their emerging professional identities. In this dual capacity, AI offers a promising means of enhancing noticing while simultaneously supporting the ongoing construction of teacher identity. While prior research has examined noticing and questioning, less attention has been given to how such dual-role designs where chatbots act both as students and mentors, can shape teachers' sense of responsibility, agency, and self-positioning as professionals. Addressing this gap, and contributing to current discussions on how AI–human collaboration may support teacher education, this study seeks to answer the following research question:

> How do dual-role AI chatbots, enacting both student and mentor roles, shape to preservice teachers' professional identity development in mathematics teacher education?

AI-based simulations in teacher education

Recent advances with AI chatbots extend these affordances by simulating authentic and dynamic interactions. Lee et al. (2024) showed that engaging with responsive AI-based chatbots improved PSTs' questioning practices and confidence in approximating real classroom exchanges. Son (2025) further demonstrated that noticing expertise in chatbot environments is closely linked to the quality of teaching moves enacted. Complementing these findings, Galiç et al. (2025) argued that AI-student interactions can surface teachers' reasoning strategies across epistemic and communicative dimensions, offering new opportunities for rehearsing adaptive responses. The value of such simulations depends on how PSTs frame and enacts their goals. When the chatbot did not provide responses, PSTs had fewer opportunities to refine their questioning, underscoring that authentic, back-and-forth interaction is essential (Lee et al., 2024). This points to the potential of dual-role designs where AI not only enacts a student but also provides mentor-style feedback. Mentor interactions can guide teachers to articulate reasoning, identify strengths and weaknesses, and reconsider alternatives in ways that foster reflection and professional identity development (Zhang et al., 2024). By integrating student enactment with mentor scaffolding, dual-role AI simulations create a cyclical process of doing and reflecting. This design situates PSTs in low-risk environments to

practice responding to misconceptions, while simultaneously receiving structured feedback that supports identity shifts from mathematics learners to mathematics teachers.

Methods

The study involved sixteen PSTs enrolled in undergraduate mathematics teacher education programs in Türkiye (n=8) and Austria (n=8). Participants were in senior year of the program. As part of a post-survey, participants reported their prior experiences with AI chatbots (e.g., ChatGPT, DeepSeek), which included educational uses such as lesson planning, homework support, translation, and the exploration of student misconceptions. Their attitudes toward AI in education were generally positive, with several highlighting the potential of chatbots to strengthen teacher–student interaction, support novice teachers in anticipating classroom challenges, and enhance creativity in designing mathematical tasks.

The study followed a three-week asynchronous design in which PSTs engaged individually with dual-role AI simulations. Each week, all participants followed the same sequence within the following cycle: (1) introduction to a short video representing a mathematical misconception; (2) interaction with an AI-student, where participants conversed with a custom chatbot simulating a secondary-level student holding the misconception and were required to diagnose the error and explain the concept; (3) writing an individual reflection on their teaching response; and (4) interaction with an AI-mentor, where a mentor chatbot provided feedback, prompting participants to reconsider their pedagogical reasoning and articulate alternative teaching strategies. Participants reported spending about 40 minutes per cycle. The students' misconception concerns an operational understanding of the equal sign, and the student examples are drawn from Carpenter et al. (2003), Stephens (2006), and Stephens et al. (2021).

The design aligns with calls in mathematics education research to scaffold both the acting and reflecting dimensions of professional practice (e.g., Son, 2025; Zhuang & Zhang, 2025). For this paper, data were drawn exclusively from a post-survey administered at the end of the implementation. The survey included open-ended questions on PSTs' perceptions of the chatbot-supported intervention. The following questions guided data collection:

1. During your interaction with the chatbot, have you felt as though you were communicating with a real student? Please explain these moments.
2. Over the three weeks, have you noticed any changes in how you describe the chatbot's responses? If so, can you provide an example?
3. Over the three weeks, have you noticed any changes in how you interpret the chatbot's responses? If so, can you provide an example?
4. Over the three weeks, have you noticed any changes in how you respond to the chatbot? If so, can you provide an example?
5. Do you think chatbots contribute to the process of teacher training in the student role? Please explain your answer and provide examples.
6. Do you think the feedback provided by chatbots in the mentor role contributes to the teacher training process? Please explain your answer and provide examples.

Responses were collected in Turkish and German and later translated into English for analysis. The questions included in the questionnaire were developed by the authors. The first three questions focused on teacher noticing, the fourth on teaching practice, the fifth on the student role of AI, and the final question on the mentor role of AI. Accordingly, themes were generated for each question. After data collection, one author conducted the initial analysis of the responses for each question. An inductive content analysis approach was adopted (Merriam & Tisdell, 2015). Codes were generated from participants' responses and refined through iterative comparison. The resulting codes were then independently reviewed by two additional authors. Following a discussion meeting, consensus was reached on the final coding scheme. Emerging codes and themes are presented in tabular form together with illustrative quotations from the participants.

Findings

Building on post-survey data, four thematic domains emerged from the analysis: noticing, practicing, student role, and mentor role (Table 1). Each domain encompasses several dimensions that capture how PSTs perceived the contribution of chatbots to teacher education.

Table 1: Codes emerging from the analysis of pre-service teachers' perceptions on chatbot interactions

Themes		Codes				
Noticing	Attending	Clarity in description			Identifying specific details	
	Interpreting	Increasing ease of interpretation	Student-like reasoning & questioning		Improved response consistency	From surface to deeper interpretation
	Responding	Questioning strategy	Clarity of responses	Managing student's pacing	Adapting to student level	Use of analogies & metaphors
Practicing		Preparation for classroom responses	Diagnosing students' error	Safe zone for practicing	Developing teaching strategy	Limitations & concerns
Student role		Perceived as a real student				Unrealistic aspects such as detailed answers
Mentor role		Guidance on questioning strategies	Alternative teaching methods & strategies	Identification of strengths & weaknesses	Providing affective support & motivation	Limitations & concerns

Participants reported progressive changes in how they attended to and interpreted chatbot responses. In terms of attending, several noted increased clarity and precision in their descriptions, as one explained, "*I have noticed changes in the way I describe the chatbot's responses over the past three weeks. Initially, I was perhaps a bit more general and less precise, which often resulted in broader responses. I have since gone into more specific details to get more targeted and relevant answers.*" Within interpreting, participants emphasised growing predictability and ease: "*It gradually became easier to interpret because I could predict how he would respond and how I would interpret him*." Others described instances where the chatbot itself contributed ideas: "*Over the past three weeks, I have indeed noticed changes in how I interpret the chatbot's responses. Initially, I was more focused on directly accessing specific information, whereas now I pay more attention to how the responses are structured and worded*" which prompted a shift from surface-level content extraction to deeper

structural interpretation. In their own responses, participants highlighted the adoption of questioning strategies to elicit student reasoning: "*Rather than answering myself, I tried to prompt the chatbot to think by asking questions*." They also reported greater clarity in their own responses, adaptation to the chatbot's pace, adjustments to perceived student levels, and strategic use of analogies and metaphors to support understanding.

Engagement with chatbots in the student role was largely regarded as valuable rehearsal for classroom practice. Participants emphasised the opportunity to anticipate misconceptions and prepare for how to respond: "*Providing us with scenarios we might encounter in advance gives me information about how to respond or guide students if these situations arise during class*." Interactions with AI-student and AI-mentor facilitated the identification of errors and offered a safe environment for practicing teaching moves without consequences for real learners. Several participants noted that such experiences encouraged them to refine teaching strategies and engage in self-reflection. However, limitations were acknowledged. One participant observed that the chatbot does not sound pedagogically: *"AI Chatbots have serious mistakes happen no matter what the subject."* Other participant questioned the scalability of such tools in larger, more diverse classroom contexts: "*I don't think it will always contribute to the education process because it can be effective in individual studies, but I don't think it can be effective in a crowded class with different profiles.*"

Almost all participants described the experience of AI-student as realistic, pointing to instances where the chatbot displayed answers and thinking: "*AI student's explanations were similar to student explanations, especially the mistakes he made in the first activity were exactly the same as student mistakes*." At the same time, concerns were raised that highly detailed or polished answers did not align with authentic student behaviour: "*Giving detailed answers is not what a real student would do.*"

The mentor role was generally perceived as supportive for professional learning. Participants valued the guidance on questioning strategies, as illustrated by one comment: "*Having a mentor to guide me through this process encouraged me to ask questions in the future*." Others emphasized the usefulness of alternative strategies, the identification of strengths and weaknesses, and affective support that enhanced their confidence: "*This boosted my self-confidence. I will take his suggestions into my future classes*." Nevertheless, two participants critically noted that the feedback was at times superficial or difficult to apply in practice, even if it was perceived as objective: "*The chatbot, however, is unbiased and can therefore provide objective feedback.*" They explicitly stated that the feedback was not sufficient, explaining that it lacked subject-specific depth and concrete, context-sensitive guidance they could readily transfer to classroom practice.

Discussion

In this study, we explore how engaging with dual-role AI chatbots, acting both as students and mentors, supported PSTs' professional identity development during a three-week intervention in mathematics teacher education. AI-generated simulations did not simply provide additional practice but also appeared to function as a context in which teacher noticing and professional identity development could develop. As participants engaged with the AI, many framed their experiences in terms of what they were noticing about students' thinking and what they were learning about

themselves as teachers. Such accounts point to the potential of AI-generated simulations to support the emergence of professional identities by offering authentic-like opportunities for teacher noticing. The interplay between these experiences and participants' evolving self-understandings highlights how AI tools may serve as complementary spaces for early professional growth.

The findings suggest several promising directions for teacher education, while also revealing limitations that need to be addressed in future research. A key outcome concerns the development of professional identity. Several participants reported that they began to feel "like a teacher" when describing and responding to chatbot contributions, while also perceiving the chatbot as behaving "like a real student." This sense of authenticity is particularly encouraging, as it indicates the potential of chatbots to serve as virtual students that allow PSTs to rehearse their pedagogical roles in low-stakes environments. Moreover, participants demonstrated increased self-awareness across the dimensions of attending, interpreting, and responding. At the same time, some participants noted that overly detailed or polished responses reduced the realism of the simulations. This aligns with findings from Zhuang and Zhang (2025), who showed that while PSTs valued AI-generated student responses for practicing questioning, they also criticised their limited authenticity and lack of unpredictability, which reduced transferability to real classrooms. These findings resonate with prior evidence that PSTs frequently struggle with interpreting students' mathematical reasoning and making instructional decisions (Parodi et al., 2023). Unlike such limitations, our design provided structured opportunities for practicing these skills through dual role chatbot interactions, thereby supporting the development of noticing as well as professional identity. Future designs should therefore carefully craft prompts that more closely mirror genuine student misconceptions and diverse communication styles.

An interesting point in our study is that one participant explicitly stated, "I tried to adjust my responses based on the mentor comments I received. I'm not sure how effective I was, but I think the feedback I received was valuable. I hope to use it in my own teaching." Since participants were not able to track their improvement across weeks, and the feedback did not directly compare their weekly teaching strategies, the absence of systematic developmental tracking limited the extent to which PSTs could observe their growth over time. This limitation may also have affected their self-awareness of professional identity development. Therefore, we highlight this as a limitation of our study. Future research should integrate longitudinal designs in which chatbot feedback explicitly emphasises changes in participants' responses across cycles.

PSTs described the chatbot interactions as a safe environment in which to rehearse responding to unexpected student questions and misconceptions. They valued the opportunity to identify student errors and to experiment with different teaching strategies without the risks inherent in real classroom contexts. However, participants also noted that these experiences may not transfer seamlessly to large and heterogeneous classrooms. In line with this, Zhuang and Zhang (2025) reported that while simulated student-teacher interactions were highly rated for usefulness in developing questioning strategies, their limitations in capturing the complexity of real student thinking posed challenges for authentic classroom transfer. Future designs could extend simulations beyond one-to-one interactions by incorporating multiple "student voices," thereby allowing PSTs to practice orchestrating discussions in group settings. On the other hand, Jacobs and Empson (2025) found that as teachers' noticing expertise increases, whole-class discussions are more likely to focus on students' thinking.

Building on this finding, it can be argued that enhancing noticing expertise through AI-based simulations may positively contribute to PSTs' ability to orchestrate classroom discussions.

Participants reported that adaptive feedback helped them identify both strengths and weaknesses, develop new strategies, and gain confidence in their teaching practices. Such personalised guidance was described as motivating and instrumental for professional identity development. Still, some PSTs found the feedback insufficiently specific, indicating a need for more tailored and pedagogically grounded responses. This resonates with Bastian et al.'s (2025) observation that although PSTs valued chatbot feedback, they also requested more subject-specific guidance. The study demonstrates the promise of AI chatbots as both virtual students and mentors in supporting teacher education. They provide opportunities for developing noticing skills, rehearsing responses to student misconceptions, and reflecting on professional strengths and weaknesses. At the same time, the study highlights critical limitations, particularly regarding the realism of student simulations, the specificity of feedback, and the scalability of one-to-one tutoring scenarios. Addressing these challenges will be essential to ensure that AI tools function not as replacements, but as pedagogically sound supports that complement and enhance the professional development of future teachers. Future studies could investigate how variation in teacher education contexts shapes participants' initial engagement with the chatbot, as these contextual factors may differentially influence the nature and development of teacher noticing across settings.

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Finding what is not yet there: Student teachers' perceived learning outcomes from action research in school

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My study is a qualitative investigation of 4 student teachers' perceived learning outcomes from a compulsory research and development (R&D)-assignment in mathematics education as part of their initial teacher education. The student teachers participated in a research project aimed at enhancing supervision of and learning outcomes of the R&D-work through collaboration with the practice field. Key features of the supervision model were tripartite supervision including the practice teacher, choosing action research as design and establishing an umbrella theme. A reflective log and semi-structured individual interviews were collected the semester after the R&D-work and analysed using the constant comparative method. The study sheds light on how close collaborations with the practice field can give opportunities for student teachers to learn professional development skills while simultaneously gaining in-depth knowledge in mathematics education.

Keywords: Research and development assignment, initial teacher training, action research, mathematics teacher education, tripartite supervision.

Introduction and background

The starting point of this article is the reform of Norwegian primary teacher education programs, from 4-year programs to 5-year integrated master's degrees programs in 2017. The reform was inspired by Finnish teacher education (Eklund et al. 2019) but also influenced by the general debate in the field of education on the need for schools and teachers to become self-improving lifelong learners, with one possible outlook being inquiry as a stance as opposed to inquiry as a project (Cochran-Smith et al., 2009). The new reform regulations state that student teachers (ST) should learn "to analyze and take a critical approach to national and international research and use this knowledge when practicing the profession" and to "apply […] relevant methods from research and development to continually develop their own and the school's collective practices, and be able to conduct smaller research projects with supervision" (Ministry of Education, 2016, p. 3). The new program includes a compulsory R&D-assignment in the spring semester of the third year and a master's thesis in the fifth year, both being opportunities for student teachers to acquire professional developmental skills. Arising from this context there was a need for knowledge on how STs can learn professional development skills. Goldshaft and colleagues (2022) argue that for STs to develop research skills and turn their knowledge into professional R&D-competence, they need to practice their skills and apply their knowledge to relevant contexts. Teacher education has received criticism for a lack of continuity between the work in practice placement and the university coursework, contributing to an impression of teacher education not being relevant enough (Ulvik et al., 2017, Eklund et al., 2019). According to Ulvik and colleagues (2017) action research is presented in the literature as a possible way to bridge the experienced gap, connecting theory and practice. Ulvik and Riese (2015) point to STs having enough time and space to reflect in depth together with others as the most important condition for

gaining professional development skills from action research projects, with STs experiences being mostly positive despite facing challenges in the start-up phase. Some STs in Ulvik and Riese's (2015) study experienced a lack of knowledge and interest in action research from their practice teachers, aligning with Cochran-Smith and colleagues (2009) findings. Antonsen and colleagues (2022) report on STs doing an action research projects in the third year R&D-assignment gaining competence that they used to reflect on and improve their own practice. However, the practice teachers did not support the ST's work as intended by the university, due to lack of experience with action research. The STs in my study were part of the *Learning, Assessment and Boundary crossing in Teacher Education* research project (LAB-TEd) running 2019 to 2023, doing their R&D-assignment in spring 2020. Part of the LAB-TEd scope was to develop new model(s) for supervision of the R&D-assignment, with the aim that STs' work would become professionally oriented and practice-based through enhanced cooperation between the university and the practice school, at the same time contributing to professional development in school. The implemented model of supervision has clear distinctions from the supervision offered to the rest of the cohort. Leading me to ask:

What qualities did the learning environment offer to support the STs in their work on the R&D-assignment, and what learning outcomes did the STs experience from doing their R&D-work?

Context of the study

Norwegian primary teacher education programs are governed by national regulations with an overall design of integrated subject and didactical knowledge (Ministry of Education, 2016). Practice placements are integral, with 110 days distributed throughout the 5 years, and STs are expected to plan and teach lessons individually and in groups in accordance with the mentor's normal workload in all years. The R&D-assignment is 15 ECTs, normally organized with group supervision, lectures and course workshops arranged as solely on-campus activities in the weeks before and after a 5-week group practice placement. Due to obstacles of program organization, the supervision-groups are not used for practice placement. STs are encouraged to collect data for their R&D-assignment during the practice placement, however they have limited knowledge of the practice context and practice teachers' involvement with the STs R&D-work is based on personal interest and willingness. My study includes four third year STs specializing in mathematics education in grades 1- 4 (with 30 completed ECTs in mathematics education), two mathematics teacher educators and one pedagogy teacher educator in university and one teacher educator in school (practice teacher) holding a master's degree in mathematics education, all participating in LAB-TEd. The main approach of LAB-TEd was Developmental Work Research employing third generation activity theory and the expansive learning cycle as theoretical tools (Engeström, 2015). Each semester all participants met in two full day Change Laboratory sessions organised alongside the formal education program, giving an organizational unit for developing the collaborations between university educators, practice teachers and the student teachers (Postholm, 2025). Due to limited space, I give no further details on CHAT, developmental work research or expansive learning, for a thorough treatment see Engeström (2015).

The implemented model of supervision of the R&D-assignment

As part of the LAB-TEd intervention the four STs were placed in a separate practice group doubling as their supervision-group. Supervision meetings were held as triadic group supervision whenever

possible, with the STs, the three teacher educators and the practice teacher all attending. As all four STs were to collect data in the same 1. grade group of pupils who worked on number sense and early arithmetic during the 5 weeks, it was important to find a common focus for the R&D-work to reduce the load on the pupils. In a process spanning two change labs and a school-based supervision meeting in Autumn 2019, the participants reached a joint agreement on development of sociomathematical norms (Yackel & Cobb, 1996) as an "umbrella theme" (Postholm et al., 2023, Postholm 2025) - combining the STs wishes for mathematics education focus and the schools need for improving the teaching practice, supported by teacher educators' translation of everyday descriptions into theoretical concepts. Action research was chosen as design of the R&D-work, giving a systematic approach for the professional development work (Postholm, 2020). Two half-day supervision meetings in school during the practice placement supported STs' work on interpreting their so-far collected data-material and discussed STs plans and ideas for further actions within their different projects. Ella and Matilda worked jointly on establishing the norm of mathematical efficiency, that it is worthwhile to look for solutions requiring minimal work effort, using carefully selected and adapted Quick image tasks in whole group discussions. Samuel worked on establishing pupils' abilities and inclinations to talk about, explain and discuss their mathematical work with each other, as opposed to with teacher(s). Pupils worked in a small group setting on figural pattern tasks with multiple possibilities for pattern continuation. Parallel, he held whole class discussions involving the pupils in establishing joint communication rules to be used in group work. Eva worked on establishing drawing as a tool to support thinking and finding a solution. She promoted drawing as a tool through explicit focusing on pupils' examples of drawing in whole group sessions and had pupils work in pairs and individually on addition tasks with contexts lending themselves to be solved by drawing as problem solving. The writing process was supervised on the university campus, including the practice teacher when possible, and she read the R&D-assignments before submission.

Data material and analysis

The data material consists of a written reflection log and a qualitative semi-structured individual interviews with each ST in the semester after completing the R&D assignment. The reflection log was written some days before the individual interviews, with the intent to support the STs recollection of the R&D-process. The log entries were 852, 505, 868 and 452 words. The interviews were conducted by one of the mathematics teacher educators and the practice teacher. The interview guide had questions on the STs perceived learning outcome of the R&D-work, focusing on professional development, and on their perceived learning outcome regarding establishing sociomathematical norms in a classroom. The interviews were 42, 54, 55 and 56 minutes of audio recording, producing 9.5, 12.5, 12 and 13 pages of transcriptions, respectively. The recordings were first transcribed using an AI-based automatic transcription service offered by my university, I then listened through each file while simultaneously correcting and completing transcription. Transcripts were analysed using constant comparative analysis (Strauss & Corbin, 1998) in a three-step process. During clean-up of the transcripts possible categories emerged, for example student teacher's perception of action research. In step two I made a table with each possible category from step one in separate columns. I then, phrase by phrase, considered the relevance for the research question, placing phrases in the table by comparing to the content of existing columns, adding new columns as needed. This resulted in a

total of 8 columns. In the third step the 8 tentative categories were discussed with the pedagogy teacher educator, who had read the transcripts beforehand. This led to realizing one category was not in the scope of the research question, five categories were merged three and two together, two categories kept but renamed. The final four categories were: action research – an ideal for a teacher, a community with a shared object, security in a situation of professional uncertainty and possibility for in depth-focus on mathematics education. Analysis of the reflection logs were done in the continuation of the interview analysis, deepening understanding of the categories, but without new categories emerging. Quotes in the following text are from both log entries and interviews and are translated by me. The four STs provided informed consent, and I have given them pseudonyms.

Results

Action research – an ideal way of being a teacher

The STs report to have little to no knowledge of or experience doing professional developmental work before they started work on the R&D-assignment. The familiarization with professional development through the R&D-work has impacted their awareness, some being very pronounced on how their own thinking has shifted.

> Matilda: I think I have changed my focus a little bit myself, on how important it is to have a professional development focus, because things can in some way always become better in a classroom. To be conscious in your own classroom, how can we improve things. How can we do it systematically, if that is our aim. Maybe, reflecting more on, how did this really go, what works and what can improve. I feel like I have learnt more about this.

The change in focus gained applies not only to professional development of the teaching of mathematics but is a more general view on how a teacher can think and go about their work.

> Samuel: To enter a classroom and discover a problem, and then find a goal of what to achieve, ok, this is how it is now, but this is how I want it to be. Then I will try this action to reach the goal. This is what you do as a teacher very often.

The STs see action research as how to behave as a teacher. During the interviews, the ST's use of action research concepts in statements regarding teaching work and professional development was very pronounced.

> Matilda: We spent a lot of time understanding what action research really is, what an action could be, what we wanted to achieve, and trying to figure out how we could see if there was any development happening. But it sorted itself, once we got started.

The action research methodology was difficult to understand, and they report grappling with the action research methodology ahead of practice placement. Still, they all state that the experience of using action research has contributed to a deeper insight into the features of action research methodology, an important learning outcome that they will implement in their work as in-service teachers. Ella, but not only her, seems to have grasped how action research is data driven. One needs information about the context where the research is to be done before anything can be planned, and every cycle of research needs to consider what happened in the previous cycle.

> Ella: The actions are difficult to plan before we start. They will come along as the work goes ahead. You don't know the effect before it happens.

The STs emphasize how they perceived their work on the R&D-project to have significance, not only for themselves, but for the practice teacher and school, promoting a different motivation to invest time and energy into it. This contrasts with how other university assignments can be regarded as something they just need to do. Samuel is noticeably clear in stating that this way of doing the R&D-work should be a minimum standard, and Eva comments on how she thinks they get something the other STs are not, namely doing genuine professional development work.

> Samuel: It was in fact what we were working on during the practice placement. So beforehand we talked about…yes just deciding the aim and everything, it was a cooperation with [practice teacher] about what would work in this group, what we could do. So even there, there was a clear connection to the practice field, and when the placement began, then what we work on is what we do research on.

The STs experienced the R&D-work as doing practice work, not as separate entities to be handled in separate situations during the practice placement, but rather that doing the R&D-work is practice, and that was what was to be worked on during the placement period.

A community with a shared object

The STs refer to conversations during the R&D-work as important for their learning outcome.

> Samuel: I think it must be all the conversations both between us students, about what we did, […] It was so nice after the lesson, we [STs] sat down and just talked about what we had done in class. What did we think, how did it work? And when we also could have the conversation afterwards with you [practice teacher], who have more knowledge than us, then it worked very nicely, I think.

They all mention, either in the interview or in the reflection log, the importance of the cooperation inside the practice group and how the umbrella theme made this possible. They report on planning lessons where actions would take place together, making sure that each ST had enough time and space to do their interventions and data-collection, helping to take observational notes for each other during lessons, and after finishing each lesson reviewing it systematically together as a group. Their work has been deeply intertwined, giving them a sense of community.

> Eva: The work on the R&D-assignment feels like a collaboration, and that the assignment I have written somehow is everyone's assignment. The whole process was done together with supervisors, the students and the practice school worked together.

The ST's sense of community extends to include all the supervisors of the R&D-assignment, however the STs especially emphasize the role of the practice teacher. How she consistently engaged in their R&D-work, giving advice on the different tasks STs planned on using and how to go about the actual work they needed to do in the classroom was vital.

> Matilda: We were the first who got to say what we wanted to write about. And we had a lot of supervision, but still we were the ones to produce everything. We have gotten help to figure out how we could do it, I feel. When we were working on a new intervention, then we were always first, then we got help to revise what we had thought about. And maybe that leads to the feeling that this is nevertheless our assignments.

As shown in Mathildas quote, the ST's simultaneously have a feeling of ownership of their R&D-work, pointing to how they participated in reaching the joint agreement of the umbrella topic on

establishing sociomathematical norms. Despite valuing the group supervision sessions and what they deem has been "close supervision", the STs find that individual supervision in the later stages allowing them more time and space to focus on their own writing is preferable.

Security in a situation of professional uncertainty

The STs give accounts of a feeling of safety in the process of the R&D-work, with several factors contributing to this feeling. Being in the same practice group in the same practice class for several years, freed up time spent otherwise on becoming acquainted and gave them freedom to try out teaching they otherwise would not have done.

> Eva: The fact that we know the pupils as well as we do makes it easier for us to try out something new in the teaching, to dare to explore more and to challenge ourselves.

Several of them mention how they now look for opportunities to get feedback on their teaching work, whereas they before would find similar situations to be unsettling. However, they report on feelings of unease about how to write the R&D-assignment.

> Matilda: We had never…we looked at old bachelor theses, but nobody had done it the way we did. And then it was…yes, how do vi start to write it? It became clearer to us as the supervision went on, but ahead of time it was difficult.

Doing R&D work has a built-in element of not knowing, but for the STs the stakes are even higher being graded on their written exposition of the work they undertake. They found a meeting plan for the whole semester, giving advice on which parts of the written assignment would be discussed in each meeting, helpful for organizing their writing. Being new to professional development work they felt was challenging, yet something they could overcome with help from what they perceived good supervision.

Possibility for in-depth focus on mathematics education

The STs point to the importance of the practice teacher having the same overarching approach to and understanding of what type of mathematics instruction is beneficial for pupils as themselves.

> Eva: We observed 7. grade, and it was very traditional teaching. In a way…make everyone see that […] that is really the main theme of our assignments, we have not focused on the answer. We have focused on how you got to the answer, and maybe that is more important than if it is correct or not.

The STs and the practice teacher being in accordance with each other on viewing the process of doing mathematics as more important than the actual answer was important for the STs. It enabled them to plan and conduct their R&D-work with a more process oriented and inquiring approach to mathematics teaching, in line with what they themselves hope they will do as in-service teachers. The STs see their learning outcome as having value beyond the mathematics classroom, Matilda for instance draws a parallel to establishing norms for the general learning environment in a classroom, stating that "the thinking" is the same whatever the subject. The R&D-work has allowed the STs a deepened understanding of the concept of sociomathematical norms itself through working on establishing a chosen norm and investigating what happens when such norms are developing in a classroom.

Ella: And what we kind of got out of mathematical efficiency was that they should find a way to solve a problem such that they don't need to spend so long time on, and that it should be as easy as possible to do. So, we did that by using quick-images. And we had talked about which questions to ask that could lead them to what we called a "clever" strategy and how to respond to their strategies.

In Ella and Mathildas work on implementing the norm of mathematical efficiency in the classroom research and knowledge on teacher strategies for eliciting pupils mathematical thinking became central. Eva's R&D-work on the norm of using drawing as a tool for mathematical problem solving made a striking impact. The situation changed from no pupils drawing initially, to 14 out of 17 drawing in their problem solving in the final mapping.

Eva: They started using it, but they used it to illustrate, so they drew nice flowers. It was very clear that they drew afterwards. […] In the end mapping, then they drew iconic drawings […] they talked about how a circle is det same as…or how you can draw a circle, you don't need to draw a flower to make it a flower kind of.

The STs became proficient users of concepts related to their R&D-work, like Eva pointing to not only how the drawings could be classified as different, but also how one needs to keep in mind how the pupil utilized their drawing in the problem-solving process.

Discussion

In keeping with both Antonsen and colleagues (2022) and Ulvik and Riese (2015), the STs in my study report on a positive outcome of their work on action research. The action research concepts have become ingrained in their way of talking about teaching, both in mathematics and more generally. But more than that, the STs report on a shift in their way of thinking and acting in the direction of taking inquiry as a stance (Cochran-Smith et al., 2009). In working on the R&D-assignment the STs have built a strong community within supporting each other's work. What might at first glance seem like a practical measure, finding and agreeing on an umbrella theme, has been imperative for the STs forming a professional community. The three sociomathematical norms worked on by the STs were quite different in nature and required them to familiarize themselves with diverse parts of mathematical educational topics and literature. However, the umbrella theme led to the STs working from a joint knowledge base from the start, while at the same time having a large degree of freedom to find their own focus. Contrasting findings from other studies (Antonsen et al., 2022, Ulvik and Riese, 2015) our practice teacher engaged fully in the R&D-work, taking part in the tripartite supervision and allowing the R&D-work to become a main objective during the practice placement. She provided systematic and rich opportunities for the STs to reflect on their work, becoming a more knowledgeable other for the STs. I argue that close collaborations with the practice field can give opportunities for student teachers to learn professional development skills while simultaneously gaining in-depth knowledge in mathematics education. Furthermore, it seems that research into the possibilities of STs doing action research in school as an integral part of initial mathematics teacher education, is not explored to its full potential in the research field.

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Connecting critical thinking to the learning of mathematics teaching practice

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At CERME, issues important for learning the practice of mathematics teaching are discussed in several working groups. For example, in TWG19, there are ongoing discussions about how to frame teaching. How mathematics teaching is framed has consequences for what knowledge teachers need and how they should develop that knowledge. At CERME14, two of the emerging themes in TWG19 were "paying attention to students and their contributions" and "talking about students without labelling them" in ways that are not beneficial for them and their learning. In this paper, these themes and three empirical examples gathered from small-scale studies about planning for inclusion are used for exploring critical thinking as a tool for learning the practice of mathematics teaching, which contributes to the ETC theme of connecting the learning of mathematics teaching to practice by highlighting critical thinking as a bridge between theoretical understanding and classroom realities.

Keywords: Critical thinking, mathematics teaching practice, teacher education.

Although each group has its own focus, there is a common ground for TWG 18, 19, and 20 that are included in the ETC Connecting the Learning of Mathematics Teaching to Practice – an interest in mathematics teacher practices including the work to be done by teachers and student teachers to create good opportunities for students to learn mathematics. A common interest is also development of competencies needed for that work – or "learning of practice" as stated in the call for paper for this ETC. Learning of practices is clarified with learning to "do" mathematics teaching. However, there are reasons to think of whether it is enough to learn how to *do* mathematics teaching. Firstly, what it means to learn a practice depends on what is meant by practice. Secondly, there are indications that it is not enough to *know* how to "do" for teaching to be that way. For example, in a study about planning in mathematics teaching it became clear that teachers sometimes made decisions that went against their knowledge about how to do mathematics teaching. In some cases, the reason seemed to be structures and opinions of other actors, such as students, parents, and school leaders (Grundén, 2020). These results might shed light on other aspects of learning the practice than learning to "do" mathematics teaching. Despite whether learning a practice is thought of as learning to do, or if something more is needed, to grasp the complexity of teaching the learning must cover several aspects.

At CERME14 discussions in TWG19 were summed up in some themes, two of which are particularly interesting in relation to learning of practice: "paying attention to students and their contributions" and "talking about students without labelling them" in ways that are not beneficial for them and their learning (Mosvold et al., in press). In this paper, these issues and "empirical seeds" from three studies are used for illuminating critical thinking – understood as questioning assumptions, seeing things from different viewpoints, and taking informed action (Brookfield, 2012) – as a cornerstone in learning the practice of mathematics teaching.

Empirical seeds for thinking

To link the discussion about learning the practice of mathematics teaching to the complex everyday life of teachers work, three glimpses taken from three studies are presented in this section. Each glimpse ends up in a question that was raised from the results of the three studies. The studies are recent small-scale studies at three schools in Sweden with an overarching interest in planning mathematics teaching with student diversity in mind. The schools were chosen for the studies because the teachers at the schools were part of development projects that aimed at giving all students the opportunity to develop as far as possible. The first question is from a qualitative survey with 24 teachers in which mathematics teachers from two schools answered open questions about what differences they consider when planning and in what ways they adapt their teaching to diversity. The second question was raised from analysis of letters that 15 teachers from one school have written in a project they called "Good teaching for all students." In the letter they wrote about what good teaching meant for them. The third question is from one 60 minutes long focus group interview with five mathematics teachers discussing in what ways they adapt content, process, and product based on cultural-, language-, social, and cognitive diversity among their students. Material from each of the three studies was analyzed to identify and make sense of patterns of meaning in line with Braun and Clarke's reflexive thematic analysis (2021).

Does it matter who the student is?

Preliminary results from the survey (Grundén, 2025) show that the differences the mathematics teachers at the two schools mainly consider are language; diagnoses, including ADHD and ASD; and previous knowledge. When it comes to language, the teachers seem to divide between those who have Swedish as their mother tongue and those who don´t, those who understand Swedish good and those who do not. The results indicate that it does not matter what language a student has as their mother tongue. In the answers relating to previous knowledge the dichotomy of "those who can and those cannot" emerged which raises thoughts that the students are labelled in ways that do not benefit their learning. The adjustments made based on the differences were mainly on an organizational level, for example group sizes, time spent on instructions, and starting all lessons in the same way. The question about adjustment was not formulated so that teachers were asked to write anything about *why* they made each adjustment. Nevertheless, analysis of the answers indicates that adjustments are made based on general ideas about what benefits most students, sometimes based on the labels weak and strong students. Hence, it seems reasonable to believe that individual characteristics of students do not matter when adjustments are made.

Valuing diversity – or not?

In three focus group discussions at a school where teachers and school leaders explicitly talk about including all students and embrace their differences , mathematics teachers were challenged to think about and discuss how they use cultural-, language-, social-, and cognitive diversity among their students to think about content, process, and product in their planning for mathematics teaching (Grundén & Roos, 2025). Results show that norms – both general and specific for mathematics – and a deficit perspective are often the starting point for planning. In addition, students in the conversation about planning are sometimes considered as non-knowers, which means that they are not perceived

as someone who can possess knowledge and pass it on to other people (Fricker, 2007) and their presence and participation in knowledge production are not valued (Grundén & Roos, 2025). A conclusion in the study is that despite the teachers' good intentions there are underlying structures where diversity is not valued (Grundén & Roos, 2025).

Where is the subject?

When 15 teachers at a school wrote individual letters about what they mean by "Good teaching for all students" the most prominent themes were: *students should be seen*, *students should feel safe*, and *teaching should be varied*. A common theme was also that the same structure should permeate all teaching which was exemplified by all lessons starting and ending in the same way regardless of subject, teacher or age of students. In many letters there were formulations that reminded of each other, which gave the impression that the teachers at the school shared ideas about what "good teaching" is. What was striking was what was *not* in the letters – references to content and subject, which means that the mathematics was gone (Grundén, 2025).

Paying attention to and valuing students

Although the empirical seeds are gathered from small-scale studies' results which are not yet published, they can be used for thinking about what it takes to learn the practice of mathematics teaching. The seeds provide indications so that possible answers to the questions *Does it matter who the student is?, Valuing diversity – or not?*, and *Where is the subject?* may be: *"No, it doesn´t always matter who the student is"; "Diversity is not always valued"*; and *"The subject risks disappearing when focusing on student well-being"*. The empirical seeds and the hypothetical answers to the questions raised can be related to two of the themes that were used to summarize the discussions at TWG19 at CERME14: and "talking about students without labelling them". The theme "paying attention to students and their contributions" was created based on discussion that arose from presentations about noticing from different perspectives and about listening from a perspective of openness and curiosity (Mosvold et al., in press). The empirical seeds above indicate that it does not matter who the student is, and diversity is not valued when the teachers are planning. Hence, despite teachers being part of developmental project and expressed intentions of "good teaching for all students", they are examples of when students and their contribution are not given attention. The theme "talking about students without labelling them" was formed based on discussions about students, their knowledge, and their needs that arose during many of the sessions. Often discussed during the conference was if "students *have* needs" or if "students *are in* need of" (Mosvold et al., in press), a discussion that is also alive in the special education field (e.g., Roos, 2023). The former expression indicates that the student owns the problem – there is a label on the student – while in the latter the teacher owns the problem – the student is in need of adjustments or special education. The empirical seeds can also be related to this. Not valuing diversity and talking about students in broad categories such *as know mathematics/doesn't know mathematics* or *speak Swedish/doesn´t speak Swedish (well enough)* speaks against not labelling students in ways that are not beneficial for them.

The empirical seeds and TWG19-discussions at CERME14 have shared features and can contribute to knowledge about what it takes to learn the practice of mathematics teaching. The examples above indicate that it is not enough for teacher education and development programs to focus on how to do

mathematics teaching. The examples indicate that there are – at least at these schools – assumptions about how it should be, i.e., norms that permeate mathematics teaching and seems to hamper teachers to attend to and value student diversity. One of the examples also indicates that when teachers focus on meeting students in diverse needs, the subject – in this case mathematics – falls into the background. These examples contribute to the understanding that learning of mathematics teaching practice is as complex as teaching within the practice.

Learning of mathematics teaching practice

To know what it takes to learn the practice of mathematics teaching there is a need to elaborate in the term "practice". The term is often used in mathematics education research, but what is meant by practice varies. However, within the field there has, according to Skott et al. (2018), been an evolvement in the meaning – from a focus on individuals' actions and behaviors (and later also knowledge, beliefs and intentions) to practice as a social phenomenon. When practice is seen as a social phenomenon, a common ground seems to be that practice means that *people act and interact* exemplified, for example, by Boaler (2003) who states that classroom practices are "the recurrent activities and norms that develop in classrooms over time, in which teachers and students engage" (p. 3). Grootenboer and Edwards-Groves (2013) have a slightly different description: "Practices both constitute and are constituted by the particular words used, the particular things done, and the particular relationships which exist in the interactions between the people and things involved". While Boaler adds *norms* to the description, Grootenboer and Edwards-Groves add *words used*, which means that in these two descriptions there is more to it than just people acting and interacting. In addition, both Boaler (2003) and Grootenboer and Edwards-Groves (2013) highlight that practices change over time. In this paper, I join Boaler, Grootenboer and Edwards-Groves, and others who see *practice of mathematics teaching* as an evolving social phenomenon in which people act and interact, use language both individually and together and are influenced by structures and norms. This means that actions and interactions within the practice are more or less situated since some actions are more habitual and related to "this is the way mathematics teaching is done". Kemmis and Grootenboer (2008) emphasize that practices are shaped by social and material conditions, and that collaborative reflection among colleagues is crucial for questioning assumptions and enabling meaningful change in teaching.

What it means to learn a practice depends on how *practice* is defined and how knowledge and learning are viewed. Regardless of definitions, mathematics teaching as a practice implies an interaction between students, teachers and mathematics, often visualized in various forms of the didactical triangle. With this foundation as a ground, it would be possible to say that learning a practice is learning how to organize situations so that it is possible for students to learn the mathematical content that is relevant. When speaking about learning a practice based on how *practice* is viewed in this paper, it poses challenges. The situatedness means that it is not enough to learn how to do mathematics teaching in what we see as the practice of mathematics teaching today. A teacher and a student teacher also need to develop tools to rethink what you have learned to meet what will be the practice of mathematics teaching tomorrow. In addition, an awareness of norms and structures and how they influence people, actions, and interactions in the practice is crucial.

Becoming and developing as a mathematics teacher

Learning the practice of mathematics teaching is part of becoming and developing as a teacher. Darling-Hammond (2023) emphasizes how new programs in which theory and practice are more intertwined focus more on prospective teachers developing capacity to systematically inquire learning and teaching. Accordingly, these programs provide a better understanding of teaching as complex and not possible to learn completely through simplistic formulas during teacher education. Prospective teachers develop their ability to see beyond one's own perspective and make thoughtful decisions (Darling-Hammond, 2023). Although Darling-Hammond focuses on the dichotomy of theory and practice and seems to represent a different view of practice than I do in this paper there are elements, such as systematic inquiries, understanding of teaching as complex, taking multiple perspectives into account, and thoughtful decision-making that can be beneficial in thinking about how to meet the challenges that arouse from thinking of practices as situated and evolving.

These elements from Darling-Hammond are also found in literature about critical thinking (e.g., Brookfield, 2012; hooks, 2010). According to hooks (2010) curiosity is at the core of critical thinking which is visible in a quotation from her: "the heartbeat of critical thinking is the longing to know – to understand how life works" (p. 7). Being curious means that one must go beyond what is most visible and see things from other perspectives and uncover core underlying "truths" (hooks, 2010). For this paper, the interpretation is that hooks does not mean "truth" as definitive in a positivist spirit, but "truth" as situated and contextual building on structures in a society. These thoughts might contribute to discussions about prospective teachers learning the practice of mathematics. By developing their critical thinking, students and teachers could meet challenges in the evolving practice with openness and curiosity Brookfield (2012) summarizes what it takes for critical thinking to occur: hunting assumptions, checking assumptions, seeing things from different viewpoints, and taking informed actions. The actions should according to Brookfield (2012) build on assumptions that "are justifiable according to some notion of goodness or desirability (Brookfield, 2012, p. 15), and hence, thinking critically in this way could contribute to just and equitable mathematics education where mathematics still is at the core.

For prospective teachers to develop as critical thinkers, they need to experience occasions where they are challenged during their teacher education. All students have experiences from their own schooling that makes them have assumptions about knowledge, learning, and teaching mathematics. During their practicum they will also come to schools where there are prevailing assumptions like the ones in the three empirical seeds. Hence, the teaching needs to be designed so that the students recognize, and question assumptions about knowledge in the specific discipline (Brookfield, 2012), which means that they must practice the key features of critical thinking: hunting assumptions, checking assumptions, seeing things from different viewpoints, and taking informed actions.

When hunting assumptions, Brookfield (2012) states that there are three kinds of assumptions that need to be explored: *paradigmatic assumptions* often thought of as valid renderings of reality, *prescriptive assumptions* that concern what should happen in certain situations, and *casual assumptions* that are assumptions about how things work and can be changed. The casual assumptions have to do with "if I do this, that will happen", sometimes based on past experiences. Examples of

the different kinds of assumptions in relation to mathematics teaching could be: mathematics is neutral (paradigmatic), using problem-solving when teaching mathematics is good (prescriptive), and using number line when number sense is taught will promote students' understanding (casual).

When teachers pay attention to students and their contributions the papers presented in TWG19 over the years and the empirical seeds presented above indicate that there are all three kinds of assumptions at play. An example of paradigmatic assumptions concerns knowledge, for example about what it means to know mathematics and who can contribute to learning in the classroom. Most certain, no teachers would be satisfied with the somewhat incisive and provocative answers – it doesn't matter who the student is, diversity is not valued, and mathematics is gone – to the questions raised from the three studies. However, the answers might provide an opportunity to examine and challenge assumptions underlying the answers. What assumptions are possible when it comes to language, previous knowledge and diagnoses in mathematics classrooms? What assumptions are there about students as knowers? And what are the assumptions about what is most important for a teacher? Hunting assumptions implies being creative and thinking of everything that could be an assumption and doing it unconditionally and not connected to one's own thinking might help finding the non-politically correct assumptions. According to Brookfield (2012) the hardest assumptions to get to grips with are the ones you have yourself and hence, working with colleagues in an environment where everyone is comfortable challenging and questioning each other is helpful. When all conceivable assumptions are on the table they need to be checked if they are valid.

Checking validity does most often not mean telling if assumptions are right or wrong but exploring the accuracy of an assumption taking the context and conditions into account. Assumptions in mathematics teaching practice are more or less accurate depending on, for example, research results about student learning, cultural and historical aspects, human rights and ethical and moral concerns, and laws and regulations of school system.

Seeing things from different viewpoints is another key to critical thinking. In issues regarding mathematics teaching, the teachers as well as the student´s perspective should be taken into consideration. The empirical seeds above contribute with aspects worth considering and to see these from a students' perspective, a teacher or a student teacher can, for example, ask themselves what it feels like to be student in a classroom where assumptions permeating the decisions the teacher makes saying, for example, that all students can contribute to the learning environment, or that to be perceived as a knower in mathematics classroom you must speak Swedish well enough.

The last step of the critical thinking process – taking informed action – becomes a reality when decisions about teaching are made, both in the process of planning and on the spot in the classroom.

Reflections

The teachers participating in the three research projects worked at schools where there was a stated ambition to develop teaching so that all students had the best conditions to learn. Teachers at the schools had been involved in several projects about differentiated instruction, special needs, and inclusion, and in many cases expressed a desire to provide teaching that benefits all students. Despite the good intentions and investments in the projects, questions were raised about taking students'

contribution into account and how students were talked about. There were also signs that the mathematical content was being overlooked. In summary, it can be said that the empirical examples indicate that there are assumptions, norms and structures that work against the teachers' expressed intentions. Thinking of mathematics teaching practice as social and evolving where awareness of structures and norms and their influence on people, actions, and interactions is crucial raises the idea that critical thinking can contribute to greater awareness of possible assumptions, accuracy of assumptions and different perspectives, which in turn would contribute to well-informed decisions. Thinking critically about aspects of one's teaching does not necessarily mean that everything should change, but that decisions made about teaching are well-founded, considering several different perspectives. Hence, including possibilities for prospective teachers and teachers to develop their critical thinking during teacher education and in continuing professional development is a cornerstone in learning the practice of mathematics teaching. Consequently, learning the practice would imply not only how to *do* mathematics teaching but also how to *think about* it.

Future directions

Future work will focus on how critical thinking can be integrated into teacher education and professional development. This includes developing strategies to expose and challenge assumptions and different perspectives. In addition, teacher education and professional development programs should address how to build environments where critical thinking is a natural part of teachers' everyday life. Moreover, as researchers we need to engage in critical thinking by making our own assumptions visible, examining them, and considering different perspectives. Discussions during this conference and the upcoming CERME conference provide excellent opportunities for such reflection, contributing to the development of the practice of mathematics education research.

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Developing pre-service teachers through STEAM competencies

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In this century, research has documented various cognitive difficulties in STEM learning, prompting debates and proposals for integrated approaches. We designed activities for pre-service secondary teachers to address science and mathematics concepts collaboratively. This paper analyzes an activity centered on mathematical modelling and its historical roots. Participants developed skills in visualization, prediction, and reasoning, though some struggled with identifying variables and building algebraic models. These findings highlight the value of STEAM in teacher education.

Keywords: STEAM, modelling, teacher education.

Introduction

The scientific community has long debated how to implement integrated STEM (Science, Technology, Engineering, and Mathematics) teaching approaches at the pre-university level. One clear indication of this is the recurring presence, across multiple editions of CERME, of a working group devoted to Mathematics in the Context of STEM Education. Previous studies suggest that the effective integration of these disciplines has proven more complex than initially anticipated (Chalmers et al., 2017; Kelley et al., 2016), revealing limited interconnection among the sciences, a lack of cross-cutting pedagogical strategies, and a scarcity of genuinely interdisciplinary activities.

This landscape raises a central challenge in teacher education: how can pre-service mathematics teachers be prepared to design and implement effective interdisciplinary STEM learning approaches? In response, various STEM teaching frameworks have been proposed that seek to connect disciplines more meaningfully in educational practice. For example, Oliveira et al. (2019) showed that pre-service teachers experience difficulties in synthesizing across areas (physics and statistics), but also recognize the benefits of interdisciplinary collaboration, the centrality of mathematics, and the supportive role of technology. English (2015, 2016) and Maaß et al. (2016) advocate incorporating mathematical modelling in STEM, identifying three pillars from the perspective of mathematics education, including a sustainability-oriented vision: (1) the development of 21st-century skills (critical thinking, creativity, collaboration), (2) authentic modelling processes, and (3) education for responsible citizenship capable of addressing real-world problems.

Building on these premises, we designed and implemented a programme of teaching activities for pre-service secondary education teachers with a STEAM focus, integrating the Arts into STEM in line with UNESCO initiatives (2017). These activities engage with physics and mathematics concepts in collaborative and contextualized environments, aiming to foster interdisciplinarity. A previous study reported an initial trial of this type of proposal (Camacho-Machín et al., 2024). In 2025, we continued this experimentation by refining the collaborative methodology and adding new integrated STEAM activities, maintaining a focus on mathematics education and sustainability within a broader set of five activity blocks centered on mathematical modelling. This article focuses on one of these

activities, involving experimentation with physical objects based on a classical problem posed by l'Hôpital (1696) in the first calculus book. In this way, the activity connects the history of mathematics with hands-on experimentation using a prototype of the pulley described therein. The study is framed by a sociocultural perspective on communities of practice and by the distinction between horizontal and vertical mathematization, which together provide both a teacher-education lens and an analytical lens on pre-service teachers' participation in a STEAM modelling task.

Against this backdrop, the aim of this research is to analyze how the implementation of activities following a STEAM design fosters the development of key skills in pre-service mathematics teachers and how these processes contribute to their training in integrated STEM competencies. In particular, the study examines the development of skills such as scientific visualization and intuition, anticipation and prediction of phenomena, conjecture formulation, reasoning and proof, as well as sensitivity to contradictions arising from unexpected results. More specifically, the study is guided by the following research questions: (1) How do pre-service teachers participate in a classroom community of practice while engaging in a STEM modelling task based on a historical pulley problem? (2) How do their modelling processes evolve from horizontal to vertical mathematization throughout the activity? (3) What opportunities and challenges for developing integrated STEM competencies emerge from this design in the context of mathematics teacher education?

By analyzing a classroom experience with pre-service teachers through this dual lens, the study seeks to show how a STEAM approach centered on mathematical modelling can help connect theoretical learning with teaching practice in mathematics, thereby addressing the central problem identified.

Conceptual framework

In this study, we adopt a dual and complementary theoretical framework that articulates: (i) A sociocultural perspective on communities of practice, which provides the context for understanding how pre-service teachers engage in realistic mathematical practices, is enacted through ACODESA, and guides instructional design by taking team-work dynamics into account. (ii) The theory of mathematization, based on the distinction between horizontal and vertical mathematization to analyze modelling processes, which serves as the analytical lens for interpreting the cognitive and modelling processes that emerge during the activity.

From a sociocultural perspective, we understand learning as participation in a community of practice, in which participants share a joint enterprise, mutual engagement, and a shared repertoire (English, 2015, 2016; Kelley & Knowles, 2016; Maaß et al., 2016). This approach justifies the collaborative design of the activity and makes it possible to analyze how pre-service teachers engage in situated mathematical practices: discussing, arguing, negotiating meanings, and jointly constructing solutions.

The ACODESA methodology (Hitt, 2007) operationalizes these principles in the classroom through cycles of collaborative work, scientific debate, and self-reflection, culminating in the institutionalization of knowledge by the teacher. In our study, the sequence of individual–pair–team work (the 1–2–4 structure) and the final whole-class discussion organize the work of the teams as small communities of practice, in which pre-service teachers not only solve a modelling problem but also experience ways of working that they can later transfer to their own classrooms.

Furthermore, the activity design aligns with sustainability competence frameworks in higher education (Brundiers et al., 2021; Wiek et al., 2011) and with UNESCO (2017) guidelines for integrating sustainability into STEAM education, as well as with recent work on integrated problem-solving competency in mathematics teacher education for sustainability (Guerrero-Ortiz, Camacho-Machín, & Hitt, in press). The community of practice that emerges in the classroom provides a context for working on competencies such as systems thinking, anticipation, complex problem solving, and collaboration, while mathematical modelling functions as a central tool for addressing socially meaningful problems. In this way, the proposal brings together the development of professional teaching competencies, education for sustainability, and the integration of disciplines within a STEM perspective (Camacho-Machín et al., 2024).

To analyze the cognitive processes developed by participants during the activity, we adopt Freudenthal's (1991) distinction between horizontal and vertical mathematization. Horizontal mathematization refers to the transition from a concrete real-world situation to a mathematical model, drawing on visualization, intuition, and the initial formulation of conjectures. At this stage, skills such as visualizing phenomena and scientific intuition (Fischbein, 2002; Zimmermann, 1991), the ability to anticipate and predict behavior, and conjecturing based on empirical observations of the physical device or the simulation are particularly relevant. Vertical mathematization, in turn, involves refining and deepening the model within the mathematical system itself, for example by generalizing, formalizing, or proving results. Here, processes such as sensitivity to contradiction (recognizing and questioning anomalous or inconsistent results), rigorous argumentation, proof, and deeper mathematical investigation of the problem are emphasized (Hitt, 2004). This distinction allows us to describe how participants move from qualitative descriptions and spontaneous sketches to explicit functions, derivatives, and the analysis of extreme points.

This dual framework is complemented by work that highlights the pivotal role of mathematical modelling in STEM contexts (Camacho-Machín et al., 2024; English, 2015, 2016; Maaß et al., 2016), where modelling connects physical phenomena, technological tools, and algebraic reasoning. In our case, the historical pulley problem and its reconstruction provide a setting in which pre-service teachers must identify variables, establish covariational relationships, construct geometric–algebraic models, and validate them using GeoGebra. We understand the development of the activity as the intersection between a community of practice in formation and a process of horizontal and vertical mathematization enacted around a modelling task with historical and interdisciplinary significance.

Methodology

The study adopts a qualitative, interpretive approach through an educational case study. It consists of a teaching intervention implemented in a real teacher education context, with the aim of describing and understanding in depth the learning processes experienced by participants during the implementation of a STEAM modelling activity. Given the exploratory nature of the study, no a priori testable hypotheses are formulated; instead, the research questions focus on how pre-service teachers develop certain skills (visualization, conjecturing, argumentation, sensitivity to contradiction, etc.) throughout the experience and how they respond to the integration of tools, disciplines, and methodologies in the activity.

The design of the intervention is grounded in the dual theoretical framework adopted: communities of practice and the ACODESA approach structure the organization of collaborative work, while the distinction between horizontal and vertical mathematization guides the analysis of modelling processes. In this sense, forming work teams and engaging them in the joint resolution of a contextualized modelling task is intended to reproduce key features of a community of practice and to foster collective reflection on the task. The implementation followed these steps: (i) initial individual work, where each student tackles the problem and formulates first ideas or conjectures; (ii) pair discussion (1–2) to contrast and enrich their approaches; (iii) work in teams of four (2–4), where a common solution or strategy is consolidated; (iv) whole-class discussion, moderated by the instructor, to share solutions, compare approaches, and clarify concepts; and (v) an individual self-reflection phase, in which each pre-service teacher revisits and reconstructs their understanding after the collaborative work and the institutionalization of knowledge.

Methods

The experimentation was carried out with students enrolled in a master's program in secondary mathematics teacher education. The course comprised 14 two-hour sessions, three of which were devoted to the proposed situation. This part of the course corresponds to the final segment of the theoretical component of the program, when students are expected to engage in problem solving and task design for upper secondary education. The sample consisted of 19 pre-service teachers (S01, S02… S19). Each participant was given a printed copy of the activity (Tables 1 and 2). In addition, each team was provided with a rod, a string, and a pulley (all of varying sizes). Each team was to construct a device like the one proposed by l'Hôpital and answer related questions (Table 2).

Table 1: Situation presented to the pre-service teachers

The pulley problem

In 1696, the Marquis de l'Hôpital published the first calculus book, *Analyse des infiniment petits*. We present a problem from that book (pp. 51–52):

"Let a pulley F hang freely from the end of a cord CF attached at C, with a weight D suspended from the cord DFB, which passes over pulley F and is attached at B, so that points C and B lie on the same horizontal line CB. Assume the pulley and cords have no weight; the question is where the weight D will come to rest."

The task is to determine the path followed by the weight and to find the point at which it stops.

The figure used by l'Hôpital:

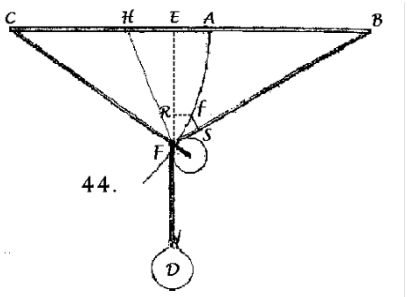

Table 2: Questions attached to the scenario

Answer the following questions.

1) After analyzing the phenomenon using the materials provided, identify the variables at play that can be used to provide a solution to the problem. A suggested first approach is to use the following data: CB bar of 1 meter, CF rope of 40 cm, DFB rope of 1 meter. This process will allow you to anticipate, predict, and conjecture before moving on to the algebraic modelling stage.
2) List the variables for a covariational analysis (one independent variable and one dependent variable).
3) Carry out a modelling process that can be used to find an algebraic model that addresses the scenario.
4) You may use GeoGebra to create a geometric model that facilitates solving the problem. Compare it with the algebraic model.
5) Based on all the above, write down the solution to the proposed problem.
6) Include any additional considerations you think are relevant to enrich this scenario.

For data analysis, several sources of evidence were collected: individual written responses to the activity (at two different times), team reports with consolidated solutions, GeoGebra files produced by pairs or teams, classroom observations carried out by the researchers, whole-group reflection documents prepared by the research team, video recordings of discussions, and institutionalization documents prepared by the instructor. The data were treated through qualitative coding guided by the dual theoretical framework: episodes and products were identified that related to horizontal mathematization and vertical mathematization, as well as features of communities of practice in team dynamics. The information from the different sources was then triangulated to reconstruct how participants' ideas evolved over the course of the activity and to answer the research questions.

Discussion

From the outset, during the individual work phase, participants showed a clear interest in understanding the scenario. Two elements of the problem statement generated early comprehension difficulties: the diagram included in the historical text and the condition "We assume that the pulley and the ropes have no weight" (Table 1). This idealised assumption conflicted with their physical intuition about the expected behavior of the system and hindered their ability to anticipate the motion of the weight, as the task required. As a result, several participants sought support from peers, initiating an early process of horizontal mathematization.

In response to these shared difficulties, the Research Team (RT) projected an enlarged version of the original diagram on the board. Observing that participants began to break the silence expected in the individual phase—verbalising their doubts and comparing interpretations—the RT reorganised the dynamics of the session: teams were given the materials to construct the device and collaborative work began. Three teams spent additional minutes discussing their initial intuitions before assembling the prototype, while the other two chose to experiment immediately and discuss afterwards based on what they observed. By the end of the session, individual responses (without GeoGebra work) had been collected; in the next session, teams compared their spontaneous representations and early ideas. This comparison between individual productions and collective discussion continued to nourish horizontal mathematization and activated the collaborative dynamics anticipated in the design.

Individual responses showed persistent difficulties with variables: some students chose quantities that were constant in the setup (e.g., mass or rope length), while others did not relate dependent and independent quantities or used vague labels such as *position* or *point*. From the standpoint of horizontal mathematization, these findings highlight that translating a real situation into a mathematical model (the mathematization process) is far from trivial and reinforce the relevance of including activities that develop scientific intuition, visualisation, and explicit treatment of variables in teacher education.

After constructing and manipulating the apparatus, and once the allotted time had elapsed, the final individual responses were collected. Fourteen participants proposed a model: two produced empirical descriptions of the trajectory; two relied on prior physics knowledge (a system of equilibrium equations or a damped oscillation model); the remaining participants approached the problem using geometric–algebraic methods (Pythagoras' theorem, the law of cosines, circle equations, or trigonometric ratios). Their proposals included parametric approaches—where the angle between the

rope and the bar served as the independent variable, though some added constants mistakenly as "variables"—and functional approaches, defining the independent variable as the horizontal projection of the weight and the dependent variable as its height or its distance to point B. This diversity of perspectives enriched pair and group discussions, fostering the confrontation of conjectures. Returning to the physical setup and to GeoGebra allowed participants to validate or revise their models, developing sensitivity to contradiction and supporting vertical mathematization. Team reports documented the conclusions reached through these collaborative exchanges.

Classroom observations and written reports showed that the use of GeoGebra unfolded in two phases: first, constructing a virtual replica of the device; second, validating the proposed models. Because the physical assembly was not always precise, the more accurate simulation enabled some participants to discard initial erroneous conjectures and others to confirm their observations. One team even returned to the physical prototype after building the digital model to explore the effect of pulley mass, which helped them understand why l'Hôpital omitted gravity from the original formulation and resolved the question that emerged at the beginning of the activity.

Clear differences emerged in the second phase. Team 1 focused on fitting a function to the point-trace obtained from the physical device, without revisiting the algebraic analyses developed earlier; in contrast, the other four teams reached consensus, explicitly formulated a function, and determined the equilibrium point by locating its minimum or maximum (Figure 1). Team 5 submitted both approaches. Individual difficulties persisted and influenced team outcomes: for example, Team 1 did not correct the initial mismatch between dependent and independent variables. This reinforces the need for a whole-class phase followed by individual reflection.

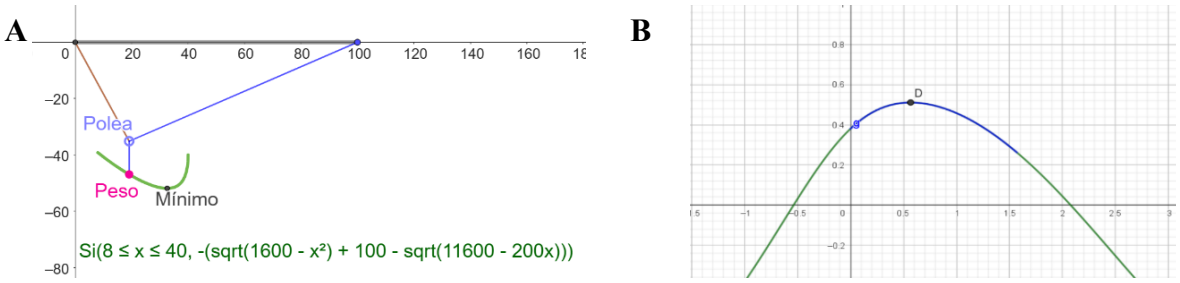

Figure 1: (A) Model validation; (B) Superposition of functional and geometric representations

In contrast, Teams 3 and 4 not only reached consensus but also integrated individual contributions. Team 3 formulated a parametric equation with the angle as the independent variable and incorporated a physical discussion, reworking an early proposal on damped oscillation into a compound pendulum model. Team 4 extended the situation by experimentally analyzing the role of pulley mass and how it modified the behavior observed in GeoGebra.

Conclusions

In light of the framework of horizontal and vertical mathematization, the findings show that the STEAM activity supported the progressive development of modelling competencies in pre-service teachers. Initial difficulties emerged in identifying relevant variables and establishing dependency relationships, and early attempts relied more on intuition than on formalization. However, as the activity unfolded, participants advanced from qualitative sketches to explicit geometric–algebraic

models and showed episodes of sensitivity to contradiction and conjecture revision, illustrating a transition from horizontal to vertical mathematization.

Regarding the integration of tools and disciplines, the combination of the physical artefact, the historical problem by l'Hôpital, and dynamic modelling in GeoGebra provided a rich environment for connecting real phenomena, visual representations, and symbolic formalization. The manipulable device and the digital simulation enabled anticipation and prediction, made the equilibrium point visible, and allowed participants to contrast and refine their models. Some teams even extended the task—for instance, analyzing the effect of pulley mass—illustrating how a well-designed STEAM activity can open space for situated mathematical inquiry.

From the sociocultural perspective of communities of practice and the ACODESA methodology, the 1–2–4 dynamics and the whole-class discussion enabled four of the five teams to function as genuine communities of practice: they shared a joint enterprise, negotiated meanings, and built collective solutions integrating individual contributions. The team that struggled to consolidate its work demonstrates, however, that such structures do not automatically ensure equitable participation and highlight the need to continue supporting role distribution, group cohesion, and collaborative reflection in teacher education.

Overall, this study provides evidence that implementing mathematical modelling processes within a STEAM framework—supported by a dual theoretical lens combining sociocultural perspectives and mathematization—constitutes a promising approach for linking mathematics learning to teaching practice. The designed activity allowed pre-service teachers to experience teaching practices grounded in authentic problems, collaboration, and integrated use of technologies, which they can later transfer to their own classrooms. Future work should deepen these proposals, incorporating new interdisciplinary situations and exploring in greater detail how communities of practice and mathematization trajectories evolve throughout initial teacher education.

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Developing an inquiry-based teaching practice: Guided reinvention through activities and teaching strategies

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Inquiry-based mathematics teaching is a highly recommended teaching approach in the Norwegian curriculum and is also supported in the research literature on mathematics education. However, teachers who are used to a more traditional teaching approach struggle to teach mathematics inquiry based. This paper explores one teacher's development of inquiry-based teaching practices, drawing on the theory of Realistic Mathematics Education (RME). Building on a pilot study where particular activities and teaching strategies were tried out, the paper focuses on the teachers' experiences from the study. Analysing data from reflective discussions with the teacher, the paper discusses how the use of concrete teaching strategies together with well-structured activities can guide the teacher in developing an inquiry-based teaching practice aiming for students' robust mathematics learning.

Keywords: Inquiry-based teaching, realistic mathematics education, guided reinvention, mathematics learning.

Introduction

In Norway, the current curriculum emphasises inquiry-based mathematics teaching, aiming for robust mathematics learning for all students (Ministry of Education and Research, 2019). There is a huge research literature supporting inquiry-based mathematics teaching for students' robust learning (Horn & Garner, 2022; Schoenfeld, 1992, 2018). This literature focuses on teaching practises aiming for students' development of their own mathematical thinking and learning through work on open-ended tasks and whole class discussions. However, although inquiry-based teaching is not new in mathematics education (Artigue & Blomhøj, 2013), this teaching approach represents a new course in the Norwegian curriculum and hence also a new approach to mathematics teaching for many teachers in Norway. Research on the implementation of the curriculum reports that teachers are struggling to teach mathematics inquiry-based (Kacerja & Gustavsen, 2023; Pedersen & Haavold, 2024). Teachers describe that they feel they are losing control of the students' learning of mathematics when they try to teach inquiry-based. Teachers also report a lack of competence in inquiry-based teaching practices. Doing research on inquiry-based teaching practices, it is therefore important to acknowledge the struggle teachers report about this way of teaching. This paper aims to explore teachers' development of robust teaching practices for inquiry-based mathematics teaching in the context of Realistic Mathematics Education (RME). The paper draws on data from a pilot study where one teacher tried out selected activities and teaching strategies, building on RME, aiming for development of inquiry-based teaching. The paper argues that teachers, in development inquiry-based teaching practices for students' robust mathematics learning, can be guided by a well-structured teaching material. The focus of the paper thus connects to the aim of the conference, asking about what foundational things mathematics teachers need to learn to do as they learn to teach, about what is involved in learning mathematics teaching, and how this connects to how to design and carry out professional education for the learning of practice.

Background literature

Discussing the history of inquiry-based mathematics education, Artigue and Blomhøj (2013) point back to Dewey's ideas as a starting point. He talked about reflective inquiry and the importance of students' learning attached to real-life situations, hands-on activities, and the learners' experience in the implementation of inquiry-based learning. However, despite a long history and a huge research literature advocating inquiry-based teaching, in schools, mathematics teaching has mostly been dominated by the more traditional and teacher-led approach for a long time. Ball (1993) argues that the conception of the mathematical content in teaching inquiry-based mathematics is more uncertain for teachers than teaching mathematics in a traditional approach. Developing an inquiry-based teaching practice can therefore be challenging when teachers are used to a traditional teaching practice. Ball (1993) focuses on the pedagogical complexities a teacher should handle in developing an inquiry-based teaching practice. One of these complexities is "[to develop] a practice that respects the integrity both of mathematics as a discipline *and* of children as mathematical thinkers" (Ball, 1993, p. 376). Aiming for students' learning of mathematics in inquiry-based teaching, Ball (1993) emphasises that the teacher both needs to be present in the teaching and listen to students' thinking, concurrent to have a focus on the mathematical horizon to know where the students are going. Having this complexity in teaching inquiry-based, Lubienski (2002) argues that teachers need to learn about this complexity, aiming to develop good practices. Horn and Garner (2022) point out that this learning of an inquiry-based teaching practice for teachers entails more than just learning a particular way of teaching. Teachers also need a shift in their concepts of teaching and learn to teach inquiry-based mathematics, therefore requires both conceptual and cultural change for teachers.

Studying support of novice teachers' learning of inquiry-based teaching, Lampert et al. (2013) also focus on the pedagogical complexities a teacher meets. Seeing the teacher as the guide for students' learning, Lampert et al. (2013) argue that the teacher's role in giving students responses must do "double duty". Besides building relationships with the students and involving them as sense-makers in classroom activities, the teachers also need "to push them toward higher order learning goals" (Lampert et al., 2013, p. 227). Beside this, to work on development of an inquiry-based teaching practice, Lampert et al. (2013) builds on three sites which is seen as important; practices, principles, and mathematical knowledge, emphasising that these three; "must be used in relation to one another, not in isolation. Furthermore, they must be used in relationships among teacher, students, and the content to be learned" (Lampert et al., 2013, p. 228). To explore teachers' development of inquiry-based teaching and the difficulties they meet, this paper addresses the research question: How can teachers develop robust teaching practices in inquiry-based mathematics through the use of mathematics activities and teaching strategies, building on RME?

Theoretical framework: Guided reinvention

In this paper, inquiry-based mathematics teaching is seen through the lens of Realistic Mathematics Education (RME). As a theory of learning, RME aims to build students' deep learning and development of emergent mathematics (Gravemeijer & Stephan, 2002). The theory can be explained through four heuristics: *mathematizing, context, emergent models*, and *guided reinvention* (Stephan et al., 2014). Seeing inquiry-based teaching in the context of this theory, mathematics in RME is all

about processes of mathematizing and hence students' inquiry of patterns and connections in mathematics. Rooting the mathematics in realisable contexts, the idea in RME is that the students will develop their mathematical thinking by moving gradually from the informal mathematics in the context towards more formal mathematics. In such learning processes, emergent models are crucial for the students' development and learning of mathematics. Starting as a concrete model connected to a context, the models will gradually emerge into more abstract models for more formal thinking. Through such processes of mathematizing, the aim is that students will develop their thinking by reinventing the mathematics (Van den Heuvel-Panhuizen, 2003). The teacher's role in this is to guide and support the students in their learning process, enabling a guided reinvention of the mathematics for the students. Focusing on the process of guided reinvention for inquiry-based teaching, the teacher needs to be open up for each student's thinking, and at the same time hold on to the mathematical horizon (Stephan et al., 2014). The teacher's guiding role also involves leading and orchestrating the work of inquiry-based teaching. This includes preparing for the development of socio-mathematical norms, which opens for inquiry-based teaching with access to participation and discussions of the mathematics for all students (Stephan et al., 2014).

Methodology

This paper builds on a pilot study focusing on one teacher's development of inquiry-based teaching practices. The study was organised as participatory action research. I, as the researcher, collaborated with one mathematics teacher to try out selected RME activities and teaching strategies, aiming for the development of inquiry-based teaching. Both the activities and the teaching strategies build on the four heuristics of RME. All material is developed by Manchester Metropolitan University (Manchester Metropolitan University, 2025).

The teaching material was tried out over eight weeks in two parallel grade 6th classes at one Norwegian school. The teacher normally taught mathematics in the two classes, but neither the students nor the teacher was used to inquiry-based teaching. The teacher and I selected together what activities to try out, which I translated into Norwegian. Parallel to the activities were also teaching strategies tried out. The teacher planned the details of the lesson and was also the one who taught the lessons. My role was a participant observer. I could ask questions and take part in classroom discussions, both in whole group teaching and in small group discussions. After each day of teaching, the teacher and I had reflective discussions, which were all audio-recorded. These reflective discussions constitute the data source for this paper. The conversations focused on the teacher's reflections on the students' work on the activities and the teacher's teaching practices in each lesson, and potential development throughout the period. The RME material was selected because of the particular structure of the activities and teaching strategies. We wanted to try out how the teacher could be supported in inquiry-based teaching by this material. Focusing on the area of polygons, all the activities were structured around contexts that opened for the square unit as an emergent model for the students' mathematizing of area. The activities were organised in a particular order with a gradual progression, presenting a variety of different approaches to use the square unit as a model. The teaching strategies consists of eight concrete teaching practices or actions the teacher can use to guide the students in their processes of guided reinvention and mathematizing. Four of the strategies were focusing on promoting mathematical discussion: "Can you say what he/she said?", "Make time

for thinking", "Remain neutral", and "Convince us". The other four focused on supporting students' mathematizing: "Go back to the context", "Where can you see…?", "Say what you see", "What is the same, what is different"?

The paper draws on preliminary analysis of the teacher's reflections and utterances from the reflective discussions. At the same time, because of the close collaboration between the teacher and me, I had to be clear about my own voice. Since I also took part in the conversations, the teacher's reflections would potentially be influenced by my view and role as a researcher. Focusing on the teacher's development of inquiry-based teaching in the context of RME, the audio recorded conversations were analysed by searching for references to the four RME heuristics: context, emergent models, mathematizing, and guided reinvention, in the teachers talk. For instance, references to mathematizing were evident in the teacher's talk about students' inquiry when they were searching for and comparing different strategies to solve a task. References to emergent models was noticed when the teacher talked about students' different strategies of counting squares in a grid. A reference to guided reinvention was when the teacher talked directly about her use of the teaching strategies, but also when she described how a particular activity guided one student's reinvention. Following the ethical guidelines, the study was reported to the Norwegian Agency for Shared Services in Education and Research (Sikt), and all participants voluntarily consented to participation. Anonymity of the teacher was assured.

Analysis

In this section, I present the results from the analysis, structured in two main sections. The first section focuses on how the teacher experienced the use of the activities and the teaching strategies for orchestrating inquiry-based teaching practices in the classroom. The second focuses on how the activities and the teaching strategies supported the teacher in developing inquiry-based teaching with a targeted focus on the mathematics.

Inquiry-based teaching practices in the classroom

Throughout all the reflective discussions, the teacher was clear about the use of the activities opened for an inquiry-based approach in the classroom. She said she experienced that the activities involved the students in exploring and discussing the mathematics. This description of exploring can be seen in the context of students' mathematizing in RME. The teacher explained how she saw the RME activities enabling the students to think differently and solve problems differently, compared to what she called the "usual" activities. She thought this was because the RME activities did not ask for one right answer, but instead focused on the students' strategies in solving the problem. As a result of this, the teacher had noticed that the students had started to share and compare their different strategies. She described how these discussions had involved all students, regardless of their different knowledge backgrounds in mathematics. According to the teacher, no strategy was evaluated as better than others. She referred to the Pricing Slates activity (figure 1) as an example of how cutting pieces of paper became an often-used strategy for all students to compare different areas.

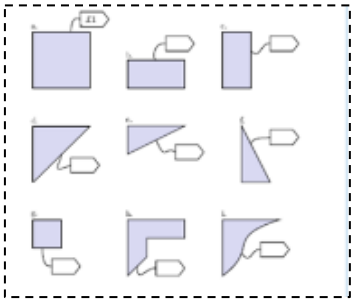

Figure 1: Examples of activities opening for more than one answer

Focusing on mathematizing, the RME activities can be argued to open for inquiry and discussions of the mathematics, and more student-centred lessons because of the structure.

The analysis also revealed references to mathematizing in how the teacher talked about the use of the teaching strategies. Already in the first reflective discussion, the teacher was clear about that she had planned to use particularly two of the teaching strategies: 'Make time for thinking' and 'Remain neutral'. She argued that giving the students time for thinking, but also opportunities to dwell on the activities as important in inquiry-based mathematics. Reflecting on her use of these strategies in a later conversation, the teacher confirmed that the strategies had influenced the students' opportunities for inquiry. She pointed out how 'Make time for thinking' had resulted in more of the students getting involved in discussions and inquiry. At the same time, using this strategy had also been important for her to manage to dwell on different activities in teaching. This refers to guided reinvention in RME and the teacher's role in guiding the students' inquiry. She described the strategy as a difficult practice to stand in, but being prepared to use it made it more manageable in the moments of the lessons. In one of the last reflective discussions, looking back at her own teaching, the teacher argued that her use of 'Make time to think', together with 'Remain neutral' had resulted in more opportunities for the students' inquiry. In classroom discussions, 'Remain neutral' had helped her particularly not to evaluate the students' answers or to focus on one right answer. References to the teacher's role in guided reinvention were also evident when the teacher described difficulties in using the activities and teaching strategies, particularly in situations when the students did not take part in discussions and inquiry work. She described situations when some students rejected the invitation to participate. Reflecting on this, the teacher thought this could be because the students were not familiar with this approach of mathematics teaching. In such situations, she described that she ended up taking over and leading discussions herself. The teacher reflected on how she could have used the teaching strategies more focused, but also that she needed to practice developing a more flexible use of the strategies in her teaching.

Inquiry with a targeted focus on the mathematics

The teacher also reflected on how the structure of the activities supported a targeted focus on mathematics in the students' inquiry work. Referring to the role of context in RME, the teacher described how the students could use the price as a mediating tool for solving problems on area in some of the activities. As in the Pricing slate-activity (figure 1), area was not asked about directly, but instead the students had to compare different areas based on the price of one square unit. The teacher also referred to how this activity followed a previously introduced context on slate mining.

The teacher then pointed out to the role of different approaches to the activities. She referred to the next activity, Jewellery designs (figure 2), which asked for the prices of different shapes of glass jewellery. Like in the Pricing slate-activity, a unit price was presented. The teacher pointed out the differences between the activities. In the Pricing slate-activity, the price was given for a big square unit, whereas in Jewellery designs, the price was given for a small square unit. Referring to RME, the teacher recognised that both activities had the square unit as a model for students' thinking, but the different presentations invited different approaches to solve the problem.

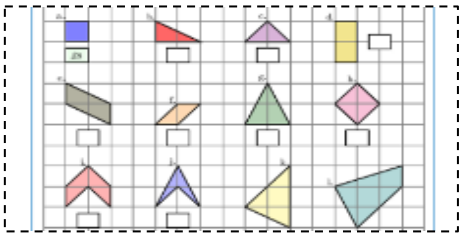

Figure 2: Activity Jewellery Design

The teacher also pointed out that on the Pricing slate activity, the figures were presented on a clean sheet, but on the Jewellery activity, the figures were presented on a grid. She described how this had invited the students for different strategies in the lessons. On the Pricing slate activity, the students solved the problem by dividing the square unit into smaller pieces and cutting the figures into pieces. In Jewellery design, most of the students had counted the number of square units. The teacher gave her reflections about how the different activities all focused on area, but with different approaches. She thought that the different structures of the activities both gave students access to different strategies to solve the tasks but also opened up for the development of the different strategies. The teacher also reflected on how the structure of the activities together with the different approaches altogether was helpful for her to focus on the mathematics in her questions and in orchestrating of classroom discussions. The teacher's reflections and descriptions of the structure here all connect to emerging models in RME, but at the same time guided reinvention. Starting from the concrete model of the square unit, the students can develop their thinking about area when the models gradually emerge into more abstract models of formal thinking. The process is guided through the structure of the activities.

Discussion

Going back to the starting point of this paper, teachers describe that they are struggling to teach inquiry-based mathematics. Seeing inquiry-based teaching as important for students' learning of mathematics, this paper asks the research question: How can teachers develop robust teaching practices in inquiry-based mathematics through the use of mathematics activities and teaching strategies, building on RME? The analysis of the teacher's reflections shows that the use of the RME material, the activities, and the teaching strategies together supported the teacher in developing inquiry-based teaching practices with a targeted focus on the mathematics. The structure of the activities opened for the students' inquiry of the mathematics, and the teaching strategies supported the teacher in her orchestration of an inquiry-based teaching. Besides this, the role of emergent models in the different activities was important to ensure a targeted focus on the mathematics in the students'

inquiry work. The structure of the activities and the different approaches of using the unit squares guided the students' reinvention of the mathematics.

However, the structure of activities was not just guiding the students' inquiry processes. The teacher reflected on how the different approaches to using square units in the activities helped her to see different approaches to area. Stephan et al. (2014) argue that in the process of guided reinvention for inquiry-based teaching, the teacher needs to understand the students' thinking but also hold on to the mathematical horizon. This is in line with Ball (1993), who argues that developing inquiry-based teaching emphasises the importance of the integrity of both mathematics as a discipline and of children as mathematical thinkers. The different approaches to the use of square units in the activities, but also the progression of this throughout the different activities, may have enabled the teacher to develop her understanding of students thinking. We can thus argue that the use of emerging models in the activities guided the teacher in a process of guided reinvention of inquiry-based teaching.

However, although a well-structured teaching material can support the teacher in developing robust teaching practices for inquiry-based teaching, inquiry-based teaching also requires the students' participation and engagement. The teacher described how she ended up taking over classroom discussions when the students did not respond as she wanted. Development of inquiry-based teaching also needs the development of socio-mathematical norms, which opens up for inquiry-based mathematics practices (Stephan et al., 2014). The teacher needs to prepare a classroom practice where all students participate, classroom discussions, and students sharing strategies and ideas are acknowledged as accepted classroom practices for learning mathematics. This may require a shift in social norms of communication and participation, both for the students and the teacher. Horn and Garner (2022) talk about how inquiry-based teaching entails more than just learning a particular way of teaching. They argue that the development of inquiry-based mathematics teaching requires both conceptual and cultural change for teachers. The teacher in the study reflected on the need to practice more on using the different teaching strategies. To support teachers' learning of inquiry-based teaching, Lampert et al. (2013) emphasise the importance of rehearsals to practice the act of teaching inquiry-based. This is both to make sure the mathematics is there, but also that the students are invited into the targeted discussions, leading to the development of their mathematical thinking. In the study, the teacher did not just practice the teaching strategies and use of activities. She had the opportunity to talk about and reflect on her own practice together with me as a researcher. Working together with teachers to develop inquiry-based teaching practices can therefore be helpful to understand more about the complexities the teachers face.

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Preparing mathematics teachers for collaborative professional reflection on classroom situations – research into PSTs' feedback cycles on analyses of PST-generated classroom cartoons

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The goal of building up mathematics teachers' professional competences in collaborative criterion-based reflection on classroom situations raises the question how such competences can be fostered in pre-service teacher (PST) education. Having PSTs elaborate classroom cartoons which require criterion-based analysis offers the possibility of PST-centred analysis- and feedback cycles, in which the PSTs who have created the classroom cartoons provide feedback to the analysing PSTs. Little is known empirically about the learning potentials and learning processes related to such PST-centred analysis- and feedback cycles. Consequently, this explorative study focuses on this research need. In the sample criterion domain of analysing how representations are dealt with in classroom situations, we analysed more than 100 PST-centred analysis cycles related to 36 PST-created classroom cartoons. The results indicate high learning potentials and possibilities of further reflection support.

Keywords: Noticing, analysis of classroom situations, pre-service teacher-centred analysis and feedback cycles, pre-service teacher-generated classroom cartoons, mathematics pre-service teachers.

Introduction

Integrating vignette-based work on classroom situations in the mathematics pre-service teacher (PST) education context requires research into the potentials of profession-related learning opportunities and related learning processes. The professional development goal of strengthening pre-service teachers in professional competences of collaborative criterion-based reflection on classroom situations can be translated into learning opportunities, in which PSTs have to analyse classroom situations represented in cartoon vignettes and receive criteria-based feedback by their peer PSTs who have purposefully designed the classroom cartoon. Like this, not only the use of theory-based criteria for analysing classroom situations is in the centre, but also collaborative work among the PST peers is an intended focus. Eliciting the PST's development of related competences can be expected to be important for their later professional practices. However, research into specific learning potentials and learning processes related to such profession-related learning opportunities is scarce, so that this study can respond to a substantial research need by exploring the quality of PST-centred analysis and feedback cycles based on classroom cartoon vignettes designed by PSTs according to focus criteria. The evidence was analysed by interpretive methods and yields insight into learning potentials and possibilities of improving the PST activity.

In the following, we will (1) introduce the theoretical background, (2) deduce research questions, (3) inform about design and methods, (4) report key results, which will be (5) discussed in a concluding section.

Theoretical background

As a theoretical framework which affords analysing pre-service teachers' (PSTs') analyses of classroom situations, we refer to *teacher noticing* (e.g., Sherin et al., 2011; Amador et al., 2021; Fernández et al., 2018). In the domain of mathematics education, the *knowledge-based reasoning* component of mathematics teachers' noticing plays a key role, as the use of professional knowledge, e.g., knowledge related to criteria for high-quality use of learning opportunities in the classroom (e.g., Kuntze et al., 2021) can be expected to play a key role for the teachers' situation interpretation and possible follow-up (re-)actions (cf. also Kersting et al., 2012). In prior publications, we have referred to this competence aspect of noticing by using the term teachers' *competence of analysing* (Kuntze & Friesen, 2018; Friesen & Kuntze, 2016), several of the corresponding studies have used representations of practice (Buchbinder & Kuntze, 2018), so called vignettes, which showed classroom situations with analysis requirements. Such vignettes not only represent practice contexts in the form of classroom situations, but they also stand for elements of theory, which have played a role during the design of the vignette or during its selection from larger sets of empirical data. When PSTs are asked to design classroom cartoons (cf. e.g., Kuntze & Krummenauer, 2023, 2025) with analysis requirements related to specific theory-based criteria, the resulting vignettes can be – in the case of successful design – expected to represent theory, through the built-in analysis requirements.

If peer PSTs are asked to analyse such a PST-designed classroom cartoon vignette, it might happen that their noticing resp. analysis does not focus on the built-in criterion (cf. corresponding findings of unsuccessful noticing in e.g., Friesen & Kuntze, 2021). In this case, feedback by the PST who has designed the vignette might be a help for the analysing PSTs in their activities of building up competence in criterion-based analysis of classroom situations, provided that the feedback draws the attention to considering the relevant criterion in the situation analysis. Moreover, such an exchange among PSTs can help to develop the PSTs' professional competences in collaborative criterion-based reflection on classroom situations – competences which can be expected to be valuable for their later professional life. Quality aspects and the effectiveness of PST-centred *analysis and feedback cycles* around the analysis of PST-designed vignettes, however, have to our knowledge hardly been explored so far in empirical studies. Consequently, for insights into the potentials of such profession-related learning opportunities in mathematics teacher education, corresponding research in needed and the explorative study presented here focuses on this research need.

Analysing how representations are being dealt with in classroom situations

Building on prior research into teachers' analysis of how representations of mathematical objects are being dealt with in classroom interaction (e.g., Dreher & Kuntze, 2015; Friesen & Kuntze, 2016, 2021), this study choses the corresponding theory-based criteria: In the seminar (cf. Kuntze & Krummenauer, 2023), PSTs learned about key theory elements related to representations, representation registers and conversions between representation registers (Duval, 2006) and analysed classroom situations against these criteria: Due to their abstract nature, mathematical objects are only accessible through different representations; in different ways of representing, so-called representation registers, different properties of the mathematical objects may become more or less accessible, so that conversions between different representation registers play a key role for solving

problems (Lesh et al., 1987). As these conversions can be complex for students, providing targeted help to students related to conversions is a key feature of instructional quality in mathematics classroom situations, across a large variety of contents. As a result of this universality, these theory-based analysis criteria are very suitable for creating profession-related learning opportunities based on PST-designed classroom cartoon vignettes and for corresponding analysis and feedback cycles.

Research questions

As highlighted above, there is a need of empirical research related to learning opportunities associated with PST-designed classroom cartoon vignettes and their use in peer PST analysis as well as feedback cycles centred in these peer PST analyses. For finding out about the related learning potential, it is of particular interest to explore to which extent the analysis and feedback – i.e., the exchange in the analysis and feedback cycle related to the vignette – are related to the target criteria, in this case criterion knowledge related to how representations of mathematical objects are being dealt with in the interaction shown in the vignette. In order to explore learning potentials related to PST-designed classroom cartoon vignettes and related to the corresponding follow-up analysis and feedback cycle, the following research questions are in the foreground if this study:

(1) To which extent do the PST-designed vignettes contain analysis requirements related to the target criteria? (2) To which extent do the peer PST analyses of these PST-designed vignettes refer to the target analysis criteria? (3) To which extent does the PSTs' feedback to peer PSTs' analyses refer to the target analysis criteria?

Design and methods

This explorative study aims at answering the research questions through an interpretive approach to PSTs' classroom cartoon vignettes, their written analyses and their written responses/feedback to other PSTs' analyses. This data was generated in the framework of semester-long vignette-based seminars on dealing with representations of mathematical objects (described in more detail in Kuntze & Krummenauer, 2023): After a short introduction to the relevant theory elements, PSTs analysed vignettes, with reflection in the seminar plenary, then they were asked to choose a content and learning material and to design a vignette for their peer students, in which representations were dealt with in a non-optimal way. In this phase of vignette development, the designing PSTs received feedback from their teacher educators. On the base of initial text vignettes, the PSTs then represented their vignettes in the form of classroom cartoons, with the help of a refined version of the digital DIVER CREATE tool (www.coreflect.eu for information about an earlier version). Figure 1 gives an overview-like impression of three classroom cartoon vignettes in order to showcase the character of this PST-designed material. The PSTs added analysis prompts to their vignettes recalling the analysis criteria, the prompts were thus relatively similar across the different vignettes.

Then, in a Moodle-based environment, seminar-overarching (virtual) small groups were formed for the analysis and feedback cycles. In a first step, the group members analysed the vignettes of their peers in their group and uploaded their analyses to Moodle. At this stage, the PSTs could not see their peers' analyses, only the peer PSTs' analyses of the vignette they had designed. In a second step, PSTs had to respond to their peers' analyses of their vignette, related to the analysis criteria the

seminar focused on. In a third step, all analyses and feedback were made visible to all participants and they were asked to reflect on all the analyses of their peers and the feedback responses in their individual portfolios.

Figure 1: Three examples of PST-generated classroom cartoons with analysis requirements related to dealing with representations of mathematical objects (task material graphical elements hidden for potential copyright issues; graphical cartoon elements: © coReflect@maths)

This design hence aimed at individual analysis in the first steps, group interaction was rather mediated through the structure of the asynchronous work in the Moodle environment. The parallelised structure of the learning environment affords an analysis of the vignettes and of the associated analysis and feedback cycles.

There were 36 participants, who were grouped in 9 groups comprising of 2 up to 5 PSTs (the small-size groups resulted from unexpected absence or inactivity of PSTs). As each PST had to analyse all vignettes from the peers in their groups and received feedback, there were 102 complete analysis and feedback cycles in total which could be analysed. As each PST had designed a classroom cartoon, these analysis and feedback cycles were related to 36 designed classroom cartoon vignettes.

The interpretive analysis was done in a top-down, criterion-based way (content analysis, Mayring, 2015). The coding concentrated on three-element-data units comprising of PST-designed vignette, its analysis by a different PST, and the feedback/response by the designing PST. For the interpretation of the findings, it may be relevant to recall that there is a tree structure behind the data units (Figure 2), as analyses depend on a joint vignette and as PST group effects might play a role for the quality of analyses and the related feedback.

Corresponding to the first research question related to the vignettes, a first, dichotomous "entry code" controlled whether the vignette contained analysis requirements in the criterion domain of the seminar, i.e., related to representations. An additional, more explorative code focused on whether there were built-in additional, potentially distracting analysis requirements related to different criteria which might have the potential of having to be analysed in addition to the seminar-related target criteria. By these means, the initial overview and control coding of the vignettes could assure that there were representations-related analysis requirements in the classroom cartoon, and whether different, potentially distracting analysis requirements played a role in the vignettes.

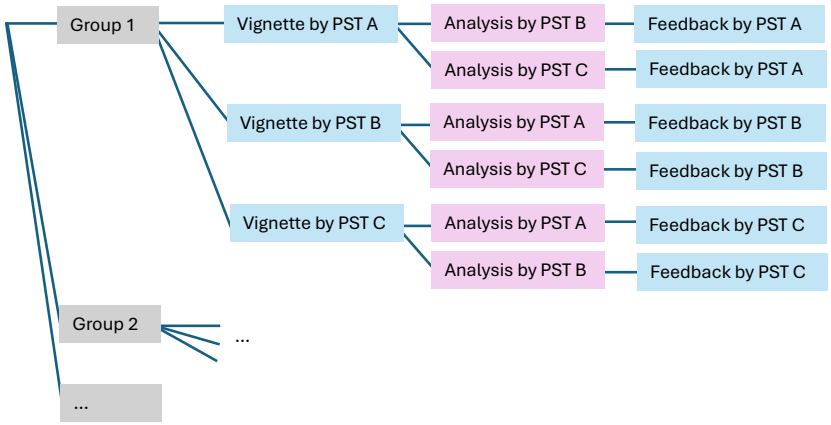

Figure 2: Tree structure of units of analysis

The second coding step aims at responding to the second research question focused on the PSTs' analyses of the PST-generated classroom cartoons. This coding focused on whether the target criteria and related situation aspects were addressed.

The third coding step corresponds to the third research question and concentrated on the vignette-generating PSTs' feedback to the corresponding analysis. Again, it was coded whether the feedback is criterion-focused in its response to the peer PST's analysis.

Consequently, the coding yields a set of codes for each unit of analysis and affords a first, indicator-like overview of the quality of the learning potential, at the micro-level of each unit of analysis, and at the macro-level of the whole sample.

Results

The first research question concentrated on whether the PST-designed vignettes contained analysis requirements related to the target criteria. The coding based on the interpretive analysis of the classroom cartoon vignettes yielded that in all vignettes, there were analysis requirements related to non-optimal aspects of dealing with representations of mathematical objects. In most cases, difficulties to be analysed arose from missing connections between different registers of representation, which may cause challenges for students' understanding. The results thus warrant that all cases of analysis and feedback cycles, i.e., the units of analysis, can be included in the following steps of the coding.

The second research question focused on whether peer PST analyses of the vignettes referred to the target analysis criteria. The coding showed that in 22,5% of all units of analysis, the peer PST's

answer did not yield evidence of reference to the target criteria. Here is an example of a peer PST analysis with a short summary of key results of the interpretive analysis:

"- The teacher uses the notion 'heaps', whereas in the task, the notion 'towers' is used.
- The pupil appears to have difficulty in seeing the relationship of towers/heaps and the number of cubes with the division task.
- The teacher does not support establishing the link between towers/heaps or the number of cubes with the division task.
- Foreground more active trying out and action.
- Make it possible to the pupils to search themselves for the divisors and enact this with material in a supportive way, for making clear dividend and divisor.
- Moreover, I would adapt the language to the specific task."

The answer mentions issues related to the non-optimal use of representations ("The teacher does not support establishing the link between …"). However, the analysis merely implicitly mentions to which extent representation registers remain largely disconnected in the vignette and what could be improved with this respect. The vignette teachers' language use appears almost as a further analysis criterion, instead of analysing it as a sub-feature of the way how representations are being dealt with, for instance. Suggestions such as "Foreground more active trying out and action" remain unexplained.

In order to find out about the learning potentials related to the analysis and feedback cycles, the third research question explores whether the PSTs' feedback to the peer PSTs' analyses refers to the target analysis criteria. According to the coding, this was the case in 77,5% of the cases, i.e., of the units of analysis. The feedback to the sample PST analysis presented above was the following:

"Hi [name],
you have noticed the problem with the notions used (heaps-towers). The teacher uses a meaning-related reasoning language ('4 heaps of five') without having introduced it beforehand or explaining it. Further, you have noticed that it is difficult for the pupil to translate the possibilities found on the enactive level to the symbolic level (division tasks).

It can be added that the teacher indeed partly relates to the pupil's [representation] register, e.g., by saying 'You have now just noted how high the heap is but the number of heaps is missing.' Despite of this in this moment, as you also have explained, connections by the teacher, in order to make the pupil aware of where exactly in the cube towers the division task can be found, are missing. Hence, I would not say that the active manipulating and trying out with the material has to continue to be in the foreground, as this is largely clear to the pupil. As you have mentioned, it should rather be made clear through corresponding stimuli how, in the material, dividend and divisor can be recognised. […]"

This feedback relates back to the peer teachers' analysis several times. By positively acknowledging what the peer teacher has noticed (in the view of the PST who gives feedback), the observations are somewhat reframed according to the focus criteria, with the apparent aim of making the analysis more exact: The (enactive) register of the pupil is named as such and identified as disconnected, even if the vignette teacher "partly relates to the pupil's register". However, the vignette teacher misses to

provide “stimuli how, in the material [i.e., in the pupil’s representation register], dividend and divisor can be recognised”. The feedback hence details the analysis with more exactness, which is a base for a critique of the analysing PST’s unexplained suggestion to “foreground more active trying out”.

Discussion, conclusions, and outlook

The example feedback cycle shown above gives insight into the learning potential that can be associated with PST-centred analysis and feedback cycles related to PST-designed classroom cartoon vignettes: The PSTs’ analysis can be deepened and PSTs can work towards shared standards of exactness in their situation analysis. Teacher (re)action alternatives can be discussed against the background of the target criteria. The PST who has designed the vignette and gives feedback is in a role of expert of her/his classroom situation vignette, as s/he has designed it based on the target criteria. However still, responding to the peer PST’s analyses is also a learning opportunity for the designing PST, as possibly different perspectives can interact through the analysis-feedback cycles.

Both the frequency of analyses and feedback responses using the target criteria and the sample analysis and feedback cycle underscore the high potential for profession-related learning of the activity. Certainly, more research is needed into the learning processes of mathematics PSTs in this vignette-based professional learning method. Exploration of how help could be provided to PSTs who did not relate their analyses or feedback to the target criteria is needed and indicates a pathway for follow-up research. Moreover, the potential role of the PST group, in particular, might have an impact on the profession-related progress of those who participate in this collaborative learning format, even under the structured conditions of the analysis-feedback cycles presented in this study. Vignette-based learning settings will continue to be in the foreground of the coReflect@maths project work.

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The use of lesson plays in learning to teach programming

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In this paper, I explore the use of lesson plays in a programming task as an approximation to practice in mathematics teacher education. As programming has become a significant part of the Norwegian mathematics curriculum, it is important to help our pre-service teachers find ways to integrate it into their mathematics teaching. I suggest writing lesson plays as one way of doing this. That means they write a script of an imagined discussion between themselves as teachers, and some imagined pupils working on a programming task in a mathematics lesson. In my study, I have examined the types of questions third-year pre-service teachers ask the imagined pupils and whether they are able to maintain the mathematical content evident in the lesson play.

Keywords: Mathematics teacher education, programming, lesson plays, approximation to practice.

Introduction

In 2020, programming was introduced into the Norwegian primary and lower secondary school curriculum in what is known as "Kunnskapsløftet 2020" ("The knowledge promotion"). It was introduced as part of the core element "Exploration and problem solving" in mathematics, and there are competence aims related to programming for each year from the 2nd to the 10th year in mathematics (Kunnskapsdepartementet, 2019). In addition, competence aims related to programming are present in other subjects, such as arts and science, indicating that programming plays a significant role in the Norwegian curricula. This means that every teacher who teaches maths in grades 1 to 10 must deal with programming in some shape or form during the school year. For many teachers and pre-service teachers (PSTs), this is a new and challenging topic with which they have little to no experience, and therefore, are unsure about how to incorporate it into their teaching (Bocconi et al., 2018). Hence, it is important that teacher educators focus on how we can help student teachers incorporate programming into their mathematics teaching.

One way suggested in research to help teacher students learn to teach is to use *approximations of practice* (Grossman et al., 2009). Using an approximation of practice means reducing the complexity of a teaching activity while rehearsing core practices of teaching. This has also been suggested as a means of bridging the theory-practice gap (McDonald et al., 2013). One such approximation of practice is writing imagined discussions called lesson plays (Marmur & Zazkis, 2022). In a lesson play, "part of a lesson is presented in dialogue format" (Zazkis et al., 2009) and teacher educators can draw attention to specific parts of a lesson in a very detailed way.

In this paper, I want to investigate the use of lesson plays on programming in mathematics teaching. In particular, I am interested in how PSTs incorporate both mathematics and programming into their lesson plays. To examine this, I have used lesson plays with third-year PSTs on the mathematical topic of geometry, with tasks in the programming environment "Scratch". The use of lesson plays in conjunction with programming activities is a topic that has not been thoroughly researched

previously, and it is a field I will contribute to by investigating how PSTs incorporate programming and mathematics when writing lesson plays. To do this, I will answer the research question:

"How do third-year PSTs integrate mathematics and programming when writing lesson plays on programming?"

I study lesson plays written by third-year PSTs in a Norwegian teacher education programme using an inductive content analysis to identify the types of questions asked and how they support pupils in reaching the mathematical goal of the task.

Theory

As opposed to writing a lesson plan, the writing of lesson plays lets the PSTs prepare for teaching while focusing on *how* the content will be communicated in the classroom (Zazkis et al., 2009). The precise, mathematical language and how we justify ideas are brought to the forefront. At the same time, PSTs are given opportunities to focus on pedagogical aspects while writing a lesson play. What questions should you ask, and how will the pupils respond? What parts of the tasks do you think the pupils might struggle with, and how do you help them? And as a lesson play is a representation of *one* specific imagined interaction, the PSTs have to make some choices, which provide opportunities for reflection on whether or not that was the right choice (Zazkis et al., 2009).

The use of lesson plays has been explored in many different settings. Arnesen and Valenta (2023) examined the use of lesson plays in reasoning and proving and found that students incorporated several teaching practices related to proving. Zazkis et al. (2009) used lesson plays to highlight particular difficulties for pupils, and Marmur and Zazkis (2022) used lesson plays to explore a claim concerning fractions. The use of lesson plays with programming in mathematics is a topic I have not found any studies on to date.

As programming has been included in mathematics curricula in several countries (Bocconi et al., 2018), an important question is how to combine the two subjects so that pupils can learn both mathematics and programming. Resnick et al. (2009) noted that the difficulty of learning to program hindered the potential of using programming in mathematics. Brennan (2015) points to the tension between the accessibility of the tool and the process of learning through it (Benton et al., 2016). Previous studies have shown contradictory results on whether programming can act as a mathematical environment. Some studies showed evidence of positive impact, while others did not detect any such evidence (Yelland, 1995, as cited in Forsström & Kaufmann, 2018). The same contradictions are also found in other studies (ibid).

Although studies have highlighted the challenges of integrating mathematics and programming, there is also evidence that specific areas of mathematics can benefit from the use of programming (Benton et al., 2018). Indeed, "programming in schools has been shown to have the potential to develop higher levels of mathematical thinking" (Benton et al., 2016, p. 28). Examples of such topics are algebra and geometry (Noss, 1968, 1897, as cited in Benton et al., 2018), multiplicative reasoning, mathematical abstraction and problem solving (Clements, 2000, as cited in Benton et al., 2016).

Scratch is a free-to-use programming environment that allows kids to learn programming through the use of visual blocks (Benton et al., 2018). The use of Scratch has become very popular for beginner

programmers, as it is very visual and allows users to build large programs without learning much syntax. By adding the "pen" feature, pupils can explore geometric shapes and their properties. An example of how you can build a simple program that draws a regular polygon of the user's choosing can be seen in Figure 1. Here, the cat, "Scratchy", has drawn a hexagon.

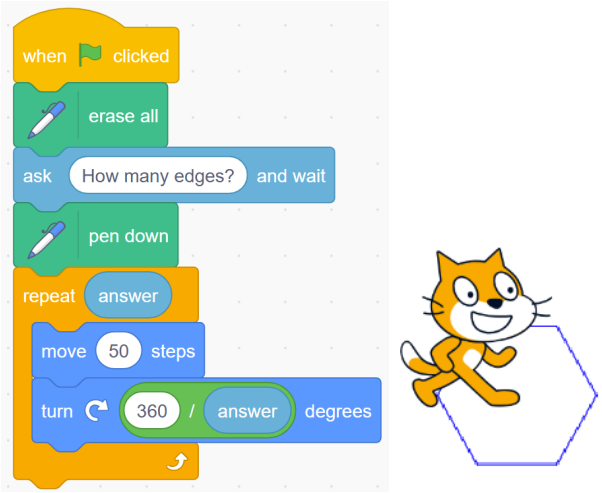

Figure 1: An example of a program in Scratch and the corresponding output

Methods

Participants and data collection

The participants of this study were third-year PSTs in a primary school teacher education programme in Norway. They have chosen mathematics as a specialisation, which means that they will write their master's thesis in either mathematics, special needs pedagogy or social pedagogy. The data material was collected from a written assignment in their mathematics course, with the theme "digital tools in teaching".

The task asked the PSTs to imagine themselves as teachers in a fifth-grade classroom, teaching a lesson on geometry and programming in Scratch, as shown in Figure 2. In fifth grade, the pupils are 10 to 11 years old.

The data consisted of 7 lesson plays written by 14 PSTs in pairs. Before this assignment, the PSTs had written a lesson play on reasoning and proving and had attended two lessons on programming in Scratch. In those lessons, the PSTs were taught the basics of programming in Scratch, as well as an introduction to PRIMM (Sentance & Waite, 2017), a model for teaching programming that focuses on discussions around code rather than writing code and discussing syntax. In geometry, the PSTs learned the van Hiele levels of geometric thought (van Hiele, 1959) in their second year, with a short refresher in their third-year course.

The task

The task the PSTs were given was as follows (translated to English):

Imagine you are a teacher in fifth-grade classroom and are planning to have a lesson in geometry and Scratch. Previously you have been working on the properties of different quadrilaterals and are now going to continue that work by making a program in Scratch.

Your goal for the lesson is to investigate the properties of geometric figures and the pupils have gotten the following task:

1. Make a program where Scratchy draws a parallelogram. Use a loop in the program.
2. Modify the program so the user of the program can give input on how long the sides should be, and one of the angles.

a) Do the task yourselves.
b) Write an imagined discussion (lesson play) between you - the teacher, and two pupils who are working on the task. Picture that you enter the conversation while they are working on the second task. What problems might they have encountered? How can you make them focus on the mathematical content? What questions would be useful to pose? The discussion should be 1,5 to 2,5 pages long.

Figure 2: The task given to the PSTs

Ethical perspectives

All the PSTs who participated in this study completed a consent form in which it was clearly stated that neither participating nor not participating would affect their assessment in the course. All the lesson plays were anonymised when they were downloaded from the learning environment where they were submitted. If some of the fictitious pupils' names could be considered identifying, these were changed by the.

Method of analysis

In analysing the lesson plays, an inductive content analysis was used. The lesson plays were compiled into a single document and uploaded to NVivo. First, I identified all the questions the PSTs asked the pupils and assigned a code to each question. Then I looked for themes and similarities, and grouped similar codes into categories. The gathering of codes resulted in six categories: Mathematics, Programming, Mathematics and programming, Ask for justification, Referring to the task and Checking in. Examples for each category can be found in the Findings section. Lastly, I looked at each lesson play in turn to see how the different types of questions shaped the conversations and helped lead the pupils towards the mathematical goal.

Findings

In the seven lesson plays, 68 questions were identified. From these 68 questions, the six categories mentioned above emerged.

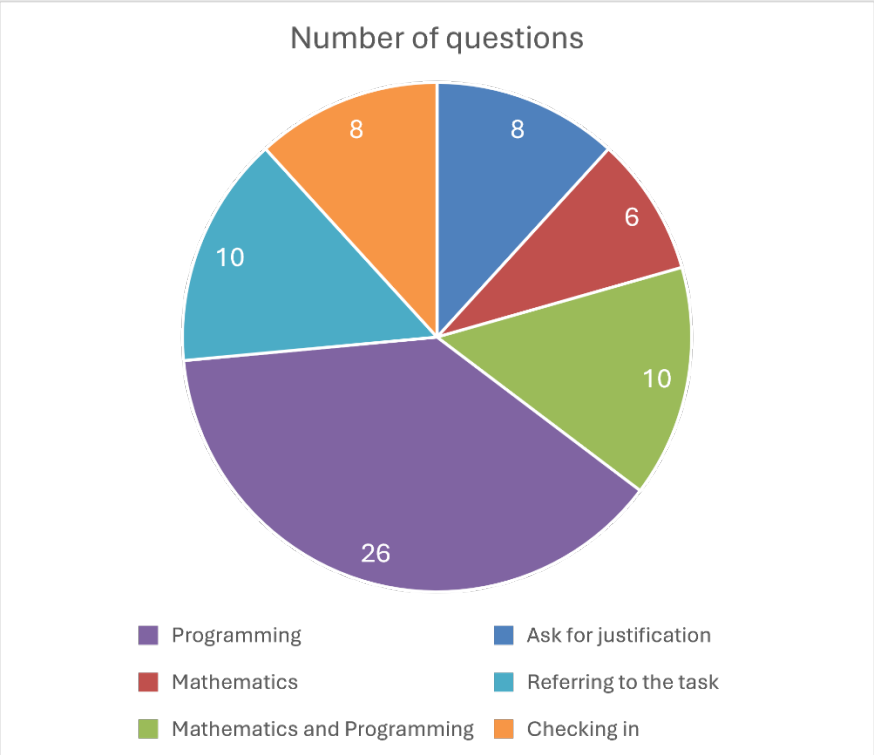

Figure 3: Overview of the categories of questions from the data

Four of the main categories consisted of questions that focused on the content of the lesson. Either the mathematical content, the more technical parts of the programming task, or a mix of both mathematical and programming content.

The largest category was the category concerning **programming** ideas. Here, the questions typically referred to which blocks the pupils thought would work and how, or to what they had told the pen to do. For instance, "Have you found out which blocks you can use?", "What does this block do?" or "What commands do you have on the pen?"

Then there were the questions that focused solely on the **mathematical** aspects of the task. In this category, the PSTs highlighted the properties of the parallelogram that the pupils were making, as a way to help them figure out the next step. Examples are "How many degrees are in a parallelogram?", "How many edges are there?" or "How do you want to explore the mathematical properties of the parallelogram?"

In addition, some questions combined both the **mathematical content** and the **programming** ideas. Here, the questions focused either on the properties of the parallelogram and connected it to parts of the program or clarified where in the program the pupils had defined or placed key variables. They asked, "How many angles should be inside the loop?", "How many times are you going to draw this parallelogram?" or "Can you point to where angle 1 and angle 2 are in the parallelogram you have here now?"

Lastly, the PSTs asked the pupils for **justification**. Not justifications for mathematical statements, but why the program did what it did, or why they thought specific things happened in the program. They asked, "What could be the reason for this?" relating to why the pen used the wrong variable, or

“How can we know this?” when the pupils claimed that a variable was connected to a certain question in Scratch, or “Why do you think that?” when a pupil claimed that the angle would be correct.

In addition to the questions that highlighted the content in the task, the PSTs posed questions that contributed more to starting the conversation or sussing out what they had done so far or what they might need help with.

They **referred to the task** to see what the pupils had done so far, whether it could help them in the next part of the task, or whether they could use specific elements of one task in the other. Examples of this were, “What did you do in the first task?”, “What are your thoughts so far?” and “How can you use this knowledge in the second task?”

Lastly, the PSTs **checked in** to see how the pupils were doing, whether they found the task difficult, or if they were figuring it out. All were variations of “How is it going?” and appeared at the beginning of a lesson play.

The only category that appeared in all seven of the lesson plays was programming. In addition, all the lesson plays had one or more questions concerning mathematics or mathematics in combination with programming, except for one. This one lesson play stood out as both shorter than the other six and as a conversation in which the pupils did most of the talking. The prompts from the teacher mostly helped the pupils figure out the correct order for the blocks in the program without much focus on the mathematics.

As the goal for the imagined lesson was to investigate the properties of geometric figures, it is interesting to see how the questions posed help support the pupils in reaching this goal. Generally, the questions that supported the pupils in reaching the goal were those concerning mathematics or a combination of mathematics and programming. These questions pointed to the size of the angles, how the angles appeared in pairs, and how they only needed information about half the parallelogram to actually draw it. The questions concerning programming did not necessarily help the pupils towards the mathematical goal. Still, they helped the pupils to move forward in the code so they could once again focus on the mathematics. Here is an excerpt from one of the lesson plays:

Teacher:	Let’s start with the angles. How many degrees do we have in a parallelogram?
Pupil 1:	400?
Teacher:	How many edges are there?
Pupil 1:	4… in a quadrilateral there are 360 degrees, then the parallelogram will have the same!
Teacher:	Right, and if you’re going to make a loop where you can do the same thing two times, how can you do that?
Pupil 1:	Half? So half of 360 degrees is… 180!
Teacher:	Yes! And how many angles do you need to have inside the loop?
Pupil 1:	2, that together will be 180! (keeps working)

The teacher starts by asking about the mathematical content to help the pupils get started with the program and point them towards the properties of the parallelogram. Then they follow up with a question about the technical part of the code and, lastly, ask about the angles in regard to the loop, combining the mathematics and the programming.

Discussion

In this paper, I aimed to highlight a possible approach to utilising lesson plays in programming and integrating programming into mathematics teaching by examining how third-year PSTs incorporate mathematics and programming when writing lesson plays. My hope when giving the PSTs the task was that the mathematical content would be clearly visible in the lesson plays. Studies have shown that learning to program can hinder the mathematical content (Resnick et al., 2009). Still, in some cases, the use of programming can lead to higher levels of mathematical thinking (Benton et al., 2016). The seven lesson plays in my study showed that, even though it can be challenging, it is possible for PSTs to bring the mathematical content to the forefront.

When examining the data, questions focusing on content accounted for more than half of the questions asked. Although the largest category, by far, is the questions about programming ideas, the questions concerning mathematical content or a mix of mathematical content and programming ideas account for 25% of the questions. All lesson plays except one had at least one question concerning either mathematics or a combination of mathematics and programming. So, the PSTs are able to keep the focus on both the mathematical content and the programming part of the task. This finding aligns with Benton et al. (2018), who mention geometry as one of the mathematical areas in which programming can benefit learning.

Zazkis et al. (2009) highlight the opportunity to work on both the mathematical and pedagogical aspects of a task when writing lesson plays. The same can be said of the PSTs in this study. All the lesson plays had aspects of both content and pedagogy. They gave the PSTs opportunities to focus on how they could help pupils struggling with both mathematical and more code-centred issues in their imagined conversations.

The results of this study imply that the use of lesson plays on programming in mathematics teacher education can be beneficial for PSTs, as it allows them to focus on both the mathematical content and programming ideas, as well as the pedagogical aspects of a task. Still, further development of the task given might be fruitful. A next step could be to ask the PSTs to also include screenshots of the code they imagine the pupils have made. In addition, ask the PSTs to reflect on the moves they make during the lesson play and discuss with each other to help them focus even more on how they can incorporate mathematical content while working on a programming task.

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Opportunities for preservice teachers' training arising on the design of scale 3D models and associated activities adapted to visual disabilities

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In collaboration with institutions supporting people with visual impairment, the project Geo3DTouch addresses the creation of adapted resources to provide them an inclusive and quality education. One of its research lines also aims to contribute to preservice teacher's training by means of the design and creation of adapted 3D models and associated activities that contribute to the development of the spatial orientation of Primary School students with visual disabilities. We use the didactic-mathematical knowledge model (Godino, 2024), combined with the orientations provided by Gonzato et al. (2011) about how to develop Primary School students' spatial visualization and orientation skills, as a reference framework to describe the several opportunities to learn mathematics teaching that arose from the associated preservice teacher's training experience, paying special attention to the cognitive and mediational facets, as well as to the common pure mathematical knowledge.

Keywords: Preservice teachers, didactic-mathematical knowledge, spatial orientation, 3D models, visual impairment.

Introduction

The Geo3DTouch project addresses the necessity of creating adapted resources to provide an accessible and quality education to visual impaired students at different study levels, from Primary School to the University, by means of the use of emerging technologies such as 3D printing, virtual or augmented reality, as well as artificial intelligence. One of the project's research lines aims to contribute to preservice teacher's (PSTs) training by means of their implication on the design and creation of adapted 3D models with Tinkercad together with the proposal of some associated activities that contribute to the development of the spatial orientation of Primary School students with visual disabilities. We focus here on analysing and discussing the opportunities to learn mathematics teaching that arose during an associated PSTs training experience.

The creation of this type of 3D materials is supported by recent technical studies on the creation of resources that contribute to improve orientation and mobility of the community (Holloway et al., 2018; Holloway et al., 2023, Nagassa et al., 2023). They outline several advantages that visually impaired people finds when information is provided through 3D models or icons compared to the usual 2.5D maps or labels, some of them related with the relative height, the better recall of features that make it easier to construct a detailed mental imagery or directional awareness at vertical connectors in multilevel models. These technical specifications should be considered by PSTs during the training experience to properly design and create their inclusive educational 3D resources. They should also take into account the orientations provided by Gonzato et al. (2011) about how to develop

Primary School students' spatial orientation skills to determine the kind of 3D resource they would create and to design the associated activities.

The sample consist of 84 PSTs, working collaboratively in groups of four, studying the subject of Teaching and Learning Geometry in the 3rd year of the Primary Education Teacher Degree at the Primary Education Teacher Degree at the Faculty of Teacher Training of the University of Santiago de Compostela. We carried out a qualitative study of the data collected by the 3D design and a portfolio, along with the participant observation of the researchers. The portfolio includes information about the design process and the details of the proposed spatial orientation activities. We use the didactic-mathematical knowledge model from the onto-semiotic approach (Godino, 2024) as a guideline to analyze the mathematics teaching learning opportunities, in particular, due to the nature and requirements of our training experience, we reduce our analysis to the common knowledge – mathematics of the educational level where PSTs teach, i.e., Primary School – and the cognitive – knowledge of how their students learn, reason, understand and progress in mathematics learning – and mediational – knowledge of appropriate resources to foster students learning – facets. We characterize this common and didactic-mathematical knowledge by using the approach provided by Gonzato et al. (2011). They consider the visualization and spatial orientation skills as a collection of abilities connected with spatial reasoning and provide a unified classification of the tasks presented at different research works to develop them, together with some prototypical examples and some remarks on the knowledge that arises during their resolution.

Our aim is to address to the following research questions: Which is the common, cognitive and mediational facets knowledge that emerges from our PSTs' training experience? Which other teaching-learning opportunities could arise by slightly modifying the intervention design?

Background

In the following lines we briefly present the didactic-mathematical knowledge model (Godino et al., 2016) from the onto-semiotic approach which is the general framework we used for the analysis and reflection about the teaching-learning opportunities. After that, we introduce the reader to the orientations provided by Gonzato et al. (2011) about how to develop Primary School students' visualization and spatial orientation skills, which were used by PSTs as training material and by the researchers to characterize the common and didactic-mathematical knowledge (cognitive and mediational facets).

Didactic-mathematical knowledge model

The onto-semiotic approach (Godino, 2024) is a theoretical framework that integrates various approaches and theoretical models used in Mathematics Education research. From its point of view, the teacher should be able to analyse the mathematics activity that arises during the resolution of a problem. Thus, it states that mathematics teachers must understand the mathematics of the educational level in which they teach – common knowledge –, as well as the mathematics of the next educational level – extended knowledge –. However, they indicate that this pure mathematical knowledge is not sufficient for teachers to organize, implement, and evaluate the teaching and learning processes of mathematics, the factors that influence these processes are complex, and a

deeper understanding of mathematics teaching, that they call didactic-mathematical knowledge, is also necessary (Godino et al., 2016).

They present the didactic-mathematical knowledge model overlapped with the pure mathematical knowledge (common and expanded) and structured in six different facets (epistemic, cognitive, affective, instructional, mediational, ecological). The epistemic facet refers to knowledge of mathematical meanings and the relationships between meanings, objects (concepts, propositions, procedures, arguments), and mathematical processes; while the cognitive facet involves knowing how students learn, reason, and understand mathematics; and the mediational facet considers knowledge of appropriate resources (technological, material and temporal) to enhance learning. The remaining facets include the relationships of mathematical content with curricular factors (ecological); good task organization, difficulties resolution, and control of classroom interactions (instructional); and, finally, knowledge of students' affective, emotional, attitudinal aspects and beliefs (affective).

Orientations to develop primary school students' visualization and spatial orientation skills

Against the lack of agreement of the community on the characterization of visualization and spatial orientation skills, Gonzato et al. (2011) consider them as a collection of abilities connected with spatial reasoning and create a unified classification of the tasks, presented at different research works, to develop them. They provide a classification into three different categories – static orientation of the subject and objects, 3-dimensional objects' perspective interpretation, subject orientation in real spaces –, together with some prototypical examples and some remarks on the knowledge that arises during their resolution.

The first category refers to the static orientation of the subject itself, with respect to other subject/object or the orientation between objects. It involves the control of the body schema (up/down, right/left, front/back) and its projection on another subject/object.

The second one includes techniques to represent and interpret different representations of a 3-dimensional object. We used the following classification of the activities provided by the authors to classify the common knowledge that emerged during the 3D modelling process:

- The initial stimulus or the available information for the task resolution: presence of the object or absence – previously observed or/and presented on the plane –.
- The principal action taken to solve the task: conversion between representations 2D/3D; rotation; fold/unfold; composition/decomposition or parts counting.
- The kind of response required: construction; drawing; identification; verbal or others.

Finally, the third one is characterized as recognition, description, construction, transformation, interpretation and representation of real-life space or displacements. The activities classification depends on three different criteria:

- The initial stimulus: real space; model/representation of the real space or both.
- The principal action taken, that depends on the initial stimulus as follows: static exploration of the real space or with movement; interpretation of graphic information by means of a model or linking the information of the real space and the model.

- The kind of response required: representation of the space – by means of a plane drawing or the creation of 3D scale models – or itineraries; localization of an object in a map, plane or 3D scale model with or without coordinates; verbal description of positions or itineraries and, finally; the physical response – by means of orientation of a real space representation with respect to the reality, localization of an object and displacements –.

This classification and the following remarks, related with the knowledge arising during the activities' resolution, are used to characterize the cognitive and mediational facets knowledge:

R1. At first educational levels, we should start with body action activities, and the use of 3-dimensional scale models should precede activities with 2-dimensional representations.

R2. The spatial localization stages by using of a model should be carried out as follows: macro-localization of the most characteristic objects, orientation of the plane or coordination of viewpoints, and micro-localization.

R3. There are two huge difficulties that usually appear at the interpretation of maps or planes: the coordination between the orientation of the representation and the real space and the understanding of the symbolic information, since the representation is a simplified model, not an exact model in a reduced scale.

R4. During the description of an itinerary by using a real space representation it necessary to dissociate two reference systems, the one linked to the student body schema and the other linked to the body schema of the imaginary person that moves along the itinerary.

Task specifications

The 84 PSTs were organized into working groups of 4 members that collaborated on the design and creation of their adapted 3D resource and the associated activities that should contribute to the development of the spatial orientation in real spaces of visually impaired Primary School students.

They received some previous training lessons (theoretical and practical) that introduce them into the orientations provided by Gonzato et al. (2011) and the use of the 3D modelling platform Tinkercad. Moreover, each member of the working group was in charge of one of the following technical guidelines for the proper design of the adapted materials (Baldrich Caselles et al., 2024; Holloway et al., 2018; Holloway et al., 2023; Nagassa et al., 2023).

Additionally, some regular support was provided to the working groups at some specific moments, for instance, at the beginning, we gave them ideas of real spaces that could be interesting to model – inner spaces such as classrooms and multi-story buildings or outer spaces like parks, squares or neighbourhoods – taken from Gonzato et al. (2011); Galvez (1985) and Nagassa et al. (2023). They also received feedback on the viability of their 3D resources and activities proposals from a specialist, a Primary School teacher of visually impaired students. Moreover, the PSTs could ask for additional support during the whole training experience, and they demanded it through tutorials.

Finally, the working groups prepared autonomously their 3D model and a portfolio that includes information about the modelling process and the details of the proposed spatial orientation in real spaces activities. We provide the details of the portfolio activities that were used to respond to the research questions.

P1. Explain into detail and illustrate the process you follow to determine the global dimensions of your 3D scale model and other additional proportional adjustments needed.

P2. Describe the associated activities in detail, such that every Primary School teacher could reproduce them. Classify them into the category subject orientation in real spaces and provide the corresponding remarks according to Gonzato et al. (2011).

Research method

The sample consist of 84 PSTs organized in working groups of 4 members. The training experience took part on the subject Teaching and Learning Geometry at the 3rd year of the Primary Education Teacher Degree (Faculty of Teacher Training, University of Santiago de Compostela).

We carried out a qualitative analysis of the common and didactic-mathematical knowledge – cognitive and mediational facets – that emerged during the design and creation of the 3D model and the formulation of the associated activities.

The common knowledge was determined by analysing the responses to the portfolio activity P1 by means of the classification provided by Gonzato et al. (2011) for the category 3-dimensional objects' perspective interpretation. We complement the portfolio information with the 3D design and the participant observation of the researchers.

In the case of the cognitive and mediational facets we analyse the responses to the portfolio activity P2. We first classify PST's activities by using the criteria provided by Gonzato et al. (2011) for the category subject orientation in real spaces, then, we check if they had classified them properly and provided correct remarks, such as R1-R4, among others.

Results

We provide some illustrative examples of PSTs common, cognitive and mediational facets knowledge that emerges from our PSTs' training experience. After that we discuss the results presented and purpose some slight modifications of the training experience that can improve the learning of mathematics teaching practice.

First, we briefly provide a classification of the 22 3D models that were designed and created by the working groups. We have got 11 inner spaces – 10 Primary School classrooms and 1 Primary School 3D multilevel plane – and 11 outer spaces – 3 squares, 2 Primary School playgrounds, 2 labyrinths, 2 parks, 1 beach, 1 neighbourhood –. The 11 classrooms' models correspond to 3 different types of designs – illustrated at Table 1–:

- Faithful: proportional representation of the real space and furniture, preserving almost all their real-life geometry complexity.
- Faithful –Lego type: proportional representation of different types of furniture elements that fit in a Lego type base, also preserving almost all their real-life geometry complexity.
- Iconic: simplified representation of different types of furniture by means of the composition of a few amounts of basic geometric bodies such as prisms and cylinders.

We now illustrate some of the PSTs common knowledge that emerged during the modelling process. Along the working groups that provided a faithful representation of a classroom we distinguish two

categories that differ on the action(s) taken. All the groups received the same type of initial stimulus, that is the representation of the real space given by several photos taken from different perspectives of the classroom, as illustrated in Figure 1, without having the opportunity to visit the real space; and all of them construct a 3D scale model of the room and furniture. However, some of them convert directly the 2D representation by photos to the 3D model and others need an auxiliar 2D representation of the classroom – see Figure 2 –. The working groups modelling outer spaces decided to choose as initial stimulus a 2D representation (top view) taken from Google Maps/Earth and, to produce the 3D scale model, they combine that information with the 3D view provided by Maps/Earth and/or photos.

We observe that the initial stimulus available conditioned the common knowledge that arose, so that, as a new opportunity to contribute to the development of PSTs' cognitive facet, it would be interesting to incorporate a space at the PSTs training experience where they could discuss these results.

Table 1: Different types of classrooms' 3D scale models

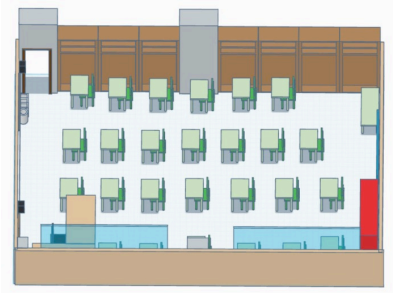
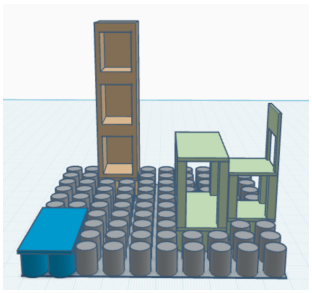
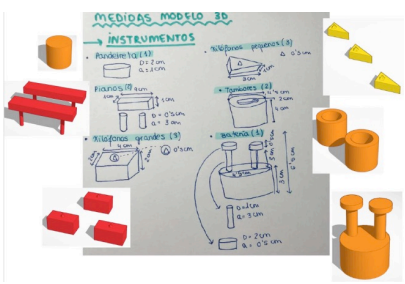
Faithful: classroom + furniture	Faithful – Lego type: furniture	Iconic: furniture
		

Figure 1: Illustration of a classroom representation provided to students

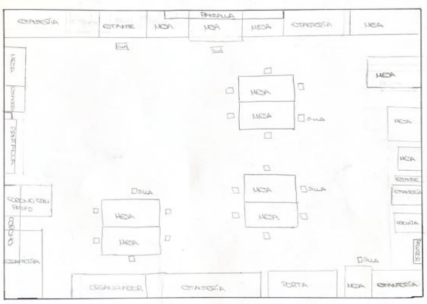


Figure 2: Illustration of an auxiliar 2D representation of a classroom

Regarding the cognitive and mediational facets, we choose the neighbourhood 3D scale model as a good example to illustrate it. This working group decided to create two different representations of a neighbourhood at the city of Lugo, a 2.5-dimensional map (adapted 2-dimensional map to visually impaired students) and a 3D scale model, both illustrated at Figure 2. They provided appropriate activities – free exploration of the 3D scale model, localization of a place by means of 3D icons, find an itinerary between 2 places of the 3D scale model, repeat the activities at the 2.5-dimensional map – and include appropriate remarks that highlighted that the use of 3-dimensional scale models should precede activities with 2-dimensional representations and the difficulty on understanding the symbolic representations on reality.

Figure 3: 2.5-dimensional and 3-dimensional representation of Augas Ferreas neighbourhood in Lugo

Conclusions

From this preliminary analysis of the didactic-mathematical knowledge that emerged during the development of the students' task, cognitive and mediational facet as well as the common knowledge as their baseline, we conclude that it has a lot of mathematics teaching training potential. Moreover, we can take advantage of it by including extra questions for preservice teachers on the task, such as some regarding the mathematical knowledge that arises during the modelling process, but also by redefining the current questions.

Acknowledgment

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Mathematics teacher noticing: A situated Taxicab lab experience connecting research and practice-based teacher development

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This paper investigates how an extracurricular Taxicab-geometry lab for secondary students functioned as a practice-based professional learning space for two in-service teachers. Observing an expert teacher, participating teachers developed professional noticing of student thinking and moves that support mathematical learning. Using the Attending–Interpreting–Deciding framework to analyse observation notes, debriefing dialogues, and reflective questionnaires, we found evidence that trajectory-based tasks, dynamic digital tools, and specific orchestration moves catalysed teacher noticing. Findings suggest that observing expert practice in authentic settings can shift teachers' professional stance toward designing principles for their own classrooms.

Keywords: Teacher noticing, practice-based teacher education, inquiry-based learning, authentic tasks, Taxicab geometry.

Introduction

Noticing and interpreting classroom events informs instructional decisions. As teacher education increasingly emphasises practice-based use of representations, decompositions, and approximations of practice (Grossman et al., 2009), reliance on video or other mediated representations limits opportunities. In-the-moment noticing preserves contingency, affect, and embodied interaction that mediated representations cannot capture. This study examines the development of two in-service teachers professional noticing while observing an extracurricular Taxicab geometry lab led by an expert high school mathematics teacher. The lab served as a live professional development context where observers could witness inquiry-based instruction and student mathematical thinking in real time. Teachers contingently observed those teaching and learning and how and why their interactions unfolded as they did.

Theoretical framework

Professional noticing is widely conceptualized through the AID framework: Attending to particular events, Interpreting them using professional knowledge, and Deciding how to respond pedagogically (Jacobs et al., 2010). This knowledge-based reasoning perspective positions noticing as a learnable practice that develops through deliberate attention to classroom phenomena. Mason (2002) emphasizes noticing as disciplined awareness, cultivated through intentionality, systematic documentation, and reflective exchange, that transforms lived classroom experience into shared professional knowledge.

While AID provides the primary analytical structure for this study, we enrich it with an enactivist perspective that recognizes noticing as co-emergent and relational (Coles & Helliwell, 2025). Within this stance, mutual noticing (co-emergent awareness among actors) and meta-noticing (noticing one's noticing) become analytically salient, providing a composite lens that captures multidirectional flows of awareness across roles and links them to in-the-moment pedagogical choice (Jacobs et al., 2010).

This perspective acknowledges that what teachers notice arises from interactions among participants and that observers can develop awareness of their own noticing processes. This complementary lens helps capture the bidirectional nature of observation: teachers simultaneously notice student thinking and the pedagogical moves that make such thinking visible.

Research on pre-service teacher noticing has emphasized the importance of task design in focusing attention on mathematically significant moments (Ivars et al., 2019). Authentic, trajectory-based tasks, that is those that engage students in extended new mathematical investigation with clear learning progressions, create rich episodes for observer attention. When combined with dynamic digital tools, such as GeoGebra in our experience, that externalize student thinking, such tasks generate observable evidence of reasoning that supports both student learning and teacher noticing.

Guided by these frameworks, we ask:

RQ1: How does observing an extracurricular lab activate professional noticing in teacher observers, particularly regarding student mathematical thinking and instructional orchestration?

RQ2: Which design features and instructional moves promote observer noticing and appear transferable to teachers' own practice?

Methods

The study took place during the 2024-2025 academic year as part of an extracurricular mathematics enrichment program in southern Italy. The lab focused on Taxicab geometry, an accessible non-Euclidean geometry where distance is measured along grid lines rather than crow flies, creating opportunities for productive cognitive conflict and conceptual development. Four kinds of participants were involved: the Teacher-Leader (TL); Observer Teachers (O1, O2), two in-service secondary mathematics teachers who accompanied students and observed all sessions with an explicit focus on student thinking and pedagogical moves; Researcher (R), a mathematics education researcher who designed the experiment with TL documented sessions and facilitated debriefing; Students (S), fifteen secondary students. The observers had volunteered to participate as part of informal professional development, understanding their role as both student supervisors and professional learners. While not enrolled in a formal program, they had expressed interest in inquiry-based approaches and were committed to considering implementation in their own classrooms.

Instructional design

The lab employed Inquiry-Based Learning (IBL) principles (Artigue & Blomhøj, 2013), engaging students in extended exploration before formal exposition. Content design foregrounded a continuous comparison between Euclidean and Taxicab geometries (Krause, 1986). Euclidean distance (dE) and Taxicab distance (dT) between two points P(x_1,y_1) and Q(x_2,y_2) are defined as: (i) dE(P,Q) = √(〖(x_2-x_1)〗 ^2+ 〖(y_2-y_1)〗 ^2); (ii) dT(P,Q)=|x_2-x_1 |+|y_2-y_1 |.

Tasks were situated in realistic urban contexts (route planning, locating equidistant points) and designed to generate cognitive conflict between everyday intuition and mathematical definition. For example, while circles are round in Euclidean geometry, they appear as squares in Taxicab geometry, a surprising result that emerges from the distance definition.

Each learning episode opened with a key problematising question, continued with pair/small-group investigation, and culminated in whole-class consolidation mediated by the TL. The instructional sequence was so declined: (1) Launch: Open-ended problem posed to whole class; (2) Exploration: Small-group work with analogue (paper, mini-whiteboards) and digital tools (GeoGebra, Google Maps); (3) Interim sharing: Brief whole-class exchanges where groups shared emerging ideas; (4) Consolidation/formalisation: Teacher-facilitated synthesis and formalization. This structure was designed both to support student learning and to create observable episodes for teacher noticing.

Digital tools served as dynamic environments for exploration, visualisation, and conceptual modelling. Parameter manipulation made visible how definitions shape geometric objects under different metrics, supported conjecture-testing, and enabled sharing of reasoning in collaborative settings, thus fostering accessible, participatory learning (Miranda & Saliceto, 2024; Romero Albaladejo & García López, 2024).

Data collection and method of data analyses

We collected multiple data sources across six lab sessions: Teacher-Leader field notes: simultaneous observations of student strategies, difficulties, and group dynamics; Researcher observation notes: structured field notes focusing on TL instructional moves and observer engagement; Debriefing dialogues: audio-recorded conversations between TL and observers immediately following sessions; Observer questionnaire: open-ended reflection prompts completed by O1 and O2 at program end: What did you notice about student mathematical thinking? What instructional moves seemed particularly effective? How might you adapt this approach in your own classroom? What challenged your previous assumptions about teaching geometry?

Following Jacobs et al. (2010), we applied the AID framework post hoc to identify and characterize teacher noticing in the data. Our analysis focused specifically on observers' noticing, tracking two directional flows: O→S (Observer to Student): What observers attended to in student thinking, how they interpreted it, and what instructional implications they drew; O→TL (Observer to Teacher-Leader): What observers attended to in the Teacher-Leader's orchestration, how they interpreted its effectiveness, and what design principles they identified.

The analyses articulated into the following four phases:

Phase 1 - Episode Selection: The researcher and Teacher-Leader independently reviewed field notes and debriefing transcripts to identify moments where observers explicitly articulated attention to student thinking or instructional moves. Episodes were selected when observers: (a) described specific student actions or statements, (b) offered interpretation of student understanding or TL pedagogical reasoning, or (c) connected observations to their own practice. This yielded 12 candidate episodes.

Phase 2 - AID Coding: For each episode, we coded observer utterances using three categories: Attending (A): Explicit reference to observable student behaviour, artifacts, or TL actions (e.g., "Group 5 kept trying even when stuck"); Interpreting (I): Inference about student understanding, affect, or pedagogical rationale (e.g., "They seem motivated by small successes"); Deciding (D): Statement of intended action or design principle for future teaching (e.g., "I want to give more time

before showing formulas"); Utterances could receive multiple codes when observers moved between levels. We noted prospective versus retrospective framing and whether statements referred to student thinking (O→S) or pedagogy (O→TL).

Phase 3 - Cross-Episode Analysis: We examined patterns across episodes, identifying recurring design features that catalysed noticing and aspects of practice observers found transferable. The two authors coded independently, then resolved discrepancies through discussion, achieving consensus on all final codes.

Phase 4 - Validation: we shared preliminary findings with O1 and O2 through member checking, asking whether our interpretations aligned with their experience. Both confirmed the analysis and provided additional context that enriched interpretation.

Results

We present four episodes illustrating how observers' noticing developed across the lab, organized by the primary focus (O→S or O→TL) and developmental progression. All episodes involve both observers, though O1 contributed more extensively to recorded debriefings.

Episode 1 — Autonomous Discovery of the Formula (O→S, O→TL)

After students worked in groups to find patterns for calculating Taxicab distance, the following exchange occurred:

O1: The Taxicab distance formula... they worked it out on their own? That's remarkable!
TL: Yes, they derived it through measuring specific cases and generalizing.
O1: Now that they have both formulae available, they can work more confidently. They can check one against the other.

O1's surprise marks the moment's salience; her expectation was violated. She attended (A) to the absence of teacher exposition and to students' independent generation of the formula. She interpreted (I) this outcome as evidence of genuine understanding ("worked it out") and increased strategic flexibility (‘check one against the other’). The prospective decision (D) is implicit: lesson designs should “suspend immediate exposition” to allow student discovery. This represents O→TL noticing, focusing on the Teacher-Leader's orchestration choice to defer formalization. The episode also suggests emerging meta-noticing: O1 identified not just what happened but which instructional feature enabled it, extended exploration time, authentic tasks, interim-whole class exchanges. This is a portable design principle that transcends specific content.

Episode 2 — Curiosity as Instructional Lever (O→TL)

Following a session where students investigated whether perpendicular bisectors exist in Taxicab geometry:

O1: It’s really good, the way you presented it. In this way you manage to stimulate curiosity. […] Usually, in school activities, we start straight away by searching for the solution and getting operational. Here, instead, you manage to place curiosity and exploration at the centre, to stimulate them, making them an integral part of the experience.

O1 attended (A) to a structural departure from conventional practice: the TL's problem-posing approach that opened questions rather than immediately pursuing solutions. She interpreted (I) this move as deliberately activating epistemic curiosity and positioning exploration as legitimate mathematical work, not preliminary to real mathematics. The comparison to routine practice ("usually we start straight away") signals awareness of contrast. The prospective decision (D) is explicit: "making curiosity integral" becomes an aspirational principle. This O→TL noticing identifies a transferable pedagogical stance, one that treats mathematical questioning as valuable activity rather than merely instrumental to answers. O1's framing shifts from describing what happened to articulating why it matters professionally, a movement toward actionable principle rather than mere observation. Mutual noticing is evident: curiosity enacted in interaction between TL and students becomes visible to the observer and re-orients her professional thinking. A meta-noticing moment occurs as she names facilitation features (question-eliciting launches, orchestrated exploration, distributed participation) and treats them as portable categories for action.

Episode 3a — Motivational scaffolding for fragile learners (TL→S)

TL: Group 5 appears the most fragile mathematically, yet they tackle each challenge with great tenacity. The methodology adopted, pair exploration followed by whole-class sharing [], is essential to prevent discouragement; indeed, each episode seems to motivate them further." [Teacher-Leader's note]

This entry documents the Teacher-Leader's own noticing (TL→S), providing essential context for understanding observers' subsequent noticing of the same group. The TL attended (A) to intertwined signals of mathematical uncertainty and persistent engagement. She interpreted (I) the inquiry structure itself, particularly interim sharing, as scaffolding that converted partial progress into motivation. Her decision (D) was to maintain and refine this structure for this group. We include this episode to establish baseline evidence: the TL's orchestration was deliberately equity-oriented, creating conditions observers would later notice.

Episode 3b — Observer recognition of equity-oriented scaffolding (O→S and O→TL)

O1: They seem a little less confident […] but they keep trying. [...]there's potential, but we need to keep working on it.

TL: Yes, Group 5 appears mathematically fragile, yet they tackle each challenge with great tenacity… The methodology adopted—pair exploration followed by socialisation—is essential to prevent discouragement. They seem a bit less confident, but they keep trying. And that matters.

O1: [...] With facilitation like yours, you make them participants: they feel inside the trajectory, not left behind.

This episode illustrates bidirectional and layered noticing. O1 attended (A) to the same student group the TL had identified, noticing affective signals ("less confident") alongside behavioural persistence ("keep trying"). She interpreted (I) potential as present but requiring instructional support, a deficit-oriented reading initially. The TL's response refocused attention on orchestration as the enabling mechanism. O1's follow-up statement shows shifted interpretation: the structure itself positions students as legitimate participants. This represents simultaneous O→S and O→TL noticing, seeing student engagement as produced by pedagogical design rather than as individual trait. The prospective decision (D), implicit but present, is to adopt "facilitation that makes students feel inside

the trajectory," an equity principle that reframes access as instructional responsibility rather than student characteristic. This episode also demonstrates mutual noticing (Coles & Helliwell, 2025): the TL's framing shaped what became salient for O1, whose articulation in turn validated and refined the TL's pedagogical reasoning.

Episode 4: Declarative noticing in questionnaire responses

O1:	This lab opened new perspectives on how to approach geometry in the classroom [...]GeoGebra helped make concepts visible that are usually abstract. I want to try this trajectory approach, letting students discover before I explain
O2:	I use inquiry approaches to some degree, and this observation confirms their validity. The digital tools make student thinking visible in ways traditional methods cannot. I'm thinking about how to structure similar explorations in my curriculum.

These written responses represent declarative rather than in-the-moment noticing—articulation of principles and intentions rather than real-time decisions. Both teachers attended (retrospectively) to methodological novelty and tool affordances. They interpreted the approach as validating (O2) or expanding (O1) their existing repertoire. Their decisions are future-oriented: explicit intentions to implement trajectory-based design and digital tools. Significantly, both teachers specified transferable elements: extended exploration time, visualization through technology, and deferred formalization. They described not just what they observed but what they could enact.

Across episodes, noticing is co-emergent. For TL, AID unfolds in situ: students' unexpected moves orient orchestration, which in turn reshapes what becomes salient for students. For the observers, noticing is primarily declarative and prospective, organised around two directional flows: O→S and O→TL. Read through Mason's temporal orientations, observers' noticing is future-oriented: it moves beyond description towards prospective design, marking a shift from observation to readiness for action—without over-claiming enactment.

Discussion

Our findings demonstrate that observing authentic classroom practice can activate multiple dimensions of teacher noticing when structured purposefully (RQ1). The observers developed increasingly attention to both student mathematical thinking (O→S) and instructional orchestration (O→TL), with these strands mutually reinforcing. Early episodes show primarily TL-focused noticing: observers attended to pedagogical moves (deferred exposition, question-driven launches) and interpreted them as deliberately positioning students as mathematical sense-makers. This aligns with Grossman et al.'s (2009) conception of decomposing practice, observers learned to identify and name specific teaching moves rather than evaluating teaching holistically. Later episodes reveal more integrated noticing: observers simultaneously tracked student reasoning and the instructional conditions enabling it. Episode 3b exemplifies this integration, O1 noticed both Group 5's persistence and the scaffolding structure that sustained it, recognizing pedagogy as productive of student engagement rather than merely responding to it. This progression suggests that authentic observation contexts support development from atomized noticing (discrete moves) toward a noticing of the relationships between pedagogy and learning. Unlike video-based professional development, real-time observation allowed observers to experience how teaching unfolds temporally and contingently.

Three design elements supported noticing (RQ2) and emerged as transferable principles: (1) Trajectory-based authentic tasks: the Taxicab problems created extended investigation where student thinking developed visibly over time. This temporal unfolding made reasoning trajectories inspectable: observers could see not just final understanding but pathways toward it (Ivars et al., 2019). The cognitive conflict embedded in tasks generated observable moments of productive struggle that focused observer attention; (2) the use of dynamic digital tools: GeoGebra and Google Maps externalized student reasoning, making thinking visible and publicly sharable. Observers noted how technology supported their noticing — literally seeing students' conjectures, tests, and revisions. This addresses a known challenge in noticing research: the difficulty of accessing student thinking (Jacobs et al., 2010). Digital tools function as inscriptional devices that render cognition observable. (3) Structured exploration sequence: the consistent instructional rhythm (launch, exploration, interim sharing, consolidation) created predictable observation opportunities. Observers knew when to attend to small-group reasoning versus whole-class argumentation and productive discussions (Stein et al., 2008). Deferred exposition gave observers extended time to focus on student sense-making before pedagogical synthesis redirected attention. Interim sharing episodes were particularly generative, as observers witnessed the TL transforming student contributions into collective knowledge.

Critically, questionnaire responses explicitly identified "extended exploration," "making thinking visible," and "delaying formulas" as elements they intended to implement. The responses, read as declarative noticing, corroborate this movement by articulating intentions for use and stabilising a shared vocabulary (“visualisation”, “methodology”, “new perspectives”) that travels into planning. Thus, the study documents a shift from observation towards readiness for action, while remaining cautious about claims of enactment. This suggests that observation of well-designed instruction can build teachers' pedagogical repertoire (Ball & Forzani, 2009).

Conclusion

This study demonstrates that observing expert practice in authentic classroom settings can develop teacher noticing in ways that connect professional vision to actionable pedagogical principles. When coupled with well-designed tasks, dynamic tools that externalize thinking, and structured opportunities for reflection, observation becomes a powerful form of practice-based teacher education. While we cannot claim these intentions become enacted practice, the study specifies plausible mechanisms by which observation can shift professional stance and readiness for action. It positions teacher noticing not merely as perceptual skill but as gateway to transformed practice, learning to see teaching differently as a prelude to teach differently. While most noticing research engages teachers with video or written artifacts, this study demonstrates learning-to-notice in real-time classroom observation. This authentic context preserves contingency, affect, and embodied interaction that mediated representations cannot capture. Tracking both O→S and O→TL flows revealed how observers developed integrated understanding of pedagogy-learning relationships. Another relevant contribution to teacher noticing is the transition from noticing to transferable principles: progression from description toward actionable design principles. This movement, from I noticed X to X happened because of move Y to I could implement Y in my classroom, suggests how noticing can become generative for practice (Mason, 2002). Of course, this is a single-site, extracurricular case with only two observers; facilitation quality and the novelty of content may have

amplified effects that are harder to reproduce under curricular constraints. Future research should follow observers into their classrooms to examine how noticing translates to instructional change and examine how observer noticing differs with teaching experience and prior professional development.

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Unpacking an example of a practice-based teacher education sequence of tasks

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Practice-Based Teacher Education (PBTE) has been the focus of reflection in multiple studies over the past two decades. However, despite the existence of various definitions of this approach to teacher preparation, its use remains ambiguous. In this paper, we analyze a task sequence designed to foster the development of professional competence in posing school problems. Our objective is to characterize how the different tasks of this sequence relate to the pedagogies intrinsic to PBTE, connecting them to core practices of teaching. This contribution seeks to offer an example of how to avoid ambiguity when characterizing task sequences for teacher education as PBTE.

Keywords: Teacher education, practice-based education, core practices, problem posing.

Introduction

Practice-Based Teacher Education (PBTE) has emerged as an approach to teacher preparation that has been the subject of growing research interest in recent decades (e.g., Forzani, 2014; Charalambous & Delaney, 2020). Ball and Cohen (1999) define PBTE as training focused on learning professional performance, centered on the core activities of the profession, and involving the investigation of critical problems in teaching. Nevertheless, PBTE remains an amorphous term (Forzani, 2014), given its recurrent yet vague usage across studies. While this presents a limitation from a research perspective, it also opens up a space for significant creativity within the mathematics teacher education community (Herbst & Chazan, 2023). In an effort to conceptualize PBTE, several studies have sought to define the core practices of teaching for PBTE curricula (e.g., Forzani, 2014; Matsumoto-Royo & Ramírez-Montoya, 2021). These core practices constitute strategies, routines, and central activities of teaching that are addressed in order to prepare teachers. Core practices are structured through pedagogies of teacher preparation (Matsumoto-Royo & Ramírez-Montoya, 2021), which may, in turn, be articulated through tasks and sequences of tasks.

In particular, this study aligns with the growing recognition that teachers must be competent in posing school problems (Montes et al., 2024). This competency requires the integration of professional knowledge with teachers' instructional intentions in order to select, adapt, and design problems for classroom use. Accordingly, the core practices examined in this study are linked to posing school problems.

The aim of this theoretical contribution is to examine in greater depth how to argue that a sequence exhibits an orientation aligned with PBTE. To that end, this paper presents a task sequence designed to develop professional competence in posing school problems within an initial teacher education program. We analyze the design of the sequence, discussing the arguments that connect each stage to teaching practice and teacher learning. We then associate these arguments with the pedagogies employed to address them, as well as with the methodological decisions underpinning the design.

Background

Practice-based teacher education (PBTE) is defined as "professional teacher training centered around the basic fundamentals of teaching, with the goal of creating novice teachers prepared to tackle issues of equity in the classroom through powerful and equitable learning opportunities for all students" (Moody, Kuo, & Eslami, 2022). Across different approaches (e.g., Teaching Works, 2021; Matsumoto-Royo & Ramírez-Montoya, 2021), three types of pedagogies have been identified: (i) pedagogies linked to the use of representations to observe and analyze practice, "to provide novice teachers with diverse examples of what teaching practice looks like in action"; (ii) pedagogies associated with the use of approximations to simulate teaching practice, drawing on various documents (e.g., curriculum guidelines, transcripts); and (iii) pedagogies centered on authentic practice, involving the design and implementation of tasks, as well as reflection and coaching on teachers' own practice. For each of these pedagogies, diverse methodologies are identified (see Table 1 for some examples).

Table 1: Pedagogies in PBTE and associated methodologies (adapted from Teaching Works, 2021)

Using representations to see and analyze practice	**Using approximations to practice practice**	**Supporting novices to enact practice**
Using video to see and analyze practice	Using video to practice practice	
Using student work, portraits, and other representations to learn about children	Using transcripts to practice practice	Designing and using field tasks
Using transcripts to see and analyze practice	Using student work to practice practice	
Teacher educator modeling of teaching practice	Coached rehearsal	Coaching strategies
	Peer run throughs	
	Simulated student interactions	

These three pedagogical approaches focus on presenting examples of different types of classroom situations to be analyzed, simulations to recreate teaching practice, and the development of practices linked to the teaching profession with varying degrees of proximity to actual practice, ranging from activities with peers in training to practice in real-world contexts.

Each of the methodologies listed in Table 1 connects to teaching practice through arguments of representation, decomposition, and approximation to practice (Grossman, Compton, et al., 2009). Representation arguments aim to make elements of practice visible, whereas decomposition arguments involve breaking down aspects of practice into distinct components that need to be learned. Finally, approximation arguments offer opportunities to enact teaching practices with varying degrees of authenticity and realism.

These connections to teaching practice focus on central elements of teaching itself, namely, "teaching practices that are essential for novices to become capable at before they are permitted to assume independent responsibility for a classroom" (Forzani, 2014). The Teaching Works project (2021) identifies 19 high-leverage practices that are fundamental to teaching. In their systematic review, Matsumoto-Royo and Ramírez-Montoya (2021) identify 26 core practices used in research articles on teacher education, of which only six are associated with activities outside the classroom. Among the 26 core practices, both *Eliciting and interpreting individual students' thinking* and *Leading a group discussion* were the most frequently cited in the reviewed studies. Less frequently reported core practices included *Planning for teaching*, *Implementing organizational routines*, *Using methods to check or assess students' understanding*, and *Explaining and modeling content, practices, and strategies*. There was no mention to after-classroom practices.

An example under discussion

This section will present an example of a teacher education task sequence, with a discussion of the practice-based nature of each task. Before doing so, some contextual elements will be provided to situate the development of the sequence. As far as the goal of the paper is to discuss the PBTE foundation of task design, no excerpts of the implementation will be presented. As far as exhaustive catalogs of core practices couldn't be found, we propose, linked to each task, core practices of teaching present in literature research, and, if there is need, a description of the possible core practice of teaching involved.

Context

The task sequence presented here was implemented during the second semester of a teacher education program in Peru, spanning several class sessions. In this context, eleven students were enrolled, of whom only nine consented to provide data for this study. The course, entitled "Development of Logical–Mathematical Skills for Children (Primary Education)", is the first course in the program focusing on mathematics education. It is taught by the first author of this paper, who combines her role as teacher educator with her work as a primary school teacher. The portion of the course in which the sequence is situated focuses on the topic of area and perimeter. The task sequence forms part of a doctoral dissertation on the development of professional competence in posing school problems. The sequence comprises four sessions, plus an information-gathering activity in a school, as will be detailed below.

Throughout the process, the teacher educator acts as a model of good teaching practice for her students (Rojas et al., 2021), which already constitutes a methodological stance that connects the students' future teaching practice to the development of the tasks. This stance is based on a modeling argument, understood as a representation of teaching practice.

Foundational task

Prior to engaging with the core of the task itself, the student teachers receive mathematical training on issues related to measurement, including a brief review of the school content to be addressed. In addition, they are introduced to selected references from mathematics education regarding measurement strategies such as the composition and recomposition of figures. Within this approach,

special emphasis is placed on the mathematical foundations of area calculation, focusing particularly on strategies of reconfiguration (Duval, 2006). The type of problems students are asked to solve at this stage are not intended for primary school pupils; rather, they are based on the argument that teachers must possess a level of mathematical knowledge substantially more complex than that of their students. For example, mathematically precise and rigorous definitions of "polygon" are provided—definitions that would not be communicated to primary school children but are potentially useful for a deeper understanding of the mathematical content to be taught.

The link between this period of mathematical grounding and teaching practice lies in the argument that effective mathematics instruction requires a solid foundation in mathematics itself—one that is coherent and consistent with the fundamental principles of the discipline (Wu, 2011)—as well as a didactical–mathematical foundation. This argument can be extended to elements of mathematics education more broadly, on the premise that high-quality teaching practice must be underpinned by insights from mathematics education research.

First task

Following this mathematical and didactical grounding on the teaching of area calculation, with emphasis on composition and recomposition of figures, participants engaged in a video analysis task. In this task, they viewed selected video excerpts showing a teacher working on area calculation through problem solving (problems created, selected, or adapted by the teacher) and were prompted to reflect on the excerpts. The clips were drawn from a previous study (Castillo & Montes, 2023) in order to provide prospective teachers with models of good practice. The video analysis focused both on discussing the management of learning around area calculation through problem solving and on reflecting upon the problems themselves.

In this case, the connection to practice is grounded in reflecting on another teacher's practice to give meaning to the didactical and mathematical knowledge acquired and to provide models of practice that might inspire the participants' future teaching. The core practices involved in this task are session planning and management—particularly the analysis of representations of practice—linked to situations involving problem solving.

Second task

Phase 1. The proposed task consisted of individually creating problems freely (Stoyanova & Ellerton, 2006), with the instruction that these problems should be designed for sixth-grade pupils (ages 11–13). This activity aimed to foster the development of professional competence in posing school problems (Montes et al., 2024) as a key element of future professional practice.

Phase 2. The subsequent task involved group discussion of the problems and the selection of four problems from among those created, followed by their refinement into a structured sequence, with the announcement that these would later be implemented in a real classroom. This process aimed, on the one hand, to developing professional competence in posing school problems—particularly the component involving problem transformation (Montes et al., 2024)—and, on the other hand, to highlighting the value of peer negotiation, recognizing that collaborative work is a cornerstone of professional development (Borko & Potari, 2024).

In this case, the link to practice is based on two complementary arguments of professional utility. In the first phase, the focus is on the need to develop competencies specifically associated with mathematics teaching, cultivating a vision of mathematics that will shape future teaching practice. In the second phase, in addition to addressing content-specific aspects, the aim is to foster an understanding of teaching as a collaborative endeavor, thereby encouraging future teachers to engage in dynamics that enhance teaching skills beyond content itself. In terms of teacher education methodologies, this represents an approach to the practice of problem posing that leads naturally into the real practice of lesson planning, since the problems were to be implemented.

Intended third task

At this point, the four developed problems were taken into a sixth-grade classroom where the first author of this study teaches, with the pupils asked to solve them using structured protocols that would later inform subsequent tasks. Although the intention was for the prospective teachers to attend and manage the session themselves, institutional permissions ultimately made this impossible.

In this third intended task, the goal was to create a controlled immersion environment in which the prospective teachers could engage with problem solving not only as an activity of preparation but also as a process of classroom management, attending to both mathematical and non-mathematical variables that might arise predictably or contingently. When institutional permissions to have the prospective teachers attend the class could not be obtained—but parental consent for pupil participation was secured—pupil work was collected instead to serve as the basis for the next task.

Third task

The third task ultimately consisted of analyzing, first in small groups and then collaboratively, the pupils' protocols documenting their solutions to the four problems. The protocols presented to the prospective teachers contained two columns: one where the pupils indicated what they had done, and another where they explained why they had done it. The goal of the session was to analyze how the pupils approached the problems, with the aim of enabling the prospective teachers—having been agents in designing the problems—to identify relevant aspects of learning that had been activated. Finally, students were invited to consider whether modifications to the problem statements might be appropriate in light of the difficulties identified.

Here the connection to practice takes several forms. On the one hand, the prospective teachers are working with authentic pupil work produced in the classroom of an in-service teacher on tasks they themselves had designed. This brings into play the professional competence of "noticing" (Weyers et al., 2024), assumed to be necessary for professional practice even though it is not made explicit to the students. On the other hand, it continues to advance the development of professional competence in posing school problems.

Fourth task

The final, synthesizing task included an initial individual component and a subsequent group reflection on the work carried out. This reflection was guided by the questions shown in Figure 1.

What did you learn from the activity of posing problems?

What difficulties did you experience when posing your problems?

What difficulties did you experience when posing the problems?

What types of difficulties did you observe in the students' responses you analyzed? How did you consider addressing them?

What was your intention when reformulating the problem?

Do you think your problem addresses any difficulties that students present? If so, which ones?

What skill would you be assessing in students when they solve the problem?

In what aspects did you feel most comfortable when posing problems?

In what areas do you feel you need further training to improve your ability to pose problems?

Figure 1: Questions for reflection in the last session

This session aimed, first, to create a space for prospective teachers to consolidate their learning related to professional competence in posing school problems. The guiding questions prompted the elicitation of various aspects of the posing process. Second, this approach incorporated a metacognitive perspective, inviting prospective teachers not only to reflect on the act of creation of problems but also on their own role within that process.

This fourth task focused on developing and promoting the analysis of practice, a frequently addressed core practice, from a metacognitive perspective that implies reflecting on one's own practice—even if it is only simulated or approximated. Methodologically, it relied solely on individual reflection and the elicitation and collective discussion of that reflection.

Final reflections

Preparing teachers to learn how to develop their teaching practice is, or should be, the primary goal of any teacher education program. Given the ambiguity with which the notion of PBTE is interpreted (Forzani, 2014), it appears necessary to provide clear characterizations of how different teacher education initiatives (e.g. tasks, task sequences) not only respond to this notion but also articulate themselves coherently and in connection with the central elements of this type of approach to teacher preparation. Figure 2 presents a synthesis of the task sequence discussed in this paper in terms of the pedagogies and methodologies employed, the core practices addressed, the role of the teacher educator, as well as the mathematics teaching professional orientation.

	Task 1	**Task 2**	**Intended Task 3**	**Task 3**	**Task 4**	**Teacher Educators' role**
Pedagogy	Represent.	Approximation Enactment	Enactment	Represent. Approximation	Enactment	Represent.

				Enactment		
Method	Using video to see and analyze practice	Designing field tasks (problems)	Teaching in a controlled scenario	Using student work to analyze learning (re)Designing field tasks	Reflecting about own approximation to practice	Modeling practice
Core Practice of Teaching	Planning for teaching Classroom management	Planning for teaching Communicating with pairs	Managing real classroom scenario	Assessing students learning Planning for teaching	(self)Assessment	All the visible ones
Mathematical professional orientation	Develop teachers skills to pose school mathematical problems, entailing creation of problems, session managing, and assessment of the session, including self-assessment					

Figure 2: Synthesis of the central aspects of the sequence

The theoretical contribution presented here inherits many of its limitations from the existing research literature. As no exhaustive catalogues of core practices could be found, which constrains the analytical potential of PBTE perspective. This should be explored in future research on PBTE, in order to provide a clear organization of what can be considered under that label. Core practices are generally linked either to classroom activity or to planning; there are few references to core practices occurring after classroom instruction. Therefore, we consider it necessary to characterize in greater depth the practice of reflecting on instruction, with a particular focus on teachers' self-analysis.

Finally, we argue that it is important to consider whether the mathematical and didactical–mathematical grounding of teaching can itself be characterized as a core practice, given the need to invite teachers into an ongoing process of continuous professional development.

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From approximation to real practice: Focusing on the 'unexpected' in pre-service mathematics teachers' scripts

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Scripting tasks are widely used as an approximation of practice in mathematics teacher education. This study investigates how pre-service mathematics teachers (PMTs) revise their scripts after teaching. In this context, 28 Turkish PMTs initially anticipated classroom disruptions in their pre-teaching scripts and later revised them based on their teaching experiences in public middle schools, focusing on unexpected elements such as mathematical errors and contradictions. The findings indicate that the majority of PMTs (16 out of 28) adhered to their scripted interruptions while refining their instructional strategies to enhance student understanding and minimize unexpected disruptions.

Keywords: Preservice teacher education, instructional activities and practices, scripting.

Introduction

Allen and Wright (2014) found that teaching practicums in pre-service teacher education did not necessarily help the participants blend theory and practice together without intentional support and explicit theory-practice connection prompts from the mentors. Theory-enriched practical teaching experiences in which theory and practice are connected with deliberate teaching actions leverage the teaching capabilities of pre-service teachers (Oonk et al., 2015). Among others (e.g., lesson studies and teaching practicums) in which theory and practice are leveraged together, mathematics education research has used scripting tasks as a tool to characterize the resources pre-service mathematics teachers (PMTs) use in their approximations of teaching practices (Campbell & Baldinger, 2022; Shaughnessy & Boerst, 2018; Shaughnessy et al., 2021; Zazkis & Herbst, 2018). Typically, individuals are provided with a fictional dialogue and asked to write a script, envisioning how the dialogue might continue among various agents (e.g., teacher, and students) (Zazkis & Herbst, 2018).

Although prior research has provided valuable insights into teachers' practices through scripting tasks (Campbell & Baldinger, 2022; Shaughnessy & Boerst, 2018; Shaughnessy et al., 2021), further studies are needed to explore how actual teaching experiences influence the revision of scripts, particularly in response to teaching moments involving the 'unexpected,' such as student errors and misconceptions, as opportunities to bridge theory and practice. Addressing the 'unexpected' requires teachers to make real-time adjustments during their teaching, which may shape teachers' perspectives on how interactions should unfold in revised scripts. However, it remains unclear how teachers draw upon their actual teaching experiences to refine their scripts, particularly by reflecting on classroom interactions. The current study aims to characterize how PMTs refine their scripts after their teaching experiences in real school settings. The research question guiding this study is: How do PMTs draw on their teaching experiences in real school settings to refine their scripts?

Background

Prior research has utilized various classroom interactions contextualized within scripting tasks to characterize the resources individuals have at their disposal for addressing students' responses, including errors, non-standard reasoning, and correct answers accompanied by incorrect reasoning (Campbell & Baldinger, 2022; Shaughnessy & Boerst, 2018; Shaughnessy et al., 2021). Teaching around the 'unexpected' can occur in a broader context and does not necessarily arise from student errors or misconceptions (Mason, 2015). For instance, while a teacher intends to describe the procedural steps for computing the arithmetic mean during a lesson, a student may inquire about the meaning of the term 'arithmetic.' Although such a question may disrupt the flow of the lesson, it prompts an exploration of the meaning of a mathematical term, potentially fostering new mathematical understanding. Stockero and van Zoest (2013) describe such an instance in a teaching activity as a pivotal teaching moment "in which an interruption in the flow of the lesson provides the teacher an opportunity to modify instruction in order to extend or change the nature of students' mathematical understanding" (p. 127).

Stockero and van Zoest (2013) characterized mathematics teachers' in-the-moment decisions in response to pivotal teaching moments, such as student errors and confusions. Their study found that teachers made a variety of decisions, including extending mathematical ideas or making connections, pursuing student thinking, emphasizing the meaning of mathematics, acknowledging but continuing as planned, or ignoring and dismissing the pivotal moment (see p. 133). In contrast to in-the-moment decisions, teachers may engage in more deliberate self-assessment after their instructional practice to enhance student learning in future teaching activities. According to Artzt et al. (2015) –

> teachers who evaluate their lessons in terms of students' understanding of concepts and procedures and dispositions toward mathematics, as well as the effects of their instruction on these outcomes, are likely to revise their lessons in ways that will enhance student learning. (pp. 85-86)

Kohler et al. (2008) found that student teachers implemented a range of instructional alterations following post-teaching, including but were not limited to modifications to learning objectives, instructional methods, and student tasks. These changes align with the framework proposed by Artzt et al. (2015), which emphasizes the interconnected processes of evaluating and revising instructional practices. According to Artzt et al. (2015), effective post-teaching reflection informs the subsequent step of revising instructional plans to enhance learning outcomes, demonstrating how reflective practices can guide meaningful instructional adaptations.

While scripting tasks provide approximations of practice, actual teaching experiences offer a richer context for reflection and adaptation. By integrating insights from classroom interactions and reflective evaluations, PMTs can refine their instructional approaches to address both anticipated and unexpected teaching moments, fostering deeper student learning and understanding. From this perspective, the theoretical foundation highlights the pivotal role of scripting tasks, in-the-moment decisions, and reflective practices in shaping mathematics teachers' instructional strategies.

Methods

The participants in this study were 28 Turkish PMTs (15 female, 13 male) in their fourth year of the program. They were enrolled in an elective methods course focused on assessing middle school students' in-class learning. All participants were also taking a mandatory student teaching methods course and completing their teaching practicum in public middle schools. Typically, they taught one or two class hours per week from Week 4 to Week 12 of the fall academic calendar when the present study took place.

As part of their midterm project in the elective methods course, the participants were asked to write a script envisioning their teaching in Week 8 or Week 9. In their pre-teaching reflection, drawing on Stockero and Van Zoest (2013), the participants reflected on the types of interruptions they anticipated and explained how they would respond if those interruptions occurred during their teaching. They also provided their lesson plans. The participants were free to include any part of their lesson in their scripts, from warm-up activities to closure, as long as they envisioned at least 10 minutes of instruction. As part of the elective methods course, the PMTs were introduced to pivotal teaching moments (Stockero & Van Zoest, 2013) (extending, incorrect mathematics, sense-making, contradiction, and confusion) and the variability in teacher responses (extends mathematics and/or makes connections, pursues student thinking, emphasize meaning of the mathematics, acknowledges, but continues as planned, and ignores or dismisses) to such moments (see p. 133).

After submitting their pre-teaching scripts, the participants conducted their lessons. Within three days of teaching, they were asked to revise their scripts, incorporating reflections on their real classroom experiences, and submit a post-teaching script. The PMTs were expected to document both their anticipated and actual interruptions (pivotal teaching moments) in their pre- and post-teaching scripts as part of their reflections.

Following data collection, the participants' scripts were anonymized and labeled as PMT-1, PMT-2, …, PMT-28. A data segmenting method (Patton, 2002) was used to break down their teaching activities into meaningful units, focusing on the interruptions (pivotal teaching moments) they identified. Through a close reading of their reflections, both their anticipated and experienced interruptions were analyzed, with particular attention to how they responded to those moments, drawing on the teacher actions characterized by Stockero and Van Zoest (2013). An analytical code list of pivotal teaching moments was created, which served as the basis for deductive analysis (Patton, 2002). In this context, teacher actions characterized by Stockero and Van Zoest was refined by emphasizing how the PMTs revised their instructional scripts. For instance, examining their pre-teaching script, it was identified whether the scriptwriter, in their post-teaching script, allowed for the same interruption, extended the repertoire of the interruption, added unexpected interruptions, or avoided the envisioned or experienced interruption. The categories of how interruptions in the pre-teaching scripts informed the post-teaching scripts focused on the PMTs' evolving understanding of student difficulties. An inductive analysis (Patton, 2002) was used when the PMTs referred to any types of disruptions apart from the codebook of pivotal teaching moments (Stockero & Van Zoest, 2013).

Individual segments served as units of analysis, enabling the identification of different types of teacher actions employed by each scriptwriter. In other words, if a participant responded to interruptions in varying ways across different segments, each response was counted separately and included in the analysis.

Results

This section examines how participants incorporated the 'unexpected' into their post-teaching scripts, building on their pre-teaching scripts. The subsections are organized in order of the most to least frequently observed categories in the scripts.

Adhering to scripted interruptions while enhancing student understanding to avoid unexpected disruptions

Sixteen PMTs adhered to their pre-teaching scripts, incorporating envisaged interruptions into their post-teaching scripts. Yet, they aimed to improve student understanding to avoid unexpected interruptions in their post-teaching scripts, making enhancements involving adapting instructional strategies, breaking tasks into smaller steps, introducing simpler tasks, and other modifications to support learning (e.g., altering learning tools accompanied by tasks). For instance, PMT-14 initially expected some calculation errors while solving the task (Figure 1). The participant's focus was on mathematical contradictions. However, she encountered an unanticipated mathematical confusion while teaching fifth-grade students. She said:

> When we started solving it, one student seemed confused and asked, "How come, teacher? Was Mr. Mehmet not home?" The student focused on the visual provided in the problem and assumed that Mr. Mehmet would pick up his son from home. This could be an example of the type of mathematical interruption I encountered.

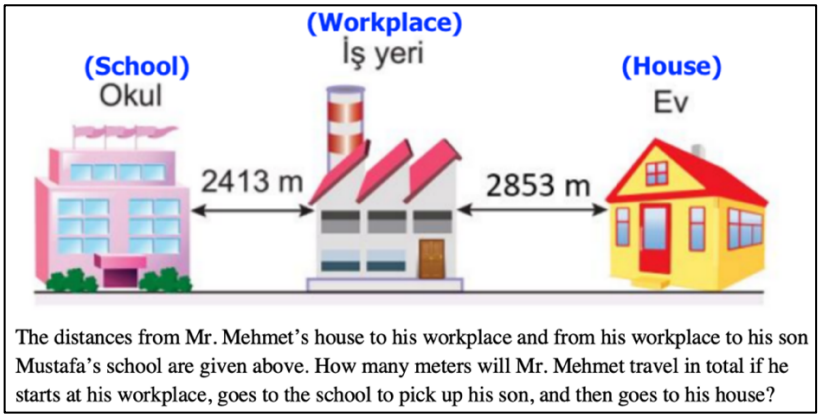

Figure 1: The task PMT-14 utilized in her pre-teaching script

In her revised script, PMT-14 aimed to facilitate students' understanding of the problem context by breaking the task into simpler calculation problems. She believed that this simplification would lead to a better comprehension of the task. She described her revision as follows.

> In my revised scenario, I created two additional questions based on the question from my previous scenario. These new questions are simpler than the original one. Since students had difficulty understanding the question I used in the earlier scenario, I included these two questions in my

revised scenario. Through the first and second questions, students will be able to grasp the difference between traveling a distance once and traveling it twice. In the first question, I directly asked about the distance from one place to another. In the second question, to help students better understand the part they struggled with in the previous scenario, I created a new question about the total distance travelled when going to a place and then returning from it.

After solving prerequisite problems on identifying the distance between two locations and calculating the distance required for a round trip, PMT-14 eliminated likely confusion about the problem situation in the third task. From there, she shifted her focus to calculation errors around mathematics contradiction, aligning with her previously envisaged script.

Addressing unanticipated interruptions in post-scripts

Seven PMTs responded to unanticipated interruptions based on their teaching. Namely, they encountered classroom situations they had not anticipated. In their post-teaching scripts, they incorporated these unanticipated events and described how they addressed them. These adaptations reflected the participants' responsiveness to actual classroom interactions, as they modified their instructional approaches to manage unexpected student errors, misconceptions, or reasoning difficulties that emerged during teaching. For instance, PMT-4 addressed an unanticipated interruption during her actual teaching. In her reflection, she stated: "I included a mistake in my new scenario, inspired by the mistake of a student who simplified without taking the opposite of the second fraction while doing division and then found the result wrong by doing multiplication." In her post-script, she referred to the same interruption, where a fifth-grade student made an incorrect simplification while dividing two rational numbers. Yet, PMT-4 employed a teacher-centered approach, reminding the student of the procedural rule for dividing two rational numbers. She provided a counterexample, and later stated: "However, we know that if we divide only two [rational] numbers that are the same, the result will be 1." Despite this, she did not provide the correct answer to the question. Instead, her focus appeared to center on preventing incorrect simplifications when dividing two rational numbers.

Extending the repertoire of the expected interruptions around the 'unexpected' in post-scripts

Seven PMTs broadened their understanding of expected interruptions by incorporating insights gained from the unexpected issues they encountered during teaching. They deepened and broadened their original expectations by integrating variations of anticipated difficulties informed by their teaching experiences. Rather than replacing or abandoning their original anticipations, they expanded their understanding of possible student challenges, capturing a wider range of related errors or reasoning issues that became apparent in practice. For instance, PMT-2 encountered a new version of expected interruptions. She expressed it as follows.

I encountered a [new] calculation error. While performing a subtraction operation, one of the students forgot to decrease the tens place by one after borrowing from the neighboring place, which resulted in an incorrect outcome. This error caused the remaining number to be larger than the minuend. Additionally, although the student experienced difficulties in comparing numbers similar to the [pre-teaching] scenario, the reasons for their difficulty during my teaching were different…

In her post-teaching script, PMT-2 addressed the same interruption she envisioned in her pre-teaching script. Yet, drawing on her teaching activity, PMT-2 extended the repertoire of calculation and comparison errors in subtracting numbers. In her post-teaching script, PMT-2 directed the student through identifying and correcting errors in subtraction and number comparison by counterexamples by saying: "Now you subtracted approximately 8000 from a number approximately 10000 and your answer was something with 12000, do you think this is possible?" From there, she emphasized the importance of starting comparisons from the highest place value and encouraged using the right mathematical terminology, such as "minuend," to clarify concepts.

Facilitating mathematical communication in post-scripts, drawing on the 'unexpected'

Four PMTs focused on enhancing mathematical communication in their post-scripts, adopting a sociocultural perspective to promote collective learning. For instance, in her pre-teaching script, PMT-1 envisioned students who provided lengthy statements, emphasized their reasoning, and constructed clear arguments. PMT-1 noted that students faced challenges in explaining their mathematical reasoning, as indicated by her statement: "Students had difficulty explaining their solutions and expressing themselves. Since they were fifth-grade students, they were not comfortable explaining their thinking." In response to these challenges, PMT-1 revised her teaching script by incorporating strategies aimed at addressing mathematical miscommunication. For instance, she stated:

To maintain the students' self-confidence while solving problems, rather than immediately allowing another student to take over when the solution was incorrect or incomplete, I used expressions like, 'Does anyone want to help your friend solve?'… Instead of directly stating whether an answer was right or wrong, I introduced simplifying questions to guide students in identifying their mistakes and gaps in understanding.

In her post-teaching script, PMT-1 emphasized step-by-step reasoning and encouraged active participation to enhance students' understanding and communication. For instance, she guided students like Berkay through identifying place values by breaking the process into smaller steps: "Let's say it in order, Berkay. Ones, tens, hundreds..." PMT-1 also used reflective questions like "What does the question ask us?" to help students employ problem-solving strategies, foregrounding clear communication and deeper mathematical understanding.

Replacing anticipated interruptions with unexpected ones in post-scripts

Two PMTs revised their scripts by substituting the anticipated interruptions with the unexpected challenges they encountered during their teaching. For instance, in her pre-teaching script, PMT-23 anticipated an interruption from a seventh-grade student questioning the meaning of "absolute" as in "absolute value." PMT-23 initially paused the discussion and later addressed the question by eliciting ideas from the class. She connected the term to both its colloquial and mathematical meanings. In her pre-teaching script, PMT-23 provided an opportunity for another student to address the interruption. On the other hand, in her post-teaching script, she responded to the student's confusion through direct instruction. Despite this difference, in both scripts, she consistently encouraged student participation in performing arithmetic calculations. In her revised script, PMT-23 de-emphasized the isolated

meaning of “absolute” and instead prioritized explaining “absolute value” as a single meaning unit. She adjusted her script to reflect the real classroom dynamics she encountered, replacing the anticipated interruption in the pre-teaching scenario with the actual interruption observed during her lesson. Reflecting on this shift, she stated:

In the first scenario, I anticipated an interruption with the question “What does absolute mean?” from students. Yet, I experienced it with the question “What is absolute value?” In the first scenario, students made more mistakes about absolute value. I did not emphasize this much in the revised scenario because, during the lesson, I realized that students did not have many misconceptions about absolute value.

Final remarks and implications

This study contributes to the literature on the integration of scripting tasks and real teaching experiences in teacher education by illuminating how PMTs revise their instructional scripts in response to both anticipated and unanticipated classroom events. Consistent with the work of Artzt et al. (2015) and Kohler et al. (2008), the findings demonstrate that many PMTs actively engage in reflective adaptation by simplifying tasks, scaffolding learning, and modifying instructional sequences to better support student understanding. These actions reflect a growing capacity for responsive instructional decision-making that aligns with prior research emphasizing the importance of post-teaching reflection for fostering adaptive expertise (Artzt et al., 2015).

Some PMTs leveraged the “unexpected” as a resource for fostering mathematical communication and classroom discourse, some of which resonates with sociocultural perspectives on teaching that emphasize dialogic engagement and collective reasoning. This highlights the potential for teacher educators to design learning experiences that intentionally connect scripting tasks with opportunities for PMTs to reflect on and respond to authentic classroom interactions, thereby cultivating professional noticing and instructional flexibility (Mason, 2015). Yet, the present study does not foreground the level of using theory in teaching practice. Drawing on prior studies (Allen & Wright, 2014; Oonk et al., 2015), offering more opportunities where PMTs’ theoretical ideas about pivotal teaching moments are surfaced and contextualized through video-based tasks and structured guides in scripts prior to lived experiences may offer a more robust picture of PMTs’ theory-enriched practical approaches around unexpectedness in teaching.

It is important to highlight that sources of the lived experiences of unexpectedness in teaching vary and can stem from mathematical, pedagogical, behavioral, affective, and institutional roots (Mason, 2015). From this pedagogical standpoint, the results of the present study may have some implications for the design of teacher education programs around teacher noticing skills through scripting tasks. Characterizing how pre-service teachers’ response to different sources of the unexpectedness, in particular, may offer snapshots regarding the cognitive and cultural entrenchments of teaching of mathematics, which in return opens new pedagogical calibrations for improving teaching quality around the noticed and unnoticed aspects. Integrating scripting tasks with structured post-teaching reflections may also enhance PMTs’ adaptive expertise by fostering connections between pre-teaching expectations and actual classroom experiences. Through this process, teacher educators can support the development of professional noticing skills and promote flexible instructional decision-

making (Mason, 2015). Finally, considering PMTs' theory use in teaching varies and may be insufficient without intentional scaffolding (Oonk et al., 2015), guided discussions on pivotal teaching moments may help PMTs cultivate a deeper understanding of how to navigate diverse classroom interactions in ways that enhance student learning.

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Data-based decision making in mathematics: Designing a professional development programme for teachers in challenging contexts

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Across many educational systems, professional development (PD) is considered a main driver for instructional improvement and educational equality. This paper discusses the conceptual design of a PD programme for mathematics teachers working in socio-economically disadvantaged schools. It examines how didactical training in problem-solving can be combined with enhancing teachers' capacity for interpreting student assessment data and engaging in collaborative data use. The PD follows a practice-oriented structure. Teachers are thereby introduced to heuristic strategies, metacognitive scaffolds, and approaches to data-based decision making. The three-unit structure integrates input, classroom application, and reflection, following established principles of effective professional learning.

Keywords: Professional development, assessment literacy, problem solving, at risk students.

Introduction

Across international contexts, the persistent gap in mathematical achievement between socio-economically advantaged and disadvantaged students represents a central educational challenge (OECD, 2023). Comparative assessments such as PISA, TIMSS or national large-scale monitoring programmes provide essential data for identifying these inequalities and guiding instructional improvement. In Germany, national low-stakes formative assessments in grade eight have revealed that a substantial proportion of students perform below minimum standards, particularly in schools serving socio-economically disadvantaged communities. Yet, data from such assessments frequently remain underused for instructional development. Teachers often hold sceptical or negative attitudes towards assessment data, sometimes based on misconceptions about its function, which leads to limited data use (Hawlitschek et al., 2024). Similar challenges have been reported internationally, where teachers' data literacy and confidence in interpreting student results are recognised as crucial but underdeveloped dimensions of professional practice. Research shows that collaborative data-based decision making can improve teaching and learning (Schildkamp et al., 2018).

The German initiative discussed in this paper builds upon this international discourse by designing a professional development (PD) programme specifically targeted at teachers in disadvantaged schools. In these contexts, mathematics is frequently taught by less-qualified or out-of-field teachers, who may lack sufficient pedagogical content knowledge to design cognitively demanding lessons (Richter et al., 2024). To address these needs, this paper presents a PD programme that combines enhancing teachers' capacity to foster problem-solving in mathematics lessons, strengthening their ability to interpret and make use of assessment data, and promoting data-based decision making within schools, while creating learning opportunities that are closely tied to teachers' everyday practice. The programme is organised into three units with phases of classroom application and reflection in between. While the paper takes Germany as its context, it addresses an internationally relevant

question: how can professional development support teachers in using data as a resource for specific improvement rather than as a tool of accountability? The programme is still in its design phase and scheduled for implementation across Germany in 2026. Thus, this paper discusses how PD can be conceptually designed to link problem-solving and data-based decision making. By situating the programme within the international PD and data use discourse, the paper aims to contribute to a broader conversation about the integration of data literacy in mathematics teacher education.

The following section introduces design principles of effective professional education, reviews research on fostering problem-solving with low-achieving students and outlines approaches to data-based instructional development.

Professional development principles

Effective PD builds on several well-established design principles identified in international research. First, a strong content focus is essential as PD that directly engages with subject matter content and student learning processes is linked to improvements in teacher practices (Desimone, 2009). Kunter et al. (2013) identified pedagogical content knowledge as an important predictor for the progress of students as it is necessary for teachers to be able to create cognitively activating learning opportunities. While content knowledge itself is essential for building pedagogical content knowledge in the first place, content knowledge alone is not sufficient for student success (Kunter et al., 2013). At the organisational level, studies suggest that the sufficient duration of the programme is an important factor for it to be successful. Teachers need time to reflect on what they have learned and to put it into practice. Additionally, coherence, active learning and collective participation are criteria for effective PD. This means that PD should connect to teachers' prior experiences, needs and beliefs, and actively involve them in the process. Collaboration among teachers plays a central role because changing established routines requires joint reflection and dialogue within a professional community (Desimone, 2009).

Building on these principles, PD needs to strengthen teachers' pedagogical content knowledge (Kunter et al., 2013). Within German educational standards, problem-solving has been established as a process-oriented cross-cutting goal of mathematics education, to ensure that students can apply mathematical competences meaningfully in new contexts. However, especially low-achieving students show great difficulties in working on open-ended tasks and solving non-routine problems. They often lack self-regulation and strategies to structure their solution process (Herold-Blasius & Rott, 2022). Teachers often respond to this by giving step-by-step instructions rather than letting the students find their own solutions. Though, research shows that low-achieving students benefit from competence-oriented approaches (Scherer, 2020). This underlines the need for explicit instructional support that combines metacognitive scaffolding, such as a solution plan, with the systematic use and reflection of heuristic strategies. Solution plans are usually derived from Pólya's (1973) problem-solving cycle. They neither serve as fixed algorithms nor ensure that a correct solution will be reached. Rather, they provide a sequence of steps that guide students through the process, offering structure and orientation in what can otherwise be an unorganised and challenging problem-solving situation (Liljedahl et al., 2016). One example is a solution plan which consists of five steps: understanding the problem, choosing an approach, performing, explaining the results, and checking.

It can be used for both problem-solving and modelling and has been regarded as a valuable support in the problem-solving process by grade six students in a qualitative interview study (Greefrath, 2015). One approach to introducing heuristic tools and strategies in the classroom are example-tasks that can be solved by using specific heuristics and can therefore later on be used as mental representation of the specific heuristic (Liljedahl et al., 2016).

While supporting students in problem-solving requires teachers to strengthen their pedagogical content knowledge, it is equally important that teachers are able to identify their students' learning needs. For this purpose, the systematic use of data can serve as a valuable tool. Data-based decision making enables teachers to ground their instructional planning in evidence rather than intuition and to monitor the effects of their teaching over time. Across educational systems, however, the purposes of data use as well as potential pressure arising from it, vary considerably. In some countries, such as England (Downey & Kelly, 2013) and the United States (Wayman et al., 2013) data are strongly linked to external accountability and performance targets, which can create tension between formative and summative purposes. Teachers in these contexts often face pressure to demonstrate student progress and, in some cases, job security or school reputation depends on test results. As a consequence, teachers may experience data use as a form of surveillance or control rather than as a learning opportunity. This again might lead to teachers feeling compelled to manipulate data or to narrow down their teaching to what is tested (Downey & Kelly, 2013; Wayman et al., 2013). An approach from the Netherland shows, how data can be used constructively. Schildkamp et al. (2018) conducted research on data teams that have been established in schools to support teachers in working collaboratively with assessment data. These teams typically follow a structured cycle in which teachers identify a problem, collect and analyse relevant data, interpret the results, and develop and evaluate targeted interventions. Though, the success of such teams depends on a range of enabling and constraining factors. These include trust within the school, leadership support, and data literacy of the school staff (Schildkamp et al., 2018). Like in the Netherlands, there has also been a promising data literacy intervention in New Zealand, showing effects on student outcome, with increasing effect sizes in those schools that sustained the established reforms long-term (Lai & McNaughton, 2013).

In Germany, however, the use of data for instructional and school development is still relatively limited, and the willingness among the staff to engage in cooperation varies considerably across schools and school types. Several reasons have been identified for the limited use of data such as the results of the national formative assessment in grade eight. Teachers often perceive little benefit for their daily instruction, hold sceptical attitudes towards the intention of the assessments, or are unsure of how to translate the results into specific instructional development (Hawlitschek et al., 2024). Nevertheless, there have also been promising efforts in Germany to strengthen teachers' acceptance of and competence in data use through targeted PD. Vogel et al. (2016) designed PD units that combine the technical interpretation of formative assessment results with transparency about the design and intentions of the assessment test, aiming to improve the acceptance among teachers. More recently, Schnitzler et al. (2024) extended these ideas to the field of reading competence, showing that carefully designed PD can increase teachers' willingness to use assessment data for instructional development. These initiatives demonstrate that PD can play a significant role in helping teachers to

perceive data as a meaningful resource for improving teaching and learning rather than as an external instrument of control.

Building on the presented research, the conceptual design discussed in the following chapter explores how PD for mathematics teachers can combine enhancement of pedagogical content knowledge with the development of data literacy.

Conceptual design of the professional development programme

Based on the theoretical considerations above, the PD presented here exemplifies how problem solving and data-based decision making can be made accessible for teachers working in disadvantaged schools. The conceptual structure of the PD follows design principles identified in international research (Desimone, 2009). These principles highlight the need for content focus, sufficient duration, coherence, active learning and collective participation. In practice, this means that each PD unit should connect to teachers' prior knowledge and beliefs, offer authentic classroom cases as starting points, and allow teachers to collaboratively test, discuss, and reflect on new approaches. To enable large-scale implementation across Germany, the PD will be carried out by qualified PD facilitators rather than by the design team themselves.

The PD is structured in three separate units, each lasting three hours. To ensure sufficient duration, while acknowledging teachers' limited time resources, there is a two-month break between the units, providing time for classroom application and reflection. The first unit of the PD focuses on strengthening the teachers' abilities to foster problem-solving in their classrooms. Low-achieving students experience difficulties in approaching non-routine tasks, which oftentimes causes teachers to mechanically explain contents step-by-step and to assign repetitive tasks instead of enhancing process-oriented competences, although research shows that low-achieving students benefit particularly from competence-oriented open approaches (Scherer, 2020). Explicit scaffolding and systematic use of heuristic strategies are therefore needed to help students engage productively with non-routine tasks (Herold-Blasius & Rott, 2022). Consequently, this PD introduces teachers to a solution plan, which is based on Pólya's (1973) problem-solving cycle and provides structure without forcing students into rigid algorithms (Greefrath, 2015). These approaches are embedded into a six-hours lesson plan that situates mathematical core competences, such as percentage calculation, proportional reasoning, averages, estimation, measurements, and data interpretation, in open tasks to promote flexible application. Teachers trial this lesson plan in their classrooms in between the PD units, ensuring that the professional learning is directly connected to their practice.

The second unit of the programme addresses the interpretation and instructional use of formative assessment data. As shown by earlier studies, teachers often hold sceptical attitudes towards assessments, which limits the extent to which results are used for lesson planning and school development (Hawlitschek et al., 2024). Yet, international experiences highlight that data use can improve teaching and student learning (Lai & McNaughton, 2013; Schildkamp et al., 2018). In response, this PD provides input on the aims, design, and interpretation of formative assessment, guiding teachers in interpreting the meaning of the solution frequencies of each task and the comparative results across classes, schools with the same location type, and the state results. To strengthen relevance, teachers also engage in analysing the results of a diagnostic problem-solving

test, designed in the style of a national formative assessment test, allowing them to practise data interpretation in connection to the didactical content of the first unit.

The third unit extends this focus to data-based decision making at the school level. Research on data teams in the Netherlands demonstrates that collaborative processes of analysing results, identifying learning needs, and developing targeted measure can improve school development (Schildkamp et al., 2018). Based on these insights, the presented PD elaborates a process for schools in which year-level teams discuss striking results, share findings in the mathematics teachers-conference, and develop school-wide measures within the general teachers-conference.

The following section reflects on potential research directions arising from this conceptual model and presents a possible evaluation of its implementation.

Accompanying research

In order to link the conceptual design with the research discourse on PD effectiveness, the accompanying study framework is designed along two dimensions: teacher beliefs and student outcomes (see Figure 1). On the teacher level, the focus lies on changes in beliefs about formative assessment data and perceived value of competence-oriented approaches for supporting low-achieving students as the attitude towards data use influences the actual data use in instructional and school development. On the student level, attention is directed to problem-solving performance and metacognitive awareness, to explore whether potential improvements in instructional practice translate into improved student learning. A control group, consisting of students whose teachers are not part of the PD shows whether possible results can be attributed to the PD rather than other factors, such as the repeated testing.

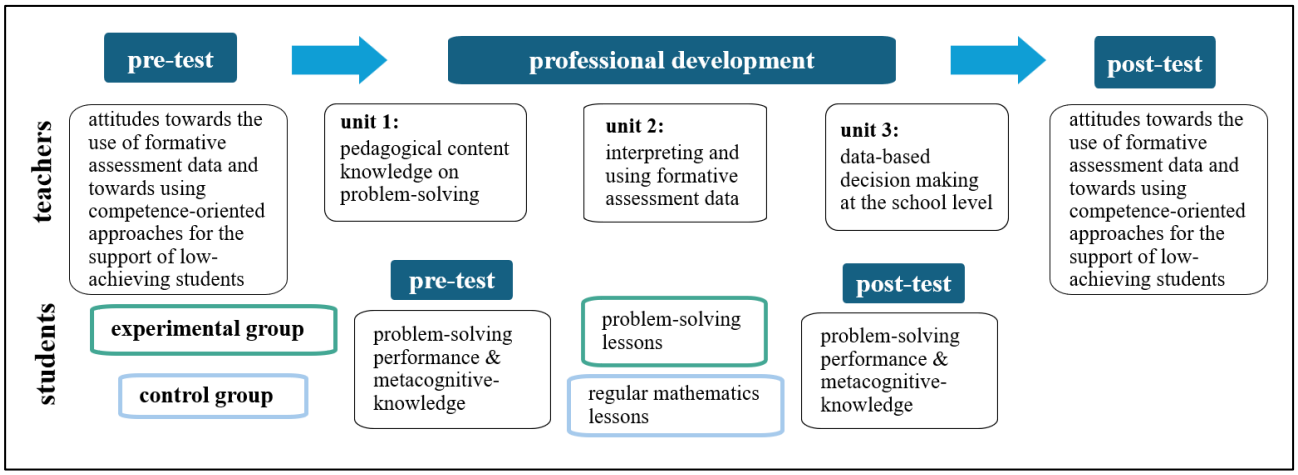

Figure 1: Evaluation model

This evaluation model aims to answer the following research questions:

1. To what extent does the PD lead to an improvement in students' problem-solving competences?
2. To what extent does the PD change teachers' beliefs towards the use of formative assessment?
3. To what extent does the PD change teachers' beliefs towards the value of process-oriented competences for low-achieving students?

With the design and evaluation framework outlined above as a foundation, the following section discusses the limitations and challenges of the PD design.

Limitations and conclusion

While the programme design builds on established principles and prior evidence, its implementation will inevitably face challenges. Teachers' participation and engagement are likely to be decisive factors for the effectiveness of the PD, as sustainable change requires active involvement in practical application and reflection (Desimone, 2009). As the potential change in teachers' beliefs are assessed through self-reports, a bias cannot be ruled out. At the same time, structural conditions such as leadership support and willingness to collaborate within the staff will influence whether data use becomes embedded in school culture (Schildkamp et al., 2018). Therefore, it cannot be determined whether a potential self-reported change in beliefs would actually translate into the implementation of data-based decision making in schools. Another methodological limitation concerns the challenge of capturing the transfer from teacher learning to student outcomes. Establishing the link between teacher learning, instructional change and student achievement has been identified as a general challenge in PD research (Desimone, 2009). While the PD aims to enhance teachers' instructional practices and data literacy, it cannot be predicted whether teachers translate their gained knowledge and potentially changed beliefs into changes in classroom practice. Even if such changes occur, it remains uncertain to what extent they will lead to measurable improvements in students' problem-solving performance within the relatively short timeframe of the intervention. Furthermore, the effect of the PD is influenced by the quality of its implementation by the facilitators. However, the intermediary level of facilitators is necessary to enable large-scale implementation of this PD in socio-economically disadvantaged schools all over Germany.

The professional development programme outlined in this paper serves as one example on how professional education can be designed to potentially support the learning of practice in schools facing socio-economic challenges. In linking the development of problem-solving competences with the systematic use of assessment data, it addresses both the subject-specific and organisational dimensions of teaching practice. Despite the presented challenges, the PD has the potential not only to foster problem-solving competences among low-achieving students but also to strengthen a constructive culture of data-based decision making in disadvantaged schools. By targeting teachers in socio-economically disadvantaged contexts, it directly addresses the urgent need to improve mathematics education for students who are at particular risk of underachievement. Although the programme is still in its design phase, its structure demonstrates how PD can move beyond abstract training to actively engage teachers in the processes that are central to their practice. The planned evaluation will be crucial to assess its influence on both teacher beliefs and student outcomes and will show whether this programme can make an impact with regard to reducing educational inequality in Germany.

Acknowledgement

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Connecting theory and practice in initial teacher education: Representing word problems and their teaching

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This paper focuses on possible ways to support prospective mathematics teachers in connecting theory and practice. It presents an empirical qualitative study that involves 46 prospective primary school teachers in the second semester of the fourth year of their five-year initial teacher education programme. It shows how cartoons with dialogues representing diverse perspectives on teaching of simple additive word problems can help prospective teachers reflect on a theory that has proven difficult for them to comprehend and connect to teaching practice: the classification of simple additive word problems.

Keywords: Grade 1, professional education, mathematics teachers, student teachers.

Introduction

One of the challenges of initial teacher education programmes consists in balancing theory and practice and connecting them in a way that would meaningfully support prospective teachers in their endeavour to understand what practice is really about and how to navigate it. One promising approach to ensuring the meaningful connection between theory and practice takes the form of practice-based pedagogies, specifically, pedagogies utilising systematic use of representations, decomposition, and approximations of school practice (Charalambous & Delaney, 2020). In terms of design, representations and approximations can take various forms with various degrees of authenticity and actors, using various artefacts and background events for various educational purposes (Skilling & Stylianides, 2020).

This paper focuses on cartoon-based dialogical representations (Herbst et al., 2011), namely individual pictures capturing a dialogue related to the content taught at school. Such a dialogue can be informed by authentic practice and serve as a means of scaffolding teacher noticing, for instance, when teachers analysing a video-recorded lesson do not focus on mathematics-specific phenomena but on general pedagogy only (Samková, 2021). It can also represent a piece of theory that a teacher educator aims to discuss with prospective teachers, for instance, the structural nature of mathematics and its consequences for learning (Samková, 2023).

This qualitative study with prospective primary school teachers as participants introduces another way of how a cartoon-based dialogical representation can be designed and included in initial teacher education: by responding to a difficulty in connecting a certain piece of theory and practice that prospective teachers struggle with and representing its essence. During a reflective mathematics teaching practicum that preceded this study, I noticed such a difficulty related to the classification of simple additive word problems. With the aim to strengthen the connection between theory and practice for this particular topic, I designed cartoons representing both theory and practice related to this topic and presented them to prospective teachers with a set of indicative questions. They

responded first individually in written form and then in a broad joint discussion. The discussion led to an in-depth elaboration of the classification and of its specific relation to practice made by the participants. This paper describes the process in detail and addresses the following research question: *Having cartoons representing diverse perspectives on teaching of simple additive word problems, what particular connections between theory and practice can be found in written responses given to these cartoons by prospective primary school teachers?*

Simple additive word problems within local educational context

The study focuses on *simple additive word problems*, namely problems that can be calculated by a single addition operation within the numeric range of up to 10. These problems are taught to primary school students at the very beginning of their schooling and are essential for mathematical learning: they serve as an introduction to problem solving as well as modelling and contribute to strengthening conceptual knowledge about numbers and arithmetic operations (Verschaffel et al., 2020). From a theoretical perspective, simple additive word problems are usually classified into four types labelled (in alphabetical order) as *change*, *combine, compare*, and *equalise* (Vincente et al., 2022). The meaning of addition is usually introduced to primary school students using *combine*, as it refers to the part-part-whole interpretation. The other types appear later, with *equalise* being the last and less frequent one. Some scholars (e.g., Thevenot & Oakhill, 2006) highlight the dynamics of the *change* problems and label them as *dynamic*, as opposed to *static* problems in which no changes of quantities appear over time.

In the Czech Republic, where the study took place, primary school teachers are generalists who teach all school subjects to students between 6 and 11 years of age. Their initial teacher education usually takes the form of a five-year study programme led by universities, and most applicants enter this programme without any teaching experience. Within the programme at our university, we work with a classification of simple additive word problems consisting of *static*, *dynamic*, and *comparison* problems (in the sense of *combine*, *change*, and *compare* types, respectively); *equalise* is not included. I will use this particular terminology since the terms also appear in collected data.

Czech primary school students start solving simple additive word problems formally as soon as they can read and write, that is, usually in the second half of grade 1. At first, they are taught to read the assignment, draw a picture that would help them discover the calculation, write the calculation, check the result, and formulate the answer. During grade 2, students also learn to write what is known and what needs to be calculated and do it before drawing the picture. The pictures are usually based on accordingly arranged coloured dots (see Figure 1). Primary school textbooks usually start with static word problems, followed soon by dynamic problems, and much later by comparison problems.

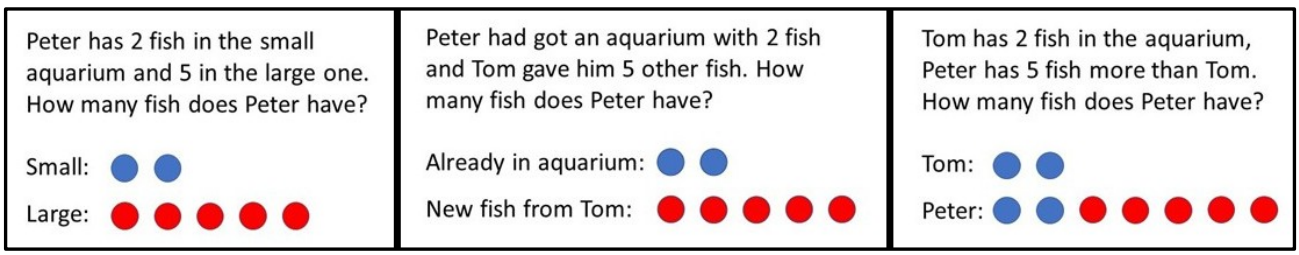

Figure 1: Sample tasks with sample pictures (left to right: static, dynamic, comparison problems)

Participants of the study and their background

The participants were 46 prospective primary school teachers in the second semester of the fourth year of the study programme at our university, divided into two equal groups for seminars. They are labelled N01 to N46. So far, the participants had studied the topic of word problems several times but with limited time allocation (see Table 1) that was usually reserved for all types of word problems altogether (additive, multiplicative, simple, complex, routine, non-routine, various numeric ranges).

Table 1: The topic of word problems within the study programme so far

Study year	What was studied specifically about word problems *[time allocation]*
1st year, 2nd sem.	Step-by-step solutions in a way taught at primary school: including pictures and excluding equations; initial encounter with the classification of simple word problems. *[2-hour lecture + 4-hour seminar]*
2nd year, 2nd sem.	Stages of solution process, effective management of students, characteristics and classification of individual types of word problems, methods of representation. *[2-hour lecture + 4-hour seminar]*
4th year, 1st sem.	Observing or teaching individual lessons (one lesson per week) where word problems were often part of the lesson plan; immediate one-hour joint reflection on these lessons. *[in 9 consecutive weeks]*

In seminars belonging to the content course in the first study year, I observed a certain discrepancy many participants exhibited when solving simple additive word problems. On the one hand, they felt confident in solving these problems, as they were able to "see" the results immediately. They did not need pictures or writing anything down, just counted in their heads and provided the result. This apparent simplicity of additive word problems seemed to be the reason why it was difficult for them to downsize to the grade 1 student perspective and try to derive a picture that would fit the task and indicate the calculation. Instead, many of them followed the reverse order of steps: Say the result, then write down the calculation leading to it, and, at the end, add any picture that seems fitting the numbers (but not necessarily the structure of the problem).

The course on didactics of mathematics in the second study year brought the topic of word problems to light again, but I do not have details on it as the course was completely led by my colleagues. However, during joint reflections within the teaching practicum in the fourth study year, I noticed participants who struggled to connect the classification of word problems with what was happening in the classrooms and reasoned about theory separately from practice.

Design of the study

The study took place in the second semester of the fourth study year, during a compulsory didactic seminar. The purpose of this seminar is to build on joint reflections from the teaching practicum in the previous semester, summarise key topics in mathematics teaching and learning, and discuss their mutual connections using various representations and approximations of practice. The content of the seminar is partly given by its attendees, who at the beginning of the semester indicate which particular topics they consider difficult and would like to include them in the seminar. The topic of word problems and their classifications was indicated by more than a third of the participants.

My plan was to design a cartoon-based representation of theory and practice related to simple additive word problems and include it in three consecutive weeks of the seminar as follows. First week: Let participants summarise what they know about the classification of simple additive word problems (10 min), then invite them to individually respond in written form to a worksheet with the representation (45 min). Time in-between: Analyse the responses and prepare an outline of a discussion based on them to include all relevant opinions and perspectives. Second week: Orchestrate the discussion, then invite the participants to revisit the classification and complete it from various theoretical and practice-based perspectives (90 min). Third week: Invite the participants to extend the classification analogically to multiplicative problems (30 min).

The representation consisted of two single-picture cartoons. To represent theory and highlight the classification of word problems, I drew inspiration from the work of Durkin et al. (2017) and Van Dooren et al. (2011) and included comparisons of various word problems. The first cartoon contrasts a static problem and a comparison problem (Figure 2), the second one a static problem and a dynamic problem (Figure 3). To represent practice, the dialogue bubbles represent the perspectives of lesson planning (Alice), calculations (Simon), task modifications (Dan), and pictures as graphical representations of the problem (Gabi). In the worksheet, each cartoon was accompanied by the same set of questions: *(1) Which teachers in the picture are right? Which ones are wrong? Why? (2) How would you include the tasks in the frame in your own teaching? (3) How would you solve the tasks in the frame yourself?* In this way, various perspectives were activated: others' teaching, own teaching, and own solving. Collected responses to these questions became data materials for the study, together with my written field notes from the seminar. I analysed data qualitatively, using open coding (Miles et al., 2014), with focus on aspects related to theory, practice, and their possible interconnections.

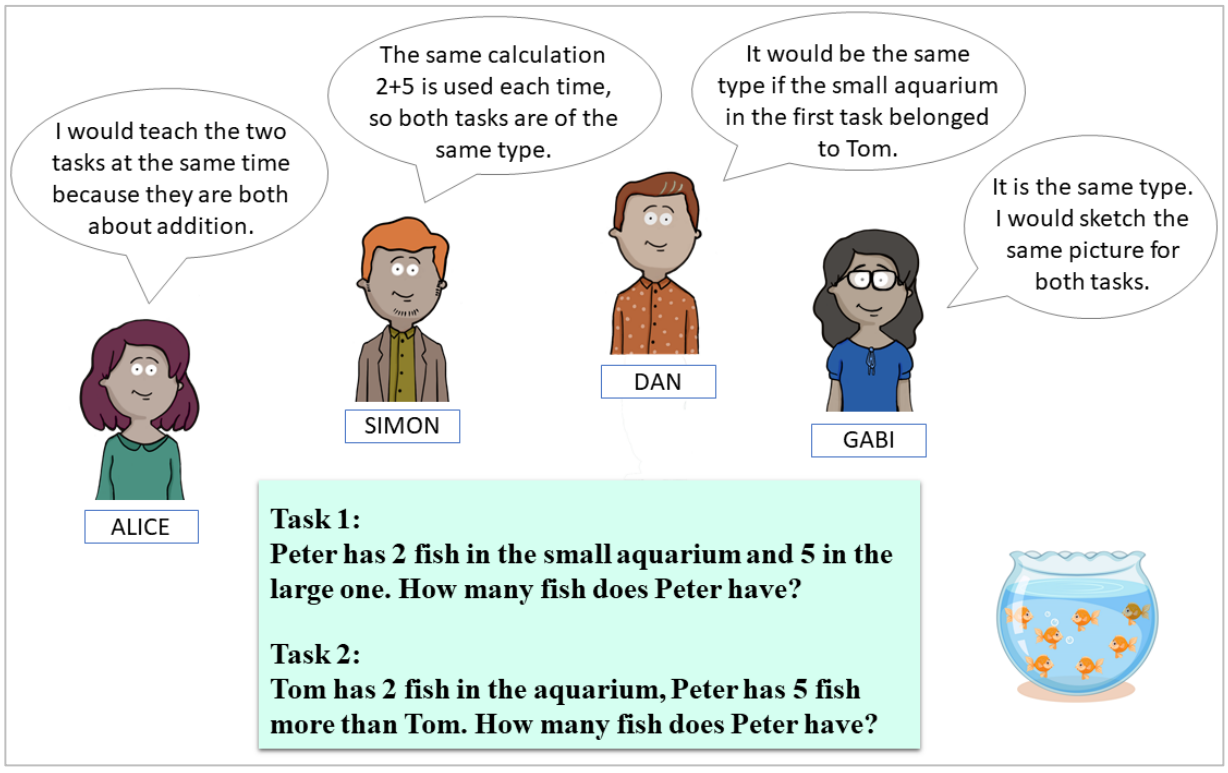

Figure 2: Static and comparison tasks; graphical elements (coReflect@maths, 2022; Freepik, 2025)

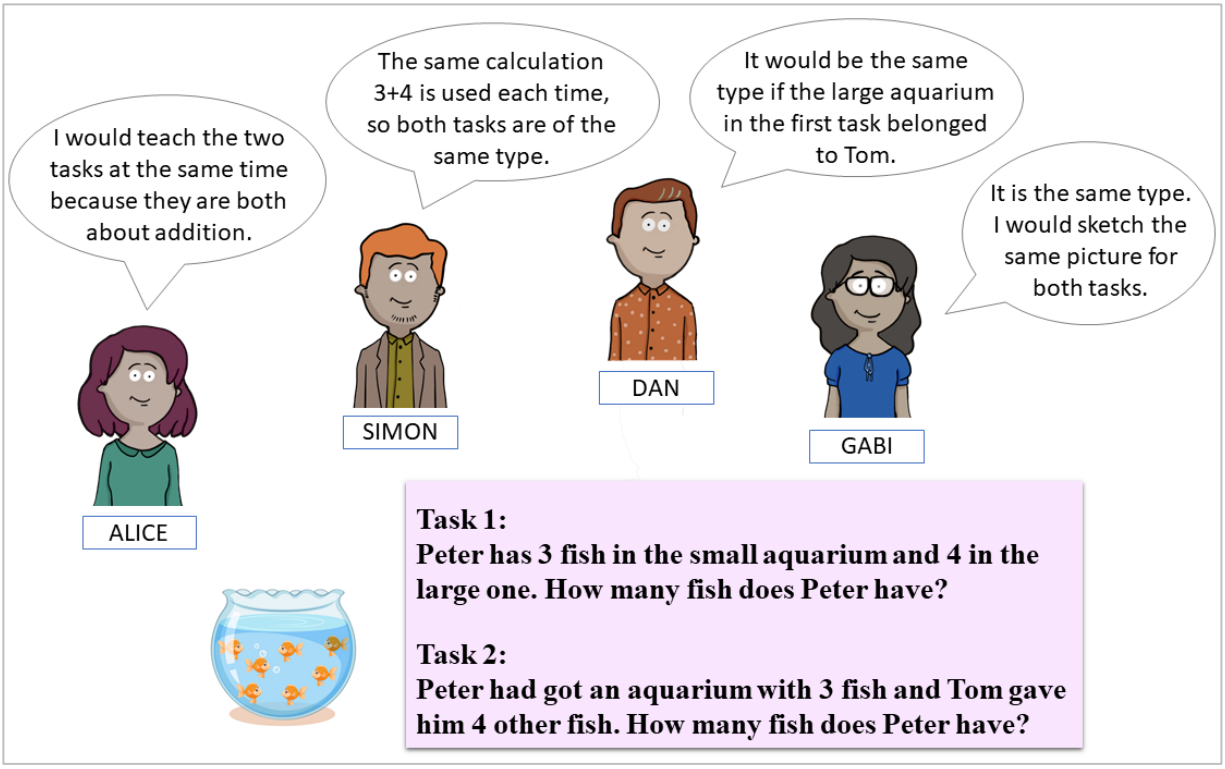

Figure 3: Static and dynamic tasks; graphical elements (coReflect@maths, 2022; Freepik, 2025)

Findings

In the first week, before working on the worksheet, each group of participants gave their initial version of the classification of additive word problems solved by addition. In both groups, the comparison type was missing. I did not point this out, just noted that we would revisit the classification next week and see if anything is missing. In the worksheets, the responses turned out completely opposite. In the responses to Figure 2, all participants recognised that the two tasks were not of the same type, although only N15 and N19 explicitly referred to the comparison process. In the responses to Figure 3, all but three respondents (N23, N37, N38) expressed their opinion that the two tasks were of the same type; only N23 explicitly distinguished between static and dynamic types.

Four main categories appeared in data analysis, two of them belonging to excerpts that properly compared the tasks in frames and properly assessed the opinions in bubbles (either with an explicit reference to the classification, or just with an implicit display of proper understanding of what are the relevant aspects distinguishing the tasks). The third category was for arguments based on irrelevant surface aspects that indicated a lack of conceptual understanding. The fourth category was for vague reasoning that sometimes appeared in data. Below, illustrative data excerpts are listed for each of the categories.

Reasoning with an explicit connection to the classification:

N15_Fig2: Yes, both problems are solved by addition, but in problem 2 we compare. We need different pictures to illustrate. Problem type 2 is included later at school. It is more challenging for students to imagine problem 2.

N19_Fig2: Both problems can be solved by addition, but the problems are different. In the first one, we ask about and count only Peter, so it is simpler and more direct. In the

second problem, we count also with Tom and compare more and less. (In the first problem we count/mention only 7 fish, in the second 9 fish.)

N23_Fig3: The same calculation, yes, but they differ in type – in one we own something, in the other the state changes → static and dynamic problems.

Reasoning based on relevant aspects but without an explicit connection to the classification:

N42_Fig3: I would teach problem #1 first. It's a very basic calculation. I would teach problem #2 later because we need to realise the relationship between Tom and Peter and that the number of Peter's fish depends on the number of Tom's fish.

N30_Fig2: Even though the same calculation is used each time, these are not the same types of tasks, because in the first task, 1 person has 2 aquariums next to each other and it is a simple addition, but in the second task, children have to realise what it means that "Peter has 5 more".

N38_Fig2: Simon is wrong. Unlike task 1, in task 2 we encounter the phrase "more than". Alice is right. The tasks demonstrate that various tasks can be solved using addition. Gabi is wrong. It is not possible to draw the same picture in task 2 as in task 1, since we do not yet know the number of fishes in one of the aquariums.

N38_Fig3: Simon is wrong. In the second problem (unlike the first problem), we are again adding something to an existing number.

Reasoning based on irrelevant aspects:

N21_Fig2: Both problems are solved by addition, but I would not solve them at the same time. First, I would teach problem 1, which involves one person who has 2 aquariums. In problem 2, there are 2 people and 2 aquariums, I would include it in the lesson only after the word problems of type 1 have been mastered.

N19_Fig3: Simon is right, in the end all the fish (in both examples) belong to Peter.

N21_Fig3: Both tasks are solved by addition and involve just Peter's fish, so it's the same type.

N14_Fig3: They are of the same type. Not because they use the same calculation, but because we ADD some number each time.

N26_Fig3: It's not "x more/less", so it's the same type of task.

Vague, non-specific reasoning:

N17_Fig2: I would calculate these tasks together on the board, frontal teaching, so that all students could see the difference between them.

In the second week, I let the participants respond to the two cartoons in classroom discussion and orchestrated the discussion to cover all aspects represented by the cartoons, while keeping in mind to include (anonymously) all thought directions that appeared in written responses. I did not contribute to the discussion with answers, only with questions. During the discussion, a draft of the classification appeared written on the board and was reflected in relation to the tasks from the cartoons. This activity resulted in adding the comparison type to the classification, and also in detailing the description for each of the types and voicing the differences between the types from various perspectives (solution process, lesson planning, instruction, typical student mistakes, samples of suitable and unsuitable graphical representations, etc.). After finalising the overview for additive word problems solved by addition, the participants supplemented the classification with analogical problems solved by subtraction. Other aspects influencing the difficulty of additive word problems were also talked about, for instance, the role of signal words and how a reformulation with an anti-signal would look like for different problems. In the third week, a detailed classification of simple multiplicative problems with samples was deduced by analogy, the activity orchestrated in a similar way.

Discussion and conclusion

From a design perspective, this study offers a new design of an educational and/or assessment instrument for initial teacher education. This design takes the form of a cartoon-based representation of practice that responds to an actual difficulty in connecting theory and practice that prospective teachers struggle with. In this way, the instrument captures content meaningful to study participants (Skilling & Stylianides, 2020). The cartoon format of the instrument allows for the inclusion of multiple levels of abstraction and multiple perspectives on practice-based situations mediated by two types of comparisons: a simultaneous comparison of various opinions on aspects that are key for teaching simple additive word problems (in bubbles), and a simultaneous comparison of various types of word problems (in coloured frames). The findings show, in agreement with (Durkin et al., 2017), that such comparisons could activate and support conceptual knowledge. In our case it was prospective teachers' knowledge of the concept "additive word problem" and, subsequently, also "multiplicative word problem". Furthermore, the opinions in bubbles represent snippets of lesson planning, problem solving, and problem posing, and the indicative questions offer perspectives of own solving, own teaching, and others' teaching. Altogether, it makes teaching practices more visible to prospective teachers (Charalambous & Delaney, 2020). For teacher educators, the instrument brings valuable information about teacher knowledge in practice-related situations, by clearly distinguishing those who can handle and sort word problems based on relevant aspects from those who focus on irrelevant surface aspects only.

The question underlying the entire study is about balance between theory and practice, and how they can inform each other. Participants of the study first attended theory courses and then practice, but many of them stored their knowledge about theory separated from knowledge they use in practice-related situations. The comparison type of additive problems was missing in the classification they provided before the instrument was administered, but all of them recognised it as different from the static type when responding to the practice-based dialogue included in the instrument. On the contrary, the dynamic type was present in the provided classification but mostly absent in responses to the instrument. The study shows how these separated approaches can be reconnected and made to complement each other meaningfully. Through the cartoon-based instrument representing both theory and practice, and a follow-up intensive reflection, the practice-related situation can help prospective teachers understand the reason, meaning, and coherence of the corresponding theory, and, at the same time, the theory can help clarify causes and consequences in corresponding classroom situations.

As for the topic of grade 1 additive word problems, some may find it unnecessary to go into such detail about such simple tasks. The reason is that these tasks not only partake in the development of conceptual knowledge about numbers and operations (Verschaffel et al., 2020) but also serve as the very introduction to problem solving and modelling (Van Dooren et al., 2011). Thus, the degree to which they are mastered during grade 1 affects the future success of students in these domains.

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Investigating mathematics teacher educator practices in teaching practicum course

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While the role of teacher education is considered as a critical element for the education of preservice teachers, there is less known about what teacher educators do in their professional practices. This study examines the role and practices of a mathematics teacher educator in a practicum course. The qualitative research design used for analyzing planning, instruction, mentoring, and evaluation processes of a teacher educator in the context of a practicum course. The findings on mathematics teacher educators' professional practices in designing and enacting the practicum to connect theory and practice are grouped around three main themes: structuring practicum, supporting preservice learning, and teacher educator's professional positioning. The research offers important insights into how the mathematics-specific dimensions of teacher educator practices, and the specialized practice of the teacher educator can influence the professional development of preservice teachers.

Keywords: Teacher education, mathematics teacher educator, teaching practicum, MTE practices.

Introduction

One of the most critical components of mathematics teacher education is teaching practicum. This process provides a space where prospective teachers integrate theory and practice and deepen their professional preparation through classroom observations and teaching experiences. However, knowledge about the practices of mathematics teacher educators (MTEs), who are central to this process, and how these practices shape prospective teachers' professional learning is limited. While the existing literature contains numerous studies on teaching practice (Korthagen, 2017; Zeichner, 2010), detailed analysis of discipline-specific practices—especially in the context of mathematics teaching—remains limited. However, the teaching practice course is a unique learning space where not only teaching skills but also teacher identity, Pedagogical Content Knowledge (PCK), and critical thinking are developed (Ball et al., 2008; Grossman et al., 2009). This study aims to understand which elements are salient in the process of preservice teachers learning to teach mathematics by examining the practices of MTE within a teaching practice course. Specifically, it examines how MTE structures the practicum course, supports preservice teachers' learning to teach, and establishes its own expert position within this process. This study highlights how the teaching practicum process offers opportunities for learning and rethinking not only for preservice teachers but also for teacher educators. In this context, the study aims to contribute to the teacher education literature by examining how mathematics teacher educators' practices in designing and enacting the practicum connect theory and practice, offering a discipline-specific perspective for understanding the dynamic nature of mathematics teaching practice.

Mathematics Teacher Educator (MTE)

Mathematics teacher educators (MTEs) are professionals involved in the preparation of prospective and current mathematics teachers. Typically employed in university settings, MTEs are equipped with mathematical content knowledge, PCK, didactic approaches, and curriculum knowledge, and they also utilize learning and teaching theories and research methods (Jaworski, 2008). More broadly, teacher educators are described as "teachers' teachers" (Lanier & Little, 1984), emphasizing the multifaceted and context-dependent nature of the profession (Dengerink et al., 2015).

In Turkiye, MTEs mostly work in university education faculties and provide preservice teachers (PSTs) with both mathematical knowledge and pedagogical formation. An effective teacher educator is expected to integrate this knowledge and prepare them for classroom practice (Sen & Erisen, 2002). It is emphasized that faculty members should not only provide theoretical knowledge but also engage candidates in active learning processes, guide their teaching practices, and sustain their professional development through research (Eristi, 2013; Hangul, 2018). University and school-based educators contribute in different ways, but the need for stronger collaboration between these two groups has been expressed (Hangul, 2018). The knowledge and identities of MTEs, the roles they assume, the strategies they employ, and the transitions they make between academic and professional communities still require further research (Beswick & Chapman, 2015; Beswick & Goos, 2018; Jaworski, 2008; Masingila, et al., 2018; Ozmantar & Agac, 2023). Therefore, it is important to examine their professional roles and practices.

Teaching practicum in mathematics teacher education

Teaching practicum is a critical process in the professional development of pre-service teachers, combining theory and practice. During this process, candidates gain classroom experience, while MTEs assume various roles and guide them (Loughran, 2006). In the context of mathematics education, internships are a significant opportunity for candidates to develop their pedagogical knowledge and skills (Zeichner, 2010). In Turkiye, this process is implemented within the scope of the Teaching Practicum course determined by the Ministry of National Education (MoNE) and the Turkish Council of Higher Education (YÖK). This course allows PST to apply the knowledge they acquired at university in the classroom environment under the guidance of mentor teachers. Pre-service mathematics teachers gain school experience under the guidance of university-based instructors and mentor teachers, and this process is a fundamental element of teacher preparation programs. According to the 2021 MoNE directive, two-semester practicum process aim to prepare candidates for the profession by applying their pedagogical and subject-matter knowledge in the classroom. The guideline also emphasizes the importance of resources and coordination between schools and education faculties to support the success of the practicum (MoNE, 2021, p. 1049). However, this guideline does not provide details on how to implement context-specific experiences.

Methods

This study was designed as a qualitative case study (Yin, 2009). The participant, Yasmin (a pseudonym), is an experienced mathematics teacher educator (MTE) with a doctorate in mathematics education and 14 years of teaching practice. Data was collected from 11 weeks of lesson observations

conducted over one semester and a semi-structured interview conducted via Zoom at the end of the semester. Observation notes from feedback sessions held after PSTs' lessons in practice schools were also used to support the interviews. These three-part interview questions focused on the participant's background, her approaches and practices as MTEs, her teaching practicum experiences, and her professional roles (Lunenberg et al., 2014). Although the questions did not explicitly ask about her design, she explained her design decisions while describing what she does in the practicum. Classroom observations in this study were used not only to provide context for the interviews but also to contribute data directly to the analysis process by revealing how MTE demonstrated their practices in real time and supported PST learning.

Data were analyzed through a thematic analysis process (Braun & Clarke, 2006). Initially, open coding was conducted, and through constant comparison, the codes were refined and organized into categories. This process led to the development of three overarching themes:

(1) Structuring the Practicum as a Course, which included the subthemes *defining the purpose of practicum; providing information about practicum; perceptions of the role of mentors; the MTE's classroom practices and approaches; and being a learner in becoming an expert on offering practicum*; (2) Designing Preservice Teachers' Learning**,** which comprised the subthemes *giving PSTs feedback on lesson plans regarding CK and PCK; fostering reflective practitioners; knowledge about PSTs; knowledge about school cultures; and knowledge about mentors*; and (3) Expertise and Professional Positioning of the MTE**,** which involved the subthemes *being specialized in teacher education; in-action learning and situated learning; and being a researcher and doing research*. These themes captured the mathematics-specific practices of the MTE in the practicum context and illuminated how she connected theory and practice in preservice mathematics teacher preparation.

Findings

From the data analysis, the teacher educator's practices in the teaching practicum course had emerged under three main themes: Designing and Structuring the Practicum, Supporting Preservice Teachers' Learning, and Expertise and Professional Positioning of the MTE. These themes reveal how the teacher educator contributed to the candidates' learning of teaching mathematics at different levels.

Structuring the Practicum as a course

This theme reveals how the teaching practicum course was structured and conducted to support mathematics PST learning to teach mathematics. Findings include defining the aims and scope of the practicum, information provided to candidates, perceptions of the role of mentors, in-class practices, and approaches developed by the MTEs based on their own experiences. Yasmin defined the *purpose of the practicum* in a multidimensional way. In the interview, she emphasized that the learning process of PSTs should not be limited to theoretical knowledge but should also involve active learning in a real classroom environment, referring to the purpose of the practicum as in-action teacher learning. Observation notes from the first session also illustrate how she positioned the practicum as a form of in-action and situated learning. For example, she told PSTs, "When you go to your practicum school, you are no longer a student but a teacher," emphasizing their active engagement in real classroom settings. She further highlighted that "Because this is an individual learning adventure. It's situated,"

and encouraged them to document meaningful moments from school observations such as students' questions and classroom interactions. In this process, she also attached great importance to the development of the professional identities of them. Furthermore, as the second code for this theme, *providing information about practicum*, Yasmin emphasized teaching practicums being not only about gaining theoretical knowledge, but also about understanding the institutional culture. She stressed that it is important for PST to learn how the school functions and the expectations. Yasmin stated that each school has its own unique teaching approach and that interns need to adapt to these differences. Observation notes also show how Yasmin familiarized PSTs with the institutional culture and everyday functioning of schools. For instance, she explained the importance of following school routines such as exam weeks and internal administrative processes, noting that "Exam weeks in high schools and the internal affairs of the school must be followed." She also encouraged PSTs to attend to formal procedures by reminding them to "Ask your mentors if we are listed in MEBBIS," referring to checking whether they are officially listed in the practicum school's digital administrative system (the national teacher placement database). According to Yasmin mentors provide the concept of being a teacher. The third code, perceptions of the role of mentors, emerged because Yasmin emphasized the critical role of mentors in the development of PST, and stated that their learning process does not depend solely on them, it is a collaboration. Observation notes show Yasmin explaining the mentor's role with concrete examples in the classroom. She emphasized that effective mentoring requires mathematical and pedagogical knowledge (CK and PCK), stating, "If teachers have a big idea in mind-a recipe-the lesson will be more organized; otherwise, the materials will be missing something." She also stated, "For learning to occur, the mentor must encourage student teachers to reflect," implying that the mentor should be a guide who supports reflective thinking. In the context of Turkiye, as discussed before, the practicum has also a university course offered by the teacher educator. The fourth code, *MTE's classroom practices and approaches: how she offers the course,* is Yasmin's description of various classroom practices and approaches to support the PSTs' learning to teach process in her class. In the interview, she mentioned how she gave importance to have PST analyse mathematical student thinking. Yasmin's observation notes also show her emphasis on analyzing student thinking. She asked the PSTs, "Follow the lesson and note the questions and problems students ask that you find interesting; we'll discuss the pedagogical aspects of the questions you like." She also stated that she had PSTs analyse the curriculum and encouraged them to evaluate the curriculum critically. Among Yasmin's classroom practices, lesson plan preparation and evaluation also have an important place. Observation notes indicated that Yasmin provided immediate feedback to PSTs by having them discuss their lesson plans in class, indicating that lesson plan evaluation was a regular and structured part of the practicum. The last code, being a learner, is related with her reflection on how she became an expert on offering the practicum course. "This is the first time I have taught a practicum course at this university, and no one has shown me how to teach it, I have learned my own lessons over time. I am building up experience." Her statements show that Yasmin's process of specializing in presenting a practicum course is based on her own learning experiences as an MTE. This shows that Yasmin is constantly seeking development as a teacher educator in overcoming the difficulties she encountered while preparing for the practicum course.

Designing preservice teachers' learning

This theme focuses on how the MTE plans and supports preservice teachers' learning. Findings indicate that feedback, particularly regarding content knowledge (CK) and PCK, fostering reflective thinking skills, and sharing knowledge about school, students, and mentors play a significant role. Yasmin emphasized the importance of preservice teacher in the first code of this theme, *Giving PST feedback on lesson plans regarding CK and PCK*, focusing on core concepts in mathematics in their lesson plans. For example, when she noticed that a PST was not sufficiently emphasizing the concept of function, she asked him to restructure your lesson by focusing on polynomial functions. This statement encouraged PSTs to recognize that polynomial functions are a fundamental concept in mathematics and to shape their lessons around such "big ideas," rather than focusing solely on solving problems during their teaching experience. As mentioned in the second code, *Aiming to foster reflective practitioners*, Yasmin aims to foster reflective teachers who question and develop their own practice. She supported this process by requiring PSTs to keep a journal during the practicum and told them that her goal is to enable them to become reflective practitioners. This practice helps candidates analyse their experiences and continuously improve their teaching processes. The third code of this theme*, Knowledge about preservice teachers (PSTs)*, reflects Yasmin's close observation of the different characteristics and performances of her students and her efforts to guide their development. In the interview, she explained that many PSTs had achieved the level she expected and described them as the best in the class. She also highlighted that some students performed beyond her expectations, noting that certain candidates, such as Denise and Melissa (pseudonyms), were a positive surprise. At the same time, she acknowledged the difficulties experienced by some students. The fourth code, *Knowledge about school cultures*, explain how Yasmin helps PST teachers better prepare for the profession by conveying the expectations of different school types. For example, she noted that some schools require detailed mathematical knowledge when hiring teachers. Similarly, she explained that different skills are expected of teachers in private schools. Using these examples, she highlighted the cultural and structural differences between public and private schools, while also emphasizing the schools' high expectations and the challenges of finding qualified teachers. In the fifth code, *Knowledge about mentors*, Yasmin expressed the differences in quality among mentors. She emphasized that a good mentor should challenge prospective teachers to think, saying, "Teacher Hilda encourages prospective teachers to think deeply." She explained that effective mentoring requires high school teachers to consider the mathematical and pedagogical perspectives of prospective teachers, and to align their guidance with PSTs' level of disciplinary understanding. These findings demonstrate that Yasmin views the mentoring process as a critical element in the development of prospective teachers.

Expertise and professional positioning of the MTE

This theme highlights the professional expertise and academic positioning of the mathematics teacher educator. The findings focus on the MTE's specialization in teacher education, her perspective of learning as a situated and in-action process, and the ways in which her researcher identity is reflected in the practicum. The first theme, *Being specialized in teacher education,* underscores Yasmin's perceptions of her responsibilities as a teacher educator as a core component of her professional identity, highlighting her commitment to nurturing future educators and her conviction in the

transformational potential of good teaching. As a teacher educator, Yasmin feels a deep sense of purpose in her job. Being an expert in teacher education isn't just a job for her; it's an important part of who she is. She mentioned that she did not learn how to become a teacher educator. Moreover, she expressed that her identity as a teacher educator has developed through what she has accumulated with her experiences. In the second code of this theme, *In-action learning and situated learning* she mentioned that she can see the in-action version of PCK in the practicum. The purpose of the practicum is in-action teacher learning. Yasmin described the practice lesson as a process where PCK can be observed, and teacher learning takes place in context. She supported this view by giving examples, such as explaining a concept through multiple representations or suggesting a simulation in action, emphasizing that different representations and methods can strengthen learning in practice. In the observation notes, it was also seen that Yasmin shared her epistemological view of making sense of knowledge in action with the PSTs in the classroom; during the lesson, she emphasized the aspect of learning that is shaped by context and action by saying, “Knowledge is static, but knowing is dynamic.” The last code in this theme was *Being a researcher and doing research.* When I asked Yasmin how her own research field contributed to the implementation of the practicum course, she stated that she understood the needs and potential of PSTs better thanks to research. "In other words, thanks to research, I can determine the stones of the bridge between the teacher they are now and the teacher they will be." In addition, in the interview, she stated that the definition of in-action comes from research and emphasized that her understanding of learning in action is also based on her research. On the one hand, she talks in detail about her ongoing research with mentors and the performance and expectations of the mentors throughout her project. These findings reveal that Yasmin's researcher identity plays an important role in shaping teacher education practices.

Conclusion and discussion

This study reveals how the multilayered role of the mathematics teacher educator (MTE) in the teaching practicum course influences both the learning processes of PSTs and how the MTE constructs her own professional identity. First, structuring the practicum as a course involves strategic planning that facilitates preservice teachers' transition from theory to practice. This finding aligns with literature emphasizing the importance of the institutional context and structured learning opportunities in teacher preparation processes (Zeichner, 2010; Grossman et al., 2009). Yasmin's multidimensional approach to the practicum process demonstrates that PSTs are positioned not merely as consumers of information but as learning agents. It has been observed that the mentoring process is decisive for the development of preservice teachers' professional identity, but that the quality of this process can vary significantly depending on differences among mentors (Hudson, 2013). Under the theme of supporting preservice teachers' learning, feedback processes structured around PCK allow preservice teachers to integrate CK with practical skills. Yasmin's emphasis on reflective thinking and critical planning supports candidates in developing deep pedagogical understanding, a goal of teacher education (Loughran, 2006). Furthermore, providing information about the school context and interacting with mentors increases candidates' awareness of professional reality, strengthening the theory-practice bridge (Darling-Hammond, 2014). Finally, Yasmin's professional positioning as an MTE indicates that teacher education is not merely a teaching activity but also a learning and research process. Despite the challenges encountered during the mastery

process, her approach based on action learning and in-situ development provides a powerful example of the continuity of professional learning. These findings demonstrate the importance of a professional model that integrates both practitioner and researcher identities of teacher educators (Cochran-Smith & Lytle, 1999).

Implications

These results indicate that practicum courses can better assist preservice teachers when they are deliberately structured to integrate mathematical knowledge, classroom application, and avenues for guided reflection. The research underscores the necessity for teacher educators to articulate their pedagogical reasoning through in-action modeling and systematic feedback. Additionally, enhancing collaboration between mentors and MTEs seems essential for synchronizing expectations and facilitating the professional development of PSTs. Lastly, these insights show how important it is to get MTEs ready to use reflective, practice-based methods that combine research, teaching skills, and knowledge of the situation. Future research could evaluate the consistency and impact of practicum experiences by comparatively examining how different teacher educators structure their practicum courses. Additionally, the impact of mentoring practices on pre-service teachers' learning should be analyzed more deeply.

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Developing pre-service teachers' teaching practices through classroom experiments

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This paper investigates how pre-service teachers' (PSTs), through engaging in classroom experiments that connect university coursework and their in-field teaching, can develop more student-centered teaching practices. Whereas coursework often emphasizes theoretical knowledge and internships foster routine practices, classroom experiments connect both by creating opportunities for deliberate practice. Yet, utilizing these opportunities for deliberate practice is mediated by the pedagogical discourses that the PSTs prioritize in their teaching, where disconnected discourses can result in tasks of teaching that are less informative to guide teaching practices and hence, provide less potential for practice development. This finding underscores the need for also considering the many discourses PSTs participate in when designing activities that aim to connect coursework and in-field teaching.

Keywords: Teaching practice, pre-service teacher education, theory-practice divide, design research.

Introduction

In many pre-service teacher education programs, there is a persisting divide between university coursework and in-field learning during internships at school. University coursework emphasizes learning of theory ("application of theory"-model, Korthagen & Kessels, 1999) to develop beginning teachers' knowledge about how to effectively master the demands of teaching mathematics (Professional Content Knowledge). This theoretical learning often remains disconnected from the classroom contexts, limiting its impact on pre-service teachers' (PSTs) teaching practices in class (Depaepe et al., 2013). In contrast, during their internships, PSTs engage in apprenticeship learning where they learn from school mentors' experiences and best practices. Apprenticeship learning often results in tacit teaching practices which are routine and cannot be reflected or modified using theoretical knowledge (e.g., Christiansen et al., 2023). This divide between theory and practice is exacerbated by institutional constraints of teacher education programs, such as time constraints that prevent frequent internship visits.

Approaches to address this divide often focus on developing noticing through authentic artifacts, rehearsing core teaching practices (e.g. Matsumoto-Royo et al., 2021), or connecting noticing to internship learning (e.g. Kilic, 2018). Yet, these approaches do not automatically translate into changes in PSTs' teaching practices (Barnhart et al., 2025), which raises the need to further investigate how to support PSTs to bridge the theory-practice divide in their in-field teaching practices. Accordingly, in the context of classroom experiments as proven approach to bridge theory and practice (Hiebert et al., 2003), I investigate the research question: *How do PTSs resolve the divide between coursework and in-field teaching when experimenting with more student-centered teaching practices?*

This question will be investigated through the eyes of Jonas and Elisa, who reflect on a teaching experiment they implemented during their in-field teaching, as part of their assignments during university coursework.

Implementing practice-based approaches in Dutch pre-service teacher education

This paper uses a sociocultural framework to investigate PSTs learning in terms of development of teaching practices. I understand teaching practices, in the most general sense, as the intentional activities of helping people attain specific learning goals (Ball & Forzani, 2009). In the context of classroom mathematics learning and teaching, teaching practices can be further specified through the *Tasks of Mathematics Teaching*, where these tasks describe the practical knowledge and decision-making process for eliciting specific practices (Ball et al., 2008; Shure, et al., 2025). In other words, tasks of teaching can specify the intentions behind teaching practices. Accordingly, to capture both, intentionality and the activities to address them, teaching practices are here captured as the pair <*Task of Teaching, activities to achieve Task*>, and written in italics.

To develop their teaching practices between in-field teaching and university coursework, PSTs need to negotiate different discourses, with different values and endorsed narratives about what counts as "good" teaching (Thompson, et al., 2013). PSTs move between several discourses (Sfard, 2008), such as the mathematical discourses they experience during their university studies, and the pedagogical discourses at university or at schools. Pedagogical discourses are here understood as the keywords, narratives, and visual mediators about how to learn and teach, including the different roles and responsibilities of teachers and students (Heyd-Metzuyanim, 2019). PSTs participate in this discourse, for instance, through their university coursework in educational sciences or mathematics education. Notably, mathematical and pedagogical discourses at university might be different from those at schools, with different narratives about what counts as successful learning. Often, school discourses are framed as "traditional" or "teacher-centered", while university courses are described as "ambitious" or "reform oriented" (Heyd-Metzuyanim, 2019). While this distinction might not be as clear cut in reality, theses framings still usefully allow to conceptualize the theory-practice divide: PSTs need to navigate between these many discourses between schools and university and align their teaching practices with the dominant narratives used to evaluate and understand them.

Together, the <*task, practice*> pair and discourses allow for characterizing the development of PST's teaching practices between university coursework and in-field teaching. During in-field teaching, a specific practice is enacted in a complex process guided by PSTs' intentions for creating learning (tasks of teaching), through a series of activities that are informed and valued by the mathematical and pedagogical discourses they relate to. During reflection of teaching practices, PSTs mobilize the different discourses they are part of to represent their teaching practices and may collaboratively develop these discourses further to create narratives about alternative practices.

Context: A design research study focused on developing teaching practices

This paper is located in the context of a design research study for designing mathematics education coursework for developing PSTs' teaching practices through deliberate practice, in the Dutch context of an academic teacher education master. In the Dutch context, university coursework and internship

teaching typically take place in parallel, providing optimal conditions for linking university coursework and in-field teaching. Deliberate practice is here understood as structured, goal-directed, and effortful activities 'explicitly designed to improve performance' with teaching practices (Ericsson, 2002, p. 5). Deliberate practice was facilitated through activities such as re-inventing practice(s) for scaffolding student thinking through video-analysis, scripting how to enact practices (Campbell & Baldinger, 2022) such as launching a problem task, and classroom experiments for engaging PSTs in teaching practices for eliciting and scaffolding student mathematical thinking. All these activities also aimed to establish a pedagogical discourse of ambitious mathematics teaching, that values promoting student mathematical thinking.

Here, we report on PSTs reflection of their classroom experiments (Hiebert et al., 2003). Classroom experiments take place two times per Semester (4 in total for a PST cohort) and consist of the following learning activities:

- Design of a cognitively demanding task to engage students in mathematical thinking and planning the lesson around this task
- scripting the teaching practices for how to launch the task in class, and peer reflection on these scripted episodes;
- implementing the lesson with live- or video-observation;
- post-lesson reflection in an online meeting at the same day of the lesson, together with a didactical facilitator, guided by the ALACT-model (Korthagen & Kessels, 1999);
- follow-up reflection during coursework to identify crucial didactical principles.

During the post-lesson reflection, the PSTs were pushed to reflect on critical moments where their lesson plan did not match the teaching practices enacted in the experimental lesson. In these moments, the facilitator pushed the students to identify why these practices did not match the planning, and to devise alternative teaching practices that are better suited to reach the intended goals. While it was found that this reflection supports the consolidation of didactical theory from the accompanying coursework, it is less clear how such reflections support the development of PSTs' teaching practices, including their underlying intentions (the tasks they perceive and aim to solve) and the discourses they mobilize for uncovering the essential aspects of teaching practices.

Methods: Qualitative learning process study

This multiple case study on PSTs classroom experimentation took place during a mathematics education course at a Dutch university for educating PSTs for teaching upper-secondary mathematics classrooms (Master level, 8 bi-weekly sessions of 4x45min. duration). Parallel to their university coursework, the PSTs follow an internship for 3 to 4 days a week. During the implementation of a teaching experiment, other group members and the facilitator observed the lesson through a videorecording and, if possible, in person. During the research period, there are eight PSTs enrolled in the mathematics education course investigated here. Four of these PSTs had their internship in parallel, resulting in two groups of four teachers collaborating during classroom experiments. Accordingly, in each group, there are two PSTs with internships, allowing easy access to the field.

The four post-lesson reflections of the four classroom experiments (45min duration) were video recorded via MS teams, transcribed, translated to Dutch, if necessary, pseudonymized, and supplemented with PSTs written observations of the experiment. Additionally, two implementing PSTs were interviewed before and after they implemented an experimental lesson, among them Jonas. Ethically, all PSTs participated voluntarily on the legal basis of informed consent. Classroom episodes that show the classroom or school learners have not been used as research data.

The data was analyzed qualitatively, employing thematic content analysis (Miles et al., 2020) and discursive analysis. The analysis focuses on coding the (implicit) tasks of teachings that guide PSTs teaching practices during experimentation, with deductive categories for tasks related to promoting student thinking, and inductive coding for other tasks reflected by the PSTs. Furthermore, through a subsequent discursive analysis of the PSTs' reflections of the tasks coded, we characterize the mathematical and pedagogical discourses through which PSTs reflect their teaching practices. Here, we report in-depth on the discursive analysis for the group of Jonas and Elisa, where Jonas and Elisa were chosen as they show the range of possible trajectories for developing teaching practices through classroom experiments.

Results

The following episode takes place during the second round of experimentations, where Jonas has implemented an experimental lesson. This lesson was intended to introduce the difference quotient with a graphical representation of Jonas weight during one year of dieting (fictive), as designed previously during preparatory coursework.

Jonas (T)	You're. You're right. Yeah. I should have taken there longer. I received a response. And it was a it was a right response. But there could have been more different answers. I mean they could have noticed that toppen [peaks] stijgen en dalen, increase, decrease, all that. And immediately after I received one response I. I continued and I should have allowed for more time. […] I wanted them to think about what would happen when you change the size of the interval without seeing the graph. So first I asked the question and only after I received a response. Did I show the graph. Uh to show them that if you change the interval, not only do you get a different graph, but you may get a different interval as well, or a different combined interval. So, in the beginning it was 0-10. And with different delta T It became 0-9. And that was also in one of the exercises. But this was more tangible for them. So, they could, uh, I mean it, it was a real a real diet. Sort of. […, 1 turns left out]
Jonas (T)	So I got a good response and then I and then I continued so that that was the, yeah, the mathematical thinking. And then of course the exercises that was quite some different types of exercise. To be frank, I find this chapter boring, unnecessary, and so do all teachers in the in our school they'd rather skip it, but it was decided that we all had to go through this chapter. But. And I don't think that's the students sensed that I find it an unnecessary chapter, but I sensed that some students find it the boring chapter. […][2 turns left out]
Elisa	[…] But when I did that lesson, I was looking at why are we doing this and it was a transition from and in function you have values but now you're not going to look at the exact values, but you're going to look at the difference between values and a diet is and I think Jonas, you didn't start with it, but you mentioned it about 10 minutes after that. I don't want to know how much I weigh. I want to know how much weight I lost. So, if you talk about the chapter, I think it's called the differentiaalrekenen it's about differences. So, it's not about data collection. It's

> about differences in data. And if you say you have a graph here and we want to look at the differences and then you ask, what do you see here, then you can you say there is a huge difference. There is a small difference. There is an increase, there is a decrease but the word difference that could trigger maybe some more thinking. [Jonas confirms]

From a discursive standpoint, this episode illustrates the different discourses that Jonas and Elisa are engaged in during their reflections. Jonas utilizes keywords such as "right response", "to show them", which indicate a more traditional, teacher-centered pedagogical discourse. He frames the episode in terms of narratives of 'receiving a correct response and continuing with the next step in the lesson'. Later, he more generally refers to the "exercises", and whether students sensed that the chapter covered in the lesson was "unnecessary." In his mathematical discourse, he initially emphasizes features of the graph (indicated by keywords "toppen", "stijgend", "dalend") and later the "interval" and "Delta T". Notably, he does not mention the relevance of investigating differences. At the same time, he regards the learning content as "boring", which he tried to alleviate by making the content "tangible" through the dieting context, as originally planned during coursework. These keywords and narratives are consistent with his goal to connect mathematics to the students' everyday experiences – a highly relevant task of teaching that he emphasizes frequently, also in a follow-up interview.

In terms of the tasks of teaching that guide his teaching, Jonas unfolds how he perceived the task *launching the lesson*. Accordingly, he is concerned with the practice <*launching the lesson, probing students' prior knowledge*>. The task is immediately achieved by one student providing a correct answer. Possibly, receiving a correct answer also realizes his task of connecting the mathematics to students' everyday experiences. He then proceeds to <show change of interval, explaining relevance of interval size>. While the situation would call for <*linking representation and key idea*, …> to utilize the graphical representation of the diet, he instead moves towards a more traditional pedagogical discourse and emphasizes "showing" the mathematics and subsequently doing "exercises". Notably, for him, this task <*show them change of interval, ...*> is not a mathematical task per se, as its closing condition is the performance of its associated practice itself, but not an observable student mathematical insight. Hence, by moving between pedagogical and mathematical discourses sequentially, but not connecting them, his tasks of teaching remain general, and only loosely give direction to his teaching practices.

Elisa enters this reflection from a mathematical discourse, emphasizing the "transition" from "exact values [of functions]" towards "difference between values". Later she emphasizes that the chapter Jonas refers to is not about features of functions, but about "differentiaalrekenen [differential calculus]" and "differences", through which she places this lesson in the larger context of the teaching unit. In her pedagogical discourse, she emphasizes this central key idea of the lesson, taking the perspective of the student ("I [as student] want to know"). On that basis, she adapts the task *launching the lesson* into *eliciting students' ideas about differences* with the help of the graph. Accordingly, she is concerned with the practice <*eliciting ideas about difference quotient, prompting ideas about differences with graph*>. Thus, through connecting pedagogical and mathematical discourses, she can locate the task of teaching on the layer of sequencing students' learning, which in turn enables her to consider more targeted teaching practices with goals formulated in terms of mathematical ideas.

Contribution and discussion

Even under ideal conditions, where Elisa and Jonas developed their noticing of student mathematical thinking through video-analysis, collaborated in a group of four PSTs to design an experimental lesson, and where both of them trialed this lesson in their in-field teaching, their teaching practices develop quite differently. By participating in different discourses, Elisa and Jonas perceive and prioritize different tasks of teaching. This difference in perceived tasks of teaching is highly impactful in what teaching practices Elisa and Jonas regard as suitable for responding to the classroom episode observed: as Jonas did not perceive tasks of teaching on the level of students' learning processes, he did not enact practices for *prompting student thinking*, instead proceeding with a traditional teaching practice of *explaining the learning content*. Possibly, his traditional discourse prevents him to specify tasks on the level of students' mathematical learning process, in turn preventing him to develop his teaching practices – perhaps even in the long term if he continues to participate in this discourse.

Originally, the group designed the dieting context as means to solve the task of *eliciting mathematical ideas* about the difference quotient, which is rooted in pedagogical discourses of ambitious/student-centered teaching. However, the group sees that Joans is deviating from this task in his lesson, as illustrated above. Interestingly, in Jonas own perspective, he might align with these tasks of ambitious teaching: By prioritizing the task of *creating relevance through connecting to everyday experiences*, he frames his teaching practices with a pedagogical discourse that is compatible with both, the school discourse of his internship school that emphasizes traditional teaching, while also being compatible with the university discourse that emphasizes ambitious, student-centered teaching (in which the other group members seem to be in). Accordingly, in his view, Jonas aligns his teaching practices with the goals of the classroom experiment of facilitating students' meaning making, while at the same time engaging in more traditional teaching practices of his school, while experiencing little conflict between both.

Generally, the discursive perspective reveals that teaching practices, and the tasks that guide them, are mediated by the way PSTs navigate the several pedagogical discourses they encounter between school, teacher colleagues, university coursework and peers. These findings explain recent research results showing that connecting university coursework and in-field teaching does not automatically translate into changes in PSTs teaching practices (Barnhart et al., 2025, Kilic, 2018): When navigating the various pedagogical discourses PSTs encounter between school and university, they may give priority to school pedagogical discourses, while only "pretending" to engage in ambitious discourses (Thompson et al., 2013). Alternatively, PSTs might, as Jonas' case illustrates, adopt tasks that are so global that they fit both, traditional and ambitious pedagogical discourses, but that are consequently not informative for guiding teaching practices. Hence, more research is needed to understand how PSTs negotiate, for themselves, these many discourses and how their positionings in these discourses relates to their development of student-centered teaching practices.

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Imaginations of practice: Toward a generative pedagogy for mathematics teacher learning

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This paper proposes and illustrates imaginations of practice as a generative pedagogy for in-service teacher learning, building on current scholarship on practice-based mathematics teacher education. We approach this work with the ethical conviction that supporting experienced teachers means also building on their professional expertise to push beyond what is tried and proven. We explore how imaginations of practice can support researchers, teacher educators, and teachers in creating and exploring new possibilities, while also deepening their understanding of tacit assumptions and forces underlying current pedagogies. Drawing on research from three in-service contexts, we examine how imaginative engagement with practice can disrupt habitual patterns, surface underlying values, and facilitate the co-construction of alternative pedagogical futures. We argue that imaginations of practice offer a powerful and underutilized resource for in-service teacher education and research.

Keywords: Pedagogies of practice, approximations of practice, imaginations of practice, simulations, boundary crossing.

Problematizing the role of relevance in mathematics teacher education

As mathematics educators and researchers, we are well-familiar with the story of Euclid's parallel postulate. For centuries, it occupied the efforts of many mathematicians, until in the 19th century Gauss, Lobachevsky, and Bolyai made a radical move, asking, *what if the parallel postulate is not true?* This question gave birth to non-Euclidean geometries and enriched the field in profound ways.

While our goals are far less ambitious, this ethos guides our thinking. In this conceptual paper, we revisit and reconsider one of the postulates of current mathematics teacher education, and of andragogy more generally: the postulate of *relevance for practice* (e.g., Korthagen, 2011). While acknowledging that the questions of relevance for practice cannot and should not be fully put aside in teacher education, we argue for a "both/and" approach that invites teacher inquiry of *not-yet-relevant* practices that interact with, complement and enrich relevance-driven teacher inquiry. We explore how imagination can serve as a generative pedagogy for in-service teacher education, attending to the diverse experiences and goals of practicing teachers. Three illustrative cases are used to exemplify this potential.

Although questioning relevance may appear conflicting with the conference's agenda of bringing the learning of practice to the forefront in teacher education, we consider the joint probing of what is often taken for granted, namely, the postulate of relevancy, essential for productive exchanges. In particular, we consider such probing as addressing the concern, also articulated in the call for proposals, that decades of efforts have done little to disrupt conventional mathematics teaching. This leads us to ask whether a focus on classroom teaching approaches that are readily applicable, while important, may not by itself suffice to facilitate a profound change. Moreover, in the case of in-service teachers, we wonder whether restricting activities to those that closely mirror daily routines risks

becoming repetitive, reproducing and reinforcing existing patterns, and eventually limiting teacher learning. Alongside designing for relevance, could it also be valuable to cultivate professional education activities that appear less directly tied to practice, yet open imaginative space for teachers to envision alternative ways of teaching and engaging with mathematics? What would such activities look like, and how might we trace their longer-term contributions to teachers' practice?

By raising these questions, we do not dismiss the importance of relevance but rather call for its elaboration and refinement. Moreover, we situate relevance for practice as *one* possibility rather than the *only* possibility for teacher educators, and argue that imagination can serve as an additional pedagogy of practice, broadening what is considered useful and desirable. Importantly, our ideas stem from the ethical conviction that supporting experienced teachers means not only exposing them to existing materials and approaches but also building on their professional expertise to push beyond what is tried and proven. We propose imagination as a generative pedagogy for teacher education that invites experiential forms of professional learning that are not confined by prior experiences but instead invite participants to explore possibilities and envision what mathematics teaching could become. Theoretically, this orientation challenges us to consider how imagination might expand the collective space of precedents teachers draw upon, and calls into question how the field currently conceives the ontology and epistemology of the practice of teaching.

Below, we first outline our theoretical framing of *imagination of practice*, situating it in relation to existing work on practice-based mathematics teacher education and highlighting why both *relevant* and *not-yet-relevant* orientations deserve attention. We then present examples from our design research with in-service teachers: creative writing workshops, simulation-based scenarios, and boundary encounters with mathematicians. We show in each context how engaging with practice in ways that were not immediately applicable created opportunities for reflection, reorientation, and consideration of new forms of action. We conclude by asking how the field might recognize the longer-term contributions of not-yet-relevant engagements alongside relevance-oriented ones, and by arguing that holding the two in contrast can expand what counts as practice and support more substantive transformations in mathematics teacher education.

The (implicit) epistemology and ontology of pedagogies of practice

Over the past three decades, practice-based teacher education has become a central orientation in the field, shaping program design and professional discourse well beyond its U.S. origins. Rooted in Dewey's emphasis on experience and further developed by scholars such as Shulman, Lampert, Ball, and Grossman, this movement redirected attention from knowledge about teaching to the ability to enact it. Foundational contributions, such as Ball and Forzani's (2009) argument for centering practice and Grossman et al.'s (2009) framework of representation, decomposition, and approximation of practice, supported the development and broad use of artifacts of practice and activities around them. While originally framed in pre-service contexts, these pedagogies have also been widely adopted in professional development for in-service teachers.

Pedagogies of practice (Grossman et al., 2009) carry with them certain epistemological and ontological premises, though these were not always explicitly acknowledged or problematized in the literature. Epistemologically, they privilege knowledge that is immediately relevant to classroom

action, namely, what teachers need to know and do to be competent. Ontologically, they position practice as a relatively stable entity that preexists and can therefore be represented, decomposed, and approximated. The word "approximation" especially carries this message, framed as imitating or rehearsing what already exists, echoing Zeichner's warning (2012) that "one danger we have to watch out for is the use of artifacts of teaching as scripts that undermine teachers' abilities to exercise their judgment and to adapt instruction to meet the constantly changing needs of their students and the different contexts of their work" (p. 379). Similarly, Philip (2019) argued for emphasizing the improvisation of the relational aspects of teaching, rather than the approximation and decomposition of preexisting teaching components.

Scholarship on imaginary or designed representations in mathematics teacher education suggests an ontology of practice that is less predetermined. Designed representations such as storyboarded scenarios, simulations, or script writing (Crespo et al., 2011; Friesen & Knox, 2022; Herbst et al., 2016) have shown how hypothetical situations can serve as powerful resources for teacher learning. Not being bound by what occurred in real classrooms, they allow teacher educators to bring selected aspects of practice into focus. Crespo et al. (2011), for example, argue that such artifacts should not be viewed as mirrors of classroom events but as rhetorical tools to advance specific pedagogical goals. Whereas video, grounded in real-time artifacts, often leaves intact the premise of practice as preexisting, designed representations open possibilities for practices not yet realized. Still, in actual use, imagined scenarios often function to prepare teachers for expected routines, keeping learning tightly aligned with prevailing realities. Our work extends this line of research by conceptualizing imagination not only as a rhetorical or representational device but as a pedagogy of practice in its own right. In this framing, imaginative activities are not merely supports for reasoning about practice but spaces in which new practices and settings can be actively configured.

Imaginations of practice as a generative fourth pedagogy of practice

Imagination of practice designates an orientation in mathematics teacher education that treats practice not only as already existing to be represented and learned, but also as emergent and not-yet-feasible. It invites the design of professional learning activities that suspend constraints, privilege possibility over feasibility, and open space for playful and creative engagements. Such activities move the focus beyond immediate relevance to open-ended exploration; they disrupt disciplinary boundaries; and they redirect attention from correct solutions toward emergent affordances, surfacing teachers' tacit knowledge and values.

The term *imagination* has been taken up in the literature with different ontological emphases: imagining *within* what is already unfolding (e.g., imagining joining a certain community of practice; Wenger, 1998), imagining *toward* what is likely to happen (e.g., Crespo et al., 2011), and imagining *beyond* existing constraints (e.g., Helliwell & Ng, 2022; Skovsmose et al., 2023;). This range of uses resonates with Wenger's (1998) typology of imagination as a key mode of belonging in communities of practice, which highlights three interrelated dimensions: the capacity to create images that connect local practice to broader worlds, the ability to take on new perspectives by seeing through others' eyes, and the generative work of envisioning possibilities beyond the immediate situation.

Building on the latter dimension, we propose *imagination of practice* as a fourth pedagogy of practice, alongside representation, decomposition, and approximation. This orientation treats practice not only as something already existing to be represented and rehearsed, and more broadly – learned, but also as emergent, possible, and not-yet-feasible – as inviting new ideas and generativity. In what follows, we illustrate this orientation by examining three in-service contexts.

Lenses for examining in-service contexts

Because this paper focuses on in-service mathematics teachers, we engage the conference's three focal questions from a somewhat different orientation. Although the original framing of the conference was posed broadly, it appears to address pre-service contexts in particular. Our concern, by contrast, is with how these questions unfold within the ongoing professional learning of practicing mathematics teachers. Accordingly, we reformulate the questions as follows: (1) What foundational issues do in-service mathematics teachers may continue developing as they refine their teaching?; (2) What is involved in learning mathematics teaching as in-service teachers?; (3) How can we design and sustain professional education that supports in-service teachers in the ongoing refinement of practice? In the next section, we present three examples, focusing on the third question and the role of imagination in design. In the conclusions section, we return to the other two guiding questions.

Illustrative examples

Example I: Imagining through creative writing

In response to the ongoing war in our region and its implication on education, the first author initiated a project (with two colleagues, see Alon et al., 2025) focusing on mathematics teacher resilience and responsiveness during crises. This initiative gathers Jewish and Arab teachers in leadership groups to foster cross-cultural dialogs about teaching mathematics in uncertain times and to rethink its relevance to students' lives. These workshops integrate creative writing and mindfulness to help teachers cope with their realities and to offer a hopeful space for collectively imagining what mathematics lessons might look like in utopian futures (see Helliwell & Ng, 2022). So far, we have conducted three cohorts with more than 40 teachers from diverse communities in our country.

The project is grounded in *pedagogical imagination* (Skovsmose et al., 2023), an approach to educational activism that encourages innovation in curriculum and classroom dynamics by distinguishing between what cannot be changed and what contingencies in classrooms can be rethought. As Skovsmose et al. (2023) put it: "the mathematics curriculum and the classroom routines can be recognised as being not necessities but contingencies. Pedagogical imagination is an expression of *hope*. Hope can concern social and political development, but simultaneously, it may concern a different life in the classroom" (p. 14). Inspired by Helliwell and Ng's use of this framework with creative writing tasks (2022), we recruited expert writing and mindfulness facilitators to work with teachers on re-imagining their practice. Teachers engaged in several forms of creative writing, such as continuous writing, composing letters to mathematical objects, and writing utopian scenarios. These tasks deliberately invited teachers to temporarily "forget" classroom constraints such as curricula, testing, or discipline, and instead focus on imagining mathematical encounters and experiences that are more joyful, hopeful, and relevant for students' lives. Although the facilitators

did not ask teachers to make explicit connections between the workshop and their practice, teachers persistently drew such links. One teacher reflected that attempting creative writing, something unfamiliar and not part of her professional routine, was akin to a student grappling with a new mathematical topic: both intimidating and full of discovery, a process of "reading, thinking, applying immediately, and experimenting along the way". Another teacher noted that, just as poets or novelists seek to reach their readers, teachers must find ways to reach students and make learning meaningful and engaging. Although the workshop tasks were designed primarily as imaginative explorations, teachers were invited to experiment with similar activities in their classrooms. In doing so, they sought to create spaces for students to be more generative, noting that "suddenly, through creative writing, I realize that in mathematics [the students] almost never invent things, never create." Their classroom implementations revealed both flexibility and professional judgment: rather than transferring the tasks directly, they adapted and reinterpreted them to suit their own contexts. This illustrates how imagination was proven to be a powerful pedagogical tool: through narrative and storytelling teachers opened space to re-envision their practice and carry new possibilities into their classrooms, and themselves forged connections between the workshop tasks and their practice – that is, their expertise allowed them to think about relevancy without any prompting from the facilitators.

Revisiting the "relevancy postulate", we suggest that designing a professional learning context that is not immediately relevant does not mean it will remain irrelevant. Expert teachers actively connect what they encounter and their own practice, generating relevance through their engagement.

Example II: Imagining through simulating

Teaching simulations have become a common means of embodying the ideas of practice-based teacher education, offering opportunities to approximate, analyze, and refine teaching practice in designed settings. Typically, simulations aim to emulate classroom interactions in order to rehearse specific instructional moves. Building on the work of Herbst and colleagues on digital teaching simulations (e.g., Herbst et al., 2016; 2022; Schwarts et al., 2025), recent designs developed at the Weizmann Institute of Science extend this line of research in new directions.

Compared to the creative writing tasks, the simulations are much closer to the realities of classroom teaching: they present teachers with authentic student thinking and decision points that resemble those encountered in daily practice. Because of this, at first glance our design may seem similar to more traditional forms of simulation. Yet the difference is ontological. Rather than treating practice as a fixed set of routines to be approximated, our simulations position practice as open-ended and contingent. The use of AI enables this openness, allowing the simulated environment to respond dynamically to teachers' choices and to sustain multiple possible trajectories of practice.

One example is a simulation based on the slope triangle task, in which teachers were presented with multiple student-generated ways of showing why the slope of a line is constant. The simulation highlighted contrasting strategies side by side, such as comparing ratios numerically, using geometric similarity, or relying on coordinate algebra. This juxtaposition momentarily loosened classroom constraints and encouraged teachers to imagine alternative instructional moves—for instance, what it might look like to foreground a visual argument before an algebraic one, or how students' informal

reasoning about ratios could be developed into more formal proofs. In this way, the simulation invited creative exploration before considering how such possibilities might reconnect to real teaching.

In this framing, simulations are not only approximations of classrooms but also infrastructures for the imagination of practice. By suspending constraints, inviting experimentation, and then reconnecting to feasibility, simulations enable teachers to stretch their sense of what teaching could involve.

Example III: Imagining through boundary encounters

The disconnect between mathematics as taught in secondary schools and as practiced at universities, widely known as *Klein's double discontinuity*, is well documented yet remains a persistent challenge. Evidence accumulated worldwide has refuted the enduring assumption that teachers' engagement with academic mathematics (AM) naturally "trickles down" to enrich school practice, independent of explicit connections to classroom teaching (Wasserman et al., 2023). A key challenge is that connecting teachers' learning of AM to their practice requires expertise in both AM and secondary mathematics (SM) teaching—a rare combination. One proposed solution is to draw on the combined expertise of mathematicians and in-service teachers, though their communication is often marked by deficit narratives, mutual skepticism, and even hostility (Pinto & Koichu, 2023).

The M-Cubed project (Mathematicians, Mathematics teachers, Mathematics) was established in 2019 at the Weizmann Institute of Science as a response to this challenge. Its goal is to study and promote collaboration and mutual learning between mathematicians and SM teachers. In M-Cubed, small groups of mathematicians and in-service SM teachers watch excerpts from videotaped SM lessons and jointly explore mathematical and pedagogical issues. Discussions often focus on teaching decisions that breach mathematicians' expectations. These perspectives challenge teachers' assumptions, prompting reflection on tacit aspects of their own practice. Through facilitated dialogue, participants weigh alternative choices, exposing tensions between AM and SM teaching. The research team supports these encounters through *mindful brokering* that invites both groups to articulate their perspectives and reconsider not only what occurs in classrooms but also what could occur in principle.

A central finding of M-Cubed is that relaxing the demand for immediate classroom applicability and embracing imagination of practice is crucial for learning (Pinto & Cooper, 2023). Faced with authentic teaching dilemmas, mathematicians and teachers often discover that no consensual, realistic instructional move can bridge their perspectives. In such moments, letting go of "what is" in favor of "what if" enables both sides to propose not-yet-realistic solutions as shared problem spaces for exploration. These may include adding new mathematical topics to the curriculum, changing the matriculation exam, or designing resources that do not yet exist.

This case exemplifies imagination of not-yet-realistic practice as an instance of Pólya's (1957) problem-solving heuristic: "If you cannot solve a proposed problem […] try to solve first some related problem; then you may find courage to attack your original problem again" (p. 114). If directly connecting AM to SM teaching proves too difficult, loosening some constraints of current educational realities can yield analogical yet more tractable problems that support learning. In this sense, imagination functions as a generative pedagogy that shifts interactions between SM teachers and mathematicians from tension and mistrust toward more collaborative and constructive exchange.

Conclusions

This paper argues for the recognition of imagination as a fourth pedagogy of practice, using three examples. We do not claim that our projects are the first to enact this approach; rather, our aim is to name it, to problematize its ontological underpinnings, and to situate existing work within this frame. After outlining the projects' design, we revisit the other two focal questions: what foundational issues in-service teachers continue to develop, and what is involved in that learning. First, our examples show that imagination-driven activities engage practicing teachers in foundational issues such as responsiveness, resilience, the negotiation of disciplinary boundaries, and the capacity to connect mathematics with students' lives. Second, we illustrated what imaginations of practice might involve: cultivation of agency, deepening of professional vision, creative envisioning of new pedagogical possibilities, and opportunities to reframe challenging encounters as constructive exchanges.

We acknowledge that imagination of practice is most applicable to in-service, and particularly experienced, mathematics teachers. For pre-service teachers, approximation and rehearsal may be an essential step before improvisation and imagination can be meaningfully taken up (though this sequencing has also been critiqued, see Philip, 2019). For in-service teachers, however, professional learning might use imaginative activities to enrich open-ended teacher inquiry. This paper provides an opportunity to consider how pedagogies of practice may be differentiated across career stages.

Looking ahead, we aim to better account for the longer-term contributions of not-yet-relevant engagements and to specify more precisely the role of imagination in different activities. We argue that professional learning should not be judged only by immediate applicability or resemblance to existing practice. This is another lesson we can take from mathematics: What seems irrelevant at first can become (through teachers' expertise and agency) a vital resource (for practice).

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Teaching simulations as formative assessments of teaching practice

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We focus on formative assessment within teacher preparation and look at the use of teaching simulation cycles (structured as formative assessments) to impact prospective teachers' subsequent practice in a teaching simulation. The teaching simulations focus on the teaching practices of eliciting and interpreting student thinking. Twenty prospective teachers participated in two back-to-back teaching simulation cycles with feedback in the middle. We analysed how teaching practice shifted across the two teaching simulations with a focus on eliciting student process and eliciting student understanding. Findings reveal shifts in teaching practice for 70% of the prospective teachers. Implications are discussed.

Keywords: Preservice teacher education, simulation, instructional strategies, assessment, mathematics education.

Introduction

Formative assessment of teaching practice is a critical component of teacher preparation (Grossman et al., 2009). It requires a setting that allows prospective teachers (PTs) to demonstrate the integration of knowledge and skills in teaching, thereby providing a teacher educator (TE) with a snapshot of the PT's current ability to use their knowledge and skill. The TE can then provide feedback and additional learning opportunities, in which a PT can show growth. Wiliam posits that the utility of a formative assessment is measured by the extent to which the formative assessment allows the demonstration of knowledge and skill and the extent to which the assessment supports all those involved in making decisions about subsequent work (Wiliam, 2010). The ultimate goal of formative assessment in teacher preparation is to elicit information about PTs' practice and to provide feedback and subsequent learning opportunities that impact their future enactment of practice.

We use simulations to provide interactive contexts for practicing and demonstrating skill with mathematics teaching within formative assessment cycles that offer opportunities for feedback and improvement. A simulation is an approximation of practice (Grossman et al., 2009) that places authentic demands on a PT while purposefully controlling the complexity of the work to allow for appropriate cognitive demand for the PT. Simulations are increasingly used to approximate teaching situations (Mikeska et al., 2021) and are used in other professional fields (Jeffries, 2005). Simulations allow the TE to control the mathematical content the PT encounters, the mathematical strategies and thinking represented by students, and the teaching practices necessary to accomplish the goals of the simulation. One way to use simulations is through formative cycles, which enable TEs to provide feedback to PTs that can be taken up in subsequent simulations. In this paper, we explore how PTs' teaching practice (as revealed through teaching simulations) shifts across two consecutive teaching simulation cycles.

Our simulations focus on eliciting and interpreting student thinking, teaching practices that are essential to the daily work of teaching (National Council of Teachers of Mathematics [NCTM] 2014). Teachers must gather and carefully consider information about what students know and understand to design instruction that is responsive to individual students and builds on students' strengths (Carpenter, 1996; Jacobs et al., 2010). A teacher's eliciting, particularly around alternative methods and incorrect answers, can have significant implications for students' learning. When teachers ask questions to learn about students' mathematical thinking, these questions can support teaching that affirms students' sense-making and advances their mathematical reasoning (Aguirre et al., 2013). In contrast, limited eliciting of student thinking can reinforce deficit-oriented interpretations and result in teachers pursuing less rigorous learning goals (Battey & Franke, 2015). We see eliciting student thinking as a precursor to responding to or extending a student's thinking instructionally. Thus, eliciting and interpreting student thinking is an important focus in teacher preparation.

In the simulations we design, a PT interacts with a "standardized student" (a TE taking on the role of a student using a well-defined set of decision rules for responding) around a specific piece of written work (see Shaughnessy & Boerst, 2018 for a more detailed description of how we design the simulation). Our live actor teaching simulations involve two TEs. One TE takes on the role of the standardized student (referred to as the student or standardized student). The second TE (referred to as the TE) facilitates by appraising the performance and providing formative feedback.

As shown in Figure 1, the PT's engagement in a teaching simulation cycle has four parts. First, the PT examines a student work sample and plans questions to ask the student to learn about their process and understanding of the mathematics. Second, the PT has five minutes to interact with the simulated student to elicit the student's thinking. We train the "simulated student" to take on a particular student's way of reasoning about and understanding the mathematics task. These response guidelines include both general guidance and specific responses to anticipated questions. As the interaction unfolds, the TE uses an observational tool to capture key aspects of the interaction. Third, the TE interviews the PT to learn about their interpretations of the student's thinking. Fourth, the TE engages the PT in a feedback conversation about their performance to support the PT's learning.

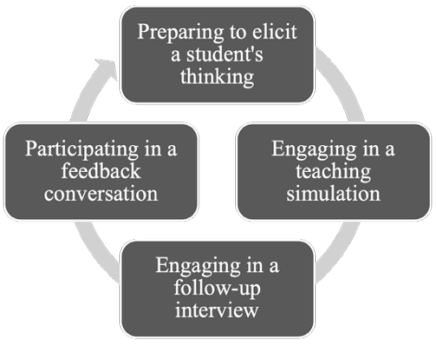

Figure 1: Teaching Simulation Cycle from a PT's Perspective (Adapted from Shaughnessy et al., 2025)

Our approach allows us to gather evidence of (and provide feedback on) a PT's knowledge and skills in eight performance areas (see Shaughnessy et al, 2024). In this analysis, we focus on 1) eliciting a student's process and 2) probing a student's understanding. The TE uses the provided observational

tool for the simulation and interview. Information from the observational tool is aligned with each of the performance areas and tied to feedback suggestions that the teacher can use.

We explore whether and how PTs' teaching practice (as revealed through the teaching simulations) shifted across two teaching simulation cycles (with similar mathematical content) to understand whether the teaching simulation cycle is functioning as a formative assessment. More specifically, we ask: How do PTs engage with eliciting the student's thinking in the first teaching simulation cycle, and does it shift (and if so, how) in the second cycle? We focus on their eliciting of the student's process and their probing of the student's understanding of the process and core mathematical ideas.

Methods

Our data comes from ongoing work with a TE in the USA. In this context, we engaged PTs in a pair of simulations focused on fraction comparison. In the first simulation, the student used common denominators to solve a fraction comparison problem. In the second simulation, the student used common numerators to solve a different fraction comparison problem. Figure 2 contains the student work samples for each student alongside the core ideas related to the process and the student's understanding. Importantly, not all aspects of the student's process are evident in the student's work, nor is the student's understanding of the process and mathematical ideas evident in the written work. Thus, it is necessary to elicit each student's thinking to learn about their processes and understandings. The TE for the course facilitated the first simulation cycle (see Figure 1), including the feedback conversation. After the first simulation cycle, the PT had 5 minutes to consider the feedback. They then engaged in the second simulation cycle facilitated by a research team member. Both simulated students were members of the research team and received training for their role.

Most PTs were in the first semester of their fourth year of undergraduate study. All PTs enrolled in the course completed the simulations, and we analyzed data from 20 PTs who consented to have their data analyzed for research purposes. The teaching simulation cycles were video recorded.

While the TE appraised the performances in the moment to support the provision of formative feedback, in this paper, we use the analyses the research team conducted using the collected video records. We first applied a detailed observational tool to each of the performances (see Shaughnessy et al., 2025 for a more detailed description of this tool). This tool focuses on specific moves made by the PT (e.g., the PT posed a question that elicited the student's process for generating an equivalent fraction for 3/7). A web-based app connects the observation checklist to rubrics that characterize the proficiency of the performance on all performance areas, including the two performance areas of interest in this study. The rubrics produce scores that range from 1 to 4 for each of the performance areas (see Table 1) with increased scoring representing increasing proficiency.

Results

In this paper, we report on PTs' proficiency with eliciting student thinking in the first simulation cycle and the shifts that were seen in performances in the second simulation cycle. We focus specifically on (a) eliciting the student's process, and (b) probing the student's understanding. We first note that 14 of the 20 PTs showed growth based on their rubric scores in one of these two areas.

Simulation Cycle 1: Common Denominator

Which fraction is greater: $\frac{3}{4}$ or $\frac{4}{6}$

$\frac{3}{4} = \frac{5}{6}$

$\frac{3}{4} > \frac{4}{6}$

Information about the student's process: The student is using "common denominators" to compare the fractions. The core ideas include: (1) the student decides to make fractions with common denominators. (2) to get fractions with common denominators, the student adds 2 to the 3 and 4 to get 5/6, (3) the student compares the numerators, and (4) the student concludes 5/6 is greater than 4/6 (not written).

Information about the student's understanding: The core ideas include: (1): the student understands that two fractions are equivalent fraction when you can add the same number to the numerator and denominator of one to get the other fraction, (2) the student understands that doing the same thing to the numerator and denominator results in an equivalent fraction, (3) the student understands that making the denominators the same means that the parts are the same size and then you consider how many parts you have to decide which fraction is greater.

Simulation Cycle 2: Common Numerator

Which fraction is greater: $\frac{3}{7}$ or $\frac{2}{5}$

$\frac{3}{7} = \frac{6}{14}$ $\frac{2}{5} = \frac{6}{15}$

$\frac{6}{14} > \frac{6}{15}$

So: $\frac{3}{7} > \frac{2}{5}$

Information about the student's process: The student is using common numerators to compare the fractions. The core ideas include: (1) the student finds the least common multiple of 3 and 3 to find a common numerator of 6, (2) to find an equivalent fraction of 6/14, the student multiplies the 3 and the 7 by 2, (3) to find an equivalent fraction of 6/15, the student multiplies the 2 and the 5 by 3, (4) the student compares the 14 and 15 to determine which fraction is larger, and (5) the student concludes 3/7 > 2/5.

Information about the student's understanding: The core ideas include: (1) the student knows that an equivalent fraction is another way of writing the same number, (2) the student does not know why their process for generating an equivalent fraction works mathematically, (3) the student understands that the common numerator method works because when you have the same number of pieces in each fraction, you can use the size of the pieces in the whole to determine which fraction is greater, and (4) the student understands that 6/14 is greater than 6/15 because you have the same number of pieces and when the whole is cut up into a small number of pieces, the pieces are larger.

Figure 2: Simulated Student Work and Core Ideas for Each Teaching Simulation Cycle

Table 1: Characterizing Performances by Performance Area: Levels of Proficiency

Level	***Eliciting the Student's Process***	***Probing the Student's Understanding***
1	Does not elicit core steps of the process	Does not probe core understandings
2	Elicits some core steps of the process	Probes some core understandings
3	Elicits all core steps of the process	Probes all core understandings
4	Elicits all core steps of the process and additional information	Probes all core understandings and additional information

Performance area: Eliciting the student's process

In the first simulation cycle, all 20 PTs demonstrated level two proficiency. They elicited some of the core steps of the student's process, but not all of them. In the second simulation cycle, five of the 20 PTs (25%) demonstrated increased proficiency with eliciting the student's process, eliciting all of the core steps of the student's process. The remaining 15 PTs (75%) continued to elicit only some steps in the student's process, demonstrating level two proficiency.

To illustrate what change looked like when it occurred, we consider the case of Ofelia, who shifted from a level two proficiency in eliciting the student's process to a level three. In the first simulation (common denominator), Ofelia elicited three of the four core steps of the process: (1) the student decided to make fractions with common denominators, (2) to get fractions with common denominators, the student added 2 to the 3 and the 4 to get 5/6, and (3) the student compared the numerators. Ofelia did not elicit that the student concluded that 5/6 is greater than 4/6, a core step of the process that is not evident in the student's work. After Ofelia engaged in the teaching simulation, the TE asked Ofelia about her interpretation of the student's process and understanding. This included the TE asking Ofelia what the student concluded when she looked at the numerators, a core step that Ofelia had not asked about. After Ofelia acknowledged that she did not have any evidence, the TE asked Ofelia what questions could have been asked to gather this information. The TE provided Ofelia with feedback on her eliciting and interpreting of the student's thinking.

In the second simulation (common numerator), Ofelia asked about all of the core steps of the process. Ofelia began by asking the student, "Can you tell me what your first step was?" After learning that the student wanted to change the fractions so that they would have something in common, Ofelia asked the student which fraction they had changed first, 3/7 or 2/5. The student explained, "I changed 3/7 to 6/14, so I multiplied the three and the seven by two. Ofelia pressed for an explanation of why the student multiplied by two, "Is there a reason that you chose two to multiply by?" The student explained, "I wanted the numerators to be six because it's the least common multiple of three and two. Through this exchange, Ofelia elicited that the student selected six as the common numerator because it is the least common multiple (a core idea) and elicited the student's process for generating an equivalent fraction for 6/14 (a second core idea). Ofelia then asked whether the student had generated an equivalent fraction for 2/5 ("and for 2/5, can you show me your procedure for that?), which is a third core idea in the process. Ofelia went on to elicit the student's process for comparing 6/14 and 6/15, "can you tell me why 6/14 is bigger than 6/15?" The student responded that "fourteenths are bigger than fifteenths." This was the fourth core step in the student's process. Ofelia did not ask a question that revealed the student had concluded 3/7 is greater than 2/5. This is part of the process is less crucial to ask about because the conclusion is written on the paper.

Performance area: Probing the student's understanding

In the first simulation cycle, 13 of the 20 PTs (65%) demonstrated level one proficiency, meaning they had not probed the student's understanding of any core ideas related to the process. Four PTs (20%) demonstrated level two proficiency, two PTs (10%) demonstrated level three proficiency, and one PT (5%) demonstrated level four proficiency.

In the second simulation, nine of the PTs (45%) demonstrated increased proficiency with probing the student's understanding. The increases in rubric levels ranged from one to three, with seven increasing by one rubric level, one increasing by two rubric levels, and one increasing by three rubric levels. Another four PTs (20%) demonstrated decreased proficiency by one rubric level. The remaining seven PTs demonstrated the same level of proficiency. Among the 13 PTs who demonstrated level one proficiency on the first simulation, five continued to demonstrate a level one proficiency, seven increased from level one to level two proficiency, and one increased three proficiency levels from a level one to a level three.

To illustrate what change looked like when it occurred, we consider the case of Daniela, who shifted from a level one proficiency with probing the student's understanding to a level two proficiency. In the first simulation (common denominator), Daniela did not ask questions about the student's understanding of the core ideas. Instead, Daniela's questions focused on the student's process. After the simulation ended, the TE asked Daniela a set of questions about her interpretation of the student's understanding. The TE provided feedback suggesting that Daniela needed to ask more about what the student did, how the student did it, and why the student did it. The TE named understandings that Daniela could have asked about, including the student's understanding of the method used to compare the fractions, and identified specific questions Daniela could have asked.

In the second simulation cycle (common numerator), Daniela pressed the student at multiple points about their understanding of core mathematical ideas. For example, Daniela asked, "So, how did you use the numerators to help you solve this?" The student explained, "Once I had my numerators, because they are the same, I can look at my denominators to figure out which fraction is bigger, and six-fourteenths is bigger." Daniela then pressed further on how the student knows that six-fourteenths is "bigger" and asked, "How do you know that six-fourteenths is bigger?" The student responded that "Fourteenths are bigger than fifteenths." Daniela proceeded, "Can you explain why 14 are bigger than 15?" The student responded, "Yeah. If I have a whole and I divide it up into 14 equal pieces, those pieces are going to be bigger than if I have a whole and I divide it up into 15 equal pieces." Through this line of questioning, the student articulated that fourteenths are greater than fifteenths because if the whole is cut up into more pieces, the pieces are smaller. However, Daniela did not ask questions focused on other important mathematical topics, such as the student's understanding of the meaning of equivalent fractions or why the process for generating equivalent fractions works.

Discussion

The importance of formative assessment in primary and secondary schooling has been well established (e.g., Wiliam, 2010), and there is increasing recognition that TEs need to be using formative assessment practices to support PTs' development as early career teachers. At the same time, there has been growing interest in approximations of practices (Grossman et al., 2009), including teaching simulations (Mikeska et al., 2021), as teacher education activities because they can be designed to enable PTs to practice teaching. As PTs engage in teaching simulations, they demonstrate the integration of their knowledge and their skills with teaching. TEs can leverage these simulations as sites for engaging in formative feedback conversations to support PTs in developing their teaching practice. The utility of a formative assessment is measured by the degree to which the

provided feedback impacts future work, in this case, teaching in a subsequent simulation. Thus, in this paper, we explored how PTs' teaching practice (as revealed through teaching simulations) shifted across two consecutive teaching simulation cycles. We focused specifically on their eliciting of the student's process and their probing of the student's understanding.

The findings were mixed. Seventy percent of the PTs showed an increase in their proficiency in one or both areas. Another 30% of the PTs did not demonstrate increased proficiency with either eliciting the student's process or probing the student's understanding. Most of the growth was seen in the PTs' probing of student understanding. We hypothesize that more limited growth in proficiency was observed on eliciting the student's process, because the tendency to "assume" steps of the process based on the written work can be challenging to overcome. These PTs may need additional opportunities to practice asking questions about the process. At the same time, the analysis (at this stage) does not account for whether the TE gave feedback in one or both areas. The TE used their professional judgement to decide what to emphasize in feedback conversations. Instances in which proficiency did not change may be explained by the feedback focusing on a different area. At the same time, developing teaching practice is challenging and some PTs may have focused on probing the student's understanding rather than eliciting the student's process. This is not to say that these PTs were discounting the importance of both areas of work, but instead, these PTs may have been able to only focus on one area in the moment. Forty-five percent of the PTs demonstrated increased proficiency with probing the student's understanding in the second simulation cycle. It was a similar mathematical context; however, the student's approach was intentionally different. We wonder to what extent content knowledge for teaching influenced PTs' uptake. To explore this hypothesis, we intend to qualitatively analyze the performances to see whether probing of the student's understanding was happening but perhaps was not focused on aspects that we had identified as core mathematical ideas.

Overall, the analysis suggests that the teaching simulation cycles hold promise as formative assessment in teacher education. More work is needed to further investigate the provision of feedback. We also want to investigate whether work on content should be done before participating in the formative assessment cycles in order to focus learning on eliciting student thinking. More generally, the field needs to consider whether teacher education activities used as formative assessments impact the subsequent teaching of PTs to better use limited time in teacher education.

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What we talk about when we talk about the mathematics laboratory: Designing professional education for inclusive practice

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This paper presents the design and implementation of a professional development (PD) programme for primary teachers, aimed at promoting inclusive mathematics teaching practices through the "mathematics laboratory", a long-standing educational framework in Italian research. Grounded in an epistemological and didactic analysis of key concepts, the programme was developed to address inclusion issues in mathematics education. Teachers' learning of the mathematics laboratory was fostered through a PD programme in which they experienced the same approach they were expected to use with their students, centred on inclusive classroom cultures, metacognitive reflection, peer collaboration, and teacher facilitation. Preliminary data from teachers' feedback indicate changes in their observations and attitudes, particularly in relation to gender bias awareness, error management, and recognition of students' active role in learning.

Keywords: Mathematics laboratory, numbers, inclusion, primary teacher professional development.

Introduction: The project context and epistemological framework

The professional development (PD) of teachers has long struggled to significantly modify conventional mathematics teaching practices, which often hinder inclusion. In response, researchers advocate for a greater focus on learning "to do" mathematics teaching, embedding professional learning within the authentic work of the classroom (Ball & Cohen, 1999; Grossman et al., 2009). This contribution addresses the ETC2026 call (question 3: "How can we design and carry out professional education for the learning of practice?"), by presenting a research-based PD programme for mathematics primary school teachers that focuses on learning inclusive mathematics teaching. Our design rationale is deeply rooted in the Italian "mathematics laboratory" (Anichini et al., 2004), which frames mathematics learning as a social, and inquiry-based activity. The PD programme presented here, titled "Inclusive Mathematics", is part of a wider project "Tackling Inequalities in Educational Outcomes: Experimental Evidence from Italian Primary Schools" aimed at designing, implementing and evaluating PD programmes on inclusive practice in the teaching of mathematics. "Inclusive Mathematics" builds on two previous projects on inclusion in mathematics education: MAT-IN (2020-21) and "Tackling the Gender Gap in Mathematics in Piedmont" (2018-19, Di Tommaso et al., 2024), which demonstrated that mathematics laboratory as a teaching practice can reduce the gap between boys' and girls' performance in mathematics, confirming other results in the field (Boaler, 2002). The project is funded by the Ministry of Education and the National Recovery and Resilience Plan, supported by the CRT Foundation and the University of Turin, aligned with sustainability goals (4, 5, and 10) by the United Nations (2015). This continuity between projects underscores our commitment to addressing inclusion as an integral pedagogical dimension of mathematics teaching. Our work aims to provide teachers with practical and theoretical tools to create

inclusive school contexts, in which all students, regardless of social background, gender, or perceived abilities, can achieve the same skills and opportunities (Forgasz & Rivera, 2012).

The mathematics laboratory: An educational framework for inclusion

The "mathematics laboratory" is a recognised educational approach in the Italian educational frameworks for teaching mathematics (Arzarello et al., 2006) through which students construct mathematical meanings in a social setting. It is not conceived as a physical space, but as "a structured set of activities involving people, structures, and ideas" (Anichini et al., 2004, p. 28), grounded on five dimensions each anchored in theoretical principles:

(A) *Construction of mathematical meanings*: Rooted in Vygotskian principles (Vygotsky, 1978), the laboratory promotes the social construction of mathematical meanings through interaction, problem-solving, and guided participation. Teacher scaffolding supports this process by making strategies visible and fostering metacognitive reflection.

(B) *Social learning in small groups*: Encourages peer-to-peer interaction, critical thinking, mutual respect, responsibility, while lowering social and emotional barriers. Its success relies on students' will to collaborate and on teachers' orchestration strategies (Anichini et al., 2004).

(C) *Use of physical and digital tools*: Tools, whether physical or digital, act as semiotic mediators of mathematical meanings. Their use supports students' conceptual development and the teacher's orchestration of representations and learning opportunities.

(D) *Mathematical discussion*: Defined as "a polyphony of voices articulated on a mathematical object" (Bartolini Bussi, 1996, p. 16). Discussion allows diverse contributions and strategies to emerge. The teacher acts as a mediator for collective meaning-making and shared understanding.

(E) *Non-competitive approach and valuing of error*: By reducing time pressure and competition, the laboratory values errors as opportunities for learning (Zan & Di Martino, 2007). It creates a psychologically safe environment that fosters affective inclusion, counteracts mathematics-related anxiety and feelings of incompetence, and empowers students, especially females, to take intellectual risks (Boaler, 2002).

The mathematics laboratory is centred on the social construction of mathematical meaning, primarily fostered through collaborative social learning. This interactive process is supported by the strategic use of tools and enriched by robust mathematical discussions. The entire educational framework is defined by a non-competitive approach that values errors as learning opportunities, creating an equitable and psychologically safe environment for all students. These features provide a robust foundation for inclusive pedagogy. Nevertheless, the very name "mathematics laboratory" often generates misunderstanding among teachers, who associate it with subject-specific laboratories conceived as separate physical spaces where students apply notions already learned. Instead, in the Italian mathematics education research, as reflected in the national curriculum Guidelines (MIUR, 2012), the mathematics laboratory is not defined by its environmental dimension but by its epistemological one: meanings are socially built through carefully designed activities, where students engage without having prior knowledge of the notions at stake. This mismatch between teachers' spontaneous representations of the laboratory and its theoretical conception in mathematics education

raises two research questions that guided our work: RQ1. How can PD programmes be designed to help teachers experience and internalise the mathematics laboratory as a professional practice? RQ2. How do teachers experience and internalise such a design in their professional practice, as reflected in their feedback?

Experimental context and research methodology

The experimental context for this study is the PD programme developed for the aforementioned project and implemented simultaneously in three Italian regions (Piedmont, Campania and Friuli Venezia Giulia) with a large and diverse cohort of volunteer teachers (4 groups and 50 teachers, all qualified to teach in primary education, either through a diploma or a degree). Teachers met nine times between December 2024 to April 2025 under the guidance of the first author, who acted as the sole didactician (Jaworski, 2006) to ensure consistency delivering the PD programme through a blended learning approach. All authors, as didacticians and as designers and facilitators of the programme, collected data on both its design and its implementation. For the design phase, data were gathered from planning meetings, where the content and format of the training resources were discussed and decided. The subsequent creation of instruments and materials for the professional development activities was also documented. These resources were then structured and made available through an online environment (https://cubeprogramme.unito.it/en), which served both as a repository of resources for teachers and as a platform for interacting with them. Importantly, the platform also enabled the systematic collection of teachers' feedback and reflections, thereby establishing a direct link between the design and the implementation phases. For the implementation phase, data consisted of teachers' reflections and interactions, collected through recordings of training sessions, teachers' reflective diaries, contributions to digital boards, and comments shared during the sessions. This body of material forms the basis for our initial qualitative analysis of the implementation.

Results

We now report insights gained from the design and from the implementation of the programme.

Design of the PD programme: Didacticians translating theory into practice

The innovative structure of the PD programme consists in being designed not only to teach about the mathematics laboratory but to have teachers experience it as learners, before implementing it as educators. To this end, the programme embodied the five key dimensions of the mathematics laboratory as described below, allowing teachers to learn through them, fostering a deep, practice-based understanding.

(A) *Construction of mathematical meanings*: Central to the PD was the "double dialectic" (Arzarello et al., 2014; Robutti, 2020), establishing a constant interplay between the didactic (teachers acting as teachers in the class with students) and meta-didactic levels (teachers acting as learners in the PD with didacticians). Teachers engaged with tasks as learners ("student worksheet" sessions) and then reflected on the pedagogical design and inclusive rationale ("methodological worksheet" sessions).

(B) *Social learning in small groups*: The programme modelled the same cooperative dynamics it aims to foster in classrooms: teachers were divided into four groups across three Italian regions (Piedmont,

Campania, Friuli-Venezia Giulia). During the sessions they worked collaboratively in small groups to face the worksheets. Collaboration was facilitated in both face-to-face sessions and online meetings, reinforcing the role of peer interaction and shared responsibility in professional learning (Anichini et al., 2004).

(C) *Use of physical and digital tools*: The PD programme provided structured resources for each activity, including manipulatives, ready-to-use "student worksheets," and detailed "methodological worksheets." In addition, innovative "video-pills" on key methodological pillars were created for both synchronous and asynchronous use, ensuring flexibility and sustainability. In detail, the video pills addressed a general account of the mathematics laboratory, mathematical discussion, argumentation, and problem solving.

(D) *Mathematical discussion*: The PD emphasised discussion at two levels: among teachers, in analysing tasks and discussing design choices, and in the classroom activities, where group work discussion and whole class discussion guide the construction of meanings along the lines offered by the narrative.

(E) *Non-competitive approach and valuing of error*: The PD programme created a safe environment where teachers could analyse misconceptions and reflect on the value of errors as learning opportunities (Zan & Di Martino, 2007). In the same way, classroom activities encouraged exploration strategies and valued errors as a natural step in meaning-making.

The choice of the content was based on the analysis of INVALSI national assessment tests at grade 2 (2013-2017), revealing that the gender gap in primary school is most pronounced in the "Numbers and Operations" domain (Di Tommaso et al., 2024). Therefore, the programme targeted grade 3 primary school teachers and focused on three core activities on this domain: *Contamille: a town to be expanded*, focusing on the base-ten number system; *Forest sprites*, exploring patterns and multiplicative structures; and *The adventures of dragon Nif*, consolidating the meaning of arithmetic operations. These activities guide students from concrete manipulation to abstract representation by integrating group work and collective discussion into a narrative that motivates students' mathematical explorations. To illustrate how didacticians designed activities truly adherent to the mathematics laboratory epistemological stance, in which mathematical meanings are collectively constructed without students having any prior knowledge of the concepts, we present here an example from the Nif dragon activity, designed to foster the construction of the meaning of division. The narrative framework specifies that the dragon grows by 12 dragon feet for each human biscuit, then the worksheet presents the following situation: *"After remembering the effect human biscuits have on dragons, Nif, who has a belly full of 14 biscuits, wonders if he has eaten enough to grow from his initial height of 30 dragon feet to that of the super-mega slide of 190 dragon feet"*. Using strategies such as distributive property (e.g., $12\times10+12\times4$, $14\times10+14\times2$) allows students to move beyond viewing multiplication as simply repeated addition. They learn to exploit its structural properties, treating it as a versatile operation in its own right. This conceptual step allows them to understand division, even without the algorithm, as a relationship between quantities. The properties of addition and multiplication (associative, commutative, distributive), previously stimulated in the activity sequence, support this transition and prepare the ground for later formalisation.

Implementation of the PD program: Preliminary findings from teacher feedback and diaries

A preliminary analysis of teacher feedback and reflections reveals a significant shift towards more inclusive and student-centred practices, directly attributable to the core characteristics of the mathematics laboratory. To illustrate these changes, we present a selection of quotations from teachers' feedback that are organized according to the five dimensions of the mathematics laboratory (Ti=teachers), selected to exemplify comments that emerged across the dataset containing words and meanings related to the five dimensions of the mathematics laboratory.

(A) *Construction of mathematical meanings*: T1 "The students' interest and active participation confirmed the value of laboratory approaches that place them at the forefront of constructing knowledge and developing awareness of their own learning journey." T2 "After answering worksheet 3b, some children invented problem texts personalizing the story of the dragon Nif." T3 "Some students asked me the following questions: "How do I write my reasoning?" "What should I write if there are no differences?" I interrupted the activity and opened a class discussion." T6 "By correcting together, we reflected on the use of multiplication [...] The different solution strategies were shared." T7 "In the end, everyone understood."

(B) *Social learning in small groups*: T3 "The pupils enjoyed this activity because they were working with a friend and because I let them know they could be creative in their representations." T4 "To complete this activity, it was necessary to create heterogeneous working pairs, both to support pupils with diagnoses and those who had previously shown difficulties." T8 "I was surprised to see the children become more adept at working in groups, discussing issues and reaching agreements. [...] The groups were reorganised to avoid putting uncooperative children together and, on the contrary, to encourage fruitful discussion." T10 "They then recovered with the help of their classmates."

(C) *Use of physical and digital tools*: T1 "I encouraged them to feel free to represent the "summary" in their preferred format, which they then explained to the class before sharing the collective one." T2 "The use of graphic representation helped overcome language barriers for children from other countries." T4 "Some chose a circular layout, with the dragon at the centre; others preferred a linear arrangement, to better understand the effects of each object." T6 "With the help of the touchscreen monitor, we manipulated Nif's image, making him grow and shrink" T8 "Manipulative activities are very exciting for pupils, relaxing them and helping them to focus their attention."

(D) *Mathematical discussion:* T1 "Group animated reading. [...] The students actively participated [...] and enthusiastically shared their reasoning related to the "drag-meter." T4 "During the collective discussion, the children who had not understood [...] received explanations from their classmates. During this discussion, one student verbalized how the option of using parentheses can help when writing calculations." T5 "The representations were compared in a group discussion, and it emerged that those in point 1 were clearer." T7 "Together, we chose the clearest and most reasonable approach. [...] A longer collective discussion than I had anticipated was necessary to find answers." T9 "During the maths discussion in class, various procedural possibilities emerged [...] Differences in approach and reasoning."

(E) *Non-competitive approach and valuing of error*: T4 “The children who had understood this detail right away helped to re-measure those who had previously made mistakes. [...] We took this opportunity to explain that [...] there is no right or wrong way, but rather a way that is more or less suitable for each of us.” T7 “The conclusion about not judging new arrivals hastily ties in with the welcome given to new pupils who arrived in the class a few weeks ago.” T8 “The characters who accompanied us on these journeys greatly motivated the students to learn mathematics.” T9 “Four children asked to repeat the reading, but I asked them to summarise the main points orally. [...] it highlighted the differences in level in their approach.”

These excerpts provide evidence that teachers’ feedback concerns all the five dimensions of the mathematics laboratory, showing how teachers internalised and experienced the PD programme by proceeding with a cross-section analysis of the dimensions, is it possible to highlight certain themes and challenges related to the project’s objectives. A recurring theme is the high level of student engagement that fostered a truly inclusive and collaborative culture (A: T6; B: T3, T4, T8, T10; C: T1, T2, T6; D: T4, T9; E: T4, T7); the hands-on, multi-sensory nature of the activities was particularly effective (B: T3; C: T1, T2, T4, T6, T8; D: T1, T9; T4, T8). The process of augmenting and defending ideas in small groups and in whole-class discussions was considered a key learning moment (A: T3, T6; B: T3, T4, T8, T10; C: T1; D: T1, T4, T5, T7, T9; E: T4, T9); teachers were often surprised by their students’ progress. However, teachers also realistically pointed out the difficulties, especially in arguing and their own difficulty as facilitators. The teachers’ comments also reveal their difficulty in recognising the difference between explaining and arguing an answer, and therefore in encouraging the latter. The data also reveal the challenges teachers face in implementing mathematics laboratory activities. These reflections are crucial for refining the PD programme. The main challenges include:

- Time: the exploratory and pressure-free nature of the laboratory sometimes conflicted with the pressures of delivering the curriculum. Teachers noted that discussions of complex issues “take a long time” and that ensuring the whole class reaches the same level of understanding “lengthens the time frame.”
- Managing group dynamics: although predominantly positive, teachers noted that in some cases “less competent students relied on the abilities of more dynamic classmates.” Managing discussions and language difficulties in groups with non-native speakers (“non-Italian-speaking children”) also presented a significant challenge.
- Differentiation/active involvement of students: another concern that emerged was meeting the needs of all students in a highly heterogeneous class. One teacher said: “Personally, I find it difficult to offer engaging stimuli for all levels of learning... especially for the more capable students who might get bored.”

These challenges were not framed by the teachers as failures, but as authentic areas for their own professional growth, underscoring the reflective nature of the programme.

Conclusion and future directions

In this study, we addressed RQ1 (how to design an effective PD) by shaping the programme following the same dimensions that define the mathematics laboratory itself. This design choice enabled teachers to experience the laboratory directly, thereby clarifying and giving substance to *what we talk*

about when we talk about the mathematics laboratory in the context of educational practices and experiencing its inclusive possibilities themselves before trying them out with their own students. Regarding RQ1, results show that structuring the programme to reflect the principles of the mathematics laboratory is an effective approach. According to teacher feedback and evaluations, this type of design directly influenced how they experienced and began to internalise this approach in their professional practice (RQ2). By becoming learners within this structure, teachers were positioned as active protagonists in their own professional development on the mathematics laboratory. Their feedback, during and at the end of the PD programme, reflects satisfaction and novelty regarding the training methods. This embodied experience is a fundamental example to support new PD approaches, showing that effective PD must reflect the teaching approach to be used in the classroom. Furthermore, teachers reported an increased focus on gender inclusivity, leading to classroom reflections on stereotypes and a more conscious use of both masculine and feminine terms; this is clearly visible in the teachers' use of masculine and feminine forms in the Italian language, which does not have a neuter form. As for the second RQ, teachers' feedback confirms the potential of the mathematics laboratory, while also pointing to challenges around addressing diverse learning needs, scaffolding argumentation, and time management. Rather than being weaknesses, these aspects provide critical insights for its further development and call for research into strategies that support teachers and students in navigating complex practices such as argumentation, effective ways of engaging learners with different levels of preparation, and teacher beliefs that still frame the mathematics laboratory as an "extra" beyond the standard mandatory curriculum. At the same time, the positive results obtained with the mathematics laboratory, as evidenced by this and previous research, suggest clear direction for education policy. Overall, the study shows that a laboratory-based design can firmly ground professional development in the same principles that sustain mathematics teaching, while teachers' feedback confirms both the value of the mathematics laboratory and the need to tackle its challenges. These outcomes consolidate the mathematics laboratory as a powerful framework for professional education and open directions for further research on its long-term impact and refinement.

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Facilitators' role in supporting core practices in a lesson-study context

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This study explores the role of facilitators in two lesson-study contexts promoting core practices of mathematics teaching. We adopt Wegerif's dialogic approach to learning and draw evidence from data generated within two lesson-study cases in Greece and Denmark that target 'supporting student reasoning' and 'teaching towards a goal' respectively. Our focus is on how and to what extent the facilitators support teachers' opportunities for developing these core practices in each case. Our analysis shows how facilitators' heuristics supported or not supported teachers' focus on key features of the core practices, especially features related to mathematics. It also indicates the importance of studying the facilitators' role further through more cases than our two individual ones.

Keywords: Lesson study in mathematics, core practices, facilitators' role, dialogic approach, professional development.

Introduction

Research agrees that the external facilitator (or koshi, knowledgeable other, external support etc.) plays a crucial role in lesson study (Seino & Foster, 2021; Takahashi, 2014) although the role varies across different countries and cultures (Skott, 2024). The role is explored to some extent, but there is a lack of research in particular on how and to what extent external facilitators can support teachers' professional development (PD) in a lesson-study context (Clivaz et al., 2023). This paper aims to contribute to this gap by exploring interactions between external facilitators and teachers.

When PD is considered, lesson-study research tends to focus on the knowledge teachers learn, related for instance to the mathematical knowledge for teaching framework (e.g. Clivaz et al., 2023). Only rarely is the focus on what they learn to do differently when teaching mathematics. Departing from teacher education, a new strand of research focuses on the work teachers do and not primarily on the knowledge they need to teach well (Grossman et al., 2018). This strand, called *practice-based teacher education,* focuses on the practices that are the *core* of teaching. Such practices occur frequently, require professional judgement and are learnable by novice teachers. One example is to elicit students' mathematical thinking. Lesson study focuses also on practices in classrooms through three phases: jointly planning a research lesson (incl. setting goals), conducting the lesson and observing student activity, and jointly reflecting on the lesson based on observations (Seino & Foster, 2021). Therefore, lesson study seems to be a promising context for teachers' learning of core practices.

In this paper, we combine a focus on core practices with an exploration of how and to what extent external facilitators can support mathematics teachers' development of such practices. We do so by comparing two lesson-study cases from Denmark and Greece. The Danish facilitator led the lesson-study process and collaborated with the teachers, advising them on issues related to the mathematical content and teaching. However, the Greek facilitator was only allowed to facilitate the lesson-study process, not to advise on such issues. Aiming to explore the impact of these two types of support on

teacher learning, we address the question: How and to what extent can facilitators support teachers' opportunities to develop core practices in a Danish lesson-study context compared to a Greek one?

Research on the role of facilitators and core practices

While the role of external facilitators is well-defined in countries, where lesson study is an integral part of the educational system, such as in Japan (Seino & Foster, 2021; Takahashi, 2014), the role is more blurred in countries unaccustomed with lesson study. In those countries, often outside East Asia, there is a tendency not to include an external facilitator (Seleznyov, 2018) and if one is included, their role is typical limited to facilitate the lesson-study process (Seino & Foster, 2021). This explains our use of the term facilitator. However, studies on adaptations of lesson study outside East Asia stress the importance of facilitators (or other external knowledge sources), also at the adaptation's mature stages, in particular to help teachers identify pressing problems to be analysed during reflections, challenge their perspectives on the content and teaching, and establish strong connections between mathematics, teaching practices and student thinking (Skott, 2024). Especially the latter resonates with features stressed by Seino and Foster (2021) and Takahashi (2014) as important in Japanese facilitators' final comments. To further explore those potentials, our two cases seem promising as they distinguish facilitation of the lesson-study process from such more complex facilitator heuristics.

Recently, a new tendency has emerged to describe teacher education in terms of core practices in the US (Grossman et al., 2018), but also in a number of other countries (Grossman & Fraefel, 2024). This paradigmatic shift from focusing on the knowledge needed to teach effectively to focusing on the practices, that are at the core of teachers' work, builds on a view of teaching as a work that requires professional judgement, reflection and complex thinking. It also implies that teaching is a specialised form of work with stable practices that can be further specified (Ball & Forzani, 2009). We draw on Grossman et al.'s (2018) definition of core practices "as those that are fundamental to teaching and grounded in disciplinary and pedagogical goals that teachers enact to support student learning" (p.2).

Core practices are complex to learn. This has been a major concern of the new field and Grossman et al.'s (2018) identification of three elements has been widely used with pre-service teachers: to use representations of practice (e.g., videos of teaching), to decompose core practices into meaningful parts, and to approximate practice by teaching such parts in a less complex context than normal classrooms (e.g., rehearsals). However, when doing lesson-study with in-service teachers these elements take on different meanings. Here no practice-representations are needed as the object is live teaching, and it seems obvious to study a specific core practice or its smaller parts. Moreover, the research lesson can be seen as an approximation to practice, if its complexity is reduced by a collaborative approach and shared responsibility. To ensure these two – to focus on a smaller part of a core practice and reduce the complexity of the research lesson – seem to be important for facilitators.

We focus on two core practices: supporting student reasoning and teaching towards a goal. The first is the focus of the Greek case. Drawing on research into student reasoning and its teaching (e.g. Niss & Højgaard, 2019), the aim was to support teachers: a) to recognize salient features of students' mathematical reasoning, b) to consider the design and enactment of tasks promoting different aspects of reasoning, and c) to reflect on the possibility of promoting students' reasoning in teaching. The second is the focus of the Danish case. Building on two sets of core practices from TeachingWorks

and Washington's U-ACT program (see Grossman et al., 2018, p. 164-173) the core practice is described as: a) to set effective goals based on external standards and the class's prerequisites, b) to establish and sequence interim subgoals to help a steady progress towards the overall goals and c) to decide which aspects of student talk to be attentive to and which of their ideas to engage the class in.

Theoretical approach

We are inspired by Wegerif and his colleagues' dialogic approach to learning (e.g., Wegerif, 2011). Their approach is orientated towards student learning in class, while we use it to understand the way facilitators contribute to create learning opportunities for teachers in a lesson-study context. Taking the Bakhtinian perspective that learning is a phenomenon that emerges through dialogic interactions, Wegerif claims that "when humans enter into dialogue there is a new space of meaning that opens up between them and includes them within it" (2011, p. 180). He calls this space a *dialogic space*. For our purpose, an example is the space that emerges when a group of teachers exploring a teaching problem together, listen to each other, ask questions to obtain more insight and change "their minds as a result of seeing the problem as if through the eyes of the others" (p. 180). In a dialogic space, it is crucial to bring more than one perspective or voice into relation with each other, as individuals' creation of meaning is dependent on this. Thus, for such meaning to emerge, a so-called *dialogic gap* between different voices is required. If more than two voices are brought into relation, Wegerif calls it a *dialogic discourse*, otherwise a *monologic discourse*. Kim (2024) shows that three heuristics are key to create a dialogic space: *opening* (i.e., exploring a question or statement that invites different views)*, widening* (i.e., inviting more voices) and *deepening* (i.e., examining underlying beliefs and assumptions). Both Wegerif and Kim show that a teacher can use these heuristics to create a dialogic space in a way that support the learning of individuals or of a community of learners.

As teachers' development of core practices require their active involvement, dialogic interactions among teachers and with facilitators seems to be critical. Therefore, we use these three heuristics to analyse if and how facilitators contribute to create dialogic spaces and how their contributions influence the emergence of opportunities for each lesson-study group to learn the core practices.

Methods

The Greek case

The Greek lesson study took place in the context of the European project LESSAM (*Lesson Study as a vehicle for improving achievement in mathematics*) with the aim to promote students' mathematical reasoning. It involved mathematics teachers from lower secondary schools in Belgium, Cyprus, Netherlands and Greece, and it aimed to address the impact of teachers' participation in lesson-study groups on their PD in relation to supporting mathematical reasoning in their classrooms. In Greece, 29 teachers from 10 schools were organised in 10 lesson-study groups (two-four participants) at their schools. Prior to the lesson-study activities, teachers and facilitators were provided a handbook, developed by the project team, including guidelines for the design and enactment of reasoning tasks and a detailed description of the key ideas and the procedures of lesson study (e.g., setting research goal, selecting case students). The teachers also received support from a member of the project team who played the role of a facilitator. The facilitators were expected to guide teachers through the

lesson-study process, foster their discussions in ways that promote their learning, and encourage them to use the handbook but without advising them on issues related to the mathematical content or teaching. The facilitators were introduced to this role in an online session, consisting of examples of questions that they could ask to help teachers plan and reflect on their research lessons. In the school year 2022/23, the teacher-groups conducted a three-cycle model, consisting of three research lessons, each preceded by a planning session and followed by a reflection (60 min each). Each lesson was taught by one teacher while the rest acted as observers. In the paper, we focus on one group of three teachers (T1, T2, T3) facilitated by Chris. He is an experienced mathematics teacher and teacher educator with a Ph.D. in mathematics education. The second author acted as a participating observer. The data consists of transcriptions of all audio-recorded group sessions (planning and reflection). The group had to select case students and focus on their collaboration and learning. T1's aim with the first lesson in grade 9 was to explore if and how students can combine geometric and algebraic approaches to solve the task: "How many lines and which ones pass through both points (6,2) and (5,6)? Find their equations." T1 selected two case students (S1, S2) of which S1 preferred to use algebra, while S2 tried to avoid algebra and use visual approaches. T1 expected that "both will benefit from each other's perspective." (first planning).

The Danish case

The Danish lesson study was initiated by a private school in the Copenhagen area and involved all its mathematics and Danish teachers. The overall aim was to strengthen the teachers' reflections and practices related to inquiry-based teaching. In fall 2022, three groups of (three) mathematics teachers conducted one three-cycle lesson study, where each teacher alternately taught the research lesson followed by a reflection session. However, in spring 2023, each group planned and taught three consecutive and coherent research lessons, preceded by a 2-hour planning session, and between the lessons, two short (10 min) reflections and a large final one (60 min). These lesson-study activities were facilitated by Bo, an experienced teacher educator and lesson-study facilitator with a Ph.D. in mathematics education. The first author acted as a participating observer. Bo's role was to guide the lesson-study process by setting the agenda, structuring and leading discussions, but also to advice and challenge the teachers' perspectives on the mathematics and its teaching. In addition, before each lesson-study round, Bo held workshops for all the mathematics teachers focused on a specific topic and how to teach it. For this paper, we focus on the grade 3 teacher group and draw on field notes and audio recordings of all the group's lesson-study activities in spring 2023 (all transcribed, but the research lessons). The group decided to focus on students' learning of multiplying two-digit numbers by involving them in developing effective strategies. This goal aligns with the national standards.

Analytical strategies

The two lesson-study cases are critical since both facilitators are experienced and skilled, and we can therefore directly compare the different conditions on their roles. For each case, we selected an episode that was representative of how the facilitator interacted with the group and that was productive in terms of the emergence of opportunities to learn the specific core practice compared to other episodes. We analysed these two episodes in two steps. First, we analysed them for emergence of learning opportunities by using the previously described core practices as analytical lenses.

Second, we used the dialogic approach to analyse if and how the facilitators used the heuristics to create dialogic spaces and the extent to which it influenced the teachers' opportunities to learn.

Results

The Greek case: Focusing on relational aspects of student communication

The excerpt below (from the first reflection) shows Chris' typical ways of interacting dialogically with the teachers. Initially, Chris creates a dialogic gap by inviting all the teachers' perspectives on the two case students' collaboration.

Chris [to all]: Based on your observation of the group, can we now identify some factors that may have had a positive or negative impact [on their collaboration]?

Teacher 1: As I observed, S1 was not feeling well during the lesson. I don't know if she didn't like how the lesson was going, but for a person who doesn't feel well, to see something that doesn't work for her makes it worse. S1 is an athlete, and she is usually under a lot of pressure. Thus, she did not present her solution clearly so as to understand. It seemed wrong but many times her solutions look like quite complex but when she provided me an example, I realised that her approach is correct, there is a structure. S2 did something with circles - possibly it could have been done in a simpler way. Often S2 goes to the extremes in order to avoid algebra.

Chris [to all]: Yes, but T1 aimed to bring these two student approaches closer. I would like to hear comments from all of you.

Teacher 2: I think S1 and S2 might be strong personalities and probably react. They are teenagers. One idea is to put each other next time in a group with a student with … let's say thinking more neutrally.

Teacher 3: I would have established groups of four students, not only friends sitting at the same desk. This enriches the contributions in the group work … this way we might have had different results.

Chris: Did you see any improvements in these students' reasoning?

Teacher 1: We now just opened a window to have a look at their thinking. I am not sure that the two [case] students could developed and shared their different perspectives of the solution in an hour or even in two or three ... I will ask S1 today to explain to me her solution so as to understand it and then see if at some point the two approaches can come together. But it was not possible to achieve during the lesson. In the long run it can be done, maybe this is a start. Now I see how difficult it is.

Initially, Chris opens a dialogic space by posing an exploring question and he widens it by inviting all the teachers' perspectives on T1's approach. Throughout the except, Chris uses the pronouns 'we', 'your', 'you' to imply that he expects all teachers' views (widening). The teachers provide different perspectives on the factors influencing the students' collaboration. T1 stresses students' emotions and the complexity of their solutions. T2 emphasises their personalities and inflexible views as a barrier, while T3 suggests organising the group work differently. Next, Chris tries to deepen their reflections by examining (observable and underlying) improvements in the students' reasoning. He seems to encourage the teachers to recognize salient features of the students' reasoning (a critical part of the core practice). T1's final recognition of the complexity to understand and combine the students' diverse ways of reasoning, suggests that he learnt something about teaching reasoning.

However, the teachers did not address the mathematical parts of the two students' reasoning. They talked about the students' idiosyncratic reasoning as an emerging factor that hinders both the students' communication and their opportunity to intervene. Due to Chris' limited role, he was not able to address the students' reasoning directly. Chris thus contributed to create a dialogic space and engage

the teachers in reflections, where they shared and discussed their perspectives on relational aspects of the two students' collaboration. The dialogic space did offer them learning opportunities, as evidenced by T1's awareness of the importance of recognising salient features in students' reasoning and the possibilities of improving their collaboration. However, the limits on Chris's support seems to hinder the emergence of opportunities for them to deepen the complexity of students' reasoning.

The Danish case: Focusing on teaching towards a goal

The episode consists of one typical exchange from the planning. Before the excerpt, T1 stresses that it is difficult for the teachers to establish a progression in the students' mathematical thinking that can support the students in obtaining the goal in course of the three lessons. When T2 raises an issue about how to orchestrate a class discussion, Bo uses this opportunity to also address T1's concern:

Bo:	But you can consider this, you have used it before, right? We have seen different student solutions, right? You chose, for instance, three different ones. Then the students present these [in a class discussion] so that their solutions become more and more sophisticated. Because then you control the direction. You want different ideas, but you also want a progression.
Teacher 2:	But should all the [student] groups present their work? [T3 mumbles no]
Teacher 1:	What do you mean by sophisticated?
Bo:	It is important that all the students see that if you decompose in tens and ones, then you can take a shortcut […] For example if you decompose 15 in 10 and 5 … it is something special to decompose a number in tens and ones. Then you can control that in any case they [the students] will see it [the decomposition strategy].

Bo uses the initial dialogic gap (the issues raised by T1 and T2) to open a dialogic space by reminding the teachers of their previous discussion about it, thus acknowledging their previous voices. He then deepens the space by examining and suggesting a way to structure a class discussion and ends by widening it, so it also relates to T1's concern, by suggesting how the teachers in a class discussion can control the direction of students' thinking towards the goal by making them share still more sophisticated solutions. Bo implies which student ideas to engage the class in, and hereby he creates opportunities for the teachers to learn about a crucial part (part c) of the core practice: teaching towards a goal. T1's next question indicates that the teachers are unfamiliar with assessing whether one student approach to multiply two-digit numbers is more sophisticated than another. Taking this teacher's voice seriously, Bo indirectly responds by deepening and examining, what a sophisticated strategy might look like. He suggests that the decomposition strategy is special (i.e., sophisticated) and that all students must see it. He thus implies that this strategy should be a stepping stone or an interim subgoal for the teachers, when conducting a class discussion. Hereby, Bo creates an opportunity for the teachers to learn about another crucial part (part b) of teaching towards a goal.

The teachers would not have managed on their own to decide which student ideas to engage the class in and to establish content-related interim subgoals – two critical parts of teaching towards a goal. They tend to "give them [the students] a method" (T3, final reflection). In this reflection, T3 also highlights the value of such interim goals after having taught the research lesson: "It is good that you have these stepping stones … such that you know as a teacher that now we are here and not where we were before … Also if you suddenly find yourself in deep water, you can quickly fall back on the fact that it's about how to decompose the numbers appropriately. Then you're on safe ground again". This episode thus shows how Bo creates new and important learning opportunities for the teachers

related to teaching towards a goal by opening, widening and deepening their dialogue. Bo contributes to create a dialogic space as the dialogue opens for new meanings among its participants.

Comparison of the two cases and concluding remarks

In both critical cases, the facilitators used all three heuristics of the dialogic approach to create a dialogic space among the teachers and to engage them in reflections on pedagogical and mathematical issues such as relational aspects of students' collaboration (Chris) and teaching towards a goal by using interim subgoals to direct students' thinking (Bo). Creating such dialogic spaces is a notable achievement for the facilitators. As Kim notes (2024) "Creating dialogic space is a challenging and intricate task" that "is established over time and often requires repeated practice" (p. 129). Consistent with Kim (2024), the facilitators skilfully asked reflective and authentic questions to engage teachers in critical thinking and reflection, and the teachers appeared comfortable posing questions, sharing their ideas, and interacting constructively with one another. In both cases, the facilitators contributed to create learning opportunities for the teachers. However, the difference between their roles seems to influence the extent to which the facilitators supported the two core practices in each case. As both core practices are closely related to the teaching of mathematics, Christ was very limited in his support as he was not allowed to explicitly ask the teachers about the mathematical content of the two case students' reasoning. So, he mostly challenged the teachers' general pedagogical views. In contrast, all of Bo's contributions related to the mathematical content and its teaching, and he supported the teachers on a deep level in developing the core practice of teaching towards a goal. We also see that this difference is reflected in the teachers' learning opportunities: the Greek teachers' opportunities were limited and not closely linked to key aspects of teaching mathematical reasoning, whereas the Danish teachers' opportunities were deeper, closely aligned with the core practice in question as evident in their final reflections on the lesson study.

This study suggests that a dialogic approach is well-suitable for examining facilitator-teacher interactions in a lesson-study context. Our findings, that both the Danish and Greek teachers needed support to reflect on matters closely related to mathematics and its teaching, align with prior research on lesson study (e.g., Skott, 2024). Such studies emphasise the facilitator's role in linking mathematics, classroom practice and student learning (Seino & Foster, 2021; Skott, 2024; Takahashi, 2014), a point also evident in our analysis. Consistent with Skott (2024), another key aspect of this role is to challenge teachers' perspective and create dialogic gaps that can foster deeper reflection. This small study also contributes an exploration of how to support teachers' development of core practices in a lesson-study context. So far, this is an unexplored area (Grossman et al., 2018). Our study shows that lesson study is a promising and an important context for developing such practices as these practices and their evolution are a natural focus of the dialogue in the planning and reflection sessions. Grossman's three elements also proved important in the lesson-study context: Decomposing a core practices, as Bo did, appeared to serve as a key heuristic for the facilitator as well as reducing the complexity of the research lessons by effectively transforming them into a joint exploration. Chris managed this process by reinforcing the collaborative nature of lesson study by using the pronoun 'we' to refer to the group's collective work. Regarding the third element about practice-representation, Chris could have used the two students' products to enhance their reflections. Our findings suggest a promising research direction: to investigate systematically the types of facilitators' heuristics and

their impact on teacher learning in a lesson-study context. To conclude, our study shows that lesson study is a promising context for developing teachers' core practices. However, this requires that the facilitator not only guide the lesson-study processes and create a dialogic space. The facilitator must also create a space, that invites teachers' different and deep reflections on the content and its teaching. Future research should explore the facilitator's role further through additional data and cases.

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Study and research paths for teacher education: Connecting theory with practice

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This paper focuses on analysing the proposal of study and research paths for teacher education (SRPs-TE) within the framework of the Anthropological Theory of the Didactic, examining how they can bridge theory and practice in mathematics teacher education. SRPs-TE revolve around generating professional questions and guiding inquiry processes, followed by analysis and reflection using tools transferred from research. We distinguish two types of SRPs-TE: Type A, adapted from school-based SRPs, and Type B, explicitly created for teacher education. Drawing on cases such as combinatorics, we illustrate how Type A devices favour the transfer of tools to classroom practice, while Type B fosters the construction of epistemological reference frameworks. The study highlights how SRPs-TE help teachers reconstruct school mathematics, adopt epistemological and didactic tools, and critically reflect on professional practice.

Keywords: Teacher education, anthropological theory of the didactic, study and research paths, combinatorics, modelling.

Introduction

From the perspective of the Anthropological Theory of the Didactic (ATD), current educational systems largely remain within the paradigm of *visiting works* (PVW) (Chevallard, 2015), where teaching is defined by a set of established knowledge contents that the teacher presents for students to study them. In contrast, the ATD proposes the paradigm *questioning the world* (PQW), where teaching is organised around open questions and inquiry prevails over mere visit of works. The PVW frames teaching as the transmission of already-produced knowledge, whereas the PQW promotes the study of genuine questions that require the production of mathematical and didactic organisations. Within the ATD, the didactics of mathematics is conceived as a discipline that questions the nature of school mathematics and its institutional conditions.

A didactic device aligned with this paradigm is the *study and research path* (SRP) (Bosch, 2019; Chevallard, 2015), organised around a generating question whose investigation leads to derived questions, to explore possible answers, and construct new praxeologies to elaborate their own answer. This collective process entails the study of new knowledge and tools, which are mobilised as instruments for addressing the initial question. In teacher education, this methodology has been developed into *SRP for teacher education* (SRPs-TE) (Barquero et al., 2019; Ruiz-Olarría, 2015).

SRPs-TE start from questions related to the teaching profession and often involve designing, implementing, and analysing an SRP as a possible response. As explained in Ruiz-Olarría (2015), a distinctive feature of SRPs-TE is their modular structure, with five stages. *Module* 0 introduces the generating question ($Q_{0\text{-TE}}$), usually related to a professional concern such as '*How can a specific content be taught?*'. *Module* 1 asks students-teachers to experience an SRP as a possible response to

$Q_{0\text{-TE}}$, under the position of student. *Module* 2 focuses on analysing the experienced SRP through the introduction of tools transferred from research to teacher education. In *Module* 3, teachers work on an adaptation of an SRP and its implementing into a specific school institution, while collecting data for later analysis. Finally, *Module* 4 focuses on analysing the implementation, identifying the potential and limitations of the SRP as a response to $Q_{0\text{-TE}}$, and elaborating other elements of response.

Some authors have proposed classifications of SRP-TE according to their aims and modalities (e.g., Sierra and Gascón, 2018). In this paper, however, we focus on the nature of the SRP proposal in Module 1 and distinguish between two types of SRP-TE:

> Type A refers to SRPs originally designed for primary or secondary students that have been adapted for their implementation in teacher education, to address specific professional questions. This type allows the construction of epistemological and didactic models that intervene in the teaching profession.
>
> Type B refers to SRPs that have been designed explicitly for teacher education, in the context of Module 2. In this case, the questions proposed in Module 1 are significantly more complex than those usually posed in school contexts, and they mainly serve to build epistemological reference models of the mathematical content at play.

Both types share an essential feature: fostering the reconstruction of school mathematics to be taught, by questioning it and revisiting its *raisons d'être*. In this sense, SRPs-TE can be seen as proposals that extend the scope of the PQW into the professional domain of teacher education. The aim of this paper is to analyse how SRPs-TE can bridge theory with practice, offering mathematics teachers the opportunity to adopt, use, and develop tools transferred from research to practice, thereby addressing the epistemological, didactic and ecological dimensions of teacher education (Barquero et al., 2018). More specifically, we address the following research questions:

RQ1: What kinds of tools can help teachers question school mathematics and its teaching, without disconnecting the analysis from classroom practice, while addressing real professional questions?

RQ2: What is the potential of SRPs-TE to bridge theory and practice in mathematics teacher education through their two modalities (Type A and Type B)?

To address these questions, we first review recent studies based on SRPs-TE, which provide insights into the diversity of contexts and approaches in which they have been implemented, as well as the kinds of tools they make available for teachers in training. Building on this overview, we then present two SRPs-TE, one of each type, to illustrate how SRPs-TE can function as bridges between theory and practice in mathematics teacher education.

The SRPs-TE have been implemented by the first author, the first with prospective secondary teachers and the second in second year mathematics course for prospective primary teachers. In each case, she guided the inquiry and supported the articulation of justifications and praxeological analysis.

Reviewing previous proposals of SRPs-TE

Table 1 synthesises some of the SRPs-TE that have been implemented so far. Similar overview tables can be found in previous works (e.g., Barquero et al., 2019; Barquero et al., 2022), where a broader

range of examples is reported. Our aim here is not to provide an exhaustive list but rather to highlight a selection of representative and well-documented cases that illustrate the two types.

Table 1: Overview of some implemented SRPs-TE

	SRP-TE	Context	Professional question $Q_{0\text{-TE}}$	Focus of the SRP
A	Hakamata et al., 2024	Pre-service Primary and Secondary TE	*How to teach inferential statistics in primary and secondary school?*	Experience an inquiry in data collection and treatment, informal inference and data modelling. *Q_0: What is hidden inside the bottle?*
	Barquero et al., 2019	Pre-service Primary TE	*How to introduce modelling in primary school? How to design and manage modelling activities?*	Algebraic and geometric modelling. Designing and analysing "cake boxes", questioning the relation between the dimensions of the sheet and the resulting box.
	Gibel & Blanquart-Henry, 2017	Pre-service Primary TE	*How to teach plane geometry? How to design and manage activities involving figures as geometrical models?*	Reproduction of quadrilaterals (particularly rhombuses) in the meso-space (courtyard). *Q_0: How to reproduce a figure?*
B	Sierra & Gascón, 2018	Pre-service Primary TE	*How to teach the rationale and use of numeral systems?*	Explore additive, hybrid, and place-value systems; linking the construction of numeral systems with its use to write and operate numbers. *Q_0: Why do people use several alphabets but one numeral system?*
	Lerma et al., 2021	Pre-service Early childhood and Primary TE	*How to teach logical knowledge in early years?*	Work on the categorization and classification of collections; criteria for its comparison. *Q_0: How to organise a 'Lost and Found' room in a school?*
	Verbisck et al., 2022	In-service Secondary mathematics TE	*How to teach statistics in secondary school?*	Transform traditional exercises into open-ended inquiries; integrate real-world issues and interdisciplinary links; address the didactic phenomenon of the invisibility of data treatment

SRP-TE on *padlocks* from a modelling perspective (Type A)

This SRP-TE, originally developed for secondary students (Vásquez et al., 2021), was later adapted to teacher education, where prospective secondary teachers experienced a similar design. The SRP-TE starts with the discussion of *$Q_{0\text{-TE}}$: How to design and manage modelling activities?* To address this issue, prospective teachers experienced the SRP first-hand (Module 1) in the position of learners, through a concrete and motivating situation.

The starting point was the presentation of a collection of five padlocks (Figure 1), each with different ways of introducing codes. The generating question was *Q_0: Which padlock is the most secure?* This question triggered an initial whole-class discussion in which several hypotheses emerged, such as linking security to the physical robustness of the padlock, to the number of keys available, or to the number of possible codes. Building on the latter idea, the study was steered towards a derived question, *Q_1: How many codes does each padlock admit?*

Figure 1: Padlocks presented in the SRP

To investigate this question, the padlocks were distributed among the working teams, each responsible for analysing one specific type. Teams of students worked on different padlocks, developing and testing techniques for representing, systematising, and counting the possible codes. Tree diagrams, systematic listings, additive and multiplicative strategies, as well as grouping approaches, progressively emerged as tools to tackle the problem. The collective validation of these techniques enabled participants to compare methods, refine them, and generalise their results. In this process, algebraic expressions and combinatorial formulas gradually appeared as models that condensed the different counting strategies. This also enabled the community to provide a consensual answer to Q_0, while generating further derived questions, such as the effect of imposing restrictions on the codes or how to classify padlocks according to the underlying combinatorial structure.

Participants' initial epistemological references showed a clear form of applicacionism: several groups immediately resorted to known combinatorial formulas and applied them directly. This revealed a dominant institutional epistemology grounded in the automatic use of established techniques, where formulas appear as ready-made tools rather than as generalisations emerging from a modelling process.

In this adaptation of the SRP-TE, the study did not continue beyond this initial stage. Since the prospective secondary teachers were already familiar with the mathematics underlying the new challenges, these derived questions were not addressed, as they did not provide additional learning opportunities. Therefore, the activity moved directly to Module 2. The instructor shifted the focus to the mathematical and didactic analysis of the praxeologies involved. Students identified and compared different systems and models (e.g., padlocks as physical systems, combinatorial models with and without repetition, product and grouping models, tree diagrams, systematic listings) and analysed how these evolved during the study: progressing from exhaustive listings and visual representations to numerical formulas and eventually to general algebraic expressions. The use of the physical padlocks as manipulatives played a key role as concrete models of more general counting problems, and their own construction from an initial system turned the activity into a rich modelling process. This work was accompanied by an explicit analysis of combinatorics, distinguishing types of tasks, techniques and theoretical assumptions.

Finally, participants extended the experience in Module 3, designing new inquiry situations for secondary classrooms by reusing the same structure in other mathematical domains such as proportionality, statistics, probability, or geometry. Although the prospective teachers did not implement their own designed proposals in real classrooms (Module 4), one session was devoted to examining and discussing examples of work produced by secondary students. This discussion took place after participants had been introduced to the complete set of didactic analysis tools, which allowed them to interpret and reflect on students' productions in a more informed way.

SRP-TE about *counting* for primary school teachers (Type B)

The same padlock-based SRP was also adapted and implemented with prospective primary teachers, to address a different professional question: $Q_{0\text{-TE}}$: *How can we teach counting strategies?* The experiment involved 57 second-year students organised in teams of four. The design again began with a collection of padlocks and the generating question Q_0: *Which padlock is the most secure?*

In Module 1, participants' work centred on exploring ways of organising and enumerating the code systems associated with each padlock. Unlike in the secondary context, where the mathematics was already well known, here the activity created genuine challenges: students had to discover and articulate counting techniques for themselves. The process revealed typical difficulties, such as keeping track of the elements in large collections, avoiding duplications, or justifying the completeness of their enumerations. The physical padlocks played an essential role, not only as concrete supports but also as models of more general counting problems.

Because these challenges were mathematically meaningful for the participants, the study naturally extended beyond the initial stage. Starting from a highly complex collection, students were first required to organise the elements, then to enumerate them systematically, and finally to recount the total number of possibilities. This progression led to the introduction of additional constraints (e.g., codes without repetition, padlocks with more variables), the classification of padlocks according to the structure of their code-counting strategy, and the generalisation of counting techniques. Through this process, prospective teachers experienced how new tasks and models emerge from inquiry and how counting strategies can be gradually formalised into algebraic combinatorial formulas.

In Module 2, participants did not engage in a full praxeological analysis but they were able to construct an epistemological reference framework. This framework integrated the diversity of counting strategies that emerged and the difficulties linked to the enumeration and structuring of the collection to be counted (Briand, 1999). All these strategies were collectively analysed in class, providing a richer perspective on how to support students in developing tools to tackle counting tasks. In this case, the SRP-TE concluded at this stage, as Modules 3 and 4 were not carried out.

We observed that the participants' initial epistemological references were strongly shaped by their school experience. This became evident when the instructor addressed several groups with questions such as "*Why do we multiply here?*" or "*What makes multiplication work in this situation?*". Many groups struggled to provide any technological or functional justification, revealing their reliance on memorised procedures whose reasons of being had never been made explicit. This highlighted the predominance of an institutional epistemology centred on unquestioned praxis, where operations are applied without explicit justification. The collective reflection that followed emphasised the importance of learning to question those mathematical gestures taken for granted and constituted an entry point for introducing a research-informed logos capable of supporting more explicit and theoretically grounded justifications.

Discussion

The comparison between the two implementations presented in this paper highlights how a similar design can take on different functions depending mainly on participants' prior mathematical knowledge. In primary teacher education, the SRP-TE worked as a Type B device, since it was explicitly designed for teacher education and confronted prospective teachers with the challenge of organising and enumerating highly complex collections. This generated the need to reconstruct counting praxeologies and to discover broader problem-solving strategies, which were progressively articulated and formalised into general techniques. In contrast, in secondary teacher education the same SRP-TE functioned as a Type A device: it was adapted from a school-oriented design and, since

the mathematical content was already well mastered by the participants, the focus shifted directly to Module 2. This allowed them to concentrate on the analysis of modelling, the generalisation to algebraic expressions, and the study of didactic phenomena within the framework of the ATD.

One important contribution of SRPs-TE lies in the "mathematical phenomena" they make visible. In the Type B example, the activity allowed prospective teachers to experience the enumeration difficulties that arise when organising and enumerating complex collections (difficulties that would otherwise remain hidden). In the Type A example, the same design revealed phenomena related to modelling, such as the evanescence of mathematical models or the lack of an institutionalised logos. These observations invite reflection on the conditions under which mathematical knowledge is sustained, transformed, or disappears in school practice, and illustrate that some mathematical phenomena are more likely to emerge depending on the type of SRP-TE implemented.

Another key contribution of SRPs-TE is the variety of professional tools they allow prospective teachers to develop. Among them, we can highlight question–answer maps and praxeological analysis, which enable teachers to problematise school mathematics and its *raisons d'être*. Among the didactic tools, SRPs-TE mobilise the notion of milieu, didactic variables, and didactic contract (Brousseau, 1998) as well as the management of inquiry processes. At the same time, these devices foster the development of professional competences such as the ability to analyse, assess, and design lessons, and to engage in critical reflection on practice.

An essential dimension of SRPs-TE is the dialectic between theory and practice through the ecological analysis. On the one hand, participants engage with genuine problems that generate the need for new techniques; on the other hand, they reflect on the *raison d'être* of these techniques, analyse the institutional conditions of their emergence or disappearance, and consider their role in the curriculum. In this way, SRPs-TE create a shared medium for teacher educators and future teachers to discuss of the conditions and constraints of classroom practice and to explore possible alternatives.

Finally, some challenges and limitations need to be acknowledged. In particular, the complete implementation of SRPs-TE, especially Modules 3 and 4, is often difficult to achieve within the constraints of pre-service teacher education. Nevertheless, our experience confirms that the initial modules (0, 1, and 2) are crucial for creating a favourable environment for discussion, where prospective teachers can engage in the dialectic between theory and practice and begin to appropriate professional tools transferred from research. In this respect, Type B devices show their strength by confronting prospective teachers with complex problems that foster the construction of epistemological reference frameworks, even when the process cannot always be extended to classroom implementation. Type A devices offer the advantage that students' productions from real classrooms can still be brought into the analysis, serving as a substitute for Module 4 and providing valuable material for collective analysis and reflection.

Concluding remarks

We consider that the contribution of SRPs-TE consists mainly of the choice of paradigmatic study sequences in the form of inquiry processes with multiple aims: first, to detach from the school mathematics and pedagogical dominant perspectives by experiencing an inquiry process that rarely

occurs at school; second, to adopt the teaching sequence as an epistemological model of a specific broad domain of school mathematics; third, to use this model to analyse existing teaching proposals, adapt and implement them; and finally, to identify the conditions that make these teaching sequences viable and the constraints that limit them. The proposal of SRPs-TE is not unique in using learning situations as the central focus of teacher education, where trainee teachers experience, analyse, and adapt them to specific school conditions. Researchers such as Alfaro (2023), Rojas et al. (2023), Strømskag (2018) and Blomhøj et al. (2022) base their work on comparable training strategies, where students are also invited to experience tasks or learning situations before analysing them. A serious work of characterising, comparing, and articulating (or contrasting) the existing variety of this kind of teacher education proposals is needed, and we consider this a future line of research.

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When the mathematical concept is the pedagogical argument: Teachers' productive pedagogical-mathematical discourse

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This article explores a possible conceptualization of productive pedagogical–mathematical discourse in mathematics teacher communities as a basis for on-the-job learning. We extend the productive pedagogical discourse framework (Lefstein et al., 2020) by adding a seventh feature, disciplinary mathematical discourse, and by treating the seven features as a dynamic network. In a qualitative, discourse-analytic study, two case studies are presented. In the first, talk consolidated into a type of discourse we consider as pedagogical–mathematical as well as productive. In the second, a potential for a similar pattern of talk was identified but not fully realized. We conclude that adopting such pattern of pedagogical–mathematical discourse, in which the mathematical concept constitutes the pedagogical argument, provides concrete guidance for one type of on-the-job teacher learning.

Keywords: Mathematical discourse, pedagogical discourse, on-the-job conversation, teacher learning.

Objective

While most research on mathematics teacher learning is focused on formal professional development (PD) activities, there has been a growing recognition that many teacher learning opportunities occur (perhaps mainly) within the routines of daily work, across formal and informal contexts (Ehrenfeld, 2022; Horn, 2005; Lefstein et al., 2020). As part of these calls to shift the field's attention to teacher learning opportunities within "on-the-job" conversation, scholars suggested aspects of professional features considered "productive" within the field of mathematics education (Horn & Garner, 2022; Van Es & Sherin, 2008) and outside of it (Lefstein et al., 2020; Little, 2003), each illuminates different types of teachers' opportunities for learning, obscuring others. One robust framework, that integrates such dimensions of talk is "productive pedagogical discourse" (PPD; Lefstein et al. 2020). It recognizes the centrality of workplace embedded professional learning and defines it as collaborative discourse with six characteristics, in which teachers address classroom-based problems of practice, analyze practice using rich representations of teaching, share diverse perspectives, and develop inquiry stances and new pedagogical understandings, while combining collegial support with constructive critique. Nevertheless, Lefstein et al.'s (2020) framework attends to features of productive discourse without explicitly addressing the disciplinary context of the subject matter. Inspired by prior work that highlight the importance of the discipline in pedagogical talk (Ball et al., 2008; Horn & Garner, 2022; Van Es & Sherin, 2008), the present exploratory study proposes to extend the robust PPD framework for the context of mathematics education by adding a seventh feature, disciplinary (mathematical) discourse. That is, the talk that engages with the content of the discipline. This feature can be seen as a reflection in discourse of Mathematical Knowledge of Teaching (MKT; Ball et al, 2008). In this paper, when we want to get more specific, we use the term mathematical discourse to refer to talk that engages with explicit attention to mathematical concepts

(which can be seen as a reflection in discourse of Ball et al.'s (2008) notion of specialized content knowledge). Importantly, we treat the seven features not as a closed list of components but as a dynamic network of relations through which productive pedagogical discourse is constituted which features are not merely present or absent but interconnected with one another.

The overarching aim of this study is to articulate and understand ways in which professional conversations of mathematics teachers can be considered as productive pedagogical-mathematical discourse (PPMD). To this end, we investigate two meetings of a school-based community composed of six elementary mathematics teachers, who discussed two classroom situations, first, around the notion of the estimation, and second, around geometrical concepts. Our research questions are: (RQ1) *What are the interactions between disciplinary discourse around the mathematical concepts and the six features of pedagogically productive discourse?* (RQ2) *In what ways do the different interaction types between disciplinary and pedagogical discourse constitute productive pedagogical-mathematical discourse?*

Conceptual framework

Productive pedagogical discourse

In this study, we adopt the notion of pedagogically productive discourse (Lefstein et al., 2020) as an analytic lens for understanding the character of discourse in teacher teams and for identifying how such discourse contributes to professional learning opportunities. This form of talk is argued to afford teachers with opportunities to examine their instructional work in depth, process classroom experiences, and develop adaptive expertise. In elaborating what PPD is productive for, Lefstein et al. argued it can support the cultivation of professional judgment, the expansion of teachers' instructional repertoires, and the development of interpretive sensitivity toward classroom events.

At the core of the framework, to identify pedagogically productive discourse, the authors proposed six core dimensions: (1) a focus on real problems of practice arising from classrooms; (2) evidence based pedagogical reasoning and; (3) grounding in rich representations of practice; (4) multiple voices and perspectives within the discourse; (5) encouragement of revisiting assumptions and generating creative, flexible approaches to content, students, and practice; and (6) a balance of collegial support with principled critique. Beyond this, our analysis examines how disciplinary mathematical discourse is interwoven with them. This extension results in a preliminary conceptualization of PPMD as emergent patterns of interactions between the seven features, a conceptualization we wish to further explore in this paper and beyond, in our future work.

Practical Rationality

The Practical Rationality framework (Herbst & Chazan, 2011, 2023) is a descriptive theory of mathematics teaching which foregrounds the interplay among teacher agency, sense-making, and structural conditions in teaching and teacher learning. In brief, Practical Rationality explains mathematics teacher reasoning and actions based on two main elements: the norms of instructional situations, and the teacher reasoning and actions, mainly when actions depart from the situational norm. In this paper, we draw on a specific element of the framework that was articulated in Herbst and Chazan (2011): a representation of teachers' arguments about instruction, adapted from

Toulmin's argumentation layout, in which *circumstances* function as data and the *suggested teaching action* as the claim, supported by *warrants* and *backings* (see Figure 1; Herbst & Chazan, 2011; Toulmin, 1969).

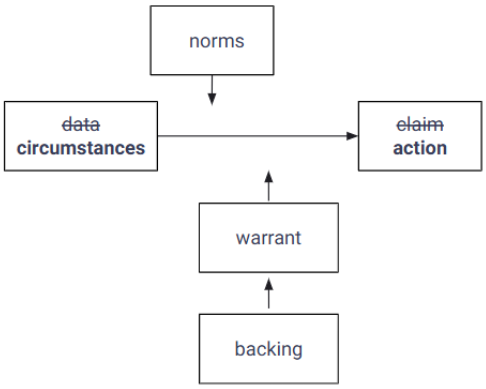

Figure 1: A practical argument using Toulmin's layout (adapted from Herbst and Chazan, 2011)

Methodology

This exploratory study employs a qualitative, discourse analytic approach to investigate and understand the learning opportunities embedded in productive pedagogical-mathematical discourse (PPMD) within a school-based professional learning community (PLC) of mathematics teachers. The study population comprises six elementary mathematics teachers, two of whom are special education teachers, and one who serves as the mathematics team coordinator and PLC facilitator. We acknowledge that prominent frameworks in mathematics education explicitly use disciplinary discourse; for example, the noticing framework centers on identifying, interpreting, and responding to students' thinking during instruction (Van Es & Sherin, 2008). For another example, a pedagogical judgment framework by Horn and Garner (2022), examines teachers' conversations by integrating action, reasoning, and responsibilities. Lefstein et al.'s (2020) framework of pedagogically productive discourse adds and delineates additional features of collective pedagogical discourse as a foundation for professional learning, however, it does so without specific attention to the discipline. In this study, we propose an extension to the PPD framework in which pedagogical disciplinary discourse serves as a supportive anchor for professional conversations.

Data collection

This study is part of a broader research project, in which our team collaborates with an Israeli PLC Network to examine the relations between different sites of ongoing teacher learning through conversations. Data for this study were gathered in a Hebrew-speaking primary school in central Israel, where 14 meetings of the mathematics PLC were held across the year. Thirteen meetings (all but the first) were video and audio recorded in full using two cameras and two microphones from different angles, enabling precise documentation of events and discourse. Each meeting lasted approximately 45 minutes. In most meetings, the teachers discussed both mathematical and non-mathematical topics. For the exploration of PPMD, we chose episodes from two meetings (Week 3 and Week 6) which were selected through a data reduction process: constructing content logs for all meetings, marking segments that addressed problems of practice, and finally, choosing two episodes that seem to illustrate numerous features of both PPD and mathematical discourse.

Data analysis

The data reduction process resulted in two discourse units: the first, approximately six minutes long, centered on the concept of "estimation." The second, approximately ten minutes long, included talk around the mathematical concepts "angles," "diagonals," and "sides." For the analysis of the two units, we drew on the Practical Rationality version of Toulmin's model (Toulmin, 1969), as applied in the work of Herbst and Chazan (2011). This model provides a framework for analyzing the structure of arguments and the justifications for suggested pedagogical actions. The model helps us situate argumentative moves within the seven PPMD dimension, by portraying the second and fourth features as the core of the visual representation: (2) evidence based pedagogical reasoning; and (4) multiple perspectives within the discourse.

Operationally, to answer RQ1 (*What are the interactions?)*, we coded warrants and backings as follows: boxes with solid borders mark elements explicitly articulated in the discourse; boxes with thin dashed borders indicate elements that were implicit in participants' talk (i.e., analytically inferred but not stated); boxes with thick dashed borders (used only in the second case study) denote warrants or backings we, as authors, proposed to strengthen the mathematical grounding, illustrating completions that were not present in the talk nor inferred (see Figure 2). This aspect of the visual representation is not part of the original work of Herbst and Chazan (2011). While RQ1 is more descriptive in nature, RQ2 is more interpretative. To answer RQ2 (*In what ways do the different interactions constitute PPMD?*), we interpreted the unique interaction patterns that were found (or found missing) in RQ1, suggesting how they might constitute as learning opportunities, which lead to developing a more concrete notion of PPMD. Reliability of interpretation was ensured through collaborative analysis among the authors and through discussions with other researchers.

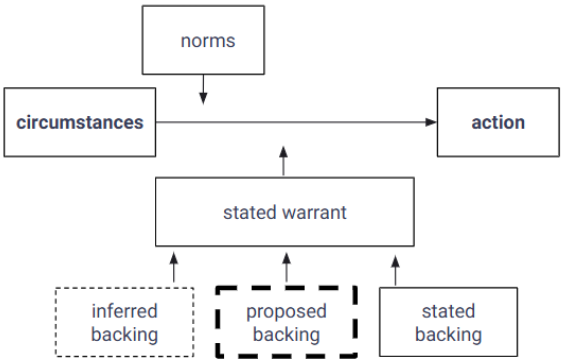

Figure 2: Adapted Toulmin-based argument mapping

Findings

This section comprises two parts. First, for each case we code the talk using Toulmin's argument model. Episode 1, centered on the concept of estimation, exhibited a teacher who built on the definition of estimation as a warrant and as a backing for her suggested pedagogical actions. We consider this discursive pattern as a form of PPMD. Episode 2, focused on geometric notions, showed a partially realized potential for such discourse pattern. Accordingly, we mark points where mathematical warrants and backings could have been made explicit to surface that potential. Finally,

we unpack the relations between pedagogical and mathematical discourse, using the estimation episode to illustrate how these links generate professional learning opportunities.

Episode 1: Using the structure of the mathematical concept as a warrant

Episode 1 is a 06:47 minute-long conversation from week 3 PLC meeting. The meeting started with Noa (the leading teacher) presenting to the other four teachers, a task she gave that same day in Grade 2. Within the task, the students (and now the PLC teachers) had to guess how many chocolate coins are within a jar in a projected photo (see Figure 3). The leading teacher then invited them to try estimate themselves. After a round of estimations, she added five hints, one by one (e.g., less than 30, odd number, without the digit '1'), allowing them to change their answer as they go, eventually realizing it is 25. Following the presentation of the task, the moderator Noa shared that in her lesson that same day a child was crying because of the task, saying "I do not succeed in anything". After sharing this story, Noa asked the group what they think about the task.

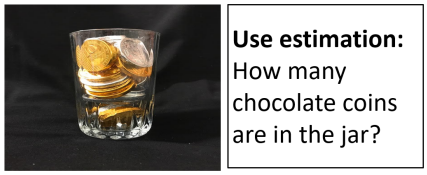

Figure 3: Estimation task, chocolate coins in a jar

From the discourse two central perspectives arose. The first perspective of the teacher Yael emphasized the suggested action of splitting the activity into two steps. At the first suggested step, students would need to understand that being close to the real number of coins is considered a correct answer (i.e., a good estimation). Then, Yael suggested that only in the second step, the teacher should explain that now the hints will focus us on the exact number. The second perspective, mostly represented by the teacher Hadas, opposed Yael's focus on splitting the task into two parts since the answer 25 is in some ways better. Instead, Hadas emphasized that the goal of estimation is to get as close as possible to the right number, approving the current task design as "good teaching".

The following brief 10-turns transcript is intended to illustrate for the readers the gist of the two perspectives that developed as response to Noa's prompt.

70	Yael:	It's not about who guesses the correct number; at first it's about who is closest to the correct number.
71	Hadas:	Yes, but in the end.
72	Yael:	I'm a bit insisting on this, but estimation is approximation.
73	Hadas:	Right.
74	Yael:	Afterwards, the second task is the exact number.
75	Noa:	Precisely.
76	Yael:	And that's what can frustrate a child, because if I was close…
77	Hadas:	But how great it is if I hit the number exactly?
78	Yael:	But even if I was close, I was right!
79	Hadas:	But they asked for an estimation, and estimation is to approximate, it is not necessarily that you don't have to hit the right number…

Figure 4 describes the structure of teacher`s Yael arguments across the 6:47 minutes episode, which focus on the situation of the crying student in class. As we mentioned, the action Yael proposed was to make clearer the division of the task into two parts: a first step where only estimation is needed,

and a second step, where there are intentional cues of exact number. Her opinion was based on three main warrants: The first, which illustrates what we mean by using the structure of the mathematical concept as a warrant, saying "Because in the estimation you are asked 'approximately'" and backed her explanation by saying "But if I was close I was right because estimation is to approximate." The second socio-emotional warrant was based on the need to strengthen the feeling of success for students who were close to the correct answer when she said: "Full credit should also be given to those who were close." The third warrant concerned her analysis of the task design and its two stages which require different guidance, first, a general estimation, and then narrowing the estimation using hints. As Teacher Yael explained: "In the first stage it's estimation; you don't have to hit the exact number, and then the hints are intended to focus us on the precise number." Each of these warrants was accompanied by supporting backings from Yael, as presented in the model (see Figure 4).

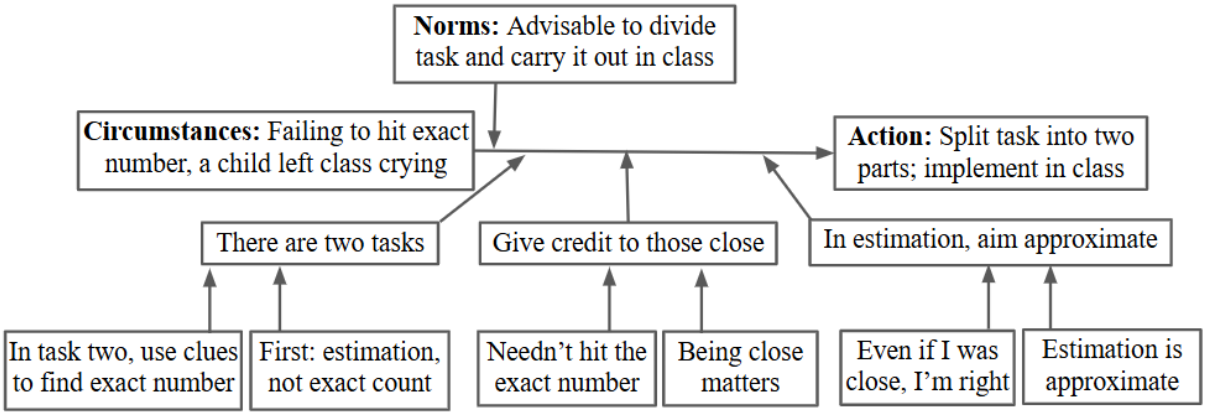

Figure 4: Analysis of Yael's argument: Concept of estimation

Episode 2: Reading together a classroom transcript about sorting quadrilaterals

We now shift to present an analysis of from a different conversation 30 minutes into Week 6 meeting. Before the Episode, Noa the moderator asked the participants to read together, in turns, a short transcript (about five minutes) from a fifth-grade geometry lesson on sorting quadrilaterals and use a hand-out with a list of teaching practices to explore it. In the classroom transcript, students just moved from working in small groups to a whole class discussion in which each group suggested ways to classify the shapes. The first group suggested sorting the quadrilaterals by types of angles. The teacher asked what they had sorted by and the students replied acute and obtuse angles, reported finding four subsets of quadrilaterals. The second group suggested a criterion based on diagonals. The teacher in response clarified that this criterion does not distinguish cases, since all quadrilaterals have diagonals (we will refer to this response again below). Finally, the third group focused on properties of sides: such as parallel, non-parallel, and opposite, and the class examined examples accordingly.

Within the PLC, after the reading, the moderator Noa clarified the aim of the discussion: "Now we have a task, I want you to try to determine which practices the teacher used from the list and which practices we can suggest she use." During the discussion, a critique of the teaching emerged among the teachers towards the teacher's facilitation of classroom discussion. The PLC teachers raised doubts about students' understanding of core concepts (e.g., diagonal; parallel/opposite sides) and they suggested the teacher could pause the task and clarify the concepts. For example, the teacher

Yael said, “I’m saying, even if your plan says so, pause for a moment, the children don’t understand. Let’s see whether they understand what a diagonal is, let’s see whether they understand what the concepts are”.

Figure 5 depicts the argument structure of the teachers around the mathematical concept of diagonals in the quadrilateral-sorting task mixed with our own proposed warrants and backing, if the conversations were to follow the pattern from the prior episode (i.e., structures of concepts are used as warrants and backings) which we considered as PPMD pattern. The circumstances are that despite signs of misunderstanding, the whole-class discussion continued without a pause. The collective proposed action was to pause briefly, define what a diagonal is, then resume sorting. The analytically proposed warrant is that classification of quadrilaterals rests on properties of diagonals rather than on their mere existence. Stated backing from the community appears in Teacher Yael’s comment, “we could try to help them see they were wrong in a different way, do not just say you are wrong, she did not even tell them why they were wrong or what,” which underscores the need for reasoned feedback and for making the mathematical rationale explicit. Additional analytically proposed backing (by the authors) is to specify the disciplinary properties that would serve as diagonal based criteria. The model illustrates a pathway toward pedagogical and mathematical discourse and the potential deepening of pedagogical arguments towards a PPMD where the warrants are rooted in the mathematical concepts (see Figure 5).

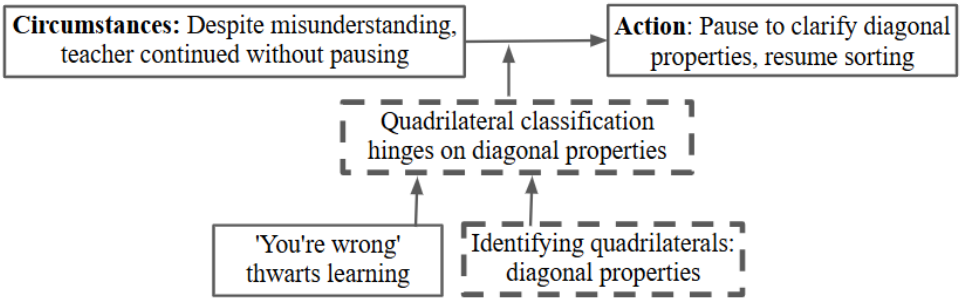

Figure 5: Teachers’ argument analysis: Concept of diagonals

Discussion

Answering RQ1, we examined how the mathematical discourse in these two episodes interacted with the six features of PPD. We found that the two episodes we analyzed here included, to certain extent, all features of PPD side by side with mathematical discourses. However, these interactions were different. In terms of PPD, both episodes (1) focused on real classroom problems of practice, (2) included pedagogical reasoning, (3) were grounding in rich representations of practice (task design with a story from the classroom in the case of the estimation episode, and enacted transcript in the case of the geometry concepts), (4) included multiple perspectives and disagreements, (5) generated ideas for new pedagogical actions, and (6) balanced support with critique. The nuanced difference that the analysis brought forth is that in the estimation segment, the disciplinary meaning of estimation linked PPD features to content, that was at the center of the reasoning, perspectives, and suggested pedagogical suggestions, through the warrants and backings. RQ2 asked in what ways different interaction types between mathematical discourse and pedagogical discourse constitute PPMD. Our

interpretation of the interactions found in RQ1, is that richer suggested actions and reasoning (and thus richer learning opportunities) emerged in the estimation episode, when mathematical discourse became a central resource that linked the PPD features to content. We argued that the unique configuration in which disciplinary meanings are made explicit and tied to reasons, perspectives, and suggested actions, is a concrete and useful pattern of PPMD. Based on these preliminary episodes, we suggest that further understanding such patterns of PPMD would allow us to guide rich mathematics teacher learning opportunities in different contexts of teachers' professional conversations. We hope that such concrete patterns can inform the design of teacher learning (for example, in PLCs) so that conversations intentionally support such learning opportunities.

Acknowledgment

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TEACHERS

Three first-time mathematics mentor teachers' mentoring perspectives while attending professional development for area specialised mentoring

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Mentors and their practices influence preservice teachers' learning to teach deeply. As being an effective teacher does not guarantee being an effective mentor, professional development opportunities should be provided to mentors. Mentoring perspectives of three first-time mathematics mentor teachers were examined as they attend an online professional development prepared according to the Mentoring for Area Specialised Teaching (MAST) Model. Pre, middle and post interviews conducted with participant to be analysed to reveal change in their mentoring perspectives. Perspectives of all participants changed in the direction of more mathematics teaching focused mentoring practices while the change in their mentoring practices were centred around different aspects. By this study, we explored how mathematics mentor teachers unpacked their pedagogical content knowledge to guide preservice teachers for learning to teach.

Keywords: Teaching practicum, mentoring, professional development, pedagogical content knowledge.

Introduction

Teacher education is a continuous process that blends theoretical knowledge with expertise gained through practice. In this process, neither a lack of theoretical knowledge nor a lack of practice can be overlooked (Darling-Hammond, 2000). One of the most important stages of preservice teacher education is the teaching practicum, as it serves as a bridge between theory and practice for preservice teachers (PSTs) (Azkiyah & Mukminin, 2017). In this context, PSTs in Türkiye attend practice schools for two semesters, each consisting of 12 weeks and 6 hours of training, during their senior year. In practicum schools, mentor teachers are responsible for supporting PSTs in their learning to teach process.

In the literature, mentoring is defined as an experience within the teaching profession but distinct from teaching itself (Tillema et al., 2015) because there are many points where mentoring differs from teaching (Achinstein & Athanases, 2005). At this point, it is clear that mentor teachers need professional support while working with PSTs in the process of learning to teach. In Türkiye, the Ministry of National Education provides 36 hours of training to teachers who will become mentor teachers, and this training focuses on the general mentoring practices. However, it has been observed that this three-day program is not addressing any aspect of area specialised teaching. In other words, it is observed that the development of mentor teachers is still supported by non-area specialised knowledge transfer or theory-practice connection approaches. Since aim of mentor teachers to improve PSTs' knowledge and skills for teaching mathematics, it's imperative to consider PCK.

Mathematical Knowledge for Teaching (Ball et al., 2008) defines PCK in three parts: knowledge of content and students (KCS), knowledge of content and teaching (KCT) and knowledge of content and curriculum (KCC). The mentoring program for mathematics mentor teachers should address all three of these knowledge types.

For this research, we studied perspectives of mentor teachers attending an online professional development: Mentor Teacher Professional Development (MeT-PD). This program was developed for mathematics, physics, chemistry and biology teachers who were mentor teachers for high school PSTs for their disciplines. Mentor teachers were grouped based on their disciplines because the purpose was to improve participants' area specialised mentoring practices. The MeT-PD program was designed based on the Mentoring for Area Specialised Teaching (MAST) model. Aslan-Tutak et al. (2023) developed the MAST Model, about effective mentoring in area specialised teaching, by considering the Situated Learning Theory (Lave & Wenger, 1991). The researchers developed the model by collecting data from three stakeholders in the practicum process: mentor teachers, PSTs, and teacher educators. The MAST Model has three dimensions: Triad Collaboration, General Pedagogy, and Area Specialised Teaching. The first dimension of the model, *Triad Collaboration*, concerns the practicum process requirements and the role and characteristics of mentor teachers within this process. The second dimension, *General Pedagogy*, is about the general pedagogy of teaching practices that mentor teachers in all areas conduct with their PSTs. The last dimension, *Area-Specialised Teaching*, covers specific mentoring practices that science and mathematics mentor teachers conduct with their PSTs.

The MeT-PD can be considered as an opportunity for mentor teachers to learn mentoring for 'learning to teach'. From mentor teacher perspective it's a learning process, *unpacking their pedagogical content knowledge* for guiding PSTs. To examine how mentor teachers' perspectives changes while attending the MeT-PD, we decided to focus on three of the participants who were first-time mentoring PSTs. The aim of this paper is to examine these three teachers' learning to mentor processes through mentoring practices before and after completing professional development. Thus, the research question is as follows: *How do three first-time mentor teachers' mentoring perspectives change during the Mentor Teacher Professional Development (MeT-PD)?*

Methods

This paper aims to examine the changes in the first-year mentor teachers' mentoring perspectives before and after completing the MeT-PD. In line with this aim, a qualitative multiple case study was conducted; a case study is an approach researchers use to examine a bounded system to develop an in-depth understanding of a case (Creswell, 2013). In this study, three first-year mathematics mentor teachers who participated in the MeT-PD were chosen as the three cases. The MeT-PD was developed by researchers based on two theoretical frameworks: online professional development (Elliott, 2017) and area specialised mentoring model (MAST) The MeT-PD consists of 10 sessions in total, and it was conducted in the 2023-2024 Fall semester. The first session was conducted face-to-face considering suggestions for online PD, to form a community of mentors with topic of general roles and responsibilities of mentor teachers in Turkish setting. Participants also discussed about how to manage internship process as well as how to help PSTs to feel belonging to school. The next eight

sessions conducted via Zoom for one hour each week with a focus topic on area specialised mentoring. The final session included a reflection-discussion component to reinforce participants' learning and deepen their thinking. The program was conducted by the authors of this paper.

Participants

A total of 11 mathematics mentor teachers from three cities participated in the MeT-PD. Four mentor teachers worked at private high schools, while the rest worked at public high schools. Six of the 11 mentor teachers were female, while five were male. The mentor teachers' professional experience ranged from eight to 25 years, while their mentoring experience ranges from zero to 21 years. For this study, three mentor teachers, *Lily*, *Sage* and *Erika*, with no prior mentoring experience, were selected using a purposive sampling method. Two of the mentor teachers included in the study worked at private high schools in Istanbul, while one worked at a public high school in Trabzon. The participants' years of teaching experience range from nine to 22.

Data collection

The data was collected through video-recorded online semi-structured interviews at three different times. The semi-structured interviews were prepared to reveal the experiences and perspectives of mentor teachers. The pre-interviews were conducted at the beginning of the MeT-PD to learn what mentor teachers planned to do with PSTs during the semester, in addition to their motivation for being a mentor. The mid-interviews were conducted around the middle of the semester, after the first teaching practices of PSTs, and mentor teachers were asked about their experiences with PSTs to reveal their practices and perspectives. One participant, Sage, missed two sessions of MeT-PD due to a family emergency and she also missed the time of mid-interviews. To capture her experiences mid-interview questions were directed to her at the beginning of post-interview. The post-interviews were conducted at the end of the MeT-PD. In addition to asking about the practices they carried out throughout the semester, participants were asked to describe their experiences regarding their interactions with mentees and mentoring practices described in the third dimension of the MAST model. The purpose of using MAST Model for the last interview was to examine whether discussions during MeT-PD were used by participants.

Data analysis

Once the recordings were transcribed, the researchers read the transcribed interviews in detail to familiarise themselves with the data. Considering the iterative and cyclical nature of qualitative analysis, we used open coding and axial coding strategies to analyse data (Leplaa et al., 2025; Vollstedt & Rezat, 2019). First interview findings reveal participants plans and expectations while the mid interview addressed how they started mentoring. The open codes generated from post interview were deductively aligned with the MAST Model to provide detailed information about mentoring practices. To ensure validity and reliability, the first two authors completed the analyses individually by using MAXQDA. Sample codes will be shared in findings section along with explanations for mentoring perspectives. Researchers gathered to compare the matched open codes, achieving consensus in the matching process.

Findings

The findings will be shared in three according to phases of the interviews (pre, mid and post) to depict change in participants mentoring perspectives.

Mentor teachers' perspectives before attending MeT-PD

Lily has been teaching for 22 years, but this was her first year in mentoring because there was another teacher at her school taking the responsibility of mentoring mathematics PSTs. Lily expressed being very excited to have the opportunity to be an official mentor. In previous years, there were PSTs observing her instruction, but it's her first time being responsible for managing the PST process of learning to teach. The analysis of her responses to pre-interview questions regarding her plans for mentoring shows her focus on managing the practicum process with the partnering university along with her views on how PST improve their skills.

> Lily: I will provide students (PSTs) with a complete learning experience…They need to learn to pull themselves together without panicking.

As she was describing her plans, she mentioned about classroom management and some general pedagogical aspects without mentioning any aspect of PCK.

The second mentor teacher, Sage has been a mathematics teacher for nine years, and it was her first year of being a mentor officially in her school, a private high school offering both the Turkish national curriculum and Advanced Placement (AP) math courses. He has been teaching mostly AP Calculus courses in addition to some 9th-grade mathematics classes. During the pre-interview, he mentioned his learning process of being a mathematics teacher and the resources provided by his current school in regard to guiding PSTs to improve themselves.

> Sage: So, to be a good teacher, you need to have mentors who not only have mathematical knowledge but also whom you trust as people.

At the beginning of the MeT-PD, her focus was on management of the internship process as well as her expectations from PSTs to be ready to learn.

The third mentor teacher, Erika was an experienced mathematics teacher who has worked at a high school for high-achieving students for about 11 years. At the time of the study, she was working at a mixed-ability private high school for two years. Unlike Lily and Sage, she had no experience with PST at the schools that she worked. This may explain reason for her concentrating on expectations from PSTs and mentor roles during the pre-interview. She also discussed mentoring experience being a learning opportunity for herself while focusing PST experiences around introducing high school student characteristics.

> Erika: There was no need for students to do point test on graph but she also felt during class. I had actually told one part of the question her before, but I didn't want to give it to her like that, ready-made. So something like let them think about it a bit. Don't take it from me right away.

She was also the only participant to discuss mentoring practices for area specialised teaching by paying importance to improving PSTs PCK regarding teaching mathematical process skills.

Mentor teacher practices during MeT-PD

To capture the process of change in participants mentoring practices, the second interviews conducted around the middle of MeT-PD. After PSTs had done some practice teaching, there was a remarkable change in what Lily discussed. During the mid-interview, Lily shared how she provided various teaching experiences to PSTs while providing self-evaluation and reflection along with her feedback.

> Lily: Then, at the end (after PSTs' teaching practices), we critiqued them. First, I said, let's critique yourselves, see what was wrong. Then I said, let your friends critique you. They critiqued each other, then I told them what I had noted.

It was interesting to see that she would provide many teaching experiences to PSTs yet have limited direction for PST reflection. She focused on general pedagogical ideas and classroom management. Considering elements of PCK, it was clear that during her guidance of PSTs Lily couldn't address neither KCT nor KCS. Her well-intention mentoring practices centred around "being and acting like a teacher".

Erika's answers to mid-interview questions were coded to understand her experiences as a first-time mentor teacher. Upon completing half of the MeT-PD, she started to discuss about her understanding on how PST learn to teach. This finding was not surprising because Erika never had experience with PSTs. This experience of learning to become a mentor teacher might influence her to reflect on her practices to guide PST. Similar to her plan at the beginning of the practicum process, she emphasized discussing high school students characteristics and their mathematical thinking with PST.

Mentor teacher practices by the end of MeT-PD

After MeT-PD is completed, individual interviews conducted with participants to examine their mentoring practices. First two interviews were consisted of broad questions regarding their mentoring experiences. For the post interview, questions reflecting MAST Model themes were used in order to examine in which areas their mentoring practices changed. The following list summarizes findings from all three teachers' post-interviews with MAST Model analysis. The frequency of codes was reported in parenthesis as f1 for Lily, f2 for Sage and f3 for Erica. By using MAST model coding, it becomes possible to recognize similarities and differences among participants.

Table 1: MAST model frequencies from mentor teacher post-interviews

Dimension I	f1	f2	f3	Dimension II	f1	f2	f3	Dimension III	f1	f2	f3
Practicum Requirements	8	14	0	Teaching Profession	16	0	2	Content Knowledge	0	14	12
Expectations from Mentees	8	14	0	Students' Characteristics	0	16	20	Mentoring Practices about Teaching	36	22	34
Role of Mentor and Mentoring Perspective	10	18	16	Pedagogy of Teaching	18	5	20	Mentor-Mentee Interaction	46	40	68

Expectations from Supervisors	2	0	0	Assessment	10	6	0	Curriculum Knowledge	16	11	4
Mentor Characteristics and Selection	0	0	2	Classroom Management	8	0	2	Instructional Materials and Technology	6	18	10
								Student Ideas	0	8	4

Lily mostly mentioned how PSTs did 'practice teaching'. As her mentoring practices centred around providing many hours of practice, the codes from MAST Model were *Teaching in the Authentic Context* (*III.2.a)* and *Designing and Implementing Tasks (III.2.b)*. This finding is also in parallel with her emphasis on *Feedback* (*III.3.e)* in terms of *Mentor-Mentee Interactions*. As each theme was asked separately, she also explained how she discussed curriculum and instructional tools with PSTs. Even though she provided many hours of classroom practice, her feedback upon PSTs' practice teaching focused on general pedagogy (student characteristics and assessment), along with curriculum issues, with limited discussion on mathematical skills or student ideas.

As it was explained before, due to family emergencies, Sage couldn't attend two sessions of MeT-PD, and he missed the time for the mid-interview. Thus, the post-interview questions were a combination of mid-interview questions (general description of practices) and post-interview questions (MAST Model). Sage was still concerned about the management aspect of practicum process, but it should be noted that she started to discuss about learning from PSTs, having a reciprocal environment for learning.

While Sage did not mention any plans for specialised mathematics teaching mentoring practices at the beginning of the MeT-PD, his statements by the end of the program focused on practices to support PSTs' pedagogical content knowledge. He addressed *Mentoring Practices about Teaching* with emphasis on *Mentor-Mentee Interactions* of *Mentor Sharing (III.3.b)* and *Reflection (III.3.d)*.

> Sage: So, after all the teaching practices, this was the first question I asked anyway. How do you think the practice went? If you were to teach it again, how would you do it? She revised our thing (the lesson plan) a bit for the second lesson presentation and did it in a way that was connected to it. I gave her feedback on it. We talked about it, discussed it, and so on. So, it changed 2-3 times (the lesson plan).

It is important to note that, according to Sage, she worked with PST on *Curriculum Knowledge (III.4)* and *Instructional Materials and Technology (III.5)*. It was one of her major duties to support PST in learning about AP Calculus courses and preparing tasks and lesson plans for these courses. He would model using Desmos, asking PSTs to prepare tasks and implement them. Her discussions with PST centred around understanding mathematics as a teacher and teaching it with well-designed instructional materials.

The third participant Erika was intriguing case to examine whether she kept her mentoring practices focused on student characteristics or not. Different than other two participants, Erika's answers centred around *Evaluation*, *Reflection*, *Revision* (*III.2.c)* and *Teaching for Mathematical Skills* (*III.2.e)* codes.

Erika: We usually started by asking them for a self-assessment (after the teaching practices). Before that, in the lectures I gave, I always asked the candidates, please evaluate me. I wanted to encourage them a little.

She also discussed how she approached these practices by emphasizing *Mentor Sharing (III.3.b)* and *Reflection (III.3.d*). It should also be noted that she included all five themes, including *Curriculum*, *Instructional Materials (III. 4)* and *Student Ideas (III.6)*.

Discussion

The purpose of this research was to examine changes in first-time mentor teachers' perspectives while attending MeT-PD, which was designed to support mathematics mentor teachers in guiding PST learning to teach endeavour. There are changes in how all three participants practised mentoring with PST. Yet, based on findings, we can also discuss how different approaches were adopted by these participants. It was important to note that Erika was centring her practices on understanding student ideas and reflection. Sage's perspective indicates that his practices focused on designing instructional tasks and discussing mathematics with PSTs. While Lily provided a limited opportunity for focused reflection, rather many more hours of teaching practice.

Answers from all these mentor teachers indicate carefully planned mentoring practices carried out as they proceed in MeT-PD. Lily's approach resembles the repeated simulation practice of pilots. Providing many hours of practice also means limited time for reflection from both mentor teachers and PSTs. Lily may be planning on PSTs to discover some aspects of teaching as they practice more and more. On the other hand, Erika and Sage show mentoring perspectives with reflection and feedback. Different approaches of these mentors, focusing on student characteristics and instructional materials, can be explained by Shulman's definition of pedagogical content knowledge encompassing student ideas and representation of ideas (Shulman, 1986, p.9). Furthermore, we may argue that Erika and Sage were implementing MeT-PD mentoring practices by anchoring aspects of PCK; knowledge of content and students, knowledge of content and teaching, respectively (Ball et al., 2008). Mentor teachers were *guiding PSTs in learning to teach*, which means supporting their PCK development. Erika's mentoring practices were anchored around students' mathematical ideas while sharing insights for lesson plan preparation, reflection and feedback after practice teaching. Sage's mentoring practices, on the other hand, seemed to be anchored around representing mathematics, especially advanced mathematics, while she was guiding PSTs through feedback and reflection. As Ball and her colleagues (2008) defined teachers to unpack their mathematical knowledge for teaching, we realize that mentor teachers were *unpacking their pedagogical content knowledge* for guiding PSTs.

At the beginning of the program, these three mentor teachers were giving similar answers. All three participants actively participated in discussions during MeT-PD, but there were differences in their perspectives by the end of the program. Mentor teachers aim to model the learning to teach process for PSTs, which means that their own teaching characteristics would influence how they implement mentoring practices. The differences in their practices may be explained by their PCK readiness and their school environment. As a further study, PSTs' experiences may also be included to support findings from mentor teachers' interviews. Also, mentor teachers' feedback sessions with PST might be examined to deepen the analysis of how mentor teachers put PCK in action for guiding PST learning to teach endeavour.

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Teachers' instructional visions and self-efficacy as meta-design principles to foster high-quality mathematics instruction

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Professional development initiatives in mathematics education often struggle to achieve shifts in teachers' instructional practices. This methodological paper argues that two theoretical constructs - teachers' instructional visions and teachers' self-efficacy - can be used as meta-design principles to support such shifts within an educational design research study. Teachers' instructional visions provide a normative orientation towards high-quality instruction, while self-efficacy addresses teachers' perceived possibilities for action in their situated practice. This paper extends the emerging conversation on teachers' learning in professional development, giving an example of how the process can become more sustainable. The contribution lies in advancing methodological awareness of how to design and carry out teacher professional development interventions, bridging the gap between aspirational goals and teachers' situated practices.

Keywords: Professional development, educational design research, teachers' instructional visions, self-efficacy, design principles.

Introduction

Research (Cobb & Jackson, 2021; Mosvold & Fauskanger, 2014) emphasizes that high-quality mathematics instruction cannot be achieved solely through new tasks or materials but necessitates shifts in teachers' instructional practices and in their underlying conceptions of what constitutes high-quality teaching. Munter's (2014) construct of teachers' instructional visions has been used to document teachers' evolving perspectives in the implementation of high-quality mathematics instruction. While these approaches provide important insights into what changes occur, there remains a methodological need to also consider how theoretical constructs can be used in the design phase to guide teachers' learning and shifts in instructional practice. Recent work stresses the importance of focusing teachers' learning processes within professional development initiatives (Brehmer & Ryve, 2024).

Educational design research (EDR) offers a promising response to that by providing a methodological frame for iteratively designing, enacting, and refining principles in close collaboration with practitioners (Bakker, 2018). Within such a methodological frame, theoretical constructs can serve not only as analytical lenses but also as overarching meta-principles for the design and enactment of teachers' learning processes in instructional practice. In this paper, we propose to take a step further by using Munter's (2014) construct of teachers' instructional vision together with teachers' self-efficacy (Bandura, 1997) to address teachers' learning processes.

The aim of this paper is methodological: to examine how constructs can be used in an intervention process with a particular focus on an educational design research study. Specifically, the paper addresses the following research question:

How can teachers' instructional visions and self-efficacy be used as meta-design principles in an educational design research study aiming to foster high-quality mathematics instruction?

By addressing this question, the paper contributes to methodological awareness of how to design and carry out professional development.

Professional development

Research on mathematics teacher professional development is extensive, and several key features of effective professional development have been identified, including content focus, active learning, coherence, sufficient duration, and collective participation (Desimone, 2009). These features underpin established professional development formats, such as lesson study, professional learning communities, and coaching. Nevertheless, studies of professional development initiatives report mixed results, both in terms of effectiveness and in the focuses adopted (e.g. Goldsmith et al., 2014; Pellegrini et al., 2021). This suggests that while the identified features are important, they are not sufficient to fully explain how mathematics teachers' professional development impacts classroom practice.

In addition to identified factors that influence the effectiveness of professional development, with attention to both the content and the form of the initiatives, more recent studies increasingly emphasize the importance of focusing on teachers' learning (Brehmer & Ryve, 2024; Goldsmith et al., 2014; Kennedy, 2016). To build a more comprehensive knowledge base, further research is required that investigates the learning process directly, rather than concentrating solely on the overall effectiveness of professional development programs. While many studies examine student learning outcomes, comparatively little is known about how teachers themselves learn (Kennedy, 2016). There is still a lack of genuine insight into what teachers actually do within professional development initiatives, what motivates them, and how their professional learning unfolds in practice.

In line with this, Munther (2014) emphasizes that sustainable professional development requires a shared and evolving vision of high-quality mathematics instruction, developed collaboratively over time rather than imposed from outside. Teachers' learning is complex and cannot be reduced to merely adopting new instructional strategies. It involves rethinking what counts as effective teaching.

The importance of motivation as a factor for teacher learning is highlighted by several scholars. Kennedy (2016) emphasizes that while professional development may be mandatory, teacher learning is not. For professional development to lead to sustainable change, it must engage teachers intellectually and create a sense of meaning. In line with this, Kennedy (2019) argues that successful professional development is closely tied to teacher engagement and highlights intellectual engagement, meaningfulness, motivation, and participation as central factors for positive outcomes. However, Ryve & Lindvall (2019) argue that teacher engagement is strongly shaped by the position teachers are assigned within the development process. Teachers are often positioned as implementers of ready-made ideas, which can undermine engagement. To foster genuine commitment, the goals of the program must be coherent with teachers' own intentions. As coherence determines whether development efforts translate into changes in practice, it cannot be taken for granted but must be actively supported.

Another important dimension in understanding teachers' learning is research design. Kennedy (2016) advocates for designs that recognize the gradual and incremental nature of teacher learning, allowing new ideas to be progressively integrated into ongoing instruction. While studies of this kind exist, they are not yet predominant in the field. Building such designs into future research is therefore essential for capturing the dynamic processes of teacher learning and the complex ways in which professional development interacts with classroom practice.

Motivation, engagement, and coherence, while important, are not on their own sufficient to drive development. Nor is a carefully designed format enough. Mechanisms that actually bring about development are also needed. Brehmer and Ryve (2024) argue that previously identified features, though significant, do not in themselves constitute catalysts for learning. For teachers' experiences, knowledge, and beliefs to contribute to the development of practice, opportunities for active reflection are required moments where teachers can process and reconsider their understandings in relation to practice. In this sense, enactment and reflection emerge as additional crucial components for fostering positive outcomes.

In sum, while prior research has identified important features of effective professional development, less is known about the processes through which teachers actually learn and how such learning supports changes in instructional practice. To build a more comprehensive knowledge base, research is needed that examines the dynamics of teacher learning within professional development - focusing on motivation, engagement, enactment, and reflection, as well as on research designs that capture both incremental and fundamental shifts in teachers' instructional practice.

Educational design research

Educational design research (EDR) is particularly well suited for addressing complex educational problems in a holistic manner (Plomp & Nieveen, 2010). Rather than merely describing how things are or have been, EDR explores how they could or should be, making it, in Bakker's (2014) words, a kind of "science fiction" that envisions and tests possible futures. EDR typically engages with educational challenges where the desired forms of learning have not yet been observed in naturalistic settings, requiring the design and study of new, purposefully engineered environments (Bakker, 2018). As such, it involves the systematic design, development, and evaluation of interventions that not only aim to solve practical problems but also contribute to theoretical and methodological understanding of educational design and learning processes (Plomp & Nieveen, 2010).

While EDR itself is not a "method" but a methodological framework, it assumes a design-based and iterative engagement with practice and places methodological demands on both the generation and analysis of data. Design principles developed by researchers and teachers are not prescriptive formulas nor guarantees of success. Rather, they constitute hypothetical and context-sensitive knowledge that can inform and guide design in related settings (Bakker, 2018). In this way, their generalisability lies not in universal applicability, but in their theoretical transferability. That is, their ability to inform reasoning and decision-making in similar, though not identical, educational contexts. Design principles may operate both on a pragmatic level, directly connected to classroom practice, and on a meta level where theoretical constructs make a frame or an overarching guideline (Bakker, 2018).

Constructs to be used as meta-design principles

Against this background, the combination of two constructs - teachers' instructional visions (Munter, 2014) and self-efficacy (Bandura, 1997) offers a productive lens. Together, they provide insight into what teachers aspire to do and what they believe they are capable of doing in their specific contexts. In this paper, we do not propose a new theoretical construct. Instead, we combine them as complementary constructs.

In what follows, we outline each construct and then describe how they are integrated to inform the design of an EDR study.

Teachers' instructional visions

The concept of vision refers to the images and ideas teachers hold about what high-quality mathematics instruction should look like. Drawing on the work of Munter (2014) the construct of teachers' instructional visions is developed to both characterize teachers' current perspectives and to trace their evolving understandings over time.

The construct defines teachers' instructional visions as more than individual preferences; instead, they are understood as socially negotiated constructs that gain clarity and purpose through professional discourse (Munter, 2014). As Munter (2014) argues, effective professional development is rarely achieved without a shared and meaningful vision of high-quality instruction. Importantly, such a vision cannot be imposed or simply chosen. Rather, it develops as teachers engage collaboratively over time, making sense of teaching and learning together.

Building on Hammerness, Munter (2014) describes teachers' visions as evolving images of ideal teaching that extend beyond their immediate practice and reflect the gap between current routines and future aspirations. This distinction makes visions especially useful in interventions, where the goal is not only to understand current practice but to support its development.

The construct operationalises visions of high-quality mathematics instruction across three core dimensions: the role of the teacher, classroom discourse, and mathematical tasks. The framework offers different levels within each dimension to capture variation in practice. The construct does not prescribe shift but supports a reflective practice culture. However, the construct, while not a trajectory for teacher learning in itself, can be paired with a theory of action (Munter, 2014) - such as self-efficacy - to guide planning toward more coherent and sustainable professional learning.

Teachers' self-efficacy

Bandura's (1997) construct of self-efficacy provides an action-oriented perspective that makes it possible to anticipate and promote teacher learning and shifts. Self-efficacy beliefs are teachers' perceived capabilities to organize and execute actions required to achieve desired instructional outcomes.

This construct is essential not only for structuring intervention cycles but also for making decisions about how and when to introduce new instructional practices. In line with Bandura's (1997) work, teacher self-efficacy is understood to be developed and strengthened through four core mechanisms: mastery experiences, vicarious experiences, verbal persuasion, and emotional/physiological states

(Bandura, 1997). These mechanisms guide how professional development activities are designed and implemented. Mastery experiences are supported through repeated opportunities for teachers to successfully plan and enact instruction. Vicarious experiences are made possible when teachers observe their colleagues managing complex instructional moments, such as facilitating student reasoning. Verbal persuasion is incorporated through peer feedback and ongoing coaching that sustains teachers' growing competence. Attention is also given to emotional and physiological states, by fostering a collaborative culture that reduces stress and builds confidence. This theoretical foundation supports the view that a shift in instructional practice must be accompanied by a strengthened belief in both one's own ability and in students' capability to participate meaningfully in practice.

As such, self-efficacy is a practical construct for supporting teachers' learning trajectories. It allows an intervention to be responsive to where teachers are currently positioned in their practice and what supports they may need to progress. The construct thus complements Munter's (2014) trajectory-based construct of instructional vision by offering insight into how teachers might move between levels, not just in what they envision, but in what they perceive as doable in their everyday context.

In this way, the constructs of teachers' instructional vision and self-efficacy enable a systematic approach to teacher learning within an intervention and helps connect the study's goals with the practical realities and potential of the participating teachers.

Teachers' instructional visions and self-efficacy as meta-design principles

The constructs of instructional visions (Munter, 2014) and self-efficacy (Bandura, 1997) will function as meta-design principles that guide an entire EDR process. As Bakker (2018) emphasizes, meta-design principles operate at a higher level of abstraction than pragmatic, classroom-focused design principles. By consciously activating the constructs throughout the process, the aim to foster high-quality mathematics instruction is theoretically grounded while remaining closely connected to teachers' situated realities.

Meta-design principle 1: The construct of teachers' instructional vision provides a normative orientation towards high-quality instruction.

Meta-design principle 2: The construct of self-efficacy addresses teachers' perceived possibilities for action in their situated practice.

Instructional visions are understood as teachers' evolving ideas about high-quality instruction. These visions are neither static nor individually held but negotiated in dialogue with colleagues and researchers. Through structured tools such as Munter's interview protocols, teachers articulate what they value in mathematics classrooms in terms of mathematical tasks, classroom discourse, and the teacher's roles. Returning to these dimensions across the cycles provides coherence and reduces the risk of isolated change that is difficult to sustain. Teachers' instructional visions thus create a collective interpretive lens that frames the design work and provides a foundation for shared professional goals. By systematically contrasting these visions with what research characterizes as high-quality mathematics instruction, teachers become aware of potential gaps between their own horizons and established theoretical perspectives. This makes theory actionable rather than abstract.

Research does not prescribe a ready-made model to implement, but instead informs what plausible next steps might be from the teachers' current position. Munter's framework contains several levels that orient the work towards high-quality instruction, and testing new ideas in practice continually generates new questions, making this iterative movement between theory and practice an important source of emerging professional knowledge.

Self-efficacy, in turn, directs attention to how teacher learning activities can be structured so that teachers build confidence in their ability to enact high-quality mathematics instruction. Bandura's (1997) sources of efficacy together influence whether a person keeps going when facing challenges. Designing for teacher learning requires carefully constructed opportunities for teachers to experience success, observe peers, receive meaningful feedback, and reflect on their progress. This orientation aligns closely with teachers' situated realities, as it emphasizes gradual movement rather than abrupt shifts, allowing confidence to grow through stepwise accomplishments grounded in both theory and practice. As teachers accumulate evidence of progress through mastery experiences as well as vicarious experiences, their self-efficacy is strengthened. In this way, learning unfolds across the levels of Munter's framework, not as a linear staircase but as multiple perspectives that together create coherence about what high-quality mathematics instruction entails. Teachers often move back and forth between the three dimensions, and the diversity of participants' instructional horizons can open up new possibilities. If a colleague can imagine a certain instructional move, others may be encouraged to challenge their own assumptions.

The step from meta-design principles to specific design principles is not straightforward. Based on the literature on high-quality mathematics instruction, design principles are selected as practice-oriented tools that translate abstract visions of high-quality instruction into concrete entry points for teacher learning. By "entry points" we mean manageable aspects of classroom practice that teachers can experiment with, observe, and reflect upon. For example, an entry point could focus on eliciting student reasoning or using specific talk-moves to structure discussions. As teachers try out these entry points in practice, new insights often emerge. They may identify strategies that need refining, recognize the need to develop tasks that better support student reasoning, or find additional dimensions to explore. The design principles are not prescribed solutions but allows teachers to move flexibly across Munter's three dimensions of high-quality instruction and their current practice.

An iterative process, where testing, reflecting, and learning from practice are central and structures the professional development work. Within this process, various activities can be used to support teacher learning, such as analysing classroom episodes, video clips from their own practice, or peer observations, engaging in collaborative planning and rehearsal, or documenting reflections through fictive lesson observations, in logbooks or other artifacts. Crucially, these activities are not the contribution in themselves. Their purpose and impact arise only when they are consistently aligned with the meta-design principles of instructional vision and self-efficacy. It is through this alignment that teachers deepen their understanding of high-quality mathematics instruction while simultaneously building confidence to enact it in their own classrooms. Activities are meaningful when they are integrated, revisited, and adapted in light of the meta-principles, rather than treated as isolated exercises.

In this way, the professional development process is more than a set of discrete activities. It becomes a coherent, iterative learning journey in which teachers' evolving instructional visions and self-efficacy drive sustained instructional change, generating insights that feed back into both theory and practice.

Towards methodological awareness of how to design and carry out a professional development

The methodological proposal presented in this paper describes how teachers' visions (Munter, 2014) and self-efficacy (Bandura, 1997) can be used as meta-design principles when designing professional development interventions fostering high-quality mathematics instruction. The proposal assumes that meaningful instructional shift requires more than new methods or materials, it demands that teachers' own pedagogical resources, such as visions and self-efficacy, to be brought into the intervention process.

The proposal addresses the knowledge gap identified by Kennedy (2016) and aligns with the key features for effective professional development outlined by Desimone (2009), while advancing a deeper understanding of these features through their enactment in the design of an intervention. By designing activities that offer mastery experiences, vicarious learning, verbal persuasion, and attention to emotional and physiological states, the intervention fosters active learning and collective participation, which are two of the features highlighted, providing teachers opportunities to try out, reflect on, and shape their own professional development and practice. Drawing on Brehmer and Ryves' (2024) reasoning regarding the importance of catalysts for teachers' learning, such as reflection, the view of the process as beneficial is supported. The content focus and the coherence features are addressed by letting teachers' vision, their own instructional practice and the teachers' actual starting position form the foundation and guide the professional development initiative, bringing coherence and supporting a realistic and stepwise shift. It responds to Ryve and Lindvall's (2019) call to ensure that professional initiatives are coherent with teachers' own intentions, and thereby supporting their engagement and motivation, as well as Desimone's (2009) last feature of sufficient duration. This not only supports immediate coherence, but also creates the conditions for ownership and for change to persist beyond the timeframe of the intervention. Here, ownership refers to the knowledge that teachers develop and establish, enabling them to carry it into future practices.

The strength lies in, independent mathematical content, allowing teachers' visions and self-efficacy to influence all parts of the professional development initiative, so that teachers to gain a deeper understanding of high-quality mathematics instruction, promoting sustainability and contributing to developing reflective practice and teachers' awareness, but above all, it helps establish self-confidence to implement shifts in their instructional practice.

In summary, if these resources this methodological contribution proposes are activated, they open opportunities for professional development that not only initiates a shift but also sustains over time. The contribution lies in advancing methodological awareness of how to design and carry out a development, bridging the gap between aspirational goals and teachers' situated practices. Further empirical studies are required to test its validity and to examine how these meta-design principles can

be enacted in practice. This proposal is to be regarded as a first step towards a framework that combines normative ambitions with pragmatic needs.

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Supporting early career teachers in developing students' mathematical reasoning

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This study examines the alignment of one early career primary teacher's perceptions about supporting students' reasoning and the extent to which these perceptions were actualized in his discourse-based instructional practice. With decades of research underscoring the importance of reasoning and discourse, enacting research-based recommendations does not always happen. This qualitative case study explored the relationship between an early career teacher's belief and their practice. Findings highlight contradictions in their abilities and tensions when trying to facilitate reasoning-centred discourse protocols. Implications emphasise the need to shift system-wide structures to strengthen teachers' capacity to promote reasoning-centred mathematics instruction.

Keywords: Primary education, mathematics teachers, discussion teaching technique.

Introduction

Understanding the role and nature of teaching and learning mathematics continues to be an important area of study and a primary area of work in K-12 schools (NCSM 2019; NCTM, 2014). However, in some classrooms, the focus of learning mathematics centres on finding answers using rote methods (Liljedahl, 2021), which can lead to seeing mathematics as only the application of learned skills and the recollection of memorized facts (Smith & Stein, 2011). In doing so, teachers can miss the importance of developing students' mathematical reasoning, which is central to learning and doing mathematics (Lannin et al., 2011). For early career teachers, much of their time and attention is spent on classroom management and implementing engaging explorations rather on more diagnostic teaching approaches (Schoenfeld, 2011), which better develops students' reasoning. This means understanding how teachers' thinking about mathematical reasoning, as well as how they develop students' mathematical reasoning, is necessary for those who support early career teachers as they help these teachers learn about and apply pedagogies that promote reasoning.

Theoretical framework

Sociocultural theory positions mathematical reasoning and discourse as socially mediated practices wherein knowledge is constructed through interaction and language (Vygotsky, 1978; Wertsch, 1991). Building on this view, the socio-mathematical norms (Cobb, 2007) and teachers' pedagogical moves that support students in articulating, challenging, and refining ideas (Chapin et al., 2009) all support students in developing their mathematical reasoning (Jeannotte & Kieran, 2017). Together, these perspectives serve as the theoretical framework for this study in that they frame the work of the teacher as central to developing students mathematical reasoning through whole class discourse.

Review of literature

Despite the role of discourse in learning mathematics (Sfard, 2008; Truxaw, 2020), mathematics often has the greatest disparities in participation and achievement (Boaler, 2016). Several obstacles stand in the way of creating rich and accessible learning of mathematics including instruction that only focuses on procedures and students' lack of mathematical confidence. Creating opportunities that address these obstacles are grounded in a classroom culture that purposefully develops students' mathematical thinking. Such opportunities assure students know how and what to share in a safe space, help students orient themselves to each other's ideas, and ensure students believe they are critical contributors to the learning (Kazemi & Hintz, 2014). That is, the mathematical norms created in the classroom, and adopted within the learning community, are paramount.

Communicating mathematically, which includes whole class discourse, is an integral intellectual activity for learning mathematics (Sfard, 2008). For this paper, discourse in mathematics will be understood to be a dialogically collaborative and complex social interaction in which students develop a richer understanding of mathematical concepts with the support of their peers and the teacher as needed (Bennett, 2018). These social interactions are one of the most important aspects of productive mathematical discourse as they help students learn complex knowledge and skills (Smith & Stein, 2011). It is because of these rich and collaborative social interactions students can refine and integrate information, hear and use new perspectives from others, share their own ideas, and participate in the cohesive and collaborative development of mathematical knowledge.

An underlying yet vital component of mathematical discourse is reasoning. According to Lannin and colleagues (2011), mathematical reasoning is the "essence of mathematical activity" (p.7) and is fundamentally about developing claims and supporting them with the proper evidence (Jeaonnette & Kieran, 2017). This means that reasoning in mathematics is used as a means of understanding mathematical relationships, knowing how and why these relationships exist, and thus developing conceptual understanding; it helps students develop problem solving skills and transfer their understandings into other real-world contexts (Boaler, 2016). Thus, reasoning is foundational to appropriately engaging in mathematics; without this reasoning, there is no mathematics.

However, supporting students' mathematical reasoning can be a challenge for teachers (Lannin et al., 2011). The complexity in mathematical reasoning, both structurally and with respect to the various processes they may take to develop (Jeannotte & Kieran, 2017), creates additional challenges for teachers as they also need to decide on what to attend to during instructional episodes (Schoenfeld, 2011). And while numerous pedagogical practices can be used to support students' reasoning (Bennett, 2018; Liljedahl, 2021; Smith & Stein, 2011), research has shown that many teachers struggle to maintain the cognitive engagement necessary for mathematical reasoning to flourish and encounter difficulty in facilitating discussions using students' strategies and ideas (Liljedahl, 2021; Smith & Stein, 2011). Furthermore, teachers' beliefs may not align with their pedagogical practices in the classroom, which can then impact student learning (Sabarwal et al., 2022), but their self-confidence, which is a personal belief on their ability to be successful, can grow through focused professional learning experiences (Childers et al., 2023). For this reason, this case study examined one early carer teacher's perspective on using a discourse protocol to support students' mathematical

reasoning to see the extent to which their perspectives align with their practice. As such, the research question for this study sought to answer: to what extent do teachers' perspectives about developing students' mathematical reasoning align with their instructional practice?

Methods

This qualitative case study (McMillan, 2016) examined an early career teacher's perspective on discourse as a central pedagogy to development students' mathematical reasoning and the extent to which these perspectives were enacted in their practice. While this study is part of a larger study examining the same topic but with teachers in various stages of their career, data will be shared only as it relates to the one early career teacher to consider how the learning of mathematics teaching is connected to practice for those relatively new to the profession. The research team consisted of a higher education researcher, a research assistant, and a classroom teacher who also served as a mentor to the participant teacher. The research team conducted two interviews, one of which was prior to the use of the discourse protocol and one about half way through the study. The teacher was also observed facilitating discussions using the discourse protocol three times and additional field notes were written to further triangulate the data. The interviews allowed the team to have a baseline understanding of the teacher's perspectives on reasoning whereas the observations allowed the team to see the extent to which the perspectives aligned with their pedagogical practices. The field notes served as a resource to better understand the classroom context, the learning environment, and the culture for doing mathematics created by the participating teacher.

Participants

For the purpose of this study, one classroom teacher was selected, who will be called Kevin, as he was the only early career teacher in the larger study. Kevin was in his third year of teaching and taught grade five. The school in which this study takes place is located in the mountain-west of the United States and it is situated in a smaller urban community, but the neighborhood and micro-community it serves has substantial social needs and significant financial insecurities; nearly 50 percent of the students come from homes that are in poverty. The school has approximately 500 students across all primary grades and less than ten percent are English Language learners.

Procedures and data analysis

To begin, the research team conducted a semi-structured interview with Kevin to better understand his perspectives on mathematical reasoning and the instructional strategies he uses. This interview was followed up with three observations of Kevin's instructional practice in supporting reasoning, which occurred over the course of three months. During all three observations, Kevin was using a discourse protocol, called a reasoning talk (Bennett, 2018). This structured discourse protocol focuses students' attention to what information mathematically matters in the question and why that information is important before they begin to solve it. Teachers prompt students to attend to key information and them ask them to clarify, justify, or otherwise support their claim to make their reasoning explicit and public. Often, this protocol is used at the start of a lesson to focus students' thinking on the learning that was to come later in the lesson. In Bennett (2018), teachers were more aware of how to focus on developing students' reasoning through the application of this protocol. To

support Kevin in developing and implementing reasoning talks, additional instructional coaching and mentoring support from Mike and the research team were provided to Kevin over the course of the project; typically, once every two to three weeks after school. Finally, another semi-structured interview was conducted with Kevin's mentor and colleague, Mike, at the end of the study.

To analyse the data, the research team independently read the transcripts from the interview and open coded for instances describing teachers' perceptions of supporting students' mathematical reasoning (Guest & MacQueen, 2012). Main categories from each primary grade level were then united into common themes in the data; memos about the specific themes were written to provide further explanation of the meaning of data. To conclude analysis, themes in the data were extrapolated to explain teachers' perspectives around reasoning and discourse in learning mathematics.

Limitations

A primary limitation of this study revolves around the fact that it included only one primary teacher. It is clear that this teacher's perspectives on the role and nature of discourse to promote students' reasoning is not a representative sample. However, with the focus on how the learning of mathematics teaching is connected to practice for those relatively new to the profession, Kevin was the only upper primary teacher in the study that met this criterion. It is also probable that other interactions that were not studied impacted the teacher's perceptions about the teaching and learning of mathematics as a discipline, and about the role of reasoning within this discipline. As such, this reduces the richness of the data and thus the interpretations that can be made despite promising findings for future research.

Findings

Findings will be discussed with respect to Kevin's interview and then the data collected from classroom observations and other field notes from his classroom to better compare and contrast Kevin's perspectives about discourse as a means of supporting students' reasoning and how these perspectives are then manifested in his instructional practice. Collectively, a more robust understanding of how Kevin's perspectives and vision of himself align with the reality of his instructional practice.

Interview

During the interview there were three primary themes that emerged, but for the purpose of this paper, only the first theme will be discussed. This theme centred on the teacher's contradictory beliefs in their self-confidence. Kevin claimed he had a high level of self-confidence in supporting reasoning through discussions, and clearly talked about ways he supports students' reasoning in discussions, but also adamantly claimed he lacked the expertise to facilitate discussions to support reasoning.

With respect to his high levels of self-confidence in supporting reasoning, Kevin stated he was "pretty confident in leading [students] in discussions," which he viewed as the primary means of supporting reasoning. Kevin talked about using such things as questioning stems to help students frame questions during discussion and his specific attention to identifying misconceptions as two of the more important tools and techniques to facilitating discussions. He also talked about the importance of being patient when facilitating discussions; it was important to give students "the time and space to be able to work it out on their own."

While Kevin expressed a willingness to try new approaches, he also emphasized the importance of adhering to the curriculum's pacing. Essentially, if it was going to take too much time to have a discussion, then he would move on. In fact, he shared that he has a reputation at the school, despite being a newer teacher, for maintaining strict schedules for instruction. He said, with a smile, that other teachers call him "father time" because he always needs to keep the pacing of the lessons on time, which then allows him to adhere to the curriculum pacing schedule, too. But this was also contrasted with the belief that deviating from the textbook can be a good thing. "Sometimes I will come up with [something] and I think I could improve on using their work to start a discussion, instead of what the textbook has provided." With that said, Kevin seemed prouder of his nick name and the implications it brings.

In contrast, Kevin also indicated that he usually does not act on instructional moments that would allow for him to better focus on reasoning. "Teachers need confidence in applying that discourse or the reasoning. It is a bold step, to step out. We humans want to be comfortable, too." Kevin credited his lack of confidence to not being an expert in facilitating discussions. This means he simultaneously held conflicting beliefs; he both had the ability to engage students in discussion to support reasoning but also was reluctant to do so because he was not an expert in facilitating discussions.

Observations of instructional practice

Kevin always presented himself in a calm, friendly, yet stern manner to the students. He was very clear with his expectations for behaviour and engagement and had clear procedures that students followed for transitions during the instructional time; "I'm waiting for a few to show they are ready" and "I can see some of us do not have our minds focused, yet." The classroom was arranged in a U-shaped design wherein the 24 students were seated at their own desk, but often adjacent to at least one other student making it easier for students to share ideas and collaborate with each other.

During all three observations, Kevin co-planned the reasoning talk with either his mentor or with one of the members of the research team. This was to ensure that the question posed to the students would allow for students' mathematical reasoning to readily come forth. And while not related to the findings for this paper, this support was necessary because Kevin often struggled framing questions in such a way to allow for multiple ways to think about the problem. Additionally, Kevin was encouraged to listen for the justification in students' responses; the "why" or "because" of their claim or statement. Figure 1 is an example of a modified story problem prompt and the question asked to elicit students' reasoning as it relates to using a bar model to represent a mathematical situation.

Story problem prompt	Question to elicit reasoning
Sheena bought a sandwich and two pretzels. She paid with a $20 bill. How much change should she get back?	If you were to use a bar model for this, what would you expect to see on it to know that it was set up correctly?

Figure 1: Example of Kevin's reasoning talk prompt guiding question to support reasoning

During all three observations, and despite using appropriate story problem prompts and questions that had the potential for students to share their mathematical thinking, the instructional episodes were

highly teacher centred. Kevin would pose the question as intended, and after varying wait times lasting at least 60 seconds, Kevin would ask for responses from the students. During each instructional episode, three to five students would have their hand up to respond. But when called upon, students would provide a response but without a justification. As an example, and in response to the prompt and question in Figure 1, one student responded with "You would have a top box and a bottom box" but Kevin did not ask why there would be two. Another student stated "we need to split the bottom box into two," thinking about the partition for the two different items purchased, and then asked "but how would we split that up?" To which Kevin responded with probably around here (pointing to a place on the bar model) rather than asking the student why that was an important thing to think about. At one point a student did provide a "because" to their response, and Kevin stated "I like your justification" but did not clarify what part of it actually mattered or was good. In doing so, the class missed out on the opportunity to connect the teacher's praise to the specific construction of the student's reasoning. Additionally, in all observations, it seemed like Kevin was very aware of the pacing of the lesson. Even if a student was in the process of explaining an idea, Kevin would end the conversation in order to continue with the next part of the lesson; "I see where you are going there but we are going to move on." This suggests his nickname of "father time" was accurate; his desire to not fall behind and to complete the full lesson as outlined in the textbook was a constant driving force in his instructional decision-making process.

It should be clearly stated there were notable and encouraging moments of promise with Kevin. At times he would probe student thinking, "tell me more about what you mean by that," or he would accentuate students' responses to bring attention to them for other students, "say that again you said something really important I want to make sure everyone heard." Likewise, his classroom management skills and procedures were highly effective and he was always patient and compassionate with students which created a strong sense of trust between the teacher and students. This attention to developing a strong culture for learning is paramount and is a precursor to creating a strong culture for discourse. So, while his current practices for facilitating discourse were still fragile, it seems he is well positioned to create a space that puts a stronger emphasis on reasoning and sense making. If anything, his self-awareness of where he needs more assistance is highly promising.

Discussion and implications

Developing the skills necessary for effectively implementing discourse-based pedagogies that build and support students' mathematical reasoning, like reasoning talks, are complex and not necessarily easy for teachers to adopt. And while teachers are developing these skills, they may hold contradictory perspectives with respect to their abilities, as also found by Sabarwal and colleagues (2022). Schoenfeld (2011) highlights, this may be due to the fact that novice teachers typically spend little time on diagnostic teaching practices that probes students' thinking and then uses and builds upon that student-generated thinking to drive instruction. Reasoning talks, by design, are a diagnostic teaching practice which can explain why implementing them can be a challenge for novice teachers. This may mean that building a knowledge base for teaching (Shulman, 1986) in teacher preparation programs around discourse rich instructional episodes should become more of a priority to help accelerate early career teachers' skills in these areas. Boerst and colleagues (2011) outlined the complexity of this work and the scaffolds needed to move "approximations of practice" closer to the

reality of teaching. As evident in this study, and even with mentor support, Kevin struggled to trust himself in less scripted situations that required greater student autonomy and voice. So while Kevin wanted to move into more dynamic and diagnostic teaching practices (Schoenfeld, 2011), there was a lack of self-confidence to do so. However, this confidence can be developed through practices such as lesson study (Elliott, 2019), co-teaching models (Steele, et al., 2021) and focused instructional coaching within the context of the teachers' classroom. So, while coaching supports were provided to Kevin, they focused primarily on the technical aspects of developing and facilitating a reasoning talk and less on the nuanced pedagogical moves to elevate students' reasoning skills. That is, implementing a particular instructional strategy consists of both technical knowledge, which is the "how" to do something, and didacticial expertise, which requires a greater level of professional insight and understanding that is only learned over time through careful attention to practice and reflection. This does not mean that early career teachers like Kevin cannot develop this level of understanding, but that instructional supports from mentors, administrators, or coaches should also focus on making the decision making process around the strategy visible (Knight, 2021). Teachers like Kevin improve schools and make a difference, but developing students' reasoning is not easy. This means that to develop students' reasoning in mathematics, helping early career teachers build their self-confidence through opportunities that help them reflect on their own practice matters.

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Turn-taking in preservice teachers' collective proportional reasoning involving multimodal full-body interaction

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This case study investigates preservice teachers' (PSTs) turn-taking in collective proportional reasoning through multimodal full-body interaction within a large-format ratio triangle. We analyse how pairs of PSTs negotiated roles and shared ideas via bodily coordination, with reasoning unfolding through the interplay of speech, gaze, movement, spatial positioning, and material use. One pair exemplified relationally linked collective reasoning, explicitly building on each other's contributions through coordinated speech, movement, bodily creativity (e.g., twists), gaze, and material interaction with a beanbag to manage turn-taking. In contrast, the other pair's turn-taking was fragmented and individually anchored, as reasoning unfolded through operation-mirroring with sparse gaze and bodily engagement limited to logistical movements, resulting in less integrated collective reasoning. The study highlights turn-taking as a feature of multimodal collective reasoning.

Keywords: Design research, teacher education, collective reasoning, turn-taking, proportionality.

Introduction and research question

Designing activities and facilitating reasoning are two aspects involved in learning mathematics teaching. This paper addresses how we can give preservice teachers (PSTs) opportunities to design collective proportional reasoning activities for their future students that foster creative reasoning through collaborative practical engagement. Collective reasoning refers to the joint construction of interconnected mathematical arguments that emerge through collaborative dialogue and interaction (Sumpter, 2016). A key aspect in collective reasoning is turn-taking. Following Gibson (2005), turn-taking is a fundamental structure of human interaction, which involves the rapid alternation between speaker and listener roles and serves not only to manage participation, but also to distribute and advance reasoning among group members. Although several studies have explored turn-taking in teacher-led talk (e.g., Radford et al., 2011) or verbal student dialogue (e.g., Ingram et al., 2011), less is known about how turn-taking and multimodal interaction intertwine in peer-led tasks. A deeper understanding of how these dimensions operate together can illuminate the ways mathematical ideas are co-constructed, coordinated, and negotiated in collaborative multimodal settings. This case study contributes to this focus by examining the question: What characterises PSTs' turn-taking in collective proportional reasoning using a large-format ratio triangle (LFRT)?

Theoretical background

Hu and Chen (2022) examined students' turn-taking patterns during dialogic collaborative problem solving and identified four latent profiles: *turn-receivers*, *turn-usurpers*, *turn-claimers*, and *turn-balancers*. Their findings highlight how most students access the conversational floor by receiving it from the previous speaker, while more assertive strategies such as usurping and claiming turns were positively associated with greater interaction intensity and a more balanced distribution of participation. Importantly, the study underscores the role of equitable access to the conversational floor in fostering respectful and inclusive peer dialogue.

Fink (2020) examined how a group of 9th-grade students shifted between individual turn-taking and collaborative play during a small-group algebra task, demonstrating that these two modes supported complementary forms of productive participation. Turn-taking slowed the pace of the task and ensured distributed access to the conversational floor, allowing individual contributions to surface clearly. In contrast, collaborative play enabled students to co-construct ideas, with responsibility for resolving issues shared across the group rather than assigned to individuals. Fink's (2020) findings highlight the interdependence of these participation structures in sustaining meaningful group work. While her study focuses on verbal interactions, it offers a useful conceptual lens for the present analysis, which explores how similar shifts between structured turn-taking and fluid, co-produced reasoning occur in multimodal settings.

Ingram et al. (2011) investigated turn-taking in secondary mathematics classrooms, identifying specific situations in which students self-select as the next speaker without teacher sanction – namely when asking questions, initiating repair, or responding to undirected teacher prompts. While overall turn-taking remained tightly controlled by the teacher, these moments of self-selection often led to spontaneous discussions and arguments among students. Although such peer-led exchanges were generally viewed as productive for classroom dynamics, the authors note uncertainty about their effectiveness in fostering mathematical understanding. Importantly, the study highlights subtle forms of student agency within a highly regulated discourse structure. For the present study, which explores multimodal turn-taking in peer-led, embodied mathematical tasks, these findings raise critical questions about how self-selection and shared control unfold when turn-taking is enacted not only through speech but also through gesture, movement, and spatial alignment. Understanding how students claim and negotiate turns across these modalities may offer deeper insight into the conditions that support meaningful, student-driven reasoning.

Two articles outside the field of mathematics education are also relevant for our research. Kendrick et al. (2023) investigated the multimodal nature of turn-taking in face-to-face interaction, highlighting the central role of visual cues – particularly gaze and manual gestures – in coordinating conversational transitions. Their findings show that while manual gestures significantly speed up transitions, gaze direction alone does not, and transitions are often inhibited when gaze is averted or gestures left incomplete. On this basis, the authors propose an integrated model of turn-taking that accounts for both linguistic and visual gestural resources. Mondada (2007) examined pointing gestures in work meetings, showing that they function not only as referential tools but also as integral components of turn-taking. Her analysis demonstrates how pointing can project the end of a turn, signal self-selection, and anticipate the next speaker, thereby contributing to the orderly management of interaction. Together with related work, this research provides a foundation for understanding how turn-taking supports equitable and effective collaborative learning. Yet most studies focus on verbal interaction or teacher-controlled settings. The present study extends this perspective by analysing multimodal turn-taking among PSTs engaged in collaborative proportional reasoning with an LFRT taped on the classroom floor.

Method

This case study reports on a design research project conducted within a mathematics didactics course attended by fourth-year students in five-year teacher education programmes qualifying them to teach mathematics for grades 1-7 and 5-10 in Norway. The project involved researchers (the authors) and volunteer PSTs as participants. The project was organised into annual cycles of implementation and testing, evaluation and refinement, beginning in 2024 (Bjørnebye et al., 2025). Each cycle spanned

two consecutive days, and only PSTs who attended both days were included in the data analysis. The 2025 cycle reported in this paper involved 22 PSTs. The first day began with an introductory lecture on the use of the LFRT, along with a presentation of the following design principles (DPs) guiding this research: DP1 – aligning mathematical content within creative bodily interaction and movement (Skulmowski & Rey, 2018); DP2 – creative problem solving (Lithner, 2017); DP3 – collective reasoning (Sumpter, 2016); and DP4 – abstraction through symbolic representations and transformation across contexts. Following this, the PSTs worked in five randomly assigned groups to develop multiple solutions to open and closed proportional reasoning tasks using a small-format ratio triangle (SFRT) scheme (cf. DP2–DP4; Figure 1). In the present study, we examine open tasks in which the groups themselves designed and solved, providing valuable opportunities for learning about teaching through practical engagement in proportional reasoning.

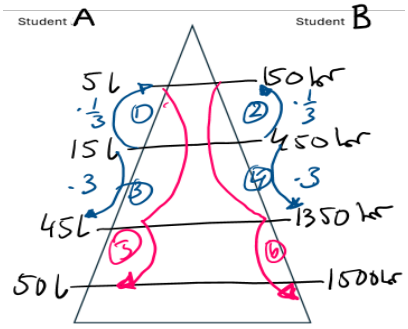

Figure 1: SFRT scheme

Figure 1 presents the SFRT scheme of the student-designed task, **"**How much does 50 L of soda cost if 15 L cost 450 kr (kroner)?", with alternating contributions from Students A and B. Blue arrows (steps 1–4) indicate multiplicative scalar reasoning, while red arrows (steps 5–6) represent building-up by addition or building-down by subtraction. The six reasoning steps, expressed symbolically, are: (15L, 450 kr) · 1/3 → (15L,450 kr) · 3 → (45L,1350 kr) · 3→ (45L,1350 kr) +(5L, 150 kr) → (50L, 1500 kr).

Building on insights from the first cycle of the project, where Bjørnebye et al. (2025) introduced the LFRT, the second cycle implemented design modifications aimed at more closely examining the collective dimension of the reasoning process (cf. DP3). As illustrated in the SFRT scheme in Figure 1, one key adjustment was to structure the tasks so that only two students per group participated actively in the reasoning dialogue, creating a more focused context for analysing peer interaction and turn-taking. The five groups documented ten solutions to their self-designed tasks using the SFRT scheme, eight of which were later enacted in the LFRT and recorded from three camera angles. The excerpts in this case study are drawn from this video material (cf. DP1). Following a deviant case selection strategy (Seawright & Gerring, 2008), the excerpts were chosen to illustrate contrasting patterns of turn-taking within the multimodal interaction of collective proportional reasoning. The transcript was translated from Norwegian into English, with non-verbal actions and editorial annotations (e.g., clarifications) enclosed in brackets. To foreground the multimodal interaction, selected still images from the video material accompany the transcript.

The analytical focus is on how turn-taking is coordinated through multiple modalities during collective reasoning in the LFRT. We examined bodily actions such as gaze, orientation, and movement. Material artefacts also played a role in structuring interaction. Items like tennis balls and

beanbags were made available to participants. However, their use was not explicitly instructed. For instance, a participant might establish eye contact or orient their body towards a peer while holding a beanbag to signal engagement. Full-body movements (e.g., sideways jumps or leg-crossing steps) were used to physically enact proportional operations like scaling up and down. In our analysis, we considered how such artefacts could potentially facilitate – or, at times, hinder – turn-taking, support speaker transitions, and sustain rhythmic coordination within the group dialogue. Post-it notes were employed to anchor percentage values and amounts within the LFRT, serving as spatial reference points for the reasoning process. These modalities were analysed in relation to argumentation patterns. Drawing on Sumpter's (2016) definition of collective reasoning, we focused on instances where PSTs' verbal justifications and embodied actions were coordinated to contribute to a shared line of thought. Within this form of reasoning, we distinguished between relationally linked collective reasoning, where the responding speaker explicitly connected their reasoning to a peer's contribution (e.g., "Because you halved 50 percent, I must halve 360 and I get …"), and operation-mirroring collective reasoning, where the responding speaker carried out the corresponding mathematical operation without making the relational link explicit (e.g., "I halve 360 and I get …"). Building on this distinction, we analysed how the coordination of bodily actions, artefacts, and speech shaped turn-taking in ways that could support or constrain collective proportional reasoning.

Results and analysis

We present two excerpts of PST pairs enacting their group's SFRT-based solutions within the LFRT.

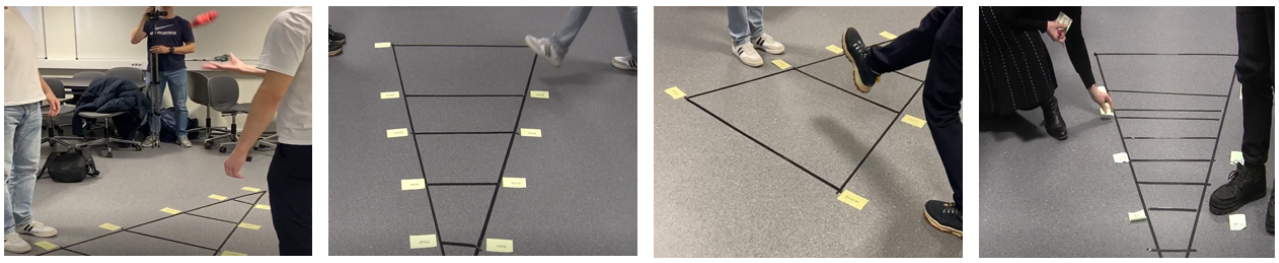

Figure 2: a) Turn-taking via red beanbag; b) Leg-crossing in scaling down; c) Mirroring leg-crossing in scaling down; d) Stepping into the LFRT, placing a note

Excerpt 1: This group designed the task: *"How much does 50 litres of soda cost if 15 litres cost 450 kroner?"* Their first solution is visualised in the SFRT scheme (Figure 1), while this excerpt captures the group's second solution, enacted within the LFRT. All percentage and value notes used in the solution are pre-placed in the LFRT. Student A, holding a beanbag, and Student B stands outside the LFRT as a third PST introduces the task (Figure 2a).

#1	**A:** [Runs into the LFRT, jumps and lands on the 15-litre note] – Then I'll start with 15 litres – [Makes eye contact with B and throws the red beanbag]
#2	**B:** [Makes eye contact with A and catches the beanbag, enters the LFRT] – Since we are told in the task that 15 litres cost 450 kroner, I'll start with – [Jumps to the 450-kroner note] – 450 kroner. [Makes eye contact with A and throws the beanbag]
#3	**A:** [Makes eye contact with B and catches the beanbag.] Then we divide by three – [Performs leg-crossing steps to the 5-litre label] – and land on 5 litres. [Makes eye contact with B and throws the beanbag]
#4	**B:** [Makes eye contact with A and catches the beanbag] – And since you divide 15 litres by three, I must divide 450 kroner by three, and I get – [Performs leg-crossing steps to the 150-kroner note] – 150 kroner. [Makes eye contact with A and throws the beanbag]

#5 **A:** [Makes eye contact with B and catches the beanbag] – Then we can divide by five again – [Short left-leg sideways jumps to the 1-litre note] – and land on 1 litre. [Makes eye contact with B and throws the beanbag]

#6 **B:** [Makes eye contact with A and catches the beanbag] – Since you divided 5 litres by five again, I have to divide 150 kroner by five, and I land on – [Short left-leg sideways jumps to the 30-kroner note] – 30 kroner. [Makes eye contact with A and throws the beanbag]

#7 **A:** [Makes eye contact with B and catches the beanbag] – For simplicity, we then multiply 1 litre by 100 – [Short, rapid steps before jumping onto the 100-litre note] – and land on 100 litres. [Makes eye contact with B and throws the beanbag]

#8 **B:** [Makes eye contact with A and catches the beanbag] – Since you multiplied 1 litre by 100, I must multiply 30 kroner by 100, and I get – [Short, rapid steps before jumping onto 3000-kroner note] – 3000 kroner. [Makes eye contact with A and throws the beanbag]

#9 **A:** [Makes eye contact with B and catches the beanbag] – Then we can divide 100 litres by two – [Leg-crossing steps (Figure 2b) followed by a jump onto the 50-litre note] – and we end up with 50 litres. [Makes eye contact with B and throws the beanbag]

#10 **B:** [Makes eye contact with A and catches the beanbag] – Since you divided 100 litres by two, I must divide 3000 kroner by two – [Leg-crossing steps (Figure 2b)) followed by a jump onto the 1500-kroner note] – and I land on 1500 kroner. [Makes eye contact with A and throws the beanbag]

#11 **A:** [Makes eye contact with B and catches the beanbag] – Then 50 litres of soda cost... [Makes eye contact with B and throws the beanbag]

#12 **B:** [Makes eye contact with A and catches the beanbag] – 1500 kroner! [Throws the beanbag into the air]

#13 **A & B:** [Clap their hands together] – Yeah!

Excerpt 2: The 10.5-litres and 35-miles-notes from their solution are placed in the LFRT. Student C and D hold a set of pre-made notes in their hands, while another PST introduces the task: "A vehicle drives 35 miles on 10.5 L of petrol. How far can it travel on 18 L?"

#14 **C:** [Standing outside the LFRT, switching between looking at the LFRT and at the hand] – To figure that out, we'll double the amount of petrol. Double 10.5 litres is 21 litres. [Walks into the LFRT, places a 21-litres-note, steps back out]

#15 **D:** [Standing by the 35-miles-note, switching between looking at the LFRT and at the hand] – And double 35 miles is – [Walks backwards] – 70 miles. [Places a 70-miles-note across the 21-litres-note]

#16 **C:** [Standing outside the LFRT, looking at the LFRT] – We decided to divide the amount of litres by 7. 21 litres divided by 7 is 3 litres. [Looks at the hand, turns around away from the LFRT, rearranging the notes, turns around, walks into the LFRT] We'll place it here [Places a 3-litres-note, then steps out again]

#17 **D:** [Looking at the LFRT] – And 70 miles divided by 7 is – [Takes a large step forward] – 10 miles. [Places a 10-miles-note across the 3-litres-note]

#18 **C:** [Standing outside the LFRT, looking at the LFRT] – So now we have 21 litres [Points to the 21-litres-note] – and we know what corresponds to 30 litres [Incorrect statement]. Then we do a subtraction: 21 minus 3 litres [Points at the corresponding notes] – that gives us 18 litres. [Walks into the LFRT, places an 18-litres-note (Figure 2c), steps out]

#19 **D:** [Looking at the LFRT] – And if you subtract 3 from 21, I must subtract 10 from 70. That brings me to – [Takes a big backward step] – 60. [Places a 60-miles-note across the 18-litres-note]

#20 **C:** [Standing outside the LFRT, looking at the LFRT] – That means you use 18 litres of petrol to drive 60 miles.

#21 **D:** [Standing beside the 60-miles note, looking at the LFRT] – Yes, you can go 60 miles on 18 litres of petrol.

Excerpt 1 displays a rich repertoire of multimodal interaction within the LFRT. Students A and B use choreographed movements to physically enact proportional operations like dividing by three (#3–#4) and multiplying by 100 (#7 & #8). These actions of multiplicative scalar reasoning are anchored in the placement of value notes, spatially grounding the participants' bodily reasoning (cf. DP 1; Skulmowski & Rey, 2018). Mirrored movements are evident as the PSTs coordinate leg-crossing steps (#3 & #4; #9 & #10), left-leg jumps (#5 & #6), and rapid steps (#7 & #8) to scale values up and down. Transitions between operations are rhythmically structured through coordinated eye contact and the passing of the beanbag, which regulates speaker shifts. Turn-taking is fluently and symmetrically enacted: each movement sequence initiated by Student A is mirrored by Student B, with the responding speaker explicitly linking both action and verbal reasoning to the initiating speaker's preceding move. This verbal–embodied linkage exemplifies relationally linked collective reasoning, in which justifications build directly on a peer's contribution and bodily synchrony reinforces proportional equivalence across modalities.

Excerpt 2, by contrast, reveals a more fragmented and individually anchored form of interaction. While students C and D use pre-made notes to represent fuel-to-distance ratios and arrange them logically within the LFRT, bodily engagement is limited and only loosely connected to the reasoning process. Student C repeatedly steps in and out of the LFRT (#14, #16, #18), with movement serving primarily logistical purposes rather than functioning as an integrated part of the explanation. Although the participants maintain mathematical alignment – doubling, dividing by seven, and subtracting values – these operations are not reinforced through mirrored actions or shared motor sequences. Turn-taking often takes the form of operation-mirroring collective reasoning, in which the responding speaker (Student D) carries out the corresponding operation without explicitly linking it to the initiating speaker's reasoning (Student C). Mutual gaze is infrequent, and coordination relies more on verbal statements than embodied cues, resulting in reasoning that is more individually driven and supported by fewer multimodal or affective resources.

Discussion and concluding remarks

This case study explores how bodily movement, use of tools, gaze, gesture, and spatial positioning function as multimodal resources for managing turn-taking in collective reasoning on proportional problems. Following Gibson's (2005) conception of turn-taking as a universal structure of human interaction that organises participation and reasoning and Hu and Chen's (2022) emphasis on equitable access to the conversational floor as a foundation for inclusive peer dialogue, our study design assigned two PSTs to alternate turns in the proportional reasoning task. This structure ensured that speaking opportunities were evenly distributed and mutually coordinated, while still allowing participants to negotiate specific roles and contributions through bodily, spatial, and material interaction. Excerpt 1 exemplifies highly integrated multimodal turn-taking, where movement, artefact placement, and relationally linked collective reasoning blend to support shared problem solving. Here, the responding speaker explicitly builds on the initiating peer's reasoning, linking verbal justifications and bodily actions to sustain a shared line of thought. The reasoning and turn-taking flowed fluently when percentage and value notes were pre-placed in the LFRT, providing a shared visual structure that anchored interaction. Additionally, the use of a beanbag reinforced eye contact and signalled turn shifts, facilitating smooth transitions between speakers and embedding reasoning in a rhythmic, embodied flow. In contrast, Excerpt 2 illustrates a more verbally anchored,

less embodied approach, characterised by operation-mirroring collective reasoning. In this pattern, the responding speaker performed the relevant mathematical operation without explicitly linking it to the initiating peer's reasoning, resulting in weaker integration between verbal and embodied modalities. Taken together, the excerpts underscore the importance of understanding turn-taking not merely as a verbal structure but as a multimodal and relational practice – shaped by how participants inhabit, coordinate, and communicate within a shared mathematical space (cf. Mondada, 2007).

Fink's (2020) distinction between structured turn-taking and more improvisational collaborative play – though based on verbal discourse – offers a useful conceptual lens for interpreting the multimodal participation structures observed in our study. In her analysis of 9th-grade students working on an algebra task, Fink (2020) found that these two modes supported complementary forms of group work: turn-taking allowed individual voices to surface clearly and slowed the pace for deliberate reasoning, while collaborative play facilitated co-construction of ideas and fluid transitions of responsibility. Similarly, in our study, PSTs alternated between clearly demarcated turn-taking – signalled by gaze, gesture, and artefact exchange – and stretches of more fluid, co-produced reasoning in which speaker roles were blurred and multimodal actions unfolded dynamically. These shifts contributed to a shared mathematical space where reasoning could emerge collectively through embodied interaction. Ingram et al. (2011) identified how students in teacher-led mathematics classrooms occasionally self-selected as speakers – particularly during questioning, repair, or open prompts – demonstrating subtle agency within a regulated discourse. While these peer-initiated turns spurred productive interaction, their role in supporting mathematical understanding remained ambiguous. In our peer-led, embodied setting, turn-taking was negotiated not only through speech but also through gestures, gazes, and spatial movements.

The work of Kendrick et al. (2023) and Mondada (2007) deepens our understanding of these embodied strategies by demonstrating how multimodal cues – especially gestures and gaze – function within turn-taking. Kendrick et al. (2023) show that manual gestures significantly accelerate speaker transitions, while incomplete gestures or gaze aversion may disrupt them. This resonates with our findings in Excerpt 1, where the pair of PSTs relied on mutual eye contact and clearly completed gestures – such as sideways jumps or rotations – as cues for turn transitions and alignment. Mondada's (2007) analysis of pointing gestures as turn-taking resources further aligns with our data: the PSTs used directional movement or physical referencing of notes to signal their intent to speak or yield the floor. These multimodal forms of turn management allowed for fluid, co-produced reasoning without reliance on verbal negotiation alone. Together, this body of research highlights how turn-taking – when expanded beyond speech to include gesture, movement, and spatial arrangement – becomes a richly layered resource for collective mathematical reasoning. Our findings extend this perspective by demonstrating the potential of such multimodal coordination unfolding in peer-led, embodied settings, offering insight into how learners build and align their mathematical arguments through shared physical and communicative practices.

In conclusion, this case study contributes to our understanding of turn-taking as a multimodal practice within collective reasoning (Sumpter, 2016), revealing dimensions of peer collaboration that are rarely foregrounded in typical classroom settings. Developing embodied multimodal turn-taking practices is valuable for PSTs, as such interactions can inform the design of engaging, inclusive, and motivating mathematical learning experiences for their future students.

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Teachers' conceptual orientation does not yet guarantee productive practices with content-specific conceptual focus: Vignette-based study

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Teachers' procedural and conceptual orientations influence their practices to enhance students' conceptual understanding. However, not all teachers who want to achieve conceptual learning goals necessarily plan, value and enact productive practices. The vignette-based study (with n = 125 mathematics middle school teachers) provides quantitative evidence for this disconnect. Teachers were asked to (1) openly articulate their own enhancement practice for a given student case in a vignette and (2) to evaluate other teaching practices. The deductive-inductive coding revealed three groups of teachers' own practices, distinct in their intensity of conceptual orientation and content-specific focus. These three groups also show significant differences in their evaluations of other teaching practices. We conclude that professional learning opportunities should also focus on content-specific realizations by unpacking the conceptual learning goals.

Keywords: Mathematics teachers, productive practices, conceptual orientation, content-specific conceptual focus.

Introduction: Quantitative evidence required about connections between teachers' orientations and practices

Mathematics teachers' overarching beliefs and orientations have repeatedly been identified as crucial precondition for the productivity of their teaching practices (Schoenfeld, 2010). For example, many countries report a persisting dominance of procedural over conceptual orientations, which hinders teachers' planning and value conceptually focused practices optimized to enhance their students' conceptual understanding (Beswick, 2007; Prediger et al., 2023). However, case studies indicated that not all teachers with *conceptual orientation* (which we define as the generic ambition to strive for conceptual learning goals) necessarily plan and value *productive practices* (which we define as those practices which were identified as most probably achieving these goals) (e.g., Friesen & Kuntze, 2020). So far, however, there is a lack of quantitative evidence about the connection between teachers' procedural and conceptual orientations and the practices they plan or positively assess. In this paper, we present first findings of a vignette-based study with the following research questions:

(RQ1) What practices do teachers articulate for dealing with a student's difficulties, what procedural and conceptual orientation and what content-specific focus characterize these practices?

(RQ2) How do teacher groups with different own practices (with underlying orientations and content-specific focus) differ in their assessments of given practices?

Theoretical framework

Orientations and content-specific knowledge as backgrounds for teaching practices

The study is conducted within the theoretical framework of content-related teacher expertise (Bromme, 1992; Prediger et al., 2023) in which *teaching practices* are defined as recurrent actions for meeting typical *tasks of teaching*, in our case, dealing with student difficulties. This includes practices for tasks of teaching such as specifying learning goals, monitoring learning progress, and enhancing students' understanding. In the framework, practices are theoretically and empirically characterized by underlying orientations and focused categories (Prediger et al., 2023). Methodologically, practices are reconstructed as a recurrent pattern, i.e., groups of individual actions.

The practices that teachers plan, enact or positively assess in reflections can be substantially shaped by their orientations. *Orientations* are defined as beliefs and pedagogical attitudes about mathematics and its teaching and learning that implicitly or explicitly guide the teacher's perception (see Schoenfeld, 2010, p. 29). Two sets of orientations have been shown to be particularly crucial for dealing with students' mathematical difficulties: *Long-term orientations* aim at fostering students' learning progress, whereas *short-term orientations* mainly optimize the task completion, sometimes even at the expense of learning progress, e.g., by unproductive support or task simplifications that reduce students' learning opportunity (Watson & de Geest, 2005; Prediger, 2023). *Procedural orientations* are defined as the ambition to strive for mainly procedural learning goals, and *conceptual orientations* for mainly conceptual learning goals (Beswick, 2007; Prediger et al., 2023).

Teaching practices are considered productive when they start from careful analyses of students' difficulties (monitoring students' learning progress), are directed towards clearly specified next learning goals for the student (which require unpacked learning goals) and constitute a probable contribution to achieving the unpacked next learning goals (enhancing students' understanding). We talk about a *procedurally (or conceptually, resp.) focused practice* when a procedural (or conceptual, resp.) learning goal is set and pursued, and about *practices without focus* when the learning goals are too vaguely articulated for pursuing them in targeted ways (Prediger et al., 2023).

Focused practices require content-related pedagogical content knowledge about the unpacked sub learning goals in view (as shown also for working with representations, Friesen & Kuntze, 2020). For example, our study vignette (in Figure 1, taken from Prediger, 2023) addresses the mathematical content of multi-digit subtraction in Grade 5. The case of Suleika is typical for students with mathematics difficulties: she superficially conducts the procedure correctly in the first task without carry but shows difficulties in the second task with carries. Her explanation of her strategy reveals missing place value understanding, as she decomposes 443 = 400 + 400 + 300. Typical *procedurally focused practices* for dealing with her difficulties refer to training the procedure with carries. In contrast, typical *conceptually focused practices* refer to enhancing Suleika's place value understanding for decomposing numbers, with positional property, additive decompositions, and then debundling tens into ones (Fuson et al., 1997), e.g., with base-ten blocks. A *conceptually oriented practice without explicit content-related focus* might, e.g., refer to place value tables without articulating what students should learn from it. It documents an ambition to strive for conceptual understanding but is not yet explicit about how a well-chosen focus can be pursued.

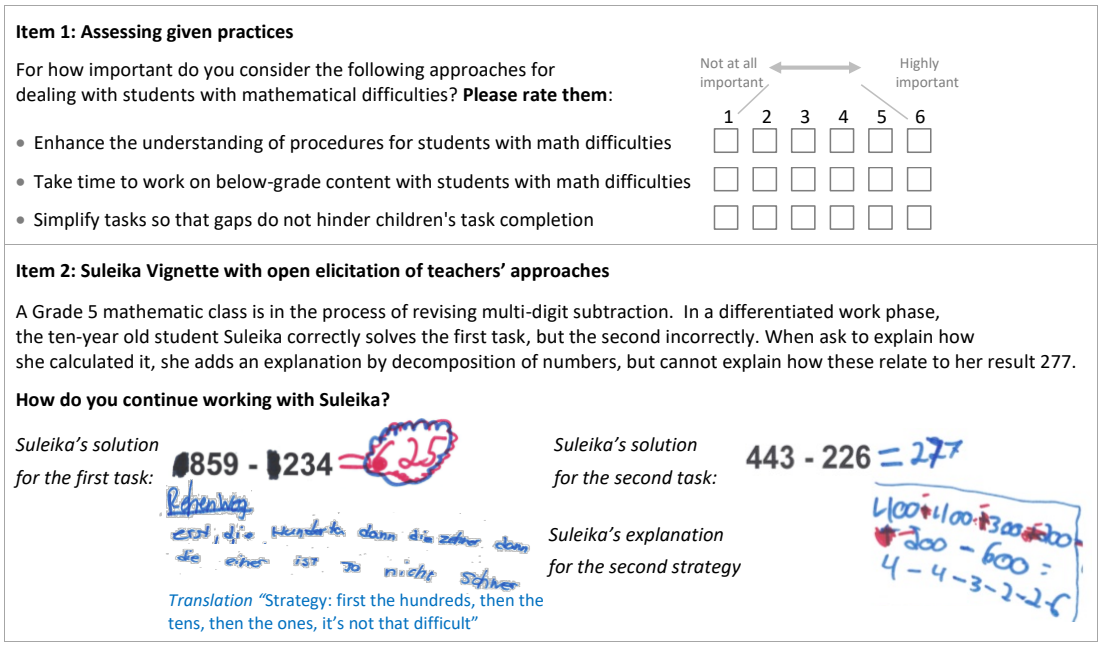

Figure 1: Items used for the current vignette-based study

The example illustrates how conceptual understanding of numbers is needed for building understanding of procedures (Fuson et al., 1997). Purely procedurally focused teachers might not assess this understanding as necessary. Unfocused conceptually oriented teachers might not see the necessity of working on positional property and addition decomposition before unbundling tens into ones.

Some teachers hesitate to go back to the foundations of place value understanding in Grade 5 because this is below-grade content (that should have been learned in Grade 2/3). In a short-term orientation, time pressure makes it rational to continue without working on these foundations, e.g., by simplifying the tasks so that students' gaps in understanding do not hinder them in the task completion for actual content. However, this again only guarantees easier task completion, but not sustainable learning progress (Prediger, 2023). In contrast, teachers who hold a long-term orientation are aware that the procedure cannot be reliably conducted without these below-grade conceptual foundations.

Methods

In our vignette-based study, we investigate how the teachers' actions can be grouped into recurrent patterns that are characterized as practices with underlying orientations and focus (RQ1), and how these group adherences relate to differences in teachers' assessment of given practices (RQ2).

Procedures and instruments of data collection

The data for this paper was collected in a Moodle survey at the beginning of the first online session of a one-year professional development (PD) program *Mastering Math* on fostering basic arithmetic concepts and skills for students with mathematical difficulties in Grade 5 (Prediger et al., 2023). The items used for treating research questions RQ1 and RQ2 are documented in Figure 1. In Item 1, teachers were asked to assess three given practices with respect to their importance for dealing with students with mathematical difficulties. The open Item 2 used the vignette of Suleika (Prediger,

2023). Teachers were asked to describe their practices, i.e., how they plan to continue the work with Suleika to deal with her difficulties.

Sample of data collection

The sample of our vignette-based study consisted of n = 125 mathematics teachers who participated in the first session of a one-year PD program, completed Item 2 and gave informed consent for their anonymized data to be used for research purposes. n = 74 of them completed also Item 1. All teachers taught mathematics in Grade 5/6, but not in Grades 1-4. In our programs, about one third of teachers usually teach without a teaching certificate in mathematics. The median of teaching experience is usually 8 years, with a range from 1 year to 38 years.

Methods for data analysis

For *research question RQ1*, teachers' descriptions of own actions in the vignette-based Item 2 were analyzed in a deductive-inductive code formation process (Mayring, 2015) in three steps:

In Step 1, 23 categories were deductively derived from the literature and inductively enriched from the data to capture the entire range of responses (see Figure 2, anchor examples provided in Figure 3 below) with respect to procedural orientation, conceptual orientation and mixed or unclear orientations. In a second dimension, they were classified as being mainly generic or indicating also a content-specific focus with unpacked learning goals (being a subgroup of the first).

In Step 2, the 125 descriptions of individual actions were coded by all categories that occurred in the respective descriptions of individual actions. In Step 3, the coded descriptions of individual actions were grouped to identify recurrent patterns. In this way, we identified three practices as typical groups of actions (marked in grey in Figure 2), which cover more than 90% of the coded actions.

For treating *research question RQ2* in Step 4, the teachers' assessment of the given practices in Item 1 was analyzed quantitatively (by mean and standard deviation), and the three groups built in the analysis of Item 2 were compared with respect to their assessments in Item 1.

Hypotheses about potential group differences were tested in pairwise t-tests at a significance level of 5% (and tendencies of significance at the 10% level are also reported as the sample size is small).

	Orientation in general (ambition to reach learning goals)	**Content-specific focus (with unpacked learning goals)**
Procedural orientation	Calculation strategies Calculation error Written algorithm Training without prior communication Revise calculations	Order of calculation steps Respecting positions in written algorithm Dealing with carries in written algorithm Tasks without carries
Mixed orientation	(Multi digit informal) subtraction Hint student to error Teachers' explanation Students report of the solution strategy Reducing size of numbers	Transition to tens
Conceptual orientation	Enhance understanding of place values Place value table Other representations (no concrete one) Inquiry learning	Base-ten blocks (or other manipulatives for place value structures) Additive decomposition into hundreds, tens, ones Bundling and debundling of tens / hundreds Positional property

Figure 2: Codes for the orientations and foci on teachers' written descriptions of actions

Results

Groups of actions described for dealing with a student's difficulties (RQ1)

In Figure 3, three practices are inferred as groups of teacher actions are listed and exemplified by anchor examples. The anchor examples illustrate that a teachers' action can rely on both, conceptual and procedural categories (as Beate Müller in the second example), so the operationalizations contain "at least one" and "no occurrence of" conditions for the occurrences. In total, we could determine three different groups of actions that cover 90% of the answers (the others mainly referred to mixed categories).

The green *practice with content-specific conceptual focus* includes individual actions in which at least one category had a content-specific conceptual focus, while also procedurally focused or more generic categories were mentioned. Ayla Esen's description serves as an example for this group. She mentions the mixed category transition to tens, and then substantiates it with two conceptually focused categories that unpack more concisely Suleika's next conceptual learning goals. 22% of the teachers articulated such a content-specific conceptually focused action.

The violet *practice with overall conceptual orientation but no content-specific focus*, comprises those actions in which at least one category documented a conceptual orientation, but without reference to any category with a content-specific conceptual focus. In the anchor example description by Beate Müller, very general conceptual goals and support means are mentioned, and a generic procedural goal. However, it is typical for this practice that no unpacking of the more concise conceptual learning goals takes place. 33% of the teachers' actions were allocated to this group.

Practices as groups of actions with ... (with relative frequencies)	**Operationalization by category occurrences** Action allocated to group when ...	**Anchor example** (synthesized, with pseudonyms)	**Coded categories in the anchor example** [in brackets: classification of category according to Figure 2] (in **bold:** relevant for allocation to group)
... content-specific conceptual focus **23%**	... at least one category had a content-specific conceptual focus, even if also procedurally focused or more generic categories were also mentioned	*"As Suleika evidently shows difficulties with the transition to tens and hundreds, I would try to enhance her understanding of the decimal system, with decomposing in manipulatives (thousand cubes, hundred flats, ...). Only the procedure would not be sufficient"* (Ayla Esen).	Transition to tens [mixed focus] **Base-ten blocks for place value structures** [conceptual focus] **Additive decomposition into hundreds, tens, ones** [conceptual focus]
... generic conceptual orientation but no content-specific focus **26%**	... at least one category documented conceptual orientations, but no category for a content-specific conceptual focus	*"I would choose another representation, such as the place value table or the written algorithm, to make her understand it. For this, I would refer to her individually repeatedly"* (Beate Müller).	**Place value table** [conceptual generic] **Enhance understanding of place values** [conceptual generic] Written algorithm [procedural generic]
... procedural focus and explicit learning goals **42%**	... no conceptual orientation was mentioned, at least one procedural orientation (all contained at least one category of procedural focus)	*"I make her write the digits under each other according to positions and train the sentence 'from 7 to 6 does not work, so we go to 10, then to 6. This borrowed 10 must be subtracted in the next step"* (Alex Bojko).	**Revise calculations** [procedural generic] **Respecting positions in written algorithm** [procedural focus] **Dealing with carries** [procedural focus] Transition to tens [mixed focus] **Teachers' explanation** [procedural generic]

Figure 3: Three practices inferred as groups of actions for dealing with Suleika's difficulties with anchor examples

Finally, the *practice with procedural focus* comprises all actions for which no conceptual orientation was mentioned, at least one category in procedural orientation and perhaps additional mixed orientations. Remarkably, the large majority addressed at least one category with content-specific procedural focus (and all by definition, no content-specific conceptual focus category). In the anchor example by Alex Bojko, procedural generic categories occur, but are also explicitly unpacked procedural learning goals. 35% of the teachers articulated such a procedurally focused action.

Differences in assessment of given practices according to group adherence (RQ2)

Research question RQ2 asks for potential differences between the three teacher groups in their assessments of three given practices. Figure 4 depicts the mean importance assigned to these practices by the different teacher groups from Figure 3.

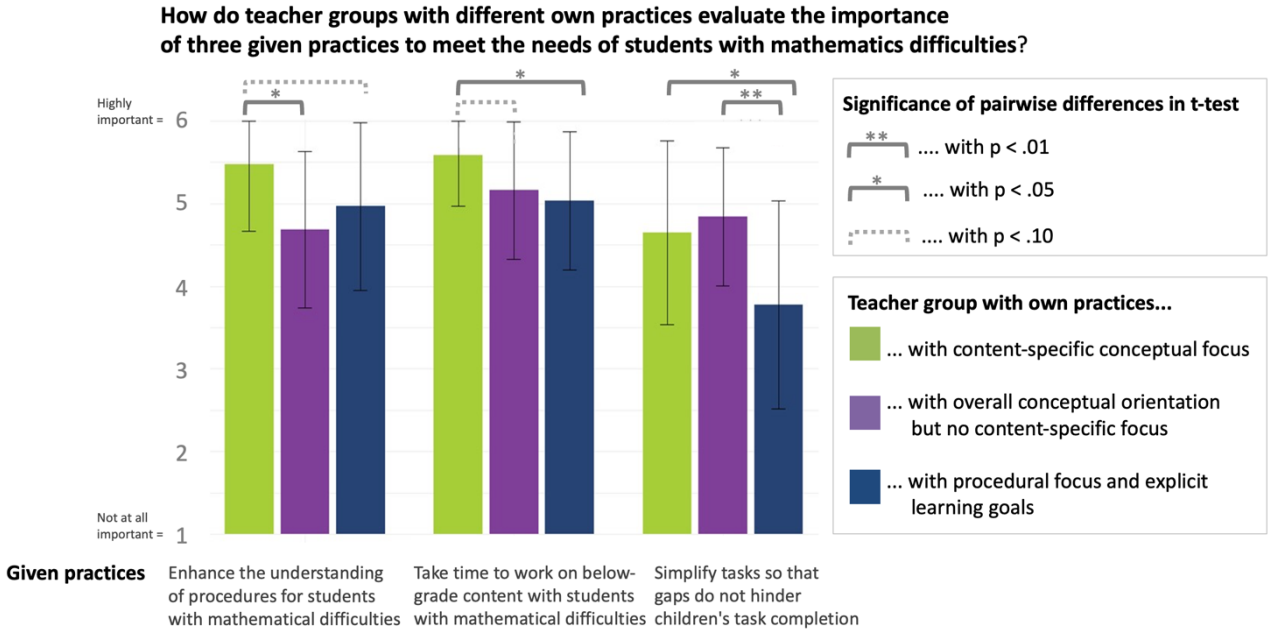

Figure 4: Comparing teacher groups' evaluations of importance for three given practices

With respect to the given practice "Enhance the understanding of procedures", the teacher group with content-specifically conceptually focused own actions also evaluated this practice as significantly more important than the group with overall conceptually oriented unfocused actions (the asterisk marks a significant difference in the t-test with $p < .05$), and higher importance that the group with procedurally focused own actions (for which only a tendency at the 10% level was found). Thus, teachers with conceptual overall ambitions without unpacked learning goals find understanding of procedures slightly less relevant for the particular target group of students with mathematical difficulties.

To the given practice "Take time to work on below-grade content", the teacher group with content-specifically conceptually focused own actions assigns higher importance than the group with overall conceptually oriented unfocused actions (only at a 10% level) and significantly more than the group with procedurally focused own actions ($p < .05$).

The order of assigned importance is different for the practice "Simplify the task so that gaps do not hinder task completion": The group with procedurally focused own actions evaluates this practice with lowest importance (most in line with the PD program goals), significantly lower than the group with conceptually focused actions ($p < .05$) and highly significantly lower than the group with conceptually oriented but unfocused actions ($p < .01$).

Discussion

In many studies, a high proportion of teachers have been found to highly value and enact practices with procedural focus (e.g., Beswick, 2007). In our sample, procedurally focused practices for dealing with students' difficulties are not the majority, as only 35% of our teachers spontaneously articulated actions belonging to this practice. Most of the teachers adhering to this group highly evaluated the given practice to enhance understanding of procedures, even for the particular target group of students with mathematical difficulties. These findings reveal that a sole procedural focus is less dominant than in earlier studies.

In contrast, the majority of teachers in our study (55%) articulated actions with an ambition to work towards conceptual learning goals. However, teachers' conceptual orientation alone does not seem to guarantee productive practices with content-specific conceptual focus: only 22% unpacked conceptual learning goals, whereas 33% articulated only the general ambition without any unpacked learning goals, which makes is much less probable to achieve them. Of course, these findings must be interpreted cautiously because some teachers might enact highly focused practices in classrooms even if they do not express them in the vignette, so they should be deepened in future studies by capturing really enacted practices. However, the frequencies underline the relevance of distinguishing between these two groups in the future.

The difference between these two groups also becomes visible when comparing their assessment of given practices (Figure 4). Assigned importance for "enhancing understanding of procedures" aligns with setting concise procedural or conceptual learning goals, and the teacher group with procedurally focused actions attributes more importance to it than the group with conceptually oriented unfocused actions, and the group with conceptually focused actions significantly more than the second. Unequal assessment of importance also occurs for the practice "taking time to work on below-grade content": Those who can unpack and focus conceptual sub-learning goals see stronger need to spend time on below-grade conceptual foundations, and significantly more than the group with procedural focus shows that the decisions of what content to prioritize are associated with the unpacking of conceptual goals. Finally, the group with unfocused conceptual actions is the group that most values the short-term compensation practice "simplifying tasks" to assure task completion rather than learning progress. Procedurally focused actions align more with scepticism against this short-term practice, even more than conceptually focused actions in our sample. This unexpected finding requires further investigation, as short-term orientation and procedural orientation seem to be separable, in our items regard, in contrast to existing case studies (Prediger, 2023; Watson & de Geest, 2005).

Evidently, the findings must be interpreted with respect to the methodological limitations of the study. Teachers' assessments of importance are captured in single-item questions without assuring reliability in multi-item scales. The sample size is relatively small for generalizing findings about

connections, and the teacher group might be positively biased as volunteers in a PD program. Future studies should extend the item base and sample sizes. And they should investigate the transferability of the findings to other mathematical contents, e.g., those with which teachers are more familiar.

However, already the current findings can have important consequences for the (re-)design of PD courses: As a conceptual orientation is not necessarily sufficient for planning and assessing productive practices with conceptual focus, future PD courses should provide systematic professional learning opportunities not only for initiating shifts from procedural to conceptual orientations (Beswick, 2007), but also from general conceptual ambitions to the ability to realize it with content-specific substantiations. This requires offering content-specific pedagogical content knowledge for unpacking the content learning goals in view (Friesen & Kuntze, 2020; Prediger et al., 2023).

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Changes in early-career mathematics teachers' noticing during a reflection-based professional development program

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This study analyses changes in the noticing of early-career mathematics teachers during a professional development program based on reflection on practice. Five teachers participated in a three-month course that combined the analysis of professional practice, modelling tasks, and task design. Data included written products, video recordings, and transcripts, analysed through a qualitative approach. Results show a collective progression from initial descriptive and evaluative comments toward more advanced levels of noticing, where teachers interpreted students' mathematical thinking and proposed pedagogical alternatives. The trajectory of one participant illustrates how reflective and situated activities can foster meaningful growth in teachers' noticing competence.

Keywords: Noticing, early-career teacher, professional development, reflection on practice.

Introduction

The ability of mathematics teachers to attend to and interpret students' thinking has been identified as a key element in decision-making that support effective teaching (National Council of Teachers of Mathematics [NCTM], 2014). In this regard, over the last two decades research in mathematics education has increasingly focused on the construct of teacher noticing (Dindyal et al., 2021; Weyers et al., 2024), generally understood as the practice of attending to instructional events, interpreting them, and deciding how to respond (Jacobs et al., 2020). This construct is closely linked to teaching expertise (Stockero, 2021) and is also recognized as a competence that can be taught, practiced, and developed (Amador, 2020). Within this body of research, most studies have focused on analyzing the noticing of both practicing and prospective teachers (Weyers et al., 2024). However, although the literature on noticing is extensive, relatively little is known about how early-career teachers transform their noticing as they transition from initial preparation to their first years of professional practice (e.g., Bastian et al., 2022; Lee & Santagata, 2020), or about the influences of programs designed to foster such development. Therefore, this study seeks to examine changes in the noticing of early-career mathematics teachers as they participate in a professional development program (PDP) oriented toward reflection on professional practice.

Becoming a teacher is considered the first step in the process of professional development (Toker & Aksoy, 2024). Studies have shown that, during their early years, teachers often experience difficulties in interpreting classroom phenomena, particularly students' mathematical thinking (Bastian et al., 2022; Orr & Bieda, 2025). Research on noticing in the early career stage has revealed diverse perspectives. For instance, Copur-Gencturk and Han (2025) found that as novice teachers confront the demands of the profession, their patterns of noticing tend to differ from those observed in more experienced teachers. Guler et al. (2023) reported significant changes in the noticing of novice teachers after participating in a PDP based on e-mentoring. Similarly, Lee and Santagata (2020), in a

longitudinal study, found that having applicable knowledge of students' mathematical thinking from the beginning of one's career supports teachers' professional growth in both knowledge and practice, provided that the school context is aligned with a student-centered vision of teaching.

To contribute to this line of inquiry, the present study addresses the following research question: *How does participation in a professional development program, based on reflection on professional practice, influence changes in the noticing of early-career mathematics teachers?* Addressing this question contributes to the field of teacher noticing in early-career mathematics teachers by providing empirical evidence of how this teacher noticing competence can develop during the transition from initial preparation to the first years of professional practice. This study advances understanding of the formative elements that foster a more sophisticated enactment of the teacher noticing competence, positioning reflection on professional practice as a central component of early professional development.

Conceptual framework: Teacher noticing and professional practices

In the literature, various conceptualizations of teacher noticing have been proposed (Dindyal et al., 2021). In this study, we understand noticing as consisting of three main aspects: (a) identifying what is important in a teaching situation; (b) using available knowledge about the context to reason about that situation; and (c) establishing connections between specific events and broader principles of teaching and learning (van Es & Sherin, 2008). From this perspective, van Es (2011) proposed the *Learning to Notice Framework* (see Table 1), which describes four progressive levels of noticing: basic (level 1), mixed (level 2), focused (level 3), and extended (level 4). These levels make it possible to characterize both *what* teachers notice and *how* they notice. In this study, we draw on this framework to analyze changes in the noticing of early-career mathematics teachers.

Table 1: Summary of learning to notice framework (adapted from van Es, 2011)

	Level 1 Baseline	Level 2 Mixed	Level 3 Focus	Level 4 Extended
What Teacher Notice	Attend to whole class environment, behavior, and learning and to teacher pedagogy.	Focus mainly on teacher pedagogy, with emerging attention to students' mathematical thinking and behavior.	Attend to particular students' mathematical thinking.	Attend to the relationship between particular students' mathematical thinking and between teaching strategies and student mathematical thinking.
How Teacher Notice	Form a general impression with mainly descriptive or evaluative comments, offering little or no supporting evidence.	Form general impressions and highlight key events, providing mainly evaluative with some interpretative comments supported by specific evidence.	Highlight noteworthy events, provide interpretive comments, use specific events and interactions as evidence, and elaborate on them.	Highlight key events, provide interpretations, support with evidence, elaborate on interactions, connect events to teaching principles, and propose alternative pedagogical solutions.

Professional practice is conceived as a network of situated actions and decisions that mediate students' learning. In mathematics, the selection of tasks and the forms of communication established

in the classroom are particularly relevant, as they shape what students can learn (Ponte & Quaresma, 2016). From a broader perspective, professional practice integrates technical, intellectual, and identity dimensions, and it is learned through progressive approximations to authentic teaching contexts (Grossman et al., 2009). At the same time, it constitutes a space for systematic reflection that makes it possible to question routines and transform teaching. In this study, we conceive professional practice as the context in which early-career teachers' noticing patterns are shaped and transformed, in dialogue with their experiences and processes of reflection.

Method

A case study (Cohen et al., 2018) was conducted with five mathematics teachers enrolled in the third term of a professional master's degree program in Guerrero–Mexico (each term consists of three months). These teachers were selected because they met the criterion of being in the early years of their professional careers, with an average of 1.5 years of teaching experience after completing their bachelor's degree in mathematics. From February to April 2025, the participants were enrolled in the following courses: *Analysis of Professional Practice* (APP), *Modeling to Problematize Linearity* (MPL), and *Task Design* (TD) (see Figure 1). The PDP was designed to integrate these courses through a sequence of interconnected activities aimed at promoting reflection on professional practice and fostering the development of teacher noticing competence, as represented in the figure by the arrows.

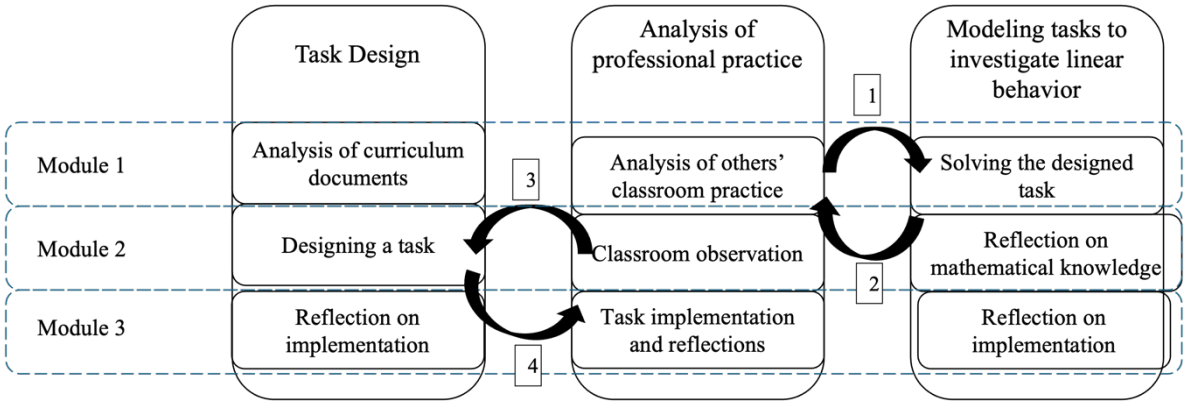

Figure 1: Relationship among the master's degree program courses

The following section describes the tasks carried out in the APP course, coordinated by the first author of this paper, and its connection with the other courses. Although APP was articulated with the remaining courses, the analysis presented in this paper focuses exclusively on the discussions held during the APP sessions. These sessions involved eight group discussions about each assigned task, which were video-recorded and constitute the main data source for this study. In Module 1, within APP, the participating teachers analyzed a video of an external teacher's lesson on linear functions, responding to the question: *What do you notice?* At the same time, in the MT course, they worked on tasks designed to study linearity; these sessions were also recorded (Arrow 1). Subsequently, in APP, they revisited the video recordings of their modeling sessions to discuss what they noticed, now from their own perspective as students solving the tasks (Arrow 2). In Module 2, the teachers observed a class taught by an external teacher at a local high school and took observation notes with the purpose of designing a new instructional task based on that experience, while in parallel, the TD course was

dedicated to refining the details of that design (Arrow 3). In Module 3, they implemented the designed task with the same group they had previously observed and recorded their experiences in field journals (Arrow 4). Finally, they completed an integrative product that articulated reflections derived from the three courses.

The data sources included in this report were: (a) video recordings of the eight group sessions from the APP course; (b) full transcripts of those sessions; and (c) the final written reports produced by each teacher. The analysis was conducted through qualitative content analysis (Mayring, 2000), using van Es's (2011) noticing framework, which comprises four levels and two analytical categories, what teachers notice and how they notice. These categories made it possible to identify both the foci and the forms of attention demonstrated by the teachers when analyzing different moments of professional practice. To ensure the consistency of the process, two of the authors independently coded 20% of the corpus (videos and transcripts simultaneously) for the calibration of the analytical framework, the refinement of coding criteria, and the establishment of intersubjective agreement before proceeding with the analysis of the entire dataset (Miles et al., 2014). The results were then compared, and discrepancies were discussed among the three authors until reaching consensus. Afterward, each author independently reviewed the agreed codes to apply them to the full dataset.

Findings

To address the research question of this study, the findings derived from the analysis are presented below. First, we provide an overview of the changes in the noticing of early-career mathematics teachers, based on written data and session transcripts from the first and final sessions of the APP course. Second, as an illustrative example, we present the trajectory of one teacher across the different levels of noticing throughout the PDP. We selected this case because it showed the most advanced progression.

Table 2 shows the changes in the levels of noticing of the five participating teachers between the first and eighth sessions of the course. In the initial session, four of them (T1, T2, T3, and T5) were at Level 1, characterized by general impressions and predominantly descriptive comments about the situation. An example of this type of response is T1's observation: "It is a recorded lesson in which the topic of functions is being worked on, specifically a linear function." This type of comment reflects a basic attention to the topic of the class, without further attempts at interpretation or connections to students' mathematical thinking.

One teacher (T4) reached Level 2, characterized by more focused attention. In her case, both in writing and oral discussion, she identified specific events, although without deepening her analysis. In contrast to these initial results, by the eighth session important progress in noticing was observed. Four teachers (T1, T3, T4, and T5) reached Level 3, which involves identifying relevant events accompanied by interpretive comments supported by evidence from classroom interactions. In addition, one participant (T2) reached Level 4, which implies not only interpreting based on specific evidence but also establishing connections between observed events and broader principles of teaching and learning, even proposing pedagogical alternatives. These results reveal a collective shift toward higher levels of noticing throughout the course, demonstrating the progressive development of this professional competence among the participants.

Table 2: Changes in levels of noticing

Teacher (T1)	First session	Eighth session
T1	Leve 1	Level 3
T2	Level 1	Level 4
T3	Level 1	Level 3
T4	Level 2	Level 3
T5	Level 1	Level 3

An illustrative example

This example presents the changes achieved by Teacher 2 (T2), who progressed to an extended level of noticing. In the first session, when asked what she had noticed in the video, her comments focused on evaluative judgments, first about students' behaviour and then about the teacher's pedagogy, without deepening her analysis of students' mathematical thinking.

T2: Well, at the beginning I noticed that when he [the teacher in the video] started speaking, only the students in the front were paying attention, while those in the back were each in their own world. What I did like was that he used an example from everyday life. I think the example of the watermelon is more practical because it is something they buy. Another thing I liked is that, for example, I have made the mistake of always telling them line [referring to the linear function], as advancing one, two, three, four, five. And he said something important—that we should also consider the intermediate values, like 1.8 kg or 2.3 kg.

This excerpt shows that the teacher initially observed students' behaviour (attention/distraction) and issued positive judgments about the teacher's strategies (use of everyday examples, to buy watermelon), which corresponds to an initial level of analysis. Although she recognized the importance of considering intermediate values in a linear function, her comment remained descriptive and evaluative, without delving into how students processed that mathematical idea.

In Session 3 (which belongs to Module 1 and Arrow 2), the course facilitator (F) raised a question about a specific event in which the discussion centred on whether the function under study was linear or not. One student had argued that "it is a linear function because it has constant variations". This generated the following dialogue:

F: What was the student referring to when she spoke of a constant variation?
T2: I understood that the weight never changed, that it always went up by 50, and another variation she found was in the length of the spring; that was also a variation that each group found. I think that's what she meant… Also, there's a point where she says it increases by the same amount. I would like to think she was thinking about direct proportionality, and direct proportionality is related to a straight line… she said it increases by the same amount.

In this excerpt, T2 demonstrates a more focused level of noticing on students' mathematical thinking. Unlike her initial comments, here she interprets the student's ideas, seeks to attribute meaning to the notion of constant variation, and connects it to broader mathematical concepts such as direct

proportionality and linearity. Thus, her analysis transcends mere description of what occurred in the video, evidencing a deeper process of interpretation.

Finally, in the presentation of the integrative product, T2 identified and described several events from her own lessons in which she attended to students' thinking, sought to interpret and explain it, and connected it with her task design to propose pedagogical alternatives. This level of analysis reflects an extended level of noticing, as she not only considered students' responses but also interpreted them in relation to the demands of the task and suggested instructional adjustments grounded in principles of mathematics teaching and learning. In her reflection, she stated:

> Regarding the development of the activity, it was considered that the students faced many difficulties and ended up being helped. For example, one of the design questions (Based on the observed increases in both weight and length, what operation should you perform to calculate the length of the spring per added gram? Explain your procedure.) was not very effective in helping them find the constant of proportionality. It was too direct and did not allow them to discover the procedure on their own. Ideally, it would have been better to guide them more gradually, first asking them to calculate the length for 75 grams, then for 50 grams, 25 grams, and finally for 5 grams. Perhaps this would have led them, on their own initiative, to seek the calculation for one gram.

This evidence highlights how the teacher moved beyond attending to events as presented in class, toward interpreting students' mathematical reasoning and relating it to her own task design, proposing alternative instructional strategies.

Discussions and conclusions

The results of this study show that participation in a professional development program based on reflection on practice fostered significant changes in the noticing of early-career mathematics teachers. Although these results are consistent with previous studies reporting advances in novice teachers' noticing (Copur-Gencturk & Han, 2025; Guler et al., 2023; Lee & Santagata, 2020), this work differs in its situated approach. Unlike Guler et al.'s (2023) e-mentoring program or Lee and Santagata's (2020) measurement-focused study, the PDP examined here was part of a master's degree program that integrated three interrelated courses: Analysis of practice, Modeling, and Task design. This structure positioned classroom experiences as the core of reflection, allowing us to document how teacher noticing competence develops through committed and practice-based participation. In particular, we observed a collective progression from initial levels, characterized by general descriptions and evaluative judgments about teaching performance or student behavior, to more advanced levels in which participants began to interpret evidence of students' mathematical thinking and, in some cases, propose grounded pedagogical alternatives. These findings align with the notion that noticing is a competence that can be developed through intentionally designed formative experiences (Jacobs et al., 2020; van Es, 2011). The case of T2 is particularly illustrative, as it reflects a developmental trajectory that moves from comments focused on discipline and student behavior to an extended form of noticing in which she was able to analyze students' mathematical reasoning and connect it to decisions about task design and implementation. This process confirms that systematic reflection on practice, supported by situated activities such as video analysis, classroom observation,

and task implementation, constitutes an effective means for fostering the development of noticing in beginning teachers (Darling-Hammond & Bransford, 2005; Santagata & Yeh, 2014). Moreover, it reinforces Stockero's (2021) assertion that experience alone does not guarantee higher levels of noticing; rather, intentional support is needed to provide opportunities to observe, interpret, and respond to students' mathematical thinking.

In conclusion, the findings provide evidence that situated and reflective professional development programs not only contribute to the acquisition of technical competencies but also strengthen novice teachers' ability to attend to, interpret, and respond to students' mathematical thinking. However, certain limitations must be acknowledged. One limitation concerns the duration of the course, which restricted the opportunity to observe broader or longer-term transformations. Future research could extend the period of analysis to further explore the factors that influence the development of noticing during the early years of teaching. In addition, a longitudinal design would help determine whether the skills observed are maintained and evolve in professional practice or are primarily a result of the training process itself. Finally, it will be important to investigate the institutional conditions that support the consolidation of these changes.

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Teachers' engagement with students' errors in mathematics: Introducing hermeneutical competence

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Errors play a central role in mathematics learning and recent research highlights the importance of how teachers recognize and interpret them. This study investigates mathematics teachers' competencies in identifying and understanding students' errors, building on findings from an Italian national project, which revealed widespread difficulties among teachers in interpreting common student errors. To address this, in this study is introduced the theoretical construct of hermeneutical competence, describing how teachers analyse students' errors and the reasoning behind them. Examples illustrating ways of examining this competence are reported, and findings suggest factors that may influence its development. By focusing on both error identification and interpretation, hermeneutical competence offers a framework for research and professional development, supporting the enhancement of instructional practice across all school levels.

Keywords: Students' errors, teacher education, hermeneutical competence, large scale assessments.

Introduction

The role of errors in mathematics teaching and learning has long been a central topic in mathematics education literature (Borasi, 1987). More recently, this interest has significantly extended to the field of teacher education (Brodie, 2014). This line of research aims to analyse how mathematics teachers engage with students' errors (Hoth et al., 2022): the degree of awareness they possess, the interpretations they provide, and the strategies they employ to capitalize on errors pedagogically, considering their disciplinary and pedagogical knowledge, beliefs, and prior experiences. The present study aligns with this perspective, seeking to deepen the understanding of mathematics teachers' competencies regarding the specific processes of identifying and interpreting students' errors. Interest in these aspects arises from a widely documented difficulty observed in the national research project *Mathematics standardized assessment as a tool for teachers' professional development – MaSt*, involving Italian teachers across various grade levels (from grade 5 to grade 13). The main objective of this study was to investigate the professional development needs of mathematics teachers in relation to national Large-Scale Assessments (LSAs) (Bolondi et al., 2025). In Italy, LSAs are administered by the public research institution INVALSI, the National Institute for the Evaluation of the Education and Training System (www.invalsi.it). They involve students in primary school (grades 2 and 5), lower secondary school (grade 8), and high secondary school (grades 10 and 13). The assessments are conducted annually, and the results are made publicly available each year. Schools receive detailed reports on their data, including comparisons with other educational institutions in the same geographical area, with the same characteristics (type of school, socio-economic status, etc.). In the aforementioned study, teachers were asked to reflect on the most common errors made by Italian students at the national level, based on items from past INVALSI assessments. It emerged that, regardless of grade level, most participating teachers found it challenging to identify the most frequently chosen incorrect responses and, consequently, to interpret these errors consistently with

the main findings in mathematics education research (Cibien, 2025; Viola et al., 2024). This difficulty highlighted the need to investigate which knowledge and competencies teachers mobilize when reflecting on students' possible errors and the reasoning behind them. To address this need, the necessity of delineating a new theoretical construct has emerged, specifically aimed at describing the competencies mathematics teachers activate in the processes of identifying and interpreting students' errors: hermeneutical competence, which will be detailed in the following section. The introduction of hermeneutical competence seeks to contribute to research in mathematics education by providing a new theoretical tool to study a fundamental and specific aspect of mathematics teachers' instructional practice. This construct differs from existing approaches due to its targeted focus on both the identification of errors and the interpretation of the reasoning and motivations underlying them. Within the context of teacher education, it also constitutes a central element to consider and foster as a key competence, to be developed during the learning-to-teach process and consolidated in the teaching practice of teachers at all grade levels. In this way, the proposal of hermeneutical competence contributes to the broader reflection on the foundational aspects that mathematics teachers need to develop as they learn to teach. The aim is also to contribute to the international discussion on teachers' engagement with students' errors, in line with studies that have employed LSAs data to investigate how teachers make sense of students' mathematical reasoning (Williams & Ryan, 2000). The focus of the paper will be Italian national LSAs INVALSI.

Theoretical background

The concept of hermeneutical competence arises from the need to describe the specific competence that mathematics teachers are expected to have in relation to two fundamental processes, identified by Wutte and Seifried (2017) as part of *Professional Error Competence* (PEC): the identification and interpretation of students' errors. Unlike the approach proposed by Wutte and Seifried, these processes are here regarded not merely as knowledge, but as competence, understood in the sense of Weinert (2001) as an integration of knowledge, skills, and attitudes mobilized to address diverse educational situations. It is based on this definition of competence that the theoretical construct of hermeneutical competence has been conceptualized, integrating elements from the two frameworks of *diagnostic competence* (Schrader, 2011) and *interpretative knowledge* (Ribeiro et al., 2013).

Diagnostic competence (Schrader, 2011) refers to the knowledge, skills, and beliefs necessary for teachers to carry out diagnostic activities, including the collection, recognition, and interpretation of information on learning conditions, processes, and outcomes, using tools such as tests, student work analyses, observations, or interviews. Its focus may be content-related, addressing mathematical content, or student-related, addressing motivation or behavior (Hoth et al., 2016) and this information allows teachers to plan interventions, inform students and parents, and guide resource allocation (Leuders et al., 2017). Specifically, diagnostic competence in error situations (Heinrichs & Kaiser, 2018) refers to interpreting students' errors using informal or semi-formal methods, rather than planning corrective actions.

Interpretative knowledge, as defined by Ribeiro and colleagues (2013), refers to the knowledge a mathematics teacher needs to attribute meaning (i.e., interpret) students' solutions. This concept is not limited to students' work containing errors but extends to any solution exhibiting ambiguities or

inaccuracies in reasoning, regardless of the correctness of the answer. Moreover, interpretative knowledge does not only intervene in the interpretation process itself but also manifests in its application to support students' learning, as occurs in the diagnostic activities described by Schrader (2011). This knowledge thus constitutes a central aspect of a mathematics teacher's instructional practice (Ribeiro et al., 2016).

Diagnostic competence in error situations (Heinrichs & Kaiser, 2018) and the process of attributing meaning to students' solutions within the framework of interpretative knowledge (Ribeiro et al., 2013) represent the crucial theoretical elements that have led to the conceptualization of hermeneutical competence, a construct consistently theorized with reference to error situations in mathematics education. Hermeneutical competence is thus defined as the practice-related competence required of mathematics teachers in order to identify and interpret students' errors on the basis of their work, whether verbal, graphical, or written. Its practice-related nature makes it dynamic, requiring teachers to account for the variety of classroom situations and the relation between hermeneutical objects and mathematical content. Within this framework, hermeneutical objects (in the sense proposed by Figal, 2022) are represented by INVALSI items, which are designed according to the National Curriculum Guidelines and grounded in key findings from mathematics education research, enabling the identification of macro-phenomena observable at the national level (Ferretti & Bolondi, 2019; Bolondi & Ferretti, 2021). Their characteristics reduce confusion between object and interpretative process (Figal, 2022), while instead providing a reliable statistical measure of competence, not subject to individual interpretations. Teachers' reflection on these items and on the macro-phenomena emerging from LSAs can foster the emergence and analysis of their hermeneutical competence. Moreover, using LSAs results within professional development programs allows for the modification and enhancement of this competence, enriching it in light of the very evidence produced. Strong hermeneutical competence entails examining and correctly interpreting LSAs items, recognizing their purpose, identifying potential student difficulties, and inferring and interpreting common national-level errors, grounded in consolidated knowledge rather than solely in subjective experience. LSAs items are thus particularly suitable as hermeneutical objects for analyzing and measuring mathematics teachers' hermeneutical competence.

The aim of this article is to explore the theoretical construct of hermeneutical competence through the analysis of teachers' responses to research questionnaires and semi-structured interviews. Rather than providing full psychometric validation, the study investigates how the construct manifests and how interactions with LSAs data may influence its development. In this way, the analysis offers qualitative insights that illuminate the complexity and multidimensionality of hermeneutical competence in teaching contexts.

Methods

The introduction of the new theoretical construct of hermeneutical competence raises the question of how it may be subjected to empirical investigation: how it might manifest itself in teachers' practices, how it can be measured, and under which strategies or conditions it may be enhanced. The need to introduce the concept of hermeneutical competence and to empirically explore its manifestations emerged from the analysis of a research questionnaire administered, within the framework of the

national *MaSt* project (Viola et al., 2024), to teachers representing different school levels from across Italy. The questionnaire was administered online, anonymously, through Google Forms, to a total of 473 teachers. The sample was large and diverse, including participants from different school levels and with different professional statuses. Specifically, the respondents comprised 155 in-service and 88 pre-service primary school teachers, 61 in-service and 25 pre-service lower secondary school teachers, 70 in-service and 74 pre-service high secondary school teachers. The questionnaire presented teachers with a set of multiple-choice items drawn from previous INVALSI assessments, appropriate to their respective school level, asking them to indicate what they considered to be the purpose of the item, as well as which incorrect option was most frequently selected by students nationwide, and to justify their choice by interpreting the causes of the error and providing an explanation for it. The questions were adapted from the validated questionnaire developed by the SIRD research group (for further details, see Arzarello & Ferretti, 2022) and the INVALSI items chosen were selected principally because they are emblematic examples of macro-phenomena identified at the national level. In particular, they were multiple-choice items characterized by very high percentages of incorrect responses (thus indicating a widespread difficulty on a national scale) and by incorrect options readily interpretable through constructs well established in the literature. Findings from this research questionnaire revealed that, regardless of the school level taught, the participating teachers experienced considerable difficulties in identifying and interpreting the most common errors made by Italian students at the national level in large-scale mathematics assessments (for further discussion, see Viola et al., 2024). Teachers' answers to the questionnaire were analyzed using a grounded theory approach (Charmaz, 1983): the response categories were developed inductively from the data, classified according to the incorrect option selected and the explanation provided (for a detailed example, see Cibien, 2025). Results from the analysis of responses to the item-related question *"What do you think has been the most popular wrong option nationwide? Why?"* showed a widespread difficulty among teachers, both in identifying the most frequently chosen incorrect response at the national level and in interpreting the possible reasoning underlying students' choices. This analysis made it possible, in a preliminary phase, to identify areas of relative weakness in teachers' hermeneutical competence, in particular referring to the process of errors' identification. To further investigate these aspects, a series of semi-structured qualitative interviews was conducted with secondary school teachers who had expressed willingness to continue participating in the research after completing the questionnaire, following the interview guide approach (Patton, 1980).The interview questions were designed based on the data collected from the questionnaire, particularly regarding the items to be addressed with the teachers and the selection of errors to focus on. The interviews are still in progress; preliminary findings are drawn from initial analyses, with 42 secondary school teachers interviewed so far. During the interviews, the teachers were once again presented with the same items previously included in the questionnaire and were asked to expand and elaborate on the reflections they had made earlier, regarding the choice of the most common incorrect option selected by students and their interpretation of it. In doing so, the aim was to gain deeper insight into the reasons underlying their choices, which in the questionnaire responses were often expressed very concisely. Following this initial discussion, teachers were then presented with protocols containing students' errors previously collected, representing various types of errors: for each incorrect option, concrete examples were provided to explore which errors teachers

regarded as most plausible and widespread at the national level. This approach allowed for the direct observation of teachers' interpretations of students' work. At the end of the protocol reflection, teachers were asked to estimate the difficulty of the items, indicating the expected percentage of correct responses at the national level, as well as to provide an estimate of the distribution of choices among the various incorrect options. This latter request aimed, in particular, to investigate the degree of confidence and awareness with which teachers had identified the most frequently selected incorrect option and how far their interpretations of the causes of errors were in line with the literature. Subsequently, teachers were shown the actual national response percentages provided by INVALSI. This step allowed a discussion both on the accuracy of their perceptions and on possible explanations for the observed results. Students' protocols and national LSA data were used to investigate how teachers' hermeneutical competence might be shaped and developed through engagement with these resources. The following section presents excerpts from interviews with secondary school teachers to illustrate how this competence manifests and changes over time.

Discussion

The discussion is based on excerpts from two interviews with high secondary school teachers. One of the items presented to the teachers was item 15 from the 2012 INVALSI assessments for Grade 10 students, which asked: "*Dividing a number by 0.2 is the same as multiplying it by…*", with the following response options: A) 1/5, B) 1/2, C) 2, and D) 5. At the national level, the correct option D was chosen by 24.4% of students; among the incorrect options, A was selected most frequently (46.1%), followed by B (20.2%) and C (7.7%). Missing or invalid responses accounted for 1.6%.

The item was discussed with an experienced teacher with more than 30 years of service (referred to as T1). In the preliminary questionnaire, T1 had indicated option B as the incorrect option most frequently chosen at the national level, without providing justification. During the interview, when asked to explain her choice, she argued that "The form 1/2 simultaneously conveys the idea of division and includes the number 2, which is given in the problem; the student takes these two elements and finds a solution. This results from a hasty reading." She also added that she initially had been uncertain between options B and C, since both recall the "given" number (2) and might therefore appear appealing to students. She was then shown several protocols representative of different types of errors (corresponding to each of the three incorrect options), but all attributable to intuitive models (Fischbein et al., 1985) whereby students tend to generalize properties of operations they already know to fractions. T1 reported that she would not have expected such errors and, in some cases (e.g., a student choosing A and justifying the choice by computing 1/5*2 = 2/10 = 0.2), stated that she could not reconstruct the reasoning underlying the students' responses. The discrepancy between the teacher's interpretation and what is documented in the literature regarding this type of reasoning suggests a hermeneutical competence still in development, both in identifying the error and in interpreting possible critical steps in the solution process linked to the underlying intuitive model.

The same item was discussed with T2, a first-year high secondary school teacher. She also identified option B as the most likely incorrect option at the national level, ruling out option A on the grounds that "it is unlikely that students would perform a conversion." She was shown the same protocols as T1 but, unlike the latter, regarded the errors as plausible. However, she did not interpret them in

relation to intuitive models; instead, she attributed them to hasty reading: “They do not read the entire text and simply perform a conversion; they write spontaneously.” After reflecting on the protocols and on the national data, T2 reported that she had not anticipated the observed distribution of distractors and acknowledged that she might have underestimated some competencies of Grade 10 students: “Between 1/5 and 1/2, I did not expect so many to know that 0.2 is 1/5; I had this bias.” In this case the reflection on the protocols appears to have contributed, at least partially, to a shift in her hermeneutical competence: although she did not initially identify or interpret the item as expected, the guided reflection prompted her to reconsider her interpretation of the errors discussed.

In conclusion, the interviews highlight a range of ways in which hermeneutical competence manifests among mathematics teachers. While some teachers appeared to struggle in identifying the incorrect options most frequently chosen at the national level and in interpreting students’ errors in relation to them, others, despite misidentifying the most common error, were able to offer plausible interpretations or revise their initial assumptions when engaging with the resources presented. These variations point to different manifestations of hermeneutical competence and show how the combined use of student protocols and national LSAs data helps make teachers’ processes visible and, at times, stimulates them. Overall, findings suggest that interviews constitute an effective means for exploring and potentially fostering this competence within mathematics teacher professional development.

Conclusion

This study outlined the process leading to the definition of the new theoretical construct of hermeneutical competence and the initial steps taken to investigate its manifestations. Questionnaire data and preliminary semi-structured interviews highlighted the complexity of how teachers identify and interpret students’ errors, confirming hermeneutical competence as a distinct, practice-related dimension of what mathematics teachers need to know and be able to do. Findings also point to the need for refined research tools to capture its multifaceted nature and for further studies on how it evolves, how it interacts with teachers’ beliefs, experiences, and knowledge, and which conditions best support its development.

From a teacher-education perspective, the study underscores the relevance of hermeneutical competence for promoting meaningful engagement with students’ mathematical reasoning. As widely documented (Brodie, 2014; Hoth et al., 2022), teachers’ approaches to students’ errors play a decisive role in shaping classroom discourse and learning opportunities. Findings suggest that structured activities involving the analysis of students’ protocols, supported by LSAs data, may constitute valuable tools for fostering this competence within initial teacher education and continuous professional development. Embedding such activities in teacher education programs can help teachers develop more informed, research-grounded interpretations of students’ errors.

In conclusion, hermeneutical competence offers a new theoretical lens for examining how teachers identify and interpret students’ mathematical errors, complementing frameworks such as interpretative knowledge (Ribeiro et al., 2013) and diagnostic competence in error situations (Heinrichs & Kaiser, 2018). Further empirical research will be essential not only to refine and validate the construct, but also to translate it into actionable guidelines and tools that can meaningfully support mathematics teacher education and professional development.

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Primary teachers' reflections on student errors in mathematics: Insights from standardized assessment results

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This paper explores Italian primary teachers' reflections on students' errors in standardized mathematics assessments, specifically using an item from the Italian National standardized test. This qualitative study investigates how teachers mobilize their mathematical and pedagogical content knowledge when analyzing possible student errors. A questionnaire was administered to 128 in-service teachers. The results highlight which aspects of mathematics teachers' specialised knowledge can emerge from their response to this questionnaire.

Keywords: Standardized assessment, teacher professional development, specialised content knowledge, in-service primary teachers.

Introduction

Alongside major international surveys such as OECD-PISA, IEA-TIMSS, and IEA-PIRLS, most countries have in recent decades implemented national standardized assessment systems in primary and secondary education (Eurydice, 2016). Over time, such assessments have gradually influenced classroom practices, affecting both teaching and learning processes (Doig, 2006), and they have gained increasing attention in mathematics education research (Suurtamm et al., 2016).

In Italy, the National Institute for the Evaluation of the Education and Training System (INVALSI) carries out national assessments with the main purpose of evaluating the Italian educational system. The INVALSI national standardized tests have been administered since 2008 and cover both primary and secondary education, specifically for grades 2, 5, 8, 10, and 13 (INVALSI, 2018). These tests are aligned with the Italian National Guidelines (MIUR, 2012) and are designed to balance the demands of standardized assessment with contemporary research paradigms in mathematics education (Ferretti et al., 2020). Such characteristics allow them to be fruitfully used in mathematics education research, for instance for the study of didactical phenomena (Spagnolo et al., 2021). When thoughtfully integrated into teacher education, findings from standardized assessments can provide teachers with tools and benchmarks for diagnostic assessment, indicate directions for formative assessment, and enhance their knowledge and practice (Ferretti et al., 2022). For the context of this study, we focused on the analysis of students' errors. Research has long shown that errors may be valuable opportunities for teaching and learning, when they are opportunely explored (Brodie, 2014). Teachers' ability to recognize and interpret students' errors has been identified as an essential component of their professional knowledge (Larraín & Kaiser, 2019). However, how best to support teachers in developing this interpretative ability remains under discussion (Larraín & Kaiser, 2019). Recent studies show how INVALSI data offer a powerful tool to investigate such ability (e.g., Viola et al., 2024).

In this paper, we investigate Mathematics Teachers' Specialised Knowledge (Carrillo-Yañez et al., 2018) in investigating teachers' reflections on students' errors in answering an INVALSI item.

The study is part of the national research project *Mathematics Standardized Assessment as a Tool for Teachers' Professional Development* (*MaSt*). The MaSt project aims to develop theoretical and operational tools to leverage the use of national mathematics assessment materials for the professional growth of teachers at all school levels. Specifically, the project seeks to promote the formative use, at both institutional and classroom levels, of INVALSI assessment materials of the effective teaching practices that may emerge from them.

Mathematics Teacher Specialised Knowledge

The study of teachers' professional knowledge in mathematics education has been significantly influenced by Shulman's (1986) foundational work, which delineated two interrelated dimensions of knowledge: Subject Matter Knowledge and Pedagogical Content Knowledge. Building upon this dichotomy, the Mathematics Teacher Specialised Knowledge (MTSK) model was subsequently developed by Carrillo-Yañez and colleagues (2018), offering a nuanced framework for understanding the specialised knowledge of mathematics teachers.

The MTSK model conceptualizes that teachers' professional knowledge is comprised of two interconnected domains: Mathematics Knowledge (MK) and Pedagogical Content Knowledge (PCK), which converge and intersect within the teacher's belief system. A distinctive feature of this model is its recognition that teaching practices are shaped not only by teachers' mathematical knowledge, but also by their conceptions and beliefs (Thompson, 1992) about the discipline, student learning, and instructional approaches. These core beliefs permeate and interact with each subdomain of knowledge, influencing both the development and classroom implementation of teachers' knowledge (Carrillo-Yañez et al., 2018).

The subdomains of Mathematics Knowledge (MK) include:

- KoT – Knowledge of Topics (such as knowledge of definitions, properties, procedures, and connections among mathematical topics).
- KSM – Knowledge of the Structure of Mathematics (knowledge of how mathematics is organized, including intra- and interconnections).
- KPM – Knowledge of Practices in Mathematics (knowledge of key mathematical processes and practices, such as proving, modelling, problem solving, and representing).

The subdomains of Pedagogical Content Knowledge (PCK) are:

- KMT – Knowledge of Mathematics Teaching (knowledge of approaches, resources, technologies, and strategies for presenting and representing mathematical content).
- KFLM – Knowledge of Features of Learning Mathematics (knowledge of learning theories, of the ways in which students interact with mathematics, and of common misconceptions or errors);
- KMLS – Knowledge of Mathematics Learning Standards (knowledge of curriculum guidelines, expected outcomes, and objectives at different school levels).

Teachers' ability to interpret and use students' errors in their teaching is influenced by both their subject knowledge and their pedagogical content knowledge (Brodie, 2014). Our research investigates which domains of teacher Content Knowledge emerge from teachers' reflections on INVALSI item. We use the MTSK model (Carrillo-Yañez et al., 2018) to address this purpose and answer the following research question: Which domains of Mathematics Teacher Specialised Knowledge (MTSK) emerge from teachers' reflections on students' errors?

Method

Questionnaire

As part of the MaSt project, a questionnaire was developed and administered to in-service primary teachers. Its' aim was to capture how teachers approach and make sense of standardized assessments within the framework of professional development. After a pilot phase, the final version was distributed to participants online. Answers were anonymously collected. This study considers the responses of 128 in-service primary school teachers (grades 1–5) from across Italy (convenience sample). In this context, mathematics education at primary school level is commonly provided by generalist teachers, who frequently undergo a process of subject-specific specialization as they gain experience and advance in their professional roles.

The questionnaire comprised three sections. The first section collected demographic and professional background information. The second focused on INVALSI mathematics items, which were selected for specific characteristics, such as being suitable to illustrate some national macro-phenomena or the percentage of correct answers. In the questionnaire for primary school teachers, only primary-level tasks were included. Each item was followed by questions regarding teachers' awareness of common students' errors, possible pedagogical and assessment uses, curricular alignment, and the influence of wording. The final section addressed broader issues, examining teachers' perceptions of INVALSI resources and their relation to everyday teaching and assessment practices.

This work focuses on teachers' reflections on a INVALSI item, described in the following paragraph.

Analysis of INVALSI item

Figure 1 represents one INVALSI item that was included in the questionnaire. The item was administered by INVALSI to Grade 5 students in Italian schools in 2018. This item has been selected as illustrates common misconceptions related to the representation of rational numbers on the number line, which have been documented in the literature (e.g., De Wolf et al., 2014). Such errors are often linked to semantic and syntactic challenges in interpreting the fraction symbol. In particular, option A may reflect students who place the fraction at the midpoint of the segment between 2 and 3; option B may indicate those who interpret $\frac{3}{2}$ as "three and a half"; and option D may correspond to students who misread $\frac{3}{2}$ as 3.2.

Analysis of national response data shows that around 17% of the students answered the item correctly, making it the item with the lowest correct response rate in the entire mathematics test that year. Each of the incorrect options were selected by approximately 23–33% of students.

Difficulties in ordering and placing rational numbers on the number line, especially when represented as fractions, have been widely reported. These challenges are often related to interpreting the value of digits and the fraction bar (Behr & Bright, 1984; Ni & Zhou, 2005; Saxe et al., 2007), as reflected in options A, B, and D. Furthermore, the item presents numbers on a graduated line, requiring students to interpret and navigate the scale, which adds further complexity. As noted by De Wolf et al. (2014), the number line makes visible the difficulties students encounter when handling different representations of rational numbers, particularly in terms of comparing and ordering fractions and decimals.

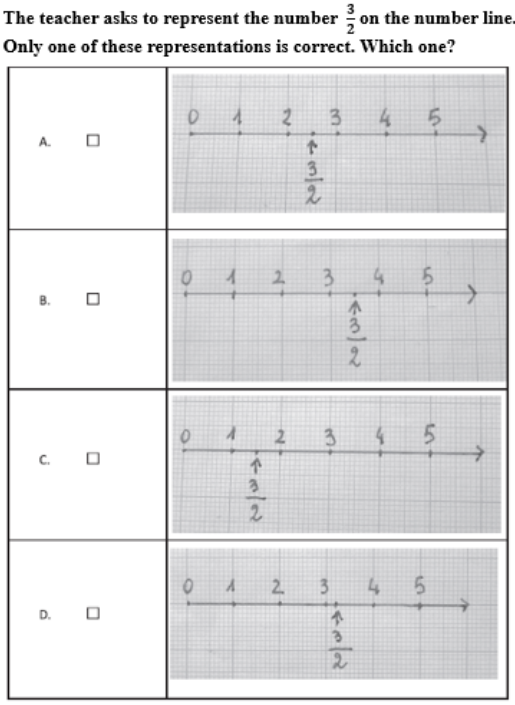

Figure 1: Item D25, INVALSI Grade 5 Mathematics Test, 2018 (translation by the authors)

As noted earlier, this item was included in the questionnaire for primary school teachers, who were asked to identify which incorrect option they believed was most frequently chosen nationally and to justify their choice.

Data analysis

Data analysis involved two phases. First, teachers' motivations for their selected answers were categorized by means of thematic analysis (Cohen et al., 2018) to study which answer the teacher selected, and which students' errors were identified. Second, answers were analyzed in order to identify whether and which subcategories of the MTSK model emerge in teachers' motivations for their selected answer. Findings from these two phases were then triangulated to gain a more comprehensive understanding of teachers' knowledge.

Results

A total of 128 teachers completed the questionnaire, of whom 86 reported having taught mathematics in a grade 5 class. Among respondents, 67 teachers (52.3%) identified option B as the most frequently

chosen, 32 teachers (25%) indicated option D, and 24 teachers (18.75%) selected option A. Seven teachers either responded "I don't know" or selected multiple options.

Analysis of teachers' reflections led to the identification of seven overarching categories of possible explanations of student errors. The largest category, "**Conversion between fraction and number line**", included 41 responses, representing about one-third of the sample. These responses highlighted students' difficulties in translating a fraction into a corresponding position on the number line. Teachers frequently referred to misinterpretations of the numerator and denominator in relation to segments of the line. For example: "D because students may associate the denominator with the number of segments and the numerator with the whole number on the line." Another teacher states: "B, because unfortunately, in Italy, even reading fractions can cause misunderstandings. When children read 'three halves,' they first look for 3 on the number line and then for its half. If we read it (as in some other countries) as 'halves of three,' the concept might be more intuitive." The category "**Conversion between fraction and decimal**" was indicated by 13 teachers. Responses in this category reflected errors arising from incorrect decimal conversion or confusion between fractional and decimal representations, in which the fraction line is not interpreted as division. Illustrative examples include: "D; many students transform $\frac{3}{2}$ into 3.2 without recalling the steps or strategies to convert fractions to decimals." The category "**Partial reading of the fraction**" included 15 responses across all distractors. Teachers in this group noted that students often focus on only one part of the fraction—typically the numerator or denominator—without recognizing the relationship between the two. Examples are: "I think the most frequently chosen option is B because the whole number 3 corresponds to the numerator." The category "**Similarity between fractions and integers on the line**" included 11 responses. Here, teachers emphasized that students often concentrate on individual digits rather than the fractional relationship, for instance: "D, because the student takes into account the numbers, they 'see' but not their ratio." The category "**Rushed reading**" contained nine responses in which teachers attributed errors to hasty or unconsidered reading of the question. An example response is: "A. A hasty reading can be misleading." The category, "**Knowledge of fractions**" comprised three responses in which teachers linked students' errors to incomplete understanding of fraction concepts, without further specification. For example: "I think D, because most students know 'everything' about fractions but do not grasp their true meaning." The final category, "**Non-standard situation**", contained a single response noting students' unfamiliarity with atypical or non-routine problems: "A, because students are not accustomed to this type of problem and textbook exercises are not presented in this way, leaving them confused." Finally, 27 teachers did not provide justifications for their choice or responded: "I don't know."

Analysis of teachers' responses considering the MTSK framework revealed links between their explanations of students' errors and their content and pedagogical knowledge. Knowledge of Topics (KoT) was evident in references to the definition and representation of fractions. When reflecting on students' errors, teachers explicitly drew on their knowledge of fractions, for example, by identifying the numerator and denominator of the fraction, as in the answer: "A, because the numerator is 3 and the denominator is 2". KoT appeared in the categories "Conversion between fraction and number," "Conversion between fraction and decimal," "Partial reading of the fraction," and "Similarity between fraction and integers." In several of these responses, teachers' topic knowledge was clearly

expressed and appeared to guide their analysis of the incorrect answers. As expected, Knowledge of Features of Learning Mathematics (KFLM) was strongly represented, as teachers recognized typical errors related to reading fractions, converting to decimals, and handling the number line. In this sense, all the over-mentioned categories are related to the KFLM. In some responses, teachers explicitly recalled typical student errors or behaviors, which helped them identify the incorrect option most frequently chosen by pupils. For example, in the response: "D, because many pupils convert $\frac{3}{2}$ into 3.2 without remembering the steps or strategies needed to convert a fraction into a decimal number and vice versa."

Knowledge of Mathematical Standard (KMS) and Knowledge of Mathematics Teaching (KMT) did not emerge explicitly, however, we think teachers' answers show some insight into these knowledge subcategories, for example, in noticing the familiarity/unfamiliarity of students with such problematic situations, and the knowledge mobilized by students in such an item.

Discussion and conclusions

The study highlights how teachers mobilize their KoT and KFLM to describe errors and name plausible causes. Given that this study draws on INVALSI test materials, it is worth noting that such insights are made possible by the careful design of distractors. The development of INVALSI tests is informed by expert knowledge and a thorough understanding of the relevant literature in mathematics education and National Guidelines. Particular attention is devoted to the construction of distractors, which play a key role in revealing students' reasoning and misconceptions. Considering the Features of Learning Mathematics, teachers identify possible misconceptions or mistakes at different levels. Some teachers attribute students' wrong choices to inappropriate behaviors, such as not reading the text carefully or focusing solely on the numerical similarity between the number in the text and the number in the figures. However, most teachers link errors to difficulties within the domain of fractions and rational numbers. Among these, some responses indicate a lack of understanding of the meaning of fractions—encompassing a cognitive dimension—while others highlight challenges in recalling the steps or strategies needed to convert fractions to decimals, thus mainly considering the procedural aspects.

In summary, this exploratory analysis of primary teachers' reflections on students' answers to an INVALSI item, indicates that teachers mainly rely on KoT and KFLM when analyzing student errors. This reflects their awareness of different student behaviors, misconceptions, and difficulties.

To foster a formative use of INVALSI materials and results—and to translate insights from students' incorrect responses into enhanced classroom practice—we believe teacher education should explicitly link error analysis to instructional decision-making and curriculum standards (KMT, KMLS). Moreover, discussing and reflecting on research results can enhance teachers' knowledge regarding the teaching-learning process (KFLM and KMT), providing teachers with information that may influence their students' error analysis. These reflections may provide valuable guidance for the development and implementation of a model for teacher educational programs. The results could also be connected to other research strands, such as those on teacher noticing and interpretative knowledge (Ribeiro et al., 2016), which would require further theoretical networking among these frameworks. Future research will focus on investigating these potential connections.

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Affective domain in mathematics teacher educators' activities

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This study investigates how mathematics teacher educators (MTEs) take affective domain into account during their educational activities. Utilizing a Grounded Theory approach to analyse the reflections of ten MTEs, the research explores the significance of affective aspects in mathematics teacher education across various settings and educational levels. The qualitative analysis of the MTEs' reflections identified four key areas: the inclusion of emotions, the MTEs' beliefs, the teachers, and the educational approach. These initial findings offer a preliminary perspective, highlighting aspects that emerge from MTEs' own reflections on their professional practices.

Keywords: Affective domain, mathematics teacher educators, educational practices.

Introduction

Mathematics Teacher Educators' knowledge and activities have emerged as a significant topic within the research agenda of mathematics education (Beswick & Goos, 2018; Goos & Beswick, 2021). The present work aims to extend this line of inquiry by exploring the role of the affective domain in MTEs' knowledge and practices. The literature clearly highlights the key role of teachers' beliefs in the teaching and learning of mathematics, particularly emphasizing the connection between beliefs and knowledge (Philipp, 2007; Ponte, 1999). Building upon this established foundation, the present study aims to expand this research focus to investigate mathematics teacher educators. In our explorative study we gather evidence about involvement of affective domain in MTEs' educational activities. We focus on the affective domain broadly. We conducted a qualitative study to identify some affective aspects that MTEs declare to consider and discuss in their teaching activities. Namely, the research question is: According to MTEs, what key areas emerge during discussions with teachers regarding the affective aspects involved in the processes of teaching and learning mathematics?

Background

The role and professional features of MTEs have been object of research in the last fifteen years. Literature provides a general definition of a MTE as someone in charge of educating mathematics teachers, who may be pre-service or in-service, and at initial or continuous teacher education: "[…] mathematics educators include educators who facilitate the learning of mathematics, as well as educators who facilitate the teaching of (or learning-to-teach) mathematics" (Zaslavsky, 2009**,** p.1). MTEs have different profiles according to their background and circumstances (Giadas et al., 2024a), basically varying regarding what they teach (mathematics contents, mathematics education contents, or both), the teachers' profile (in-service, pre-service teachers, or both), the type of training (academic courses, internships, mentorships, professional development courses or microcredits, etc.), and their relationship with research (MTEs involved in research activities or focused on teacher training). Researchers have focused on the description of mathematical and didactical knowledge of MTEs (e.g.

Goos & Beswik, 2021), often looking for a relationship between MTEs' and teachers' knowledge – see, for instance, a detailed study of the Spanish case in Giadas et al. (2024) or the recent review by Castro-Superfine et al. (2024). When considering that knowledge, as Beswick and Goos (2018) stressed, it becomes necessary to interpret it in an extensive way, that is, we should also consider MTEs' beliefs and reflections about their activity. We assume that affective aspects are a constituting domain of MTEs' knowledge, together with and permeating subject and pedagogical-content knowledge, as it was established in the MTSEK model for mathematics teachers (Ferretti et al., 2021; Martignone et al., 2022) that extend the MTSK model (Carrillo-Yáñez et al., 2018) by including beliefs and conceptions about mathematics and its teaching and learning as one domain. A characteristic of this model is that teachers' actions are strongly linked not only to their mathematical knowledge, but also to their conceptions and beliefs (Thompson, 1992) about mathematics, how it is learned, and how it should be taught. Consequently, a more or less coherent set of beliefs permeates a teacher's knowledge across each of the specialized knowledge subdomains. There are at least three different approaches in the literature to the construction of a MTE's knowledge model starting from the MTSK model, the three of them quite recent. The first one is based on observation, and it is illustrated in Pascual et al. (2021). The second one aims at a literature comparison (Escudero-Ávila et al., 2021). The third one (MTESK model) is based on interviewing MTEs, as in Ferretti et al. (2021) and Martignone et al. (2022), or the trained teachers, as in Giadas et al. (2025). We must stress that the three approaches are not contradictory, but complementary, since they analyse the same reality from different methodological perspectives and, probably, the combination of results from the three of them would provide the most complete description of the MTE's knowledge. MTESK model (Ferretti et al., 2021; Martignone et al., 2022), like the MTSK model, makes explicit the presence of MTEs' beliefs when considering MTEs' knowledge and actions. In this paper, we present the initial steps of a study designed to expand and deepen our understanding of which possible affective aspects are taken in consideration by some MTEs in their educational activities.

Methodology

Ten MTEs were considered as informants for this research. The selection of the informants was not probabilistic but depending on the ease of access to each one of them and the time they had available for participating. They comprised researcher in mathematics and in mathematics education, primary-school teachers and secondary-school mathematics teachers involved in pre-service teacher education, and in-service teacher development. Their professional engagement varied considerably, with some working exclusively in in-service teacher education and others participating in both pre-service and in-service programs. This diversity results in a heterogeneous sample, characterized by a wide range of backgrounds, teaching levels, and research interests, thereby enriching the perspectives represented in the study. A standardized open-ended interview was conducted (Patton, 1980). The MTEs were posed the following open-ended question: In your educational activities, do you discuss with teachers' affective aspects involved in the processes of teaching and learning mathematics? If so, why and what aspects usually arise? If not, why not?". They had no limited time or space to answer the question, in most of the cases the interview was orally conducted. MTEs' orally answers were recorded and transcribed by the researchers. The question was posed in the mother language of the interviewed MTEs (either Italian or Spanish), so that they were also able to answer in that

language. The interviews were translated into English by the researchers; all the translation issues were discussed before the final version. The analysis of MTE's answers was performed by using Grounded Theory to obtain the main underlying ideas and their relationship. Grounded Theory is a general research approach (Strauss & Corbin, 1997) that does not necessitate a pre-established theoretical framework, as its core tenet is that the theory inductively emerges directly from the data. In contrast to approaches that presuppose an existing theoretical structure, this methodology is not guided by preconceived hypotheses but is instead constructed from the collected and analysed empirical evidence. We employed simultaneous data collection and analysis practices with the aim of achieving repetition of patterns and themes through methodical coding. Initially, we broke the transcripts identifying natural breaks in the MTEs' speech and their possible correspondence with topic changes, thus we obtained 90 different excerpts (in a variable number from one MTE to another). In a second step, the researchers engaged in open coding, categorizing the data into emerging concepts and themes related to MTEs' reflections on their professional development, influences and professional trajectories. Triangulation was employed at this stage to reinforce the trustworthiness and rigor of the findings, as the three researchers independently coded the data and subsequently compared the interpretations. The third step involved synthesizing the identified codes into categories (first individually, then triangulated), looking for comparisons to interpret the MTEs' answers relative to our research question. Through constant comparison, researchers refined the categories and identified the relationship among them. The iterative process – analysing and comparing the data, returning to the field when necessary, and incorporating diverse viewpoints – further strengthened the robustness of the coding process. Thus, we finally reached the representation of the key ideas, classified into different codes and categories.

Results

Even when the sample size is not great, several similarities arose about effectively considering the affective domain in mathematics teacher education. After the content analysis we have identified the following four categories from MTEs' answers (percentages indicate the appearances of each category over the total number of different excerpts): *inclusion of emotions* (34.2%), *MTEs' beliefs* (34.2%), *teachers* (46.3%), and *educational approach* (78.2%). There were also some codes (around 20%) related to contextual or tangential aspects, non-substantial to the goals of this paper, and, therefore, we do not consider here. We must recall that the same excerpt can include evidence from different codes, and, thus, different categories. We now explain what type of codes constitute each category, in increasing order of frequencies.

Inclusion of emotions

This category concerns the presence (or absence) of the emotional aspects during the educational activities, as well as the MTEs' aim to plan these aspects. Most of the informants (7 cases) explicitly acknowledged a previous planning (coded as *declaration of practices*) for considering affective aspects. Here, MTE#4 affirms the inclusion of emotional aspects in his/her teaching:

> MTE#4: So the emotional aspect must be addressed and above all, I would say, it must be known to manage.

MTE#9, in planning activities, also focuses on student engagement and motivation:

MTE#9: […] to identify the didactic nodes, to predict errors and difficulties, to use them in collective discussion, and to make educational proposals that are engaging and motivating for the age group to which they refer.

However, we also found 4 cases in which the approach is spontaneous, that we coded as *unplanned*:

MTE#1: Well, sometimes it emerges, but to be truth I usually do not have this previously planned.

Finally, 2 excerpts were coded as *personal experience* since they referred to the need of founding the planning on teachers' personal background:

MTE#8: In general, I always try to point out how negative past experiences with mathematics then affect the teaching of the discipline.

MTEs' beliefs

In many answers there were explicit formulations of MTEs' beliefs about what is mathematics teacher education and how it should be undertaken. Thus, we grouped these answers in the category *MTEs' beliefs*. One excerpt referred to beliefs *about contents* and their relationship with the affective domain:

MTE#1: in some parts of the subjects, […] emotional aspects are an explicit part of the contents (as in problem solving).

Other answers (4) referred to beliefs *about students*, based on their affective approach to mathematics, as in:

MTE#8: […] it is difficult for the students to approach positively this discipline, which already prejudicially appears difficult and arid.

Or based on the emotional development as mathematical learner in different educational levels:

MTE#10: In early childhood and primary education affectivity is very important: the teacher is like a god, well, this works similarly with future teachers, and they understand it very well and put it in practice.

However, most of the answers in this category (9 excerpts) refer to the beliefs *about MTE's role and practices* in teacher education. MTEs pointed out the need of reinforcing teachers' emotional aspects:

MTE#4: [considering emotional issues] it is neither a trivial nor a marginal aspect and requires a lot of effort from the TE.

MTEs believe that they must be aware of these emotional aspects and promote a free and untethered exchange of ideas with the teachers:

MTE#10: They have and feel freedom to make me know aspects that I need to know about them, this is basic.

And, during the educational processes MTEs should make teachers empathize with students' emotions when they are asked to elicit thinking in a mathematics classroom:

MTE#7: Very often I start with a game that offers Di Martino: "Get up on your feet and play a game. I ask math questions: those who answer correctly remain standing while those who make mistakes sit. "At this point they are all clear how a student can feel when the teacher says: "Today I question".

Some of these answers also expressed beliefs about methodology, which is specifically considered in another category, but within this category we analyse them from the perspective of MTE's beliefs

about education, considering that the way MTEs conceive their role will permeate the methodological approach they use.

Teachers

Many answers (19 excerpts) remarked different aspects about the mathematics teacher who are trained (either pre- or in-service), showing the relevant consideration that MTEs conceded to work on the field. Teachers' positive and negative *personal experiences* also appear in this category to be considered as starting points of the educational activities:

MTE#2: These [negative experiences] are often part of the personal experience of prospective teachers and so they want to talk about these aspects.

MTE#8: I insist a lot on the fact that it is necessary to pass on to the children our pleasure in teaching mathematics.

Another MTE referred the diversity among *teacher profiles*, and how these differences influence the emotional aspects. For instance, when students have to take a work for paying fees:

MTE#10: there are many special students, like 25-30% they have special situations […] the majority of the students have to work for paying the tuition fees, and therefore, the affective part is fundamental.

Nevertheless, most excerpts in this category can be codified as *teachers' beliefs and attitudes*. MTEs referred to situations they detected among their trained teachers, addressing emotions as anxiety or fear (there were more negative than positive emotions pointed out):

MTE#3: We often talk about the fear of mathematics (of pupils and teachers, teachers' difficulty in working in a team, of teachers forced to teach mathematics against will, etc.

MTE#1: [detected teachers' emotions as] the so called "mathphobia" or math anxiety.

Some positive attitudes were pointed out:

MTE#10: And those students [from diversity contexts] are later much more sensitive to dealing with diversity.

One MTE also expressed his/her concern about the impact of trained teachers' emotions in their self-perception as efficient teachers:

MTE#10: In recent research I did with some colleagues we observed (still quantitatively) how the emotions towards mathematics have a strong impact on the sense of self-efficacy of future teachers.

Educational approach

This was the most frequent category (30 excerpts), and the richest, from our point of view, since it provides illustrative examples about how affect are considered during mathematics teacher education processes and the type of approaches that MTEs follow. Five MTEs' answers were about creating a *confidence climate* in the teacher education lectures, so that:

MTE#10: There is a fundamental point: if they do not feel comfortable nothing will work.

The confidence climate is the basis on which to build an educational process:

MTE#4: The teacher in training must not feel demoralized but rather must feel motivated to reconstruct his modality.

Creating that climate allows dealing with the complex reality of being a teacher:

MTE#3: Obviously there are no 'safe recipes', but the fact of talking about it in a group eases tension and reduces problems.

To reach a confidence climate also requires that the MTE promotes *motivation* among the teachers, that is another relevant observed code (4 excerpts):

MTE#1: […] in particular, motivation is an affective aspect that many of them are worried about.

Three of the excerpts pointed out *reflective practice* as a fundamental procedure in teacher education:

MTE#4: Very often I am told: "you upset me, and you took away what was certainties for me, so now I have to rebuild". I think this is an aspect not to be avoided but to be brought to light.

MTE#9: it is very important to reflect on the discipline.

It is not everything about how but also about what, so some answers pointed out *critical aspects in teaching to be discussed*, for instance:

MTE#5: [aspects that emerge when training teachers] little time available, […] inability to take any action when children are unable to solve problems.

Another 5 answers underlined the need of developing *meta-cognitive skills about teaching-learning processes*.

MTE#4: If you don't understand that you have a problem, you will never work to overcome it.

Finally, in this category, 6 excerpts remarked *teacher's role*, in some cases as a the one who should make affective aspects emerge:

MTE#7: Precisely for this reason they [affective] are aspects that cannot be ignored neither on the student side nor on the teacher side.

And, in other cases, *teacher's role* to overcome some of the obstacles derived from day-by-day work:

MTE#3: [we often talk about] teachers' difficulty in working in a team, of teachers forced to teach mathematics against will, etc.

Discussion

Our qualitative study highlighted several affective aspects considered by MTEs in their teacher education activities. Consistent with the Grounded Theory approach, the analysis of MTEs' responses (despite the convenience sample and the focus on a single question) revealed a common concern regarding the approach for dealing with the affective domain. Specifically, key emerging themes included: the inclusion of reflections about emotions (planned and unplanned in their educational activities); the MTEs' beliefs about content, students, their role and practices; the consideration of teachers' personal experiences, beliefs and profiles; and the adoption of an educational approach that involves creating a confidence climate to support motivation and enable reflective practices. Many answers point to a reflective model of teacher training, that could be related to Korthagen's (2001). This reflection is sometimes directly link to processes of deconstruction, reconstruction, and co-construction of mathematical knowledge by analysing emotional aspects (Alsina & Mulà, 2019). For this process, mutual confidence to share and to analyse emotions between MTE and teachers is a

previous requirement. There is also a shared concern about the use of negative beliefs and attitudes, which is consistent with literature about personal experiences related to mathematical background and their influence in the process of mathematical identity (for instance, Carotenuto et al., 2022). Based on their exploratory nature, we think that the results of our study are promising in helping to determine what is the role of the affective domain in the MTESK model (Martignone et al., 2022). For this reason, our future studies will focus on finding evidence about the relationship among the categories we have identified in this analysis some MTEs' knowledge domains. For instance, it seems to be a relationship among MTEs' beliefs and the KMT-teachers domain (knowledge about mathematical teaching, which is required for designing activities, choosing strategies, selecting resources, and studying teaching approaches in teacher education activities) and KFLM-teachers and KFLM-students domains (Knowledge of Features of Learning Mathematics, e.g. knowledge of learning theories and the ways students interact with mathematical content, including their common difficulties and errors) in the MTESK. Therefore, these aspects need deeper analysis.

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Tracing mathematics teacher educators' specialised knowledge through a classroom observation framework

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This study explores whether components of Mathematics Teacher Educators' (MTEs) knowledge can be identified using an observation framework designed to analyse mathematics teachers' practice. While models such as Mathematics Teacher Educators Specialised Knowledge (MTESK) conceptualise this knowledge, no observation frameworks currently capture it in practice. To address this gap, the POEMat.ES framework was examined through the lens of the MTESK model to assess how well it reflects elements of MTEs' knowledge. The analysis showed alignment with all domains of mathematical knowledge and some aspects of pedagogical content knowledge, but no references to research in mathematics education. Some POEMat.ES indicators also did not match MTESK's subdomains. Although not developed for MTEs, POEMat.ES can help capture their specialised knowledge, showing potential to support future studies of MTEs' knowledge and practice.

Keywords: Educational practice, knowledge, mathematics teacher educator, observation framework.

Introduction

In recent years, Mathematics Teacher Educators (MTEs) have emerged as a central focus of research in mathematics education, due to the key role they play in teacher training and, consequently, in improving the quality of mathematics teaching (Beswick & Goos, 2018; Castro Superfine et al., 2024). The interest in MTEs is exemplified in various initiatives aimed at exploring the role of these professionals, such as the creation of specific working groups within the framework of the International Congress on Mathematical Education (ICME) and the Congress of the European Society for Research in Mathematics Education (CERME). According to Karsenty et al. (2023), the term MTE is used to encompass various professionals responsible for teacher education (both at pre-service and in-service levels). The specific denominations they receive (e.g., didacticians, facilitators, mentors, etc.) vary according to the context in which they operate (Chapman, 2021). Currently, as shown by the TSG-35 Working Group (2024), the research community continues to ask fundamental questions: Who are the MTEs? What do we know about their knowledge, practices, and beliefs? And what theoretical frameworks are most appropriate to explore the knowledge of MTEs?

Literature presents different approaches to understand the nature of MTEs' knowledge. Some authors argue that the professional knowledge of mathematics teachers is fully contained within that of MTEs (Pérez-Montilla et al., 2025), while others suggest that, although both knowledge domains share certain foundations, they differ in scope, purpose, and complexity (Martignone et al., 2022). Despite these differing perspectives, there is general agreement that MTEs' knowledge is deep, complex, and specialised (Castro Superfine et al., 2024), which has led researchers to explore it through a variety of methodological techniques. Among these, observation of teaching practice stands out. For example, Masingila et al. (2018) and Olanoff et al. (2021) employ this technique to analyse MTEs' knowledge within communities of practice, while Pascual (2021) and Pérez-Montilla et al. (2025) apply it in the context of initial teacher education.

However, no structured observation frameworks specifically designed for analysing MTEs' practice are reported in the literature. In contrast, such frameworks are well established in research on mathematics teachers. Validated frameworks, such as the one developed by the SiMiS study group (2021) for secondary education, enable a direct analysis of classroom instruction by addressing both pedagogical and mathematical dimensions (Olfos et al., 2022). Notwithstanding, it is important to note that frameworks like POEMat.ES were originally developed to analyse teaching practice rather than to assess teachers' specialised knowledge. In the case of MTEs, there are no established frameworks for systematically observing either their practice or their specialised knowledge. Observational approaches tend to rely on retrospective video analysis, which supports in-depth reflection but lacks standardisation.

This situation invites reflection on whether observation frameworks originally developed for mathematics teachers might also be useful for analysing the MTEs' knowledge. In fact, exploring the applicability of such frameworks may offer insight into the extent to which the knowledge of teachers and MTEs overlaps or diverges. In this context, the aim of the present study is to explore the extent to which components of MTEs' specialised knowledge are present in frameworks designed to observe the teaching practice of mathematics teachers. Specifically, the research focuses on the POEMat.ES observation framework and examines it through the lens of the Mathematics Teacher Educators' Specialised Knowledge (MTESK) model. The guiding research question is: What subdomains of the MTESK model are present in the POEMat.ES framework?

Theoretical background

In the field of observing mathematics teaching practice, this study draws on the Mathematics Teaching Observation Protocol for Secondary Education (POEMat.ES, for its acronym in Spanish), developed by the SiMiS study group (2021). This framework was designed to identify relevant aspects of teaching practice in secondary school mathematics lessons through the analysis of video recordings made in the Spanish educational context.

The construction of POEMat.ES was carried out through a dual methodological approach, combining a systematic literature review on mathematics teacher knowledge and observation frameworks (top-down), with a detailed analysis of videos of real classes (bottom-up). This methodology enabled the definition and refinement of a set of indicators that capture different aspects of teaching practice. These indicators are organised around three interrelated dimensions following the idea of Karsenty and Arcavi (2017): mathematical content, didactics of mathematical content, and classroom management. This structure allows for a precise analysis of teaching practice. It is noteworthy that POEMat.ES emphasises the need to address the classroom management dimension specifically from the perspective of the mathematics classroom, and not in a general manner.

POEMat.ES was based on the premise that teaching practice is closely linked to the specialised knowledge of teachers, drawing on theoretical models as Shulman's (1986) teacher knowledge model and the Mathematics Teachers' Specialised Knowledge (MTSK) proposed by Carrillo et al. (2018). In this framework, three key dimensions are identified, each with specific indicators. The mathematical content dimension includes the use and exchange of representations, definitions, argumentation, mathematical flexibility, intra-mathematical connections, and errors made by the

teacher. The didactics of mathematical content dimension considers materials, tasks, extra-mathematical connections, the transfer of responsibility to students, discourse, and student interventions. Finally, the classroom management dimension covers density (the ratio between the time dedicated to mathematics teaching and learning and the total class time), the use of expository resources, written materials (textbooks, photocopies, etc.), and the management of disruptive behaviour. Each of these indicators is assessed using a four-level scale, structured as a Likert-type scale from 0 (lowest level) to 3 (highest level), where higher levels indicate a greater alignment with the teaching and learning standards defined by the research group.

Following the SiMiS group (2021), and conceiving teaching practice as a manifestation of specialised knowledge, this research proposes analysing MTEs' knowledge. As noted by Castro Superfine et al. (2024), the literature includes various models that attempt to describe the specialised knowledge of MTEs. Among these, the MTESK model, developed by Martignone et al. (2022), is conceived as an extension of the MTSK model. Thus, the MTESK model characterises the specialised knowledge of MTEs around three domains: Mathematical Knowledge (MK), Pedagogical Content Knowledge (PCK), and Beliefs about Mathematics and its Teaching and Learning (see Figure 1).

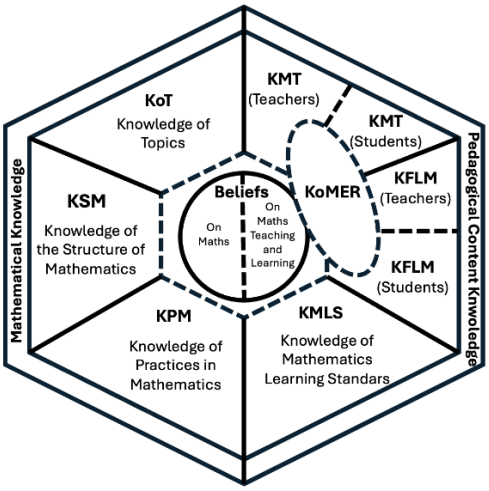

Figure 1: The MTESK model

In the MTESK model, MK maintains the structure of the MTSK and includes three subdomains of knowledge: Knowledge of Topics (KoT), Knowledge of the Structure of Mathematics (KSM), and Knowledge of Practices in Mathematics (KPM). However, Martignone et al. (2022) point out that, in the case of MTEs, this knowledge is deeper and more robust than that of mathematics teachers, thus constituting a form of metaknowledge (Beswick & Goos, 2018).

Regarding PCK, the model presents notable differences compared to the MTSK. Two approaches are distinguished within Knowledge of Mathematics Teaching (KMT) and Knowledge of Features of Learning Mathematics (KFLM): one centered on teachers and the other on students. This distinction has been described as a "double gaze" (Reyes-Bravo et al., 2024), referring to the ability of MTEs to attend simultaneously to both the classroom student and the prospective teacher. This dual focus affects not only who the teaching is directed at, but also the nature of the mathematical content being addressed, whether it relates to school mathematics or to the professional preparation of future

teachers. Additionally, Knowledge of Mathematics Education Research (KoMER) is incorporated as a subdomain that permeates both KMT and KFLM. Finally, this domain also includes Knowledge of Mathematics Learning Standards (KMLS).

Although POEMat.ES and the MTESK model were developed for different educational actors and purposes, they can be regarded as complementary. POEMat.ES focuses on the observable manifestations of teaching practice, whereas MTESK conceptualises the specialised knowledge that underpins such practice. In this study, POEMat.ES serves as the empirical lens through which teaching actions are analysed, while MTESK provides the theoretical constructs to interpret those actions as expressions of knowledge. Mapping an observation framework onto a knowledge model is theoretically meaningful because it bridges the gap between practice and cognition, revealing how professional knowledge is enacted in, and sometimes remains implicit within teaching activities. By identifying which elements of the MTESK model are observable through POEMat.ES, the study contributes to understanding the visibility and boundaries of MTEs' specialised knowledge, and problematises the complex relationship between knowing and doing in professional practice.

Methods

This study is situated within qualitative research, developed from an interpretative paradigm, and aims to explore which elements of the specialised knowledge of MTEs are reflected in the POEMat.ES observation framework, designed for analysing teaching practice in secondary education. To this end, a documentary analysis was conducted, which, following Bisquerra Alzina (2009), allows access to relevant information about a specific phenomenon, providing fundamental data that would be difficult to obtain through other methodological techniques. This type of analysis is especially useful in the exploratory phases of research, as it facilitates the formulation of hypotheses and the identification of initial conceptual patterns relevant for the subsequent development of the study.

The procedure followed involved a detailed analysis of each of the indicators included in the POEMat.ES framework, aiming at identifying the meaning and scope of each. It is worth noting that each indicator in the framework includes not only a general statement but also a detailed description and an illustrative example of how it might be applied in the analysis of professional practice. In establishing the correspondence between each indicator and the subdomains of the MTESK model, we considered the type of professional practice implied by the POEMat.ES indicator and the specific knowledge required by MTEs to perform such practice, as conceptualised in the MTESK model. In other words, the mapping was based on identifying the knowledge that underpins the professional actions described by each indicator. From this analysis, correspondences were established between the indicators and the subdomains of the MTESK model, using a deductive approach. This analysis was carried out independently by each of the researchers involved in the study to ensure a diversity of interpretations and minimise potential biases. Subsequently, a process of triangulation of the information was undertaken to compare the results obtained, discuss any discrepancies identified, and reach consensus agreements in cases of divergence (Bisquerra Alzina, 2009).

Results

Figure 2 presents the correspondence between the POEMat.ES observation framework and the subdomains of the MTESK model. As shown, POEMat.ES includes components related to all MK subdomains, while not all PCK subdomains are represented.

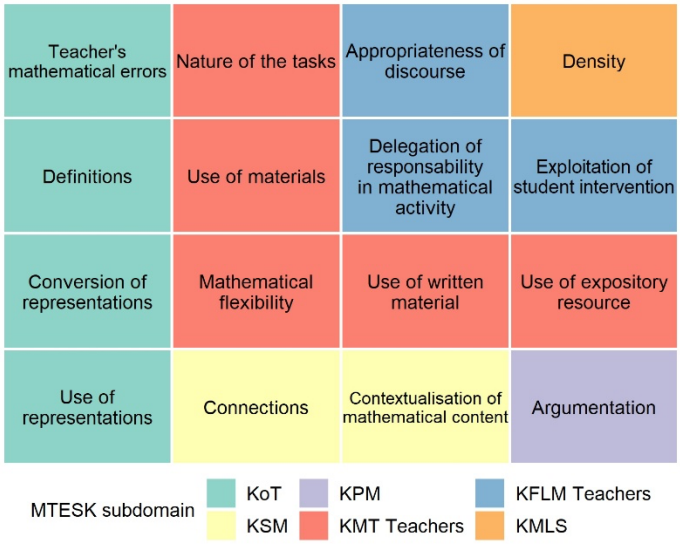

Figure 2: Relationship between the POEMat.ES framework and the MTESK model

Within the MK domain, the KoT subdomain is illustrated through POEMat.ES indicators related to the use of representations, conversions between representations, definitions, and the teacher's mathematical errors. These indicators show the MTEs' understanding of mathematical content. Moreover, mathematical errors are interpreted as evidence of possible gaps in knowledge concerning the topic addressed. The KSM subdomain appears in indicators linked to connections and the contextualisation of mathematical content. These indicators reflect the MTEs' ability to establish links between different mathematical concepts (intra-mathematical connections) as well as with content from other disciplines (extra-mathematical connections). The KPM subdomain is reflected in the indicator concerning argumentation, as it refers to the MTEs' knowledge of one of the fundamental mathematical practices: reasoning and argumentation.

Within the PCK domain, the KMT Teachers subdomain is included through indicators related to mathematical flexibility, the use of materials, the nature of the tasks proposed, the use of expository resources, and the use of written material. Most of these indicators, except for mathematical flexibility, refer to the knowledge MTEs possess regarding the use of teaching resources in the context of initial teacher education. In the case of mathematical flexibility, the practice observed in POEMat.ES reflects MTEs' ability to approach a single mathematical task through multiple representations or strategies and to encourage prospective teachers to analyse and compare these alternatives. This capacity illustrates how a MTE designs and adapts instructional approaches to make diverse methods of teaching mathematics explicit—an essential feature of the KMT Teachers subdomain. The KFLM Teachers subdomain appears in indicators related to the delegation of responsibility for mathematical activity, the suitability of discourse, and the exploration of student interventions. These aspects reveal the MTEs' knowledge about the learning processes of prospective

teachers in initial training contexts. The KMLS subdomain is included through the indicator concerning density, which measures the MTEs' understanding of how to organise mathematical content in the initial teacher education classroom.

Conversely, the indicator concerning the management of disruptive behavior does not show any correspondence with subdomains of the MTESK model. Unlike the interpretation proposed by the SiMiS group (2021), the MTESK model does not consider the ability to resolve conflicts effectively as a form of specialised knowledge specific to MTEs, but rather as a competency shared with other teacher educators, and even with mathematics teachers themselves.

Discussion and conclusions

This research shows that a connection can be established between an observation framework designed for mathematics teachers and a model of MTEs' knowledge (specifically, between the POEMat.ES framework and the MTESK model). When mapping the indicators, it becomes clear that, aside from certain elements of general pedagogical knowledge (shared by both teachers and MTEs), most of the POEMat.ES indicators clearly reflect components of MTEs' specialised knowledge. However, certain MTESK subdomains, namely KoMER, KMT Student, and KFLM Student, are not represented in the framework, which highlights the need to develop specific observation frameworks or indicators that can fully capture the distinctive aspects of MTEs' knowledge and practice. This is to be expected, given that POEMat.ES was designed to evaluate the practice of mathematics teachers, who work with a single type of students and do not typically engage with educational research as part of their professional role. At the same time, these findings reinforce the notion of the MTEs' "double gaze" (Reyes-Bravo et al., 2024) and confirm that the knowledge base of MTEs and mathematics teachers does not fully overlap (Chapman, 2021). Nevertheless, the absence of some subdomains in the mapping may also suggest that certain dimensions of MTEs' knowledge are not easily captured through observation alone. Such knowledge may be enacted in more implicit or contextualised ways that do not leave direct traces in classroom interaction. For this reason, the partial mismatches identified are not interpreted as methodological shortcomings but as evidence of both the distinct analytical purposes of the two frameworks and the challenge of making some forms of professional knowledge visible through observation. While POEMat.ES captures enacted aspects of teaching practice that are directly observable, MTESK conceptualises underlying knowledge structures that often remain implicit in professional activity.

Future research should explore the use of other observation frameworks and alternative models of MTEs' knowledge (beyond POEMat.ES and MTESK) to assess whether these frameworks can effectively capture MTEs' knowledge, and to what extent this knowledge aligns with or diverges from that of mathematics teachers. Developing a dedicated observation framework for MTEs holds significant methodological potential: it would provide a systematic, reliable framework for analysing their professional practice, thus advancing research beyond prior studies like those of Pascual et al. (2021) and Pérez-Montilla et al. (2025), which relied on knowledge models without the support of a structured observation framework. Such a framework would facilitate deeper, more consistent insights into the unique professional knowledge of MTEs, ultimately improving both research and practice in this field. Beyond its analytical contribution, this study also offers implications for the

professional learning of MTEs. Making explicit which aspects of their specialised knowledge become visible through teaching practice can foster self-reflection and guide targeted professional development. In this sense, observation frameworks such as POEMat.ES, when interpreted through conceptual models like MTESK, can inform not only the analysis but also the learning of practice, supporting MTEs in recognising and refining the knowledge enacted in their professional activity.

Furthermore, combining different observation frameworks and theoretical frameworks, as proposed in this study, allows researchers to gain some distance from the limitations or biases of any single approach. This, in turn, supports more comprehensive and nuanced answers to research questions, in line with the suggestions of Beswick and Goos (2018). Ultimately, this line of work contributes to addressing the core questions raised by the TSG-35 Working Group (2024) regarding the knowledge and practices of MTEs.

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Pre-service teachers engaging with a collaborative problem-solving model: Bridging cognitive and socio-emotional dimensions

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This study examines pre-service teachers' reflections on their problem-solving process using the Creative Collaborative Mathematical Reasoning (CCMR) model as a lens. The model emphasizes patterns of interaction in collaborative problem solving—bi-directional and one-directional—which served as a lens for pre-service teachers to articulate their collaboration. Findings show that the model helped pre-service teachers verbalize their roles and recognize how cognitive and affective dynamics shaped collaboration. One pair demonstrated bi-directional interaction through shared heuristics and engagement, while another revealed misaligned heuristics and resources that led to parallel processes and frustration. These contrasting cases illustrate how socio-emotional dynamics shaped collaboration, suggesting that the CCMR model can help pre-service teachers reflect on and improve CPS facilitation in future classrooms.

Keywords: Collaborative problem-solving, socio-emotional dimension, interaction patterns.

Introduction and theory

Research shows that collaborative problem solving (CPS) can deepen mathematical learning by fostering explanation, negotiation, and mutual understanding (Webb et al., 2021). CPS has been defined as two or more individuals engaging in a process where they "attempt to solve a problem by sharing the understanding and effort required to come to a solution and pooling their knowledge, skills and resources to reach that solution" (OECD, 2017, p. 6). CPS also cultivates related skills such as communication, reasoning, and collaboration. However, far less is known about how pre-service teachers (PSTs) can be prepared to reflect on and enact these collaborative dynamics effectively in their future classrooms.

Facilitating CPS is far from straightforward. Supporting students' participation, fostering mutual understanding, and managing interpersonal dynamics (Hesse et al., 2015) require teachers to develop nuanced knowledge of how collaboration unfolds in classrooms. PSTs need to gain experience with connecting theoretical insights from teacher education with the practical demands of classroom interaction (Cavanagh & Garvey, 2012). Norwegian teacher education emphasises practice-based learning and the integration of theoretical and experience-based knowledge (UHR, 2018). In mathematics teacher education, this shift has led to a focus on teachers' core practices as key to professional development (Mosvold et al., 2018).

Building on this practice-oriented perspective, PSTs need analytical tools that help them recognise and interpret students' CPS and use such insights to promote in-depth learning. Although the Creative Collaborative Mathematical Reasoning (CCMR) model in Figure 1 (Hansen & Naalsund, 2025) was built from students' interactions and learning processes, the intention of the model was to support teachers' noticing of their students' collaborative dynamics. Hence, the model provides PSTs with a structured lens for analysing classroom collaboration from a teacher's perspective. Engaging with the

model enables them to connect theory and practice by developing professional awareness of how collaborative reasoning emerges and how instructional decisions may influence such dynamics. Developed from classroom-based studies of student pairs' interaction patterns in CPS (Hansen, 2022; Hansen & Naalsund, 2022), the model integrates three dimensions: mathematical reasoning, collaborative processes, and exercised agency. When addressing 'interaction' in this study, we refer to the intertwinement of these three dimensions. The model has not previously been tested with teachers or PSTs. This study explores its potential in teacher education by having five PSTs use the CCMR model to reflect on their own CPS.

Central to the CCMR model is the framework of *Creative Mathematically founded Reasoning* (CMR), which distinguishes between imitative reasoning—reproducing memorized facts and procedures—and creative reasoning, where solvers construct novel, plausible, and mathematically anchored arguments (Lithner, 2017). Creative reasoning is seen as a gatekeeper to in-depth mathematical learning, as it involves making claims, exploring alternative strategies, justifying solutions, and grounding ideas in mathematical properties. Through this process, PSTs can experience developing a mathematical voice rooted in sense-making and exploration. A second dimension of the CCMR model is *collaboration*, understood as the processes of *building* (making suggestions and accepting ideas), *monitoring* (questioning and explaining ideas), and *repairing* (negotiating and correcting ideas) shared understanding (Roschelle & Teasley, 1995). In collaborative activity, PSTs engage in turn-taking, negotiation, and resolving conflicting interpretations. Such interactions not only support the co-construction of knowledge but also offer PSTs valuable insights into how joint understanding emerges and shifts through dialogue, and how moments of conflict or agreement can serve as opportunities for reasoning together. The third dimension is the key to understanding interaction pattern and is *exercised agency*. Agency is defined as "the way in which he or she acts, or refrains from acting, and the way in which her or his action contributes to the joint action of the group in which he or she is participating" (Gresalfi et al., 2009, p. 53).

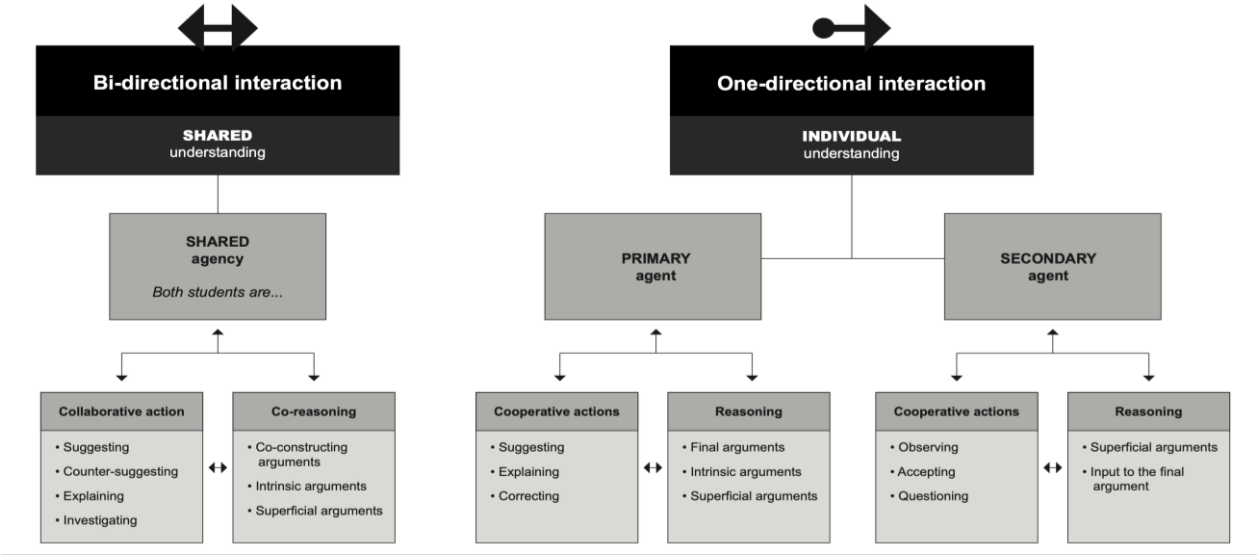

Figure 1: The CCMR-model (Hansen & Naalsund, 2025, p. 13)

The CCMR model (Figure 1) categorizes dyadic interactions into two overarching patterns, *bi-directional* and *one-directional*, which were identified from empirical data and assimilated with existing literature (Hansen & Naalsund, 2025). In a *bi-directional interaction,* both students actively

contribute with suggestions and explanations, engage collaboratively, and co-construct arguments. In this interaction pattern, students typically exercise shared agency, meaning that arguments are jointly built and would not exist without the contributions of both participants. Such interactions involve all collaborative processes—building, monitoring, and repairing shared understanding—and are often grounded in reasoning anchored in intrinsic mathematical properties. This dynamic is generally regarded as highly productive for learning, as it fosters deep mathematical argumentation and mutual understanding. In contrast, *one-directional interaction* occurs when one student predominantly drives the problem-solving process, making decisions and providing most of the mathematical explanations, even if both students remain engaged. Here, one student acts as the primary agent, formulating the arguments, while the other takes on a secondary role by offering suggestions or questions that shape, but do not drive, the reasoning. The secondary agent often relies on imitative rather than creative reasoning, which may limit individual learning opportunities. Although this pattern can still support progress, it will generally provide less learning opportunities than bi-directional interaction, and it may prove difficult to shift once established (Hansen & Naalsund, 2022). Recognizing how agency distributes across participants may help PSTs reflect on their own roles in shaping collaboration and notice whose contributions are taken up, ignored, or rejected. This awareness is crucial for fostering equitable participation and meaningful engagement in classrooms, leading us to pose the following research question: *What insights does the CCMR model enable four pre-service teachers to articulate about their collaborative problem solving?*

CPS encompasses attention to both *cognitive* and *affective* dimensions (Erdoğan et al., 2025). A *cognitive* dimension is "students' efforts to deepen their conceptual understanding through thinking, analyzing, reasoning, and joint knowledge construction" (Erdoğan et al., 2025, p. 5). An *affective* dimension refers to the dynamic interplay of beliefs, attitudes, emotions, and motivations/interests that influence how individuals engage with mathematical tasks (Schindler & Bakker, 2020). From a cognitive perspective, problem-solving competence has been characterized through frameworks such as Schoenfeld's (1985) four categories: *resources* (the mathematical knowledge, facts, and procedures that solvers can draw upon), *heuristics* (problem-solving strategies that guide exploration of unfamiliar problems), *control* (monitoring and regulating one's problem-solving process, such as deciding when to abandon an unproductive strategy or how to allocate time and effort), and *belief systems* (individuals' views about mathematics and themselves as learners, shaping choices and approaches during problem solving). While Schoenfeld (1985) initially positioned beliefs within the cognitive domain, later research has emphasized the affective nature of beliefs, also intertwined with attitudes, emotions, and motivation (Schindler & Bakker, 2020). Beliefs thus serve as a bridge between cognition and affect, highlighting that knowledge, strategies, and regulation are dynamically linked with emotional experiences.

Some studies adopt a broad perspective on affect, including concepts such as beliefs (students' views of mathematics or problem solving), attitudes (stable affective responses in certain situations), motivation/interest (preferred engagement with a particular object), and emotions (feelings such as joy or frustration in temporary situations) (Schindler & Bakker, 2020). In contrast, Erdoğan et al. (2025) focus specifically on the *socio-emotional* dimensions of CPS, arguing that emotions are socially situated and critically shape participation and engagement in collaborative learning. By highlighting the interplay between emotions and social relations, they show how these factors

influence opportunities for productive collaboration in mathematics. This study adopts a *socio-emotional perspective*, emphasizing how *emotions* emerge dynamically through interaction in collaborative mathematical activity. We attend to particular *emotional expressions*—such as joy, curiosity, satisfaction, frustration, fear, or anxiety—but our primary focus is on how these elements intertwine with pre-service teachers' articulation of their CPS. This approach provides a holistic view of how socio-emotional factors shape and are shaped by engagement in CPS.

Method

The data collection process comprised three stages. First, five PSTs were introduced to the CCMR model through a short introductory text in Norwegian. Secondly, the model was presented in more detail by the first author the following day. Thereafter, the PSTs were divided into two pairs—Mateo and Karl, and Joel and Miriam—seated at opposite sides of the classroom. The remaining female PST was assigned the role of teacher. She was instructed to encourage mathematical reasoning and prompt the pairs to attend to one another's thinking. The pairs were asked to solve the following task: "Create a straight line $y = mx + c$. Then create another straight line in a way that the corresponding graphs are perpendicular. Formulate a rule for when two straight lines are perpendicular and test the rule for other straight lines."

The first and second author was present while the PSTs solved the task, making observations without interacting with the PSTs. The two groups were video recorded, capturing their writing, gestures, facial expressions, and the groups' interactions with the PST-teacher. In the third stage, taking place immediately after the problem-solving session, the first and second author conducted a group interview with the PSTs. In these interviews, the participants shared their reflections drawing on the CCMR model. The interview did not follow a predefined guide but was structured as an open conversation centered on the PSTs' experiences with applying the CCMR model. The first author initiated the dialogue with a broad question—inviting the participants to elaborate on their reflections—and the discussion developed organically from their responses. The video recordings were not shown during the interview but were used afterwards to contextualize and substantiate the PSTs' comments. In this way, the interview and video data complemented one another, capturing both participants' interpretations and their interactional practices. The interview data forms the empirical basis of this paper. While data from the PST-teacher is not included in this paper, her observations supported the interpretation of the pairs' interaction patterns.

The analytical process began with the first author transcribing the interview. Subsequently, all three authors reviewed the video recordings and repeatedly listened to the interview before engaging in a data-driven thematic analysis of the transcripts. Of the three variants of thematic analysis outlined by Braun et al. (2019), our approach aligns most closely with the reflexive model, which emphasizes the researcher's active role in knowledge production. In this model, themes are constructed from meaning-based patterns that emerge as researchers explore and interpret the data. The purpose of reflexive thematic analysis is not to produce an objective or definitive summary of the data, nor to minimize researcher influence, but rather to generate a "coherent and compelling interpretation of the data, grounded in the data" (Braun et al., 2019, p. 848). To articulate the results of our analysis, we adopted the terminology on problem solving competence from Schoenfeld (1985) and Erdoğan et al.'s (2025) socio-emotional dimensions of CPS. Finally, we selected examples, re-analyzed them, and related them to our research questions and relevant literature.

Results and discussion

The CPS of the two pairs revealed contrasting cognitive and affective processes. This section presents their conversations through the lens of the CCMR model. Mateo and Karl described their collaboration largely as bi-directional, explaining their respective roles as follows:

> Mateo: At the beginning, we used trial-and-error strategies (…) But then, all of a sudden, Karl provided the solution. (…) I took the role of a secondary agent, until I tried to join and understand his reasoning, and be able to argue that it was correct as well.
>
> Karl: Yes, (…) you could say that it was kind of a one-directional interaction when I reached the answer, but since Mateo reacts the way he does… Then it becomes more like a bi-directional interaction. Because it's not like… «Explain to me why all of this is happening». He doesn't let go, when he sees a relationship, he asks and somehow takes ownership as well.

Mateo and Karl explained that one key reason for their ability to maintain a bi-directional interaction throughout the process was their mutual desire for a shared understanding and sense of ownership. They listened to each other and employed similar heuristics, which facilitated collaboration. From the very first line of Mateo's utterance, he described a feeling of "falling into a secondary role", which can be interpreted as an emotional response to the group dynamics—potentially reflecting uncertainty but also a willingness to participate. Further, Mateo's utterance may showcase a belief that participation in mathematical problem solving requires not only listening but also understanding one another and being able to justify the solution. Karl acknowledged his role as the primary agent, which initially created a one-directional interaction, but noted that the interaction shifted toward bi-directional as Mateo increasingly engaged by monitoring, negotiating the solution process, and reasoning mathematically about relationships. Karl's response reflects recognition of Mateo's curiosity, initiative, and sense of ownership, which contributed to their mutual pursuance of the problem. Based on more examples, their expressions indicate a shared agency, where the socio-emotions reveal a positive engagement—curiosity for the peer's ideas and motivation to co-construct a shared understanding.

Thus, Mateo and Karl provided rich insights into how the CCMR-model helped them articulate which form of interaction they were in, how they built on each other's thoughts and input, and how they both participated in this process. They also highlighted challenges in their mutual process:

> Karl: We agree on many things, but it was hard to articulate what we actually agreed upon, (…) Perhaps because we haven't worked with GeoGebra and with angles in a while, (…), These concepts made it more difficult to understand each other (…). Because, we agreed on many things, still it was hard to verbalize it.
>
> Mateo: It was hard to verbalize it… But eventually, it [our collaboration] became… amazingly enough. We had to meet each other's needs in order to collaborate.

They both struggled to articulate their thoughts and arguments, seemingly due to the mathematical content and the concepts at play. They remembered concepts that they knew were relevant, but of which they could not yet internalize and verbalize the meaning. Thus, their affective expressions started with uncertainty and perhaps frustration regarding their own resources (knowledge about GeoGebra and related mathematical concepts). The expressed emotions evolved from frustration toward satisfaction as they saw value in their shared sense-making way of collaboration. While Mateo and Karl's collaboration largely reflected the bi-directional pattern described in the CCMR model, Miriam and Joel's interaction revealed a distinct form of collaboration that challenged the model's

existing categories, suggesting that additional nuances of collaborative engagement might exist beyond the current framework. They explained their CPS in the following way:

> Joel: I have a tendency to try and experiment and test ideas somewhat blindly. And Miriam had a need to understand what I did in all stages of the process. So, we spent much time explaining why I was thinking the way I was (…). We were quite frustrated actually, for quite some time.
>
> Miriam: I am left feeling frustrated about the collaboration (…). I feel that my annoyance over how badly we work together has built up. (…) He throws out utterances and is unable to explain it to me, and I can't follow what is happening. I use all my capacity trying to understand him and get passive (…).

Miriam and Joel did not work well together, and clearly, they had different needs in a CPS process. Whereas Miriam sought to understand all stages of the process by monitoring and controlling Joel's reasoning, Joel preferred an initial stage of trial and error as his use of heuristics. This discrepancy created frustration and irritation, which in turned seemed to affect their CPS throughout the session and made them both demotivated and without a feeling of ownership. Their reflections revealed a form of interaction that could not be fully captured by the CCMR model's one- and bi-directional distinction, as both participants appeared to act as primary agents pursuing parallel lines of reasoning. Miriam and Joel both felt that they were in a one-directional interaction throughout the problem solving process. The CCMR-model enabled them to articulate factors contributing to their way of interacting:

> Joel: The reason for my frustration is that I felt it became a one-directional interaction where I was pressured into taking on a primary role, since Miriam just asked questions regarding what I was doing and how I was thinking. I didn't want to be a primary agent. I wanted it to be in a bi-directional interaction, or taking on a secondary role since I felt I didn't have much control (…). As a secondary agent, I feel it's easier to create a bi-directional interaction than it is as a primary agent. Now, I was just stuck as a primary agent and wasn't able to include Miriam in my reasoning and share my argumentation.
>
> Miriam: (…) Since I don't understand him [Joel], so if I was to take on a leading role, I would be primary. We can't change the way of collaborating. Since I don't follow your reasoning, I won't be able to build further on it. We then have to switch and start a new thinking process.

Joel felt pressured into taking on the role of primary agent, expected to lead the process and justify his thinking—a role he did not want and felt unprepared for. As Miriam continued to demand explanations, his frustration grew. Miriam, in turn, felt she used all her capacity systematizing Joel's thought process, which she considered necessary. Joel attempted to articulate his strategies (control, heuristics, resources), but the imbalance in roles limited opportunities for co-constructing knowledge. Miriam tracked and systematized Joel's reasoning, engaging cognitive aspects (control, resources) as a secondary agent. This asymmetry made it difficult to build shared understanding. Concerning affective expressions, Joel expressed frustration over the pressure to explain, while Miriam experienced strain in trying to follow his thinking. These socio-emotional dynamics, intertwined with their roles, shaped how they participated and negotiated the problem-solving process—all marked by frustration.

Concluding thoughts

This study contributes to understanding how PSTs can be supported in developing the knowledge and awareness needed to facilitate CPS in mathematics classrooms. The CCMR-model proved useful in

enabling the four PSTs to articulate and reflect on their collaborative processes, making visible both cognitive strategies (Hansen & Naalsund, 2025; Schoenfeld, 1985) and socio-emotional (Erdoğan et al., 2025) dynamics that shaped their interactions. The pairs were able to connect specific moments of their CPS to elements of the model, and in doing so, they highlighted the role of affective factors—particularly socially situated emotions—in shaping how they engaged with one another (Schindler & Bakker, 2020). These findings align with research emphasizing that emotions in collaborative settings are deeply embedded in social and contextual factors, and that such emotions influence how PSTs navigate instructional practice. By engaging with the CCMR model, the PSTs were able to connect theoretical insights with practical demands of classroom interaction (Cavanagh & Garvey, 2012).This may strengthen their ability to facilitate CPS as practicing teachers.

At the same time, our analysis revealed an interaction pattern that extends beyond the CCMR framework. In Miriam and Joel's case, their collaboration cannot easily be described as either one-directional or bi-directional. Rather, both appeared to act as primary agents, pursuing parallel problem-solving processes. While they shared a similar sense of agency—both expressing a willingness to participate and collaborate—they differed in how heuristics and resources should be employed. This misalignment prevented them from building on each other's reasoning and created frustration, but it also revealed a nuanced interplay between cognitive approaches and socio-emotional dynamics. By contrast, Joel explicitly expressed a preference for first taking on a secondary role and then "working his way into" a bi-directional interaction, reflecting a view of CPS as a gradual progression from peripheral to mutual participation. This perspective resonates with how Mateo and Karl described their own development from one-directional to bi-directional interaction, but it stands in contrast to the parallel processes observed in Joel and Miriam's case. The combination of the CCMR-model and the PSTs' affective reflections thus uncovered a form of interaction not fully captured by existing categorizations, highlighting a need for further research into interaction patterns in CPS. Future studies involving a broader group of PSTs could help substantiate these findings and contribute to the continued development of the CCMR-model.

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TPACK learning paths of pre-service teachers when designing digital mathematical escape games

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Pre-service teachers (PST) need subject-specific, didactic and pedagogical knowledge for their future teaching practice, commonly described within the TPACK framework. The 'Digital Competence Framework for Educators' (DigCompEdu) emphasizes the importance of technical and digital skills for teachers in general. One way to address relevant knowledge facets for PST is through game development-based learning, which includes designing digital escape games. Yet, questions remain about which knowledge facets PST activate when designing a digital mathematical escape game (DIMEG) and how this develops throughout the design process. In this study, one group of primary school PST designs a DIMEG within a seminar. Each seminar session was videotaped, transcribed and coded towards TPACK knowledge facets. Findings from this study show that all knowledge facets were activated, except TCK, suggesting that DIMEG is a promising approach for teacher education.

Keywords: Game based learning, gamification, teacher education.

Introduction and theoretical background

The International Computer and Information Literacy Study (ICILS) 2023 found that nearly 70% of German teachers use digital media at least once per day in their teaching, which is 46,7% more than in ICILS 2018 but less than in other countries, e.g., Denmark or Norway (Eickelmann et al., 2024). One reason may be that digital and technological competences are not sufficiently addressed in teacher education. About one third of the teachers report that in their teacher education, 'approaches to support learning and teaching with digital media' was part of their teacher education, which is significantly less than the international mean (Eickelmann et al., 2024). Yet, in the digital age, those competences are crucially important (Redecker, 2017). Thus, already pre-service teachers (PST) need to acquire subject-specific content, didactical, pedagogical, and technological knowledge for their future teaching practice (Mishra & Koehler, 2006; Redecker, 2017). Those knowledge facets are commonly described within the Technological Pedagogical Content Knowledge (TPACK) framework (Mishra & Koehler, 2006). In the following, the framework and the development of its knowledge facets are explained and illustrated in the context of 'game development-based learning' (GDBL) (Wu & Wang, 2012) with digital mathematical escape games (DIMEG).

TPACK framework and the development of knowledge facets

It defines and interconnects the seven different facets of teacher knowledge: CK (content knowledge), PK (pedagogical knowledge), PCK (pedagogical content knowledge), TK (technological knowledge), TCK (technological content knowledge), TPK (technological pedagogical knowledge), TPCK (technological pedagogical content knowledge) (Mishra & Koehler, 2006). The TPACK framework is the result of studies that focused on "how" teachers learn to teach with technologies instead of "what" they have to learn about these. Successful learning about teaching with technologies

requires the equivalent consideration of each facet (technology, pedagogy, content) to emphasize technology skills as well as to develop a comprehensive TPCK (Mishra & Koehler, 2006).

Developing the knowledge facets of the TPACK framework of PST is an important goal of teacher education and professional development (Mishra & Koehler, 2006; Shambare & Simuja, 2024). Yet, the process of constructing and developing this knowledge remains largely unknown (Shambare & Simuja, 2024). Only a few studies explore the development of the knowledge facets in different contexts. Darkwa and Agyei (2021) investigated how eight PST developed their TPACK by designing and implementing audio-visual supported lessons. They conducted a sequential multiple case study and showed an improvement in all knowledge facets, whereby the largest was in technology-related facets (TPCK, TPK, TCK and TK). They highlight that PST tried a combination of the three core knowledge facets (technology, pedagogical and content) and not only developed their TPCK. Shambare and Simuja (2024) discuss the integration of technology in rural education settings and show that non-technology-related facets scored higher in terms of mean. Therefore, they suspect that combining technological understanding with content and pedagogy is challenging. There are also studies investigating the perception of TPACK and the connections between single knowledge facets using statistical path analysis. For example, Koh et al. (2013) explored how the individual knowledge facets of teachers in Singapore are interrelated. They demonstrate that teachers view TPACK as stemming from the direct effects of PK, TK, TCK, and TPK, where TCK and TPK have stronger effects, and confirm the strong correlations between the technological facets.

Methodologically, studies focusing on the development of the knowledge facets use surveys (Koh et al., 2013) or conduct a pre-post-test design (e.g., Darkwa & Agyei, 2021). Yet, self-reported methods may not accurately reflect the actual learning (e.g., Kosiol & Ufer, 2024). Thus, in this study, we use video observation to map the development of the TPACK facets.

Game-development-based learning with escape games

The development of educational technologies based on the concept of ‘Learning-by-Design’ provides an opportunity for (pre- and in-service) teachers to learn about technology and expand their knowledge facets (Mishra & Koehler, 2006). One way to implement this approach is to design games in the spirit of GDBL. GDBL allows a constructivist learning approach, provides a meaningful and engaging student activity, enhances deep learning, is highly motivating and particularly characterized by authentic problem-based learning (Vos et al., 2011; Wu & Wang, 2012). For this contribution, PST design their own digital escape game (EG). In an EG (or escape room), the players have to escape from an unpleasant situation or rescue someone from a predicament by discovering clues and accomplishing tasks collaboratively in teams within a limited amount of time. A central challenge is figuring out what the puzzle is about and how to solve it. To overcome these, clues and hints are included (Nicholson, 2015; Stohlmann, 2023).

In sum, GDBL is an effective approach to enhance PSTs’ teaching competences and thus, knowledge facets concerning the TPACK framework. However, the activation of these facets and their development in the context of GDBL and EGs has been scarcely studied. Thus, in this study, we examine two research questions: (1) Which knowledge facets do PST activate when designing a DIMEG, and (2) How do these facets develop throughout the design process of a DIMEG*?*

Methods

Design of the study

This study was conducted within a seminar at TU Dortmund University (Germany) developed for primary school PST at the end of their bachelor's program in the winter term of 2024/2025. 14 PST participated (1 male, 13 female) and were grouped with 2 or 4 PST (a total of four groups) to design their own DIMEG. The seminar was structured into three main sections: (1) designing a DIMEG (e.g., task selection, narrative, game flow, storyboard, technical implementation), (2) game implementation with groups of primary school students in a real classroom and data collection, and (3) data analysis and writing the bachelor's thesis. The design process (sessions 1 to 6) was supported by a designer team consisting of four interdisciplinary students. They provided advice on various aspects, e.g., game or graphic design, and the technical implementation of the DIMEG within an online-based editor. Data was collected by videotaping sessions 2 to 6. Group discussions were videotaped for each group separately with four cameras and microphones. In addition, one camera captured the seminar room for the plenary phases. Session 1 was not videotaped because students mainly learned about organizational input, some theoretical background concerning mathematical problem solving, and escape games and played a DIMEG. Thus, they did not talk much with each other which is why little output concerning their knowledge activation was expected. For this contribution, video data of group A with four PST in sessions 2 to 5 is considered because their design process was especially rich and interesting concerning the activation of knowledge facets.

Data coding and analysis

First, all data were transcribed using MAXQDA 24. Second, the transcripts were coded to identify the knowledge facets activated by the PST. Therefore, a category system was developed deductively (Mayring, 2000) (Table 1) and implemented in three steps:

1) Identify phase of the seminar: The seminar consisted of two phases: (a) *plenary phases* with input or exchange of ideas with all PST, or (b) working phases of the *group*.
2) Identify main action in the phase: Independent of the phase (group vs. plenary), there was (a) *input* by the lecturer or the designer team, or (b) *work* between the PST.
3) Identify knowledge facet: Each turn in the transcript was coded according to the seven *knowledge facets* (Table 1). To assign a turn to a category, existing categories and item descriptions were considered (e.g., Herold-Blasius et al., accepted; Kosiol & Ufer, 2024).

Additionally, other content of discussion (e.g., organizational topics concerning the bachelor's thesis or private information of the participants) was coded for reasons of completeness. However, those codes are not reported here due to the research focus.

To ensure intersubjective comprehensibility and match scientific standards (e.g., objectivity and reliability) (Mayring, 2000), two researchers coded the transcripts of the group separately. To measure intercoder reliability, the Kappa coefficient is applied (Landis & Koch, 1977). As full accordance among both coders cannot be expected – all coding involves interpretation – a value of 0.7 is considered sufficient. In this study, a value of 0.7 was reached in all sessions by both independent coders (κ_2: 0.73; κ_3: 0.84; κ_4: 0.93; κ_5: 0.92). To improve the intercoder reliability, the

differences were discussed and validated consensually. This explains the increase from session to session. After coding, descriptive statistics with absolute and relative frequencies were realized, and a sequential representation of the order and distribution of the codes was generated with MAXQDA 24 to illustrate the development of knowledge facets within group A.

Table 1: Category system

Knowledge facets	Categories (and subcategories)	Examples of coded statements from PSTs' work
CK	application of your own mathematical knowledge	That's a geometric body, isn't it? At least in terms of its shape, it's not a net. (GA, S2, turn 411)
PK	learning process of students; development of the narrative and game flow	I imagined that they were in a maze or something similar. Then, somehow, they ended up there, where they had to solve various tasks. (GA, S2, turn 311)
TK	digital competence; getting to know and using the online-based escape game editor	Could we arrange for the wizard to appear in a room when the time runs out? (GA, S5, turn 160)
PCK	learning objectives; curriculum, problem-solving tasks; tips and assistance	It could be something like matching puzzle pieces to their correct geometric body. That could be another task. (GA, S2, turn 71)
TCK	knowledge about potential applications and their use to represent content	/
TPK	criteria and limits for the use of digital media in the classroom	They probably don't use research tools. (GA, S2, turn 131)
TPCK	medium-specific design process, technical implementation of tasks and use of the game with students	I think determining and comparing the volume of solids using unit cubes is quite difficult in terms of technical implementation. (GA, S2, turn 88)

Results

Activated knowledge facets when designing a DIMEG

To investigate, which knowledge facets were activated, Table 2 presents the descriptive statistics for each knowledge facet and seminar session of group A.

Table 2 shows that PST activated all knowledge facets, except for TCK, with varying frequency. In total, 642 activations of knowledge facets were coded. Also, with each session, an increasing number of different facets are activated – PCK in session 2 vs. PK, TK, PCK, TPCK in session 4. This could indicate a kind of connectivity between the facets. Moreover, technological knowledge facets occur mainly in sessions 4 and 5, whereas core knowledge facets PCK and/or PK are frequently activated

Table 2: Descriptive statistic of knowledge facets (absolute and relative frequency)

	session 2		session 3		session 4		session 5		total
CK	4	2.5 %	6	4.6 %	0	0 %	14	7.4 %	24
PK	17	10.5 %	54	41.2 %	42	26.1 %	15	8 %	128
TK	8	4.9 %	0	0 %	24	14.9 %	84	44.7 %	116
PCK	121	74.7 %	70	53.4 %	36	22.4 %	49	26.1 %	276
TCK	0	0 %	0	0 %	0	0 %	0	0 %	0
TPK	5	3.1 %	0	0 %	0	0 %	0	0 %	5
TPCK	7	4.3 %	1	0.8 %	59	36.6 %	26	13.8 %	93
total	162		131		161		188		642

throughout all sessions. In session 3, the PST do not only select and adapt (problem-solving) tasks (PCK: 53.4 %), but also focus on developing a narrative and tell a story within the DIMEG (PK: 41.2 %). In sessions 4 and 5, the designer team actively participates in the seminar, and supports PST with advice about task selection, game or graphical design, and the technical implementation of the game. In these conversations, various topics occurred covering TPCK (36.6 % resp. 13.8 %), TK (14.9 % resp. 44.7 %), and PCK (22.4 % resp. 26.1%). Overall, two codes occurred only scarcely – CK, and TPK, while TCK was not observed at all.

This analysis indicates that first, many knowledge facets can be activated by designing DIMEG, and second, the knowledge facets are not only activated sequentially, but rather in combination with each other and possibly connected.

Development of knowledge facets throughout the design process of a DIMEG

To analyze the development of the PSTs' knowledge facets throughout the DIMEG design process, all codes are represented sequentially for seminar sessions 2 and 5 (Figure 1). This chronological representation (x-axis) provides an overview of the codes (illustrated with colors according to Table 1) used in these sessions (y-axis). It shows all three steps of the coding procedure (see Methods): (1) the phase of the seminar (grey bars); (2) the input or work phase, and (3) the activated knowledge facet coded. When a code is assigned to a statement, the code's color is displayed in the corresponding column.

Table 2 shows that PST focus on different knowledge facets during the sessions. Comparing them, please note that the input corresponds to the PSTs' focus (Figure 1). In other words, the focus of input in each seminar session is reflected in the activated knowledge facets (e.g., see session 2). The PST focus on task selection, and primarily activate PCK in session 2 by raising this as a topic during the first group phase (Figure 1). The subsequent plenary phase addresses precisely the aspects raised by

the PST – task search and task analysis. This encourages them to carry on working on this topic throughout the session. We conclude that PST dealt with the topics of each session, and regularly enriched this by activating additional knowledge facets. Therefore, the changes in sessions 2 and 3 in the activated facets (Table 2) during the sessions could be related to the focus of each session.

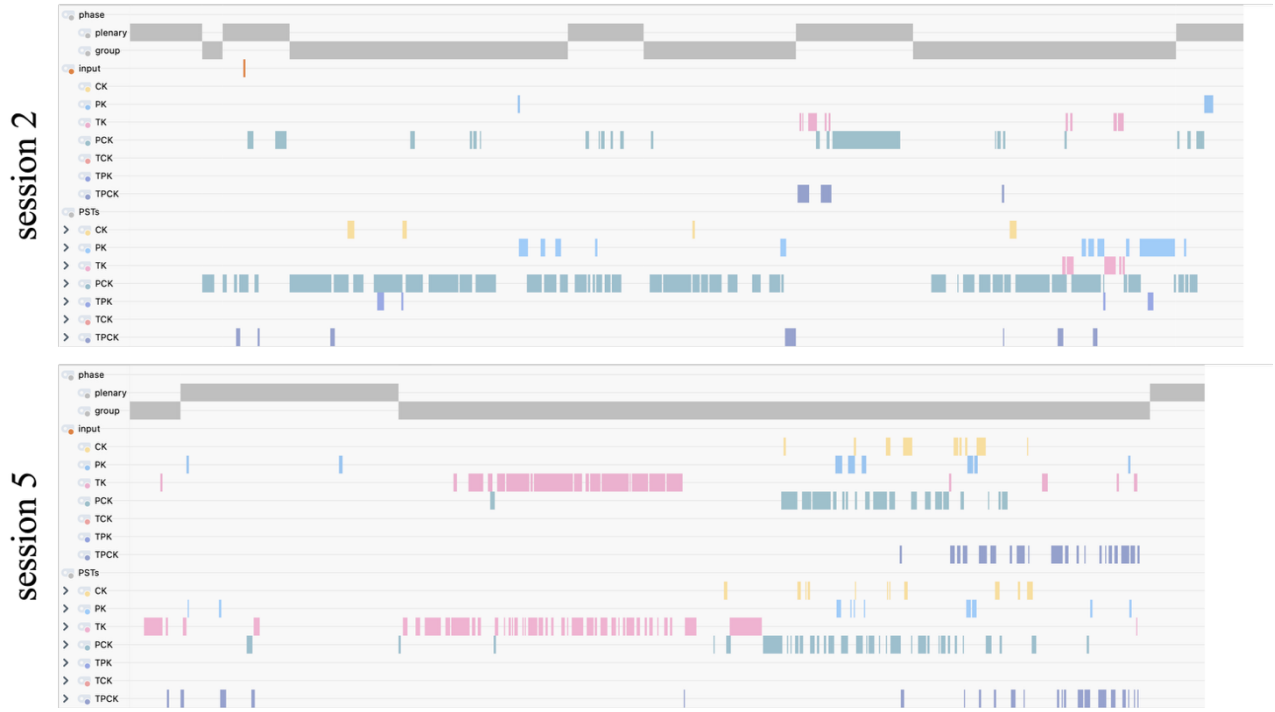

Figure 1: Activated knowledge facets in seminar sessions 2 and 5

Furthermore, the facets are not activated sequentially, but alternately (Figure 1). Moreover, PST repeatedly pick up on knowledge facets from previous sessions, and connect them with new facets. Thus, more knowledge facets are activated from session 2 to session 5. Also, the different knowledge facets are increasingly interconnected, which is indicated by the alternating change of colors, and thus, a change of the knowledge facets (Table 2 and Figure 1). In sessions 4 and 5, the designer team supported PST in the design process for their DIMEG. Various facets of knowledge are activated in those sessions. Longer sections of the same knowledge facet indicate conversations with members of the designer team. TPCK mainly occurred in session 4. Both, Table 2 and session 5 in Figure 1, indicate that this time of the design process in combination with the support of the designer team fosters the interconnection between knowledge facets.

Discussion and conclusion

This study aimed to investigate which knowledge facets PST activate when designing DIMEG and how these develop during the design process. The results indicate that designing a DIMEG allows PST to activate all knowledge facets, in this particular group, except for TPK. This supports results from earlier studies in the context of GDBL (Mishra & Koehler, 2006; Piñero Charlo et al., 2021; Wu & Wang, 2012). Yet, in sum, there are lower scores for all technology-related knowledge facets (TK, TPK, TCK and TPCK) than for all non-technology ones (CK, PK and PCK) (Table 2), which is in line with the results of Shambare and Simuja (2024). At the same time, PST activated different

knowledge facets alternately and increased the number of facets, implying an increasing connection between them.

Different and unexpected from existing studies (e.g., Koh et al., 2013), PST in this study activated TPCK, without showing TCK and with five coded activations of TPK throughout all sessions. This might indicate that other ways could lead to the acquisition of TPCK. Further qualitative and quantitative investigations are needed to explore this observation.

Four main limitations of the study should be acknowledged. First, our results are not generalizable, as this study focused on one group only. Future studies should analyze more groups of PST. Second, PST were observed and videotaped during their seminar sessions. Neither interviews nor knowledge tests were conducted, mainly due to practical reasons. Yet, this could strengthen the design of the study. Third, the coding considered what PST said in each seminar session. Thus, what they thought or discussed outside the seminar could not be recorded. Hence, PST could possibly have activated more knowledge than we could code. Fourth, by providing an online-based editor to implement the DIMEG technically, we limited the range of available technologies. In future studies, this variety could be increased by adding artificial intelligence or integrating apps such as GeoGebra.

The design process of a DIMEG provides an opportunity to address the deficit of digital skills among PST (and teachers). For students to become more digitally competent, PST and in-service teachers must first be trained in these competences. From the results of this study, we conclude that designing and implementing escape games within a seminar of teacher education is a setting that provides a rich opportunity to develop (almost) all knowledge facets of the TPACK framework. The design of escape games is an effective way to introduce PST to technology in teacher education. These skills are essential for teachers to be able to use technology in their own teaching and to change conventional mathematics teaching. It remains unclear whether this can be transferred to other digital environments, such as the development of learning or explanation videos, or other content, such as biology or chemistry. Also, it would be interesting how PST can transfer their acquired knowledge for instance to reflect the quality of other DIMEG.

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Beliefs, attitudes, and pedagogical development in Finnish primary school mathematics education

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Teachers' beliefs and attitudes towards mathematics teaching and learning play a crucial role in shaping teaching practice. Yet, how these affective orientations develop across teachers' learning histories and through professional education remains insufficiently understood. This paper examines how a university mathematics course can foster new orientations, a question that resonates with international concerns about preparing teachers for equitable and engaging mathematics learning. The paper draws on findings in the dissertation project using surveys, interviews and pedagogical interventions with Finnish pre- and in-service class teachers. Preliminary results suggest that teacher education should address not only cognitive and didactical knowledge but also affective orientations as part of the learning of practice. Reflective learning tools and pedagogical languaging approaches are highlighted as promising means to support mathematics teaching in primary schools.

Keywords: Teachers' beliefs, teachers' attitudes, mathematics education, languaging.

Introduction

International evidence highlights a steady decline in students' performance, interest and self-concept in mathematics, alongside widening disparities linked to socioeconomic, linguistic and geographic factors (OECD, 2023). Finland, once a high performer, now faces similar challenges. Primary school assessments reveal early divergences in achievement and declining student confidence. Moreover, Finnish students' perceptions of teacher support—including individual scaffolding, clear instructions and structured lessons—have decreased in recent years (OECD, 2025).

At the same time, primary school teachers face mounting challenges: increasing classroom heterogeneity, new pressures of digitalisation and relatively stable teaching practices (OECD, 2023). The Finnish education system and its policies ensure a high level of teacher qualification, requiring a master's degree and supporting a two-tier curriculum that grants considerable autonomy to schools and teachers (EDUFI, 2016). However, teachers' *affective* and *identity-related* barriers to professional growth have not been sufficiently explored (Kayan Fadlelmula, 2022; Lutovac et al., 2024). These patterns underscore the urgency of strengthening mathematics teaching in ways that address not only teachers' pedagogical knowledge but also their affective orientations—beliefs and attitudes that influence classroom climate and instructional quality.

This study proposes *languaging* as one promising response, positioning mathematics learning as a meaning-making process of concepts in which teachers' affective orientations can shape their instructional choices (Planas & Pimm, 2024). Languaging (kielentäminen in Finnish) refers to the process of expressing mathematical thinking and sharing it with others to build conceptual understanding (Perkkilä & Joutsenlahti, 2021). Yet, little research has explored how new experiences at university can support affective orientations shaped by teachers' learning histories, or how teacher education can actively foster them. By addressing these relationships, the study contributes to international discussions on preparing teachers for equitable and engaging mathematics teaching.

Theoretical background

Conceptualisation of teachers' beliefs and attitudes

Research on teachers' beliefs has shifted attention from teaching as a set of behaviours producing student outcomes to teaching as a complex, reflective practice shaped by personal beliefs and contextual constraints (Thompson, 1992). Teachers' effectiveness cannot, therefore, be understood solely in terms of achievement metrics; rather, it emerges from the interplay between beliefs, practices and the wider social context of schooling. Beliefs influence how teachers conceive mathematics, make instructional decisions, and create classroom climates, yet inconsistencies between beliefs and practice are common, reflecting methodological, contextual and knowledge-related challenges (Ernest, 1989; Hannula, 2007).

The positioning of beliefs has long been debated between cognitive and affective domains (Thompson, 1992). From a cognitive perspective, beliefs are subjective knowledge systems that guide how teachers design learning environments, while from an affective perspective they are interwoven with emotions, values and attitudes, reshaped through ongoing comparison with new experiences (Pehkonen & Törner, 1995). Recent research largely situates beliefs within belief systems, conceptualising them as basic or conscious structures that regulate teaching and learning in interaction with related constructs such as attitudes, emotions and identity (Hannula, 2020).

In this study, beliefs are understood as relatively stable cognitive lenses through which teachers interpret mathematics, its teaching and learning, and their own roles as teachers (Liljedahl & Andrà, 2020). Attitudes, by contrast, are viewed as dispositions to respond favourably or unfavourably towards mathematics-related phenomena (Ajzen, 2005), representing affective expressions of beliefs that shape teachers' responsiveness in specific contexts. This conceptualisation highlights the importance of addressing both beliefs and attitudes in teacher education, since they directly influence classroom climate, responsiveness and the enactment of pedagogical approaches.

Languaging as a pedagogical approach

The concept of languaging has its roots in sociocultural theory, which emphasises that learning is socially mediated and that language is the primary cultural tool through which thought is developed (Vygotsky, 1986). In this perspective, languaging refers not only to verbal expression but also to the use of symbolic, visual, and embodied modes for constructing and communicating mathematical meaning. Such multimodal meaning-making can be facilitated through not only spoken and written language but also nonverbal strategies such as gestures, material artifacts, diagrams, tables, graphs, and mathematical symbols (Moschkovich, 2010).

In Finland, Joutsenlahti (2003) introduced the concept of *kielentäminen* in mathematics education, highlighting its potential for both cognitive and affective development by enabling learners to articulate and reflect on their mathematical ideas in multiple ways. Over the past two decades, languaging has evolved into an increasingly central pedagogical approach in Finnish mathematics education, informing both curriculum reforms and teacher education (Joutsenlahti et al., 2018). The Finnish National Core Curriculum (EDUFI, 2016) embeds languaging within its broader emphasis on *multiliteracy* and *language-awareness*, encouraging students to justify solutions orally, visually

and in writing. Pedagogically, this approach helps teachers access students' reasoning processes, identify misconceptions, and foster participation across diverse learners (Erath et al., 2021).

Languaging extends beyond related notions such as mathematical communication or discourse, which typically prioritise spoken and written forms. It foregrounds multimodal and metacognitive processes through which students externalise and refine their understanding, drawing on symbolic, pictorial and embodied resources (Planas & Pimm, 2024). Importantly, teachers play a pivotal role in designing and facilitating these learning environments: they must interpret students' multimodal expressions while also modelling how mathematical meaning can be expressed across modes. In this way, languaging positions language not merely as a vehicle for communication, but as a developmental catalyst for inclusive, conceptually rich mathematics teaching.

Taken together, this study conceptualises languaging as a bridge between teachers' affective orientations—beliefs and attitudes about mathematics teaching—and their classroom practice, offering a pedagogical pathway for fostering student-centred approaches in mathematics education.

Research questions

This study is guided by the following research questions:

RQ1. How do Finnish pre-service and in-service primary school teachers' beliefs and attitudes towards mathematics teaching and learning develop across their learning histories?

RQ2. In what ways can a university mathematics course, grounded in languaging, foster pre-service teachers' beliefs about mathematics teaching and learning as a meaning-making and student-centred process?

Methodology

This paper draws primarily on three studies of Finnish pre-service and in-service primary school teachers who participated in a university mathematics methods course designed around languaging-based pedagogy. The course emphasised multimodal activities—verbal, written, pictorial, symbolic, and embodied—that encouraged participants to externalise and share their mathematical thinking. All studies were conducted in accordance with the ethical guidelines of Tampere University and the Finnish National Board on Research Integrity (2019). Participation was voluntary, informed consent was obtained from all participants, and data were anonymised to ensure confidentiality.

Study 1 (qualitative approach supported by descriptive statistics)

Two Finnish universities participated in the study during 2023: Cohort A (n = 56, no previous teaching experience) and Cohort B (n = 41, career changes in the open university programme). A pre-survey was conducted at the beginning of the mathematics methods courses, consisting of three open-ended questions: (a) *What does mathematics mean to you? Describe mathematics teaching and learning from your previous schooling. Begin with the metaphor: "Mathematics is like… because…"*; (b) *What kind of student were you and how do you learn best?*; and (c) *How would you describe mathematics teaching during your school years?* Predefined categories—*Journey, Tool, Structure,* and *Language type*—were drawn from prior metaphor studies to capture teachers' conceptual lenses, while additional categories (e.g., *necessity*) emerged inductively. Metaphors were coded according to their underlying conceptual structures through multiple review round to ensure reliability. Word-

cloud analysis and colour-coded mapping (positive = blue, mixed = yellow, negative = red) illustrated dominant affective orientations. Descriptive statistics summarised metaphor types and prior school experiences, supported by scatterplots and cluster visualisations to explore cross-cohort patterns.

Study 2 (convergent mixed methods)

Cohort A from Study 1 completed a post-survey after participating in the languaging-based methods course, used here as an intervention from August to November 2023. The reflection survey included Likert-scale questions addressing changes in beliefs and attitudes towards mathematics teaching and learning, as well as perceptions on what makes a good mathematics teacher. Respondents who selected *1 = changed markedly* or *2 = changed somewhat* were invited to provide written explanations. The languaging-based mathematics methods course (5 ECTS) was part of the subject modules in the class teacher programme at Tampere University for undergraduate newcomers. *Matematiikan opetus ja oppiminen* [Mathematics teaching and learning] (Joutsenlahti et al., 2018) was the main course material, yet the course emphasised peer discussion and small group activities designed to develop and express mathematical thinking in versatile ways.

Quantitative data were analysed using descriptive statistics, while qualitative data were examined through iterative thematic analysis, combining deductive and inductive coding. Deductive codes were informed by the conceptual distinction between beliefs and attitudes and by key objectives of languaging pedagogy (Joutsenlahti et al., 2018). Inductive coding allowed new categories—such as *safe atmosphere,* or *avoiding hierarchies like "math head"*—to emerge from participants' reflections.

Attitudinal polarity (positive, neutral, negative) was traced using contextualised lexical indicators (e.g., *inspired*, *confident*, *challenging*, *frustrating*). Views of a "good mathematics teacher" were compared with sensitising orientations from previous studies (Portaankorva-Koivisto & Grevholm, 2019), including subject, didactical, and pedagogical expertise.

Study 3 (qualitative multiple-case design)

Using purposive sampling, five in-service class teachers and one special teacher (teaching Grades 2–4, with 5 to 22 years of teaching experience) participated in the study from September to December 2024. Five had experienced languaging through university courses and one through self-motivated study with a colleague. Both surveys and interviews were conducted to explore teachers' beliefs and prior learning experiences in mathematics, as well as views on languaging and its impact on their beliefs. Data were thematically analysed through iterative readings that coded and grouped responses into broader themes, which were then visualised in a simple line graph to capture participants' previous learning histories. This paper reports results published in Kim et al. (2025).

Results

Pre-service teachers' learning histories and beliefs and attitudes

Survey data from 97 Finnish pre-service class teachers revealed that prior experiences strongly shaped their beliefs and attitudes towards mathematics teaching and learning. Cohort A, comprising mainly younger students with little teaching experience, tended to view mathematics as a practical tool for personal problem-solving or career advancement. By contrast, Cohort B, consisting of career-change students with diverse professional backgrounds, were more likely to associate mathematics

with broader social purposes, such as fostering transversal competences and connecting individuals to society. Despite these differences, both cohorts reported that traditional mathematics instruction dominated their school experiences, often linked with negative or mixed affective orientations. Metaphor analysis showed frequent associations of mathematics with rigidity, difficulty, and anxiety, although some participants also highlighted creativity or usefulness. These findings illustrate the enduring influence of schooling on belief formation and the persistence of transmissive instructional norms. At the same time, they show how professional and life experiences can broaden pre-service teachers' conceptualisations of mathematics beyond the classroom.

In-service teachers' learning histories and beliefs and attitudes

Survey and interview data from five in-service class teachers and one special education teacher illustrated how prior mathematics experiences shaped their beliefs and attitudes (Figure 1). Three teachers (Aino, Tuuli, Helmi) reported consistently positive or mixed-positive trajectories, while three (Mila, Elina, Lumi) described predominantly negative experiences throughout schooling. Despite these varied pathways, participants converged on two shared principles of mathematics teaching: (1) the importance of building a strong foundation of knowledge and skills for all learners, and (2) the relevance of mathematics for daily life.

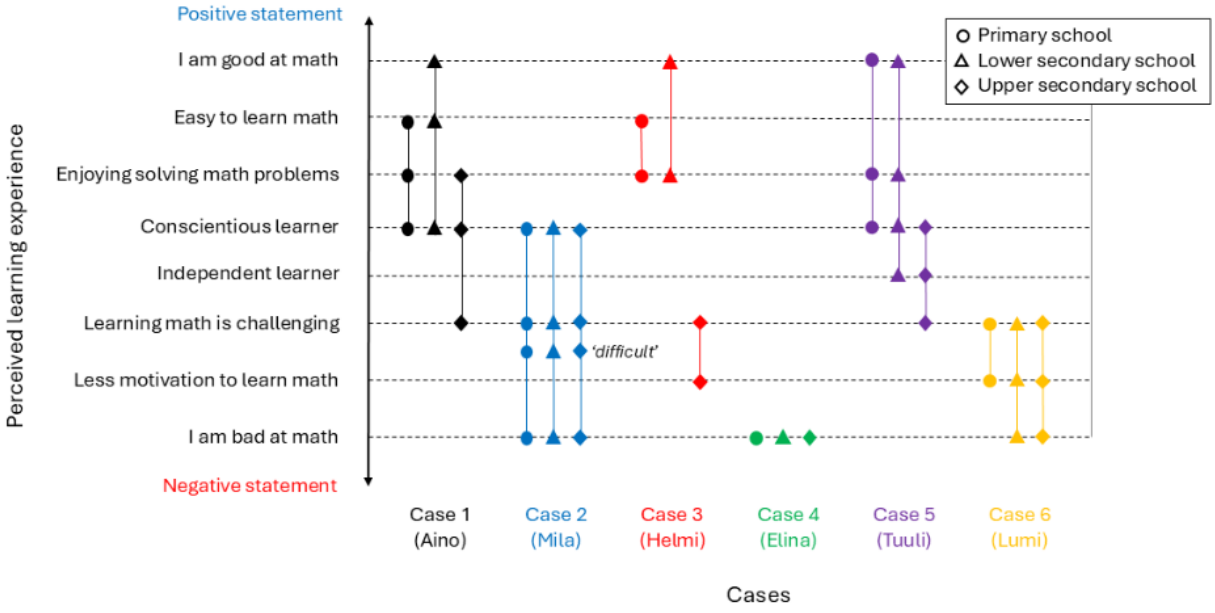

Figure 1: In-service class teachers' experiences of their own mathematics learning from primary school to upper secondary school (Kim et al., 2025, p. 11)

These trajectories illustrate how deeply embedded experiences of school mathematics continue to shape teachers' current orientations. Positive histories often reinforced confidence and a focus on procedural fluency, whereas negative or mixed histories led teachers to emphasise equity, accessibility, and conceptual understanding through varied approaches. Taken together, the findings highlight how both pre-service and in-service teachers carry forward affective legacies from their schooling, which can either support or constrain their development as mathematics educators.

Evolving beliefs and attitudes through languaging pedagogy

Among pre-service teachers, survey and reflection data showed that most participants developed more pedagogical beliefs about mathematics teaching and learning as a meaning-making process in socially engaged settings, with attitudes either becoming more positive or remaining consistently positive. While their prior schooling had been dominated by traditional teaching, the course enabled them to see mathematics as a meaning-making process rather than a fixed set of procedures. Collaborative tasks, particularly small-group activities, helped participants value diverse viewpoints and recognise the importance of supporting active learners. Reflective accounts indicated a growing aspiration to combine didactical expertise with pedagogical sensitivity, redefining the qualities of a "good mathematics teacher."

Similar patterns were evident among in-service teachers. Case studies revealed trajectories from negative or insecure prior experiences towards more positive orientations, influenced by exposure to languaging practices during teacher education and professional work. For example, Mila described mathematics as once "painful" but now "illuminating," while Helmi explained how deliberately making mistakes in class encouraged students' questioning. Aino and Tuuli reported that languaging enriched their practice by helping them connect mathematics to meaningful, functional activities. These shifts illustrate how languaging can serve as a catalyst for reinterpreting past experiences, fostering more inclusive and student-centred pedagogical approaches.

Discussion and conclusion

Based on the results, three key insights emerged.

First, the findings confirm the strong imprint of teachers' own schooling experiences on their beliefs and attitudes. Both pre-service and in-service teachers often carried forward mixed or negative memories of mathematics, dominated by transmissive practices that reinforced static and procedural conceptions. These affective legacies shaped how teachers positioned themselves as learners and teachers of mathematics, consistent with earlier studies highlighting the persistence of traditional instructional norms (Eichler et al., 2023).

Second, the study demonstrates that professional and life experiences can diversify and enrich teachers' conceptualisations of mathematics. Career-change pre-service teachers associated mathematics with broader social purposes, while in-service teachers with negative prior experiences reported renewed commitment to equity and accessibility. This suggests that affective orientations are not fixed but develop dynamically across teachers' life histories, echoing Pehkonen and Törner's (1995) notion of belief systems as evolving structures.

Third, the languaging-based methods course provided evidence of how teacher education can actively foster pedagogical beliefs—specifically, beliefs aligning with mathematics as a meaning-making process. Multimodal ways of expressing thinking and collaborative activities helped participants reframe mathematics as a meaning-making process and strengthened their pedagogical sensitivity to diverse learners. This aligns with sociocultural perspectives that position language and interaction as mediators of conceptual and affective development (Vygotsky, 1986). Importantly, teachers with both positive and negative learning histories benefited from languaging, suggesting its potential to bridge diverse affective orientations and support inclusive classroom practices.

Consequently, these findings underline the need for teacher education to address not only cognitive and didactical knowledge but also affective domains. By explicitly integrating reflective tools that address prior learning experiences and current views, teacher-education programmes can challenge static orientations and value diverse life experiences. In combination with languaging pedagogy, these approaches can cultivate classroom practices that are equitable and engaging. Future research should investigate how such interventions sustain affective growth and influence long-term instructional practices, thereby contributing to the design of mathematics teacher education that prepares teachers to meet international demands for equity, inclusion, and high-quality practice.

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Teachers' perceptions and use of talk moves following a professional development course

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This study explores how four lower secondary school teachers interpret and enact the practice of talk moves—specific teacher strategies to promote classroom discussions—following participation in a practice-based professional development (PD) course. Audio recordings of interviews, group discussion, notes from classroom observations and e-mail correspondence with the teachers were analysed, using key elements of the talk moves strategy as a framework. Findings indicate that teachers with different background and contexts perceive and use talk moves in very different ways, highlighting the need for further research on the impact of PD on teachers' practice.

Keywords: talk moves, professional development, mathematical discussions, teaching practice.

Introduction

Professional development (PD) is widely accepted as a means to enhance the quality of teaching practices (Kennedy, 2016), and teachers participate in PD every year. Likewise, there is a growing body of literature that has provided information about PD of teachers. Nonetheless, research on PD in mathematics education has focused on the effectiveness of particular programs, curricula, or professional learning approaches (Goldsmith et al., 2014). Few studies have focused on the processes or mechanisms of teachers' learning, and therefore, they say little about how teachers develop knowledge, beliefs, or instructional practices (Goldsmith et al., 2014). In a systematic review of research on PD activities for teaching practices to master particular tasks of mathematics teaching, Shure et al. (2025) revealed a recent tendency to emphasize teacher learning opportunities. Their findings highlighted the need for more detailed reports on learning opportunities for teachers.

Further, research on students' learning over the last two decades has emphasized ambitious mathematics teaching; a teaching approach that supports a high level of student participation and conceptual understanding (Lampert et al., 2013). The work of teaching in a classroom engaging students in an active construction of knowledge is a complex phenomenon (Ball, 2017). Karsenty and Brodie (2023) argue that there is much research showing success of PD programs in supporting ambitious instruction, and we need more research that articulates the difficulties. Thus, this study sheds light on the broader challenge of integrating PD into existing practices across diverse contexts.

Many PD programs focus on the facilitation of classroom discussion, based on strong research evidence such as Michaels and O`Connor (2015), Stein at al. (2008), and Kazemi and Hintz (2023). Productive mathematical discussions are those that both build on student thinking and also advance important mathematical ideas (Stein et al., 2008). "It is now widely accepted that certain kinds of well-structured talk, whether teacher guided or student directed, promote academic learning" (Michaels and O`Connor, 2015, p. 348). This recognition underscores the significance of equipping teachers with the expertise they need to facilitate such discussions effectively, but the real-time orchestration of classroom talk remains a challenge. Research suggests that teachers have found it

difficult to implement new ways of using talk in practice and that dialogic pedagogy is still rarely observed in everyday practice (Chapin et al., 2009; Davies et al., 2017). One practical tool which is designed to support teachers in orchestrating classroom discussion which builds on students' ideas is *talk moves,* highlighted in work by Chapin et al. (2009) and Kazemi and Hintz (2023).

Talk moves

Talk moves refers to strategic ways of asking questions and inviting participation in classroom discussions. Chapin et al. (2009) presented talk moves that are designed to accomplish four steps towards productive discussions: helping student clarify and share their thoughts, orient towards each other, deepen their reasoning, and engage in other's reasoning. *Revoicing* helps students to clarify and share their thoughts and entails repeating some or all of what a student has said and asking them to verify if the *revoicing* is correct or not. This Talk Move can be used to amplify or highlight an idea. *Repeating* involves asking a student to repeat or rephrase what another student in the class has said. This move can be used to emphasize important ideas, slow the pace of discussion, and help students orient to others' thinking. Chapin et al. (2009) stress that *repeating* is not supposed to be a management move, but rather is designed to invite students to contribute in a positive way. *Reasoning* both entails asking students to explain their reasoning or asking whether they agree or disagree with peers. The aim is both to enable students to deepen their reasoning and compare and engage with other students' reasoning. *Adding on* invites students to contribute further to the conversation or clarify their own thinking. Finally, *wait time* involves giving students time to think and organize their thoughts after a question is asked or a student has been called on, which can support students in engaging in other students' thinking.

Building on this framework, Kazemi and Hintz (2023) added two additional moves: *turn-and-talk* and *revise your thinking*. When using the talk move *turn-and-talk*, students are asked to share their ideas with one or two of their peers. They are oriented towards each other, while the teacher circulates the classroom, listening to their conversations, and using this information to decide whom to call on. Having time to talk to a peer helps students clarify their ideas and make it safer to share them aloud. *Revise* enables students to reconsider and adjust their ideas in response to a teacher's asking if anyone's thinking has changed during the discussion and why.

Research suggests that although the use of talk moves to orchestrate classroom dialogue may support higher levels of mathematics content learning (Chapin et al., 2009; Kazemi & Hintz, 2023; O'Connor & Michaels, 2017), teachers' uptake and perceptions of the strategy vary. For example, O`Connor and Michaels (2017) found that, while some teachers picked talk moves techniques up with ease, others, particularly those less experienced, seemed to find them difficult to use. Some teachers used the moves in a "robotic" manner rather than using them in response to classroom dynamics or the goal of the discussion. Similarly, Myklebust and Guadie's (2024) study of pre-service teachers found that although some used some talk moves in practice and rehearsal discussions (*revoicing*, *reasoning*, *wait time* and *repeating*), they struggled to engage students in complex exchanges.

Teachers' responses to and use of the moves varies across research studies. Justnes and Mosvold's (2022) study of kindergarten teachers' perceptions of the use of talk moves found that while some moves, such as *revoicing*, *repeating*, *reasoning* and *adding on*, were seen as aligned with the

kindergarten culture, others, such as *wait time* and *revising*, were not. The teachers felt that children in kindergarten generally disliked *waiting* and that *revising* was too cognitively demanding. In another study of 24 teachers' uptake of talk moves across five different schools, Michaels and O`Connor (2015) reported that not all talk moves were taken up by the teachers equally. Talk moves that intended to orient to each other (*repeating*) were not used nearly as much as other moves. Wæge and Svingen's (2023) study found the opposite pattern - teachers in this study appreciated talk moves for orienting students to each other's ideas and particularly favoured *repeating* as effective. These are examples of how research to date raises issues about how teachers perceive individual moves and how easy they find it to integrate them into their existing practice. Consequently, my research question is: *How do lower secondary mathematics teachers perceive and enact talk moves following their participation in a professional development program in ambitious mathematics teaching?*

Method

Four lower secondary school teachers from two different schools, located in a suburban area in the southeastern part of Norway, participated in this study (Ava, Anne, and Adam from Alstad school and Ben from Bergheim school). They had 1 to 7 years of teaching experience. At the beginning of the data collection, the teachers were enrolled in a PD program called *Mastering Ambitious Mathematics teaching* (MAM), a PD program aimed to support in-service teachers in developing practices that engages with students' thinking, encourages questioning, and supports reasoning and argumentation to promote understanding, learning, and motivation. The project consisted of 12 whole-day sessions over a two-year period. Teachers had been introduced to talk moves as part of the program via an article in a professional magazine (Wæge, 2015). They had practiced the use of talk moves in rehearsals (Wæge & Fauskanger, 2023) with other teachers with real school students during the PD sessions and had been asked to practice with their own classes between sessions. All participants gave informed consent, and all names of participants and schools are pseudonyms.

This paper is based on transcripts from audio-recorded individual semi-structured interviews with each teacher, one group discussion (work between sessions) with the three teachers from Alstad, field notes from classroom observations of Ava and Anne, and email correspondence with all four teachers. Interviews took place at Alstad and Bergheim school between session 7 and 8 and were designed to explore the teachers' experiences of implementing new practices in their classrooms. The group discussion took place at Alstad school between session nine and ten and focused on changes that teachers had made in their own classrooms. I observed and audio-recorded this discussion. Preliminary analysis of the Alstad teacher data highlighted their emphasis on the ease of implementing talk moves into their teaching. However, I also noticed that the teachers explained their understanding and use of the moves differently. To explore this further, I followed up with classroom observations of Ava and Anne, where I noticed that they used talk moves that they had not mentioned in their interviews. I e-mailed all four teachers individually, telling them what I had observed in their interviews and teaching and asking them to elaborate on their use of talk moves.

To analyze the data, I read through all transcripts and notes, identifying excerpts related to talk moves, categorizing them in terms of Kazemi and Hintz's (2023) seven talk moves (as outlined above). During the process, it became apparent that an additional category was needed, *talk moves in general,*

which included excerpts saying anything about teachers' enactment and perceptions of talk moves in general, not related to a specific talk move. In the second face of the analysis, I looked at each category and teacher separately, trying to catch the nuances in their interpretation and use of talk move.

Findings

As indicated above, teachers seemed to have different perceptions of what individual talk moves are and how to use them. Here, I will focus on *repeating*, *reasoning*, and *turn-and-talk*, which teachers talked about the most and reported using most frequently. Then I will turn to how the teachers have integrated MAM into practice.

Repeating

Repeating was the most frequently used and discussed move. The teachers stated that it was new to their classroom practice and easy to implement. It was used several times in both Ava's and Anne's classes. However, teachers expressed different reasons for using *repeating* as a talk move. In keeping with the program ethos, Ben emphasized that *repeating* helped students to stay alert and to develop the habit of orienting themselves towards others thinking: "They get used to working this way, and I think it helps them stay a bit alert, but also to think critically about what everyone else is saying" (Ben, individual interview). Anne also saw the value of *repeating* other students' contributions for building understanding, but in addition emphasised her view that repetition was important for learning:

> I want my students to listen to each other's explanations. Many become more "alert" and focused on their peers' reasoning when they know they might be asked to repeat it. Often, students understand each other's explanations better than mine, so it's important that they listen to one another. Repetition is also important for everyone's learning. Many in the group benefit from having things repeated several times. (Anne, e-mail)

This was also confirmed when observing Anne's class, where Anne, just like Ava, asked the students to repeat important points several times. Adam focused less on building understanding, instead describing *repeating* as useful for keeping the classroom calm and forcing students to pay attention:

> Sometimes, using the talk move "Can you repeat what was said?" helps create calm in the classroom, because students actually need to listen to what someone on the other side of the room is saying. It can also force them to be more focused, since it's embarrassing not to be able to answer "Can you repeat?" (Adam, group discussion)

Ava similarly explained that she asks students who do not pay attention to repeat: "I use it to sharpen their focus, and I often ask those I know aren't paying attention" (Ava, email). This does not align with Chapin et al.'s (2009) intention of the *repeating*, who state this is not a management move. They emphasize that it should be used in a positive way and not to catch student in not paying attention.

Reasoning

Reasoning is designed to encourage students to explain their thinking and engage with each other's reasoning, by for instance asking if they agree or not with what another student has said. However, it appeared that the teachers did not use this talk move to foster discussion in terms of engagement with

others' thinking, but rather as an invitation to students to explain their own thinking by asking 'why?'. Ben explained that "I don't feel like I use this talk move explicitly, but I always ask 'why', 'how', and 'what do you think?' as a natural part of classroom dialogue" (Ben, email). Ava, Anne and Adam also focused on students explaining their own thinking, arguing that this is challenging. Anne thought that students should be encouraged to try though: "They gain a better understanding of what they've done, why it works, or what might not be correct when they explain how they're thinking" (Anne, email).

Turn-and-talk

The teachers reported differently on their use of this talk move. While Ben and Adam said they used it regularly, Ava and Anne described difficulties with noise and focus, as described by Ava:

> I actually like 'turn-and-talk', or 'work on this in pairs', which I use from time to time. But I often find that students don't really talk to each other, or they end up talking about completely unrelated things, and it can be hard to bring them back to mathematics. (Ava, individual interview)

Adam, however, explained how he used *turn-and-talk* to get more students engaged in discussion:

> Many [students] hesitate to raise their hand to answer a teacher's question in front of the whole class. I think the reason is that they worry that their answer might be wrong, and that could be embarrassing in front of their classmates, or even the teacher. With 'turn-and-talk', this embarrassment is avoided, and many more students participate in the discussion. (Adam)

Most in keeping with the program ethos, Ben emphasized how he used *turn-and-talk* to listen and communicate with students: "It's important to walk around, listen to the discussions, offer input, and guide students—without giving them the correct answer." (Ben, email)

Integrating MAM into practice

In this section, I will show how each teacher responded to the program ethos and their reflections on talk moves in terms of achieving various teaching aims. I will start with Ben, who works at a different school than the other three. While Ava, Adam and Anne refer to MAM as a contrast to what they usually do, Ben says that it aligns with his education and the way his school teaches mathematics:

> (…) Working with open-ended tasks, right? Working with trial and error, working with inquiry, that kind of thing we've been doing for a long time here, and I'm used to it from my studies (…) So I'm actually quite used to working the way MAM wants us to work. (Ben, interview)

Ben is focusing on teaching when he talks about talk moves. As described above, Ben used *turn-and-talk* to listen and communicate with students and *repeating* to help students orient themselves towards others. He said that he used *revoicing* to support equity: "This can help more students get on board and understand the approach and background." (Ben, email). Equity is also emphasized when explaining *wait time*: "(…) when only a couple of students want to speak—I've started to wait a bit longer to see if others also want to contribute. Every thought is equally important." (Ben, email). He emphasized how *revising* can be used if students share a misconception:

> If many students in a class share a misconception, it's important to create a shift in understanding. In such cases, the whole class needs to move in the same direction together with the teacher. This

> gives students an opportunity to rethink their approach and suggest new ways to solve the problem. (Ben, email)

When asked which talk moves, he used in his teaching practices, Ben emphasized the importance of understanding students' thinking when fostering mathematical discussions:

> How and why these talk moves are used is almost more important than simply using them. I've noticed that gaining an understanding of what students are actually thinking provides a stronger foundation for classroom dialogue in general. (Ben, individual interview)

Like Ben, Anne embraced the MAM ethos of inclusion and recognises that learning is not about teachers' telling:

> It's about getting mathematical conversations started in the classroom and involving a larger number of students. (...) So in a way, it's more about helping them arrive at a way of thinking, rather than me simply telling them. (Anne, individual interview)

In contrast to Ben, the other teachers seemed to focus more on problems of practice. Anne explained why *revising* is challenging to use like this:

> I think that's because it demands more from the students—they have to explain their own thinking, listen to others, and use what others say to adjust their own reasoning. That's definitely more demanding than, for example, simply repeating what someone else said. (Anne, email)

Ava claimed to have become more aware about what she said in class during and after the course, and she seemed to embrace students' understanding and inclusion. As explained above, she expressed worries about the students' focus during *turn-and-talk* and appreciated *repeating* as a management move. She said that she used *adding on* and *wait time* to include as many students as possible. About *revising,* she explained that if her students have reasoned differently, she asks whether that had changed anything: "That way, they get to consider whether there might be multiple solutions or reflect on whether they might have made a mistake" (Ava, email).

Although Ava had not mentioned it in her interview, I noticed when observing her class that Ava used *revoicing* quite often. When asked about this, she explained:

> I repeat things so that more students can catch what's being said, and so that the student can correct me if I've misunderstood or misheard something. I often adjust the wording a bit if what the student says is hard to understand. I do that quite often, actually, and I haven't really thought of it as a MAM-related practice. (Ava, email)

Unlike the others, the only two talk moves Adam claimed to use was *repeating* and *turn-and-talk*. He said that "he should be able to work on the use of the others" (Adam, email), he did not say much about student learning. As described above, he appreciated *repeating* as a management move and *turn-and-talk* to involve more students in the discussions. About *wait time,* he said: "It feels like something you need a bit of guts to use in a classroom" (Adam, email).

Discussion and conclusion

In this study, I have examined lower secondary school mathematics teachers' perception and enactment of talk moves after attending a PD program in ambitious mathematics teaching. The results indicate, like O`Connor and Michaels (2017), that teachers perceive and implement talk moves differently, even when they have attended the same PD program and practiced using talk moves together. Looking at Ben who had experience from before with similar teaching practices, and Adam who was new to teaching mathematics, it is clear that there were differences in the way they applied MAM principles and talk moves into their existing practices. While Ben had integrated talk moves in his teaching and seemed aware of which talk move he used, in relation to the goal he wanted to achieve with them, Adam seemed unsure about using more than a couple of the moves.

The variation in how teachers use talk moves reflects the broader challenge of integrating theory and practice in PD. Teacher education and PD programs need to consider that teachers' uptake and reasons for using or choosing not to use theoretical tools will vary. Teaching practices that engage students in active construction of conceptual understanding are challenging to lead (Ball, 2017). This study reinforces an awareness that ambitious mathematics practices may be too difficult to implement for some teachers (Karsenty & Brodie, 2023). Due to the complexity of teaching ambitiously, teachers need guidance in the dilemmas and tensions their pedagogical actions cause. Teachers in different stages of professional growth might need to be offered different levels of complexity (Shure et al., 2025). Although the findings of this study are restricted to just four teachers and their experiences, it tells us something about how different teachers enact and perceive theoretical tools learned in PD. There is a need for more research on how teachers integrate PD into their practices and what PD can help teachers achieve. Further research should look closer into teachers' enactment of theoretical tools from PD in their teaching practices.

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Learning to teach mathematics playfully: When infrastructural constraints and learning goals compete in the redesign of games

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Despite their benefits, the integration of playful learning experiences in formal mathematics education is limited and faces important challenges. Preservice teachers are often more positive towards playful learning, but struggle with barriers like a lack of relevant resources. We explored how student teachers balanced learning goals and infrastructural constraints when collaboratively redesigning games for secondary mathematics classrooms in the context of an elective course on games and learning. We iteratively coded interview transcripts, lesson plans and posters and found that while some participants prioritized infrastructural constraints, others focused on pedagogical objectives, showcasing diverse approaches to designing playful learning experiences. Our findings highlight the importance of teacher education programs in fostering adaptive and resourceful design thinking and equipping educators to tailor innovative pedagogies to varied classroom contexts.

Keywords: Playful learning, teacher training, secondary education, educational games.

Introduction

Many reform efforts in mathematics education have been shown to improve student experiences and achievement (e.g., Allensworth et al., 2021). However, when looking at classroom practices at scale, there is evidence that implementation of such reforms is often superficial or faces important barriers (Ross et al., 2002). One innovation that has faced challenges in implementation is *playful learning*. Playful learning is an activity "that is designed to include one or more elements of games for the purpose of enhancing the learning process" (Plass et al., 2014, p. 6). Even though playful learning experiences can have multiple socioemotional and cognitive learning benefits for students (e.g., Barab et al., 2009; Dishon & Kafai, 2020; Plass et al., 2014), their presence in formal mathematics learning environments is rare (Matic et al., 2023).

Typically, only a small number of teachers is willing to use games in their classrooms (Bacsa-Károlyi & Fehérvári, 2024). Hesitations are present even amongst teachers who identify as experienced gamers themselves (Ketelhut & Schifter, 2011), and previous positive experiences with implementing playful learning are not enough for integrating it regularly in one's teaching practice (Huizenga et al., 2017). This suggests that the barriers to implementation are multi-faceted and persistent, preventing educators who view playful learning positively from employing it in their classrooms. Some of the barriers teachers often mention are: (a) a lack of relevant instructional resources, (b) a lack of alignment between a game and target learning goals, and (c) the inability to fit playful learning experiences in the context of formal education (Baek, 2008; Razak et al., 2012; Watson et al., 2016).

Teacher training programs can positively affect teachers' abilities to select, evaluate and integrate games in classroom teaching (Meletiou-Mavrotheris & Prodromou, 2016). Pre-service teachers in particular often have positive perceptions about playful learning but require professional development opportunities that prepare them to overcome external barriers (Bacsa-Károlyi & Fehérvári, 2025).

Because pre-service and early career teachers perceive the unavailability of resources for game-based learning as a bigger barrier to the implementation of playful learning experiences than other curricular constraints (Baek, 2008), participation in courses that invite them to learn and design playful learning experiences is important, as it can prepare student teachers to better navigate such barriers and support their subsequent experimentation with innovative pedagogies (Kim et al., 2024). Our study, which took place in a secondary math teacher training program at a university in central Europe, was guided by the question: *How might mathematics student teachers balance learning goals and infrastructural constraints when collaboratively redesigning a game for use in secondary mathematics classrooms?*

Theoretical background

We wanted to explore how student teachers interacting in one environment (that of an elective course in their teacher training program) made decisions about the design of a learning experience (lesson and game) that was meant to be used in a different environment (that of a secondary mathematics classroom). To make sense of this complex phenomenon, we employed *learning in activity* (Greeno & Engeström, 2014). *Learning in activity* is a socio-cultural and socio-historical framework for studying learning that goes beyond the individual and recognizes the role that people, socio-material resources and the environment play in guiding people's participation in goal-oriented activities. According to this framework, an activity involves a *subject* (person(s) performing the activity), the *object* of the activity (goal), and the *tools* used in the activity. An activity is also mediated by the *community* the subjects are members of, the *rules* (social rules and norms that specify how people are expected to interact and participate in this activity), and a *division of labor* (how responsibilities, power, and tasks are negotiated and distributed among participants) (Cole & Engeström, 1993).

Given our focus on collaborative design, group decisions, intended learning goals, and infrastructural constraints, the components of *subjects*, *object*, and *rules* were of particular relevance to our study. *Learning in activity* allowed us to conceptualize how two activities that were part of two different activity systems, (a) the design of a playful learning experience in an undergraduate course and (b) the intended implementation of this learning experience in a secondary math classroom, might interact with each other. We examined (one of) student groups' *object* in the first activity system (i.e., the learning goals they wanted to support), and its interplay with anticipated *rules* of the second activity system (e.g., timetables, budgets). By doing so, we were able to draw connections between the two activities and shed light on how student teachers might envision the enactment of innovative pedagogies in formal learning environments and design instructional resources to support them.

Methods

The context of our study was an elective upper-level undergraduate course on game-based learning that was offered in a teacher training program in a university in central Europe in the winter 2024/2025 semester. During the first half of the course, students explored tabletop and digital games, read select literature about the use of games for learning, and reflected and discussed readings and gameplay experiences. During the second half of the course, students worked in small groups (3–4 students) to (re)design a game and create a lesson plan for its use in secondary mathematics classrooms. The groups could choose the mathematical content, learning goals, student age group, and which game to edit or design. The instructor (first author) assigned students in groups based on

their instructional interests, and every group had at least one member who was already teaching in a school. We invited all (29) enrolled student teachers, and 21 agreed to participate in the study.

We collected participants' survey responses and written reflections, exit tickets regarding early work on their final project, the game materials and lesson plans they developed, and their poster for the end-of-semester showcase. Three of our study participants also agreed to be interviewed about their experiences in the collaborative project work. Interviews lasted approximately one hour, were audio- and video-recorded, and conducted by the second author. Given our interest in the game and lesson design choices and how groups made those choices, we focused our analysis on the work of the three student groups whose members we could interview (David, Emma, Leon— all pseudonyms).

We looked at their lesson plans, exit tickets, final posters, and interview data. First, we transcribed the audio recordings of the three interviews and translated them from their original language (German) to English. Then, we developed a coding rubric for the most common infrastructural constraints according to the literature (Table 1). We examined both interview transcripts and exit tickets for the presence of infrastructural constraints. Because the content of the exit tickets varied considerably across participants and rarely included references to such constraints, we excluded them from our analysis. We then used simultaneous and structural coding (Saldaña, 2015) at the sentence level to code the interview transcripts. Both authors used MaxQDA to code all transcripts independently at first, then met to discuss and resolve differences until we agreed on all codes applied.

Table 1: Infrastructural constraints coding rubric

	Definition	**Literature reference**
Technical infrastructure	School facilities, hardware and game availability that might support playful learning.	Razak et al. (2012), Watson et al. (2016)
Time constraints	Limited time available to plan or implement GBL lessons. Strict/inflexible school timetables.	Razak et al. (2012), Baek (2008)
Lack of game-based learning resources	Not having access to examples of lessons or games that have previously been used successfully in classrooms.	Baek (2008)
Curriculum constraints	No or limited space for innovation in the current curriculum.	Razak et al. (2012), Watson et al. (2016)
Financial cost	Limited or no budget available to buy/create games.	Baek (2008)

Once the coding for the infrastructural constraints was complete, the first author employed open coding to capture characteristics of the designed materials. She examined lesson plans, game materials, and final posters for: (a) explicitly stated learning goals, (b) the math content of the games, (c) the expected duration of gameplay, and (d) the total planned lesson time. We then used data source and researcher triangulation (Patton, 1999) to validate our interpretation of these data. For example, we examined both posters and lesson plans to identify learning goals. We addressed discrepancies

between the different data sources by prioritizing data sources that were obtained closest to the end of the semester (thus assumed to reflect participants' evolving foci) and discussed all codes applied to ensure our interpretation of the data reflected both of our understandings.

Ethical considerations and research quality

Given that the lead researcher and first author of this paper was also the instructor of the course, we took steps to ensure the different roles were ethically managed. We chose data collection methods that aligned with course objectives, study participation was blind to the instructor until all course-related activities had concluded, and interviews were managed exclusively by the second author. Our study was reviewed and approved by the Ethics Committee of our institution and we obtained informed consent from all participants.

Emerging findings

When interview participants discussed their groups' design decisions, all literature-identified infrastructural constraints were mentioned at least once across the three interviews (Table 2). The most frequently mentioned constraint in our data was *time constraints* (n = 15), followed by the *lack of game-based learning resources* (n = 6). All three interviewees mentioned instances of *technical infrastructure* and *time constraints*, whereas only two of them (David and Emma) mentioned the *lack of game-based learning resources* and *curriculum constraints*. Finally, only Emma referenced the *financial cost* associated with the implementation of playful learning experiences (n = 1).

Table 2: Total counts of infrastructural constraints mentioned by participants during interviews

	Technical infrastructure	Time constraints	Lack of GBL resources	Curriculum constraints	Financial cost	Total
David	1	12	4	1	0	18
Emma	2	2	2	1	1	8
Leon	1	1	0	0	0	2
Total	4	15	6	2	1	28

The three groups also differed on the games and lessons they developed. In what follows, we share interview data about the design process of each group and discuss how each group's priorities informed the design of the instructional materials they developed, including their games.

Case 1: Prioritizing infrastructural constraints

David's group developed a game that had shorter playtime (15–30 minutes) and the instructional unit lasted a single class period (about 50 minutes). They were very concerned with *time constraints* (mentioned 12 times in David's interview), and this played an important role in their design decisions:

> David: We wanted something in the direction of a practice game, and just, orienting yourself to something that has been used before, you can better estimate how long it will take, and that was a big issue for us.

Their game, “Mathingo”, was a *practice/revision* game on simplifying algebraic expressions for 12–13 years old students (Figure 1a). They wanted their students to engage in *backwards thinking* while practicing calculations with algebraic expressions, so they extended the typical Bingo game by asking students to first create complex algebraic expressions that were equivalent to simplified ones printed in the Mathingo playing cards. These complex expressions were then used during gameplay by the teacher, who acted as the moderator of the game. In this game, learners would experience the mathematical content primarily separately and prior to the main playtime.

Case 2: A general consideration of infrastructural constraints

Emma’s group designed a game that required longer playtime (more than 45 minutes) and a lesson plan that spanned 2 class periods. “Super Mario DKT” (Figure 1b) is a *practice/revision* game for 10–14-year-olds to practice fractions and unit conversions, and develop *strategic thinking* and *negotiation skills*. Based on Emma’s interview, her group likely took into consideration all 5 common infrastructural constraints, without any of them dominating their design choices. Their priority was to design an activity that would be easily adaptable and not tied to specific mathematical content:

> Emma: I think that was very important to us in the game, that you can adapt it very quickly. So these fractions and also what is required for the risk cards regarding the unit conversions, these are actually just the additional cards that are placed in the middle of the playing field (board). And as a teacher, I can just change these tasks and take over the entire game except for these two categories. So I think that you can adapt it again so that it is suitable for every lesson again.

In this excerpt, Emma makes a reference to *time constraints* when she says that adapting the game can be done quickly. However, the emphasis is on the adaptation itself: by changing only a few of the added cards, a teacher can adjust the mathematical content the students engage with and adapt the game for new lessons. The strong preference for adaptability could also suggest an implicit motivation to address a perceived *lack of game-based learning resources*. The mathematical content was incorporated into this game by expanding two existing game components, the “risk” and “mission” cards, with new cards. This resulted in a game in which learners would experience the mathematical content during the main playtime, and mathematics in this case was an add-on rather than embedded into the game mechanics.

Case 3: Prioritizing learning goals

Finally, Leon’s group developed “Capture the flag” (Figure 1c). This was an original *practice/revision* game for 10–12-year-olds, with a longer playtime and an instructional unit the covered 3–4 class periods. The game was intended to enhance students’ understanding of the coordinate system and support the development of *collaboration and communication* and *thinking ahead* skills. In this game, learners would experience the mathematics during playtime and as part of the game, as the coordinate system was embedded in the gameplay in the form of a map.

In his interview, Leon shared that their design was primarily driven by their target learning goals. For example, when discussing how playing “Escape from the aliens in outer space” informed the design of their own game, he highlighted that they wanted a different social experience for their students:

> Leon: We just went here and said, the mistrust of other people, we want to take that out.

In the original game, the players are divided in two groups (aliens and humans). Members of each group play independently, and players try to guess each other's identity throughout the game. While Leon's group wanted the students to practice the coordinate system, an idea that originated from this game, their experience of being suspicious of their playmates was in contrast with their goal of supporting their students' collaboration and communication skills. His group chose to keep elements from one game and introduced elements from another game to meet these goals. Unlike David and Emma, Leon referred to infrastructural constraints only twice in his interview, which suggests such constraints might have played only a small role on this group's design decisions.

(a) (b) (c)

Figure 1: The games students developed: (a) Mathingo, (b) Super Mario DKT, (c) Capture the flag

Discussion and limitations

The three groups in our study adopted diverse approaches to designing playful learning experiences, with some prioritizing infrastructural constraints and others focusing on pedagogical objectives. Still, all three groups took into account the possible rules of the activity system of a secondary math classroom, while themselves participating in a design project in the different activity system of an elective course in their teacher training program. This suggests that student teachers were able to take on different roles (teacher, teacher within a professional community, learner, educational designer), an important experience for successfully scaling instructional innovations (Clark-Wilson et al., 2015).

What is more, the degree to which the three groups in our study attended to infrastructural constraints might have influenced the learning experiences they designed. By prioritizing time constraints, David's group developed an instructional unit that could be implemented in a single class period and a game in which the mathematical content was presented as an add-on to the game, in a format that was already familiar to students and allowed easy and quick implementation. On the other hand, Leon's group, driven primarily by target learning goals, designed an instructional unit that lasted multiple class periods and a game with mathematical content embedded in the gameplay. Their approach to game design reflects some researchers' stance on the design of games for learning to embed mathematical content in game mechanics (Lameras et al., 2017), as well as empirical findings that having fun when playing a game does not necessarily support learning (Iten & Petko, 2016).

Our findings are limited by the small sample size and the context of a single elective course, which may not fully capture the broader challenges faced by teachers in varied educational settings. However, our findings also show that providing student teachers with opportunities to collaboratively design innovative learning experiences offers them space and time to practice balancing

infrastructural constraints with target learning goals. Such opportunities have the potential (a) to foster student teachers' adaptability and resourcefulness in the design of instructional materials, and (b) to result in training experiences that are better connected to the context in which they will be asked to teach. This has the potential to increase their willingness and feelings of preparedness to implement innovative pedagogies in the future.

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Vasiliki Laina: conceptualization, data collection (equal), data analysis (lead), investigation (lead), methodology, project administration (equal), supervision, writing – original draft, writing – review & editing (lead). *Jonas Mayrhofer:* data collection (equal), data analysis (supporting), investigation (supporting), project administration (equal), writing – review & editing (supporting).

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Interpretative knowledge:
The role of collective discussion with in-service teachers

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This paper investigates how in-service primary teachers mobilise Interpretative Knowledge (IK) when on fractions and division within an individual–collective–individual cycle. Findings show that while teachers generally valued error as a learning opportunity, limited content knowledge restricted their ability to interpret students' productions. However, collective discussion seems to support a shift of interpretations. Moreover, no link emerged between years of teaching experience and higher levels of IK. This evidence suggests that IK does not develop with school experience or mathematical training but may be fostered through different kinds of professional experiences, especially those that promote inclusion and openness to students' reasoning.

Keywords: Interpretative knowledge, students' production, in-service teacher education.

Introduction

Interpretative Knowledge (IK) has emerged in the last decade as a central construct in mathematics teacher education. As Di Martino et al. (2019) point out, a crucial challenge for both pre-service and in-service teacher education lies in understanding teachers' IK and, at the same time, finding ways to strengthen it within training programs. These two aspects are intertwined: activities designed to assess IK can become opportunities to enrich both teachers' mathematical understanding and their approaches to students' errors or unexpected strategies. IK refers to the ability to make sense of students' non-standard answers and to turn them into opportunities for learning (Borasi, 1994; Ribeiro et al., 2013). Research shows that even experienced teachers could fail in recognizing the mathematical potential of such productions, assessing them against their own preferred strategies (Jakobsen et al., 2014). IK does not develop spontaneously but requires targeted professional tasks exposing teachers to authentic student solutions, prompting interpretation and feedback design (Mellone et al., 2020; Ribeiro et al., 2013). Peer discussion has also proven crucial, helping teachers shift from evaluative judgments to more interpretative stances (Pacelli et al., 2020).

Previous research has mainly examined prospective teachers (Jakobsen et al., 2014; Pacelli et al., 2020; Ribeiro et al., 2013;) or mathematicians (Ribeiro et al., 2013). Studies with prospective teachers, framed within the Mathematical Knowledge for Teaching model (Ball et al., 2008), show that fragile Common Content Knowledge (CCK) often leads them to overlook mathematically valid non-standard strategies, while research with mathematicians highlights that CCK alone is insufficient and that interpreting students' reasoning requires elements of Specialized Content Knowledge (SCK). However, little is known about how in-service primary teachers mobilize IK in professional tasks. Given their combined experience with CCK, SCK and classroom-based Pedagogical Content Knowledge (PCK), investigating their interpretative work can broaden current understandings of IK and extend the range of educational actors considered in previous research. This study aims to address

this gap by examining how in-service primary teachers mobilize IK in interpretative tasks, thus contributing to a deeper conceptualization of IK in relation to the professional background of those who enact it.

Theoretical framework

The Mathematical Knowledge for Teaching (MKT) construct builds on Shulman's (1986) influential idea of Pedagogical Content Knowledge (PCK). Shulman argued that effective teaching requires more than generic pedagogical skills or advanced disciplinary knowledge; rather, it depends on specific professional knowledge combining content and pedagogy. Expanding on this concept, Ball et al. (2008) proposed a more detailed model that distinguishes between the different types of knowledge that mathematics teachers require in their professional practice. Within Content Knowledge, they differentiate between Common Content Knowledge (CCK), which is the mathematics that any educated adult might use, and Specialized Content Knowledge (SCK). The latter is unique to teaching and points to the professional ways in which teachers must understand, apply and connect mathematical ideas in their daily work. Although research has acknowledged the specificity of SCK, few attempts have been made to describe it in detail, especially in relation to using students' errors and productions as a resource for developing new mathematics. Borasi's (1994) reflection on errors as opportunities for enquiry is a step in this direction, and Interpretative Knowledge (IK) is a construct that has developed from this line of thought.

Early studies with prospective teachers (Jakobsen et al., 2014; Ribeiro et al., 2013) revealed that limited content knowledge often leads to dismissing mathematically valid strategies developed by students. According to Mellone et al. (2020), IK represents a specific kind of teacher knowledge that allows them to interpret, value, and make pedagogical and mathematical use of students' unexpected reasoning. Within the framework of MKT (Ball et al., 2008), IK can be understood as a hybrid form of knowledge situated between CCK and SCK: it requires sufficient mathematical understanding to recognize the validity of non-standard strategies (CCK), and the professional ability to unpack and mathematically interpret students' reasoning (SCK). To perform this interpretative work effectively, teachers require a broad and flexible repertoire of examples, strategies, representations and potential errors. This enables teachers to move beyond their own solution space and understand not only approaches that resemble their own, but also those that differ significantly (Jakobsen et al., 2014). Thus, unlike CCK and SCK, which refer to forms of mathematical knowledge, IK concerns how teachers use that knowledge to interpret and respond to students' productions.

In studies on IK, the authors developed a methodology originally structured in two phases (Ribeiro, et al., 2013): first, teachers were asked to solve a mathematical problem themselves, confronting its specific difficulties and revealing their preferred strategies; second, they were asked to interpret a set of real students' solutions to the same problem and to propose specific feedback aimed at advancing the students' mathematical understanding. In further studies, a third phase was later added: participants engaged in collective discussion about the mathematical aspects emerging from the students' productions, followed by a delayed written reflection in which teachers revisited and reinterpreted the previous individual work (Di Martino et al., 2019). Subsequent research (Mellone et al., 2020; Pacelli et al., 2020) placed particular emphasis on this collective phase, showing that

discussion fostered insights that did not arise during individual work and contributed to shifts in teachers' attitudes towards students' non-standard reasoning.

The study pursues two aims: first, to examine how in-service primary teachers mobilize IK when individually interpreting students' productions; second, to explore whether and how collective discussion supports the emergence or continued use of IK. Given the heterogeneity of the participants, the study also considers, in an exploratory way, whether different professional backgrounds appear to shape the ways in which IK is mobilized.

Methodology

The study

The study followed the individual–collective–individual cycle (Mellone et al., 2020; Pacelli et al., 2020), involving three phases: individual work on two tasks, a collective discussion, and a final individual reflection. Two tasks were selected for the study (Figure 1).

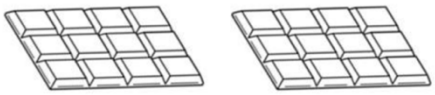
What amount of chocolate would six children get if we divide five bars equally among them?	Mario, Luca, Dario and Stefano are dividing two bars of chocolate equally between themselves.  How many squares of chocolate does each child receive?
Task 1 from Ribeiro et al. (2013, p. 4)	**Task 2 from Grade 2 INVALSI test**

Figure 1: Interpretative Tasks used in the experiment

Task 1 was the "chocolate bars" problem, widely used in IK research. For this task, we selected exactly the same set of student productions used in Ribeiro et al. (2013). This choice allows comparability with previous studies and provides a basis for examining teachers' interpretative moves. Task 2 was adapted from a Grade 2 INVALSI item. It was administered to an additional group of students to gather further examples of children's reasoning.

Data collection and analysis

Data were collected in three phases. In the first individual phase, teachers completed an online questionnaire in which they solved the tasks and interpreted a set of student productions, always justifying their position. In the collective phase, an audio-recorded discussion was facilitated by one of the authors, who invited teachers to explain their questionnaire responses and react to their peers' interpretations. Finally, in the third individual phase, a second questionnaire asked teachers to reflect on the experience and to reassess the same student productions, providing justifications.

Teachers' interpretations from phases 1 and 3 were analysed qualitatively. Each response was classified into three categories—correct, incorrect, and no interpretation—according to the position taken by the teacher toward the student's production. "Correct" indicates that teachers recognised the response as mathematically acceptable; "incorrect" refers to cases in which teachers rejected the response; "no feedback" was used when teachers reported being unable to understand or evaluate the student's reasoning. Coding was conducted collaboratively by the two authors, who discussed each

case until agreement. The collective discussion was examined to identify shifts in interpretive stance, understood as moments in which teachers reconsidered previous judgments or articulated new mathematical elements.

The sample

The study involved ten in-service primary school teachers from central Italy. The sample was heterogeneous in terms of years of service, teaching experience and educational background (Table 1). The group included both classroom and support teachers. In Italy, support teachers are special-education teachers assigned to accompany students with special educational needs across all subjects. They do not teach disciplinary content but work alongside classroom teachers during each lesson (including mathematics) supporting the targeted students and contributing to classroom activities.

Table 1: Teacher sample

Teacher	Years of service (of which in Math)	Education
D1	3 (2)	Master's Degree in Primary Teacher Education (PTE)
D2	8 support teacher (0)	Master's Degree in Psychology; Master's student in PTE
D3	11 (10)	Diploma in PTE; Degree in Arts, Music and Performing Arts
D4	4 support teacher (0)	Master's Degree in PTE
D5	6 support teacher (0)	Master's Degree in Psychology; Master's student in PTE
D6	4 (2)	Master's Degree in PTE
D7	4 (2)	Master's Degree in PTE
D8	6 (4)	Master's Degree in PTE
D9	9 (7)	Diploma in PTE; Master's student in PTE
D10	1 (1)	Diploma in PTE

In the Italian context, participants with a master's degree in Primary Teacher Education (PTE) had completed university courses in mathematics and mathematics education (et least two exams). In Italy, the master's in PTE was introduced only in 1998; before then, primary teachers qualified through a dedicated secondary school Diploma.

Data analysis

The presentation of results follows the structure of the individual–collective–individual cycle: we first examine teachers' initial interpretations (phase 1), then the contributions of the collective discussion (phase 2), and finally the reinterpretations produced after the discussion (phase 3). The analysis focuses on teachers' interpretations of Task 1 as this provides direct comparability with previous IK studies (Jakobsen et al., 2014; Ribeiro et al., 2013). Figure 2 presents the four authentic student productions for Task 1 used in the activity.

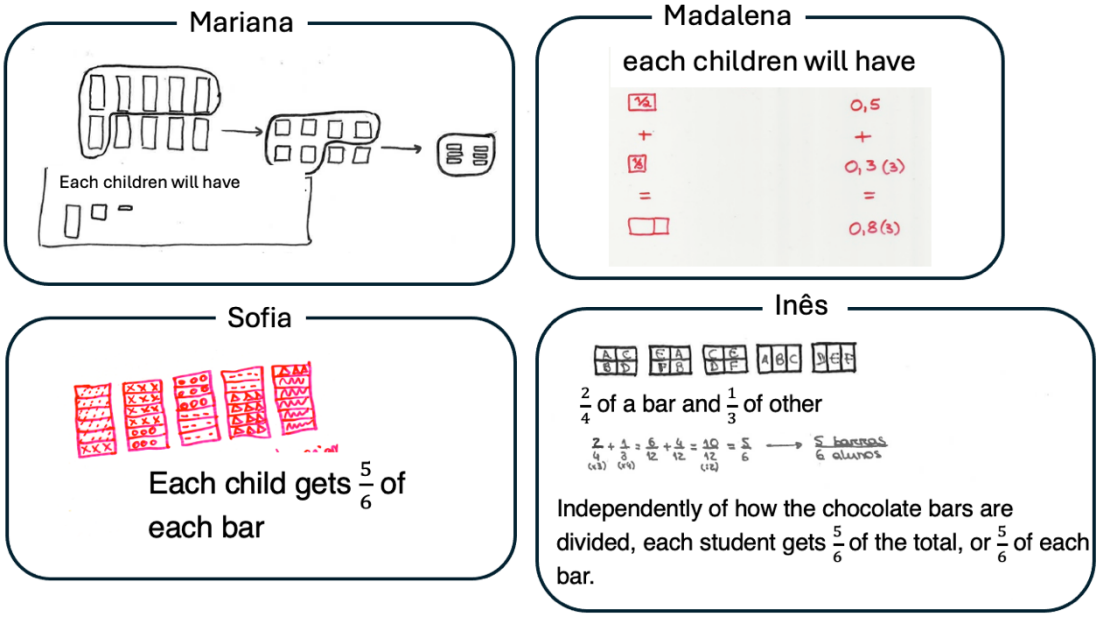


Figure 2: Student productions for Task 1 selected by Ribeiro et al. (2013)

Table 2 summarises, how each teacher positioned the responses in phases 1 and 3 using the three categories described in the methodology (correct, incorrect, no interpretation).

Table 2: Summary of teachers' responses on students' production in the individual phases

Task 1- phase 1				Task 1 – phase 3			
Student	**no feedback**	**Correct**	**Incorrect**	**Student**	**no feedback**	**Correct**	**Incorrect**
Mariana	2	7	1	Mariana	0	9	1
Madalena	2	8	0	Madalena	0	10	0
Sofia	0	7	3	Sofia	0	9	1
Ines	2	4	4	Ines	0	9	1

In phase 1, teachers were first asked to solve the task individually and then to analyse four student productions. Six out of ten teachers expressed the solution as the fraction $\frac{5}{6}$, but only one (D5) provided a complete and unambiguous response, explicitly specifying the whole by stating: 'Each child will get $\frac{5}{6}$ of a chocolate bar.' In the other cases, the fractional notation was imprecise or ambiguous, making it unclear whether $\frac{5}{6}$ referred to a single bar or to the total amount. In three cases (D4, D6, and D10) it was unclear whether $\frac{5}{6}$ referred to a single bar or to the total amount. In two further cases (D1 and D8), although the graphical representation and calculation were correct, the final wording suggested that each child would receive $\frac{5}{6}$ of all the bars. Finally, three teachers (D3, D7, and D9) expressed the solution only graphically, without using arithmetic terms.

In analysing teachers' interpretations of the four student productions in phase 1, a predominantly evaluative rather than interpretative stance emerged. Although most productions were classified as

correct, teachers' written feedback tended to focus on adjusting, correcting or rephrasing the students' answers, rather than reconstructing the underlying reasoning. In this initial phase, no teacher consistently mobilised elements of IK. Only in a few isolated comments did a more interpretive orientation surface. For example, while acknowledging the ambiguity in Sofia's verbal statement, D5 noted that her drawing "correctly shows the equal sharing of the five bars," hinting at an appreciation of the mathematical structure of the student's reasoning.

A brief look at the four productions confirms this pattern. Madalena's solution was almost unanimously accepted as correct, yet the corresponding feedback primarily restated or simplified her representation. Mariana's and Inês' drawings prompted uncertainty in some cases, but even when recognised as correct, teachers rarely elaborated on the value of the students' strategies. Sofia's work was considered correct by the majority, yet feedback often concentrated on reformulating the final expression rather than engaging with the reasoning represented in the drawing.

Overall, phase 1 reveals limited evidence of IK, even when teachers recognised the correctness of the students' productions. To understand whether and how IK may emerge or be supported, it is necessary to examine the collective discussion (phase 2). We focus on Sofia's production not only because it generated the richest debate, but also because it was the only production that all teachers provided feedback, providing a shared basis for comparing how their interpretative stances evolved during the discussion.

Only one teacher (D5) explicitly pointed out the lack of completeness, while D10 considered her answer correct. Teacher D8, although identifying the imprecision, suggested as feedback that "each child will have $\frac{5}{6}$ of the total bars," a reformulation that introduced a new misunderstanding. Two further teachers (D2 and D3) focused on the total number of pieces, proposing corrections such as '5 out of 30.' Despite these interpretive difficulties, most teachers tended to highlight positive aspects in the students' work. Their feedback often focused on adjusting or simplifying the solution, reinforcing what they considered correct, rather than using the student's production as an opportunity to elicit deeper mathematical ideas. For example, D2 commented: "The feedback I would give Sofia would be to acknowledge the correctness of the way she partitioned the bars, while guiding her to reason about the total number of pieces."

The collective discussion about Sofia's production created conditions for IK to emerge, particularly through the interaction between teachers with different initial stances. At the beginning of the discussion, several participants focused on correcting Sofia's final expression or on inconsistencies in her drawing, reiterating the evaluative stance observed in phase 1. D3, for example, argued that "she should correct the fraction, writing that each child receives 5 out of 30 pieces", reinforcing the idea that the solution should match a canonical form. A turning point occurred when D5 highlighted that Sofia's drawing was "quantitatively correct and the only one that could be generalised to other cases". This comment shifted the focus from formal accuracy to the underlying mathematical structure, drawing attention to the coordination of representations and to the role of the whole. D5's interpretation made explicit what had previously remained implicit in her phase-1 feedback, and it prompted several teachers to reconsider their views. D6 acknowledged that the drawing "actually works for any number of bars", and D8 recognised that "$\frac{5}{6}$ of a bar is not the same as $\frac{5}{6}$ of the total".

These exchanges illustrate how IK, when made visible by only one participant, can create opportunities for others to refine or expand their interpretative stance.

In the third phase, the case of Sofia's production illustrates this shift particularly well. Initially, many participants judged her answer as flawed: D2 concluded that "only the first bar shows $\frac{5}{6}$… in reality it should be $\frac{5}{30}$ of the total," while D8 explained that she would "propose my version of the solution and ask Sofia to compare it with hers." These comments reflect a focus on formal correctness and some confusion about the role of the whole. After the collective discussion, however, several teachers reconsidered their positions. D5 emphasized that, despite the misleading verbal expression, Sofia's graphical strategy was "quantitatively correct and potentially generalizable," suggesting it could be used to highlight the distinction between $\frac{5}{6}$ of one bar and $\frac{5}{6}$ of the total. In her final reflection, D8 acknowledged her earlier misinterpretation, noting that "the group helped me realize the difference between $\frac{5}{6}$ of the total and $\frac{5}{6}$ of a single bar," and proposed guiding Sofia to explore it visually.

Overall, the analysis of the three phases suggests that IK does not emerge spontaneously in individual work, even when teachers recognise the correctness of a student's solution. It becomes visible only when interpretative insights are made explicit within the group and taken up by peers. This indicates that collective discussion can support the emergence of IK, although its uptake varies across teachers.

Conclusions

This study set out to examine how in-service primary teachers mobilise IK when interpreting students' productions in fractions and division, and to explore whether and how collective discussion can support the refinement or emergence of interpretative stances. In continuity with research on prospective teachers (Jakobsen et al., 2014; Pacelli et al., 2020; Ribeiro et al., 2013), our findings confirm that limited attention to the role of the whole and to the coordination of representations may constrain teachers' ability to recognise the mathematical validity of non-standard strategies. At the same time, the case of a support teacher (D5), who displayed solid interpretative skills without evident weaknesses in her content understanding, suggests that content knowledge alone does not explain teachers' performance. This result resonates with studies on mathematicians (Ribeiro et al., 2013), which highlight that making sense of students' reasoning requires broader resources, including the ability to identify the structure of a student's argument and the mathematical potential of non-standard representations. By focusing on in-service teachers, our study contributes to the literature by showing that professional background and teaching role may also influence the development of interpretative sensitivity, and that IK does not automatically improve with professional experience. For example, one of the most experienced teachers in the group (seven years of teaching mathematics, D9) failed to recognize a central issue in the interpretation of fractions, while a support teacher (D5) with no direct responsibility for mathematics demonstrated a particularly effective interpretive approach. This pattern suggests that experience do not necessarily correlate with more advanced IK, and that teachers' professional roles may shape different dispositions toward students' reasoning. The collective discussion played a crucial role in supporting teacher IK shift, enabling them to move from evaluative judgments toward more interpretive stances and to develop new mathematical awareness, especially concerning the role of the whole. For example, in discussing Sofia's answer, teachers

moved beyond labeling it as correct or incomplete and began to see it as a rich starting point for classroom work, where comparing "$\frac{5}{6}$ of each bar" with "$\frac{5}{6}$ of the total" could foster reflection on the role of the whole in fractions. Through this process, interpretative insights expressed by one participant (D5) created opportunities for others to refine their own interpretations, suggesting that IK can become visible when made explicit within a collaborative setting.

These results, while promising, are constrained by the small and heterogeneous sample and cannot be generalised to broader populations or classroom practice. Nonetheless, they point to relevant directions for future research. In particular, it would be important to examine how IK developed through interpretative tasks translates into teachers' actual classroom practices, and how students' learning may be affected. Further investigations should also explore the influence of teachers' professional backgrounds, as the case of support teachers in our study suggests that interpretative sensitivity may arise from trajectories outside formal responsibility for mathematics teaching.

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Developing classroom observation competency in pre-service teachers: A study of a teacher education program

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This study examines how pre-service teachers develop their observation tasks over time with a didactic observation system. Initially focused on the classroom environment and general teaching, they gradually shifted toward recognising the mathematical and didactic structure of lessons. This reflects growing awareness of teaching as an integrated system involving teacher actions, student engagement, and mathematical content. While observation quantity did not change significantly, their qualitative understanding of the role of mathematical knowledge in lesson structure deepened.

Keywords: Lesson study, anthropological theory of the didactic, observation praxeology, didactic observation system, professional development.

Introduction: The role and challenges of observation in lesson study

Lesson study is a collaborative practice in which teachers plan, observe, and reflect on a lesson to improve instruction (Miyakawa & Winsløw, 2013; Minisola et al., 2024). It emphasizes both pedagogical refinement and a deeper understanding of student learning. In mathematics education, it has enhanced teachers' didactic knowledge and fostered shared insights into effective teaching. Since the 1990s, lesson study has gained international recognition, particularly through its connection to Japan's structured problem-solving lesson style (Stigler & Hiebert, 1999). Research shows that lesson study enhances teacher learning, reflection, and student outcomes by fostering professional learning communities that attend closely to students' thinking (Lewis & Hurd, 2011). Within this cycle, lesson observation is not merely a procedural step but the core activity through which the learning potential of lesson study is realized, as teachers jointly investigate how and why classroom events unfold.

Prior studies have examined how observers interpret lessons. For instance, Shimizu and Kang (2022) showed that post-lesson discussions increasingly focus teachers' attention on students' thinking and mathematical themes, while Færøyvik Karlsen (2019) found that teachers' interpretations of classroom observations influence subsequent lesson planning. Thus, the quality of lesson study critically depends on the depth and orientation of the observation conducted during the research lesson. However, little has been said about how lesson study as a whole relates to didactic or paradidactic gestures (Miyakawa & Winsløw, 2013). Understanding lesson observation as a collaborative process can illuminate the knowledge and praxeologies that observers bring into, develop through, and transform during the phases of planning, teaching, and discussion (Shinno & Yanagimoto, 2020).

In Japan, teachers' observations reflect their training and experience: novices focus on surface classroom features, while professional observation attends to both what occurs and what does not. Although some studies suggest strategies for teacher development, observation itself is rarely treated theoretically. Our research project, using the Anthropological Theory of the Didactic (ATD; Chevallard, 2019), explores how observation is shaped by institutional and praxeological factors.

Tomosada and Shinno (2025) used ATD to analyse observers' notes from a primary mathematics lesson, showing that techniques varied by didactic moment and observer experience, highlighting the need to clarify observers' praxeology. Building on this, Mizoguchi et al. (2025) surveyed pre- and in-service teachers, finding no major group differences or reference models, and proposed a hypothetical model. However, neither study examined the link between the teacher's and observers' praxeology. To address this, Tomosada et al. (submitted) analysed lesson study sessions where novice teachers observed a veteran, finding that observers focused on pedagogical aspects but largely ignored mathematical praxeologies.

These findings show the difficulty of identifying observers' praxeological equipment and highlight the need for a theory-based didactic observation system. Both pre- and in-service primary teachers often overlook mathematical knowledge in practice. This study addresses these issues by developing such a system, as described next.

Theoretical framework: A didactic observation system

This chapter provides a concise summary of the key findings from our submitted manuscript (Mizoguchi et al., submitted). To develop a didactic observation system, we outline its essential elements in ATD terms—namely, institution, positional instance, praxeology, and didactic system—and describe their respective roles, resources, and observation gestures.

ATD is a theoretical framework that describes educational activity through institutions, the positions within them, and the praxeologies these positions mobilize. It clarifies how different roles in lesson study—designers, implementers, and observers—shape the knowledge they use and the ways they conduct observation. In this study, lesson observation is understood as an "observational praxeology," involving what observers attend to, how they interpret it, and the criteria guiding their judgements. ATD distinguishes between the knowledge (logos) and *how-to* (praxis) underlying these judgements, allowing systematic analysis of which lesson aspects observers focus on and how they evaluate them. Moreover, lesson study can be viewed as an institutional process with phases such as design, implementation, and post-lesson discussion. ATD models this multilayered structure as a configuration of institutions, providing a coherent way to situate lesson observation within the broader lesson study process and trace how knowledge circulates across its phases.

Institutions

In a lesson study, various institutions are involved, and the participants can be defined as individuals acting in their respective roles. First, we define the institution of lesson study as I_{LS}. The I_{LS} is composed of the following sub-institutions: I_{LD} (Lesson Design), I_{LI+O} (Lesson Implementation and Observation), and I_{PD} (Post-Discussion). This relationship can be expressed as: $I_{LS} = I_{LD} \cup I_{LI+O} \cup I_{PD}$.

In the I_{LD}, a lesson plan is developed by a lesson design team (J) or by the individual teacher (j) (or positional instance $\hat{\jmath}$) responsible for implementing the research or open lesson. In the I_{LI+O}, the teacher ($\hat{\jmath}$) implements the lesson, generally following the lesson plan. This teaching activity is described as the management of a didactic system, denoted as $S(X; \hat{\jmath}; o)$. Simultaneously, a group of observers V of individual observers v (or positional instances $\hat{v}$) selectively observes $S(X; \hat{\jmath}; o)$ with

reference to the lesson plan. In the I_{PD}, the discussion is carried out by the design team (J) or the teacher (j) and the observers (V). The discussion may also include a knowledgeable other ($\hat{k}$), such as a university professor or a regional education board supervisor, in addition to the regular observers. It is worth noting that this knowledgeable other ($\hat{k}$) can also be a member of the observer group (V).

For the rest of our discussion, we'll focus primarily on the I_{LI+O}.

Didactic gesture of the teacher ($\hat{j}$) in the I_{LI+O}

The teacher ($\hat{j}$) generally implements the lesson according to the lesson plan. This means that the teacher follows the flow of the designed lesson to achieve its objectives. The lesson plan usually includes time allocations for each activity, and the teacher makes an effort to adhere to these timings. However, the lesson plan may be modified and implemented during the lesson if it is determined that the lesson is not proceeding as planned. Experienced teachers tend to be more flexible in handling such situations, whereas inexperienced teachers may find this more difficult. These gestures of the teacher ($\delta_{\hat{j}}$) are considered a manifestation of their didactic praxeology or praxeological equipment.

Didactic gesture of the observer ($\hat{v}$) in the I_{LI+O}

The observer ($\hat{v}$) typically reads the lesson plan beforehand to understand the teacher's ($\hat{j}$) intentions and the lesson's flow. Sometimes, common observation viewpoints are given to the group of observers (V). In both cases, the observer records the gestures of the teacher and the learners (X) based on these viewpoints and their own questions. A key point of observation is whether the lesson achieved its purpose. If it did, the observer judges what was 'excellent' and how it was achieved. Conversely, if the purpose was not achieved, the observer evaluates the reasons. These judgments are then brought into the subsequent I_{PD} discussion. These gestures of the observer ($\delta_{\hat{v}}$) are presumed to be based on a praxeology distinct from that of the teacher ($\hat{j}$). However, just as the observer judges the teacher's gestures ($\delta_{\hat{j}}$), the observer's own gestures ($\delta_{\hat{v}}$) are in turn judged by a detached entity (the reference instance $\widehat{w}$, often from the researcher's perspective), as observers do not typically reflect on their own gestures by themselves.

Components of the didactic observation system ($\mathcal{S}$)

The subject of observation is the observer group V (or $\hat{v}$) within the I_{LI+O} phase, while the object of observation is a lesson. In a typical lesson study, the lesson plan is provided by the teacher or the lesson design team. Consequently, V (or $\hat{v}$) observes the lesson by using the lesson plan as a guide. A lesson can be represented as a didactic system $S(X; \hat{j}; o)$ over a limited period (e.g., 45 to 120 minutes), where o is a didactic stake, i.e. a content or topic described in praxeological terms. Based on previous project initiatives and the arguments above, the observed lesson is not always the didactic system as a whole. Instead, what is observed depends on the individual observer. Therefore, it is logical to state that V (or $\hat{v}$) selectively observes the didactic system $S(X; \hat{j}; o)$ with reference to the lesson plan (LP). This leads to the conclusion that the didactic observation system $\mathcal{S}$ is composed of three components: the observers V, the lesson plan LP, and the selected elements of the didactic system S. The didactic observation system can thus be notated as: $\mathcal{S}\left(V, LP, S^{\dagger}(X; \hat{j}; o)\right)$. Here, the '†' symbol on the right shoulder of S indicates the selected element(s) from the didactic system.

Observational tasks and stakes

Observers (V) always have a specific task ($\mathcal{T}$) that shapes their observation. Although common 'viewpoints' are often provided in a lesson study, individual observers' tasks and thus their observational stake can vary. This task ($\mathcal{T}$) is crucial for reconstructing the observed content or topic (o). Therefore, the observer group (V) can be denoted as $V(\mathcal{T})$, and the didactic observation system is more accurately represented as: $\mathcal{S}\left(V(\mathcal{T}), LP, S^{\dagger}(X; \hat{\jmath}; o)\right)$.

Reconstructing tasks ($\mathcal{T}$) in the didactic observation system ($\mathcal{S}$)

The $\mathcal{T}$s are derived from the S by formally enumerating its subsets, as shown in Table 1.

Table 1: Observation task types ($\mathcal{T}$) based on selected elements of ($\mathcal{S}$)

$\mathcal{T}_{\{X\}}$: What gestures does X use? / What is the function of X's gestures?
$\mathcal{T}_{\{X,o\}}$: What is X's gesture toward o?
$\mathcal{T}_{\{\hat{\jmath}\}}$: What pedagogical gestures does $\hat{\jmath}$ use? / What is the function of $\hat{\jmath}$'s pedagogical gestures?
$\mathcal{T}_{\{\hat{\jmath},o\}}$: How does $\hat{\jmath}$ use didactic gestures to address o?
$\mathcal{T}_{\{X,\hat{\jmath}\}}$: What kind of pedagogical assistance would $\hat{\jmath}$ provide to X at a specific moment?
$\mathcal{T}_{\{o\}}$: How does o manifest in a specific didactic moment? / What is the nature of o in a specific didactic moment?
$\mathcal{T}_{\{X,\hat{\jmath},o\}}$: How would $\hat{\jmath}$ assist X with o at a specific didactic moment?
$\mathcal{T}_{S(X;\hat{\jmath};o)}$: How would $\hat{\jmath}$ assist X's learning of o throughout the lesson, considering the mathematical and didactic organization?

The distinction between $\mathcal{T}_{\{X,\hat{\jmath},o\}}$ and $\mathcal{T}_{S(X;\hat{\jmath};o)}$ presented in Table 1 is essential for understanding the scope of observational tasks. $\mathcal{T}_{\{X,\hat{\jmath},o\}}$ focuses on the interaction of learner (X), teacher ($\hat{\jmath}$), and content (o) at a specific moment, capturing a snapshot of the lesson's dynamics. In contrast, $\mathcal{T}_{S(X;\hat{\jmath};o)}$ views the lesson as an integrated system, examining these interactions across multiple moments to reveal overall patterns. Thus, analysing an observer's praxeological equipment begins by identifying the specific tasks ($\mathcal{T}$) within the didactic observation system ($\mathcal{S}$).

Research question

This study addresses the following research question, situated within the contextual framework to be described in the following chapter: *How do the observational tasks of pre-service teachers evolve through successive lesson observations, leading to an awareness of the didactic and mathematical organisation embedded in the lessons?*

Research context

Conducted as a component of a year-long course at the authors' affiliated university, this study centered on a four-month (first semester) program. The program was specifically designed to enhance students' lesson observation competencies through weekly 90-minute classes. The participants were four second-year university students (two males, two females) with no prior teaching practice experience. The program was structured into five distinct phases over a total of 15 classes (excluding orientation, etc.), with each session designed to be completed within one to two weeks. Each session

combined actual and video-recorded lesson observations with a subsequent discussion. Table 2 shows a concise overview of the five sessions.

Table 2: Overview of the five sessions (L/V indicates Live lesson and Video lesson)

Session	Dates (2025)	Grade & Topic	L/V	Observation Focus and Student Findings	Program Guidance/Emphasis
1	May 9 & 16	4th Grade – "Area"	V	PTs focused on classroom rules, environment, and teacher techniques, beginning to see blackboard writing as representing the learning trajectory.	The recognition of the learning trajectory was the key outcome.
2	May 23	6th Grade – "Systematic Listing (Combinations)"	V	PTs recognized questioning's role in eliciting explanation, though they focused on isolated events.	Teachers emphasized connecting observations to broader instructional goals.
3	May 30 & June 6	6th Grade – "Reciprocals"	L	PTs noted unclear goals, uneven learner understanding, and tasks lacking mathematical depth.	Teachers emphasized purposeful task design and supporting student thinking over checking correctness.
4	June 13 & 20	9th Grade – "Central Angles and Circumscribed Circles"	V	PTs initially observed as learners, noting instructional intentions (generalization and justification) and the didactic role of tools.	Guided viewing was used to help identify instructional intentions.
5	July 4 & 13	9th Grade – Babylonian Method (SRP)	L	Group-based inquiry made focusing difficult, leading to fewer notes, with limited content knowledge further constraining observation.	Brief introduction to ATD and SRP concepts, but observation-related aspects were the focus.

Data collection

Data were collected from all class meetings through field notes by researchers, written notes from students, and audio recordings of discussions, allowing a detailed analysis of the students' evolving observational competencies.

Analytical procedures and result

The tasks held by the student participants were not explicitly stated by the students themselves. Instead, the authors inferred these tasks by identifying them through multiple sources, including the students' observation notes and comments made during the sessions. In other words, these tasks reflect the students' implicit concerns or focal points, which the authors identified and interpreted based on various data collected throughout the study.

The criteria for analysing students' observation were established corresponding to the observation task types shown in Table 1. Specifically, we identified that a student demonstrated the relevant observation task if the items listed in Table 3 were included in their observation notes or statements made during the discussion.

Table 3: Criteria for identifying pre-service teachers' observations

$\mathcal{T}_{\{X\}}$	*learning motivation, student interaction, the learning environment (e.g., posted materials/displays), and students' expressions*

$\mathcal{T}_{\{X,o\}}$	*student responses and actions to questions, specific methods of solution and representation, and the focus and activities of group discussions*
$\mathcal{T}_{\{\hat{\jmath}\}}$	*bansho or boardwork (layout, color, etc.), the use of ICT and prepared materials (e.g., handouts), lesson flow/structure (problem-solving, learning rules), time management (time allocation), the teacher's positioning and movement in a classroom, teacher's instructions, teaching techniques (voice volume, positioning, and monitoring during individual work), and pre-service teachers' impressions/feedback on the instruction*
$\mathcal{T}_{\{\hat{\jmath},o\}}$	*lesson objectives and the problem/task, boardwork (writing order, keywords), individual assistance, the structure of whole-class discussion (order of calling on students, selection of solutions/methods to be taken up), questions regarding the approach/style of instruction, summary and assessment, application problems/tasks, and pre-service teachers' impressions of the lesson content and alternative instructional proposals*
$\mathcal{T}_{\{X,\hat{\jmath}\}}$	*specific individual and whole-class assistances*
$\mathcal{T}_{\{o\}}$	*analysis of targeted mathematical topic and unit planning, connections to other mathematical knowledge, mathematical background and validity, and questions regarding the mathematical content, etc.*
$\mathcal{T}_{\{X,\hat{\jmath},o\}}$	*concrete individual assistance provided in a specific didactic moment, and an analysis comparing it to the original lesson plan (e.g., modifications)*
$\mathcal{T}_{S(X;\hat{\jmath};o)}$	*observation seeking to comprehend the entire lesson through the connection of individual didactic moments across the whole flow*

Figure 1 presents a summary of these inferred tasks. The numbers within the cells indicate how many times each student recorded notes or made comments related to the corresponding observation task. It should be noted that Student A was absent for Session 3, while Student C missed Sessions 4 and 5.

PT	A					B					C					D				
Session	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
$\mathcal{T}_{\{X\}}$	5					1		1			2		1			1				
$\mathcal{T}_{\{\hat{\jmath}\}}$	9	1		1	1	5			1		6		4			4		1	2	
$\mathcal{T}_{\{X,\hat{\jmath}\}}$	2			2		1					1					6		1		2
$\mathcal{T}_{\{o\}}$				1	1				1										1	
$\mathcal{T}_{\{X,o\}}$	1	1		1			4	2	1	3		2	2				4	1	2	4
$\mathcal{T}_{\{\hat{\jmath},o\}}$	3	4		9	3	4	6	5	3	5	3		2				5	5	2	3
$\mathcal{T}_{\{X,\hat{\jmath},o\}}$																				
$\mathcal{T}_{S(X;\hat{\jmath};o)}$																				

Figure 1: Observation tasks identified for students in each session

Discussion and conclusion

Evolving observation focus in initial sessions

In Session 1, student observation notes primarily focused on learner-related aspects ($\{X\}$) (e.g., classroom norms and environment), often reflecting comparisons to their own prior experience as learners. Comments concerning instructional techniques ($\{\hat{\jmath}\}$) were limited to general classroom practices, lacking specificity to mathematical content. From Session 2 onward, notes contained fewer general comments ($\{\hat{\jmath}\}$) and exhibited a more focused approach, particularly on *bansho* (板書) or boardwork and teacher questioning ($\{\hat{\jmath}, o\}$).

This initial pattern indicates that student observers predominantly adopted the learner's perspective rather than the teacher's one. The subsequent shift observed from Session 2 suggests a developing

understanding that observing specific teacher actions (questioning, board use) can reveal the intended learning goals, reflecting a growing awareness of the lesson's didactic structure.

Developing an awareness of didactic structure and mathematical knowledge

Student observers were guided to consider not only the teacher's techniques and intentions but also how mathematical knowledge is embedded in the lesson. They gradually incorporated these elements into their notes, showing increasing attention to how mathematical knowledge structures instruction. By Sessions 4–5, they compared lessons with their own knowledge. While quantitative changes were small, notes indicated growing recognition that effective teaching involves teacher analysis, learner responses, and underlying mathematical knowledge. Initially learner-focused, notes later reflected attention to how teachers facilitate understanding and integrate mathematical knowledge. However, integrated observation tasks addressing learners, teacher techniques, and lesson structure simultaneously (e.g., $\mathcal{T}_{\{X,\hat{\jmath},o\}}$ and $\mathcal{T}_{S(X;\hat{\jmath};o)}$) were not recorded and mentioned.

The initial guidance provided to the observers mirrors findings reported in García et al. (2024), where pre-service teachers working within an ATD framework developed awareness of how didactic praxeological equipment—particularly content knowledge—shapes lesson structure. Their gradually changing notes suggest a similar developmental trajectory: observers came to recognise that mathematical knowledge is not merely part of instructional materials but a principal organiser of the lesson's didactic coherence. The changes observed from Sessions 1 to 5 indicate a shift in perspective: from an early focus grounded in the observers' own experiences as learners to a growing awareness of the teacher's role in orchestrating student understanding through the meaningful use of mathematical knowledge. This development also resonates with findings by Ding et al. (2023), who note that teachers often overlook content knowledge in their reflections, reinforcing the importance of sustained attention to mathematical substance during observation. The absence of integrated observation tasks involving the simultaneous consideration of learners, teacher actions, and the mathematical organisation of the lesson suggests that, although progress was evident, the observers had not yet developed a fully holistic analytical stance. This points to an area requiring further instructional support to foster more comprehensive didactic observation skills.

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Bridging gender stereotypes and learning outcomes: A new perspective on teacher identity

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This paper presents a theoretical contribution on the relationship between teachers' gender stereotypes and students' mathematical learning, offering a new interpretation of teachers' professional identity within this process. Previous studies have established how gender-stereotypical expectations shape students' self-concept, motivation, and achievement. Building on Radford's notion of subjectification, this work extends the discussion by suggesting that stereotypes operate both as elements influencing students' mathematical learning and as tools through which teachers construct their professional identity, thereby intertwining these two processes. Illustrative examples from teachers' accounts indicate how such stereotypes can orient classroom interactions while simultaneously contributing to identity development.

Keywords: Teacher identity, gender stereotypes, subjectification.

Introduction

Although progress has been made toward achieving gender equality, disparities in STEM (Science, Technology, Engineering, and Mathematics) fields remain pronounced, with males continuing to be overrepresented in both higher education and professional sectors (OECD, 2023). These differences are rooted in historical patterns and persist due to the interaction of internal and external factors (Leder & Forgasz, 2008). Internal influences include students' self-perceptions and emotional responses, such as low mathematical self-concept and anxiety, while external factors encompass societal stereotypes, classroom environments, and institutional policies (Pajares, 2005). Mathematics, as a key gateway to STEM pathways, plays a central role in shaping these opportunities, making equitable access to mathematical learning particularly crucial. Research in mathematics education highlights how cultural stereotypes and educational practices can reinforce gendered differences in participation and achievement (Ferretti & Giberti, 2021). Addressing these disparities is not only a matter of educational policy but also a component of broader social justice and sustainable development goals (UNESCO, 2015, unesdoc.unesco.org/library/). Gender-based assumptions tend to perpetuate systemic biases, yet inclusive approaches, equitable institutional frameworks, and shifts in social norms can challenge these patterns and foster cultural transformation (Deutsch, 2007).

In educational practice, teachers' beliefs and expectations play a central role in mediating these structural and cultural influences. Research has shown that gender-stereotypical assumptions held by teachers can shape students' engagement, motivation, and performance in mathematics (Dai et al., 2022; Robinson et al., 2011). These stereotypes operate not only as external pressures on students but also as a lens through which teachers interpret and interact with their classrooms. Emerging discussions in mathematics education research emphasize that teaching practices, including those mediated by digital tools as observed during distance learning, interact with teachers' professional identity, shaping the ways in which they enact stereotypes and respond to different student needs (Ferretti et al., 2020).

From a theoretical standpoint, the notion of subjectification provides a useful lens to understand this process. Teachers' gendered stereotypes are not merely transmitted to students; they also constitute a mechanism through which teachers develop and negotiate their professional selves (Radford, 2023). In this view, professional identity can be considered an intermediate dimension linking teachers' stereotypes and students' learning outcomes, offering an innovative. Illustrative statements from teachers, such as the recurring characterization of female students as diligent and male students as naturally gifted, serve as examples of how these stereotypes simultaneously shape classroom interactions and identity construction, even when systematic empirical data are not available. In this view, professional identity can be considered an intermediate dimension linking teachers' stereotypes and students' learning outcomes, offering an innovative lens through which to understand the reciprocal influence between teacher beliefs and student achievement. Positioning teacher identity as a critical perspective in this analysis opens new pathways for research and reflection in mathematics teacher education.

Theoretical framework

Teacher gender's stereotypes

Gender stereotypes have long influenced social roles and career aspirations, particularly in scientific fields, influencing both student experiences and teacher practices (Nosek et al., 2007). Gender stereotypes refer to *structured sets of beliefs about the personal attributes of men and women* (Dai et al., 2022, p.179), which affect expectations, interactions, and didactical decisions. In mathematics classrooms, such stereotypes often manifest as misconceptions, for example that male students possess greater innate ability, whereas female students are perceived as diligent but less naturally talented (Dersch et al., 2022). These stereotypes impact the ways teachers interact with students, the types of tasks they assign, the feedback they provide, and the attention they give, ultimately influencing students' self-concept, motivation, and performance (Eccles & Wigfield, 2002). Stereotypes are culturally and socially mediated, operating both consciously and unconsciously, and are embedded within curricular, institutional, and societal structures (Ferretti & Giberti, 2021). In this regard the stereotypes can illustrate how gendered stereotypes may shape classroom dynamics and learning opportunities, potentially contributing to the persistence of the mathematics gender gap (Robinson et al., 2011).

Teacher identity

Traditionally, research has conceptualized teacher identity primarily as a set of beliefs. Both national and international literature widely acknowledges that such beliefs influence classroom practices and teacher behaviours. In other words, beliefs are no longer regarded merely as explanatory principles for teacher practice; instead, it is essential to consider the complex interplay of factors that define the teacher's social context and shape their actions and decision-making (Ferretti et al., 2020). The notion of a thinking system, as introduced by Pepin and Roesken-Winter (2015), assumes that beliefs are best understood in relation to other constructs, such as attitudes and affective dimensions, where the various components are inseparable and embedded within specific social and group contexts. To account for the development of teachers' stereotypes in shaping their identity and their interactions with the educational context, this paper proposes framing teacher identity and stereotypes within the

Theory of Objectivation, particularly through the lens of subjectivation processes (Radford, 2023). From this perspective, stereotypes are not treated as isolated elements that contribute to an internal, subjective dimension independent of the historical and cultural context. Rather, the stereotype-identity pair is understood as co-constructed dialectically through practices in which teachers and students actively participate, situated within a historical-cultural context defined by its symbolic superstructures (D'Amore, 2017). Pre-service teachers develop professional identity as they navigate challenges, reflect on instructional decisions, and reconcile institutional expectations with personal beliefs (Ferretti & Santi, 2021). In this regard teacher identity integrates cognitive, emotional, and social dimensions, influencing instructional choices, classroom management, and responsiveness to students' needs. Teachers' professional identity influences decisions regarding task allocation, feedback, and classroom interactions, reflecting their internalized sense of professional role and guiding their instructional practices. Importantly, teacher identity interacts with stereotypes: the enactment of gendered beliefs not only affects students but also shapes teachers' sense of efficacy, professional self-concept, and role within the classroom (Ferretti et al, 2020). In this sense, teacher identity can be seen as both an outcome of experience and a framework for didactical practice.

Subjectification

Subjectification, as theorized by Radford (2023), provides a theoretical framework for examining how teacher identity is continuously constructed. Subjectification refers to the process through which individuals constitute themselves as subjects within social and cultural practices. In the educational context, teachers' beliefs, stereotypes, and classroom interactions are not merely directed toward students; they also contribute to the ongoing development of the teacher's professional self (Ferretti et al., 2020). The classroom thus functions as a site of mutual constitution: student engagement and outcomes are shaped by teachers' assumptions and actions, which in turn inform teachers' self-understanding and professional subjectivity. By framing teacher identity through the lens of subjectification, it becomes possible to understand how professional development, reflection, and social engagement contribute to the formation of the teacher-self, while simultaneously shaping the learning environment for students.

Integrating teachers' gender stereotypes, teacher identity, and subjectification provides a comprehensive framework for analyzing an aspect of the complex dynamics of students learning. Gendered stereotypes function as tools that influence classroom interactions and simultaneously contribute to the ongoing construction of teachers' professional identity (Dersch et al., 2022). Teacher identity can thus be conceptualized as an intermediate dimension linking stereotypes to student learning outcomes, offering a perspective on the mechanisms through which inequalities are reproduced or mitigated. For instance, a teacher who interprets male students as inherently talented and female students as diligent participates in subjectification processes where these stereotypes shape instructional choices and feedback, while also reinforcing the teacher's self-concept and professional positioning. Recognizing this interplay has direct implications for professional development, suggesting that interventions aimed at reducing gender disparities in mathematics should engage teachers in critically examining their own stereotypes and the effects these have on both students and their professional identity (Huang & Wang, 2021). This framework emphasizes the

importance of understanding teaching as a socially situated, identity-mediated practice, rather than merely a set of techniques or content delivery methods.

Study approach

Given the theoretical nature of this study, the methodological approach is based on a critical review and synthesis of the existing literature in mathematics education, with a particular focus on teacher identity, subjectification, and gender stereotypes. The reflection is guided by the aim of understanding how teachers' gendered beliefs can be interpreted not only as shaping students' mathematical learning but also as tools through which teachers construct their professional identity, and by considering how Radford's notion of subjectification offers a lens to connect stereotypes, teacher identity, and student outcomes. From this perspective, the analysis also seeks to explore the implications of such a framework for teacher education and the promotion of gender equity in mathematics classrooms. The primary sources for this analysis include empirical and theoretical studies, covering the period 2000-2025, allowing for the inclusion of both foundational and recent literature, that investigate the effects of teachers' gendered stereotypes on student learning outcomes (e.g. Dai et al., 2022; Robinson et al., 2011), research on teacher identity (Ferretti et al., 2020; Radford, 2023), and studies addressing math-gender misconceptions and their implications for teaching (Dersch et al., 2022). The selection criteria for the literature were based on three dimensions: relevance to mathematics education, direct investigation or conceptual discussion of teacher stereotypes and identity, and explicit attention to gender-related issues in teaching and learning. The selected studies were then coded according to these three dimensions, identifying the specific ways in which each contribution engaged with and developed each of them. The coding followed an iterative, interpretive process in which the studies were examined for theoretical constructs, methodological features, and empirical or conceptual claims aligned with these dimensions. Studies were selected not only based on this alignment, but also for the depth and clarity with which they theorized or explored the connections among stereotypes, identity development and students' learning outcomes. Both empirical studies, including those reporting quantitative and qualitative findings, and theoretical contributions, such as conceptual frameworks and reflective analyses, were considered. In addition, secondary references cited in these papers were examined to identify further insights and to ensure a comprehensive understanding of the mechanisms linking teacher stereotypes, identity development, and student learning. To illustrate the theoretical arguments, indicative examples of teacher perceptions were included, such as views that high-achieving female students are diligent, whereas high-achieving male students are perceived as naturally talented. These examples are drawn from responses from 753 teachers collected via a Mentimeter survey; however, they do not constitute a statistically representative sample. They are used illustratively to highlight patterns consistent with findings reported in the literature and to make the discussion more concrete. The aim is to show how teacher stereotypes operate simultaneously on classroom interactions and the construction of professional identity. The analysis proceeded through a process of thematic synthesis. First, relevant literature was coded according to the main constructs of interest: gender stereotypes, teacher identity, and subjectification. Second, connections between these constructs were identified, highlighting how stereotypes influence teacher self-conception and, in turn, student learning experiences. Third, illustrative examples from the literature were integrated to support the theoretical argument and to provide a bridge between conceptual discussion and

observed classroom phenomena. This approach allows for the development of a coherent theoretical framework that situates teacher identity as a mediating mechanism linking gendered stereotypes to student outcomes.

Discussion

Previous literature has demonstrated the consequences of stereotypes on students' self-concept and performance (Dai et al., 2022; Robinson et al., 2011); however, what emerges is that stereotypes simultaneously operate on teachers themselves, shaping how they perceive their role and define their identity. Radford's notion of subjectification (2023) proves particularly valuable for this interpretation. Subjectification highlights the ways in which individuals become subjects through historically and culturally mediated practices. Subjectification, in this view, positions teachers as subjects who continuously construct their identity through the tools of stereotypes, entangling their professional self with the categories they apply to students. For instance, when teachers attribute mathematical success differently to male and female students, they are not merely reproducing social stereotypes, but they are engaging in practices that define what it means to be a mathematics teacher within those cultural discourses. The illustrative data collected from 753 in-service teachers reinforce this theoretical claim. The contrast in descriptors assigned to male (Figure 1) and female (Figure 2) students, brilliance and determination for boys, diligence and self-discipline for girls, echoes the misconceptions about mathematical abilities identified by Dersch et al. (2022). Indeed, the three misconceptions identified can be summarized as the stereotypes that boys are naturally more talented in mathematics, girls succeed primarily through greater effort, and that such effort does not always result in equivalent achievement, highlighting how gender stereotypes shape teachers' perceptions and expectations. These patterns correspond to the analytical dimensions identified in the literature review, namely teachers' gendered beliefs, their implications for students' learning outcomes, and their interaction with teacher professional identity.

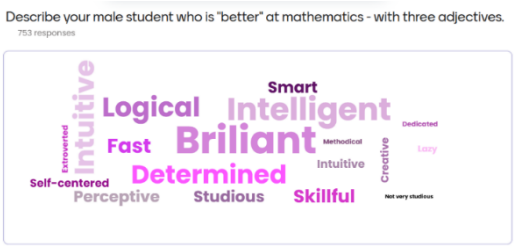

Figure 1: Main results on male students from a Mentimeter survey of 753 in-service teachers, translated by the authors

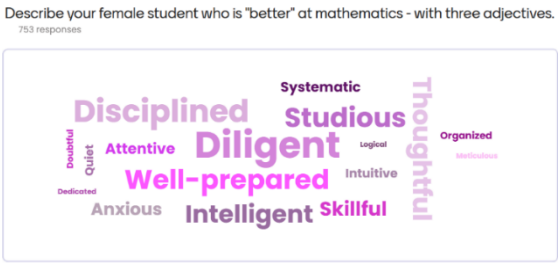

Figure 2: Main results on female students form a Mentimeter survey of 753 in-service teachers, translated by the authors

The results exemplify how stereotypes are not only beliefs about students' capacities but also tools through which teachers enact their own positioning: the teacher who recognizes brilliance defines themselves as the one capable of discerning innate talent, while the teacher who values diligence defines themselves as a guardian of effort and perseverance. Identity construction and stereotype enactment thus appear as two inseparable processes. This integration of teacher identity into the stereotype-learning nexus represents the theoretical contribution advanced here. By foregrounding identity and subjectification, it becomes possible to understand stereotypes not as unidirectional forces acting solely upon students, but as discursive practices that simultaneously shape students' opportunities to learn and teachers' becoming as professionals. Teacher identity, in this view, emerges as the hinge through which stereotypes circulate, connecting the cultural and social dimension of mathematics education (Huang & Wang, 2021) with the professional and didactical dimension (Ferretti et al., 2020). Indeed, this connection is further illuminated by the perspective of Ferretti and colleagues, who demonstrate how teachers' views, framed through stereotypical assumptions about distance mathematics education, contribute to the construction of teacher identity through processes of subjectification. Their study revealed that resistance to distance education positioned some teachers as custodians of traditional didactical practices, waiting for a return to what they perceived as more reliable and authentic conditions for teaching mathematics. By contrast, teachers who engaged more positively with digital technologies constructed themselves as adaptive professionals, capable of reshaping their practices and roles in response to an unprecedented educational crisis. In both cases, stereotypes are not merely private beliefs or pragmatic preferences. Rather, they acted as cultural tools through which teachers defined their own professional selves and positioned themselves within the mathematics classroom. This theoretical insight resonates strongly with the case of gender stereotypes. Just as attitudes toward distance education shaped teachers' subjectivities during the pandemic, so too do gendered assumptions about students' mathematical abilities participate in the ongoing constitution of teacher identity (Huang & Wang, 2021). To describe male students as "brilliant" or "logical" and female students as "diligent" or "studious" is not a neutral classification. It simultaneously positions the teacher as the authority who recognizes and legitimizes such distinctions, thereby reinforcing particular professional roles. In this way, stereotypes, whether related to teaching modalities or to students' abilities, serve as powerful cultural mediators that influence not only students' learning opportunities but also teachers' processes of subjectification and the formation of their professional identity. These results exemplify how stereotypes operate simultaneously on students and teachers, making visible the constitutive link between cultural misconceptions, classroom practices, and professional identity. This interpretation directly reflects the thematic synthesis conducted in the review, which revealed convergences across empirical and theoretical studies concerning the dual impact of stereotypes on both students and teachers, a perspective that holds significant implications for mathematics teacher education. Efforts to challenge gender stereotypes should not be limited to raising awareness of their impact on students' achievement. They must also address the constitutive role of stereotypes in shaping teachers' professional identity. Recognizing the stereotypes contribution can open possibilities for rethinking professional development, encouraging reflective practices that make explicit the cultural narratives teachers inhabit and enact. Such an approach not only fosters equity in mathematics learning but also promotes more critical and emancipatory forms of teacher subjectivity. Figure 3 shows a summary

diagram illustrating the theoretical result described in the discussion, for clarity and completeness; the diagram synthesizes the relationships previously outlined in the theoretical framework, visually representing the mediating role of teacher identity within the stereotype-learning nexus.

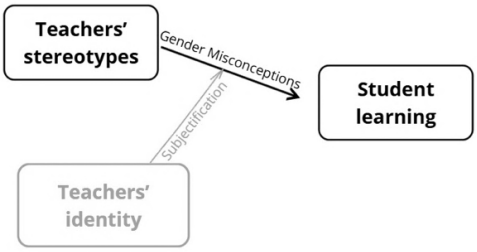

Figure 3: Illustrative diagram of the theoretical process

Conclusion

The primary contribution of this work lies in opening a new theoretical space by positioning teacher identity as an intermediate dimension linking gender stereotypes and students' mathematical learning outcomes. Drawing on the notion of subjectification, the analysis highlights that stereotypes do not act solely on students but also function as discursive tools through which teachers construct and negotiate their identity. In this sense, attributing different characteristics to male and female students is not merely a classificatory act, but a process that actively shapes teachers' professional selves. The implications for teacher education are significant: fostering reflective practices on gender stereotypes can enhance teachers' awareness of their impact not only on students but also on their own identity. Such interventions can promote gender equity in mathematics classrooms, supporting both students' learning opportunities and teachers' professional development. For future research, empirical studies are recommended to further investigate the relationship between teacher education and the mathematics gender gap, exploring concrete classroom practices and the use of assessment information, including standardized tests, as discussed by Ferretti and Santi (2021). Such research could validate and enrich the proposed theoretical framework, providing actionable insights for designing teacher education programs that are more aware of gender stereotypes and more effective in promoting equity in mathematics learning.

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Preservice secondary mathematics teachers learning to notice noteworthy events in their practicum

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This paper explores the noticing of preservice secondary mathematics teachers during their practicum, as analysed through videos of their own teaching. The study is framed within the Learning to Notice framework. It employs a qualitative-descriptive research design involving preservice teachers who (1) recorded their lessons, (2) selected relevant clips, and (3) wrote reports justifying their video selections. In these reports, they also describe and interpret the events they noticed. The findings indicate that most participants focus on aspects of their teaching at a mixed level of noticing, primarily centred on the teacher's performance, with limited attention to mathematical thinking. The study concludes that, in the early stages of teacher education, preservice teachers tend to view their own teaching performance as the most noteworthy events. It is recommended that practicum experiences gradually shift preservice teachers' attention towards students' mathematical thinking.

Keywords: Noticing, prospective teachers, fractions, use of video, internship.

Introduction

In every teaching act, it is essential to attend to what students do, how they respond, and to understand what they say to decide how to proceed with the lesson (Mason, 2002). Numerous studies in the field of mathematics education have underscored the importance of developing in preservice teachers (PTs) the competence to perceive and interpret students' mathematical thinking (Scheiner, 2023; Weyers et al., 2024). This competence, referred to as noticing, involves the ability of mathematics teachers to observe significant classroom events and interpret them in ways that support students' learning (Scheiner, 2023; van Es & Sherin, 2002).

While more experienced teachers are often able to recognize and respond to noteworthy events during a lesson, these events frequently go unnoticed or are inadequately addressed by novice teachers (Peterson & Leatham, 2009). As Stockero and Van Zoest (2013) note, teachers must make rapid decisions during instruction, without the time to systematically analyse students' thinking and without immediate support from colleagues. Moreover, due to the high cognitive demand of this task and the fact that novice teachers are still developing other fundamental aspects of their practice, they often fail to recognize or respond appropriately to these moments (Berliner, 2001).

Similarly, PTs, due to their limited classroom experience, often face challenges in focusing on students' thinking and other relevant didactic events. In response, the literature (e.g., El Mouhayar, 2025; Michalsky, 2021; Scheiner, 2023) highlights the need to integrate noticing into teacher education programs, providing PTs with opportunities to engage in practices that cultivate this essential competence, which is considered a key element of effective teaching (Mason, 2002; Weyers et al., 2024). Therefore, it is crucial to foster and examine the development of noticing competence in preservice mathematics teachers (Michalsky, 2021; van Es, 2011).

On the other hand, video recordings have become a fundamental tool in teacher education. While it was once believed that learning could only occur through shared video discussions, Sherin and Dyer (2017a, 2017b) suggest that learning to notice can also occur through recording one's own teaching and selecting clips for sharing. This study adopts this approach and investigates the following research question: *What events do preservice secondary mathematics teachers identify as noteworthy in their own lessons when analysing recordings of their teaching during their practicum?* The study highlights that one of the key competencies PTs must develop is the ability to notice noteworthy events in their classroom.

Theoretical considerations

This paper is framed within the *Learning to Notice framework* (van Es, 2011) and assumes that noticing encompasses the following dimensions: (a) selective attention, which involves identifying and directing attention to relevant elements (i.e., what teachers notice), and (b) knowledge-based reasoning, which involves using knowledge to interpret the observed events (i.e., how teachers notice) (Sherin, 2007; van Es & Sherin, 2002). Specifically, to describe the noticing levels of preservice teachers, we refer to Lee and Choy (2017), who drawing on van Es's work (2011), propose indicators that help identify different levels of attention and interpretation in noticing (see Table 1).

Table 1: Dimensions and levels of noticing (Adapted from Lee and Choy (2017))

Dimension	Level	Level indicators
Attending (What teachers notice)	Baseline	Attend to irrelevant details that do not have direct impact on student learning Attend to whole class environment, behavior, generic content and learning and to teacher pedagogy
	Mixed	Primarily attend to teacher pedagogy Begin to attend to particular aspects of mathematical concepts and the difficulties associated with them Begin to attend to particular students' mathematical thinking and behaviors
	Focus	Attend to particular aspects of mathematics and relate students' confusion to the teaching approaches Attend to particular students' mathematical thinking
	Extended	Attend to the relationship between particular students' mathematical thinking and between teaching strategies and student mathematical thinking
Interpreting (How teachers notice)	Baseline	Form general impressions of what occurred Provide descriptive and evaluative comments Provide little or no evidence to support analysis
	Mixed	Form general impressions and highlight noteworthy events or details Provide primarily evaluative with some interpretive comments Begin to refer to specific events and interactions as evidence
	Focus	Provide interpretive comments Refer to specific students' difficulties, events and interactions as evidence Elaborate on these specific students' difficulties, events and interactions

Dimension	Level	Level indicators
	Extended	Provide interpretive comments Refer to specific events and interactions as evidence Elaborate on these specific students' difficulties, events, and interactions Make connections between events and principles of teaching and learning On the basis of interpretations, propose alternative pedagogical solutions

It is important to highlight that there are key indicators that allow for distinguishing different levels of noticing. According to Amador et al. (2025) and Morales-Reyes and Zakaryan (2025), with respect to the attention dimension, at the baseline level, students' mathematical thinking is not considered. At the mixed level, attention to mathematical thinking is superficial, whereas at the focused level, it is more detailed. Finally, at the extended level, not only are specific instances of mathematical thinking analysed, but they are also related to teaching strategies and student mathematical thinking. Regarding interpretation, the baseline level is characterized by descriptions without concrete evidence, whereas at the mixed level, specific situations are presented along with evaluative comments. As the comments become more interpretive and the situations are described in more detail, the level progresses toward the focused level, reaching its highest stage when these comments lead to solutions for the observed events.

Furthermore, among the noteworthy events that a teacher may consider are moments of contingency (Rowland et al., 2015). Since these events are a natural part of classroom dynamics, teachers must be prepared to face discrepancies between what was planned and what occurs during the lesson. Regardless of how detailed the planning is, there will always be moments of uncertainty (Coskun et al., 2021).

Method

This research is qualitative and descriptive in nature and corresponds to a case study (Yin, 2018). The study focuses on 36 preservice secondary mathematics teachers enrolled in a bachelor's degree program in mathematics instruction at a Chilean university. The program spans 10 semesters, during which PTs take courses in mathematics, general pedagogy, pedagogical content knowledge, and complete four practicum experiences. At the time of data collection in 2024, the PTs were in their seventh semester, undertaking their third practicum and enrolled in a workshop course designed to engage them in video clubs and related activities that foster reflective practice.

As part of the workshop assignments, PTs were required to record lessons from their practicum, then form nine groups with their peers. Together, they analysed the videos of their recordings and selected a clip from one of the videos to address the following prompts, similar to the approach outlined by Sherin and Dyer (2017a): justify the choice of video, indicate what aspects caught their attention, interpret them, and reflect on what occurred in the class relative to the lesson plan.

For this paper, the work of a group (Group #6) that fully addressed all points of the assignment was selected. The video chosen by this group is a 10-minute and 40-second clip from a class with 13-year-old students, in which the focus is on understanding the process of multiplication and division of

fractions through concrete, pictorial, and symbolic representations. Specifically, the video shows a preservice teacher facilitating the pictorial representation of multiplying $\frac{4}{5} \cdot \frac{2}{3}$ (see Figure 1).

Figure 1: Snapshot of the video selected

For the data analysis, the indicators of noticing levels for the attention and interpretation dimensions were operationalized, using content analysis (Bardin, 1996) of the aspects that the PTs detailed in their written reports. The following section illustrates a case in detail.

Analysis and results

To illustrate the analysis process and the teacher training experience described in this paper, the report of the Group #6 is presented in detail below. The PTs indicate that they had access to three class videos (see Figure 2), corresponding to three of the four group members, as there were factors that prevented the recording, such as a lack of authorization from the educational institution and unforeseen circumstances during the sessions. Additionally, they emphasize that the selected video was submitted within the established deadline for the analysis. Furthermore, the group chose this video due to their interest in the topic covered, related to the pictorial representation of fraction multiplication. The PTs highlight that this aspect was particularly relevant to them, as the analysis of the video led to the reflection that not all of them had mastered pictorial multiplication, which made the case more engaging for them.

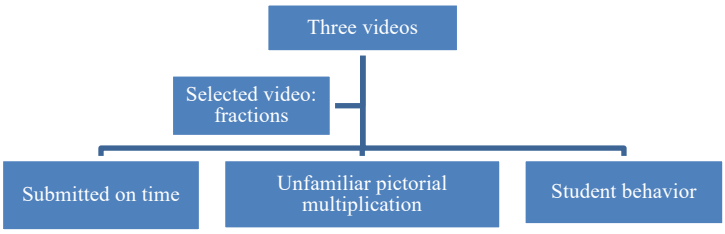

Figure 2: Justification for video selection

Regarding the evaluation of the class, the PTs agree that the video represents good pedagogical practice, and they emphasize that this evaluation was made by comparing it with their own previous experiences during their own teaching implementations. They considered it to be good practice because the teacher demonstrated a clear command of the topic, the students maintained consistent respect throughout the class, and the students actively participated in the proposed activities. In this regard, the preservice teachers expressed a desire for a future class they would teach to resemble the experience observed in the video.

Additionally, regarding what caught their attention in the selected video, the preservice teachers' information in their reports is described in Table 2.

Table 2: Justification of the chosen video and selected events – Group #6

#	Response from the student group - Mixed level of attention
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17	First and foremost, it is worth noting the teacher's mastery of voice control, as he maintains an appropriate tone for the classroom, which, as seen in the video, is quite large. The board available also has a considerable size, suitable for being used effectively in various sessions. It is important to highlight that the teacher uses more than one colour (although not visible in the video, it is evident in the editing), which demonstrates the importance of highlighting and differentiating various objects. For instance, the teacher uses colours to differentiate the pictorial representations of fractions and to help students identify when the colours overlap during the multiplication of fractions. On the other hand, it seems important to emphasize that throughout the video, the teacher uses terms that could have been better chosen, such as when the teacher mistakenly refers to the numerator as the denominator, which causes confusion among the students. Additionally, when explaining the graphical representations, the teacher specifies where to start drawing the rectangle with the lines dividing each fraction bar, whereas it could have been clearer to simply refer to the bars directly. It is also worth noting that the teacher seems to give little importance to how the rectangles are divided in the representations when pictorially representing a fraction. This could be a foundational issue for multiplying fractions, as this approach might limit a deeper understanding of the fraction representation, as no equal divisions are made, thus not accurately representing the fraction. Another aspect that caught our attention is that this class does not focus on basic mathematical knowledge, but rather on transversal knowledge, which offers students options when performing an operation such as multiplying fractions.

The aspects highlighted in Table 2 are of a general pedagogical nature (lines 1 to 6), content pedagogy (lines 7 to 12), and mathematical (lines 13 to 17). PTs primarily refer to the teacher's pedagogy and certain mathematical aspects, which suggests they are displaying a mixed level of attention.

On the other hand, regarding interpretation, the PTs focus on evaluating the teacher's actions during a moment in the video where a student is asked to participate at the board to represent the fraction $\frac{2}{3}$ in a table with three columns and five rows. The student colors the representation to show how the two overlapping fractions would appear. Table 3 shows their response.

Table 3: Interpretation of the selected event – Group #6

#	Response from the student group - Mixed level of interpretation.
18 19 20 21 22 23	Regarding the teacher's actions, the feedback provided was not bad, but perhaps not the most appropriate, as the student did not do what the teacher requested (represent the fraction 2/3 pictorially). However, the student did perform the multiplication between the fractions correctly, meaning the student advanced beyond the teacher's explanation, completing the task correctly. Additionally, the teacher could have highlighted the student's result as feedback. If the teacher's initial goal was to represent each fraction separately and then overlap them, the teacher could have done it on the other board and, at the end, pointed out that it was the same thing the student had done directly.

This interpretation is classified as a mixed level because the PTs refer to specific events and interactions as evidence, and their comments are evaluative (see Table 3). In lines 18 and 19, they state that the feedback was "not the most appropriate", and in lines 21 and 23, they make general comments such as he "could have highlighted the result" and "teacher could have done it on the other board" suggesting alternative actions the teacher could have taken but without interpreting how these actions would influence the student's learning.

Finally, the last point requested was to analyse whether what was done aligned with what was planned. These aspects are described in Table 4.

Table 4: Overall analysis of the event – Group #6

#	Response from the student group
24 25 26 27 28 29 30 31 32 33 34 35	During the class, compared to what was planned, some differences were observed in the use of time. In Moment 1, approximately 11 minutes were spent instead of the 23 minutes initially planned, which turned out to be appropriate since this segment served as an introduction to the partner activity planned by the teacher. Another relevant point is that the example used to recall the parts of fractions aligned with what was stipulated in the plan. The same example was even used to illustrate the multiplication of fractions through a pictorial representation. We consider it appropriate to use the same example for multiplication, as students often benefit from the continuity and coherence of examples throughout the class. Since the teacher finished with spare time, they could have taken the opportunity to carry out the additional example planned for the pictorial representation of fraction multiplication. This additional exercise would have allowed students to discuss the steps to follow based on the previous example, ensuring that most students understood the procedure. This, in turn, would have facilitated collaborative work during the partner activity, ensuring both students participated actively and applied the new knowledge, avoiding a situation where one worked while the other relied on their partner's knowledge.

In this description, the preservice teachers' pay attention to the differences between the planned time for activities and the actual time used in the classroom (lines 24 to 26). This shows a decision-making process, which was not the focus of this study, but they highlight their actions as teachers. In lines 30 to 33, they emphasize the activities that could have been carried out during the additional time, which were part of the planning (line 32, additional example). Additionally, in line 27, they highlight the elements that went as planned.

In this way, in the selected noteworthy event and its interpretation, Group 6 exhibits a mixed level of attending and interpreting. A similar analysis was conducted with the other eight groups; Table 5 shows the distribution of noticing levels across all nine groups.

Table 5: Levels of noticing evidenced in the reports of the nine groups

Dimension	Baseline	Mixed	Focused
Attending	2	6	1
Interpreting	3	5	1

Most groups reports were classified at the mixed level, indicating a focus on the teacher's performance with few references to students' mathematical thinking. None of the PTs reached the extended level, suggesting difficulties in connecting classroom events to principles of teaching and learning.

Discussion and conclusions

The results show that the PTs primarily focused on the actions of the teacher in the classroom. This may be because the recording was made from the back of the classroom (see Figure 1), which prevented them from observing the students' work. According to Sherin & Dyer (2017b), this could reflect a teacher-centred perspective rather than a student-centred one. Additionally, in the video clip, the preservice teachers focus on pedagogical aspects, for which the literature offers several explanations. First, it could be attributed to their limited experience as teachers (Peterson & Leatham, 2009). Second, identifying and analysing students' thinking is a challenging task (Berliner, 2001), and the types of tasks proposed in the class also play a key role. These tasks may not have been conducive to the exhibition of students' mathematical thinking (El Mouhayar, 2025).

Moreover, since PTs are still developing fundamental professional skills—such as voice modulation, appropriate use of the board, and managing student interactions—and receiving education that may be valuable in the future but is not fully understood in the early stages of their professional development, they may be uncertain about what is truly important to focus on (White & Forgarz, 2016).

On the other hand, while current trends in noticing research emphasize the importance of teachers focusing their attention on students' mathematical thinking, this paper argues that, within the context of initial teacher education, reflection on other pedagogical aspects should also be considered. Therefore, it is expected that preservice teachers' noticing competence will progressively evolve toward a focus on students' mathematical thinking as they consolidate these foundational teaching aspects. One way to guide noticing toward mathematical thinking is by incorporating specific guidelines that direct preservice teachers' attention to key aspects (e.g., El Mouhayar, 2025). Another strategy could involve asking them to anticipate, during lesson planning, moments when students may encounter difficulties or make errors, and to remain attentive to these moments to record them in their video observations (e.g., Sherin & Dyer, 2017a, 2017b).

In this regard, it is suggested that the first practicum experiences recognize as noteworthy events not only those related to mathematical content but also the broader pedagogical aspects. The metaphor of learning to drive a car is useful in describing this progression. In the early stages, the driver focuses on tasks such as using the mirrors or shifting gears. Over time, these actions become automated, allowing attention to shift toward the road and potential obstacles. Similarly, in PTs the development of attention to students' mathematical thinking requires the seamless integration of general pedagogical aspects. All these elements are crucial for PTs as they learn to teach.

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Fostering pre-service teachers' noticing of mathematical thinking through textual AI-human interactions: Opportunities and challenges

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This paper explores pre-service mathematics teachers' (PTs) responses to students' thinking during AI-based teacher-student simulated textual interactions. In this pilot study, critical events are used to facilitate teacher noticing (i.e., identifying, interpreting, and responding to students' thinking) where an AI system (i.e., Copilot) acts as a student. Data consists of eight PTs' transcripts of prompt history around a critical event, focusing on dividing an equation by x. We analyzed the transcripts using the concepts of "disregarding" and "shaping," which revealed mathematical ideas pre-service teachers responded to and disregarded when discussing with an AI system. Most PTs responded to the ideas of dividing the equation by x and factorization, while disregarding the idea that dividing by x was transferred from a previous topic. We show the variation in participants' interactions and discuss implications for fostering teachers' ability to notice critical events.

Keywords: AI-based teacher training, pre-service teachers, mathematics teacher education, teacher noticing, critical events.

Introduction

Effective mathematics teaching requires fostering teachers' responsiveness skills, such as teacher professional noticing, which enables teachers to attend to and effectively respond to students' statements, thereby fostering deeper understanding (e.g., van Es & Sherin, 2002). These noticing skills, comprised of identifying, interpreting, and responding to students' thinking, can be learned, and many teacher education programs position them as central pillars in the programs (e.g., Amador et al., 2021). Research shows that when learning to notice students' thinking, PTs manage to learn to identify students' thinking and develop their interpretations of that thinking. Yet they experience challenges in responding to the thinking they just interpreted. Often, the PTs' suggestions for responding to students' thinking are oversimplified and disconnected from the mathematics that is anchored in the students' thinking (e.g., Rotem & Ayalon, 2024). Thus, more meaningful practice opportunities are needed to support PTs' nuanced ways of responding to student thinking.

Current research suggests that GenAI can provide new prospects for simulating such practice opportunities (Dieker et al., 2024). In the rapidly growing field of using GenAI to support PTs' learning to notice students' thinking, most research examines teacher noticing using pre-post tests after the teachers interacted with the Chatbot (e.g., Galiç et al., 2025). Here our goal is to qualitatively explore how AI-based teacher-student interactions can support this endeavor, exploring the nuanced ways GenAI can support PTs to respond to student thinking. To address this goal, we developed an AI-based learning activity that simulates student-teacher dialogues around student thinking through textual interactions. The PT assumes the role of the teacher, while an AI system simulates the student. The AI acted based on common student difficulties and misconceptions across various subjects, as evident in academic literature (e.g., Welder, 2012). This pilot study is a part of a larger research

project aimed at investigating the opportunities and challenges of leveraging AI systems to support PT in noticing student thinking.

Literature background

Teacher noticing has expanded notably over the past two decades, developing well beyond its original conceptualization. One preliminary conceptualization, introduced by van Es and Sherin (2002), defines teacher noticing as identifying students' mathematical thinking during classroom situations and making knowledge-based interpretations of it. Later conceptualizations introduced responding to student thinking in ways that promote their mathematical understanding as an inherent additional skill of teacher noticing (Jacobs et al., 2010). Rotem and Ayalon (2024) use the construct of *critical events* to support PTs' learning to notice students' mathematical thinking as it unfolds during the lesson. Critical events are moments in which students' thinking becomes apparent and can serve to deepen students' mathematical understanding. They demonstrated that while PTs become better at identifying and interpreting students' thinking, PTs often struggle to respond to student statements in ways that build on previously interpreted students' thinking.

Most research on teacher noticing has been conducted using videotaped excerpts and pen-and-paper tests (e.g., Amador et al., 2021). However, teacher noticing skills are practice-based enacted skills during a lesson. As such, to fully comprehend how such skills are learned and developed, they need to be taught and assessed while the teacher practices them during teacher-student interactions. Examining teacher noticing in practice – as teachers practice their noticing skills – holds some conceptual and methodological challenges. For example, distinguishing whether a teacher's question following a student's comment is meant to scaffold the student's thinking (thus building on their comment) or gain more information about the student's intentions (thus supporting the teacher's interpretation), can be subject to the researcher's interpretations (e.g., van Es & Sherin, 2021).

Addressing this challenge, van Es and Sherin (2021) expanded their conceptualization of teacher noticing and included additional constructs: *disregarding* and *shaping*. Disregarding is the other side of the coin of identifying. While "identifying key features of instruction" (van Es & Sherin, 2021, p.19), selected features of classroom interactions are being *disregarded*, to allow the teacher to focus on aspects they find more relevant in terms of students' thinking. Shaping is defined as a set of "construct interactions and contexts that provide access to additional information" (ibid., p.19). Through shaping, teachers are taking an inquiry stance and engaging with students' thinking to gain a better understanding of their thinking. Hence, in the teacher-student interaction, some of the teacher's statements can be part of their response to support students' understanding. In contrast, other statements can, in fact, be part of the teacher's shaping and questioning of the critical event.

We build on the recent theoretical development in teacher noticing described above to address our goal and explore the nuanced ways GenAI can support PTs to respond to student thinking. To do so, we designed, implemented, and evaluated a workshop where eight PTs interacted with a GenAI system around a critical event showcasing a moment in which the teacher can build their instruction on the students' understanding. Consequently, we ask: *In what ways can engaging in AI-based textual teacher-student interactions support PTs' responding to student thinking?*

Methods

Research setting and context

This research is situated in the context of a teaching methods course at Utrecht University. All course participants hold a BSc in mathematics or a comparable degree in terms of mathematical content knowledge. The methods course, co-taught by the first author, is part of a master's program designed to prepare PTs to teach mathematics at the upper-secondary level. This course is accompanied by a one-day-per-week internship. Topics covered in the course include learning and teaching functions, derivatives, integrals, and several overarching themes, such as problem-solving and argumentation. One session was devoted to teacher noticing of critical events.

The study was conducted during a workshop held after the course concluded, to which participants voluntarily enrolled. The voluntary character of the workshop enabled a low-stakes participation and participants having completed the course while following good research practices (McGill et al., 2023). All eight course participants participated in the workshop. The workshop focused on several critical events from secondary school mathematics classrooms, providing PTs with multiple opportunities to engage with student-like thinking that emerged within these events. In this study, we focus on the case of a critical event that is often encountered as a challenge in mathematics education and can provide a rich context for teacher noticing (Rotem & Ayalon, 2024), the critical event is:

"The critical event happened in a 9th-grade mathematics lesson. While solving the equation: $x^3 = 6x^2 - 5x$, the student attempted to solve the equation by writing down the next step as: $x^2 = 6x - 5$."

The event was based on a critical event reported by a teacher who participated in the first author's previous project (Rotem & Ayalon, 2024; project webpage: http://aklim.haifa.ac.il/english). This critical event, along with the developed material containing GenAI prompts, was explicitly designed to capture a frequently occurring critical event, such as when students learn to solve polynomial equations of degree three and higher, among others. Mathematical ideas that may arise during the solving process include: recognizing that x can be zero (and is zero in this case), understanding that division by zero is undefined, and applying factorization to the trinomial expression. Conceptual ideas can also be raised, such as generalizing a previously learned solving strategy. The PTs copy-pasted the critical event and a pre-designed prompt into Microsoft Copilot, an AI assistant integrated into Microsoft 365 applications, provided by the university. Using the university's licensed Copilot application helped mitigate privacy and security concerns associated with unregulated GenAI tools.

Data collection and analysis

Data collection consists of the eight transcriptions of PTs' interactions with the Copilot application around the critical event mentioned above. All prompts were written in English, and the PTs could choose whether they wished to interact with the Copilot application in English or Dutch. Six participants interacted in English, and two in Dutch.

The unit of analysis is turn-talk. The average number of turn-talks in the AI-based teacher-student interactions is 16 turns, with a standard deviation of 3 turns. Firstly, for each turn, we identified the mathematical ideas anchored within that turn, for both the AI system (Copilot) and the PTs. Secondly,

we deductively coded the PTs' responding turns using 'disregarding' and 'shaping' (van Es & Sherin, 2021). If a mathematical idea was raised by the Copilot-student and then addressed by the PT, then that mathematical idea was considered as noticed by the PT. Mathematical ideas raised by the Copilot-student and ignored by the PT were coded as 'disregarded'. For all PTs' responses, we also examined whether these responses were questions that provided access to additional information about the students' thinking, thus 'shaping' or whether these were statements and questions that aimed to guide the Copilot-student toward the right solution strategy for the task. Table 1 depicts the coding scheme. After coding both the Copilt-student-like turns and the PT, we counted frequencies for mathematical ideas that were: a) noticed, and b) disregarded by the PTs. We also explored how much and in what ways the PTs shaped the interaction.

Table 1: Codebook - responding to the Copilot-student

Code	Definition
Noticing mathematical ideas taken by the PT	Mathematical ideas taken by the PT indicate that the PT *identifies* these mathematical ideas, *interprets* them, and makes a conscious decision to *respond to* them.
Disregarding mathematical ideas	Mathematical ideas that are raised by the Copilot-student and are not taken by the PT
Shaping	Questions providing access to additional information about the student's thinking
Directions to carry on the solution	Statements or other guiding questions (such as rhetorical questions)

Results

Figure 1 depicts the mathematical ideas that emerged throughout the eight AI-based teacher-student interactions' transcripts, while indicating noticed/disregarded mathematical ideas.

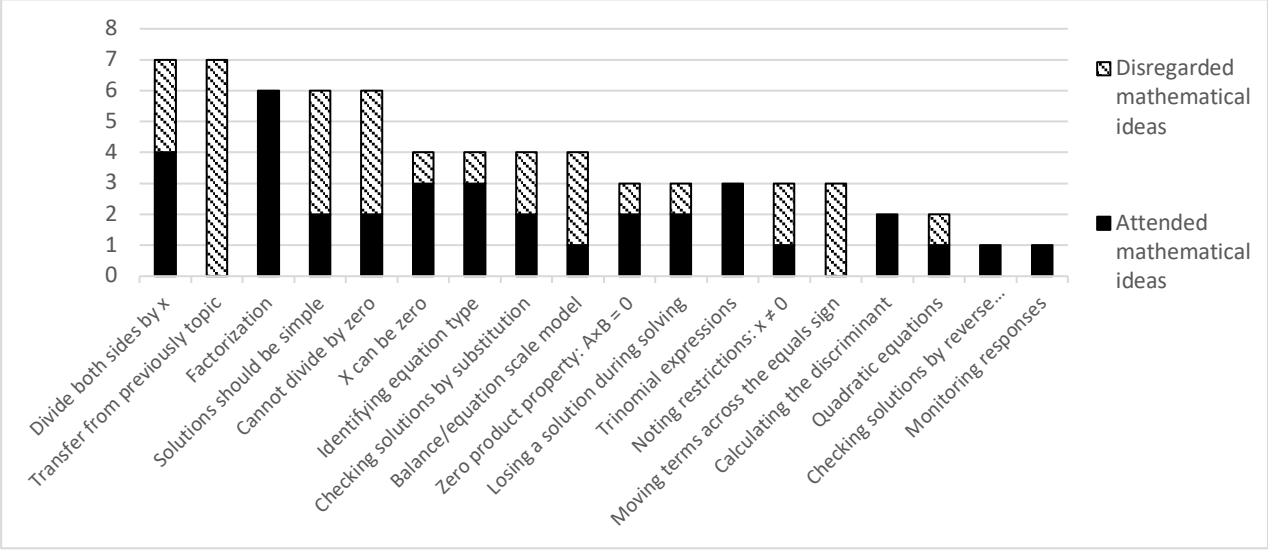

Figure 1: Mathematical ideas attended, and disregarded by the PTs during their AI-based interactions

Our data analysis as illustrated in Figure 1, shows that the most prominent mathematical ideas that were raised during the interactions were: a) dividing both side of the equation by x (7 scripts out of 8); b) transferring strategies from previously learned topic (i.e., using 'old' strategies in 'new' tasks) (7 scripts out of 8); c) factorization (6 out of 8); d) cannot divide by zero (6 out of 8), and e) solving should be easy (6 out of 8). The ideas that were predominantly noticed by the PTs in the interaction are: a) dividing both sides by x; b) factorization; c) x can be zero, and d) identifying equation type. These ideas align with the expected solution strategy. Ideas that appeared in more than half of the transcripts and were mostly disregarded by the PTs were: a) a solution strategy can be transferred from a previously learned topic; b) solutions should be simple; c) one cannot divide by zero; and d) the balance/equation scale model.

Additionally, most of the PTs' responses to the Copilot-student were to give directions for carrying out the solution, while fewer were to shape the interaction, i.e., using questions to provide access to additional information about the student's thinking. These findings are depicted in Table 2.

Table 2: PT's type of responses to the Copilot-student in the AI-based interaction

PT #	Total number of turns	# PT's turns	Giving directions to carry out a solution (# turns)	Shaping moves (# turns)
PT1	11	5	5	0
PT2	15	8	6	2
PT3	20	10	9	1
PT4	12	6	5	1
PT5	15	7	7	0
PT6	20	10	9	1
PT7	17	8	6	2
PT8	15	7	7	0

From Table 2, together with Figure 1, it emerges that the PTs mainly used the Copilot-student ideas to give direction to move forward with the task solution. Still, the AI-based teacher-student interactions provide PTs with the opportunity to practice their skills of responding to student thinking. Furthermore, the variety of mathematical ideas emerging from these transcripts demonstrates the variations in the transcripts, providing different ways of learning to notice, as we will exemplify next.

Variations among PTs' transcripts in the AI-based teacher-student interactions

We illustrate the variation between PTs' transcripts in AI-based teacher-student interactions with two examples: excerpts from PT2's and PT4's AI transcriptions. The emojis appearing in the text were part of the AI-based interaction, marked as Copilot. First is PT2's example showing 6 out of 15 turns.

Student (me, raising hand): Um, excuse me, sir? I think I messed up somewhere... I was trying to solve this equation: $x^3 = 6x^2 - 5x$. And then I thought I could just divide both sides by x, so I wrote: $x^2 = 6x - 5$. But now I'm not sure if that's right? I thought you could […] that, 'cause we did something like that in another problem.

PT2: Hi there, you didn't do it completely wrong, but you forgot something.

Copilot: Ohhh… wait, so I was kinda close? 😅 Is it because I divided both sides by x, but I didn't think about what happens if $x = 0$? Like… I just canceled the x's without checking if that's allowed. But I thought we could always divide both sides by the

same thing? So… should I just divide everything by x again but like… write "x ≠ 0" next to it? Or is there a better way to do it? Also, what did I forget exactly?

PT2: Do you remember how to solve $x^2 - 4x = 0$?

Copilot: Ohhh yeah! I think I kinda remember that one! 😅 So for $x^2 - 4x = 0$, I think we did something like… take out the x? Like factor it? So it becomes: $x(x - 4) = 0$ And then we said something like, "if two things multiply to zero, then one of them has to be zero," so we did: $x = 0$ or $x - 4 = 0$, which means $x = 4$ … Wait… is that what I was supposed to do with the $x^3 = 6x^2 - 5x$ one too?? Like factor it instead of dividing? But I thought dividing was faster… 😬

PT2: Exactly! you are on the right track. So why didn't you divide by x in the last example? As you did in the first example?

In the first turn, the Copilot-student turns to PT2 and raises two mathematical ideas: dividing both sides by x, and that a solving strategy can be transferred from previously learned lesson. The Copilot simulating the student also indicates insecurity regarding their solution strategy. PT2 first reassured the student and then hinted that something was forgotten. Then, the Copilot-student raised the idea that x can be zero, that we cannot divide by zero, the scale model of the equation, and that restrictions should be noted. PT2 referred to a similar equation, which is somewhat simpler, and *disregarded* all these mathematical ideas. By asking about a similar equation, PT2 *shaped* the critical event, as this question provided access to additional information about the student's thinking. This leads the Copilot-student to suggest factorization, a central concept needed for solving this equation.

Next is PT5's transcript with the Copilot-student. PT5 also used shaping, yet it led the conversation in a different direction. This excerpt shows turns 2-5 of the transcript (out of 15). Turn 2 is the first PT's reply to the Copilot-student request for help.

PT5: Hi, yes indeed, if you divide by x and x turns out to be zero, you did something that wasn't allowed. But maybe you can do something similarly, called binomial factoring, do you remember what that is?

Copilot: Ohhh right! Binomial factoring… I think I kinda remember that from last week? 😅 Is that the thing where you look for something that all the terms have in common and then you pull it out in front of parentheses? Like… a common factor? So for $x^3 = 6x^2 - 5x$, maybe I could factor out an x from everything? Like: $x(x^2) = x(6x - 5)$. Wait… is that right? I just put x in front of both sides because they all have x in them. But now I'm not sure what to do next… Also… is this even binomial factoring or did I just make up a new thing? 😬

PT5: Yes, indeed, like a common factor! This is similar to binomial factoring, you're almost there. What do you do when you try to solve an equation like this?

Copilot: Ohhh okay, so I was kinda close! 😅

This excerpt starts with PT5's first turn occurring right after the Copilot-student turn, which was similar to the first turn with PT2 (due to the prompt). Yet, PT5 raised the idea that x can be zero, and division by zero is then undefined (or, in her words, it is not allowed). In that respect, she built on the Copilot-student statements, while directing it to the desired solution strategy by raising the idea of factorization. The Copilot-student's factorization was done in an interesting way, which would potentially provide an opportunity for the PT to consider a possible follow-up response. Here, PT5 mentions a common factor and encourages the student to proceed with factorization. She does not give the solution, and her question seems to guide the solving process. Yet, similar to the excerpts above, many of the Copilot-student ideas were *disregarded* by the PT5, such as the idea that solving

strategies can be transferred from previously learned topics. Nonetheless, this interaction provides an opportunity for PT5 to delve into the student's thinking.

Discussion

Overall, our analysis showed that the developed activity enabled PTs to engage in responding to student-thinking-like of the GenAI. We learned that the PTs noticed some mathematical ideas introduced by the Copilot-student, mainly involving dividing both sides of the equation by x and factorizations. These ideas are central mathematical ideas in this event. Yet, while noticing some ideas, the PTs disregarded others. Notably, all PTs disregarded the Copilot-student's reasoning for why they divided the equation by x; that is, the Copilot-student claimed that they had transferred something from previously learned lessons. Possible reasons why the PTs disregarded this idea could be, first, that they perceived the idea as a conceptual aspect to which the PTs did not think they needed to respond or did not know how to respond. Another explanation is that once the Copilot-student presented a correct idea, PTs felt less compelled to return to other ideas, even if these held mathematical significance. Building on these findings, the variation that exists among PTs' transcripts in terms of the opportunities to respond to, disregard, and shape different mathematical ideas suggests that AI-based teacher-student textual interactions can provide valuable opportunities for PTs to practice responding to students' thinking.

Indeed, accessible AI systems like Copilot hold great potential as a complement to traditional learning environments, e.g., by tackling logistical barriers to teacher training and better preparing PTs for their internships, while providing teacher trainers with the ability to effortlessly check PTs' interactions (e.g., Galiç et al., 2025). Nonetheless, such activities require thoughtful design and facilitation to ensure they foster, rather than constrain, the development of teachers' noticing. Although the critical events in the workshop reflected common challenges documented in the literature and/or observed in PTs' classroom experiences, they still involved interaction with a computer. Even if participants perceived the experience as authentic, it did not constitute real teaching. Further, overreliance on AI in teacher training raises environmental and ethical concerns, as well as the risk of reinforcing a narrower focus on procedural correctness at the expense of conceptual engagement, thereby limiting the professional growth of PTs.

Like every study, this pilot study has several limitations that must be acknowledged—including the "clean" experimental setting, the AI's comprehensive knowledge base, and ongoing privacy concerns (Golda et al., 2024). Yet, the current widespread engagement of people with and support-seeking from GenAI across various domains (e.g., Brandtzaeg et al., 2022) suggests that teacher education, like other fields, could potentially benefit from GenAI. We argue that our work is a first step, as it indicates that there is substantial room for teachers to use AI systems to simulate their upcoming lessons. While considering our results as a starting point for learning about responding to students' thinking using AI systems, more research in this direction is needed. From a methodological and theoretical perspective, the dual lens of disregarding and shaping could potentially be elaborated beyond this study and provide valuable insights into analyzing teacher noticing in the midst of instruction, a field that remains under-researched despite its great importance for classroom teaching.

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The identity development across multiple contexts of an out-of-field mathematics teacher

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Teacher identity has emerged as a key lens in mathematics education research, either through studies of structured professional development (PD) programs or through studies of teachers' everyday professional lives. However, these two bodies of literature are typically fragmented, and there is less understanding of how teacher identities evolve across multiple formal and informal professional contexts. This exploratory paper addresses this gap by examining the identity development of Yael, a teacher with expertise in inclusion education, who also works as a mathematics instructor. Identity is examined through the Patterns of Participation (PoP) framework, which views it as a dynamic experience of "being, becoming, and belonging" in professional contexts. The analysis further develops this perspective by showing how Yael drew on resources from different contexts to shape her evolving identity through a complex web of interconnected learning experiences.

Keywords: Mathematics teachers, professional identity, professional development, patterns of participation, out-of-field teacher.

Introduction

Research on mathematics teachers' Professional Development (PD) typically focuses on the direct impact of PD programs on teacher learning. However, from the teacher's perspective, professional development is shaped by the interaction of multiple contexts. For this reason, we need accounts of teacher professional development and guiding frameworks that consider how seemingly fragmented activities are positioned in relation to teachers' ongoing professional growth (Ehrenfeld, 2022; Opfer & Pedder, 2011). In this exploratory paper, we investigate one approach to addressing this problem through the lens of mathematics teacher identity.

Teacher identity research has established a rich foundation across a range of educational contexts (e.g., pre-service, in-service) and methodological approaches (e.g., narrative, participative) (see Darragh, 2016; Graven & Heyd-Metzuyanim, 2019; Rushton et al., 2023 for reviews). Work in this area has emphasized narrative accounts of identity construction (e.g., Darragh, 2016) and the role of PD in shaping identity (Heyd-Metzuyanim & Shabtay, 2019). However, few studies examine how these elements intersect (Rushton et al., 2023; Skott, 2024), specifically, how teachers co-construct identity narratives (typically studied through interviews) while also negotiating their participation in professional contexts **(**typically studied through analysis of workplace interactions). Skott (2024) recently noted with regard to this that "much identity research does not acknowledge how teacher identities and local practices co-constitute one another, and that an interactionist supplement is needed" (Skott, 2024, p. 127). Put simply, while teacher identity research in mathematics education has developed considerably, we have limited accounts of how identity narratives are co-constructed in participation within and across multiple professional contexts. Understanding these cross-context processes matters because PD is rarely experienced in isolation. Teachers simultaneously participate

in multiple activities, such as formal PD, team meetings, and other informal conversations (to name a few) that may support or complicate identity development.

We find the case of out-of-field mathematics teaching particularly productive for exploring and theorizing teacher identity development across time and multiple contexts. *Out-of-field teaching* refers to situations in which teachers instruct subjects outside their area of certification or academic training (Hobbs, 2013). Recent literature adopts an asset-based perspective, highlighting the unique expertise that these teachers bring to PD contexts (Perl-Nussbaum et. al., 2023). Combining interview-based narrative analysis with observations of identity performance in PD, we ask: *How did the teacher's experiences across multiple contexts shape her professional identity as an out-of-field mathematics teacher?*

Prior literature and theoretical framework

We conceptualize identity as constructed through two interrelated dimensions: (a) narratives that are reifying, endorsable, and significant (Sfard & Prusak, 2005), and (b) participation across professional contexts that shape possibilities for being, becoming, and belonging. This is especially relevant for out-of-field teachers, whose sense of professional identity often evolves through both their positioning and participation in unfamiliar disciplinary contexts.

First, *Narrative accounts* frame identity as the set of stories people tell about themselves and others tell about them, specifically those that are "reifying, endorsable and significant" (Sfard & Prusak, 2005, p. 16). Within this view, current identity refers to present-tense stories (e.g., "I don't know math"), while designated identity refers to future-oriented stories about who one hopes to become or is expected to become (e.g., "I want to participate confidently in math talk"). Identity development can be understood as a movement toward closing the gap between designated and current identities, as particular narratives become endorsed through practice.

Second, we treat the gap between Yael's initial and emergent identity narratives as being negotiated through *participation* (what teachers do, where, and with whom), including recognition by others. Following that process-oriented view, we employ *Patterns of Participation* (PoP), which conceptualizes teacher identity as shifting experiences of being, becoming, and belonging across social practices, shaped by both individual and collective processes of "meaning-making and agency" (Skott, 2019; Skott et al., 2011, p. 32). This framing foregrounds lived experiences, emotional realities, and participation in social communities (e.g., school and PD).

Specifically, we use PoP's three dimensions as analytical categories: *being* (present professional presence and recognition), *becoming* (emergent possibilities and trajectories), and *belonging* (sense of fit within professional communities). Read together, the narrative lens tells what identity stories are in play (current vs. designated), and PoP specifies how they change through situated participation and recognition that endorse particular stories over time (Skott, 2024; see also Skott, 2019). We use this integrated framework to examine how Yael's participation across various contexts (PD, inclusion, and classroom) and the recognition she receives (or doesn't receive) influence movement between her current and designated identity narratives along these three dimensions.

Methodology

Research context

The case of Yael was drawn from a larger study in which we collaborated with a nationwide program aimed at fostering collaborative, teacher-led inquiry into teaching practices through local learning communities in Israeli schools. The PD model supported collaboration by centering authentic problems of practice, incorporating classroom artifacts such as videos or case descriptions for collective reflection, and guiding discussions with structured protocols that fostered productive professional discourse. While the broader study includes multiple teachers across various learning communities, this paper focuses on Yael as a focal case to explore identity development in depth.

Focal case

This exploratory study is designed as a qualitative case study focusing on Yael. Yael is a special education teacher (16 years of experience), who was in her third year at her present school in Israel. Holding certifications in Special Education and Science, Yael is considered an out-of-field teacher in mathematics. She works as an inclusion teacher in 3rd and 4th-grade classrooms, and she also mentors the homeroom teachers (primary classroom teachers responsible for a specific grade-level class) on behavioral challenges. In our setting, an inclusion teacher pulls out small-group mathematics support aligned with the classroom curricular goals; she also attends the school's mathematics PD meetings alongside six mathematics teachers with grade levels ranging from Grade 1 to Grade 6.

Following Yin (2009), our goal is analytical rather than statistical generalization: to refine theoretical understandings of how identity develops and unfolds across such interconnected contexts. We used purposeful sampling to select an information-rich case that illuminates identity development across professional contexts (classroom, inclusion support, and PD). Yael was selected because (a) she is an out-of-field teacher who co-teaches mathematics and participates in mathematics PD, and (b) her data corpus (introductory interview and PD meetings) offered rich episodes of participation and narration relevant to being, becoming, and belonging.

Data collection methods

Data collection in Yael's elementary school learning community included (a) an in-depth introductory interview (lasted 60 minutes) exploring Yael's professional journey and educational experiences, and (b) video-recording the math teachers' learning community weekly meetings (45 minutes) during the school year of 2024-2025. While 13 of 14 meetings were recorded (all but the first week), this exploratory study focuses on the two initial recorded meetings (weeks 2 and 3) because they contained rich interactions that foregrounded Yael's participation and early identity negotiations within the PD setting, with an eye towards continuing to a systematic analysis of all 13 recorded meetings.

Videos were captured in naturalistic settings. All meetings were recorded in Hebrew and automatically transcribed; translations to English were produced with attention to semantic accuracy. All participants provided written consent to participate in the study.

Data analysis methods

Our analysis combined inductive thematic analysis of life-history narratives with deductive PoP-informed coding of PD interactions, integrated through a narrative case synthesis.

In Step 1, we employed a thematic analysis approach (Braun & Clarke, 2006) to analyze Yael's interview, which involved reading the transcript and memos, initial coding, generating candidate themes, and finally naming themes. This analysis yielded five major themes: mathematical narratives, other professional narratives about "who I am?", educational experiences, supporting factors or communities, and challenges.

In Step 2, we analyzed the interview narratives to trace the development of Yael's professional identity across her educational and professional experiences. We identified narrative trajectories that move from stories reflecting low self-regard (whether as a student or teacher) to narratives of belonging and support within various institutional contexts, ultimately towards expressions of confidence. Regarding Yael's mathematical identity, we noticed that narratives moved from stories of being a struggling student to becoming a teacher who perceived herself, or was perceived by others, as lacking the preparation to teach math; then to narratives of belonging to supportive teams that compensate for her mathematical gaps. However, the interview revealed no explicit narrative of who she became, or hoped to become, mathematically. We coded these narratives by attending to dimensions of *being* (initial self-perception), *becoming* (emergent possibilities), and *belonging* (sense of fit within communities).

In Step 3, we conducted PoP-informed analysis of Yael's participation across PD meetings, examining evidence of those narratives through her interactions. For example, we identified instances where she explicitly positioned herself as an expert and colleagues validated her contributions, which corroborated the confidence and self-regard trajectory identified in Step 2. Notably, regarding mathematical participation, we observed her emerging as more confident and engaged in mathematical discourse around disciplinary concepts, a dimension of becoming that had not been evident in her interview narratives. This discrepancy between narrated identity (Step 2) and enacted participation (Step 3) became analytically significant.

In Step 4, we integrated findings from Steps 1-3 to construct a within-case narrative synthesis that traces Yael's trajectories of being, becoming, and belonging across professional and mathematical contexts. This integration revealed two distinct but interrelated identity dimensions: professional becoming (evidenced primarily through interview narratives and validated through PD participation) and mathematical becoming (emerging mainly through PD participation despite absence in self-narrated identity). Combining narrated identity (interviews) with enacted identity (PD participation) allowed us to surface discrepancies that signal implicit or emerging identity dimensions. To enhance transparency and analytical rigor, both authors reviewed segments of data and discussed discrepancies until a consensus was reached.

Findings

We report the two interrelated dimensions of Yael's identity development, mathematical and professional (inclusion), drawing on (1) themes from the interview, (2) PoP-informed interactional

coding of Weeks 2–3 PD meetings, and (3) a narrative synthesis that traces being becoming belonging across contexts. The interview themes reveal Yael's history of struggle and emerging recognition; the PD episodes demonstrate how belonging was negotiated in practice and how these negotiations reshaped her narrated identity.

Mathematical identity: From struggles into belonging to a disciplinary community

Yael's interview highlighted three experiences that shaped her mathematical identity: persistent difficulties with school mathematics, self-positioning as a non-expert in math teaching, and emerging support through collaborative mentoring. Yael's story illustrates her evolving identity as she navigates mathematics instruction as an out-of-field teacher. She openly acknowledged that mathematics was never her strong suit as a student, "When I struggled, they say, 'What don't you understand?' or 'What's wrong with you?'". These experiences reveal being in a long-standing difficulty with school mathematics and associated deficit self-images stemming from earlier negative schooling experiences. As a result, she habitually positioned herself in the interviews as a non-expert who avoids leading math content, admitting, "You really don't want me teaching math because I don't know math… I always get confused about everything." She repeatedly talks about herself as an inclusive teacher rather than as a teacher who teaches mathematics. Her role as a mathematics teacher was marked by hesitation and self-doubt, "If I feel like it's a topic I don't know well, I won't even take the kids out (of the class). I tell them, 'Let's stay in the classroom'".

However, as the interview progressed, Yael's narrative revealed a parallel process of identity construction. Moving from isolation toward community, she described a growing network of professional support, including both mathematics teachers at her school and a mentor from the MATIA program. MATIA is a district-based professional body that guides inclusion teachers. As part of this support, she meets regularly with both a special education mentor and a mathematics mentor. She stated: "She (math mentor) said: if you have difficulties, I'll come and mentor you at school." Yael repeatedly emphasizes collaboration, stating, "Every teacher I worked with in math knows that I need their help". She framed colleagues' routine check-ins (e.g., Yael, did you get it?') as supportive coordination. Although the intent of such check-ins is open to interpretation, Yael perceived them as expressions of collaborative support rather than judgment. Yael's *belonging* to the mathematics team fundamentally shifted her mathematical identity. As a result of these mentoring and collegial arrangements, Yael became more engaged in mathematical discussions and interpreted this increased participation as a sign of belonging.

Despite her continued hesitations about teaching mathematics, Yael's active participation in PD discussions revealed that she engages confidently with mathematical ideas. For example, in the third PD meeting, Yael joined a discussion on student strategies for estimation. She compared different students' reasoning, insisting that "we should still give credit to those who were close" because, in the context of estimation, "if I were close, I was right". This contribution suggests that she was not only engaging with the mathematical idea of estimation but also advocating for an inclusive framing that values students' approximative reasoning. This perspective illustrates how her emerging mathematical identity is intertwined with her expertise in inclusion. In this discussion, she interprets

tasks and student efforts in ways that demonstrate not only her growing confidence in engaging with mathematical ideas but also her care for students who struggle.

Taken together, these episodes illustrate how Yael's identity as a teacher evolved through ongoing negotiations of belonging. Notably, interview narratives and participation analysis complemented our understanding of how Yael perceived her place within the mathematics team, as she increasingly became part of that professional network.

Professional identity: From self-doubt to recognizing expertise

In the interview conducted with Yael, she demonstrated a clear trajectory from profound self-doubt to gradually recognizing her own capabilities. Beyond the mathematical struggles we noted above, she also shared her general negative school experience as a student,

> I wasn't one of those kids with self-confidence, the ones who say, 'I can do it.' I also didn't have anyone to encourage me; there wasn't much support around … They didn't really believe in me.

She also explained that early in her career, she didn't believe in herself and even had limited professional confidence, stating that she "wasn't encouraged too much" and that "self-confidence wasn't high." Over the progress of the interview, however, Yael shared that she noticed a shift in how colleagues related to her: "I started noticing, especially at [the current] School, that so many people came to me for advice." This recognition led her to pursue a "practicum" where she plans to mentor other teachers on how to deal with behaviors of struggling students as part of her master's degree obligations, something she "never imagined doing before."

Yael's personal history of struggle has shifted to being an expert in understanding and supporting struggling students. This shift is evident in the PD sessions, where Yael consistently positions herself as an expert in behavioral matters. For example, during a session where the facilitator presented a video of a "With Me, Next to Me, and in the Space" teaching model, an educational framework that organizes heterogeneous classroom learning through differentiated group work based on required teacher intervention levels, Yael confidently reframed the pedagogical discussion toward her area of professional strength: "I take this to the area where I have more expertise, which is behavioral problems". This self-positioning aligns with her interview reflections, where she described how colleagues often consulted her about behavioral issues. Drawing on her role as an inclusion teacher who monitors classroom dynamics during the discussion of differentiated instruction, she positions herself as an observer-analyst, noting, "we, the inclusion teachers, watch a bit from the side." This stance presents her as someone who identifies patterns others might overlook, a perspective informed by her MATIA training and her understanding of educational contexts. Such uptake endorses her positioning as a contributor with specialized expertise, strengthening Belonging.

Yael's participation in the analyzed PD sessions demonstrates the active, confident contributor she described becoming in her interview, contrasting with her earlier self-described lack of professional confidence. For example, she explicitly draws on her M.A. studies when discussing behavioral interventions, links classroom challenges to systemic issues by comparing Israeli and U.S. education systems, and grounds her comments in everyday school realities. In addition, her peers affirm her observations with comments such as *"true"*, and the facilitator builds on her remarks respectfully,

signaling recognition of her professional authority. We infer that such episodes reinforced her status as a valued member whose perspectives carry weight in shaping group discourse, strengthening her sense of Belonging.

Taken together, these findings highlight how Yael's professional identity as an out-of-field teacher, rooted in her inclusion expertise and lived experience of struggle, became a source of recognition within the PD space. Together, these two interrelated identity dimensions demonstrate how Yael negotiated her role as an out-of-field mathematics teacher by drawing on different sources of expertise, and how her story illustrates that teacher identity is not singular or static but rather composed of multiple, sometimes contrasting elements through participation within a community.

Discussion

This exploratory study employed both *narrative* and *participation* frameworks to examine how professional identity is enacted and negotiated through changing experiences of being, becoming, and belonging within professional communities. We traced how Yael, an out-of-field mathematics teacher, negotiated identity across contexts (PD, school teams, classroom/inclusion work), treating identity change as participation- and recognition-dependent (Skott, 2019; 2024).

Our findings reveal how Yael's professional identity developed across two interconnected dimensions, framed by the three notions of being, becoming, and belonging: (1) her mathematical teaching identity, which transformed from deficit-based *being* to collaborative *becoming* through supportive *belonging*, and (2) her inclusion specialist identity, where established expertise provided confident *being* that enabled *becoming* a resource for other teachers as a form of *belonging*.

As a single information-rich case, claims are bounded. Transferability lies in the mechanism documented how participation and recognition reorganize being–becoming–belonging, rather than in population-level inference. Conceptually and methodologically, we illustrated the complementarity of lenses: narrative accounts reveal what identity stories are in play, while analysis of participation specifies how they evolve in local interaction and over time, through participation and recognition (Skott, 2024). This was particularly evident in Yeal's analysis of mathematical identity, with the narratives and analysis of interviews and participation each backgrounding and foregrounding different aspects of Yeal's process of being, becoming, and belonging. Empirically, we suggest that an integrated identity lens can support future research in exploring teachers' development across multiple contexts, including the unique case of out-of-field mathematics teachers. Our preliminary findings suggest that when participation structures afford recognition and uptake, the expertise of out-of-field mathematics teachers can serve as a disciplinary resource, both for their colleagues and for their own identity development.

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Exploring pre-service primary teachers' self-efficacy through mathematics didactics portfolios

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The preparation of future primary school teachers in the Czech Republic is currently influenced by two new key policy documents: the Revision of the Framework educational programme and the Competence framework for graduate teachers. In this context, self-efficacy for teaching mathematics plays a key role. The study examines how students who have chosen to create and defend a portfolio as a form of their final state examination in mathematics education spontaneously express their self-efficacy in teaching mathematics. Qualitative research based on the analysis of 18 student portfolios using the hybrid coding method yielded three main categories of indicators of self-efficacy formation: (1) practice and reflected experience, (2) knowledge of content, (3) attitudes, emotions, math anxiety. These findings inspire us to streamline the preparation of future teachers in order to strengthen their self-efficacy for teaching mathematics.

Keywords: Student portfolio, self-efficacy, reflective practice, mathematics instruction, attitudes toward mathematics instruction.

Introduction

It is natural that, given the current political and economic changes in society, the educational requirements of citizens are changing. At every level of education, intensive efforts are being made to improve the quality of teaching in the light of the new requirements. Two major events are currently taking place in the Czech Republic – a major revision of the Framework educational programme[1] is being completed, which affects both the content of the curriculum and, much more strongly, the emphasis on how the curriculum is to be conveyed to pupils and the conditions under which learning is to take place. How future teachers are to be prepared for their profession, what competences are to be developed both at the level of undergraduate preparation and in practice, is currently declared by the Competence framework for teacher graduates.[2] Since 2023, a general framework has been in force under the guarantee of the Ministry of education, and currently expert teams are formulating subject-specific frameworks that will come into force at the end of 2025.

Many foreign and Czech studies agree that a strong and positive professional self-efficacy of teachers is one of the most important prerequisites for a successful teaching career, motivation, perseverance in the profession, and resilience to stress and negative events. Therefore, the topic of the professional self-efficacy of future teachers is also being intensively addressed in various innovative projects at faculties of education (Bártek, Dofková, 2017; Dofková, 2019). Knowing that a positive and clearly formed self-efficacy of pre-service teachers is key to successful professional growth and effective work as a teacher (Vaculík Pravdová, 2022; Juklová, 2008), we study how specifically in mathematics teaching and mathematics didactics we can prepare students for new requirements. We believe that

[1] https://revize.rvp.cz

[2] https://msmt.gov.cz/vzdelavani/kompetencni-ramec-absolventa-ucitelstvi

in this area we, the educators of teachers, have a significant opportunity to act. Several of our previous studies (Jirotková et al., 2025; Slezáková et al., 2017) show that our sustained efforts are yielding results. During undergraduate training, we are successful in building a positive attitude towards mathematics, which has a strong influence on students' self-efficacy for teaching mathematics (Jirotková et al., 2025).

We build on our research (Slezáková et al. 2017), in which we demonstrated that the very act of creating a portfolio, which students can choose to defend as an alternative to the traditional form of the state final exam in mathematics education, leads to further development of their professional competencies in teaching mathematics. Students are not given any precise instructions on how their portfolios should look, mainly so as not to limit their creativity. They are only given the following general requirements: the mathematical and didactic content of the portfolio should cover the content of traditional state exam questions, each file included should be justified, reflected upon, and should contain the student's own intellectual contribution. It is clear that students need to collect, organize, and continuously reflect on a large amount of material from their entire undergraduate studies. The freedom in creating the portfolio concept and the nowhere explicitly stated requirement to reflect on one's self-efficacy allows students to express themselves as future primary school teachers completely spontaneously. Therefore, we have decided to focus our research attention on students' teaching self-efficacy, which, if it appears in the reflections contained in the portfolios, is rather spontaneous and therefore true. Our investigation is guided by the following research questions:

RQ1. How do our students think of themselves as beginning teachers of primary mathematics at the end of their studies?

RQ2. What opportunities do we have to strengthen the quality of self-efficacy for primary mathematics teaching during their mathematics courses in undergraduate preparation?

Theoretical framework

Research on the professional self-efficacy of teachers and future teachers, or the development of it, the relationship between self-efficacy and the effectiveness of teaching, etc. has been and is being studied by many researchers. The reason is obvious. The importance of professional self-efficacy for the quality of the teaching profession is emphasized by many authors. From the Czech environment, these are (Gavora et al., 2024; Lukášová, 2015; Spilková & Zitková, 2004).

Positive and clearly formed teacher self-efficacy is crucial for successful professional growth and effective teaching. When built on repeated successes, it is resilient to any temporary setbacks and stresses. If a teacher's self-efficacy is unclear, he or she may feel insecure in his or her profession and react inappropriately to external influences (Bandura, 1999; Gavora et al., 2024; Juklová, 2008). Gavora et al. (2024) even provide research support for the claim that self-efficacy is one of the most important characteristics of a teacher that predisposes him or her to success in his or her teaching. However, Gavora (2011) comes to the realization that a teacher's good self-efficacy may not coincide with his or her actual teaching abilities and classroom performance, but a strong sense of self-efficacy is usually positively correlated with effective teacher performance.

Researchers agree on the definition of self-efficacy and define it as the belief in one's own capability to carry out a given intention, which is one component of it, and as the confidence in oneself that the teacher's actions will actually bring the intended results, which is its second component. It is perceived as a complex internal dynamic variable that influences everyday teaching and in which the teacher can simultaneously develop (Bandura, 1999; Gavora et al., 2024; Vaculík Pravdová, 2022). If a teacher has only the first component, i.e., is professionally well qualified, it does not mean that he or she will be successful as a teacher. Therefore, research studies what indicators characterize teacher self-efficacy, how it can be effectively developed and supported during pre-service training (Bandura, 1999; Juklová, 2008; Spilková & Zitková, 2024) and what are the resources that satiate it.

Self-efficacy for mathematics is domain-specific and thus not automatically transferable from other subjects. For example, a teacher may have good self-efficacy in science subjects but can feel insecure in mathematics (Philipp, 2007). Therefore, in the following we will mainly refer to authors who deal with this specific domain of self-efficacy for mathematics teaching.

Dofková and Chudý (2019) used quantitative research tools to study the readiness of future teachers to teach mathematics and identify their weak and strong areas. Charalambous et al. (2008) found in a case study that reflective practice and positive attitudes significantly enhance self-efficacy, while anxiety about mathematics weakens it (Gresham, 2008). Philipp (2007) studied the link between self-efficacy and choices of methods and motivation. Study (Bates et al., 2011) finds a high correlation between mathematical knowledge and self-efficacy, supporting theory of Bandura (1999) that mastery experiences are a key source of self-efficacy. Teachers' psychological mindset, their emotions and attitudes also play a significant role in shaping their self-efficacy.

Methodology

Most research on self-efficacy are quantitative and use the questionnaire method, such as the Mathematics Teaching Efficacy Beliefs Instrument (MTEBI, Bates et al., 2011), the Mathematics Anxiety Rating Scale (MARS, Gresham, 2008), the ZSE questionnaire assessing teacher self-efficacy improvement resources, the OSTES-Cz, a questionnaire assessing teacher self-efficacy for teaching and for classroom discipline management (Gavora et al., 2024). Qualitative research uses a case study method (Charalambous et al., 2008) based on content analysis (Swars et al., 2007), student interviews and analysis of essays "I, the future teacher" (Spilková & Zitková, 2024).

To address our research questions, we chose a qualitative research design and used hybrid coding to analyze the students' final portfolios, combining inductive and deductive approaches. We started with open coding. We categorized the codes created in vivo (e.g., supporting cooperation, working with mistakes, guiding pupils to reflect on their learning, guiding pupils to discuss) and linked them to codes obtained from professional literature that strongly influence and reinforce self-efficacy for teaching mathematics for future teachers. In this paper, we discuss the three categories of indicators in more detail. The tool of our investigation was therefore the final portfolios of the students of the final year of primary school teaching study. Since the COVID pandemic, students have been submitting their portfolios electronically, so we have an archive of the portfolios. For our research, we used 18 portfolios from 2022–2025, which are still available electronically. It should be emphasized that students who chose to create a portfolio for their final state exam (which involves

much more work than learning the answers to a set of questions) are usually those who think more deeply about their future as teachers. Portfolios are usually around 150 pages long. Due to the small research sample and the scope of the paper, we do not provide quantitative data.

Findings

We have divided the indicators that shape the self-efficacy of students into three categories.

Category 1. Practice, interaction with pupils, reflected experience

The indicator that emerged as most common in the portfolios is reflection on practice. Students have 10–12 teaching hours supervised practice of teaching mathematics at primary school. They receive feedback on each lesson preparation from the class teacher, as well as from the faculty teacher and from other students in the group. After the implementation of the student's teaching, a joint reflection of a small group of students (about 6–7) takes place, especially in terms of the didactics of mathematics. Finally, the student writes a reflection and proposes a reasoned alteration of at least some fragments of the lesson, especially those where the student's teaching could have been more effective towards the pupils. We strive to ensure that each student has the opportunity to experience his/her own successful teaching, so that he/she knows that, perhaps with the support of the class teacher and the faculty teacher, he/she is able to implement a successful lesson.

The phenomenon of reflection on practice in portfolios is most often expressed by students through the following indicators:

a) A description of own experience from teaching practice. During the teaching practice, the students realized that the new experiences made them reevaluate their ideas about teaching primary school mathematics. They mostly confirmed their belief that they had chosen their profession well. Some students even decided to teach before graduation. In reflections of practice, theoretical knowledge was linked to real-life practice and students appreciated usefulness of it. Student: "My teaching practice in a school where mathematics is taught with innovative methods is the fulfilment of my professional vision of an ideal school." Another student: "It was only during my teaching practice that I realized that I wanted to teach in a constructivist way, using many teaching aids." Others mentioned the use of mind maps and the importance of clearly articulated objectives of the lesson and the opportunities to work with their own mistakes.

b) Thinking about the pupils. The student expressed her belief that she can make her pupils experience mathematics as joyfully and enchantingly as she once did. Another student was amazed during her mathematics classroom observation at all that the children could do in mathematics, and she was strongly motivated to further develop her didactic content knowledge. Another student considered how she could develop her pupils' intrinsic motivation to explore mathematics, thereby enabling them to experience the joy of learning for themselves. Some students considered helping weaker pupils based on prior diagnosis. They also realized the importance of discovering pupils' preconcepts on which to build their understanding. They cited the following additional phenomena as important for effective mathematics teaching: enough thinking time individually for each pupil, connecting geometry to the world around them, a safe classroom environment that can easily be compromised by inappropriate work with mistakes, and a good social climate using peer learning.

c) Thinking about teaching. Detailed reflection on teaching naturally led students to reflect on their future educational style. Their practice showed them that teaching in the spirit of genetic constructivism (Kvasz, 2022) is effective for the pupils and a permanent challenge for themselves to further their self-development in both the didactics of mathematics and pedagogy. For example, they also wanted to promote pupils' communication with each other in mathematics and to shift their role as a teacher to that of a guide who shapes the important pupil's relationship with mathematics. Sometimes students reflected on their teaching by a role model in the person of a teacher they have met. One student paid tribute to a teacher whose style of teaching mathematics she liked so much that she began to rethink her pedagogical beliefs and view of the constructivism she had previously rejected.

Category 2. Content knowledge. Indicators of this category found in the portfolios in the ongoing self-reflections:

a) Reflection of materials. The students expressed satisfaction with the materials they had developed during their mathematics courses, and their usefulness had been demonstrated during the practice sessions. They perceived them as a good starting point for further enrichment, re-reflection and possibly alteration.

b) Honesty in preparation. Many students mentioned their preparation for mathematics teaching as a key part of their study. One student who had already worked as a teacher, said that making preparation was important to her for both seminars and practice, as well as now in her daily work with children. Students often mentioned the role of their own mistakes for their growth. The honesty of preparation is also evident in the students' reflection on the interconnectedness of the learning goals, their forms, and the needs of the pupils, and their awareness of preparing for teaching as an ongoing, complex process.

c) Portfolio as a way. Students saw their entire portfolio to guide them on their journey to becoming a quality mathematics teacher. It mirrored their development and helped them to set further development: "The portfolio in math didactics is a propulsion for me on my way to my future profession as a teacher, and it documents my way of thinking, which is visibly changing, and provides me with a huge reservoir of annotated materials immediately usable for teaching".

d) The usefulness of the theory. Students perceived theory as a tool that helped them to understand didactic approaches, enriched their practice and reflected on their own pedagogical practice. For example, they learned to work with mistakes as a learning opportunity, not as a failure. They realized that theory allowed them to better diagnose pupils' preconceptions. Many students saw their portfolio and theoretical study as a reservoir of ideas and a tool for future professional growth, as discussed above for the phenomenon of ongoing self-reflection. They saw the usefulness of theory in being able to translate it into practice and developed it further based on experience. Theoretical knowledge provided them with inspiration and concrete materials for future lessons, specifically they listed materials for teaching geometry or negative numbers.

Category 3. Attitudes, emotions, anxiety. This category refers to the following indicators:

a) Positive attitude towards oneself and self-esteem. Several students mentioned their self-esteem, either directly or indirectly. In a message to her "future Self ", one student addressed self-doubt and valuing her abilities. She knew it is important to stand up for herself, even though it might be difficult. Another student mentioned her professional development, which had been changing throughout her studies. It was only at the end of her studies that she became clear about what she had learned and how she wanted to teach.

b) Coping with stress. Several students mentioned how they coped with the stress of the future teaching profession. One student wrote that at first she was afraid to teach mathematics because she was weak in it herself, afraid that she would not be able to cope with a task. However, she concluded that through her honest preparation she was able to prepare herself for the mathematical content of the curriculum and that she was also able to lead her pupils on a path of discovery. This relieved her of stress and even made her happy to work with pupils in this way.

c) Collegial relations. Students often learnt from their practical experience that they would not always meet colleagues who would get along with them. They realized the importance of having good collegial relationships in the workplace. They perceived that it was beneficial if their knowledge had a solid basis in reasoning. This would also help them avoid stress during classroom observations, which they perceive as an opportunity for professional discussions at school.

d) Satisfaction and enthusiasm for their career choice. Several students expressed their relationship with their future profession in strongly emotional terms: "I could not have chosen better" and described their desire to observe and influence children's development: "I am looking forward to teaching geometry to my class as I really enjoy it. I would like to pass on my enthusiasm to others."

Discussion and conclusion

We answered question RQ1. *How do students, future teachers, think about themselves as beginning mathematics teachers at the end of their studies?* in the Findings section. In line with the research by Charalambous et al. (2008), we conclude that students' self-efficacy is most influenced by thoroughly reflected practice, where they receive support from a mentor in connecting theoretical knowledge of mathematics with didactic principles of teaching mathematics (Bates et al., 2011; Newton et al., 2012). Another strong phenomenon mentioned is positive attitudes towards mathematics, including positive emotions, as described by Charalambous et al. (2008) and Bleicher (2004). These findings provide us with an answer to RQ2. *What options do we have for strengthening the quality of self-efficacy for mathematics teaching during mathematics and mathematics education courses in undergraduate training?* It is necessary to use all available means to cultivate positive attitudes towards mathematics in students (Charalambous et al., 2008; Bleicher, 2004) and build their confidence in their ability to transform good didactical content knowledge into effective teaching of pupils (Philipp, 2007). We formulate several recommendations, which we continuously incorporate into our teaching:

- to combine as much content-based education with teaching practice as possible. We are succeeding in this, for example, by focusing the new subject Child and Mathematics on guiding students in mathematics clubs and on formative assessment in mathematics.

- to offer optional content of seminars,

- to offer students our support at the beginning of their internship. We offer graduates of the primary school teaching program so-called Cafés, where we share good practices and address their current problems.

- to provide ongoing support to all teachers with whom we collaborate in the context of student teaching practice.

We are aware that by analyzing final portfolios, we only get a picture of one group of students who choose the more demanding path of completing their studies with a state final exam. These students are usually identified as more capable ones. As a result, we do not obtain a picture of the self-efficacy of the entire range of students, future primary school teachers. To obtain an overall picture of certain sub-items, such as the relationship to mathematics and incentives that change the relationship to mathematics, math anxiety among beginning students, and the causes of change, we use other tools – questionnaires and essays as part of long-term monitoring (Jirotková et al., 2025). However, even a smaller group of students and qualitative research through portfolio analysis provide us with feedback, which is valuable since the students' responses in the portfolios are completely spontaneous.

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Looking back after the first year of teaching: What changed in teachers' perspectives on analogies

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Within the study reported in this paper, we return to our more than year-old research on teachers' awareness of the existence of analogies in the conceptual structure of mathematics and biology. An entire school year after completing the research, we contacted again the secondary school teachers of mathematics and biology who, as preservice teachers at the very end of their professional training, had participated in that research. We asked them to reflect on their perspectives on analogies and their use in teaching practice, and now we investigate how the very first year of full-time teaching influenced these perspectives. The study has a qualitative design, with the system of qualitative categories from the previous research serving as a framework for theoretical coding. In the results section, we show how teachers' perception of analogies evolved after their first school year.

Keywords: Beginning teachers, mathematics education, biology education, conceptual structure, analogies.

Introduction

Efforts to integrate science, technology, engineering, and mathematics education (STEM) have received considerable attention worldwide in past decades. This is a natural tendency, as these fields of human activity have much in common. Understanding the relationships between them can facilitate mutually effective knowledge transfers and, as a consequence, higher quality educational outcomes (English, 2017; Holmlund et al., 2018). What remains to be resolved is how to prepare teachers for such an integration, especially in countries where STEM education does not have a tradition and individual STEM subjects are taught separately in schools as well as in teacher trainings. As the Czech Republic belongs to these countries, we have started investigating this issue in more detail.

Going into the very core of STEM (Lyons, 2020), we directed our attention to the integration of mathematics and biology. And, as secondary school teachers in our country are specialists in two selected school subjects, we drew on the opportunity to work with teacher trainees specialising specifically in the two core STEM subjects. In a way that was a bit atypical for STEM related research, we decided not to focus on presenting or studying authentic situations suitable for the integration. Instead, we considered higher levels of conceptual understanding (de Jong & Ferguson-Hessler, 1996; Star & Stylianides, 2013) and focused on integration informed and mediated by analogies in the conceptual structure of mathematics and biology. Our effort resulted in a qualitative empirical study (Samková et al., 2025) conducted with teacher trainees at the end of their five-year training, in which the trainees solved several analogical tasks, commented on analogies they noticed in them, and gave suggestions on how such analogies could be used in school practice.

As teacher trainees in our country do not have much teaching experience, the suggestions on how analogies could be used in school practice were mainly based on theories or observational experience.

Therefore, we were interested in how the suggestions would stand up to long-term full-time teaching practice, whether the activities conducted during teacher training before the full-time practice could enable self-learning of the teachers during their teaching practice, that is, whether they can activate teaching as a means of teacher learning (Charalambous & Delaney, 2020). This interest led to a follow-up study which is presented in this paper. An entire school year after the previous study (Samková et al., 2025), we contacted again study participants who had been training for secondary school level. We found out which ones had been teaching full-time mathematics and biology, and asked them to reflect on their perspectives on analogies and on using analogies in teaching practice. Now we investigate how the very first year of full-time teaching influenced the perspectives, addressing the following research questions:

RQs: How do mathematics and biology teachers perceive analogies after a year of full-time practice? How do they report on implementing analogies in their own teaching?

The study has a qualitative design, with the system of qualitative categories from the previous study serving as a framework for theoretical coding.

Theoretical background

Analogies in the conceptual structure of mathematics and biology

The inherent integrity of mathematical knowledge resides in understanding mathematical ideas, objects, and their properties within an organised, meaningful system referred to as a conceptual structure of mathematics (Richland et al., 2012). Such integration of mathematical concepts also consists in their hierarchical classification, in which partial concepts form certain subsets of a set of more general concepts based on their properties (de Villiers, 1994). For example, in arithmetic, natural numbers are understood as a special case of rational numbers, or in geometry, a square is a special case of a rectangle, so the organised system of these planar shapes is the hierarchy of quadrilaterals (Fujita & Jones, 2007).

In biology, organisms are classified in a similar way. For example, a cat is a certain type of carnivorans, or a dandelion is a special case of eudicots. Living organisms form groups called taxa that are sorted according to their properties in taxonomic hierarchy (Brower & Schuh, 2021). Such system connects and integrates biological concepts, thus becoming the basis of the conceptual structure of biology (Momsen et al., 2022).

As the structure of mathematical and biological concepts is similar, it means that there are certain parallels between the relationships of individual concepts in both fields, and those can be perceived as analogies in the sense of cognitive psychology (Gentner & Smith, 2013). From the perspective of schooling, analogies constitute an appealing tool. They enable learners to transfer knowledge between fields with a similar conceptual structure and thus contribute to effective integration of the disciplines (Gray & Holyoak, 2021).

System of categories for noticing analogies and for their possible use in school practice

In the study (Samková et al., 2025), several vignettes with dialogue bubbles and related questions were used to represent the analogies. Hypothetical children in the vignettes commented on analogical

situations from mathematics and biology, and their statements were also analogous to each other across the vignettes. The vignettes were placed in data collection instrument (on pages 1 to 4; see Figure 1) and accompanied by the following questions.

Q1: Do you see analogy between the cartoons on pages 1 to 4? Which one?
Q2: How could such analogies be used in school practice?

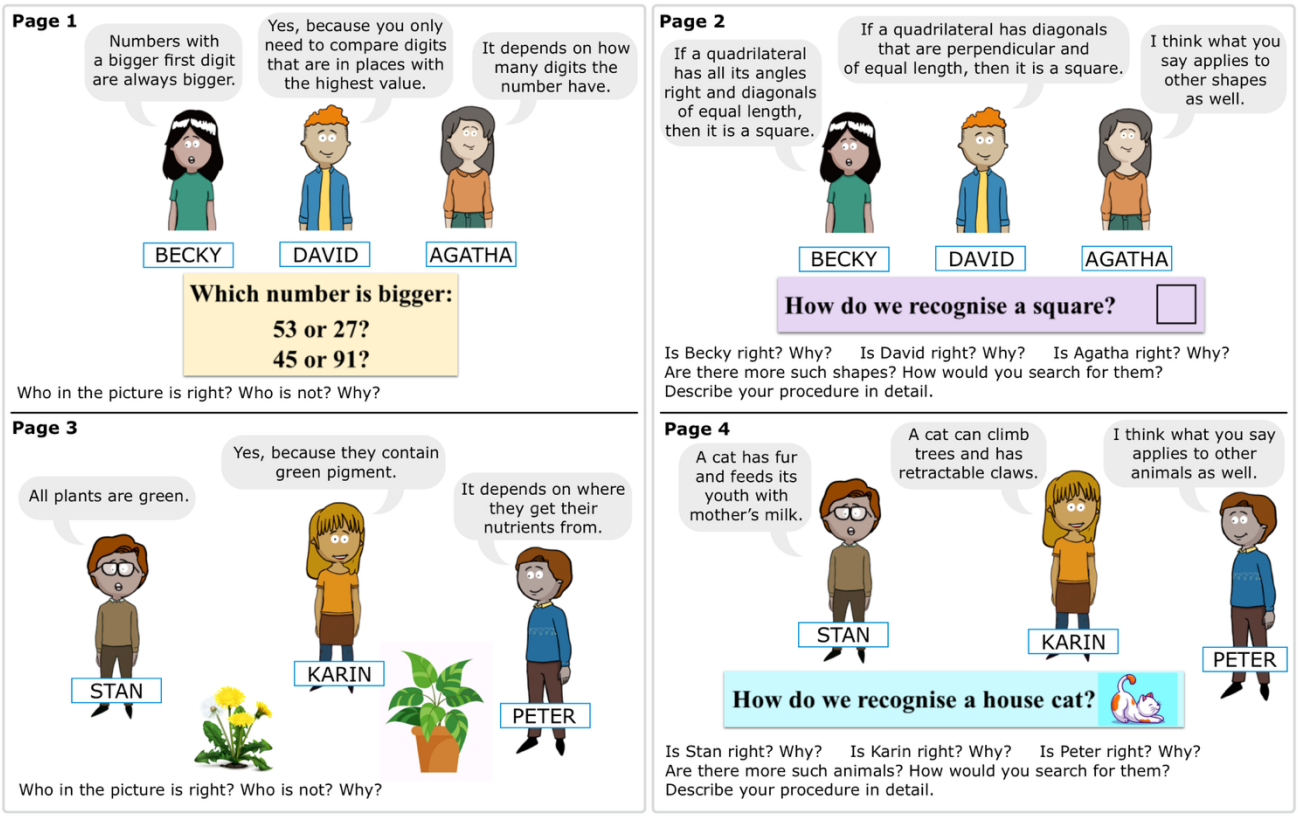


Figure 1: Vignettes on pages 1 to 4 of the data collection instrument in (Samková et al., 2025); graphical elements: (coReflect@maths, 2022; Freepik, 2024)

The analysis of the responses to these questions resulted in establishing categories characterising noticing analogies per se (questions Q1) and suggestions for the use of analogies in school practice (question Q2). As categories for noticing analogies, we had obtained A1 – *surface similarities* (noticed similarities were not related to any mathematical/biology concepts nor their structure), A2 – *conceptual similarities description* (noticed similarities were related to concepts and their structure), and A3 – *conceptual similarities principles* (noticed similarities were related to concepts and the reasoning was based on a general concept or principle). As categories for the use of analogies in school practice, we had obtained S1 – *general pedagogy* (reasoning only in general pedagogical terms), S2 – *local use* (perceiving analogies within a specific discipline), S3 – *interdisciplinary use* (adopting analogies across different disciplines), and S4 – *transdisciplinary use* (using analogies across different disciplines with overlap into real-world problems).

This order of categories expresses an increasing intensity of connections between the subject contents of the two core STEM disciplines (Lyons, 2020) as well as an increasing quality of conceptual understanding (de Jong & Ferguson-Hessler, 1996; Star & Stylianides, 2013). In parallel, the

categories form a theoretical underpinning situated at the intersection of conceptual structures of mathematics and biology. This connects the domains through analogies and so contributes to the debate on teacher competence – an issue pointed out as key but unresolved in the light of STEM teaching and learning (Skilling & English, 2025). We will use these categories as a reference framework when coding new data; an overview of the framework is shown in Figure 2.

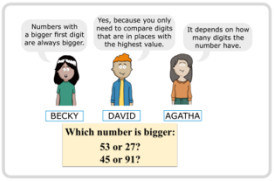
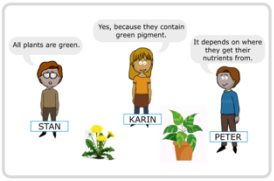
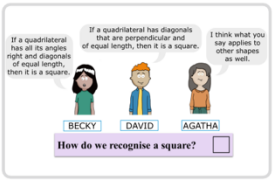
Cartoons	Questions	Focus of the categories	Structure of the categories	
 	Q1: Do you see analogy between the cartoons on pages 1 to 4? Which one?	Noticing analogies and reasoning about them	surface similarities	A1
			conceptual similarities description	A2
			conceptual similarities principles	A3
	Q2: How could such analogies be used in school practice?	Suggestions for school practice	general pedagogy	S1
			local use	S2
			interdisciplinary use	S3
			transdisciplinary use	S4

Figure 2: An overview of the reference framework taken over from the results of (Samková et al., 2025); graphical elements: (coReflect@maths, 2022; Freepik, 2024)

Methods

Participants

Among the 10 secondary school level trainees in the previous study (Samková et al., 2025), we found three who graduated from university at the end of the academic year 2023/24, spent the entire school year 2024/25 teaching mathematics and biology in secondary school, and were willing to participate in the follow-up study. All of them were teaching as full-time teachers, with approximately 10 hours a week of mathematics teaching and a similar amount of biology teaching. Other participants from the former inquiry (Samková et al., 2025) did not take part in the current survey because they had not yet completed their university studies, were not teaching mathematics or biology in 2024/25, or apologized for not participating in the survey.

In (Samková et al., 2025), these three participants had code names S06, S07, and S09. However, we will rename them T06, T07, and T09, to avoid potential confusion between labels for participants and labels for 'suggestion' code categories.

Data collection and data analysis

After contacting these three participants, we sent them back the worksheets they had completed more than a year ago, asked them to study the materials again, recall possible reasoning behind their answers at the time, and respond in writing to the three following questions:

Q3: Do you see any new analogies between the cartoons on pages 1 to 4?
Q4: Do you encounter analogies in your teaching practice? Give specific examples.
Q5: After a year of your teaching, what is your opinion on the use of analogies in school practice?

As data materials for the follow-up study, we used both the responses to questions Q1, Q2 collected within the previous study, and the current responses to questions **Q3**, **Q4**, **Q5**. To distinguish clearly between the former and the latter questions, the latter ones are in **bold**.

We analysed these data materials qualitatively and applied theoretical coding (Holton, 2007) based on the framework from Figure 2.

Findings

Question **Q3** was linked to the original question Q1, and therefore the answers to it were analysed from the perspective of noticing analogies and reasoning about them. Similarly, questions **Q4** and **Q5** are a continuation of question Q2, so the answers to them were distinguished within the context of suggestions for school practice.

The assignment of categories to individual answers is summarized in Table 1. To distinguish clearly between the categories assigned already in (Samková et al., 2025) and those assigned now, the latter ones are in **bold**.

Table 1: Categories assigned to individual participants for each question

	Noticing analogies and reasoning about them		Suggestions for school practice		
	Q1	**Q3**	Q2	**Q4**	**Q5**
T06	A3	**A3**	S3	**S3**	**S3**
T07	A2	**A3**	S1	**S3**	**blank**
T09	A1	**A1**	S1	**S4**	**S4**

We will illustrate these findings by data excerpts below; each excerpt is labelled with 'Participant (question, assigned category' info).

As for noticing analogies and reasoning about them, participants T06 and T09 got the responses to questions Q1 and **Q3** equally categorised before their full-time practice and after it, as illustrated by the example of the participant T06:

T06 (Q1, A3): 1, 2 => gradually developing and refining statements
2, 4 => moving from the general to the more specific.
T06 (**Q3**, **A3**): pp. 1, 3, 4 – no
p. 2 – There may be a problem with the classification of algae in Standa's case => Especially in elementary and lower secondary schools, algae may be included in plants, although they are not classified as such in the new system.

In contrast to that, participant T07 now perceives analogies in a new way. The response to question **Q3** was assigned a higher category compared to the initial reaction to question Q1:

T07 (Q1, A2): It is always a matter of searching for and sorting the properties of individual objects.
T07 (**Q3**, **A3**): A common property or properties are always searched for that would describe the given object as exactly as possible. It is necessary to always realize that the given property must be as exhaustive as possible to achieve a complete and, if possible, indisputable definition of the given object.

As for suggestions for the use of analogies in school practice, the participant T06 got the responses to questions Q2, **Q4**, and **Q5** all categorized identically as interdisciplinary use of analogies (category S3/**S3**). However, the other two participants' suggestions given after the full-time practice were more profound than before the practice, they both obtained higher categories.

T07 (Q2, S1): Such tasks encourage students to think critically and sort information.

T07 (**Q4**, **S3**): I often encounter analogies, for example, I use analogies in mathematics when explaining the difference between a number and a digit. I compare it to a letter and a word. I compare it to a letter and a word. Digits are for us letters from which a word, i.e., a number, can be formed. As in the Czech language, there may be situations where a digit itself is a number, just as a single letter can be a word.

Data collected from respondent T09 after full-time practice also contain suggestive ideas for the use of analogies in school practice and show examples from various areas of school education and connections to the real-world situations. This is something new compared to the respondent's answer to the former question Q2:

T09 (Q2, S1): When searching for multiple different procedures that lead to the same solution. This allows each student to learn the method of solution that is closest to them and that they find easiest.

T09 (**Q5**, **S4**): It always gives students confidence when they can rely on something they already know when learning new content. At the same time, every diagram, graph, or picture I draw for students to help them better understand the content is an analogy to real phenomena.

It is excellent when analogies can be found between mathematics and other subjects or practical applications. Mathematics is a big bogeyman for many children, and when they see that they can use something they have learned in mathematics in an analogous way outside of mathematics itself, learning mathematics often begins to make much more sense to them.

I also try to start new topics in biology by looking for the same or similar characteristics (analogies) and only then do we deduce how the given groups differ, for example, in what ways they are more perfect.

Discussion and conclusion

We presented the findings of our study in connection with the findings of our previous work (Samková et al., 2025), and therefore our response to the current research question also reflects our earlier conclusions. We observed how mathematics and biology teachers perceive analogies after a year of their full-time practice. We found that two of our participants (T06, T09) considered them in practically the same way as they did at the end of their teacher training, and one in a deeper way (T07). However, their experience with analogies and suggestions for use of analogies in teaching practice indicate a significant shift in the case of participants T07 and T09: an enhancement from only general pedagogical suggestions to suggestions that can be considered an interdisciplinary (T07) or even transdisciplinary (T09) character. In that sense, the full-time teaching practice of our participants confirmed its ability to function as a cornerstone of teacher learning (Charalambous & Delaney, 2020) when properly supported by activities during initial teacher training.

We are aware that our findings are limited by the small amount of data obtained, which is due to the involvement of only a small group of participants as explained above. However, despite this constraint, our qualitative analysis corresponds with the finding that analogies can play a significant role in the integration of disciplines and contribute to interdisciplinary and transdisciplinary level of

such integration (Gentner & Smith, 2013; Gray & Holyoak, 2021). We conclude this based on the reflections on the use of analogies in teaching, in which participants linked the content of the core subjects of STEM: mathematics and biology (Lyons, 2020), thus demonstrating a deeper level of integration of STEM disciplines as such and regarding school practice (English, 2017). The identified shift in participants' perception of the relevance of analogies for education can be understood as an enhancement of teachers' competencies. However, our data does not explicitly indicate the reason for such a change. We can only speculate about it, for example, understanding it as an indirect consequence of the participants' one-year experience with teaching mathematics and biology.

We may say that our study does not provide explicit suggestions for increasing teacher competence in STEM teaching and learning, as called for by Skilling and English (2025). However, we bring to the discussion a specific way of linking the hierarchical understanding of mathematical and biological objects (Brower & Schuh, 2021; de Villiers, 1994; Fujita & Jones, 2007), and thus the analogies between the conceptual structure of both fields (Momsen et al., 2022; Richland et al., 2012). At the same time, we implement this link in the form of educational vignettes and accompanying tasks that can be used in university courses for preservice teachers or in teacher training programs. Specifically, educational vignettes and supplementary questions used to collect data for this study can be applied. They can serve as a basis for discussion in the training and, consequently, help develop attendees' understanding. Such extension of teacher training courses can also foster awareness of analogies between educational contents, thereby encouraging the development of conceptual understanding among teachers (de Jong & Ferguson-Hessler, 1996; Star & Stylianides, 2013) and, consequently, among their students. Our study took place within a pedagogical seminar at the end of the initial teacher education program. However, similar activities can also be implemented within content courses at the beginning of the program (as, e.g., in Samková et al., 2021).

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Investigating mathematics teachers' evaluations of mathematical activities: An algebra task example

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This study aims to enhance mathematics teachers' activity evaluation skills through a qualitative case study within a TUBITAK 4005 project. 12 teachers employed the rubric of Bozkurt et al. (2023) to evaluate Driscoll's (1999) Crossing the River classroom video under the dimensions of "Teaching Process" and "Mathematical Potential," providing written justifications. Two experts independently scored the video to establish reference points. Findings showed that more than half of the teachers correctly assessed "Depth, Complexity, Drawing Conclusions, Student Engagement, and Effective Use of Materials," while accuracy was lower for other components. Misinterpretations often stemmed from incomplete observation, misunderstanding of rubric criteria, or confusion between dimensions. The study emphasizes the importance of clear and systematic training on rubric use and strong observational skills in teacher education.

Keywords: Teacher professional development, mathematics education, evaluation of task, teacher evaluation skills.

Introduction

Mathematical activities are tools that enable students to explore concepts, develop problem-solving skills, and engage actively in learning processes (Hiebert et al., 2003). Activities play a critical role in guiding teachers' classroom practices and shaping students' learning experiences. Therefore, the ability of mathematics teachers to accurately evaluate classroom activities is an important factor in enhancing instructional quality (Bardy, 2024). Although there are studies in the literature examining teachers' activity evaluation processes, most of them have focused on a limited number of observation dimensions or general performance criteria (Bozkurt et al., 2023). In particular, studies focusing on the evaluation of components such as mathematical depth, complexity, student engagement, and drawing conclusions are quite limited. In this study, the task evaluation skills of teachers working at the middle school level will be examined. The study identifies teachers' accurate and inaccurate evaluations and analyses the reasons behind incorrect assessments. In this context, the research questions of the study are as follows:

1. To what extent are teachers able to accurately evaluate the components of a task?
2. For what reasons do teachers incorrectly evaluate certain components of a task?

Evaluation of task

The evaluation of mathematical tasks is regarded as a critical instructional practice that significantly shapes both teaching quality and student learning outcomes. Task evaluation requires teachers to analyse the cognitive demands of tasks and their alignment with instructional goals, thereby playing

a vital role in enhancing the effectiveness of mathematics teaching (Bardy, 2024). Previous studies have highlighted that teachers' competence in task evaluation is closely related to their Mathematical Knowledge for Teaching (MKT). Teachers with stronger MKT are better able to judge the appropriateness and complexity of tasks, which in turn supports more effective instructional decisions. To support teachers in this process, several frameworks have been proposed in the literature. The Mathematical Tasks Framework (MTF) and the Levels of Cognitive Demand provide structured approaches for analysing tasks in terms of their potential to foster higher-order thinking and deep mathematical understanding (Markle & McGarvey, 2025). Despite the existence of such frameworks, studies show that teachers continue to face challenges in evaluating tasks accurately. These challenges often arise due to limited time, insufficient training, or differing interpretations of task features.

Given these findings, scholars argue that teacher education programs should place a stronger emphasis on the development of task evaluation skills. Embedding structured frameworks such as the MTF into pre-service and in-service teacher education can help teachers make more informed choices regarding the selection, design, and assessment of mathematical tasks, thereby contributing to more effective mathematics instruction (Markle & McGarvey, 2025).

Methods

This study employed a qualitative case study design (Creswell, 2014) to examine mathematics teachers' skills in evaluating a specific algebra task within a clearly defined setting.

Participant and context

The participants comprised 12 in-service middle school mathematics teachers who were enrolled in a professional development programme funded by the TUBITAK 4005 project (Project No. 224B621). All teachers voluntarily and possessed at least three years of teaching experience. The project aims to support the development of teachers' activity-evaluation skills and instructional decision-making. Thus, teachers' participation in this programme provided the specific contextual foundation for the present study.

Data collection

Data were collected using the mathematical activity evaluation rubric developed by Bozkurt et al. (2022). The rubric evaluates activity implementation across 11 sub-dimensions (see Table 1). To apply this rubric, a specific mathematical task, the Crossing the River problem (Driscoll, 1999) was utilized. The task requires determining the minimum number of trips to transport adults and children across a river under specific constraints (e.g., boat capacity and driver rules). Teachers were asked to determine the minimum number of trips for various scenarios and to generalize an algebraic expression for y adults and 2 children.

A Turkish adaptation of the problem was prepared by field experts and a video recording of a teacher's classroom implementation of this adapted activity was used as the stimulus for data collection. After viewing the video, the 12 participating teachers evaluated the implementation based on each sub-dimension of the rubric and provided written justifications for their ratings. This

procedure was designed to reveal teachers' perceptions, understanding of the criteria and approaches to evaluating mathematical activity implementations.

Data analyses and reliability

The collected data were analysed using both quantitative and qualitative methods.

Quantitative Analysis: Teachers' ratings for each sub-dimension were coded according to the four levels defined in the rubric (0 = very low, 1 = low, 2 = medium, 3 = high). Each sub-dimension corresponded directly to one component in the framework presented in Table 1. Therefore, the numerical scores served as quantitative indicators of how strongly the evidence in the task aligned with the operational definitions of the components. These ratings were then compared with the reference codes independently established by the field experts. A rating was classified as correct when the teacher's score matched the expert's assigned level for the same sub-dimension. Based on this criterion, correct and incorrect evaluations were identified, and the number, percentages, and distributions of correct responses for each sub-dimension were calculated. Furthermore, tendencies in how teachers evaluated the different rubric components were examined through cross-dimensional comparisons. In this step average scores, distributions and correctness rates across components were compared to identify which aspects teachers evaluated more consistently with expert judgments and which components presented greater difficulty.

Table 1: Definitions for evaluation of activity implementation components

Components	Definitions
Attracting student attention	Entails guiding their engagement, sparking genuine interest and highlighting the activity's relevance to enhance motivation and commitment
The organization of physical conditions	Refers to the strategic arrangement of classroom furniture and resources to support optimal student engagement and activity implementation
Working in accordance with instructions	Refers to students' consistent alignment with and adherence to provided guidelines during an activity
Effective use of materials	Ensures that materials used in activities are effective tools for achieving mathematical outcomes, simultaneously facilitating task execution and enhancing understanding
Segmentation and time management	Refers to the structured organization and optimal use of time during various stages of an activity.
Classroom management	Involves the teacher's active monitoring of student behaviors to ensure alignment with tasks and promptly intervening with appropriate strategies when disruptions or off-task behaviors arise.
Student engagement	Encapsulates the depth and breadth of students' active involvement in activities.

Drawing conclusions	Emphasizes the clear linkage between students' works and the intended mathematical outcomes, while ensuring these outcomes are understood and reflected upon by the students.
Depth	Signifies the extent of understanding associated with concepts, principles, and generalizations underlying mathematical rules.
Complexity	Refers to the understanding of relationships formed across time, between different perspectives, and among various disciplines.
Mathematical focus	Refers to the clear and central emphasis of an activity on specific mathematical concepts or skills.

Based on the dimensions and operational definitions presented in Table 1, teachers' judgments were quantified using the four-level rating scale ranging from 0 to 3. In this scale, 0 indicates that the criterion is not demonstrated in the task, 1 reflects limited or superficial evidence of the component, 2 denotes partial but meaningful alignment with the criterion and 3 represents a clear and consistent demonstration of the component in accordance with the rubric. These categories provided a standardized framework for analysing the degree to which teachers' evaluations aligned with expert reference ratings across dimensions and for interpreting performance differences between components.

Qualitative Analysis: In addition to assigning scores, teachers were asked to provide written justifications for their ratings. These explanations were analysed through descriptive analysis based on the rubric criteria (e.g., attracting student attention, segmentation and time management, student engagement, depth, mathematical focus). The initial examination of incorrect ratings revealed three main sources of mismatch with expert judgments: (1) not evaluating the instructional process holistically, (2) limited understanding of the scope of the specific criteria, (3) difficulty distinguishing between conceptually close components. Following the descriptive stage, teachers' written justifications were subjected to thematic analysis. This analysis revealed recurring patterns such as focusing on surface-level behavioural cues instead of identifying the intended mathematical focus, interpreting student engagement as mere participation rather than depth of involvement and conceptualising classroom management primarily as discipline rather than instructional regulation.

To ensure reliability, two independent researchers evaluated the classroom implementation using the same rubric and discrepancies were resolved through consensus-building discussions. Validity was supported by use of a previously validated rubric grounded in established theoretical frameworks. Requiring teachers to provide written justifications and comparing these with expert evaluations further strengthened the construct validity of the study and enhanced the reliability of the findings.

Results and discussion

In this section, the evaluations of the mathematical tasks conducted by two subject-matter experts are presented first to establish a reference point. Subsequently, the findings are organized to address the

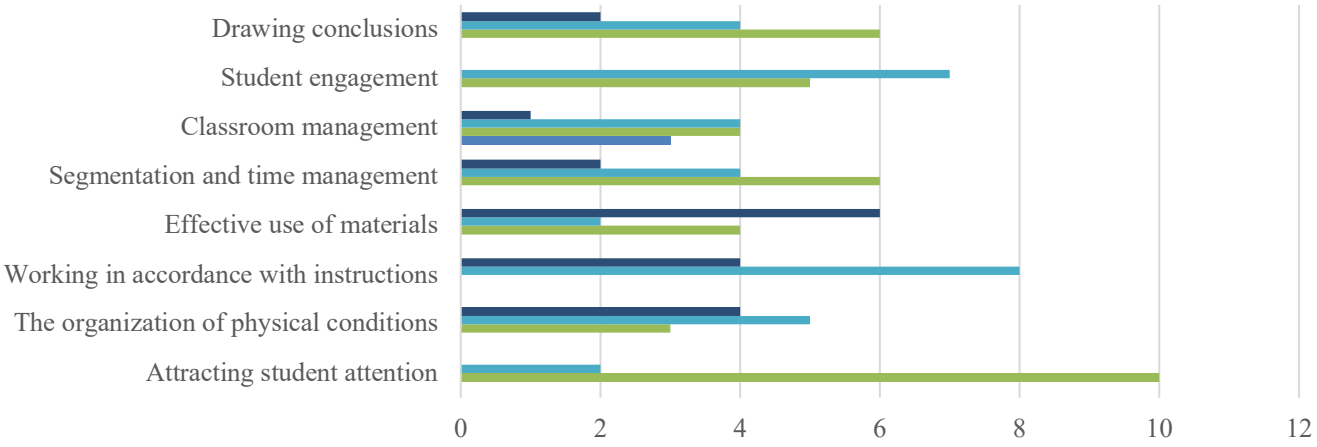
two research questions: first, examining the accuracy of teachers' evaluations (RQ1) and second exploring the underlying reasons for their incorrect assessments (RQ2).

Reference evaluations provided by subject-matter experts

The expert evaluations indicate varying levels of effectiveness across different dimensions of the mathematical activity. The organization of physical conditions, working in accordance with instructions, effective use of materials, and depth were rated as high (3), indicating that the activity was implemented smoothly in these areas. Attracting student attention, segmentation and time management, student engagement, complexity, and mathematical focus were assessed as medium (2); while some students participated actively, the implementation was not consistently effective across all cases. The drawing conclusions dimension was rated low (1), suggesting that students' ability to make mathematical inferences required additional support. Classroom management received a very low (0) rating, highlighting significant shortcomings in guiding student behaviour and maintaining classroom order.

Teachers' assessment of the activity: Correct and incorrect ratings

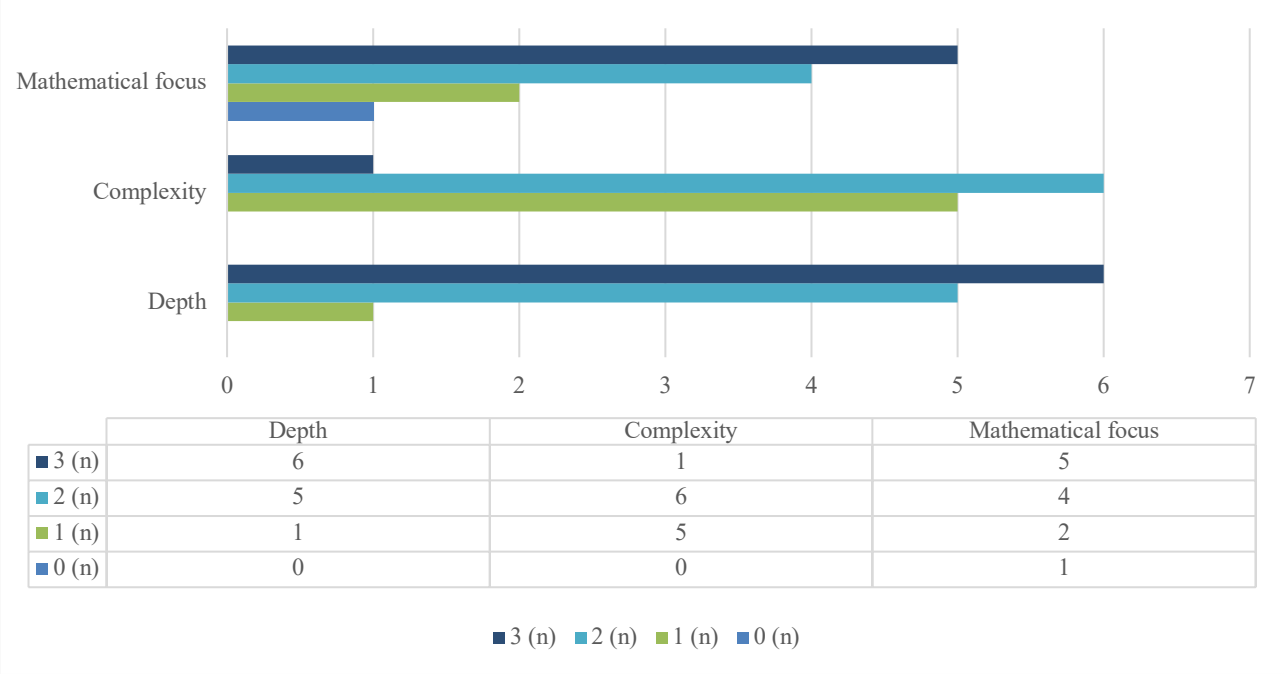
To address the first research question, the study examined the extent to which teachers accurately evaluated the task components. The findings indicate that teachers were able to make highly accurate evaluations in cognitively oriented dimensions, while experiencing difficulties in pedagogical-practical ones. The teachers' evaluations are detailed in Figures 1 and 2.



	Attracting student attention	The organization of physical conditions	Working in accordance with instructions	Effective use of materials	Segmentation and time management	Classroom management	Student engagement	Drawing conclusions
3 (n)	0	4	4	6	2	1	0	2
2 (n)	2	5	8	2	4	4	7	4
1 (n)	10	3	0	4	6	4	5	6
0 (n)	0	0	0	0	0	3	0	0

Figure 1: Results of teaching process evaluation

Teachers showed high accuracy in drawing conclusions (Expert Rating:1), demonstrating strong skills in correctly identifying that the activity needed improvement in making inferences. Sherin and van Es (2009) identified making sense of students' ideas as central to professional noticing, and these findings support that view. Teachers also made accurate evaluations of student engagement (Expert Rating:2), indicating their ability to recognize active participation levels. Jacobs et al. (2010) argued that attention to students' ideas enhances noticing, though sustaining engagement within classroom management remains a challenge. In effective use of materials (Expert Rating:3), over half of the teachers gave accurate evaluations, showing awareness of linking materials with instruction. However, consistent with Blömeke et al. (2015) such evaluations became more complex for teachers when influenced by other classroom factors. By contrast, in attracting student attention (Expert Rating:2), organization of physical conditions (Expert Rating:3), working in accordance with instructions (Expert Rating:3), and classroom management (Expert Rating:0), more than half of the teachers were inaccurate. This indicates difficulties in judging pedagogical–practical aspects. For instance, despite the expert rating of 0 for classroom management, many teachers failed to notice the significant shortcomings in maintaining order.



	Depth	Complexity	Mathematical focus
3 (n)	6	1	5
2 (n)	5	6	4
1 (n)	1	5	2
0 (n)	0	0	1

Figure 2: Results of mathematical potential evaluation

More than 50% of teachers made accurate evaluations in terms of depth (Expert Rating:3), showing awareness of analysing mathematical content. Similarly, Jacobs et al. (2010) reported that teachers are relatively successful in noticing students' mathematical thinking. In complexity (Expert Rating:2), over half of the teachers also made accurate evaluations, reflecting their ability to grasp multi-step problems. Overall, findings indicate that teachers are more successful in cognitive dimensions (depth, complexity, drawing conclusions, student engagement, materials use) but face challenges in practical ones.

Reasons for teachers' misjudgments in evaluating certain task components

Addressing the second research question, this section analyses why teachers evaluated certain component incorrectly. Figure 3 and 4 present specific examples of teachers' written justifications, labelled as correct (r) and incorrect (w) based on their alignment with expert codes.

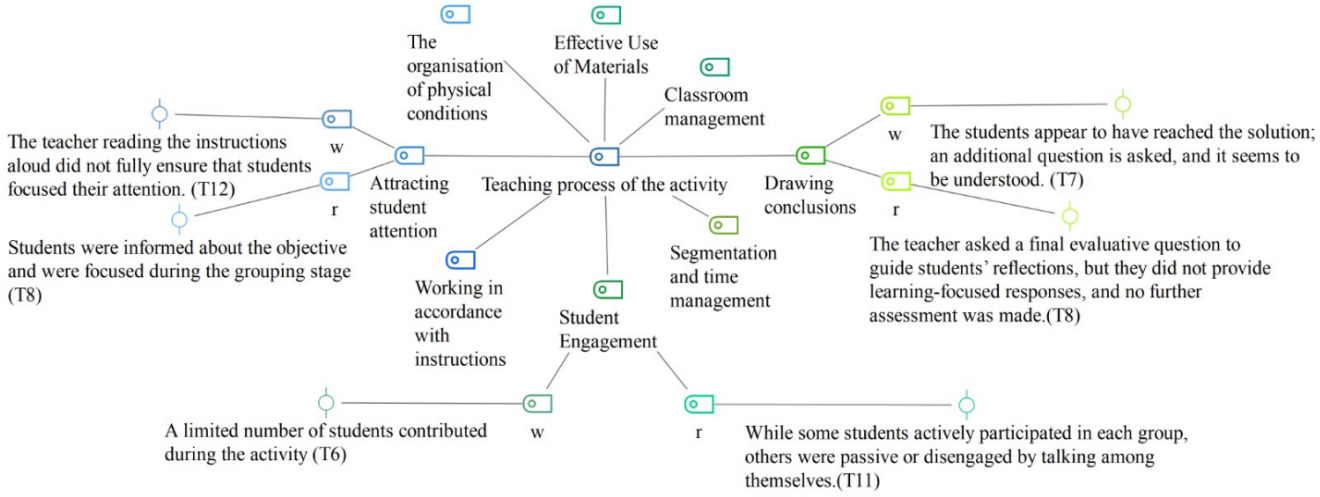


Figure 3: Examples of teachers' justifications regarding the teaching process of the activity

As illustrated in Figure 3, incorrect evaluations often stemmed from a lack of holistic observation. For example, T12 focused solely on the teacher's action ("reading instructions aloud") rather than the students' actual reaction, whereas correct evaluations (T8, T11) aligned observations with evidence. These patterns reflect what Sherin and van Es (2009) describe as selective attention, where teachers prioritize salient behaviours without considering their instructional relevance. Similarly, Jacobs et al (2010) found that teachers often confuse behavioural cues with learning-oriented evidence when analytic categories are not internalized. The misjudgements observed here also echo Blömeke et al (2015) view that incomplete interpretation rather than lack of perception leads to inaccurate evaluations. Overall, errors in Figure 3 suggest that teachers' noticing was event focused rather than conceptually grounded.

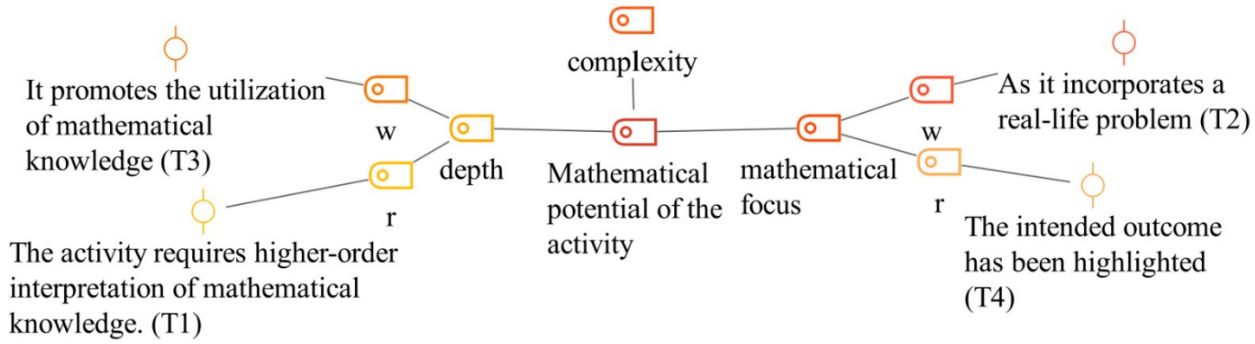


Figure 4: Examples of teachers' justifications regarding the mathematical potential of the activity

As shown in Figure 4, misjudgements in mathematical dimensions often resulted from teachers focusing on surface features (e.g., "real-life context") rather than the intended mathematical idea. While correct evaluations (T4) identified the task's mathematical aim, incorrect ones (T2) relied on

general impressions. This difficulty parallels research showing that teachers often recognize visible task features but struggle to evaluate the mathematical concepts underlying them (Bardy, 2024; Stein & Smith, 1998). It also aligns with Wilkins (2022), who notes that without structured frameworks, teachers may conflate engagement with mathematical demand. Consistent with Jacobs et al. (2010), teachers in this study appeared to struggle with differentiating between closely related mathematical criteria, leading to conceptually imprecise judgments.

Concluding remarks

This study shows that teachers are generally strong in cognitively oriented dimensions, such as depth, complexity, and drawing conclusions, but face challenges in pedagogical–practical areas like classroom management and sustaining student attention. Strengthening teachers' skills in linking content knowledge with effective instructional strategies and classroom practices may enhance both student engagement and mathematical learning outcomes.

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Trained teachers' implementation of a specific way of teaching combinatorics: Factors influencing implementation fidelity

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The paper examines the implementation fidelity of Teaching Combinatorics through Isomorphism (TCI), a constructivist approach designed for Czech secondary schools. TCI builds on carefully sequenced tasks, student autonomy, and the use of isomorphic tasks. Within a design-based research framework, four experienced teachers implemented TCI. Data from lesson observations, recordings, interviews, and reflections revealed two groups: teachers implementing with high fidelity, aligned with constructivist norms and student-centred practices, and those with lower fidelity, constrained by traditional methods and curriculum demands. Key fidelity factors were teachers' prior experience with constructivist teaching, their confidence in TCI, and their collaboration with the researcher. Barriers included time pressure and reliance on formula-based teaching. The study highlights the central role of teacher practice in the effectiveness of design-based interventions.

Keywords: Combinatorics, combinatorial models and operations, isomorphic tasks.

Introduction

This study is part of a dissertation project focused on researching the implementation and effects of a constructivist approach to teaching combinatorics, based on isomorphic tasks (here, TCI), in Czech secondary grammar schools. The teaching concept was developed, drawing on findings from both national and international research. The research was conducted as a design-based study, with the teaching methodology gradually refined over three cycles based on feedback from four cooperating teachers. Through both quantitative and qualitative analyses of various data types, the teaching was evaluated in terms of its impact on students' ability to solve combinatorial tasks and the fidelity of implementation. Results for the former were published elsewhere (Zenkl & Vondrová, 2023).

The focus of this study is on the latter; we explore the fidelity of TCI implementation enacted by four cooperating teachers and the factors influencing such fidelity. In doing so, we aim to fill a research gap in the published design-based intervention studies, which recognise the critical role of teachers but provide limited insight into their actual work and how faithfully they implement interventions.

Theoretical background

Teaching combinatorics

Combinatorics belongs among the topics that present students with problems, often stemming from their over-reliance on formulae (e.g., Batanero et al., 1997; Eizenberg & Zaslavsky, 2004). Research on its teaching is limited but consistent: reliance on procedures without sufficient understanding leads to their uncritical and ill-conceived use (Lockwood et al., 2018; Wasserman & Galarza, 2019). Students often struggle to grasp patterns of combinatorial operations if formulas are introduced too early and then merely practised (Batanero et al., 1997; Lockwood, 2014). To address this, some recommend teaching combinatorics without formulae (Godino et al., 2005) and engaging students in solving combinatorial tasks, which fosters rich mathematical thinking (Lockwood et al., 2018; Uptegrove, 2015). A gradual discovery of patterns across years has also proved effective (Maher et

al., 2010; Uptegrove, 2015). Suggested instructional tools include solving isomorphic tasks, decomposition into sub-problems, and systematic listing strategies (Eizenberg & Zaslavsky, 2004; Godino et al., 2005; Lockwood & Gibson, 2016; Maher et al., 2010; Uptegrove, 2015).

Building on this research and his knowledge of local context, the first author developed Teaching Combinatorics through Isomorphism (TCI), grounded in genetic constructivism (Kvasz & Pilous, 2020). TCI rests on three principles: (1) carefully sequenced tasks supporting various strategies development and discovery and understanding of factorial and binomial coefficients, (2) students' active and collaborative work with autonomy and peer discussion, and (3) the flexible use of a rich repertoire of solving strategies such as systematic listing, isomorphic tasks, visualisation, and decomposition into sub-problems.

Intervention in mathematics education and teachers' role

An important task of mathematics education research is to develop teaching materials and examine their effectiveness in real classrooms. This is typically achieved through intervention studies of various kinds. While randomized controlled trials represent one form of intervention, design-based research offers a more feasible and flexible approach, particularly for doctoral projects. Design-based research aims to create tools and processes that enable teachers, students, textbook authors, and mathematics educators to address practical problems (Plomp, 2010). It generally proceeds through three stages: preliminary research (literature review and theoretical framing), development (design and refinement of the innovation), and assessment (evaluation, reflection, and recommendations for further improvement), often carried out across multiple iterative cycles (Plomp, 2010).

A critical factor in such interventions is the fidelity of implementation—that is, the extent to which an innovation is enacted in accordance with its core principles and the intentions of its authors (Century & Cassata, 2016). The effectiveness of an intervention depends not only on its intrinsic quality but also on the acceptance and enactment by its key actors: teachers and students. Our focus in this study is on the teachers.

Teachers may participate in intervention studies in different ways, ranging from simply enacting lesson plans provided by researchers to contributing directly to the design of instructional materials. Prior work emphasises that teacher–researcher collaboration increases the likelihood of high implementation fidelity and strengthens the applicability of innovations in authentic classrooms (Anderson & Shattuck, 2012). At the same time, even when teachers implement materials designed by others, they inevitably adapt them to fit their local classroom contexts (ibid). For this reason, structured teaching plans and targeted training are often provided to support teachers' adherence to the intended instructional approach.

Research questions

Despite the importance of implementation fidelity, Morina and Vondrová (2025, p. 37) note:

> Ultimately, our literature review revealed that while intervention studies in design-based research recognise the critical role of teachers in applying the new curriculum for effective intervention, they offer limited insight into the actual work performed by these teachers, the fidelity of their implementation and their perception of the implemented way of teaching.

Our aim is to contribute to this matter and explore the question of four teachers' role in a design-based study on the teaching of combinatorics. We answer the following research questions: How did the teachers succeed in implementing TCI in accordance with its core principles? What prerequisites support the successful implementation of TCI, and what barriers hinder it?

Methods

The study is part of a dissertation project that followed a design-based research methodology to integrate a constructivist teaching approach for combinatorics at upper secondary schools in Czechia. Some results were published elsewhere (Zenkl & Vondrová, 2023), in this study we focus on the fidelity of implementation of TCI in the cycles of design-based research.

Participants

The concept of TCI was developed by the first author, who enhanced its classroom implementability by repeatedly piloting it in his own teaching. The intervention was then carried out in three cycles by four cooperating teachers across four classes. Although these teachers were not involved in the initial design of the materials, their feedback and the analysis of data from each cycle informed subsequent refinements. The core principles of TCI, however, remained unchanged throughout.

The participants were four experienced secondary mathematics teachers: Agatha (17 years of experience), Bedrich (18 years), Cecilia (23 years), and Dagmar (8 years). They volunteered to participate after attending the first author's presentations on teaching combinatorics. Their involvement included collecting student data, providing feedback on the materials, and participating in interviews. All four also completed initial training in TCI and its instructional materials.

Data and data analysis

This study draws on selected data from a broader project, specifically classroom observations, audio recordings of lessons, teacher interviews, and teachers' reflective notes on their lessons (Table 1).

Table 1: Number of collected data

	Agatha	Bedrich	Cecilia	Dagmar	Total
Classroom observations	2	2	2	2	8
Recordings of lessons	17	14	13	3	47
Teacher interviews	3	2	3	3	11
Teachers' reflective notes (number of standard pages)	10	8	13	12	43

Studies employ different tools to measure implementation fidelity. For example, Clements et al. (2011) used a Likert-scale questionnaire completed by 106 teachers. Because our study focused on only four teachers, our primary instrument for addressing RQ1 was an observation sheet designed to capture how each lesson aligned with the three core principles of TCI. For each principle, we articulated indicators of compliance as well as evidence of departures.

To answer RQ2, we also drew on the teachers' reflection notes and interviews in which teachers reflected on their lessons. These interviews were used to identify reasons behind their instructional decisions (e.g., omitting prepared tasks, emphasising only one solving strategy) and to elicit their perceptions of students' work and the quality of their understanding of the subject matter.

Data were reviewed iteratively, with particular attention to triangulating evidence across multiple sources. The first author conducted the coding and categorisation, while the second author – the thesis supervisor – reviewed and discussed the emerging analyses. This process helped refine the coding scheme, resolve ambiguities, and ensure that interpretations were well grounded in the data.

In the qualitative analysis, we first conducted a deductive analysis of the observation sheets to trace compliance with, or breaches of, the TCI principles—for instance, which TCI tasks were omitted, whether collaboration was encouraged, and whether multiple solving strategies were sought. At the same time, an inductive analysis was undertaken to identify factors that might have shaped the way TCI was implemented (e.g., students' reactions during the lesson, the manner in which the teacher introduced or assigned tasks). It must be emphasised that our focus was not on the frequency of particular events but on the nature of the observed phenomena. For example, rather than counting how often a teacher omitted a task, we examined *which* tasks were omitted and *why*. Next, both deductive and inductive approaches were applied to the teachers' written reflections and interview transcripts to identify probable causes of the breaches of and compliance with the TCI principles.

The analysis was conducted separately for each case (each cycle), with insights from earlier cases informing and refining the analysis of subsequent ones. The findings presented in this study stem from the overarching synthesis of all cycles.

Results

Fidelity of implementation

The analysis of implementation fidelity led us to distinguish two groups of teachers: those exhibiting 'high fidelity' and those showing 'lower fidelity'.

Agatha and Dagmar exhibited a *high degree of fidelity* in their implementation of TCI. Both established a safe classroom climate and adopted the role of moderators, leaving the initiative for task solving to the students. They refrained from presenting formulas or solving procedures, instead supporting students in developing diverse strategies. Students collaborated in pairs or larger groups, explained their solutions to one another, and engaged in substantive whole-class discussions.

Agatha departed from TCI in two respects: she incorporated additional task sources and introduced a classification of tasks according to combinatorial operation, which was not envisaged in the original materials. These deviations occurred only in the final phase, after students had had opportunities to develop their own strategies. Her stated intention was to enable students to consolidate knowledge and to situate combinatorics within the broader framework of arrangement, combination, and permutation. Dagmar's departure from fidelity was minimal; although she did not employ every task from the materials, she consistently adhered to the use of isomorphism of tasks to generalise.

Both teachers also enhanced TCI in distinct ways. Agatha refined the formulation of tasks, established links between combinatorics and other mathematical domains such as statistics, and pursued substantial differentiation in her lessons. Dagmar employed a spiral approach, whereby students revisited selected combinatorial tasks over time and progressively deepened their understanding.

In contrast, Bedrich and Cecilia demonstrated *a lower level of fidelity*. Bedrich's students worked individually or in pairs; although they solved tasks independently of the teacher, they did not exchange solutions with peers, resulting in limited discussion of alternative strategies. Cecilia's

students engaged actively in group work and discussion, seeking to justify their solutions. However, to save time, she introduced new concepts and relationships after a single task had been solved, without allowing students to generalise their experience across a broader set of tasks.

Neither Bedrich nor Cecilia employed isomorphic tasks to foster generalisation, nor did they exploit the potential of tasks graded in difficulty for differentiation, both of which are central features of TCI. Furthermore, both were selective in their use of the tasks provided: Cecilia aimed to expedite coverage of curriculum requirements, while Bedrich' main goal was to maintain students' awareness of combinatorics prior to its continuation in the following school year, not to cover the subject in full.

Factors influencing the implementation of the intervention

The analysis uncovered several factors influencing the way TCI was implemented by the teachers.

The first factor is *the teacher's adoption of socio-mathematical norms pertinent to constructivist teaching and their experience with such teaching*. Both teachers in the 'high-fidelity group' and Cecilia from the 'lower-fidelity group' had experience with constructivist mathematics teaching. Our lesson observations confirmed what the teachers claimed in the interviews, that is their adherence to constructivist teaching. The TCI lessons were perceived by the students and teachers as something natural and not differing from other mathematics lessons. Bedrich also stated that he preferred a constructivist approach. However, when observing his lessons, it was apparent that students were not used to working in groups or collaboration, and were accustomed to frontal teaching.

The second factor is *the teacher's confidence in the meaningfulness and effectiveness of TCI.* All teachers except Bedrich showed a high confidence, both in interviews and in teaching. Bedrich strictly adhered to the curriculum, which emphasised combinatorial terms and formulas. He skipped many tasks, and it seems that he did not fully grasp the principles of TCI. This stands in stark contrast to Dagmar who only used the provided materials which illustrates her trust in TCI.

The third factor is *proactive collaboration between the teacher and the researcher*, which naturally led to their better grasp of TCI principles. All teachers except Bedrich were active when collaborating with the researcher (the first author). Agatha and Cecilia recorded their lessons and wrote thorough feedback soon afterwards. They attended many training sessions and interviews and replied promptly to e-mail questions. Dagmar recorded three lessons only, wrote feedback only later, and only for some lessons, and replied to e-mails with delay. On the other hand, she answered questions at meetings in detail, and attended numerous training sessions. Bedrich's lower level of collaboration was evident in his lack of communication, delayed or repeated responses to e-mails, and incomplete data collection. His feedback was written retrospectively with little detail.

Each of the above may become a barrier to the faithful implementation of TCI. Some more were identified. The first is *a lack of time.* Bedrich and Cecilia felt limited by time during the intervention and adapted their teaching accordingly (severely limiting the core principles of TCI). Even though the decision on how much time to devote to a topic lies with the teacher in Czech schools, there might be causes outside their influence. Let us add, that if the conceptual thread present in the series of tasks is not disrupted, omission need not be detrimental (see Dagmar's case). A *lack of familiarity with the materials* may have influenced an inappropriate selection of tasks to be omitted in Bedrich's case. As the traditional way of teaching combinatorics in Czechia stresses formulas and names of

combinatorial operations (Zenkl, 2021), *fear of not fulfilling curriculum requirements* can become a barrier as regards TCI. This was identified as a barrier for the two teachers in the 'lower-fidelity group', which, in turn, may become a barrier to students' understanding in combinatorics.

Discussion

The fidelity of implementation of a teaching approach to its core principles and its authors' intentions is a key concern (Jankvist et al., 2022). The study revealed two groups of teachers: those who implemented TCI with high fidelity and those who did so only partially. High fidelity tended to occur in classrooms already familiar with constructivist practices such as group work, student initiative, and sharing solution strategies. Lower fidelity was found in classrooms where frontal teaching and limited collaboration were typical. This supports findings that successful implementation depends not only on the quality of the intervention but also on the alignment of teacher practice with its principles (Windschitl, 2002). Importantly, some deviations from the intended design did not undermine the intervention and could even enhance it. For example, Dagmar's spiral use of isomorphic tasks encouraged the gradual discovery of new relationships, echoing work on the benefits of revisiting combinatorial tasks over time (Maher et al., 2010; Uptegrove, 2015). By contrast, premature introduction of formulas constrained students' opportunities to construct meaning, a risk documented in research on students' overreliance on formulae (Lockwood, 2014).

For RQ2, three key prerequisites emerged. First, teachers' prior experience with constructivist methods proved crucial. This aligns with earlier studies showing that teachers rarely change their instructional style completely and partial implementation is more common (Dunn et al., 2018). For teachers unaccustomed to constructivist methods, the transition can be challenging and time-consuming (Windschitl, 2002). Bedrich's case shows that a short intervention is insufficient for such a shift. Second, teachers' conviction in the meaningfulness and effectiveness of TCI was essential. Where this conviction was weak, teachers tended to revert to traditional approaches, though even partial adoption of individual principles—such as using isomorphic tasks or reducing reliance on operation labels—can positively innovate teaching. Third, proactive collaboration with the researcher supported teachers' understanding of TCI principles, consistent with findings on the importance of teacher–researcher partnerships in design-based research (Anderson & Shattuck, 2012).

When these prerequisites were not met, they became barriers. Time pressure, often exacerbated by adherence to curriculum requirements, was a common constraint. While formula-driven teaching may appear efficient in the short term, it risks fostering superficial learning and encourages students to treat combinatorics as the mechanical fitting of problems to formulas (Lockwood, 2014). Skipping tasks was another response to time constraints; when done selectively, this was not harmful, but poor task selection undermined conceptual progression. Finally, reduced student motivation, particularly in classrooms accustomed to frontal teaching, further limited the uptake of TCI principles.

Limitations, implications and conclusions

This study has several limitations. The cooperating teachers were not involved in the original design of TCI and only partly in its adaptations, which may have reduced their sense of ownership. The sample was small and self-selected, consisting of teachers already inclined toward constructivist methods (Dunn et al., 2018), and their limited time constrained meetings, feedback, and lesson recording (Windschitl, 2002).

Nevertheless, the findings suggest clear implications. Teacher education should not only present the principles of TCI but also support changes in socio-mathematical norms, time management, and curriculum alignment. Combinatorial terminology should be introduced only at the end of teaching to consolidate knowledge without undermining reasoning (Hejný, 2012; Lockwood, 2014), though this requires better task sequences than those currently found in Czech textbooks (Zenkl, 2021).

To sum up, implementation fidelity depended on teachers' prior constructivist experience, confidence in TCI, and collaboration with the researcher. High-fidelity cases promoted student reasoning, while low-fidelity ones were hindered by time pressure, reliance on formulas, and limited student motivation. Thus, teacher practice is decisive for the success of interventions, and sustained support, training, and appropriate curricular resources are essential (Windschitl, 2002).

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