Chinese teachers' professional noticing of students' reasoning in the context of Lakatos-style proving activity

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Despite the pedagogical values of Lakatos-style proving activity in school mathematics, little is known about how teachers react to students' reasoning in this kind of activity. As an attempt to unpack the underlying decision-making processes, this study examines teachers' noticing of students' justification and refutation of conjectures in the context of Lakatos-style proving activity. Twelve Chinese pre-service and in-service secondary mathematics teachers participated in semi-structured, vignette-based interviews where they were presented with realistic classroom scenarios that were based on actual classroom episodes reported in the literature and covered various aspects of Lakatos-style reasoning in the context of geometry. Findings show that teachers can better notice students' justifications than students' refutations, and better notice students' valid arguments than invalid arguments. Key themes are identified to characterise their noticing of various student arguments.

Keywords: Proof, Lakatos, Justification, Refutation, Professional noticing.

Introduction

In the form of fictional classroom discussion, Lakatos (1976) described how mathematicians constructed and utilized mathematical knowledge through a zig-zag reasoning process. Some aspects of Lakatos-style reasoning, such as conscious guessing and the zig-zag path of reasoning, were suggested by mathematical educators (e.g., Lampert, 1990) to be applied in some school mathematical activities for engaging students in authentic mathematics. Some empirical studies also showed that school-age students can perform in line with Lakatos-style reasoning, but most of them paid attention to these students' proving processes (e.g., Komatsu, 2012; Reid, 2002), while few studies focused on the perspective of teachers. It remains unclear how teachers deal with various types of students' responses throughout different phases of Lakatos-style proving. To address this research gap, one promising attempt is to investigate teachers' professional noticing by focusing on how they pay attention to and make sense of particular instructional situations (Jacobs et al., 2010). This study tries to explore: How do teachers notice students' justifications and refutations of conjectures in the context of Lakatos-style proving activity?

Theoretical framework

Lakatos-style proving process

Drawing upon Lakatos' (1976) book and some later studies which attempt to implement his approach (e.g., Komatsu, 2016; Reid, 2002), we summarised five phases of the Lakatos-style proving process to reflect the core idea of Lakatos-style reasoning, and to ensure the feasibility and practicality of applying this framework to design Lakatos-style proving situations. Specifically, in Phase 1, a conjecture is formulated through conscious guessing. Then the conjecture is tested through examination of supportive examples (Phase 2). Proof may be constructed to further validate the conjecture (Phase 3). Still, counterexamples may emerge and refute the conjecture (Phase 4), and thus refinement of the conjecture is needed (Phase 5).

Justification and refutation scheme

We utilized a framework of students' justification and refutation schemes to conceptualise how students justify and refute conjectures in diverse ways, reflecting different degrees of mathematical sophistication. This framework was constructed based on previous research on students' proof schemes (Balacheff, 1988; Harel & Sowder, 1998; Lee, 2016; Stylianides & Stylianides, 2009). There are four levels of justification scheme. Students with the lowest level of scheme, Naïve empirical justification, believe that examining few examples which are easy to check (e.g., examining the example "x = 1" for validating the conjecture "for any natural number, 2x is an even number") can produce valid mathematical generalization. Students with a higher level (Crucial experiment justification) also accept example-based validation, but they believe that the examples need to be strategically identified following some rationales (e.g., examining a set of odd numbers "x = 1, 3, 5, 7, 9…" for the above-mentioned conjecture). By contrast, students with Nonempirical justification believe that it is not sufficient to validate a conjecture based on a subset of examples, but unlike students with the most advanced level (Deductive justification), they may not recognize that deductive inferences can produce valid generalizations. Regarding refutation schemes, there are also four levels. Students with the least advanced level (Naïve refutation scheme) regard counterexample(s) as exception(s), and still consider the conjecture as true. Students with Empirical refutation think it is insufficient to refute based on a counterexample, and more counterexamples are needed to convince them that the conjecture is refuted. With a more advanced level, Single counterexample refutation, students believe that it is sufficient to refute a conjecture based on a counterexample. Students with the most advanced level (General counterexample refutation) accept the sufficient role of one counterexample in refutations, and meanwhile, they believe that identifying the common properties of counterexamples can support the refutation of the conjecture.

Noticing

Researchers on teacher noticing have adopted various conceptualizations of this emerging construct. Some researchers (e.g., Es & Sherin, 2008) thought it involves two components, *Attending* and *Interpreting*. By contrast, many researchers like Jacobs et al. (2010) additionally considered a third component (i.e., *Deciding*), given that this component works with another two components in an interrelated way to determine teachers' responses to students' mathematical thinking. Specifically, to process complex classroom situations, teachers selectively *attend* to students' strategies which they think as noteworthy in particular instructional events; they *interpret* students' understanding reflected in these strategies and *decide intended responses* (as opposed to executing actual responses) to students based on their interpretations. In the present research, we particularly adopt Jacobs et al.'s (2010) conceptualization because it is more related to our research focus on students' reasoning.

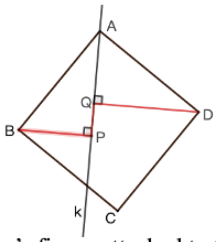
Research methods

Data were drawn from semi-structured interviews with twelve Chinese teachers. For a diversity of teacher profiles, these participants included four pre-service teachers, four novice teachers with an average of 2.75 years of teaching experience, and four experienced teachers with an average of 17.75

years of teaching experience in junior high school for students aged 12-15. They were recruited through convenience sampling.

The one-hour interviews were conducted online and based on comic-style vignettes. Participants were presented with a set of classroom vignette episodes showing how students solved a geometric proof task (see Figure 1) throughout five phases of Lakatos-style proving. In this study, we particularly focused on teacher noticing of students' justification and refutation in Lakatos-style proving activities, trying to investigate how teachers attend to and make sense of students' uses of examples and counterexamples which potentially can be building blocks for students' further refinement of conjectures or proofs. Given this focus, this paper focused on teachers' responses to 4 episodes about justification and 4 about the refutation of the same conjecture (i.e., irrespective of the position of line k, PQ = DQ − BP). These 8 episodes respectively reflected each level of Justification and Refutation schemes, covering Phase 2-4 in the Lakatos-style proving process.

Draw a line k that passes through a point A of square ABCD and passes through the inside of square ABCD. Draw lines BP and DQ perpendicular to line k from points B and D, respectively. What is the length relation between BP, DQ, and QP? Prove your conjecture.



(The teacher's figure attached to the task)

Figure 1: The geometric proof task (Komatsu et al., 2014) presented in the vignette episodes

To make the episode contents more authentic, these episodes were adapted from real classroom episodes reported in Komatsu et al.'s (2014) empirical study with the guidance of the above-mentioned frameworks (Skilling & Stylianides, 2019). They were designed in a comic style that can provide participants sufficiently realistically but also abstract enough information, aiming to help them pay attention to critical aspects of classroom practices, and allow them to form their interpretations (Herbst et al., 2011). The episode contents were designed in a format of classroom discussion to simulate Lakatos-style discourse. Figure 2 shows two translated sample episodes.

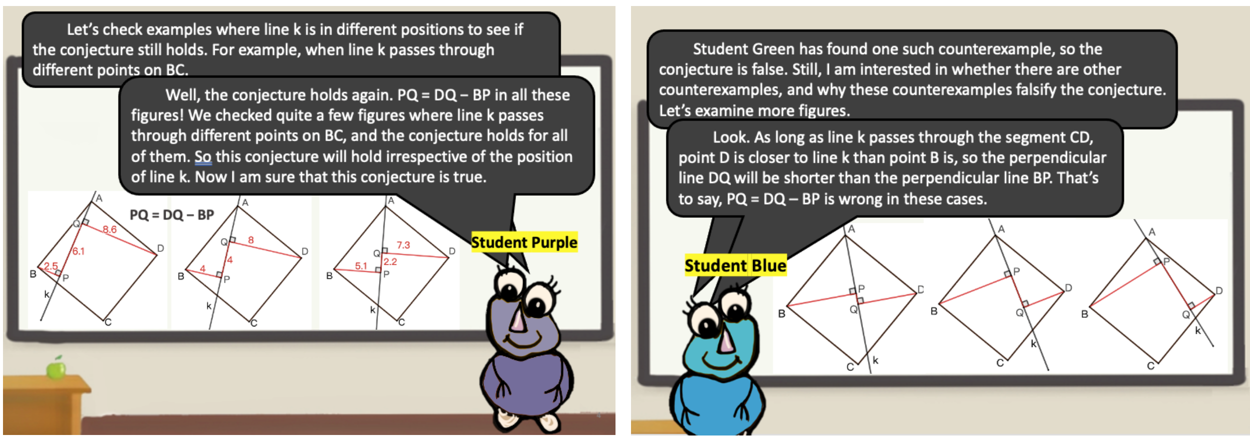


Figure 2: Crucial experiment justification (Left) and General counterexample refutation (Right)

After seeing each episode, participants were asked to describe (i) the student's thinking and/or actions that they attended to, (ii) how they interpreted the student's understandings, and (iii) how would they respond to the student, corresponding to the Attending, Interpreting, and Deciding aspects of teacher noticing. Following Jacobs et al.'s (2010) coding scheme of teacher noticing, teacher responses were coded based on the robustness of evidence that showed teachers' consideration of the mathematically important details in students' proving strategies. Specifically, responses about the Attending aspect were coded on a 2-point scale: Evidence (1), and Lack of evidence (0), and responses about the Interpreting and Deciding aspects were coded on a 3-point scale: Robust evidence (2), Limited evidence (1), and Lack of evidence (0). After coding teachers' responses into different categories, emerging themes were identified from each category to capture its characteristics (Corbin & Strauss, 1990). Double counting was applied for responses that related to more than one theme.

Findings

Overview of teacher noticing

To capture participants' noticing of students' thinking in their justification and refutation, mean scores of the Attending, Interpreting, and Deciding aspects were calculated for each of 8 episodes (i.e., 4 types of student justification and 4 types of student refutation). Given that the two lower levels and the two higher levels of Justification and Refutation schemes show students' invalid and valid reasoning respectively, I calculated the average scores of each two levels to achieve a more stable measure of teachers' noticing of (in)valid justification/refutation arguments (see Table 1).

Table 1: Means (SD) for teaches' scores of noticing students' justification and refutation

Component skill (Scale)	Justification			Refutation		
	Invalid	**Valid**	**Overall**	**Invalid**	**Valid**	**Overall**
Attending (0-1)	0.54 (0.51)	0.96 (0.20)	0.75 (0.44)	0.46 (0.51)	0.63 (0.49)	0.54 (0.50)
Interpreting (0-2)	1.29 (0.55)	1.29 (0.46)	1.29 (0.50)	1.13 (0.68)	1.13 (0.54)	1.13 (0.61)
Deciding (0-2)	1.17 (0.56)	1.42 (0.65)	1.31 (0.59)	1.08 (0.65)	1.50 (0.66)	1.29 (0.68)

Participants had on average higher scores in noticing students' thinking in justification than in refutation, across each aspect of teacher noticing–Attending (0.75 vs. 0.54), Interpreting (1.29 vs. 1.13), and Deciding (1.31 vs. 1.29). Although participants showed the same averages (1.29) in interpreting students' valid and invalid justification, they showed higher averages in attending to (0.96 vs. 0.54) and deciding how to respond to (1.42 vs. 1.17) students' valid justification than invalid justification. A similar pattern existed when participants were noticing students' refutation.

Table 2 shows the number of each category of teacher responses. Such results supplemented teachers' mean scores (Table 1) and gave us a more complete picture of teachers' noticing of students' thinking in justification and refutation.

Attending to students' strategies

For attending to students' strategies, more than half of teacher responses showed evidence, no matter for which subgroup of student arguments (see Table 2). In particular, almost all 12 participants gave evidence of attending to students' thinking in Naïve empirical justification (10), Nonempirical justification (11), Deductive justification (12), and Single counterexample refutation (11). Yet, a notable number of participants still provided a lack of evidence for other types of thinking.

Table 2: Number of each category of teacher responses to students' justification and refutation

	Justification (Total: 48 teacher responses)		**Refutation** (Total: 48 teacher responses)	
	Invalid (Total: 24)	**Valid** (Total: 24)	**Invalid** (Total: 24)	**Valid** (Total: 24)
Attending	Evidence (N=13) Lack of evidence (N=11)	Evidence (N=23) Lack of evidence (N=1)	Evidence (N=13) Lack of evidence (N=11)	Evidence (N=15) Lack of evidence (N=9)
Interpreting	Robust evidence (N=8) Limited evidence (N=15) Lack of evidence (N=1)	Robust evidence (N=7) Limited evidence (N=17)	Robust evidence (N=7) Limited evidence (N=13) Lack of evidence (N=4)	Robust evidence (N=5) Limited evidence (N=17) Lack of evidence (N=2)
Deciding	Robust evidence (N=6) Limited evidence (N=16) Lack of evidence (N=2)	Robust evidence (N=12) Limited evidence (N=10) Lack of evidence (N=2)	Robust evidence (N=6) Limited evidence (N=14) Lack of evidence (N=4)	Robust evidence (N=13) Limited evidence (N=9) Lack of evidence (N=2)

A common characteristic of lack-of-evidence responses was that they omitted some mathematically important details reflected in the strategy. This is especially evident when comparing teacher responses to students' Crucial experiment justification (8/12) and other types of justification (0/12 or 1/12), and thus may somewhat explain teachers' lower average scores in attending to invalid justifications than valid justifications (see Table 1). To illustrate, commenting on a student's Crucial experiment justification, 7 teachers mentioned students' testing of more examples, but they did not describe the unique feature of Crucial experiment justification, that is, these examples were strategically identified (i.e., in all tested examples, line k passed through the segment BC).

Some lack-of-evidence responses may include some information inconsistent with the student strategy. This is notable in teacher responses to students' invalid refutation (9/24). For example, regarding the episode about Empirical refutation, 3 teachers wrongly described a student's idea of "finding more counterexamples" as "finding all counterexamples".

Interpreting students' understandings

Compared with the evidence of attending to students' strategies, showing robust evidence of interpreting students' understandings may be more challenging for participants. Although 8 participants offered robust evidence in interpreting Naïve empirical justification, no more than 4 participants could provide robust evidence for other student arguments. This may somewhat explain teachers' slightly higher average scores in interpreting justifications (1.29) versus refutations (1.13).

A marked characteristic of limited-evidence interpretation was that it included general and/or incomplete descriptions of students' (mis)understandings. More than half of the participants demonstrated this characteristic in their interpretation except for students' Naïve empirical justification and Empirical refutation. For instance, for a student's Nonempirical justification, 5 teachers did not show their appreciation of the student's insightful idea of testing all examples to justify the conjecture. For interpreting a student's General counterexample refutation, 5 teachers did not provide details of interpreting one of its mathematically important aspects (i.e., identification of common properties of counterexamples that refute the conjecture).

Other non-robust-evidence interpretation include incorrect information and/or assumptions made up by the participants. To illustrate, for a student's Empirical refutation, a teacher thought the student was testing all counterexamples, which were inconsistent with the student's actual understandings. Another teacher showed a partially incorrect interpretation as she wrongly accepted the student's Empirical refutation, and meanwhile she correctly thought it sufficient to refute a conjecture with only one counterexample. Sometimes, teachers may make assumptions in interpretations. For example, in commenting on a student's Naïve refutation, a teacher assumed that the student ignored the counterexample because this student was stubborn. Yet, this assumption was hard to be justified based on what the student said. Overall, these shared themes across different subgroups of student arguments resulted in the same mean scores for interpreting invalid arguments and valid arguments.

Deciding how to respond based on students' understanding

Teachers may have performed better in deciding how to respond to students' valid arguments than their invalid arguments. This was supported by the fact that around half of the teacher responses for valid justifications (12/24) and valid refutations (13/24) had robust evidence, whereas only limited responses for invalid justifications (6/24) and invalid refutations (6/24) showed robust evidence.

Some teachers' responses were coded as "limited evidence" because they did not customize their intended responses to different students based on the specific features of different students' thinking. To illustrate, 11/24 limited-evidence responses for students' invalid justification offered very similar suggestions and/or next steps (e.g., simply asking students to give a proof) for students' Naïve justification and Crucial experiment justification, even though these two types of thinking indicated different levels of mathematical sophistication (as reflected in students way of identifying supportive examples). By contrast, an illustration of a customised response for a student's Crucial experiment justification was as follows. This teacher made use of the student's strategically identified examples to encourage the student to give proof. Such customised response was constructed based on the teacher's attention to and robust interpretation of the student's strategy:

> Teacher: I will appreciate the student's spirit of exploration, and I will remind him these are only some examples…The student needs to learn how to analyze…whether there are common properties among different figures (i.e., examples). If there are, can we start proving this conclusion with geometric proof? He has drawn a lot of figures, and we need to find their common properties.

13/24 teacher responses for students' invalid refutation were coded as "Limited evidence" as they were not useful to further the student's thinking or proposed unreasonable next steps for the student, according to the Justification and refutation framework. For example, some teachers accepted students' Naïve refutation and/or Empirical refutation, which were not conducive to students' development of refutation ability. A few responses were coded as "Lack of evidence", given that they only cited little reference to the student's thinking.

Discussion and Conclusion

In summary, this study constructs a picture of teachers' noticing of students' thinking in justification and refutation, which was seldom investigated in previous research. These findings can help us better understand how teachers attend to and make sense of different types of students' justification and refutation in the context of Lakatos-style proving activities. Results showed that teachers may be more able to notice students' thinking in justification than in refutation. Especially when the students gave valid arguments (regardless of whether for justification or refutation), teachers may be more likely to attend to and decide how to respond to their thinking. Furthermore, the following two phenomena merit our attention.

First, most participants perform well in attending to students' justifications, except to a student's Crucial experiment justification. 7/12 teachers noticed that the student with Crucial experiment justification tested more examples, but they failed to further point out that these examples were strategically identified. Accordingly, they interpreted students' Crucial experiment justification and Naïve empirical justification as similar (i.e., both of them validate conjectures based on several examples rather than deductive inferences), without interpreting the more advanced feature of the former one. In other words, teachers may be sensitive to whether students' reasoning is example-based, but they may have limited attention to and interpretation of students' ways of identifying supportive examples. This may partly explain why some teachers' decisions of responding to students' Crucial experiment justification and Naïve empirical justification lacks customisation. They did not identify students' mathematically important strategy details and reasoning behind strategies, and thus simply suggested students a very similar next step (i.e., trying to give proof).

Second, for students' refutations, although 11/12 teachers provided evidence of attending to a student's Single counterexample refutation, only a few teachers could give evidence about other refutation strategies. This implies that teachers may be more sensitive to the Single counterexample refutation, given that the sufficient role of a single counterexample in refuting conjectures is highlighted in China's curriculum standard (Ministry of Education of China, 2012). Interestingly, despite the wide recognition of the role of a single counterexample in refutation, some teachers still allowed or even encouraged students who had Naïve refutation or Empirical refutation to examine more counterexamples for refuting the conjecture. This may be not conducive to students' understandings of the minimally necessary and sufficient way of refutations. But as suggested by the description of General counterexample refutation, if teachers can emphasize the role of a single counterexample and meanwhile support students' investigation of more counterexamples to find out their common properties that refute the conjecture, students may have opportunities to develop this advanced refutation scheme.

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