

CERME 6

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CERME 6 – TABLE OF CONTENTS

GENERAL INTRODUCTION	XVIII
PLENARY 1	
Signs, gestures, meanings: Algebraic thinking from a cultural semiotic perspective	XXXIII
<i>Luis Radford</i>	
PLENARY 2	
Mathematics education as a network of social practices	LIV
<i>Paola Valero</i>	
SPECIAL PLENARY SESSION	
Ways of working with different theoretical approaches in mathematics education research	1
Introduction	2
<i>Tommy Dreyfus</i>	
Networking of theories: why and how?	6
<i>Angelika Bikner-Ahsbahr</i>	
People and theories	16
<i>John Monaghan</i>	
Discussion	24
WORKING GROUP 1	
Multimethod approaches to the multidimensional affect in mathematics education	26
Introduction	28
<i>Markku S. Hannula, Marilena Pantziara, Kjersti Wæge, Wolfgang Schölglmann</i>	
The effect of achievement, gender and classroom context on upper secondary students' mathematical beliefs	34
<i>Markku S. Hannula</i>	
Changing beliefs as changing perspective	44
<i>Peter Liljedahl</i>	
“Maths and me”: software analysis of narrative data about attitude towards math	54
<i>Pietro Di Martino</i>	
Students' beliefs about the use of representations in the learning of fractions	64
<i>Athanasios Gagatsis, Areti Panaoura, Eleni Deliyianni, Iliada Elia</i>	
Efficacy beliefs and ability to solve volume measurement tasks in different representations	74
<i>Paraskevi Sophocleous, Athanasios Gagatsis</i>	

Students' motivation for learning mathematics in terms of needs and goals.....	84
<i>Kjersti Wæge</i>	
Mathematical modeling, self-representation and self-regulation.....	94
<i>Areti Panaoura, Andreas Demetriou, Athanasios Gagatsis</i>	
Endorsing motivation: identification of instructional practices	104
<i>Marilena Pantziara, George Philippou</i>	
The effects of changes in the perceived classroom social culture on motivation in mathematics across transitions	114
<i>Chryso Athanasiou, George N. Philippou</i>	
“After I do more exercise, I won't feel scared anymore” Examples of personal meaning from Hong-Kong	124
<i>Maike Vollstedt</i>	
Emotional knowledge of mathematics teachers – retrospective perspectives of two case studies	134
<i>Ilana Lavy, Atara Shriki</i>	
Humour as a means to make mathematics enjoyable	144
<i>Pavel Shmakov, Markku S. Hannula</i>	
Beliefs: a theoretically unnecessary construct?	154
<i>Magnus Österholm</i>	
Categories of affect – some remarks.....	164
<i>Wolfgang Schlöglmann</i>	
 WORKING GROUP 2	
Argumentation and proof	174
Introduction.....	176
<i>Maria Alessandra Mariotti, Leanor Camargo, Patrick Gibel, Kristina Reiss</i>	
Understanding, visualizability and mathematical explanation	181
<i>Daniele Molinini</i>	
Argumentation and proof: a discussion about Toulmin's and Duval's models	191
<i>Thomas Barrier, Anne-Cécile Mathé, Viviane Durand-Guerrier</i>	
Why do we need proof.....	201
<i>Kirsti Hemmi, Clas Löfwall</i>	
Proving as a rational behaviour: Habermas' construct of rationality as a comprehensive frame for research on the teaching and learning of proof.....	211
<i>Francesca Morselli, Paolo Boero</i>	
Experimental mathematics and the teaching and learning of proof.....	221
<i>Maria G. Bartolini Bussi</i>	
Conjecturing and proving in dynamic geometry: the elaboration of some research hypotheses.....	231
<i>Anna Baccaglioni-Frank, Maria alessandra Mariotti</i>	
The algebraic manipulator of alnuset: a tool to prove	241
<i>Bettina Pedemonte</i>	
Visual proofs: an experiment	251
<i>Cristina Bardelle</i>	

Teachers' views on the role of visualisation and didactical intentions regarding proof.....	261
<i>Irene Biza, Elena Nardi, Theodossios Zachariades</i>	
Modes of argument representation for proving – the case of general proof.....	271
<i>Ruthi Barkai, Michal Tabach, Dina Tirosh, Pessia Tsamir, Tommy Dreyfus</i>	
Mathematics teachers' reasoning for refuting students' invalid claims.....	281
<i>Despina Potari, Theodossios Zachariades, Orit Zaslavsky</i>	
Student justifications in high school mathematics.....	291
<i>Ralph-Johan Back, Linda Mannila, Solveig Wallin</i>	
“Is that a proof?”: an emerging explanation for why students don't know they (just about) have one	301
<i>Manya Raman, Jim Sandefur, Geoffrey Birky, Connie Campbell, Kay Somers</i>	
“Can a proof and a counterexample coexist?” A study of students' conceptions about proof	311
<i>Andreas J. Stylianides, Thabit Al-Murani</i>	
Abduction and the explanation of anomalies: the case of proof by contradiction.....	322
<i>Samuele Antonini, Maria Alessandra Mariotti</i>	
Approaching proof in school: from guided conjecturing and proving to a story of proof construction.....	332
<i>Nadia Douek</i>	
 WORKING GROUP 3	
On “stochastic thinking”	343
<i>Introduction.....</i>	
<i>Andreas Eichler, Maria Gabriella Ottaviani, Floriane Wozniak, Dave Pratt</i>	
Chance models: building blocks for sound statistical reasoning	348
<i>Herman Callaert</i>	
Recommended knowledge of probability for secondary mathematics teachers.....	358
<i>Irini Papaieronymou</i>	
Statistical graphs produced by prospective teachers in comparing two distributions.....	368
<i>Carmen Batanero, Pedro Arteaga, Blanca Ruiz</i>	
The role of context in stochastics instruction.....	378
<i>Andreas Eichler</i>	
Does the nature and amount of posterior information affect preschooler's inferences	388
<i>Zoi Nikiforidou, Jenny Pange</i>	
Student's Causal explanations for distribution	394
<i>Theodosia Prodromou, Dave Pratt</i>	
Greek students' ability in probability problem solving	404
<i>Sofia Anastasiadou</i>	
 WORKING GROUP 4	
Algebraic thinking and mathematics education.....	413
<i>Introduction.....</i>	
<i>Janet Ainley, Giorgio T. Bagni, Lisa Hefendehl-Hebeker, Jean-Baptiste Lagrange</i>	

The effects of multiple representations-based instruction on seventh grade students' algebra performance	420
<i>Oylum Akkus, Erdinc Cakiroglu</i>	
Offering proof ideas in an algebra lesson in different classes and by different teachers.....	430
<i>Michal Ayalon, Ruhama Even</i>	
Rafael Bombelli's Algebra (1572) and a new mathematical "object": a semiotic analysis	440
<i>Giorgio T. Bagni</i>	
Cognitive configurations of pre-service teachers when solving an arithmetic-algebraic problem.....	449
<i>Walter F. Castro, Juan D. Godino</i>	
Transformation rules: a cross-domain difficulty.....	459
<i>Marie-Caroline Croset</i>	
Interrelation between anticipating thought and interpretative aspects in the use of algebraic language for the construction of proofs in elementary number theory.....	469
<i>Annalisa Cusi</i>	
Epistemography and algebra.....	479
<i>Jean-Philippe Drouhard</i>	
Sámi culture and algebra in the curriculum	489
<i>Anne Birgitte Fyhn</i>	
Problem solving without numbers – An early approach to algebra.....	499
<i>Sandra Gerhard</i>	
The ambiguity of the sign $\sqrt{\quad}$	509
<i>Bernardo Gómez, Carmen Buhlea</i>	
Behind students' spreadsheet competencies: their achievement in algebra? A study in a French vocational school.....	519
<i>Mariam Haspekian, Eric Bruillard</i>	
Developing Katy's algebraic structure sense	529
<i>Maureen Hoch, Tommy Dreyfus</i>	
Children's understandings of algebra 30 years on: what has changed?.....	539
<i>Jeremy Hodgen, Dietmar Kuchemann, Margaret Brown, Robert Coe</i>	
Presenting equality statements as diagrams	549
<i>Ian Jones</i>	
Approaching functions via multiple representations: a teaching experiment with Casyopee	559
<i>Jean-Baptiste Lagrange, Tran Kiem Minh</i>	
Equality relation and structural properties	569
<i>Carlo Marchini, Anne Cockburn, Paul Parslow-Williams, Paola Vighi</i>	
Structure of algebraic competencies	579
<i>Reinhard Oldenburg</i>	
Generalization and control in algebra	589
<i>Mabel Panizza</i>	
From area to number theory: a case study	599
<i>Maria Iatridou, Ioannis Papadopoulos</i>	
Allegories in the teaching and learning of mathematics	609
<i>Reinert A. Rinvold, Andreas Lorange</i>	

Role of an artefact of dynamic algebra in the conceptualisation of the algebraic equality	619
<i>Giampaolo Chiappini, Elisabetta Robotti, Jana Trgalova</i>	
Communicating a sense of elementary algebra to preservice primary teachers	629
<i>Franziska Siebel, Astrid Fischer</i>	
Conception of variance and invariance as a possible passage from early school mathematics to algebra.....	639
<i>Ilya Sinitsky, Bat-Sheva Ilany</i>	
Growing patterns as examples for developing a new view onto algebra and arithmetic.....	649
<i>Claudia Böttinger, Elke Söbbeke</i>	
Steps towards a structural conception of the notion of variable	659
<i>Annika M. Wille</i>	
WORKING GROUP 5	
Geometrical thinking	669
Introduction.....	
<i>Alain Kuzniak, Iliada Elia, Matthias Hattermann, Filip Roubicek</i>	
The necessity of two different types of applications in elementary geometry.....	676
<i>Boris Girnat</i>	
A French look on the Greek geometrical working space at secondary school level	686
<i>Alain Kuzniak, Laurent Vivier</i>	
A theoretical model of students' geometrical figure understanding	696
<i>Eleni Deliyianni, Iliada Elia, Athanasios Gagatsis, Annita Monoyiou, Areti Panaoura</i>	
Gestalt configurations in geometry learning	706
<i>Claudia Acuña</i>	
Investigating comparison between surfaces.....	716
<i>Paola Vighi</i>	
The effects of the concept of symmetry on learning geometry at French secondary school.....	726
<i>Caroline Bulf</i>	
The role of teaching in the development of basic concepts in geometry: how the concept of similarity and intuitive knowledge affect student's perception of similar shapes.....	736
<i>Mattheou Kallia, Spyrou Panagiotis</i>	
The geometrical reasoning of primary and secondary school students	746
<i>Georgia Panaoura, Athanasios Gagatsis</i>	
Strengthening students' understanding of 'proof' in geometry in lower secondary school	756
<i>Susumu Kunimune, Taro Fujita, Keith Jones</i>	
Written report in learning geometry: explanation and argumentation	766
<i>Silvia Semana, Leonor Santos</i>	
Multiple solutions for a problem: a tool for evaluation of mathematical thinking in geometry.....	776
<i>Anat Levav-Waynberg, Roza Leikin</i>	
The drag-mode in three dimensional dynamic geometry environments – Two studies	786
<i>Mathias Hattermann</i>	
3D geometry and learning of mathematical reasoning	796
<i>Joris Mithalal</i>	

In search of elements for a competence model in solid geometry teaching Establishment of relationships	806
<i>Edna González, Gregoria Guillén</i>	
Students' 3D geometry thinking profiles	816
<i>Marios Pittalis, Nicholas Mousoulides, Constantinos Christou</i>	
 WORKING GROUP 6	
Language and mathematics	826
<i>Candia Morgan</i>	
Introduction.....	828
<i>Candia Morgan</i>	
Imparting the language of critical thinking while teaching probability.....	833
<i>Einav Aizikovitsh, Miriam Amit</i>	
Toward an inferential approach analyzing concept formation and language processes	842
<i>Stephan Hußmann, Florian Schacht</i>	
Iconicity, objectification, and the math behind the measuring tape: An example from pipe-trades training	852
<i>Lionel LaCroix</i>	
Mathematical reflection in primary school education: theoretical foundation and empirical analysis of a case study	862
<i>Cordula Schülke, Heinz Steinbring</i>	
Surface signs of reasoning	873
<i>Nathalie Sinclair, David Pimm</i>	
A teacher's use of gesture and discourse as communicative strategies in concluding a mathematical task.....	884
<i>Raymond Bjuland, Maria Luiza Cestari, Hans Erik Borgersen</i>	
A teacher's role in whole class mathematical discussion: facilitator of performance etiquette?	894
<i>Thérèse Dooley</i>	
Use of words – Language-games in mathematics education	904
<i>Michael Meyer</i>	
Speaking of mathematics – Mathematics, every-day life and educational mathematics discourse	914
<i>Eva Riesbeck</i>	
Communicative positionings as identifications in mathematics teacher education.....	924
<i>Hans Jørgen Braathe</i>	
Teachers' collegial reflections of their own mathematics teaching processes Part 1: An analytical tool for interpreting teachers' reflections.....	934
<i>Kerstin Bräuning, Marcus Nührenbörger</i>	
Teachers' reflections of their own mathematics teaching processes Part 2: Examples of an active moderated collegial reflection.....	944
<i>Kerstin Bräuning, Marcus Nührenbörger</i>	
Internet-based dialogue: a basis for reflection in an in-service mathematics teacher education program	954
<i>Mario Sánchez</i>	

The use of algebraic language in mathematical modelling and proving in the perspective of Habermas' theory of rationality.....	964
<i>Paolo Boero, Francesca Morselli</i>	
Objects as participants in classroom interaction	974
<i>Marei Fetzer</i>	
The existence of mathematical objects in the classroom discourse	984
<i>Vicenç Font, Juan D. Godino, Núria Planas, Jorge I. Acevedo</i>	
Mathematical activity in a multi-semiotic environment	993
<i>Candia Morgan, Jehad Alshwaikh</i>	
Engaging everyday language to enhance comprehension of fraction multiplication	1003
<i>Andreas O. Kyriakides</i>	
Tensions between an everyday solution and a school solution to a measuring problem.....	1013
<i>Frode Rønning</i>	
Linguistic accomplishment of the learning-teaching processes in primary mathematics instruction.....	1023
<i>Marcus Schütte</i>	
Mathematical cognitive processes between the poles of mathematical technical terminology and the verbal expressions of pupils	1033
<i>Rose Vogel, Melanie Huth</i>	
WORKING GROUP 7	
Technologies and resources in mathematical education	1043
Introduction.....	1046
<i>Ghislaine Gueudet, Rosa Maria Bottino, Giampaolo Chiappini, Stephen Hegedus, Hans-Georg Weigand</i>	
Realisation of mers (multiple external representations) and melrs (multiple equivalent linked representations) in elementary mathematics software	1050
<i>Silke Ladel, Ulrich Kortenkamp</i>	
The impact of technological tools in the teaching and learning of integral calculus.....	1060
<i>Alejandro Lois, Liliana Milevicich</i>	
Using technology in the teaching and learning of box plots.....	1070
<i>Ulrich Kortenkamp, Katrin Rolka</i>	
Dynamical exploration of two-variable functions using virtual reality	1081
<i>Thierry Dana-Picard, Yehuda Badihi, David Zeitoun, Oren David Israeli</i>	
Designing a simulator in building trades and using it in vocational education	1091
<i>Annie Bessot, Colette Laborde</i>	
Collaborative design of mathematical activities for learning in an outdoor setting	1101
<i>Per Nilsson, Håkan Sollervall, Marcelo Milrad</i>	
Student development process of designing and implementing exploratory and learning objects	1111
<i>Chantal Buteau, Eric Muller</i>	
How can digital artefacts enhance mathematical analysis teaching and learning.....	1121
<i>Dionysis I. Diakoumopoulos</i>	

A learning environment to support mathematical generalisation in the classroom	1131
<i>Eirini Geraniou, Manolis Mavrikis, Celia Hoyles, Richard Noss</i>	
Establishing a longitudinal efficacy study using SimCalc MathWorlds®	1141
<i>Stephen Hegedus, Luis Moreno, Sara Dalton, Arden Brookstein</i>	
Interoperable Interactive Geometry for Europe – First technological and educational results and future challenges of the Intergeo project.....	1150
<i>Ulrich Kortenkamp, Axel M. Blessing, Christian Dohrmann, Yves Kreis, Paul Libbrecht, Christian Mercat</i>	
Quality process for dynamic geometry resources: the Intergeo project.....	1161
<i>Jana Trgalova, Ana Paula Jahn, Sophie Soury-Lavergne</i>	
New didactical phenomena prompted by TI-Nspire specificities The mathematical component of the instrumentation process.....	1171
<i>Michèle Artigue, Caroline Bardini</i>	
Issues in integrating cas in post-secondary education: a literature review	1181
<i>Chantal Buteau, Zsolt Lavicza, Daniel Jarvis, Neil Marshall</i>	
The long-term project “Integration of symbolic calculator in mathematics lessons” The case of calculus	1191
<i>Hans-Georg Weigand, Ewald Bichler</i>	
Enhancing functional thinking using the computer for representational transfer.....	1201
<i>Andrea Hoffkamp</i>	
The Robot Race: understanding proportionality as a function with robots in mathematics class	1211
<i>Elsa Fernandes, Eduardo Fermé, Rui Oliveira</i>	
Internet and mathematical activity within the frame of “Sub14”	1221
<i>Hélia Jacinto, Nélia Amado, Susana Carreira</i>	
A resource to spread math research problems in the classroom	1231
<i>Gilles Aldon, Viviane Durand-Guerrier</i>	
The synergy of students’ use of paper-and-pencil techniques and dynamic geometry software: a case study	1241
<i>Núria Iranzo, Josep Maria Fortuny</i>	
Students’ utilization schemes of pantographs for geometrical transformations: a first classification	1250
<i>Francesca Martignone, Samuele Antonini</i>	
The utilization of mathematics textbooks as instruments for learning	1260
<i>Sebastian Rezat</i>	
Teachers’ beliefs about the adoption of new technologies in the mathematics curriculum	1270
<i>Marilena Chrysostomou, Nicholas Mousoulides</i>	
Systemic innovations of mathematics education with dynamic worksheets as catalysts	1280
<i>Volker Ulm</i>	
A didactic engineering for teachers education courses in mathematics using ICT	1290
<i>Fabien Emprin</i>	
Geometers’ sketchpad software for non-thesis graduate students: a case study in Turkey	1300
<i>Berna Cantürk-Günhan, Deniz Özen</i>	

Leading teachers to perceive and use technologies as resources for the construction of mathematical meanings	1310
<i>Eleonora Faggiano</i>	
The teacher's use of ICT tools in the classroom after a semiotic mediation approach.....	1320
<i>Mirko Maracci, Maria Alessandra Mariotti</i>	
Establishing didactical praxeologies: teachers using digital tools in upper secondary mathematics classrooms	1330
<i>Mary Billington</i>	
Dynamic geometry software: the teacher's role in facilitating instrumental genesis	1340
<i>Nicola Bretscher</i>	
Instrumental orchestration: theory and practice.....	1349
<i>Paul Drijvers, Michiel Doorman, Peter Boon, Sjef van Gisbergen</i>	
Teaching Resources and teachers' professional development: towards a documental approach of didactics	1359
<i>Ghislaine Gueudet, Luc Trouche</i>	
An investigative lesson with dynamic geometry: a case study of key structuring features of technology integration in classroom practice.....	1369
<i>Kenneth Ruthven</i>	
Methods and tools to face research fragmentation in technology enhanced mathematics education.....	1379
<i>Rosa Maria Bottino, Michele Cerulli</i>	
The design of new digital artefacts as key factor to innovate the teaching and learning of algebra: the case of Alnuset	1389
<i>Giampaolo Chiappini, Bettina Pedemonte</i>	
Casyopée in the classroom: two different theory-driven pedagogical approaches.....	1399
<i>Mirko Maracci, Claire Cazes, Fabrice Vandebrouck, Maria Alessandra Mariotti</i>	
Navigation in geographical space	1409
<i>Christos Markopoulos, Chronis Kynigos, Efi Alexopoulou, Alexandra Koukiou</i>	
Making sense of structural aspects of equations by using algebraic-like formalism.....	1419
<i>Foteini Moustaki, Giorgos Psycharis, Chronis Kynigos</i>	
Relationship between design and usage of educational software: the case of Aplusix	1429
<i>Jana Trgalova, Hamid Chaachoua</i>	
 WORKING GROUP 8	
Questions and thoughts for researching cultural diversity and mathematics education	1439
Introduction.....	
<i>Guida de Abreu, Sarah Crafter, Núria Gorgorió</i>	
A survey of research on the mathematics teaching and learning of immigrant students.....	1443
<i>Marta Civil</i>	
Parental resources for understanding mathematical achievement in multiethnic settings.....	1453
<i>Sarah Crafter</i>	
Discussing a case study of family training in terms of communities of practices and adult education.....	1462
<i>Javier Díez-Palomar, Montserrat Prat Moratona</i>	

Understanding Ethnomathematics from its criticisms and contradictions.....	1473
<i>Maria do Carmo Domite, Alexandre Santos Pais</i>	
Using mathematics as a tool in Rwandan workplace settings: the case of taxi drivers	1484
<i>Marcel Gahamanyi, Ingrid Andersson, Christer Bergsten</i>	
Parents' experiences as mediators of their children's learning: the impact of being a parent-teacher	1494
<i>Rachael McMullen, Guida de Abreu</i>	
Batik: another way of learning mathematics	1506
<i>Lucília Teles, Margarida César</i>	
The role of Ethnomathematics within mathematics education	1517
<i>Karen François</i>	
WORKING GROUP 9	
Different theoretical perspectives and approaches in mathematics education research	1527
Introduction.....	1529
<i>Susanne Prediger, Marianna Bosch, Ivy Kidron, John Monaghan, Gérard Sensevy</i>	
Research problems emerging from a teaching episode: a dialogue between TDS and ATD	1535
<i>Michèle Artigue, Marianna Bosch, Joseph Gascón, Agnès Lenfant</i>	
Complementary networking: enriching understanding.....	1545
<i>Ferdinando Arzarello, Angelika Bikner-Ahsbabs, Cristina Sabena</i>	
Interpreting students' reasoning through the lens of two different languages of description: integration or juxtaposition?	1555
<i>Christer Bergsten, Eva Jablonka</i>	
Coordinating multimodal social semiotics and institutional perspective in studying assessment actions in mathematics classrooms.....	1565
<i>Lisa Björklund-Boistrup, Staffan Selander</i>	
Integrating different perspectives to see the front and the back: The case of explicitness	1575
<i>Uwe Gellert</i>	
The practice of (university) mathematics teaching: mediational inquiry in a community of practice or an activity system.....	1585
<i>Barbara Jaworski</i>	
An interplay of theories in the context of computer-based mathematics teaching: how it works and why	1595
<i>Helga Jungwirth</i>	
On the adoption of a model to interpret teachers' use of technology in mathematics lessons	1605
<i>Jean-Baptiste Lagrange, John Monaghan</i>	
The joint action theory in didactics: why do we need it in the case of teaching and learning mathematics?.....	1615
<i>Florence Ligozat, Maria-Luisa Schubauer-Leoni</i>	
Teacher's didactical variability and its role in mathematics education	1625
<i>Jarmila Novotná, Bernard Sarrazy</i>	
The potential to act for low achieving students as an example of combining use of different theories	1635
<i>Ingolf Schäfer</i>	

Outline of a joint action theory in didactics	1645
<i>Gérard Sensevy</i>	
The transition between mathematics studies at secondary and tertiary levels; individual and social perspectives.....	1655
<i>Erika Stadler</i>	
Combining and Coordinating theoretical perspectives in mathematics education research	1665
<i>Tine Wedege</i>	
Comparing theoretical frameworks in didactics of mathematics: the GOA-model.....	1675
<i>Carl Winslow</i>	
WORKING GROUP 10	
Mathematical curriculum and practice	1685
Introduction.....	
<i>Leonor Santos, José Carrillo, Alena Hospesova, Maha Abboud-Blanchard</i>	
Effective ‘blended’ professional development for teachers of mathematics: Design and evaluation of the “UPOLA” Program	1694
<i>Lutz Hellmig</i>	
Teachers’ efficacy beliefs and perceptions regarding the implementation of new primary mathematics curriculum.....	1704
<i>Isil Isler, Erdine Cakiroglu</i>	
Curriculum management in the context of a mathematics subject group	1714
<i>Cláudia Canha Nunes, João Pedro da Ponte</i>	
Gestures and styles of communication: are they intertwined?.....	1724
<i>Chiara André</i>	
Teachers’ subject knowledge: the number line representation	1734
<i>Maria Doritou, Eddie Gray</i>	
Communication as social interaction. Primary School Teacher Practices.....	1744
<i>Antonio Guerreiro, Lurdes Serrazina</i>	
Experimental devices in mathematics and physics standards in lower and upper secondary school, and their consequences on teacher’s practices	1751
<i>Fabrice Vandebrouck, Cecile de Hosson, Aline Robert</i>	
Professional development for teachers of mathematics: opportunities and change.....	1761
<i>Marie Joubert, Jenni Back, Els De Geest, Christine Hirst, Rosamund Sutherland</i>	
Teachers’ perception about infinity: a process or an object?.....	1771
<i>Maria Kattou, Michael Thanasia, Katerina Kontoyianni, Constantinos Christou, George Philippou</i>	
Perceptions on teaching the mathematically gifted.....	1781
<i>Katerina Kontoyianni, Maria Kattou, Polina Ioannou, Maria Erodou, Constantinos Christou, Marios Pittalis</i>	

The nature on the numbers in grade 10: A professional problem.....	1791
<i>Mirène Larguier, Alain Bronner</i>	
A European project for professional development of teachers through a research based methodology: The questions arisen at the international level, the Italian contribution, the knot of the teacher-researcher identity	1801
<i>Nicolina A. Malara, Roberto Tortora</i>	
Why is there not enough fuss about affect and meta-affect among mathematics teachers?.....	1811
<i>Manuela Moscucci</i>	
The role of subject knowledge in Primary Student teachers' approaches to teaching the topic of area	1821
<i>Carol Murphy</i>	
Developing of mathematics teachers' community: five groups, five different ways	1831
<i>Regina Reinup</i>	
Foundation knowledge for teaching: contrasting elementary and secondary mathematics.....	1841
<i>Tim Rowland</i>	
Results of a comparative study of future teachers from Australia, Germany and Hong Kong with regard to competences in argumentation and proof	1851
<i>Björn Schwarz, Gabriele Kaiser</i>	
Kate's conceptions of mathematics teaching: Influences in the first three years	1861
<i>Fay Turner</i>	
Pre-service teacher-generated analogies for function concepts	1871
<i>Behiye Ubuz, Ayşegül Eryılmaz, Utkun Aydın, Ibrahim Bayazit</i>	
Technology and mathematics teaching practices: about in-service and pre-service teachers	1880
<i>Maha Abboud-Blanchard</i>	
Teachers and triangles.....	1890
<i>Sylvia Alatorre, Mariana Saíz</i>	
Mathematics teacher education research and practice: researching inside the MICA program	1901
<i>Joyce Mgombelo, Chantal Buteau</i>	
Cognitive transformation in professional development: some case studies	1911
<i>Jorge Soto-Andrade</i>	
What do student teachers attend to?.....	1921
<i>Nad'a Stehlíková</i>	
The mathematical preparation of teachers: A focus on tasks.....	1931
<i>Gabriel J. Stylianides, Andreas J. Stylianides</i>	
Problem posing and development of pedagogical content knowledge in pre-service teacher training	1941
<i>Marie Tichá, Alena Hospesová</i>	
Sustainability of professional development	1951
<i>Stefan Zehetmeier</i>	
A collaborative project as a learning opportunity for mathematics teachers	1961
<i>Maria Helena Martinho, João Pedro da Ponte</i>	
Reflection on Practice: content and depth.....	1971
<i>Christina Martins, Leonor Santos</i>	
Developing mathematics teachers' education through personal reflection and collaborative inquiry: which kinds of tasks?.....	1981
<i>Angela Pesci</i>	

The learning of mathematics teachers working in a peer group	1991
<i>Martha Witterholt, Martin Goedhart</i>	
Use of focus groups interviews in mathematics educational research.....	2000
<i>Bodil Kleve</i>	
Analyses of interactions in a collaborative context of professional development.....	2010
<i>Maria Cinta Muñoz-Catalán, José Carrillo, Nuria Climent</i>	
Adapting the knowledge quarter in the Cypriot mathematics classroom	2020
<i>Marilena Petrou</i>	
Professional knowledge in an improvisation episode: the importance of a cognitive model.....	2030
<i>C. Miguel Ribeiro, Rute Monteiro, José Carrillo</i>	
WORKING GROUP 11	
Applications and modelling	2040
Introduction.....	2042
<i>Morten Blomhøj</i>	
Mathematical modelling in teacher education – Experiences from a modelling seminar	2046
<i>Rita Borrromeo Ferri, Werner Blum</i>	
Designing a teacher questionnaire to evaluate professional development in modelling	2056
<i>Katja Maaß, Johannes Gurlitt</i>	
Modeling in the classroom – Motives and obstacles from the teacher’s perspective.....	2066
<i>Barbara Schmidt</i>	
Modelling in mathematics teachers’ professional development.....	2076
<i>Jeroen Spandaw, Bert Zwaneveld</i>	
Modelling and formative assessment pedagogies mediating change in actions of teachers and learners in mathematics classrooms	2086
<i>Geoff Wake</i>	
Towards understanding teachers’ beliefs and affects about mathematical modelling.....	2096
<i>Jonas Bergman Ärlebäck</i>	
The use of motion sensor can lead the students to understanding the cartesian graph.....	2106
<i>Maria Lucia Lo Cicero, Filippo Spagnolo</i>	
Interacing populations in a restricted habitat – Modelling, simulation and mathematical analysis in class.....	2116
<i>Christina Roeckerath</i>	
Aspects of visualization during the exploration of “quadratic world” via the ICT Problem “fireworks”	2126
<i>Mária Lalinská, Janka Majherová</i>	
Mathematical modeling in class with and without technology.....	2136
<i>Hans-Stefan Siller, Gilbert Greefrath</i>	
The ‘ecology’ of mathematical modelling: constraints to its teaching at university level	2146
<i>Berta Barquero, Marianna Bosch, Josep Gascón</i>	
The double transposition in mathematisation at primary school	2156
<i>Richard Cabassut</i>	
Exploring the use of theoretical frameworks for modelling-oriented instructional design	2166
<i>F.J. García, L. Ruiz-Higueras</i>	

Study of a practical activity: engineering projects and their training context	2176
<i>Avenilde Romo Vázquez</i>	
Fitting models to data: the mathematising step in the modelling process	2186
<i>Lidia Serrano, Marianna Bosch, Josep Gascón</i>	
What roles can modelling play in multidisciplinary teaching.....	2196
<i>Mette Andresen</i>	
Modelling in environments without numbers – A case study.....	2206
<i>Roxana Grigoras</i>	
Modelling activities while doing experiments to discover the concept of variable.....	2216
<i>Simon Zell, Astrid Beckmann</i>	
Modeling with technology in elementary classrooms.....	2226
<i>N. Mousoulides, M. Chrysostomou, M. Pittalis, C. Christou</i>	
WORKING GROUP 12	
Advanced mathematical thinking.....	2236
Introduction.....	
<i>Roza Leikin, Claire Cazes, Joanna Mamona-Dawns, Paul Vanderlind</i>	
A theoretical model for visual-spatial thinking.....	2246
<i>Conceição Costa, José Manuel Matos, Jaime Carvalho e Silva</i>	
Secondary-tertiary transition and students’ difficulties: the example of duality	2256
<i>Martine De Vleeschouwer</i>	
Learning advanced mathematical concepts: the concept of limit	2266
<i>António Domingos</i>	
Conceptual change and connections in analysis	2276
<i>Kristina Juter</i>	
Using the onto-semiotic approach to identify and analyze mathematical meaning in a multivariate context.....	2286
<i>Mariana Montiel, Miguel R. Wilhelmi, Draga Vidakovic, Iwan Elstak</i>	
Derivatives and applications; development of one student’s understanding	2296
<i>Gerrit Roorda, Pauline Vos, Martin Goedhart</i>	
Finding the shortest path on a spherical surface: “academics” and “reactors” in a mathematics dialogue.....	2306
<i>Maria Kaisari, Tasos Patronis</i>	
Number theory in the national compulsory examination at the end of the French secondary level: between organising and operative dimensions.....	2316
<i>Véronique Battie</i>	
Defining, proving and modelling: a background for the advanced mathematical thinking.....	2326
<i>Mercedes García, Victoria Sánchez, Isabel Escudero</i>	
Necessary realignments from mental argumentation to proof presentation	2336
<i>Joanna Mamona-Downs, Martin Downs</i>	
An introduction to defining processes	2346
<i>Cécile Ouvrier-Bufferet</i>	
Problem posing by novice and experts: comparison between students and teachers	2356
<i>Cristian Voica, Ildikó Pelczer</i>	

Advanced mathematical knowledge: how is it used in teaching?.....	2366
<i>Rina Zazkis, Roza Leikin</i>	
Urging calculus students to be active learners: what works and what doesn't.....	2376
<i>Buma Abramovitz, Miryam Berezina, Boris Koichu, Ludmila Shvartsman</i>	
From numbers to limits: situations as a way to a process of abstraction.....	2386
<i>Isabelle Bloch, Imène Ghedamsi</i>	
From historical analysis to classroom work: function variation and long-term development of functional thinking.....	2396
<i>Renaud Chorlay</i>	
Experimental and mathematical control in mathematics.....	2406
<i>Nicolas Giroud</i>	
Introduction of the notions of limit and derivative of a function at a point.....	2416
<i>Ján Gunčaga</i>	
Factors influencing teacher's design of assessment material at tertiary level.....	2426
<i>Marie-Pierre Lebaud</i>	
Design of a system of teaching elements of group theory.....	2436
<i>Ildar Safuanov</i>	
WORKING GROUP 13	
Comparative studies in mathematics education.....	2446
Introduction.....	
<i>Eva Jablonka, Paul Andrews, Birgit Pepin</i>	
Comparing Hungarian and English mathematics teachers' professional motivations.....	2452
<i>Paul Andrews</i>	
Spoken mathematics as a distinguishing characteristic of mathematics classrooms in different countries.....	2463
<i>David Clarke, Xu Li Hua</i>	
Mathematical behaviors of successful students from a challenged ethnic minority.....	2473
<i>Tiruwork Mulat, Abraham Arcavi</i>	
A problem posed by J. Mason as a starting point for a Hungarian-Italian teaching experiment within a European project.....	2483
<i>Giancarlo Navarra, Nicolina A. Malara, András Ambrus</i>	
A comparison of teachers' beliefs and practices in mathematics teaching at lower secondary and upper secondary school.....	2494
<i>Hans Kristian Nilsen</i>	
Mathematical tasks and learner dispositions: A comparative perspective.....	2504
<i>Birgit Pepin</i>	
Elite mathematics students in Finland and Washington: access, collaboration, and hierarchy.....	2513
<i>Jennifer von Reis Saari</i>	
International comparative research on mathematical problem solving: Suggestions for new research directions.....	2523
<i>Constantinos Xenofontos</i>	

WORKING GROUP 14	
Early years mathematics	2533
Introduction.....	2535
<i>Patti Barber</i>	
Girls and boys in “the land of mathematics” 6 to 8 years old children’s relationship to mathematics interpreted from their drawings.....	2537
<i>Päivi Perkkilä, Eila Aarnos</i>	
“Numbers are actually not bad” Attitudes of people working in German kindergarten about mathematics in kindergarten	2547
<i>Christiane Benz</i>	
Learning mathematics within family discourses.....	2557
<i>Birgit Brandt, Kerstin Tiedemann</i>	
Orchestration of mathematical activities in the kindergarten: the role of questions.....	2567
<i>Martin Carlsen, Ingvald Erfjord, Per Sigurd Hundeland</i>	
Didactical analysis of a dice game	2577
<i>Jean-Luc Dorier, Céline Maréchal</i>	
“Tell them that we like to decide for ourselves” Children’s agency in mathematics education.....	2587
<i>Troels Lange</i>	
Exploring the relationship between justification and monitoring among kindergarten children	2597
<i>Pessia Tsamir, Dina Tirosh, Esther Levenson</i>	
Early years mathematics – The case of fractions	2607
<i>Ema Mamede</i>	
Only two more sleeps until the school holidays: referring to quantities of things at home.....	2617
<i>Tamsin Meaney</i>	
Supporting children potentially at risk in learning mathematics Findings of an early intervention study.....	2627
<i>Andrea Peter-Koop</i>	
The structure of prospective kindergarten teachers’ proportional reasoning.....	2637
<i>Demetra Pitta-Pantazi, Constantinos Christou</i>	
How can games contribute to early mathematics education? – A video-based study	2647
<i>Stephanie Schuler, Gerald Wittmann</i>	
Natural differentiation in a pattern environment (4 year old children make patterns)	2657
<i>Ewa Swoboda</i>	
Can you do it in a different way?	2667
<i>Dina Tirosh, Pessia Tsamir, Michal Tabach</i>	
 WORKING GROUP 15	
Theory and research on the role of history in mathematics education	2677
Introduction.....	2679
<i>Fulvia Furinghetti, Jean-Luc Dorier, Uffe Jankvist, Jan van Maanen, Constantinos Tzanakis</i>	
The teaching of vectors in mathematics and physics in France during the 20th century	2682
<i>Cissé Ba, Jean-Luc Dorier</i>	

Geometry teaching in Iceland in the late 1800s and the van Hiele theory.....	2692
<i>Kristín Bjarnadóttir</i>	
Introducing the normal distribution by following a teaching approach inspired by history: an example for classroom implementation in engineering education.....	2702
<i>Mónica Blanco, Marta Ginovart</i>	
Arithmetic in primary school in Brazil: end of the nineteenth century	2712
<i>David Antonio Da Costa</i>	
Historical pictures for acting on the view of mathematics.....	2722
<i>Adriano Demattè, Fulvia Furinghetti</i>	
Students' beliefs about the evolution and development of mathematics	2732
<i>Uffe Thomas Jankvist</i>	
Using history as a means for the learning of mathematics without losing sight of history: the case of differential equations	2742
<i>Tinne Hoff Kjeldsen</i>	
What works in the classroom	
Project on the history of mathematics and the collaborative teaching practice	2752
<i>Snezana Lawrence</i>	
Intuitive geometry in early 1900s Italian middle school.....	2762
<i>Marta Menghini</i>	
The appropriation of the New Math on the Technical Federal School of Parana in 1960 and 1970 decades	2771
<i>Bárbara Winiarski Diesel Novaes, Neuza Bertoni Pinto</i>	
History, heritage, and the UK mathematics classroom.....	2781
<i>Leo Rogers</i>	
Introduction of an historical and anthropological perspective in mathematics: an example in secondary school in France.....	2791
<i>Claire Tardy, Vivianne Durand-Guerrier</i>	
The implementation of the history of mathematics in the new curriculum and textbooks in Greek secondary education	2801
<i>Yannis Thomaidis, Constantinos Tzanakis</i>	

CERME 6 – GENERAL INTRODUCTION

INTRODUCTION TO CERME 6

BY BARBARA JAWORSKI PRESIDENT OF ERME EUROPEAN SOCIETY FOR RESEARCH IN MATHEMATICS EDUCATION

CERME is the two-yearly congress of ERME, the European Society for Research in Mathematics Education. CERME 6 marks more than a decade of ERME and it is important to recognise the achievements of the society over this time.

In May 1997, a group of 16 scholars from different European countries met in Osnabrück, Germany, for three days to discuss the formation of a European society in mathematics education. In true European spirit, we decided that we wanted a society which would bring together researchers from across Europe, particularly including colleagues from Eastern Europe, fostering *communication, cooperation* and *collaboration*. We wanted a conference that would explicitly provide such opportunity. We wanted especially to encourage and contribute to the education of young researchers. Thus ERME was born and began to take shape.

We decided on a two-yearly conference, or *congress* as it later became known, and the name CERME emerged – Congress of the European Society for Research in Mathematics Education. CERME should have a group structure in which researchers would have sufficient time to really get to know each other, share and discuss their research and engage in deep scholarly debate. The first CERME was planned for February 1999, at Osnabrück. The Program Committee took very seriously the aims for the conference expressed at the 1997 meeting. Seven working groups were planned and 12 hours were provided for work in a group. To avoid most of the conference time being taken up by paper presentation, it was decided there would be no oral presentations at the conference. Papers would be presented in written form before the conference with sufficient time for group participants to read the papers. The 12 hours would be spent discussing the papers and working on themes and issues suggested by the papers and the group leaders. Over the succeeding years, a group led by Konrad Krainer (Austria) and Paolo Boero (Italy) developed a plan and style for a YERME summer school (YESS). The first summer school was held in Klagenfurt, Austria in August 2002. Like CERME, the summer school was based around groups, working on papers submitted by the young researchers, each with an international “expert” as leader.

The pattern of CERME and YERME has developed so they take place in alternative years, the group structure being developed and carried forwards from one to the next. We had CERME 2 in Mariánské Lázně, Czech Republic in 2001; CERME 3 in Bellaria, Italy in 2003; and YESS 2 in Pödebrady, Czech Republic in 2004. CERME 4 took place in Saint Feliu, Spain in February, 2005 and YESS 3 in Jyväskylä, Finland in August 2006. CERME 5 was held in Cyprus in February 2007, and YESS

4 in Trabzon, Turkey in August, 2008. CERME 6 will take place in Lyon, France in 2009 and YESS 5 in Palermo, Italy in 2010. People came from these events speaking of *inspirational* experiences. It seemed clear that the events generated something that we came to call *the CERME Spirit*. Based fundamentally on the three Cs, communication, cooperation and collaboration, the CERME Spirit was about the inspiration that derives from serious scholarly tackling of ideas and concepts in key areas and of mathematics education research with colleagues from multiple nations, facilitated by the group design of the events.

However, the group design was not without its critics. Some critics felt constrained by the requirement to spend a conference, largely, in just one group. Some felt that a conference ought to offer a greater variety of opportunity to participants. Participants should be free to choose where to be at any time. However, the group work at CERME or YESS would be seriously disrupted if participants were to hop from group to group, not engaging seriously with the work in any one. Some suggested that perhaps planning could allow participants to take part in two groups, so that engagement in both could be serious. Such ideas have been considered by the ERME Board and Programme Committees but so far we have remained faithful to the initial conception. Many participants have said in evaluation of the events that the opportunity to spend serious time in one group allowed them to really get to know researchers from other countries, and that this contributed significantly to the depth of thinking that was possible.

We want to encourage wider participation to ongoing activity in our Society, with more nations contributing to hosting events and a secure financial platform for continuing our inclusive communication, cooperation and collaboration within Europe. Further details of ERME can be found at the following site: <http://ermeweb.free.fr/>

Barbara Jaworski – President of ERME

PRESENTATION OF CERME 6 BY FERDINANDO ARZARELLO, CHAIR OF THE SCIENTIFIC INTERNATIONAL COMMITTEE

As pointed out in the document written by our President, CERME is a Congress designed to foster a communicative spirit in European mathematics education according to the three Cs of ERME: communication, cooperation and collaboration. It deliberately moves away from research presentations by individuals towards collaborative group work. Its main feature is a number (15) of thematic groups whose members have worked together in a common research area.

In addition to the working group sessions, there was:

- Two plenary lectures and a panel;
- Two parallel 1 hour sessions where the participants had the opportunity of debating with the plenarists;
- A poster session;
- Final parallel sessions (repeated twice), where each group has presented its work to the interested participants;
- Policy and purpose sessions to negotiate the work and directions of ERME.

The philosophy of our Congress is based on the following two issues:

- i. We need to know more about the research which has been done and is ongoing, and the research groups and research interests in different European countries;
- ii. We need to provide opportunities for cooperation in research areas and for inter-European collaboration between researchers in joint research projects.

In organising this Conference we considered both the ERME spirit and the observations from the questionnaires filled by the participants, which mainly concerned the plenary events. Consequently, the following structure was planned:

- Two plenary lectures of 75 minutes; each plenarist had a reactor: they had 60 minutes for their two presentations, and then there was 15 minutes for questions from the floor. Moreover the interested people had the opportunity to meet the plenarists in an informal meeting in another day.
- An other event is the special 2 hour plenary of the last day, which had three participants: the aim was to discuss a topic emerging from previous CERMEs,

analysing it from different standpoints and to give people the possibility of a wide debate.

The structure of the Working Groups was essentially the same: each group had more of 12 hours for discussing its topic. In the final Sunday session each group have presented the results of its work in two consecutive one hour slots, according to the model experienced in CERME 5, which had received the approval of the participants.

I think that all of us had a very exciting week, plenty of interesting scientific and social opportunities. In particular I underline the lecture of Prof. E. Ghys — <http://www.dimensions-math.org> — and the discussion on a Project of a European Journal of Mathematical Education.

I wish to thank the local organisers, and particularly Viviane Durand-Guerrier, for the enormous work they have done to make possible the realisation of this Conference.

Ferdinando Arzarello – Chair of the scientific international committee

QUALITY AND INCLUSION IN CERME 6: A PROPOSED REVIEW

The European Society for Research in Mathematics Education (ERME), and its principal activities CERME (2-yearly Congress of ERME) and YERME (meetings of Young researchers in ERME) are committed to the three Cs: *Communication*, *Cooperation* and *Collaboration* in research in mathematics education. Over the years in which ERME has existed, the community has developed what has become known as “*The CERME Spirit*”. These words capture a practical manifestation of the objectives expressed in the three Cs. The phrase refers to an *inclusivity* of working in which people genuinely work together, in which all are welcome, and in which members work hard to ensure that all can take a full part in activity. A major factor and issue – that of the language of our work – has been addressed seriously; different groups devising their own approaches to their working language.

However, these things are not straightforward and issues arise as soon as we construct practical situations. The main example of this concerns the scientific quality of our work in mathematics education research. Of course we aspire to a high quality of scientific work, just as we aspire to operate in fully inclusive ways. Ideally we should like there to be compatibility between the two. But what does or can this look like in practice?

These issues face group leaders as soon as they set out to construct a programme of work for their group, starting with a call for papers. Responding to this call, we see that many papers are now received for all groups. This suggests that researchers in our field want to be part of CERME and offer their work to colleagues in CERME. From an inclusive point of view, all papers should be welcome and all those wishing to participate should have a place. However, from a scientific point of view, papers should be reviewed according to scientific criteria, those that are of a suitable scientific quality (according to the group leaders) should be accepted and others rejected. In practice this means that authors of rejected papers may not be able to attend the congress since funding depends on an accepted paper. The practice seems to go against principles of inclusion.

The ERME Board, and Programme Committees of CERME conferences have been aware of these issues and have addressed them by creating a two stage review process. For presentation of papers at the congress, a much more open attitude should be taken to the criteria, aiming to include as many participants as possible. At this stage, feedback to prospective participants should detail what is required for a paper to be acceptable for the scientific proceedings following the congress. Papers not meeting these requirements would not be accepted for the proceedings. Of course, it is then up to the group leaders to determine how to make the necessary decisions: what is acceptable for presentation, and what are the more strict criteria for publication? They also have to decide how to conduct the work of the group in an

inclusive way. Similarly those organising YERME events have to decide how to ensure both quality and inclusion in practice.

Our sixth CERME achieved, it therefore feels like a time to review these issues and procedures. For this purpose, a small group of interested members of ERME has agreed to survey participants in CERME 6 and seek views on the processes and issues that are involved. We have included an opportunity to comment in the evaluation questionnaire for CERME 6 and possibility to send us a personal communication (written) to express your views in more detail. We have also asked group leaders, present and past, to tell us how they have made decisions and what difficulties if any there have been.

As a result of analysing the information received we hope to write a paper for a scientific edited book on the topic of inclusion and quality. Such a paper could also act as a basis for future policy in ERME, CERME and YERME.

*Barbara Jaworski,
Ferdinando Arzarello
M. Alessandra Marriotti
Constantinos Christou
Joao Pedro da Ponte*

SCIENTIFIC PROGRAM

CERME 6 PLENARY CONFERENCES

Jan 28, 15:30 - 16:45

Luis Radford, Université Laurentienne, Ontario, Canada.

SIGNS, GESTURES, MEANINGS: ALGEBRAIC THINKING FROM A CULTURAL SEMIOTIC PERSPECTIVE.

Reactor: Heinz Steinbring (Duisburg-Essen University)

Summary. In this presentation I will deal with the ontogenesis of algebraic thinking. Drawing on a cultural semiotic perspective, informed by current anthropological and embodied theories of knowing and learning, in the first part of my talk I will comment on the shortcomings of traditional mental approaches to cognition. In tune with contemporary research in neuroscience, cultural psychology, and semiotics, I will contend that we are better off conceiving of thinking as a sensuous and sign-mediated activity embodied in the corporeality of actions, gestures, and artifacts. In the second part of my talk, I will argue that algebraic thinking can be characterized in accordance with the semiotic means to which the students resort in order to express and deal with algebraic generality. I will draw upon results obtained in the course of a 10-year longitudinal classroom research project to offer examples of students' forms of algebraic thinking. Two of the most elementary forms of algebraic thinking identified in our research are characterized by their contextual and embodied nature; they rely extensively upon rhythm and perceptual and deictic (linguistic and gestural) mechanisms of meaning production. Furthermore, keeping in line with the situated nature of the students' mathematical experience, signs here usually designate their objects in an indexical manner. These elementary forms of algebraic thinking differ from the traditional one—based on the standard alphanumeric symbolism—in that the latter relies on sign distinctions of a morphological kind. Here signs cease to designate objects in the usual indexical sense to give rise to symbolic processes of recognition and manipulation governed by sign shape.

The aforementioned conception of thinking in general and the ensuing distinction of forms of algebraic thinking shed some light on the kind of abstraction that is entailed by the use of standard algebraic symbolism. They intimate some of the conceptual shifts that the students have to make in order to gain fluency in a cultural sophisticated form of mathematical thinking. Voice, gesture, and rhythm fade away. Embodied and contextual ways of signifying are then replaced with a perceptual activity where differences and similarities are a matter of morphology, and where meaning becomes relational.

Jan 29, 9:15 - 10:30

Paola Valero, Aalborg University, Denmark.

ATTENDING TO SOCIAL CHANGES IN EUROPE: CHALLENGES FOR MATHEMATICS EDUCATION RESEARCH IN THE 21ST CENTURY

Reactor: Margarida Alexandra da Piedade Silva Cesar (Lisbon University)

Summary. Based on an analysis of mathematics education research as an academic field and on current social, political and economic transformations in many European countries, I would argue for the need to rethink and enlarge definitions and views of mathematics education as a scientific field of study in order to provide better understandings and alternatives for practice in the teaching and learning of mathematics today. I will explore the notion of the “network of mathematics education practices” as a complex, multi-layered space of social practice where the meanings of the teaching and learning of mathematics are constituted. I will illustrate the potentiality of this notion to envision possible research paths in the field. I will illustrate these with the research that my colleagues and I have been carrying on multicultural classrooms in Denmark; as well as will offer examples of other research studies in Europe and other parts of the world where I see that the discipline is gaining newer insights that could allow attending to the social changes and challenges of the 21st century.

Feb 1st, 11:00 – 13:00

SPECIAL PLENARY: WAYS OF WORKING WITH DIFFERENT THEORETICAL APPROACHES IN MATHEMATICS EDUCATION RESEARCH

Speakers: **Angelika Bikner-Ahsbabs, Bremen University, Germany**

John Monaghan, University of Leeds, United Kingdom

Chair: **Tommy Dreyfus, Tel Aviv University, Israel**

Structure : This plenary activity is planned to last 2 hours and will comprise five parts

Introduction (T. Dreyfus, 5 min)

Networking of theories – why and how? (A. Bikner-Ahsbabs, 25 min + 5 min for clarifications)

Taking the appropriate parts from a variety of theories (J. Monaghan, 25 min + 5 min for clarifications)

Questions to the floor (T. Dreyfus, 10 min)

Questions and contributions from the audience with reactions from the speakers (45 min)

Background. The development and elaboration of theoretical constructs that allow research in mathematics education to progress has long been a focus of mathematics education researchers in Europe. This focus has found its expression in many CERME working groups: some are focused around a specific theoretical approach and others allow researchers from different theoretical traditions and backgrounds to meet and discuss. More specifically, relationships between theories have been made the explicit focus of attention of the theory working group that started at CERME 4 in 2005. The present plenary activity inserts itself in this line of work of CERME, and aims to broaden the discussion about relationships between theories to include members of all CERME working groups.

Abstract by Angelika Bikner-Ahsbals: Networking of theories – why and how? Research in mathematics education addresses teaching and learning of mathematics in a wide sense. For example, theories about learning fractions may tell a lot of different things about learning fractions. Some of them are about mistakes and why some mistakes are stable. Others may tell us about how students can be motivated to learn fractions. There are theories about how fraction concepts can be built best, which students' imaginations accompany learning fractions and what abstraction processes can be observed. In addition, we have to distinguish between theories for gifted students and theories for students with special needs, etc.

These considerations already show that research objects within mathematics education are complex. This complexity has led to a large variety of theoretical approaches. Every successful new theoretical view broadens or deepens insight in a phenomenon, hence, enriches our knowledge about the phenomenon. Therefore, it seems necessary to regard the large diversity of theories as richness. However, the rich diversity of theoretical approaches engenders problems of understanding and communicating. Sometimes we find the same terms meaning different things, for example the different concepts of abstraction, mathematising and constructing. However, we also find different words for the same or similar meanings, for example reification and constructing can both mean building a new knowledge entity.

Hence, a large diversity of theories can be regarded as richness but it also causes difficulties for researchers to understand each other and for teachers and teacher trainers to make use of research results in an adequate way. These problems raise the following questions: How could researchers gainfully frame the use of the diversity of theoretical approaches? What kind of benefit can be gained through such frames? How can theories be made more useful for practitioners?

In the plenary talk, networking of theories is proposed to be a fruitful approach to frame the diversity of theories or theoretical approaches. It has been practiced and reflected on since 2005 (CERME4) within a group of researchers networking their theories. This work has already shown that networking of theories means more than creating a consistent frame to investigate a research question it is a systematic way of theory development. In the plenary talk, an example is used to clarify the meaning

and to describe some benefit of it for the research and the practice of teaching and learning mathematics.

Abstract by John Monaghan: Taking the appropriate parts from a variety of theories.

I will argue the case for ‘taking the appropriate parts from a variety of theories according to needs of the research’ rather than trying to ‘merge theories’. One part of my argument is who I and, if I may extend this, who most CERME participants are – working mathematics education researchers. Mathematics education research is demanding and does not (except for a few gifted individuals) allow researchers to become specialist philosophers, psychologists and /or sociologists; but we may find it useful to use the ideas of philosophers, psychologists and /or sociologists. Another part of my argument will concern theoretical frameworks within mathematics education and I will argue for caution with regard to attempts to merge such theories. These theories have, in general, distinct historical roots, developed in academic communities which have appropriated constructs in specific ways and the ‘grain sizes’ of their analyses often differ. Attempting to merge whole theories, as opposed to appropriating constructs, comes with a real danger of creating an ill-formed hybrid.

So will I argue that mathematics education researchers should ‘pick a little bit from this theory and a little bit from that theory’? Well, yes, I will ... but with caution! I will argue that the ‘bits we pick’ depend on the situation, the specific focus of the research in which we are engaged, and the consistency of ‘bits we pick’.

I have avoided referring to specific theories in this abstract but I will detail theories in my talk and I will also use research studies as cases to exemplify my arguments.

WORKING GROUPS

15 working groups: 7 sessions, 1 or 2 per day, duration 1h30 or 2h

Final group reports: Sunday Feb 1st, 8:30 - 10:30

Poster Session: Thursday Jan. 29 17:15 - 18:30

N.B. The posters remain during the all congress in the hall of the THEMIS. During the poster session, the authors were present.

Group 1: Affect and mathematical thinking - This includes the role of beliefs, emotions, and other affective factors

Markku Hannula, Finland (Chair); Tine Wedege, Norway; Marilena Pantziara, Cyprus.

Group 2: Argumentation and proof - This includes epistemological and historical studies, learning issues and classroom situations

Maria Alessandra Mariotti, Italy (Chair); Patrick Gibel, France; Leonor Camargo, Colombia; Kristina Reiss, Germany.

Group 3: Stochastic thinking - This includes epistemological and educational issues, pupils cognitive processes and difficulties, and curriculum issues

Andreas Eichler, Germany (Chair); Maria Gabriella Ottaviani, Italy; Dave Pratt, United kingdom; Floriane Wozniak, France.

Group 4: Algebraic thinking - This includes epistemological and educational issues, pupils cognitive processes and difficulties, and curriculum issues

Chair: Giorgio Bagni, Italy (Chair); Janet Ainley, United Kingdom; Lisa Hefendehl-Hebeker, Germany; Jean-Baptiste Lagrange, France.

Group 5: Geometrical thinking - This includes epistemological and educational issues, pupils cognitive processes and difficulties, and curriculum issues

Alain Kuzniak, France (Chair); Iliada Elia, Cyprus; Mathias Hattermann Germany; Filip Roubicek, Czech Republic.

Group 6: Mathematics and language - This includes semiotics and communication in classrooms, social processes in learning and teaching mathematics

Candia Morgan, United Kingdom (Chair); Marie-Thérèse Farrugia (Malta); Marei Fetzer (Germany); Alain Mercier, France.

Group 7: Technologies and resources in mathematical education - This includes teaching and learning environments

Ghislaine Gueudet, France (Chair); Rosa Maria Bottino, Italy; Stephen Hegedus, United States of America; Hans-Georg Weigand, Germany.

Group 8: Cultural diversity and mathematics education - This includes students' diverse backgrounds and identities, social and cultural processes involved, political issues in the educational and school policies.

Chair: Guida de Abreu, United Kingdom (Chair); Nuria Gorgorio, Spain; Sarah Crafter, United Kingdom.

Group 9: Different theoretical perspectives / approaches in research in mathematics education - This includes ways of linking theory and practice and paradigms of research in ME.

Susanne Prediger, Germany (Chair); Marianna Bosch, Spain; Ivy Kidron, Israel; John Monaghan, United Kingdom; Gérard Sensevy, France.

Group 10: From a study of teaching practices to issues in teacher education - This includes teachers' beliefs and the role of the teacher in the classroom, as well as strategies for teacher education and links between: theory and practice, research and teaching and teacher education, collaborative research.

Chair: Leonor Santos (Portugal) José Carrillo, Spain; Alena Hospesova, Czech Republic; Maha Abboud-Blanchard, France.

Group 11: Applications and modelling - This includes theoretical and empirical-based reflections on: the modelling process and necessary competencies, adequate applications and modelling examples, epistemological and curricular aspects, beliefs and attitudes, assessment and the role of technology.

Morten Blomhøj, Denmark (Chair); Susana Carreira, Portugal; Katja Maass, Germany; Geoff Wake, United Kingdom.

Group 12: Advanced mathematical thinking - This includes conceptual attainment, proof techniques, problem-solving, processes of abstraction, at the upper secondary and tertiary educational level.

Roza Leikin, Israel (Chair); Claire Cazes, France; Joanna Mamona-Downs, Greece; Paul Vanderlind, Sweden.

Group 13: Comparative Studies in Mathematics Education - It includes questions surrounding mathematics teaching and learning in the classroom, learners' and teachers' experiences and identities, and policy issues in different cultures and/or countries.

Eva Jablonka, Sweden (Chair); Paul Andrews, United Kingdom; Birgit Pepin, United Kingdom; Pasi Reinikainen, Finland.

Group 14: Early Years Mathematics . This Working Group deals with the research domain of mathematics learning and mathematics education in the early years, age 3 to 7- In the last decades the interest in this topic has increased immensely.

Götz Krummheuer, Germany (Chair); Patti Barber, United Kingdom; Demetra Pitta-Pantazi, Cyprus; Ewa Swoboda, Poland.

Group 15: Theory and research on the role of history in Mathematics Education - The integration of history of mathematics in mathematics education is a subject which has received increasing attention during the last decades.

Chair: Fulvia Furinghetti , Italy (Chair); Jean-Luc Dorier, France; Uffe Thomas Jankvist, Denmark; Costantinos Tzanakis, Greece.

YERME - YOUNG ERME

YERME is an organization aiming at creating collaboration and mutual support among young researchers of different countries in the field of mathematics education. The two main activities of YERME are:

1. YESS – YERME Summer Schools

The aims of the Summer Schools are:

- To let people from different countries meet and establish a friendly and cooperative style of work in mathematics education research;
- To let people compare and integrate their preparation in mathematics education research in a peer discussion climate with the help of highly qualified and differently oriented experts;
- To let people present their research ideas, theoretical difficulties, methodological problems, and preliminary research results, in order to get suggestions (from other participants and experts) about possible developments, different perspectives, etc. and open the way to possible connections with nearby research projects and co-operation with researchers in other countries.

YESS1 took place in Klagenfurt, Austria, 2002; YESS2 at Podebrady, Czech Republic, 2004; YESS3 at Jyväskylä, Finland, 2006 and YESS4 at Trabzon, Turkey, 2008.

YESS5 will take place in Italy (August 2010). Ph.D., Master and post-graduate students and other people entering Mathematics Education research are invited to take part in YESS summer schools.

2. YERME day

The YERME-day takes place the day before CERME. The spirit is the same as YESS. Young European researchers take part in Discussion Groups and Working Groups. The topics of these groups are close to young researchers' interests. This kind of organization allows European students to meet and start to build links between different countries. They also have the opportunity to work with experts in the research education field. The program of the YERME-Day 2009 (January, 27th and 28th) is available on the YERME Website <http://yerme.eu> .

CERME 6 – PLENARY 1

Signs, gestures, meanings: Algebraic thinking from a cultural semiotic perspective

Luis Radford, Université Laurentienne, Ontario, Canada

Reactor: Heinz Steinbring (Duisburg-Essen University)

Summary. In this presentation I will deal with the ontogenesis of algebraic thinking. Drawing on a cultural semiotic perspective, informed by current anthropological and embodied theories of knowing and learning, in the first part of my talk I will comment on the shortcomings of traditional mental approaches to cognition. In tune with contemporary research in neuroscience, cultural psychology, and semiotics, I will contend that we are better off conceiving of thinking as a sensuous and sign-mediated activity embodied in the corporeality of actions, gestures, and artifacts. In the second part of my talk, I will argue that algebraic thinking can be characterized in accordance with the semiotic means to which the students resort in order to express and deal with algebraic generality. I will draw upon results obtained in the course of a 10-year longitudinal classroom research project to offer examples of students' forms of algebraic thinking. Two of the most elementary forms of algebraic thinking identified in our research are characterized by their contextual and embodied nature; they rely extensively upon rhythm and perceptual and deictic (linguistic and gestural) mechanisms of meaning production. Furthermore, keeping in line with the situated nature of the students' mathematical experience, signs here usually designate their objects in an indexical manner. These elementary forms of algebraic thinking differ from the traditional one—based on the standard alphanumeric symbolism—in that the latter relies on sign distinctions of a morphological kind. Here signs cease to designate objects in the usual indexical sense to give rise to symbolic processes of recognition and manipulation governed by sign shape.

The aforementioned conception of thinking in general and the ensuing distinction of forms of algebraic thinking shed some light on the kind of abstraction that is entailed by the use of standard algebraic symbolism. They intimate some of the conceptual shifts that the students have to make in order to gain fluency in a cultural sophisticated form of mathematical thinking. Voice, gesture, and rhythm fade away. Embodied and contextual ways of signifying are then replaced with a perceptual activity where differences and similarities are a matter of morphology, and where meaning becomes relational.

SIGNS, GESTURES, MEANINGS: ALGEBRAIC THINKING FROM A CULTURAL SEMIOTIC PERSPECTIVE

Luis Radford

Université Laurentienne, Ontario, Canada

À la mémoire de Georges Glaeser

INTRODUCTION

To deal with algebraic thinking in a plenary session is a bit risky. Unavoidably, it conveys the feeling of something *déjà vu*—something that has been said again and again. Indeed, since the 1980s algebraic thinking has been one of the most researched areas in mathematics education. And this is so not by chance. Among the branches of mathematics that students have to learn in school, there is none more frightening than algebra. Many students in our teachers' training program at Laurentian University confess that everything was going well until they met algebra in Junior High School. As they admit, suddenly they found themselves in front of an abstract symbolic language, the meaning of which they could not grasp—a kind of hieroglyphic language that, to their dismay, has become like the Esperanto of modern sciences.

And it is the investigation of the students' legendary difficulties in understanding algebra and the search for new ways to teach this subject that has kept many researchers busy for the past three decades. The question, hence, is whether or not there is really something new to say about algebraic thinking. It looks like there is not much left to be said about it. This impression would only be strengthened if you were to do a Google search. We did one at the end of November 2008, in our preparation for this talk, and our “algebraic thinking” search returned almost 176,000 hits. However, as you go through the entries, you realize that the content does not tell you much about algebraic thinking. The content is rather about items usually included in school algebra curricula. The least that can be said is that the term “algebraic thinking” has become a catch-all phrase. This may be a token of the fact that to deal with algebraic thinking is not a simple matter. It supposes that you have some sort of theory about thinking or at least a clear idea of what you mean by thinking in general. Let us pause for a moment: What do you take “thinking” to mean?

As psychologists, philosophers, anthropologists and others are willing to acknowledge, there is no simple and direct answer to this question. As odd as it may seem, thinking is something that we continuously do. Thinking is as ubiquitous as breathing. Yet, we still do not know how we think! Commenting on the elusiveness of thinking, Dan Rappaport said: “The knowledge that thinking has conquered for humanity is vast, yet our knowledge of thinking is scant. It might seem that thinking eludes its own searching eye.” (Rappaport, 1951; quoted in Benson, 1994, p. 13). Western idealist and rationalist epistemologies have conveyed the idea that thinking is something immaterial, something purely mental, bodiless. The influence of Plato's epistemology on our understanding of thinking is perhaps greater than we are usually aware (Radford, Edwards, Arzarello, 2009).

In this article, I introduce a typology of forms of algebraic thinking based on their level of generality. The typology rests on a theoretical approach that capitalizes on the results of the 1990s algebra research agenda and supplements it by incorporating a semiotic theoretical platform. Signs lose the representational and ancillary status with which they are usually endowed in classical cognitive theories in order to become the *material* counterpart of thought. This semiotic platform opens up new possibilities for understanding algebraic signs and formulas in a nonconventional manner. Traditionally, letters and signs for operations (like “+”, “x”, etc.) have been considered *the* algebraic signs of school algebra. Alphanumeric symbolism has indeed been regarded as the semiotic system of algebra *par excellence*. Yet, from a semiotic perspective, signs can also be something very different. Words or gestures, for instance, are signs on their own — semiotically speaking, they could be as genuine algebraic signs as letters. Of course, as I will argue later in more detail, this does not mean that they are equivalent or that we can simply substitute the ones for the others. What makes semiotic systems unique and unsubstutable is their *mode of signifying*. There are things that we can signify and intend through certain signs, and things that we cannot. Try to put Pablo Neruda’s famous poem “Canción Desesperada” [“Desperate Song”] in an algebraic formula, and you will see how hopeless the task is.

In the first part of this article, I argue that the mathematical situation at hand and the embodied and other semiotic resources that are mobilized to tackle it in *analytic* ways characterize the *form* and *generality* of the algebraic thinking that is thus elicited. My claim is based not only on semiotic considerations but also on new theories of cognition that stress the fundamental role of the context, the body and the senses in the way in which we come to know. In the second part, I present some concrete examples through which the typology of forms of algebraic thinking is illustrated.

THE 1990s ALGEBRA RESEARCH AGENDA

During the discussions held in the 1980s and 1990s, either in the PME Algebra Working Groups or in other similar research meetings (Bednarz, Kieran, & Lee, 1996; Sutherland, Rojano, Bell, & Lins, 2001), it was impossible to agree upon a minimal set of characteristics of algebraic thinking. There was, however, a more or less general consensus concerning two aspects. Algebra deals with objects of an *indeterminate* nature, such as unknowns, variables, and parameters. Furthermore, in algebra, such objects are dealt with in an *analytic manner*. What this means is that in algebra, you calculate with indeterminate quantities (i.e. you add, subtract, divide, etc. unknowns and parameters) as if you knew them, as if they were specific numbers (see, e.g., Kieran 1989; 1990; Filloy & Rojano, 1984a, 1989; Cortes, Vergnaud, & Kavafian, 1990; for some epistemological analysis, see Filloy & Rojano, 1984b; Puig, 2004; Radford & Puig, 2007; Serfati, 1999).

Of course, one way or another, algebraic objects have to be designated. The general tendency in the 1990s was to associate school algebra and algebraic thinking with the use of letters. Even if at the time the idea was not universally shared (Linchevski, 1995; Balacheff, 2001), it nonetheless prevailed and is still very strong in current research on the teaching and learning of algebra. Although I do believe that it is impossible to practice

abstract algebra (e.g., Galois Theory) without some sort of sophisticated notations, I do not think that algebra and algebraic thinking can be reduced to the use of letters. As John Mason pointed out some years ago, “the manipulation of symbols is only a small part of what algebra is really about” (1990, p. 5). Letters indeed have never been either a necessary or a sufficient condition for thinking algebraically. For instance, in his *Elements*, Euclid used letters without thinking algebraically. Conversely, Chinese and Babylonian mathematicians thought algebraically without using letters (Radford, 2006).

What I am suggesting here is hence this: algebra is about dealing with indeterminacy in analytic ways. But instead of giving alphanumeric symbolism the exclusive right to designate and express indeterminacy I am claiming that there is a plurality of semiotic forms to accomplish it. This is true of the practices of elementary algebra and of advanced algebra as well—even if in the latter, alphanumeric symbolism becomes more salient.

But before I go further, let me reassure you that my idea is not to challenge the power of symbolic algebra. Rather, I am trying to convince you that it is worthwhile to entertain the idea that there are many semiotic ways (other than, and *along with*, the symbolic one) in which to express the algebraic idea of unknown, variable, parameter, etc. I deem this point important for mathematics education for the following reason. Ontogenetically speaking, there is room for a large conceptual zone where students can start thinking algebraically even if they are not yet resorting (or at least not to a great extent) to alphanumeric signs. This zone, which we may term the *zone of emergence of algebraic thinking*, has remained largely ignored, as a result of our obsession with recognizing the algebraic in the symbolic only.

SENSUOUS COGNITION

My claim about a diversity of semiotic forms for dealing with algebraic indeterminacy rests on a perspective on thinking that is squarely at odds with the mental conception of thinking that informed most of the 1990s research on mathematics education. Within this mental conception of thinking signs were often considered “symptoms” of mental activity—hence the distinction between internal and external representations. Drawing on Vygotskian psychology, from the semiotic-cultural perspective advocated here, the question of the relationship between signs and thought is thematized in a different way. First, signs are considered in a broad sense, as something encompassing written as well as oral linguistic terms, mathematical symbols, gestures, etc. (Arzarello, 2006; Ernest, 2008; Radford, 2002a). Second, signs are not considered as mere indicators of mental activity. In contrast, signs are considered as *constitutive* parts of thinking. In more precise terms, within this semiotic-cultural perspective, thinking is considered a *sensuous and sign-mediated reflective activity embodied in the corporeality of actions, gestures, and artifacts*.

The adjective *sensuous* refers to a conception of thinking that is inextricably related to the role that the human senses play in it. Thinking is a versatile and sophisticated form of sensuous action where the various senses *collaborate* in the course of a multi-sensorial experience of the world (Radford, 2009a). This multi-sensory characteristic of cognition

has been emphasized by philosophers like Arnold Gehlen (1988) and Maurice Merleau-Ponty (1945) and at its heart is the idea of the important role that the body plays in the way we come to conceptualize things. As Gallese and Lakoff recently contended,

the sensory-motor system not only provides structure to conceptual content, but also characterizes the semantic content of concepts in terms of the way that we function with our bodies in the world (Gallese and Lakoff, 2005, pp. 455–456).

In tune with such views, some researchers in our field are paying attention to the embodied nature of mathematical cognition. This is the case with Ferdinando Arzarello and the Torino Team in Italy, Rafael Núñez and Laurie Edwards in the USA, Michael Roth and the CHAT group in Canada, the Uniban research team in Brazil, etc. To mention a brief example, the Uniban research team in Brazil is investigating the role of gestures in blind children. Here gestures and tactility come to play a crucial role in understanding mathematical concepts (Figure 1).

Of course, tactility and other sensorial mediated processes are also important in non-impaired children. Ricardo Nemirovsky has suggested that instead of being mere mental processes, understanding and imagination of mathematical concepts are literally embedded in perceptuo-motor action: the “understanding of a mathematical concept spans diverse perceptuo-motor activities” (Nemirovsky, 2003, I -108), so that in this regard, “understanding is ... interwoven with motor action” (Nemirovsky, 2003, I-107).



Figure 1. Exploring area, from research conducted by Solange Ali Fernandes and Lulu Healy with blind children (Ali Fernandes, 2008).

However, thinking encompasses still much more than that. Thinking is an activity that, although performed by an “I” and the “I’s body”, is ubiquitously drawing on culture’s kit of patterns of meaning-making as well as on historically constituted concepts of an ethical, political, scientific, and aesthetic nature. Thinking is bound to the context and the culture in which it takes place. This is why it is more accurate to say that thinking in general, and algebraic thinking in particular, is a body-sign-tool mediated cognitive historical *praxis*.

LEARNING AS OBJECTIFICATION

From an educational perspective, the main question is: How do the students acquire fluency in such cognitive cultural historical *praxes*? How do they become acquainted with

the historically constituted forms of action, reflection and reasoning that those *praxes* convey? Since mathematical forms of reasoning have been forged and refined through centuries of cognitive activity, they are far from trivial for the students. It is the historical density of such *praxes*, sedimented now in compact, systemic, and highly abstract formulations, that is the basis of what Vygotsky intended with his famous distinction between “quotidian” and “scientific” concepts —regardless of how unfortunate Vygotsky’s choice of terms was.

Reflective acquaintance with cognitive historical *praxes* and their concomitant forms of action and reasoning is what learning consists of. And, as I submitted elsewhere (Radford, 2008a), it can be theorized as *processes of objectification*, that is, those social processes through which the students grasp the cultural logic with which the objects of knowledge have been endowed and become conversant with the historically constituted forms of action and thinking.

Working within this theoretical framework, where semiotics, culture and history are driving principles, in recent years my collaborators and I have been busy in implementing classroom holistic activities that can offer the students a possibility to reflect algebraically and to get acquainted with some basic ideas of algebra in different contexts —equations, pattern generalization and, recently, graph interpretation (Radford, 2000, 2002b, 2003, 2009a; 2009b; Radford, Bardini & Sabena, 2007). Our goal has been to try to understand what I previously referred to as the *zone of emergence of algebraic thinking* and forms of algebraic thinking elicited by our activities.

Let me pause this theoretical discussion here and turn now to some short examples that come from our first longitudinal research project—a project that we conducted from 1998 to 2003 and during which we accompanied four classes of students as they went from Grade 8 to Grade 12, i.e., until the completion of high school. The examples will, I hope, give an idea of our approach and the kind of analysis we conducted.

SOME CLASSROOM RESULTS

The students’ first contact with algebraic symbolism occurred when they were in Grade 8. In Grade 9 we decided to start with an activity that was intended as a means to revisit the concepts learned in the previous year. In the introductory part of the activity, the students, working in groups of three, had to draw Figure 4 and Figure 5 of the sequence shown in Figure I and to find out the number of circles in Figures 10 and 100¹. In the second part of the activity, the students were asked to write a message to a student of another Grade 9 class indicating how to find out the number of circles in any figure (“*figure quelconque*”, in the original French), and then to write an algebraic formula for the number of circles in Figure n.

¹Figures identified with Roman numbers (e.g., Figure II) refer to objects in the article, whereas figures identified with Indo-Arabic numbers (e.g., Figure 2) refer to elements of a pattern in the classroom activity given to the students.

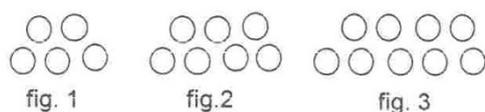


Figure I. The sequence of the introductory pattern generalization activity in Grade 9.

Factual Algebraic Thinking

Usually, the students start counting the number of circles in Figures 1, 2, and 3, and realize that, in sequences like the one shown in Figure I, the number of circles increases by the same number each time. However, as the students quickly notice, this recursive relationship between consecutive figures is not really a practical way to answer the question about “big” figures, like Figure 100.

In one of the groups (formed by Jimmy, Dan, and Frank), working on the sequence shown in Figure I, the students imagined the figures as divided into two rows:

1. Dan: (*Referring to Figure I*) Well... (*pointing to the top row*) 2 on top; there, there is 3 on the bottom...
2. Jimmy: [Figure] 2, there are 3; [Figure] 3, there are 4.
3. Dan: wait a minute. Ok (*he makes a series of gestures as he speaks; see four of the six gestures in Figure II*), Figure 1, 2 on top. Figure 2, 3 on top. Figure 3, 4. Figure 4, 5.
4. Jimmy: Figure 10, it will be 11...
5. Dan: ... 11 on top, and 12 on the bottom.
6. Jimmy: All the time it will be one more in the air.
7. Frank: [Figure] 100? 101, 102...
8. Dan: 203.

“Figure 1	2 on top”
“Figure 2	3 on top”
<i>Figure II. Dan makes a sequence of pointing gestures coordinated with words in a first process of objectification (reconstruction from the video data).</i>	

As the students’ dialogue suggests, the generalization was accomplished in two steps.

In the first step (lines 1-3), the students conceived of the figures as divided into two lines, and, drawing on perceptual observations made on the first three given figures, they were able to objectify a *regularity*: a relationship between the number of the figure and the number of circles in its rows.

The grasping of the regularity is not enough, however, to ensure the generalization. The regularity has to be generalized. And this is what the students accomplished in the following lines where they came up with a formula to find the number of circles in Figures 10 and 100. Indeed:

- In lines 4 and 5 the observed regularity of perceptually available figures was *generalized* to Figure 10, a figure that is not in the students' perceptual *field*.
- Line 6 contains a partial linguistic formulation of the general structure of the figures, as perceived by the students: "All the time there will be one in the air", i.e., for all figures of the sequence, there is always one unmatched circle on the bottom row.
- In line 7, Frank resorted to the objectified pattern structure in order to calculate the number of circles in Figure 100.

The students are equipped now with a formula to answer questions about Figure 1000, Figure 1 000 000, or whatever particular figure you may have in mind.

Now, I am talking about a formula, yet there are no letters! That's true. The algebraic formula consists, rather, in a piece of embodied action. We can call it—borrowing an expression from Vergnaud (1996) and changing it slightly—an *in-action-formula*.

A "formula" of this concrete form of algebraic thinking can better be understood as an embodied predicate with a tacit variable: indeterminacy does not reach the level of discourse. It is present through the appearance of some of its instances ("1", "2", "3", "4", "5", "10", "100"). It remains an empty space to be filled up by the eventual uttering of particular terms. We call this type of situated and concrete form of algebraic thinking that operates at the level of particular number or facts *factual*².

Despite its apparently concrete nature, factual algebraic thinking is not a simple form of mathematical reflection. On the contrary, it rests on highly evolved mechanisms of perception and a sophisticated rhythmic coordination of gestures, words, and symbols. The grasping of the regularity and the imagining of the figures in the course of the generalization results from, and remains anchored in, a profound sensuous mediated process—showing thereby the multi-modal nature of factual algebraic thinking³.

Let us turn now to the second part of the Grade 9 activity.

Contextual Algebraic Thinking

In the introduction I suggested that the mathematical task at hand and the social sign-mediated processes of perception and generalization can inform us of the form and generality of the algebraic thinking that is thus elicited. What kind of algebraic thinking will now be generated? The task requires that the students go beyond particular figures and

² The adjective *factual* stresses the idea that this generalization occurs within an elementary layer of generality—one in which the universe of discourse does not go beyond particular figures, like Figure 1000, Figure 3245, and so on.

³ In our current research with Grade 2 students these mechanisms of rhythmic coordination are also present, but they do not reach the subtle sensorial synchrony that we observe in older students as those reported here.

deal with a new object: a *general* figure. Indeterminacy must now become part of explicit discourse. Our question is: How will the students build the formula? In the absence of gestures and rhythm, to which linguistic mechanisms will the students resort?

In fact, in being asked to write a message, the students were invited to enter into a deeper level of objectification than the one of action and perception characteristic of factual algebraic thinking. Writing makes one render explicit things that may have remained on what neuropsychologists call the area of proto-attention, or what Husserl used to call the horizon of intentions (Husserl, 1954).

In Grade 8, writing a message that involves this new object “general figure” proved to be very difficult. As we reported in previous work (see, *e.g.*, Radford, 2000), the students often used particular figures (like Figure 12) as examples to convey a *generic* idea or used particular figures in a *metaphorical* sense to talk about the still unutterable generality (Radford, 2002a). Sometimes the message was not complete. Here is an example: “You add 1 [circle] on the top and 1 on the bottom.”

In Grade 9, the students felt much more comfortable with this level of generality. The following message is paradigmatic of what the students wrote: “You have to add one more circle than the number of the figure in the top row, and add one more circle than the top row to the one on the bottom.”

Of course, this procedural sentence can be seen as a *formula*. But it is very different from the one discussed in the previous section. Here, rhythm and gestures have been replaced by key descriptive terms—“top,” “bottom.” These terms are what linguists call spatial *deictics*, that is to say, words with which we describe, in a contextual way, objects in space. The indeterminate object variable is now explicitly mentioned through the term “number of the figure.” However, although different from factual algebraic thinking both in terms of the way indeterminacy is handled and the semiotic means which the students think, the new form of algebraic thinking is still contextual and “perspectival” in that it is based on a particular way of regarding something⁴. The algebraic formula is indeed a *description* of the general term, as it was to be drawn or imagined. This is why we term this form of algebraic thinking *contextual*. Here is another Grade 9 example: “# of the figure + 1 for the top row and the # of the figure + 2 for the bottom. Add the two for the total.”

Let us turn now to the last part of the Grade 9 activity.

Standard algebraic thinking

Expressing the formula in algebraic standard symbolism was much more difficult than expressing it in words, both in Grades 8 and 9, although, of course, there was some progress from one year to the next. The results mentioned in the previous section shed some light on the nature of these difficulties: previously, the students could resort to a range of semiotic resources, like pointing and iconic gestures, deictics, adverbs, etc. Those

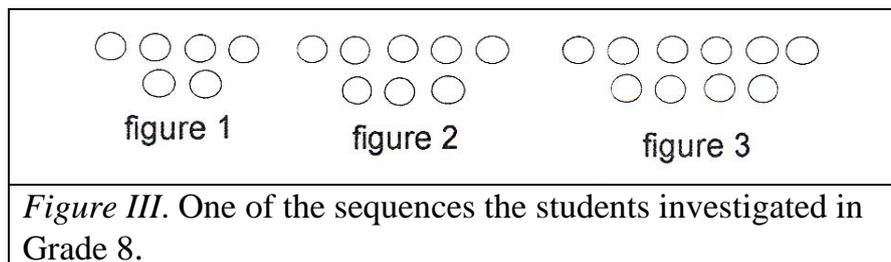
⁴ It still supposes a spatially situated relationship between the individual and the object of knowledge that gives sense to expressions like “top” and “bottom”.

rich semiotic resources do not have a place in the alphanumeric based algebraic formulas. In short, there is a drastic change in the mode of designation of the objects of discourse.

How then to designate the number of circles in a figure, in the highly condensed semiotic system of alphanumeric signs? From an ontogenetic viewpoint, direct “translation” is not something on which we can count, as we cannot count on direct translation from our native language to a new one we are just starting to learn. Direct translation presupposes that you already know the target language. In the case of the standard alphanumeric algebraic language, the situation is even worse, as this language is not even “natural.” Our standard algebraic language is *artificial*. Historical analysis shows that its construction was preceded by a good deal of efforts that ended up in dead ends and failures (Høyrup, 2008; Serfati, 2006).

In Grade 8, the students often resorted to particular examples. Thus, dealing with the sequence shown in Figure III, Dan and his group (in Grade 8, the group was formed by Dan, Frank and Sara), illustrated the formula through the case of Figure 100:

1. Dan: You add 3 on top, and 1 at the bottom.
2. Sara: That’s true if you go by the [form of the] figure.
3. Dan: You add 3 on top, and 1 at the bottom. Let’s say that n equals 100. It would be 100... you add 1, it would be 101 [on the bottom row]...
4. Frank: (Interrupting) and 103 [on the top row].



In other cases, the students often resorted to formulas that, superficially, look to be algebraic, in particular because they contain letters. Thus, in the sequence shown in Figure III, several students in Grade 8 produced the formula $n \times 2 + 4$. However, despite its appearance, the formula is *not* algebraic. It was instead obtained by trial and error. Dan and his group first tried $n \times 2 + 1$, then $n \times 2 + 2$, etc. until they obtained $n \times 2 + 4$, which seemed to work in the few cases in which they tested it. This procedure is not based on an *analytic* way of thinking about indeterminate quantities — the chief characteristic of algebraic thinking. This procedure does not even reach the sophistication of pre-algebraic arithmetic methods such as “false position.” It is rather a kind of arithmetic naïve induction⁵.

To counter these inductive arithmetic procedures, in the designing of the classroom activity, we added a question in which the students were asked to provide a formula for the

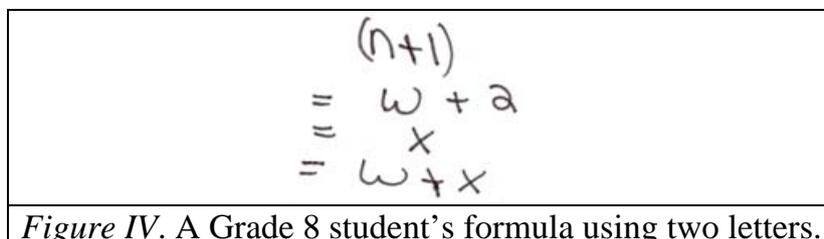
⁵ I do not have the space here to go into the details of the delicate distinction between algebraic and arithmetic formulas. For a detailed discussion, see (Radford, 2006, 2008).

number of circles on the top row of Figure n , followed by the question of finding a formula for the total of circles in Figure n . Establishing a functional relationship between the number of the figure and the number of circles on top of the figure proved very difficult. Dan and his group suggested using two letters:

Dan: (*Noticing that each figure has two more circles than the previous one*) It's plus 2 [to obtain the number of circles in the next figure], plus 2 [to obtain the number of circles in the next figure], plus 2... Unless we put 2 letters... What we would do is ... the top row would be n , and the top row would be like b . After that, you do $n + b + 2$.

In this case, the letters n and b do not designate the number of circles in the top and bottom rows of Figure n . Actually, the number of the figure is not even taken into account. The formula, rather, expresses a vague recursive relationship.

Another Grade 8 group suggested the “cascading formula” shown in Figure IV.



$$\begin{array}{l} (n+1) \\ = w + 2 \\ = x \\ = w + x \end{array}$$

Figure IV. A Grade 8 student's formula using two letters.

The first line corresponds to the number of circles on the bottom row. The result is called “ w ”. This is expressed in the second line, where it is also said that you still have to add 2 to get the number of circles on the top row. This last number is called “ x ”, as indicated in the third line of the formula. Finally, in the last line, the students are saying that you still have to add the numbers represented by “ w ” and “ x ” to obtain the total of circles in Figure n . Not bad, although still a bit far away from the standard way to write formulas within the alphanumeric semiotic system of algebra. Not bad, even if the use of several letters and their inter-connected meanings is not fully clear for the students. As one of the students from this group said to the other two members, “You mix me up with all your letters!”

The first example (Dan's) is interesting in that it shows that, although these students were able to produce an inductive formula that looked like an algebraic one (i.e., “ nx^2+4 ”), they did not produce the expected algebraic formula “ $n+3$ ” for the top row of Figure n —even if the formula “ nx^2+4 ” seems much more complex. The complexity of the formulas cannot be judged by the number of involved terms only; the complexity of the formula should also be judged in terms of the mode of designation of the objects of discourse.

The second example is interesting in that it unveils some of the tremendous difficulties that the students have to face when using letters to intend to say what they perfectly know how to express in natural language. This problem is much more complex than a simple translation. As Glaeser remarked, the need to give an immediate meaning to every intermediate result has to be refrained (1999, p. 154). Meaning, indeed, has to be put in abeyance.

In Grade 9 we still found some formulas that resembled the formulas produced in Grade 8. But more typical of Grade 9 were the formulas shown in Figure V (these formulas correspond to the sequence shown in Figure I).

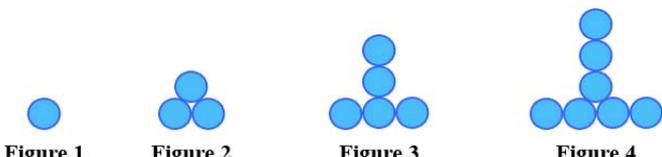
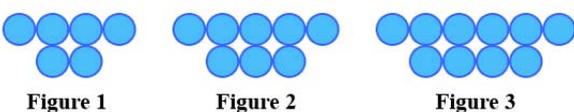
$(n+1)+(n+2)$	$(n+2)(n+1)$
<i>Figure V.</i> Left, the formula produced by Dan's group in Grade 9. Right, a variant of it produced by another Grade 9 group.	

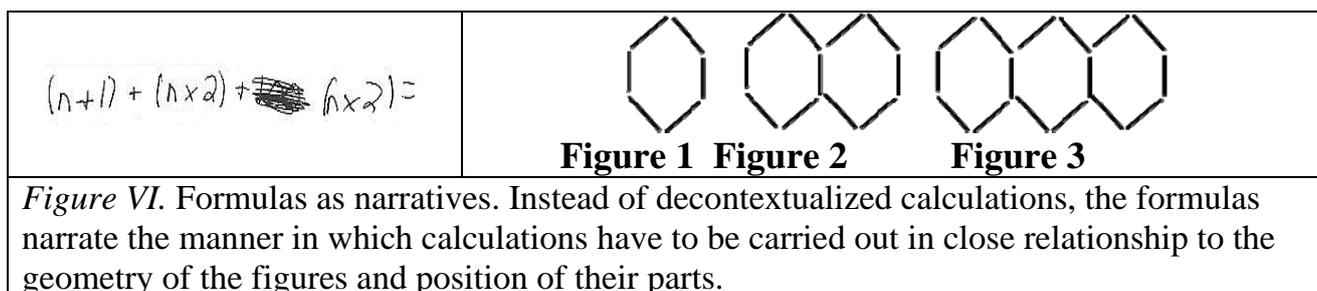
Although much better than the formulas found in Grade 8, the signs in these formulas still keep the embodied and perspectival experience of the objectification process. We easily recognize in the term “n+1” the reference to the top row, as we recognize in the term “n+2” the reference to the bottom row. In Dan's group, for instance, this embodied manner of symbolizing was made very clear:

1. Dan: No, no, well, it's that... n + 1 is the top row...
2. Frank: (*Interrupting*) Yes, I know.
3. Dan : n + 2 is the bottom row.

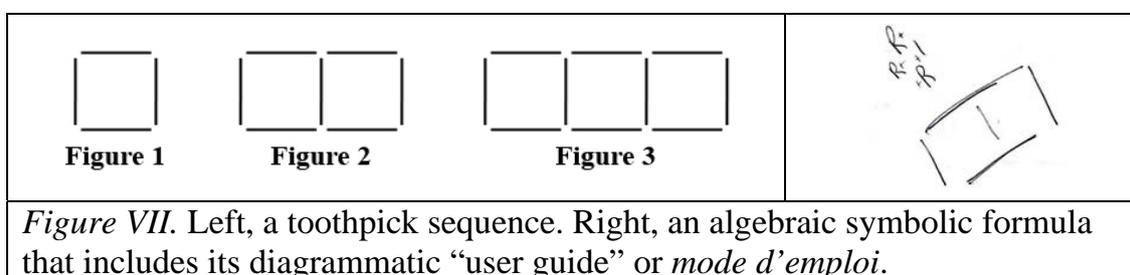
As is clear from Figure V, the students add brackets to carefully distinguish between the rows. This is why, I want to suggest, the formula is an *icon*, a kind of *geometric description* of the figure. In other terms, the formula is not an abstract symbolic calculating artifact but rather a *story* that narrates, in a highly condensed manner, the students' mathematical experience. In other words, the formula is a *narrative*. And it is the narrative dimension of the students' iconic formulas that very often makes it possible to infer from the formula the sequence to which it corresponds (see figure VI).

That which previously was distinguished through pointing gestures and linguistic deictics is now distinguished through the effect of signs and brackets. It is precisely this “perspectival” nature of the formula that leads many students to argue that brackets cannot be removed. Otherwise, they argue, it would be impossible to know what the terms of the formula mean. Yet, this is precisely what constitutes the force of algebra—the detachment from the context in order to signify things in an abstract way. The mode of designation has to move to a different layer where signs borrow their meaning not from the things they denote but from the *relational* way they mean within the context of other signs.

$n+(n-1)$	 <p style="text-align: center;"> Figure 1 Figure 2 Figure 3 Figure 4 </p>
$(n+1)+(n+2)$	 <p style="text-align: center;"> Figure 1 Figure 2 Figure 3 </p>



The narrative meaning of iconic symbolic formulas became even clearer when a fifth class was added to our project. As our project progressed, other teachers became interested in it and, to the extent that we could, we included new classes. The fifth class regrouped Grade 8 students who were recognized as having difficulties in following the rhythm of “regular” math classes. Dealing with the pattern shown in Figure VII (left) one group of students produced the formula shown in Figure VII (right).



The formula does not have the usual linear organization of standard algebraic formulas. Rather, signs signify in a spatial manner: as the students explained to us, the top “R” means that there are as many toothpicks on the top of the figure as the number of the figure. The “R” placed on the bottom of the formula means that there are as many toothpicks on the bottom of the figure as the number of the figure. The lateral “R” means that there are as many vertical toothpicks on the top of the figure as the number of the figure, but not really. There is an extra toothpick to be accounted for, placed at the right end, signified by the lateral sign “1.” The “+” signs mean that you have to add all of those things.

FROM ICONIC FORMULAS TO SYMBOLIC ONES

One of the important didactic problems is to implement classroom activities that will allow the students to endow their formulas with new abstract meanings. In more precise terms, the problem is to transform the iconic meaning of formulas into something that no longer designates concrete objects. For instance, the formula $(n+1) + (n+2)$ mentioned previously (Figure V), has to be seen in a new light. The narrative dimension of formulas has to collapse (Radford, 2002c). The embodied meaning of the formulas does not disappear. It rather gives rise to a more abstract one. Thus, in addition to signifying the sum of circles in the top and bottom rows, the terms of the formula have to be considered in relation to the signs that they contain. Resemblances and differences—these key aspects of signification in general (Radford, 2008b)— must no longer be exclusively based on spatial and

contextual considerations (such as “top” and “bottom”). In the new form of signifying, there is a shift in focus: attention has to be directed now to morphological differences, i.e., differences in terms of *letters* versus *numbers*. In short, meaning must become *relational*.

The search for the pedagogical actions allowing the students to objectify this abstract form of signifying became one of our goals, both from a theoretical and a practical viewpoint. Our strategy was based on *comparing* and *simplifying* formulas. Here is an example that deals with the sequence of squares shown in Figure VII.

The previous day, the students produced several formulas. At the beginning of the class, the teacher asked for some examples. The students mentioned two, that were written as $r \cdot 3 + 1$ and $(r+1) + r \cdot 2$, where r stands for the rank or number of the figure.

1. Teacher: I would like to compare these formulas and to see where they come from. Brian, do you want to explain the first formula to us?
2. Brian: (*Going to the blackboard*). Ok, yesterday we saw that the first figure only has 1 toothpick at the bottom (*he points to the bottom of Figure 1 on the blackboard*) and the second figure, there were 2, third figure, there were 3. So, we added the bottom and the top, and then we saw that, in the first term, there were 2 [vertical toothpicks] (*points to the vertical toothpicks of Figure 1*) and Figure 2 has 3 (*points to the vertical toothpicks of Figure 2*) therefore, it's always [the rank or number of the figure] plus 1. So we did the bottom plus the top plus the rank plus 1. And then we saw that... Well, we discussed a lot, and we saw that ... it was the rank, rank times 3 (*points towards the first term of the formula*) because it has the bottom, the top and the vertical. There was, there was, plus [one]...
3. Teacher: So you say that this (*pointing to the bottom row of the first square and colouring it with blue chalk; see Figure VIII, pic. 1*) is one r ; this is another r (*pointing to the top row of the first square and colouring it with blue chalk; see pic. 2*); and this is the third r (*pointing to the left vertical side of the first square and colouring it with blue chalk; see pic. 3*) and there remains another one [toothpick] (*pointing to the second vertical line of the first square; pic. 4*). So, (*pointing to the formula*) r times 3... I have three r here (*pointing successively to the coloured sides of the first square*) plus another one in each term (*pointing the uncoloured right vertical side of the first square*). (*Then, the teacher repeated the same set of sequence of pointing gestures on Figure 2, see Figure VIII, pics. 5-8*). This is the explanation of the formula. Now, Ron, would you please explain the second formula?

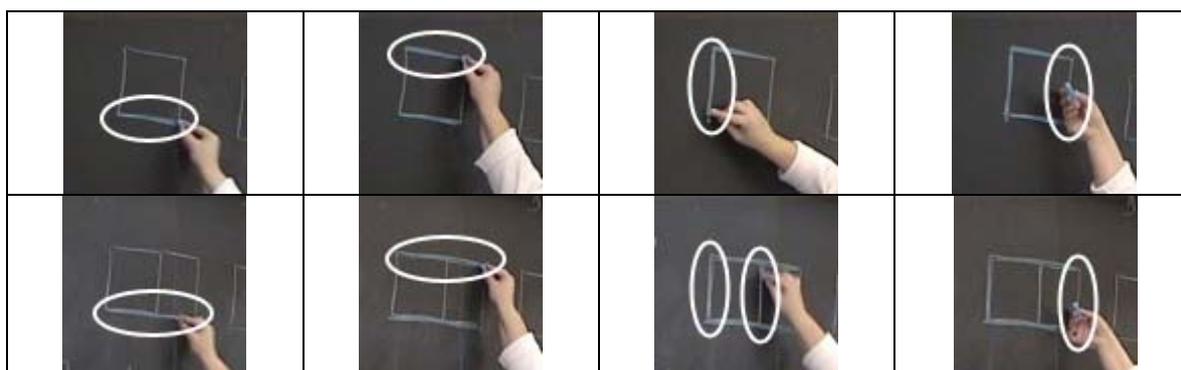


Figure VIII. Pictures 1-4 (top) show the teacher's effort to relate the terms of the formula $r \cdot 3 + 1$ to the various parts of Figure 1. Pictures 5-8 (bottom) shows the same effort but this time the focus is on Figure 2. The teachers makes apparent for the students the new way of signifying through a subtle coordination of gestures, words, drawings and coloured segments.

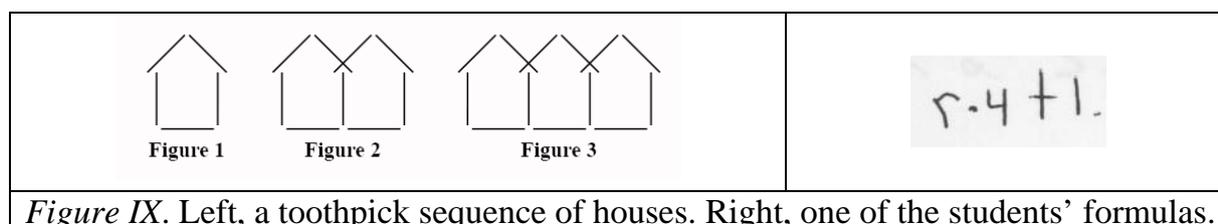
Ron went to the blackboard and explained the various elements of $(r+1) + r \cdot 2$. After that, the teacher encouraged a discussion about the formulas. Sandra—a student sitting at the end of the classroom— argued that both equations work but the first one was simpler. The teacher summarized the difference as follows:

1. Teacher: the difference is that here (*pointing to the formula $r \cdot 3 + 1$*) we put together the terms that were the same and we simplified. Since I am calculating the total number of toothpicks, I can put all together (*while talking, she emphasized the words "same", "simplified" and "total"*). It is exactly this that the first formula does. (*Smiling to the class, she says*) I think that you are ready for the next activity.

The previous formula $r \cdot 3 + 1$ looks much like Dan's formula $n \times 2 + 4$ discussed earlier. Yet, the difference is considerable. Brian's formula was not produced by trial and error. It was the result of an algebraic generalizing process where general functional relationships were first identified (*e.g.*, the number of toothpicks on top vis-à-vis the rank or number of the figure), then simplified. As Brian put it, "... it was the rank, rank times 3 *because* it has the bottom, the top and the vertical." The teacher capitalized on Brian's idea and, through a feast of clear and consecutive gestures that echoed Brian's timid gestures, coloured parts of the first two figures to make clear for all the students the relationship between the spatial-geometric parts of the terms and their corresponding rank (Figure VIII, pic. 1-8). After showing each one of the tree r on Figure 1, she linked the first part of the formula ($r \cdot 3$) to the three parts she had just coloured. She said: " r times 3... I have three r here," followed by the crucial remark that there is still "another one in each term" (which corresponds to the constant term of the formula). Her coordinated gestures and words related very well the spatial elements of the figures with the corresponding parts of the formula. The idea of putting *together* the toothpicks on the bottom, the top and the vertical ones, led to *adding* the number of the figure several times.

That day, after the general discussion, the students dealt with a sequence of houses (Figure IX). The students identified the relationship between clue elements of the figures and their rank or number:

1. Raymond: the number of toothpicks in the roof is twice the number of the figure. For the walls [which included the floor], it is twice, and another wall ...
2. Joyce: (*Interrupting*) to close the space...
3. Raymond: So, the formula is rank times 4 plus 1.



In so doing, the students entered into a new form of algebraic understanding and moved into a deep region of the zone of emergence of algebraic thinking. They moved from a referential understanding of signs (signs as referring to particular objects, like the number of toothpicks in the roof) to a morphological one—the beginning perhaps of what Kieran (1990) Kirshner (2001), Hoch & Dreyfus (2006) and others have called the structural dimension of algebra.

It is clear that the symbolic formula is no longer just iconic. Iconicity is still present, but it has receded to make room for a more concise and abstract form of signification. Naturally, the students have yet to undergo a supplementary lengthy process of objectification to become fluent with the modern form of symbolic algebraic thinking, where symbolic calculations are carried out through formal considerations only. For this to occur, new objects like x^2 and $x^2 + x$ will have to enter the universe of discourse and acquire a detached existence. It is not vain to recall here that this process was not easily achieved in the history of algebra. Thus, to distinguish magnitudes, Vieta—one of the founders of our modern algebraic symbolism—was still in the 16th century talking about “length”, “plane”, “solid”, etc.. Our modern way of referring to the now abstract monomials of algebra still reminds us of their embedded concrete beginnings. Indeed, monomials such as x^2 or x^3 read as “ x square”, “ x cube”. Our modern language hangs behind the relics of its past revealing thereby the monomials’ original geometric-spatial origin.

Synthesis and Concluding Remarks

In this article, drawing on recent conceptions of thinking offered by anthropology, semiotics and neurosciences, I suggested that thinking is a complex form of reflection mediated by the senses, the body, signs and artifacts. In this view, thinking is not a kind of Cartesian mental activity monitored by a homunculus residing somewhere in a black box of ideas and representations. As the Russian philosopher Elvald Ilyekov put it, “Thinking is not the *product* of an action but the *action itself*” (Ilyenkov, 1977, p. 35). To a large extent, thinking is indeed a material process. But thinking is also more than the processes that a sensing body can produce. Thinking is something that is intrinsically *historical* and

cultural, and the proof is that had we happened to live in Babylonian times, we would have found ourselves with body and brain structures and anatomies indistinguishable from the ones we have today. Yet, we would have been thinking mathematically, aesthetically, politically, etc. in a very *different* way. It is this distinctive historical and cultural trait of thinking that I want to convey when I say that thinking in general and algebraic thinking in particular is a body-sign-tool mediated cultural historical *praxis*.

The historical nature of cultural praxes has, as a corollary, the non-transparency of the forms of action, reflection and reasoning they convey. To become fluent in those praxes, we have to undergo lengthy processes of objectification. The creation of the conditions for those processes to occur is an educational problem. In the approach expounded here, the basic premise is that algebraic thinking cannot be confined to activities mediated by the standard alphanumeric semiotic system of algebra. From a semiotic viewpoint, there are several ways in which to analytically reason through, and to reason on, indeterminate quantities. More importantly, the mathematical situation and the semiotic resources that are mobilized to tackle it in analytic ways characterize the *form* and *generality* of the algebraic thinking that is thus elicited. Focusing on the context of pattern generalization, I suggested a classification of three forms of algebraic thinking —factual, contextual, and symbolic. As with most classifications, the borders of those categories are not necessarily well defined. Furthermore, those forms of thinking do not necessarily exclude each other. A student, for instance, can very well combine factual and symbolic forms of thinking. The typology is rather an attempt at understanding the processes that the students undergo in their contact with the forms of action, reflection and reasoning conveyed by the historically constituted praxis of school algebra.

The classroom data presented here offers a glimpse of the ontogenetic journey of our students on their route to algebraic thinking. It stresses some of the challenges that they had to overcome when passing from factual to contextual to symbolic thinking. It stresses in particular the changes to be accomplished in modes of signification. While in factual thinking, indeterminacy remains implicit and gestures, words, and rhythm constitute the semiotic substance of the students' *in-action-formulas*, in contextual algebraic thinking indeterminacy becomes an explicit object of discourse. Gestures and rhythm are replaced by linguistic deictics, adverbs, etc. Formulas are expressed in a perceptual and "perspectival" manner based on key terms like "top", "bottom", etc. Formulas, in short, are based on a particular way of seeing the sequence at hand.

Our discussion about symbolic algebraic thinking sheds some light on the meaning with which the students endow their first alphanumeric formulas. Instead of being an abstract calculating device, formulas often appear as vivid narratives. They are icons in that they offer a kind of *spatial description* of the figure and the actions to be carried out. What I called the "collapse of narratives" appears as an important step towards more encompassing ways of algebraic signification. The constitution of meaning after such a collapse deserves more research (see also Barallobres, 2007). While Russell (1976) considered the formal manipulations of signs as empty descriptions of reality, Husserl stressed the fact that such a manipulation of signs requires a shift of intention: the focus

becomes the signs themselves, but not as signs *per se*. And he insisted that the abstract manipulation of signs is supported by new meanings arising from rules resembling the rules of a game (Husserl 1970), which led him to talk about signs having a game signification.

The classroom example discussed in the last section shows how the teacher, through a complex coordination of gestures, alphanumeric formulas, and words, capitalized on the formula of one of the groups to make apparent for the whole class the idea of simplification of formulas. It was a first step, and certainly an important one in the students' ontogenetic journey.

Although I limited my account to the first two years of the 5-year journey, I hope that such an account is enough to get an idea of the students' struggles and progresses towards increasingly more encompassing forms of algebraic thinking.

Acknowledgment

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CERME 6 – PLENARY 2**Mathematics education as a network of social practices**

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Reactor: Margarida Alexandra da Piedade Silva Cesar (Lisbon University)

Summary. Based on an analysis of mathematics education research as an academic field and on current social, political and economic transformations in many European countries, I would argue for the need to rethink and enlarge definitions and views of mathematics education as a scientific field of study in order to provide better understandings and alternatives for practice in the teaching and learning of mathematics today. I will explore the notion of the “network of mathematics education practices” as a complex, multi-layered space of social practice where the meanings of the teaching and learning of mathematics are constituted. I will illustrate the potentiality of this notion to envision possible research paths in the field. I will illustrate these with the research that my colleagues and I have been carrying on multicultural classrooms in Denmark; as well as will offer examples of other research studies in Europe and other parts of the world where I see that the discipline is gaining newer insights that could allow attending to the social changes and challenges of the 21st century.

MATHEMATICS EDUCATION AS A NETWORK OF SOCIAL PRACTICES

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As academic fields advance, reflexivity on its own results and processes becomes a centre of attention and of disciplined inquiry. The growing amount of published papers and conference activities considering mathematics education, its theories, methods and results exemplify the need researchers have to make sense of the practice in which they are involved. Such type of reflexivity has always been a central part of my interest, probably due to the fact that my background in the social sciences has led me to constantly formulate questions about the type of insights on educational practices that mathematics education research offers in relation to the realities of schools and mathematics classrooms. Developing awareness on the research perspective that I adopt has, therefore, been as central to me as generating particular understandings and interpretations of the practices of teaching and learning in mathematics classrooms.

In this paper I focus on the issue of how to conceive of mathematics education as a field of research. This implies, on the one hand, examining definitions of the field as they appear in existing literature, and, on the other hand, articulating alternative views and languages to talk about the field. My intention is to provide a ground for discussing the research practices in which we engage and to which we devote a great deal of our effort and commitment. In my examination of this issue, I will contend that in the historical development of what we may identify as the field of mathematics education research, particular dominant definitions about the field of educational practices of mathematics teaching and learning have emerged. Such definitions of the educational practices have defined what the legitimate objects of study of the field of research are, and with that encompassing theories and methodologies to research the field of educational practices. As research advances, however, the definition of the field of research emerging from research practices is being pushed to its limits. I argue that the time has come to open possibilities of defining both research practices and educational practices in a way that allows tackling in serious, rigorous and systematic ways the social, cultural and political complexity of mathematics education in our contemporary societies. Opening the scope of the field does not represent a threat to the identity of the field, but rather an opportunity to engage with the enormous challenges that mathematics education practices pose to all their participants.

I start by a conceptual clarification of the language that I choose to address this issue, which entails a presentation of the underpinning ideas of my theoretical perspective. I clarify the notions of mathematics education as a field of educational practices and as a field of research practices. The distinction is useful in addressing the way in which these realms constitute each other, and of how different meanings have been ascribed

to them particularly from the second half of the 20th century when the international field of mathematics education research has been more visible and identifiable. I then move to argue that dominant definitions of the field of research and its corresponding views of its object of study are insufficient in tackling in a comprehensive manner the impact of larger contextual factors on the teaching and learning of mathematics. While research results continue to point to the influence of the “context” on actual possibilities to an effective improvement of the teaching and learning of mathematics, the field of research misses the development of scientific strategies to deal with both the understanding of those influences and the devising of strategies to deal with them in practice. As a response to this shortcoming, I play with the idea of defining mathematics education as a field of research which studies the complexities of the *network of mathematics education practices*. I define three different types of research moves or strategies that are necessary to deepening the understanding of the practices of teaching and learning of mathematics. I finalize by exemplifying these research moves with projects carried out by a growing number of mathematics educators around the world.

ANALYSING THE FIELD THROUGH ITS DISCOURSES

The increasing attention given to reflexivity in mathematics education research invites to discussions of how and why theories, methods and discourses in research are simultaneously constructed and get reproduced. In his paper during the ICMI study on what is mathematics education and what are its results, Ernest (1998) had identified the need for mathematics education research to address not only the primary objects of the field (the practices of teaching and learning mathematics), but also the secondary objects of the field (i.e., the products and processes of research practices). The growing emphasis on the effects of language and its connection to practices within the social sciences —known as the social turn— has influenced the way mathematics education researchers think about the field. Thus, it appears increasingly important to pay attention to the discourses that mathematics education research constructs about itself and the contributions and limitations of these constructions. By discourses here I understand the ways of naming and phrasing the ideas, values and norms that emerge from the constant and complex interactions among human beings while engaged in social practices. Researchers in academic fields construct particular discourses about their objects of study and their overall activity. Such discourses constitute systems of reason that regulate what is possible to think and do in a given field (Popkewitz, 2004). Thus, discourses generate both a space of possibilities as well as of limitations of what we can imagine as alternatives to existing orders.

Mathematics education as a field of research is not an exception. As researchers engage in studying the field, they not only define what characterizes legitimate practices of mathematics education. They also define the ways in which it is valid and legitimate to research those practices. I have elsewhere engaged in examinations of

the discourses generated in and by the field of mathematics education research, such as the idea of mathematics education being “powerful” (Christensen et al., 2008), the conceptions of students as mathematics learners (Valero, 2004a), and the concept of learners’ identity in mathematics (Stentoft & Valero, in press). In this paper I turn to the discourses of the field about itself. My analysis is based on a study of a variety of texts addressing mathematics education research as a field of study, such as, for example, the work of Jeremy Kilpatrick (e.g., 1992, 2006, 2008; Silver & Kilpatrick, 1994), books addressing the issue (e.g., Menghini, Furinghetti, Giacardi, & Arzarello, 2008; Sierpinska & Kilpatrick, 1998) and recent handbooks (e.g., English, 2008; Lester, 2007). Drawing on elements of critical discourse analysis (Fairclough, 1995), I focus on the dominant ways of talking that emerge from the texts as they address what mathematics education practices and mathematics education research are about. The references in my analysis serve as illustrations of the characteristics of the discourses that I am identifying.

EXAMINING “MATHEMATICS EDUCATION”

The use of the term “mathematics education” in English is ambiguous. Among others, Ernest (1998, p. 72) has argued that the term refers to “both a practice (or rather a set of practices) and a field of knowledge”. The term names the set of practices of mathematical teaching and learning, carried out mainly by practicing teachers and students, in a variety of formal and informal contexts, and where mathematical thinking and communication occurs. The term also refers to the set of practices, carried out mainly by researchers hired at colleges of education and universities, that study teaching and learning practices. A first thing to notice about the two meanings is that each one of them is addressing a field of practice. The former refers to the field of educational practices; the latter refers to the field of research practices. As fields of practice, each one of them has particular embodied, routinized activities, artifacts, ideas, values and forms of communication. They are distinct practices, though with intersections of practitioners (most often than not, researchers are themselves teachers and teachers are also researchers), interests, concerns and discourses. However, the two fields of practice are not identical. It is not my intention to go deeper into the characterization of these two fields of practice here. Suffice to say that their separateness or connection is a matter of concern for many practitioners located in each one of the fields (e.g., Ruthven & Goodchild, 2008; Sfard, 2005).

My intention with distinguishing the two fields here has to do with the relationship between the two, not in terms of how the field of research practice should illuminate and improve the field of educational practice; but rather in terms of how the definitions constructed for each of them are mutually constitutive. Let me explain, starting with a basic assumption. A theoretical perspective and an object of study are mutually constituted. It is not possible to talk about an object of study without a set of assumptions and language that recognizes and phrases a happening or a social event,

and makes it focus of attention. If this is the case, then we can think about the relationship between what is taken to be mathematics education as a field of educational practice and mathematics education as a field of research practice. My contention here is that through the development of the field of research practice, definitions of the field of educational practice have emerged.

Looking back at the history of the field of research practice through a general study of the different trends that have emerged in literature, as well as an examination of texts addressing the history of mathematics education research, there seems to emerge a common narrative about the origins of research. The interest of mathematicians and educators engaged in the teaching of the subject at different levels, particularly in relationship to teacher education, was a seed for paying systematic attention to mathematics in a learning and teaching environment (Kilpatrick, 2006). “The problems of practice” that is, the set of concerns for the predicaments of teachers’ instruction and students’ learning of mathematical topics, as formulated by Silver and Herbst (2007), have become the cornerstone of the research endeavor. The problems of practice have become the natural object of study of the field of research. They have also determined the ultimate goal of research, which is contributing to the improvement of practice. Many people defend these ideas as the essence of mathematics education research; the ideas are a central part of how many researchers define the object and aims of study (e.g., Hart, 1998). These ideas are seen by many in opposition to the idea that mathematics education research is growing as an academic field in itself, with a theoretical and methodological development that not always connects so closely with teaching and learning practices. There are also many scholars who acknowledge and actually try to understand not only the findings, but also the theoretical, methodological constructions of the field (e.g., Silver & Herbst, 2007). Of course, this debate is also fuelled by different agendas outside the field of study and the field of practice of mathematics education, such as the growing political demand for accountability of research funds and the focus on educational research to be the basis for evidence-based practice.

Independently on which side personal intentions and commitments are, two points are evident here. First, there is nothing “natural” in the definitions given to the field of research practice. The discursive construction of the object of study and the aims of research in the field correspond with the practices of researchers both in national and international communities. We actually need to denaturalize what seems to be taken for granted in the way we researchers, collectively and as individuals, talk about the field and engage with the field. Following from this, the second point is that definitions of the field of study entail definitions of the educational practices that research studies. This implies that it is not possible to assume complete independence between the social practices of teaching and learning of mathematics, from the social practices of researching them. The discourses of the field of study construe frameworks for thinking, conceiving and therefore actually engaging in the

educational practices (Popkewitz, 2004). The fields are distinct but discursively related.

Digging deeper into how the educational practices are being defined by the research practices, it is evident that definitions are historical and also situated in particular geographical settings. They are also contingent upon theories adopted to account for the problems of practice. A proper account of the complexity of the definitions exceeds the scope of this paper. Nevertheless, I will point to some salient features of the way research has been defined in general international terms. Although for many researchers the history of mathematics education research is short—in relation to the history of, say, mathematics—it is possible to find shifts in the ways of phrasing the focus of both educational and research practices. Looking at the 100-year long history of the International Commission of Mathematical Instruction (ICMI) as one international organization that has had an important role to play in promoting mathematics education research, the initial focus of the meetings, discussions and concerns of interest in the educational practices was the mathematical content. In what Bass (2008) has named ICMI’s “Klein Era”, at the beginning of the 20th century, attention was paid to issues of content and little distinction existed in fact between the gatherings of ICMI and the general meetings of the International Mathematical Union, except for the fact that the mathematical topics addressed in ICMI were more elemental mathematics. Such observation resonates with Kilpatrick’s assertion that the work of the first mathematics educators at the end of the 18th and beginning of the 19th centuries had a strong focus on the mathematical contents, although few other topics were present as well such as the history of mathematics and teaching experiments (Kilpatrick, 2006). A graphic representation of the field of educational/research practice in this time could look like this:

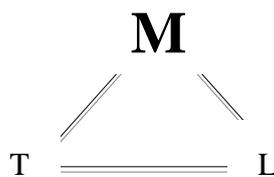


Figure 1: Mathematics at the centre of the field of practice and research

The linkage to psychology as a support discipline has been important in the construction of an empirical investigative approach towards the problems of practice. With the strengthening of parts of psychology as an experimental science and with mathematics education becoming a field in universities, mathematics education research found theoretical and methodological approaches to the inquiry of teaching and learning problems in mathematics (Lerman, 2000). The influence of the European didactic traditions have also played a major role in defining that the focus

of research is placed in the *didactic triad* constituted by the relationships between mathematics, the teacher and the student. As the 20th century advanced and more research work in the area was produced, explorations of the didactic triad had been focused on each of its elements, on the relationships among them, and on the whole complexity at stake in it. Combined with a variety of theoretical approaches to deal with the specificities of each of the elements, the didactic triad has been a basic but powerful model behind a great deal of research in the field. Saying that the didactic triad has been a model behind research in mathematics education does not intend to oversimplify and dismiss the advances of the field in understanding the complexity of the relations at the interior of the triad. There are numerous examples of particular models that have shown such complexity (e.g., Balacheff & Kaput, 1996, for the case of the role of technology in mathematics learning).

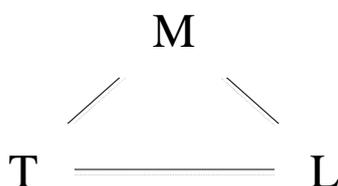


Figure 2: The didactic triad at the centre of the field of practice and research

There are several points to notice in the research and discussions about the field of research adopting this model. A first issue is the issue of the *mathematical specificity*. Mathematics education research is defined as the discipline studying “the practice of mathematics teaching and learning at all levels in (and outside) the educational system in which it is embedded” (Sierpiska & Kilpatrick, 1998, p. 29). In this field, “[...] mathematics and its specificities are inherent in the research questions from the outset. One is looking at mathematics learning and one cannot ask these questions outside of mathematics.” (p. 26). Questions, problems, theories and methods not allowing for mathematical specificity tend to be considered irrelevant, and out of the scope of mathematics education research. Second, there also seems to be an underlying assumption about the *decontextualization* of the triad. The objects of research tend to be presented in terms of students’ learning of concepts (and most often students’ misunderstanding of them), and teachers’ instruction of mathematical concepts. They are text, the content, the centre. The con-text, that surrounding accompanying and constituting the text, does not fall inside the research gaze. Therefore, except for a brief mention to the characteristics of the people involved in a study, no more grounding and information is available about the context of a given phenomenon studied. If some context is mentioned, it is not taken significantly as part of the analysis. The assumption of decontextualization goes also hand in hand with the assumption of *closure* of the didactic triad. This means that research problems are both formulated within and accounted for in the didactic triad. The

practices of teaching and learning are somehow self-contained and self-explanatory. There are plenty of examples to find in research on geometrical thinking, argumentation and proof, etc. A review of, for example, the CERME proceedings on these topics will clearly show this tendency.

As some researchers have started to consider classrooms dynamics, the classroom has appeared as a clear boundary around the triad, a clear, manageable context. One example of research contributing to the strong emergence of the classroom is the work of Cobb and collaborators during the 1990's which lead to the notion of the socio-mathematical norms (e.g., Cobb, Wood, & Yackel, 1992) which explained students' learning possibilities in terms of the continued interactions happening in the instructional practices in classrooms. In the case of Cobb and collaborators, the move from a social constructivist theory of learning to address mathematics education, to a socio-cultural theory of learning was one of the reasons for an enlarged understanding of the role of the social dynamics of the classroom in relation to individual learning. This seems to have been the case for many other researchers who started to focus on the situatedness of teaching and learning practices in classrooms and schools (e.g., Boaler, 1997).

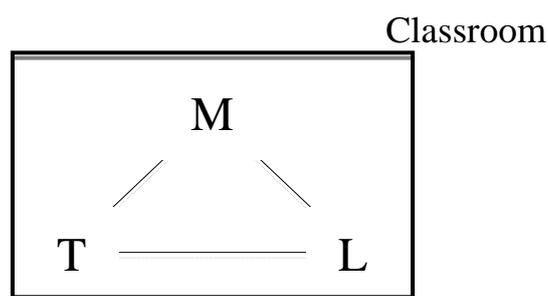


Figure 3: The didactic triad within the boundaries of the classroom

Lerman (2000, 2006) has argued that researchers in mathematics education, influenced by the language turn in the social sciences, have adopted a variety of sociological and cultural-anthropological theories for the study of the teaching and learning of mathematics. The strong social turn in the field has meant the recognition of the embeddedness of mathematical thinking, learning and teaching in larger social, cultural, economic and political structures. Research with a concern for equitable access to mathematics is an example of how such recognition has been fundamental in the generation of new research areas. In many cases, there is an attempt to stick to the formulation of problems within the didactic triad, though, from a different theoretical position. For example, part¹ of the work of Radford concerning semiotic,

¹ In few of his papers, Radford shows a broader analysis of the relation of mathematics and culture. For example, Radford and Empey (2007) present a study of social and mathematical practices outside the didactic triad. They show that “within a certain historical time period, mathematics –in its amplest sense [...] accounts for the formation of new social sensibilities –both in terms of

embodied interpretations of students' mathematical thinking give a cultural dimension to the issues of the didactic triad and show a connectedness of children's thinking and school practices with other forms of practices outside schools and classrooms (Radford, 2008). In general, it is interesting to notice that, despite the adoption of theoretical frameworks that have an understanding of the social and cultural that goes beyond the limited understanding of "social" in terms of interaction among people present in interactionist theories associated with constructivism, the focus of attention of research remains being the classroom and, within it, the didactic triad.

Some other types of research have also challenged the idea that the privileged site for research is the classroom. If mathematical thinking is a social and cultural activity it happens in other social spaces different from classrooms. The classical example of this broadening is the research by Nunes and collaborators (Nunes, Carraher, & Schliemann, 1993) which opened the space for investigations of the relationships between mathematics in school and out of school. The extensive research belonging to the ethnomathematical program has also explored mathematical practices in working and everyday life settings. Already at the beginning of the 1990's Gómez, Perry and collaborators (e.g., Gómez & Perry, 1996; Perry, Valero, Castro, Gómez, & Agudelo, 1998) had studied mathematics teachers' change and professional development within the complexity of the school organization. Such trend has also been explored by Cobb and collaborators (e.g., Cobb, McClain, Silva Lamberg, & Dean, 2003) in an attempt to connect classroom communities with their immediate organizational contexts. More recently attention has been paid to the school mathematical experiences of parents in relation to the school mathematics practices of students when coming to new countries and cultures (Civil, 2007). In general, there has been a growth in research that documents the relationship between factors outside of the classroom (in the context) and the state of affairs inside the classroom, in the didactic triad.

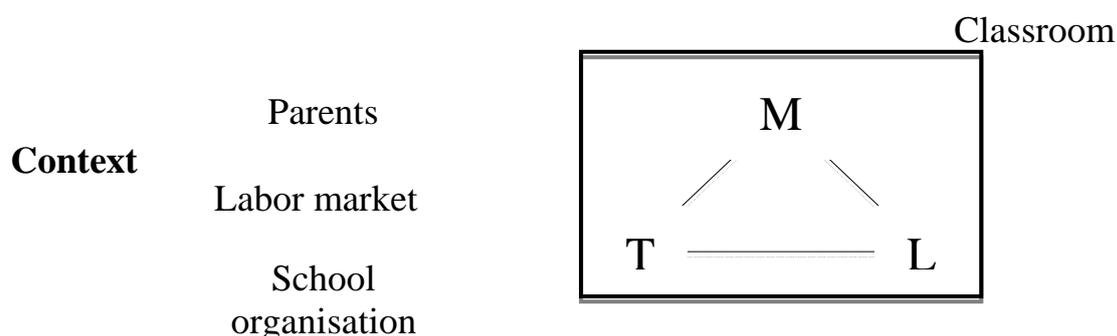


Figure 4: The didactic triad in a context

capacities to create new forms of understanding and novel forms of subjectivity" (p. 232).

In other words, the welcoming of socio-cultural theories to deal with the problems of practice has helped considering the context of those problems as a significant part of them. With such move the interpretations and understandings of the terms “mathematics” “teaching”, “learning”, and “thinking” are broadened and new phenomena, interactions and practices where mathematical elements are present start being included as legitimate objects of study. As an evidence of this we could look at different studies classifying the research published in different international journals and conference proceedings. All these studies assume that certain international journals actually represent the production in the field at any given time. Gómez (2000) argues that “mathematics education research production is centred mainly on cognitive problems and phenomena; that it has other minor areas of interest; and that it shows very little production on those themes related to the practices that influence somehow the teaching and learning of mathematics from the institutional or national point of view” (p. 2-3). In a review of literature focusing on how research addresses the significance of students’ social class for the learning of mathematics, Chassapis (2002) also argues that little and almost insignificant attention has been paid in 30 years of research production to the issue of who are the mathematics learners and how the learners’ background influences mathematical learning. This lack of attention contributes to a lack of comprehension about the social, political and cultural complexity of mathematics education and the factors involved in it. Lerman, Tsatsaroni and Xu (2006; 2002) have also produced an overview of the theories used in mathematics education research in the period 1990-2001. Their data shows that although socio-cultural theories of different types had been more used in the field, the majority of theories used in published papers are traditional psychological and mathematical theories focusing on the learners, the mathematics and the teachers. Skovsmose and Valero (2008) have also classified publications with the purpose of showing how the field gives different meanings to the term “democratic access to powerful mathematical ideas”. The concentration of research on mathematical and psychological interpretations, focusing on the study of classroom practices led them to conclude that “it is highly problematic that dominant research trends in mathematics education operate within a limited scope of the space of investigating democratic access to powerful mathematical ideas. Such a paradigmatic limitation effectively obstructs the possibilities for mathematics education to face the paradoxes of the informational society”. Time has passed and, as Lerman and collaborators show, the adoption of socio-cultural theories enlarges and thereby a sensitivity to define research objects outside the didactic triad emerges. However, the majority of research published defines problems that deal with the central elements of the didactic triad, and from theoretical perspectives focused on mathematical cognition. A recent overview for the papers published in ESM, JRME, MERJ, FLM, ZDM and PME proceedings during the year of 2007² confirms the previous findings: 25% of

² I thank Alexandre Pais for his support doing this overview.

papers choose as a focus a mathematical notion in learning or teaching; 29% of the papers address issues of teachers' dealing with mathematical contents and 31% choose the learners' understanding or thinking of mathematical notions. The issue remaining is how does the field of research practice address the complexity of the field of educational practice beyond the didactic triad?

OPENING UP THE CONTEXT OF THE DIDACTIC TRIAD

Although the research gaze of the field of research practices seems to be enlarged, still many researchers express a concern with the issue of dealing with the “context”. Let us see at this in a more detailed way. In the first place it is important to discuss the notion of context and how the field of research defines and addresses it. In the section above I shortly defined context as the surroundings of an object –the “con” accompanying a “text”. As I argued before, research approaches focusing on the didactic triad tend to ignore context, since the focus of research is the “text”. In the type of research focusing on learning and thinking mathematically within the didactic triad, some understandings of “context” are present, although in the form of the context of the mathematical contents, problems or ideas that students and/or teachers deal with. This is what Wedege (1999, p. 206) calls the *task-context*.

I also argued that socio-cultural theories in mathematics education have opened for considerations of the factors that affect a classroom situation. A *situation-context*, following Wedege's formulations above has been evident in research literature, i.e., in research addressing the immediate context of teaching and learning in the classroom. But I also argued that context can be much more than the walls of the classroom. Concerning the conceptualizations of the notion of context in socio-cultural theories, Abreu (2000) has discussed how different socio-cultural theoretical trends conceptualize context, and which implications such conceptualizations have for the study of mathematical thinking and learning. On the one hand, one can consider the *micro-social and cultural contexts* of mathematics teaching and learning by focusing on “the immediate interactional setting where face-to-face interactions take place” (p. 2). On the other hand, one could focus on the *macro-social and cultural contexts* which are the “non-immediate interactional settings loosely defined by other authors as ‘the broader socio-cultural systems’ [...] or ‘broader sociocultural milieu’” (p. 2), which frame mathematical activity in any particular interactional setting. The interesting research endeavor, however, is how theories connect micro and macro contexts in a search for relationships between how individuals make sense of mathematical ideas in the complex field of activity within larger symbolic systems. For Abreu, the issue of the micro-macro relationship is not only a matter of how particular interactions with certain cultural tools mediate thinking, but also of how social valorizations of knowledge mediate individual positioning towards that

knowledge in the creation of personal identities³. From these perspectives, context is not just like the “the bowl that contains the soup” or the “surroundings of a text”, but rather a constitutive element of the text itself. Text and context are dynamic; and they are dialectically constituted (McDermott, 1996).

In the discussion by Abreu (2000), the distinction between micro and macro context opens up for a reflection on where, on the continuum between agent and structure, mathematics education research tends to focus its research gaze. If mathematics education research is seen as a social/human field of study, it cannot escape this reflection. The classical micro-macro debate in sociology addresses the issue of whether the social world is to be understood by studying individual and their interactions or by studying social structures. Each social discipline delimits the scope of the “social” in its objects of study in particular ways. Some types of areas refer to the “social” as a broad, all-embracing functioning of human action in whole cultures and civilizations (e.g., Beck, Giddens, & Lesh, 1994). Other kinds of sociological viewpoints related to disciplines such as psychology or economics, have defined the “social” as the realm of interaction among individuals. Mathematics education researchers, in the study of the social and human phenomena of mathematical thinking, learning, teaching and education, have taken a stance in this discussion implicitly (more often than not). Mathematics education research, as characterized previously with a focus on the didactic triad, has tended to focus so much in individual *mathematical* thinking, reasoning and cognition that the “social” dimension was almost non-existing. One example of this could be mathematics education as seen from a radical constructivist perspective centered on individual reorganizations of mathematical ideas. Social constructivism and related views of learning opened for a social dimension in terms of inter-personal interactions. It is only with certain recontextualizations of socio-cultural theories that the understandings of the social move beyond the individual and inter-individual level and, as Abreu says, push for the need of establishing a connection between micro and macro levels of the social. Nevertheless, studies in mathematics education from socio-cultural perspectives have also tended to focus on micro-contexts, probably because the dominance of discourses of the field of study with a centre on the didactic triad, and with a closeness to the “problems of practice” define the legitimate problem field in terms of micro-interactions and micro-contexts. The interesting question that emerges here is whether focusing on objects and problems in a micro-sociological level is the only possibility for mathematics education research. I will return to this point.

³ The research of Guida d’Abreu offers an interesting example of the different notions of context put in operation in research on mathematical practices. From her earlier research on Brazilian sugar cane farmers to her recent work on the valorizations of mathematics among immigrant children and parents in England (Abreu, 2007), it is possible to identify the differences in theoretical perspectives concerning how to deal with the significance of context in relation to mathematical practices.

Addressing context—and with it the many factors, actors, meanings and discourses that are difficult to grasp at a micro-social level but that researchers know have a great influence on the micro settings that we choose to research—is a difficult matter. In systematic readings of literature, researchers point to the need of research that actually deals with both the micro-complexity and the macro-complexity of mathematics education. I present here a selection of studies from different types of research and theoretical orientations that illustrate this concern.

In the USA and dealing with the concern of how to expand massively the constructivist-inspired vision of school mathematics of the NCTM, Confrey (2000a, 2000b) argued that it was necessary to expand constructivism from the level of a learning theory operating at individual or classroom level, to the level of a system. She urged for a view of research that could go beyond the micro-findings of research:

[...R]esearch never anticipated all of the leaks in the bucket, nor did it bring strongly enough into relief the fact that the bucket is only a small part of a large system. It is undeniable that researchers identified critical issues [...] Despite the importance of these results, changing any one of them alone was proving insufficient to fix the problems of mathematics and science. [...] All of these changes require one to look more broadly, beyond the restricted focus of a research study. All of them require us to move beyond the level of the classroom, a move that occurs only rarely in educational research. (Confrey, 2000a, pp. 88-89)

An examination of research and development initiatives in the USA to bring democratic access of students to the goods associated with high achievement in mathematics, Rousseau and Tate (2008, p. 315) conclude:

The factors influencing democratic access in mathematics education are complex. If we look strictly at events as they occur in the classroom, without consideration of the complex forces that helped to shape those learning conditions, our understanding is only partial [and] the solutions to the problem [are] ineffectual. We must seek to reach a fuller understanding of the complex issues that shape access and opportunity to learn in mathematics so that, in turn, we can develop more effective strategies to ensure access and opportunity for all students.

In the area of teacher education, studies on the professional development of mathematics teachers and on their learning have argued and shown the importance of broadening the understanding of what is at stake when professional teachers do their work and learn. Krainer has pointed to this systematically since the end of the 1990's. More recently (2007, p. 2), he writes:

It is important to take into account that teachers' learning is a complex process and is to a large extent influenced by personal, social, organisational, cultural and political factors.

Acknowledging the multiple influences in teachers' learning, the third volume of the *International Handbook of Mathematics Teacher Education* (Krainer & Wood, 2007) is organized around chapter addressing teachers' professional learning at individual,

team, community and network levels. The book as a whole illustrates research that moves beyond individual teachers and classrooms.

The examples above represent few key studies of people who, in different research areas and during the last 10 years, have argued for a need to expand the scope of research of the field. If mathematics education research ought to tackle systematically not only the micro-contexts of mathematical teaching, learning and thinking, but also its macro-contexts and the relationship between the two types of contexts, it is evident that definitions of the field of study centered on the didactic triad and recognizing the existence of a context are not enough. I will now engage in exploring a proposal of what the field of research practices, and therefore, the field of educational practices could be thought of.

MATHEMATICS EDUCATION AS A NETWORK OF SOCIAL PRACTICES

Our understandings of mathematics education as a field of research practices need to be enlarged, and with that our understandings of the practices that are the objects of study of the research field. This idea has always been part of a concern that has emerged from my research experience in Colombia as part of the team of researchers called “una empresa docente” at the Universidad de los Andes in Bogotá, later on as part of my doctoral studies at the Danish University of Education in Denmark, and now as part of the research group in mathematics and science education at Aalborg University in Denmark.

This idea has been developing since 1999 when, in the exploration of the relationship between mathematics education and democracy, I wrote:

First, the justifications to connect mathematics education to democracy are not only found in the mathematical content, but also and mainly in the social and political factors that constitute the learning and teaching relationships in the classroom, in the school and in society. Second, and as a consequence of the latter, it is necessary to study the context of the practices and its components. By doing so, we could gain a better understanding of what mathematics education for democracy means in other instances where the social relationships that constitute and shape mathematics teaching and learning are built. Thus, a definition of the social practices of mathematics education should include not only all the institutionalized relationships among teachers, students and mathematics at the different levels of schooling, inside and outside the educational system, but also the activity of policy makers that at a national level deal with the design of curricular guidelines for the teaching of mathematics [...]; the activity of writing mathematics textbooks [...]; the complex relationships that configure the teaching of mathematics within the organizational structure of educational institutions [...]; the spaces of teacher education both in its initial [...] and further stages [...]; as well as the configuration processes of social conceptions about the role of mathematics education in society [...]. All these practices together should be potential and legitimate objects of study if we aim

at understanding and, at the same time, promoting a mathematical education for democracy. (Valero, 1999, p. 21)

My initial concern for the relationship between mathematics education and democracy within the framework of critical mathematics education proposed, among others, by Skovsmose (Skovsmose, 1994) has evolved to become a general concern for developing a socio-political approach to mathematics education. As I have argued elsewhere (Valero, 2004b, 2007), such an approach views mathematics education as social practices where power relationships among the participants in and the discourses emerging from the practices are an important constitutive dimension. In contrast to a socio-cultural perspective to read mathematics education where the issue of power is not dealt with explicitly or is hidden in the valorization of practices and meanings within semiotic systems, a socio-political approach privileges power.

The concept of the *network mathematics education practices* has been under construction for a while and it has been named slightly different in my different writings (Valero, 2002, 2007, 2009). This paper has been an opportunity more to clarify the views, assumptions and analysis behind such notion. More than a finished concept, I see the concept as being still under construction. But what does this notion refer to?

In the first place, if mathematics education practices are to be defined beyond the didactic triad and in relation to their broad context, it is necessary to define “mathematics education” not only in terms of the agents and phenomena strictly related to mathematical thinking, teaching and learning, but also in terms of the series of social practices that contribute giving meaning to the activity of people when thinking, learning and teaching mathematics, as well as when engaging in situation where mathematical elements are present. Thus, the *meaning* of mathematical thinking, teaching and learning is not exclusively related to the particular meaning of the mathematical content and concepts in learning and/or teaching situations⁴. Meaning is also related to the significance given to the mathematical rationality within a diverse series of social practices constituting educational practices in a given historical time. Behind this idea there is the clear recognition that what we understand by mathematics is far from being a unified body of knowledge determined by the practices of professional mathematicians, but rather a series of “knowledges” and “language games” bounded to a diversity of practices, all of which have a family resemblance. The recent work of Knijnik (2008) in ethnomathematics is useful here to discuss the issue of meaning and diversity of mathematics in relation to social practices. The work of Sfard (2009) in identifying the irresistible pervasiveness of *numberese*, the numerical discourses in our societies, is useful in understanding how

⁴ Skovsmose (2005) has pointed to this idea in relation to the sense that students make of mathematical ideas. For him meaning is constructed in relation to the students’ foregrounds and the role that mathematics plays in how students perceive their future possibilities in life.

numerical discourses associated with the diversity of language games of mathematics in our society constitute ways of seeing the world. If mathematics-related language games are present in many spheres of practice, the meaning of them are also constituted in relation to those practices and their discursive elements.

Second, which is the diversity of social practices where the meanings are constituted? Mathematics education as a field of educational practice can be defined as a series of social practices, carried out by different people in different sites, where the meaning of the teaching and learning of mathematics is constituted, in particular historical conditions. Those social practices are to be found not only in the classroom where teachers and students interact around mathematical content, but also in, for example:

- family practices and parents' demands to school (mathematics)
- local community practices and their educational needs
- international or national educational policymaking practices in mathematics, which structure and regulate the forms of valid knowledge, competences and achievement levels to be attained by students and teachers in mathematics
- teacher education practices
- textbook production practices
- labour market practices and expectations on the mathematical qualifications of workers
- mathematics education research practices
- mathematics research practices
- youth culture practices
- mass media practices and the construction of public views and discourses of mathematics
- practices of international comparisons of (mathematics) achievement

Many other sites of practice could be mentioned and could be identified to be relevant at a given historical time. As an example, we could consider the role of the international comparative studies that, from the time of TIMSS in the middle of the 1990's have had a great influence in national policies, local curricular changes and teachers' work. Particular meanings of what counts as mathematics education have been put forward through the impact that results of these comparisons have had on adjusting mathematics educational policies in many countries. The PISA studies have also brought with them definitions of mathematical competency that have been incorporated in several European countries. These definitions have framed what at this historical time policy makers, teachers and researchers understand by mathematical competence. The work of Jablonka (2009) evidencing this rationality is useful in seeing how the PISA rationality has permeated many other spheres of practice in mathematics education. Whether international comparisons will keep on having such a defining role in the network of mathematics education practices in the future depends on political and economic configurations of the discourses that will rule educational thinking in the years to come. As for mathematics education it is

clear that such an element has impacted in this historical time. The future also remains uncertain.

By using the idea of a network—in contrast with the use of the concept of system—I want to convey the idea that these various sites of practice, their participants, organization, rules and discourses, are sometimes loosely and sometimes tightly coupled depending on particular historical circumstances. It is not possible to assume a particular general dynamic and development of the practices, except from the idea that many of them are implicated in the construction of the multiple meanings ascribed to mathematics education in a given time and location. In this sense, this notion is different from, for example, the vision proposed by Confrey (2000b) of a constructivist learning system.

Figure 5 is an attempt to represent the network of social practices of mathematics education, as far as my two-dimensional expertise for this kind of drawings permits to grasp the idea. The “bubbles” represent a site of practice. Notice that some bubbles are empty. With this I want to convey the idea that many practices may be considered at a given time. The connecting lines may sometimes be weaker and some times be stronger. A better representation would be to imagine a 3-D constellation of bubbles that move, become bigger or smaller, and connecting in distinct ways at different times.

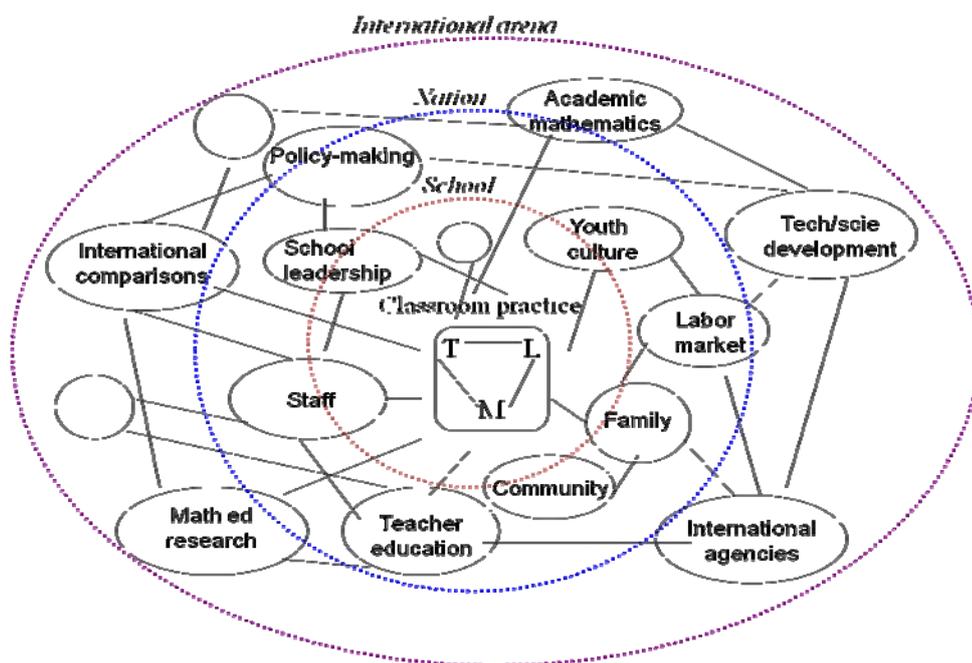


Figure 5: A representation of the “Network of mathematics education practices”

For me, defining mathematics education in terms of the network of mathematics education practices allows to evidence the cultural, social, economic, historical and political complexity of mathematics education. It also opens for envisioning a quite distinct field of research practices that, besides dealing with the objects and relationships that it has addressed until the moment, can engage in other types of research moves, with the double aim of generating deeper understandings and

interpretations of the field and of addressing the problems of practice of the multiple participants in this broader field.

If the field of research deals with the study of the field of educational practices defined in terms of the network, then three issues become evident. Firstly, the field of research and any study within it can be defined in terms of the *mathematical specificity* of it. However, the mathematical specificity of mathematics education research cannot be defined mainly in terms of the particular mathematical content, notions or competencies being addressed in the research. Rather, it has to be defined in terms of the significance of the mathematics-related practices and rationalities for the construction of the meaning of such practice, or other related practices, among its participants. When discussing research the concern of some researchers with the mathematical specificity of a given project is often expressed through questions such as: “But... would it matter if one changed the word ‘mathematics’ for the word ‘geography’ or ‘history’ in this project?” If we understand the mathematical specificity of mathematics education research in the broader terms proposed here, questions such as the one above will become completely irrelevant and would not be anymore a question to judge whether a research is a “proper” mathematics education research. If a research addresses in substantial ways the meaning and importance that different participants give to mathematics-related practices, or how mathematics-related rationalities that have an impact on the way mathematics education discourses are formed, then a research could be part of the field of mathematics education. In other words, the mathematical specificity of the field is related more to the social valorization that mathematics-related practices have in the dominant cultural, social and political order, and not to an explicit mathematical content or knowledge being researched. Such valorization is associated to the status of the field as a power/knowledge, which allows participants in mathematics-related practices to gain a positioning in relation to other people. That we study mathematics-related practices and their relation to the meaning of mathematics education has therefore a social and political significance, even if there is no apparent mathematical content involved.

Second, the study of any of the practices involved in the network has to acknowledge seriously *contextualization*. In contrast to the decontextualization that dominates in views of the field focusing on the didactic triad, researching the network of mathematics education practices invites to search for the intricate relationships between different sites of practice in constituting each other. The contextualization of mathematics education practices point to the contingency of practices and discourses when people engage in the task of giving meaning to mathematics-related ideas and practices in educational spheres or in any other sphere of human action⁵.

⁵ For an example of a study exploring the significance of contingency and complexity when researching mathematics education practices see Stenoft (2009).

Third, the view of mathematics education as a network of social practices implies that research problems do not need to be defined nor addressed within the didactic triad in a closed manner, but rather they can be formulated and tackled in the *openness* of the sites of the network. While a closed view of the field of research and practice will tend to become internalistic and provide problems and explanations within the realm of the elements involved in the didactic triad, the network of mathematics education practices highlights that the problems researchers formulate and their interpretations are always fragmented and cover only one little part of the complexity of practice.

The issue that I will engage with now is: how is it possible to do research in the “hyper-complexity” that the network of mathematics education suggests?

RESEARCH MOVES IN THE NETWORK

Whenever we do research, we perform a “move” or a strategy in the process of constructing knowledge about the objects involved in our study. It is obvious to say that these moves depend on theoretical and methodological frameworks, as well as on the traditions of the field of study. In mathematics education as a field of research practice focused mainly on the didactic triad, the most frequent research moves can be characterized as strategies addressing a very well defined research object, where the complexity of variables or factors considered is limited in order to make research projects manageable and realizable. The research move has been then a move towards an in-depth exploration of few factors and actors. The result of such move has been the production of a considerable amount of knowledge about how factors work in isolation, at the expense of how they interact together. Confrey’s quotation cited above pointed precisely to this characteristic of mathematics education research. Some people call this the “fragmentation” of the field, which could be solved by striving for unification of theories. Whether this unification is possible and desirable, and actually can contribute to address the fragmentation is an issue of debate in the community. I do not think that striving for unification is neither possible nor desirable. I agree with Lerman (2006) in the argument that the apparent “fragmentation” is a very condition of the endeavor of researching social and human processes such as mathematics education, at the historical time we are living now. Rather, I would argue that fragmentation emerging from research moves that try to cover the depth of defined problems needs to be complemented by different research moves that provide needed problematization and better insight into the social and political complexity of the multiple practices of mathematics education. In what follows, I will formulate three research moves —among many others one could think of— for researching the network of mathematics education practices.

If mathematics education practices are seen as the network I proposed, the aim of the research field would be to provide insight into not only how each single node of the network operates constructing the meaning and significance of mathematics education, but also into how different nodes interconnect at particular historical

times. A research move aiming at *covering the breadth of the social practices of mathematics education* would then “slice” and define objects of study in a different manner. It would define problems in terms of the interrelationships of different nodes in the network.

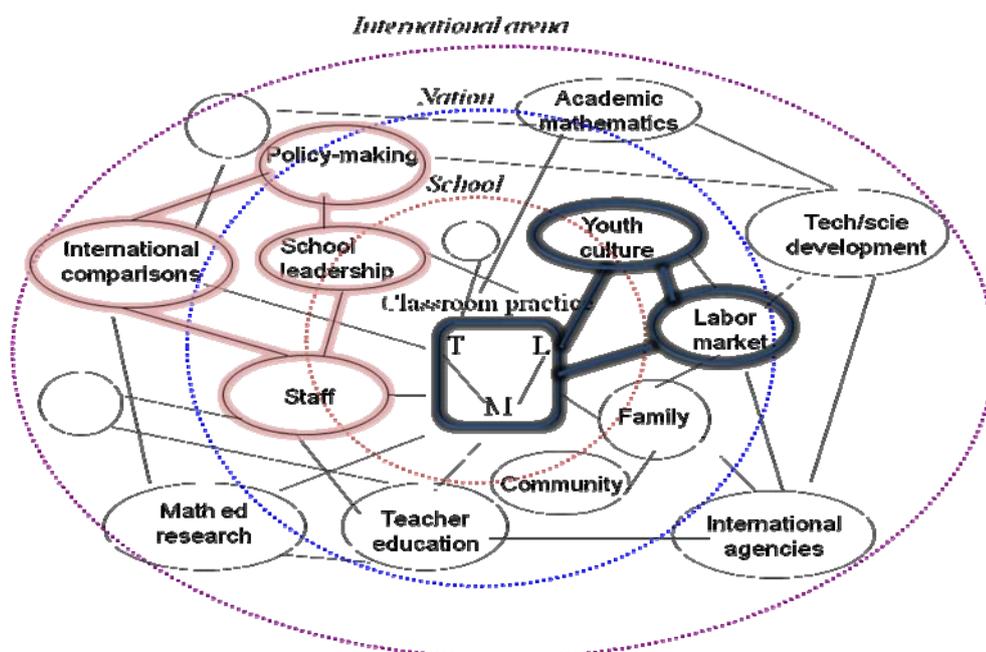


Figure 6: Defining research moves in the “Network of mathematics education practices”

The highlighted areas in the diagram above illustrate possible ways of “slicing” the network in a research move trying to gain breadth in the research. The area highlighted in the right side of the diagram would correspond to a study of, for example, how international comparisons in mathematics have affected national policy making, school leadership and demand for change to mathematics teachers in schools at the level of staff organization. The highlighted area to the right could correspond to a study on teaching and learning cultures in the classroom in relation to youth culture and demands from the labor market. The study of Zevenbergen (2005) on “Millennial” young people’s numeracies at the workplace is close to such a kind of exploration of the network.

Other examples of such a research move for breadth is Martin (2000) who examines how the systematic failure of Afro-American students in the USA is constituted in a multilayered space of individuals, schools, families and communities. He shows how the mathematical identities of the students in his study can only be seen and interpreted in this multiple, interconnected levels. The research of Alrø, Skovmose and Valero (2008) argue and document the need of expanding the lenses for researching learning possibilities and conflicts in multicultural mathematics classrooms by considering the interconnectedness of at least nine different settings of practice: students’ foregrounds, students’ identity, teachers’ perspectives of and

priorities in mathematics teaching, classroom interaction, the mathematical content, friends' priorities for participation in mathematics education, parents' expectations of mathematics education, the tools and resources available and the public discourses on diversity and education.

Another important strategy is *moving back and forth along the continuum of agency and structure* or, in other words, micro-social and macro-social units. One example of this type of move is the work of Gellert (2008), who in examining the issue of comparing and combining different theoretical frameworks, delineates a general methodology that, based on interactionist and structural theories, allows to interpret how the mathematics classroom discourses and practices are implicated in the reconstruction of social in(ex)clusion. Morgan (2009) also presents a study that, within the framework of critical discourse analysis, shows how the differential discourses of mathematical ability in curricular documents and textbooks targeted towards students with different attainment levels generate differential educational possibilities for different types of students. This study illustrates that ideas and discourses of individual mathematical ability are not only produced in the classroom, but are also produced in institutionalized practices at a level of structure that goes beyond the individual participants in mathematics education practices in classrooms. These two studies exemplify research moves, with their corresponding theoretical and methodological tools, that connect the micro and the macro contexts of mathematics education.

Yet another strategy is *moving along time to find the historical constitution of the meanings of mathematics education*. Such a move evidences the contextualization of mathematics education practices in particular social configurations. Inspired on the archaeology and genealogy of practices and discourses suggested by Foucault, Knijnik and her collaborators have been recently exploring how different central ideas in the field of mathematics education have come to be created. One example is the research by Duarte (Duarte, 2008) on how the idea of the necessity and importance of connecting school mathematics and the world out of school—or the “real” world—has emerged in the particular case of Brazilian mathematics education discourses. The study digs in the history of education in Brazil and identifies the historical moment in which the conditions for the introduction of such idea took place at the beginning of the 20th century. At the same time, the process of recontextualization of the idea in relation to mathematics education is shown through an analysis of mathematics education journals and conference proceedings in recent times. Other studies (Knijnik, Wanderer, & Duarte, 2008) examine and problematize how other ideas such as the necessity of using concrete materials have become part of the dominant discourses of mathematics education.

TOWARDS THE FUTURE

Mathematics education research has grown as a field of educational research. It has expanded in terms of the amount of results produced, the diversity of theoretical approaches and the richness of the problems addressed. Mathematics-related practices in schools and in different social spheres of action also become more and more evident to different participants in those practices. Whether mathematics education research has the potentiality for addressing in significant ways those practices and generating interesting insights about them, is a matter of how far researchers—as well as practitioners—want to engage in the exploration of the social, cultural, historical, political and even economic significance of them in the construction of society.

Enlarging the scope of the field in terms of the network of mathematics education practices poses both intellectual and ethical challenges. Researching the network of mathematics education practices through, among others, the three types of research moves I suggested here demands much more collective effort, and much more sustained interdisciplinary collaboration with colleagues with expertise in other research fields. I am well aware that, given the tighter funding possibilities for mathematics education research at this moment and the increasing publication demands from university administration, constructing research agendas in this line is an ambitious task. Nevertheless I still think that more studies in this line will help the field gaining a richer insight and understanding into the functioning of mathematics education in society. Tackling the complexity of mathematical thinking, learning, teaching and rationality in our societies is definitely an intellectually sophisticated and demanding—as well as fascinating—endeavor.

It is also an ethical challenge in that an honest concern with the betterment of practices—and with the many tortuous and disenfranchising school experiences of many children around the world—demands taking political risks that go beyond the known boundaries of established disciplines and fields of research. Moving the boundaries of a research field such as mathematics education is an ethical commitment with what our work as educators and researchers has to offer to our selves, our children and the generations to come.

I hope that the complexity that suggests the network of mathematics education practices can question the very many comfortable, good and predictable research results that pullulate in the field, and open the space for a third epoch of research concerned and committed with the relationship between mathematics, education and society. As suggested by different participants in the ICMI Centenary symposium in Rome in March 2008 (i.e., Artigue, 2008; Blomhøj, 2008, p. 172; da Ponte, 2008, p. 110; Povey & Zevenbergen, 2008, pp. 285-286) as an international community we have gained awareness of the complexity of mathematics education. The European community represented in CERME can certainly contribute in that direction. It is time to do it!

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<p>CERME 6 – SPECIAL PLENARY SESSION Ways of working with different theoretical approaches in mathematics education research</p>

The contributions on the following pages constitute a report about a CERME 6 special plenary session; they include an introduction by Tommy Dreyfus, two papers by Angelika Bikner-Ahsbabs and by John Monaghan, and a report on the discussion that followed the presentations by Angelika and John.

We thank Susanne Prediger for her help with recording the discussion.

TABLE OF CONTENTS

Introduction.....	2
<i>Tommy Dreyfus</i>	
Networking of theories: why and how?	6
<i>Angelika Bikner-Ahsbabs</i>	
People and theories	16
<i>John Monaghan</i>	
Discussion	24

WAYS OF WORKING WITH DIFFERENT THEORETICAL APPROACHES IN MATHEMATICS EDUCATION RESEARCH AN INTRODUCTION

Tommy Dreyfus

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The development and elaboration of theoretical constructs that allow research in mathematics education to progress has long been a focus of mathematics education researchers in Europe. This focus has found its expression in many CERME working groups: some are focused around a specific theoretical approach and others allow researchers from different theoretical traditions and backgrounds to meet and discuss. For example, the working group on *Argumentation and Proof* at the present (CERME 6) conference has reported on passionate discussions about different theories and their relationships (Mariotti, 2009). More specifically, relationships between theories have been made the explicit focus of attention of the theory working group that started at CERME 4 in 2005. This group has been reconvened at CERME 5 as well as at CERME 6; this year, we discussed fifteen papers, twelve of which make use of at least two theories and deal with how or why they can be connected in some way (see the part on Working Group 9 on *Different Theoretical Perspectives in Research in Mathematics Education* in these proceedings). The plenary activity from which this report emanated inserts itself in this line of work of CERME; one of its aims was to broaden the discussion about relationships between theories to include members of all CERME working groups.

The undertaking of mathematics education is very complex; this complexity is well expressed, for example, in Paola Valero's diagram (Valero, 2009). It is not without reason that the field has developed from having a curricular focus via a cognitive focus in various directions including philosophical, socio-cognitive, anthropological, ethnographic, and other perspectives, all the while producing home-grown theories to deal with all these aspects – and I am not even trying to distinguish between paradigms, theories, theoretical frameworks etc. For example, Realistic Mathematics Education has variously been characterized (including by people from the Freudenthal Institute) as a theory for mathematics education, as an instructional design theory or simply a philosophy for mathematics education.

When one reads a journal like *Educational Studies in Mathematics*, it seems at times that every paper presents a new combination of existing theories, a new theory, or at least a development of an existing theory. This raises the question how to look at and deal with the diversity of existing theories in mathematics education. Does this diversity express richness or does it express lack of focus (Steen, 1999) or even arbitrariness?

The question is made all the more urgent and difficult since theories come in different 'shapes' and 'sizes' and have different functions. Some concern the micro-genetic

analysis of a learning processes in a classroom on a time scale of seconds, others the development of an individual student over months (or even years) and still others the momentary functioning of entire education systems. The ‘mesh sizes’ of theories thus range from the individual student via groups, classes, and schools to entire educational systems; and time scales under consideration range from seconds to years. Nevertheless, in the end they all deal with the same fundamental issue: How can students learn mathematics (better)?

However, even for (roughly) the same type of issue and scale, several theories with possibly different outlooks may exist; take for example the role of the social aspects in learning processes at the scale of a lesson: Is the social unimportant since deep mathematics is learned mainly when individual students are thinking by themselves, is the social the very vehicle of learning, or is it something in between, part of the context of learning (see Kidron, Lenfant, Bikner-Ahsbahs, Artigue, & Dreyfus, 2008)? Such a fundamental difference is likely to express itself in terms of different theoretical notions and hence different means and ways to analyze data.

Quite a lot of work has been done and published over the past ten years by people aware of the issues raised by the existence and use of many different theoretical frameworks, and trying to ‘do something about them’. Approaches have been very diverse. A few group studies have been published, in which researchers have worked on a common set of data, each researcher illuminating these same data from a different perspective such as a recent special issue on Affect in Mathematics Education (Zan, Brown, Evans, & Hannula, 2006). While this constitutes an interesting learning experience for the researchers as well as for the readers, it does not help us make progress toward connecting between the theories. We should be more ambitious. Nobody is probably aiming at a grand unified theory (see, e.g., Grand Unified Theory, 2009) as are theoretical physicists - this may be impossible altogether in the social sciences, and even if it is possible, mathematics education certainly has not reached this stage. We cannot even expect our community to converge to a set of common basic notions because the very idea of common basic notions negates the option of a variety of analytic approaches, and such a variety is needed in order to understand the complex multi-scale phenomena we are dealing with.

But we do need to make efforts to realize to what extent we are doing similar things in different languages and to what extent we use the same language to do different things. And once we realize that, we may want to establish connections, eliminate redundancies and distinguish what can and needs to be distinguished. Even more importantly, we want to find points of contact between theories that are dealing with different but related areas and find a language to talk about such theories together, to link between them in ways that are robust in the sense that they can be used by other researchers. These issues are very complex because theoretical frameworks are culturally situated – we have long known this from the difficulties many of us have to

connect to and deeply appropriate the Theory of Didactic Situation (Brousseau, 1997) that has emanated from the French cultural background and grown in the environment of mathematics education in France. A recent issue of *ZDM - The International Journal on Mathematics Education* (Prediger, Arzarello, Bosch, & Lenfant, 2008) emanating from the CERME meeting at Larnaca offers a number of concrete case studies for how different research teams dealt with the fact that several theories were relevant for their study. There exist also examples from outside CERME, for example an attempt to coordinate argumentation theory and Realistic Mathematics Education to provide a microanalysis of a whole-class discussion (Whitenack & Knipping, 2002).

In the following two papers, two researchers experienced in consciously using, combining, comparing and contrasting several theoretical frameworks in the same study, will present different and possibly complementary approaches to such an undertaking. Angelika Bikner-Ahsbahs has taken the initiative of creating and coordinating a group of researchers who continue the work taking place at the CERME conferences also in-between conferences. She has coined the term networking theories to describe her view of how theories can be linked. John Monaghan presents a point of view formed outside of the CERME theory working group, on the basis of his research; this research has led him, for example, to refine the theory of abstraction in context, which has enabled him to take a step of integrating work on instrumentation with a dialectical, situated view of processes of abstraction; he has also recently connected the purely cognitive ideas of concept image and concept definition with a social view of learning mathematics. In his paper, he stresses the role of the person of the researcher when selecting (parts of) theories to network with; these two papers will be followed by some excerpts of the discussion that followed the presentations.

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NETWORKING OF THEORIES: WHY AND HOW?

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This contribution presents a short overview of the current discussion about a meta-theoretical standpoint of working with theories: the networking of theories as a practice of research. It explains some principles on which this kind of research practice is based. Based on a methodological frame, an example is worked out showing how the networking of theories can lead to deepening insight into a problem and to methodologically reflecting the process of connecting theories.

During the last four years a new kind of research practice has been investigated: the networking of theories (Bikner-Ahsbabs & Prediger, 2006; Prediger, Arzarello, Bosch & Lenfant, 2008; Prediger, Bikner-Ahsbabs & Arzarello, 2008). What does this mean? Networking of theories is regarded as a systematic way of linking theories (Bikner-Ahsbabs & Prediger, 2009). Linking theories is not a new idea. Within conceptual frameworks (Eisenhart, 1991) different theoretical approaches are used to build a consistent frame for research. In the case of design research, Cobb (2007) argues for connecting theories as a kind of “bricolage” in order to capitalize on different views. In addition, triangulation has developed as a kind of evaluation criterion for qualitative research (Schoenfeld, 2002; Denzin, 1989).

A lot of scholars in the community of mathematics educators have already triangulated different theoretical perspectives in their research projects to enhance insight. However, the networking of theories means more than that, it means going beyond triangulation and developing methodological tools for *systematically* connecting theories, theoretical approaches and theory use. To be a bit more precise, I will describe the networking of theories as a process of

- analyzing the same phenomenon in mathematics education from different theoretical perspectives or within different theories,
- reflecting the use of these different theories,
- respecting the identity of each theory,
- exhausting the possibilities for linking them, and
- linking them

Meanwhile some research has been executed which has led to the development of strategies, methods and techniques for the networking of theories and to some insights about the benefit that can be reached this way (Prediger et al., 2008). An interesting example is shown by Kidron (2008). Based on data she explains in detail why more than one theory is needed to understand limit concepts. She networks three theories analyzing the discrete continuous interplay of limits and shows how these

three theories - the concept of procept, the instrumentation approach, and the theory of abstraction in context - provide complementary insights and, hence, deepens understanding of limit concepts like the definition of the derivative. This way, Kidron is also able to show strengths, weaknesses and the limitations of the three theories.

On a product level, the networking of theories might lead to types of networked theories. However, since only first steps have been made in this direction, e.g. at CERME 4, 5, and 6 and elsewhere (ZDM 40 (2) for an overview), it is not yet clear, how these products might look. As Radford (2008) stated, the kinds of products will depend on the aims of networking, for instance, developing the identity of theories, experiencing the limits of linking theories, developing new methodological tools and new kinds of questions etc. One current result of this effort is a landscape of networking strategies that was worked out on the base of the contributions to the theory working group of CERME 5 (Prediger, Bikner-Ahsbabs & Arzarello, 2008).

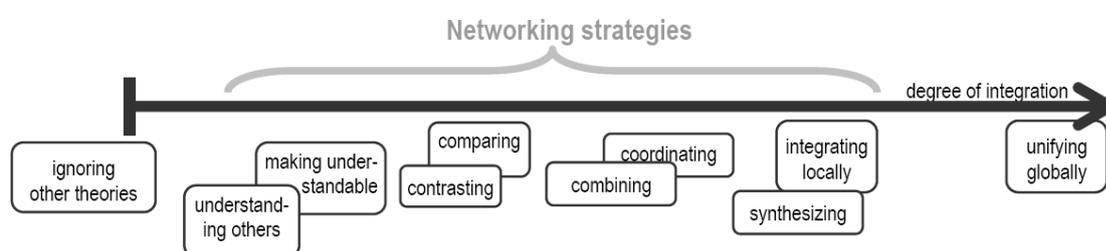


Figure 1: Networking strategies (Prediger et al., 2008)

This landscape represents a continuum of strategies for relating theories and theoretical approaches to each other including the extreme poles of non-relation between theories on the one hand and unifying them globally on the other. The term connecting theories means all kinds of building theory relations whereas networking strategies exclude the extreme poles. This landscape is ordered in complementary pairs of strategies according to their potential for integration. An example below will illuminate some of these strategies.

The idea of the networking of theories is based on some principles, the principle of

1. regarding the diversity of theories as a form of scientific richness,
2. acknowledging the specificity of theories,
3. looking for the connectivity of theories and research results,
4. developing theory and theory use to inform practice.

The first two principles acknowledge the diversity of theories in the field of mathematics education and accept diversity as a resource for scientific progress (Bikner-Ahsbabs & Prediger 2009). The third principle assumes that research in mathematics education produces much more connectivity than is visible at first sight. Related to different viewpoints, the networking of theories provides the opportunity to make these implicit aspects more explicit. The different ways of connecting

theories presented at the theory Working Group 9 at CERME 6 illustrate the value and variability of the third principle. The fourth principle does not necessarily need to be shared by all the researchers in our field; however, it helps to keep research about the networking of theories grounded in practical problems producing concepts with an empirical load that is not empty (Jungwirth, 2009).

We are all busy doing research within and about mathematics education. If research demands the use of different theories we should use them being aware that this has to be justified somehow. But why is it necessary to engage in a meta-theoretical discourse about theory use? Why do we need to reflect about linking theories?

1. WHY DO WE NEED THE NETWORKING OF THEORIES?

In order to inform practice, theories facing specific practical problems are needed. Therefore a variety of theories of middle range scope, so-called foreground theories (Mason & Waywood, 1996), have been developed, for instance different theories about abstraction (Mitchelmore & White, 2007). Furthermore, the objects of mathematics education research can be viewed from different theoretical perspectives, e.g. cognitive, semiotic, social, Thus, a variety of research perspectives and various theories have been used leading to theory development in different directions. Researchers normally know what their theory is about but often the theories' limitations remain implicit. Limitations of theories can be experienced through the failure to apply them. A systematic way to provoke these experiences is critique. It can lead to a change of view (Steinbring, 2008) but also to the development of theories in that concepts and their limitations become more precise, additional concepts are constructed or the theories' parts become interconnected more deeply. Therefore, the diversity of theories can be regarded as a resource for and a consequence of critique (see also Lerman, 2006) and is scientifically necessary.

However, the diversity of theories has also caused problems (Prediger, Bikner-Ahsbals & Arzarello, 2008), for instance a language problem and a connectivity problem. The first problem arises whenever researchers from different theoretical traditions try to talk to each other, since different theories might use the same words in different ways (e.g. social interaction in different tradition, see for example Kidron et al., 2006) or different theories use different words for the same or very similar phenomena (for example *interest-dense situation* and *a-didactic situation*, see Kidron et al., 2006). The connectivity problem is related to the question of how research results from different theoretical traditions can be connected to understand and solve practical problems.

So we need scientific ways of dealing with the diversity of theories that encounter these problems. The idea of the networking of theories might be a promising concept for this task which has the potential to induce the development of a common language among different research traditions and to investigate the ways in which theories and research results can be linked.

I will now present an example that shows how these goals can partly be achieved.

2. HOW CAN THEORIES BE NETWORKED?

In order to connect theories, a framework is needed that allows building relations among them. Radford assumes a semiosphere that comprises the collection of the semiotic parts of the different theoretical cultures within mathematics education (Radford, 2008). He explains that a semiosphere is

“an uneven multi-cultural space of meaning-making processes and understandings generated by individuals as they come to know and interact with each other.” (Radford, 2008, p. 318)

Theories within this semiosphere can be described as triplets (P, M, Q) that establish languages and allow the building of relationships between them. In these triplets, P represents the system of principles, M is a sign for a system of methodologies that can be connected to these principles in an appropriate way, and Q represents a set of paradigmatic questions related to P and M. A connection between two theories establishes a specific relation that depends on the theories' structures and the goal of this connection.

Using this frame, I will present an example of the networking of two theories illuminating the benefit of critique for developing insight into a problem. Methodological reflections will uncover five steps through which the process of networking has passed. This example refers to a data set that was used by Arzarello and Sabena (Arzarello, Bikner-Ahsbabs & Sabena, 2009). I will use it to explicitly show benefits and limits of networking practices.

An episode about the growth of the exponential function

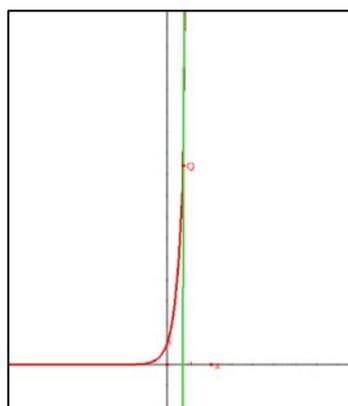


Figure 2

Two students of grade 10 are working in a pair on an exploratory activity on the exponential function and its growth. They use Cabri Géomètre to explore the graph's tangents. In this situation the teacher asks the students: What happens to the exponential function for very big x . The transcript shows the dialogue among the students G, C and the teacher.

Now I would like to invite the reader to participate in a short exercise using just a few pictures.

Figure 2 shows the computer screen the students observe.

Figure 3 presents two pairs of pictures. The left pair shows the student's gestures accompanying his utterances: his left hand goes up. The right pair illustrates the teacher's gestures accompanying his utterances: he crosses two fingers going to the right.



Figure 3: The student's gestures (left pair of pictures) and the teacher's gestures (right pair of pictures)

Please imagine for a moment what the teacher and the student are talking about. How does the student answer the question about the growth of the exponential function for very big x and how does the teacher react? – The student describes his perception of the screen meaning that the graph seems to approximate a vertical straight line. The teacher wants to show that this is wrong because every vertical straight line would be passed by the graph.

We now consider the beginning of the discussion.

- 1 G: but always for a very big this straight line (pointing at the screen), when they meet each others, there it is again...that is it approximates the, the function very well, because...
- 2 T: what straight line, sorry?
- 3 G: this ... (pointing at the screen) this, for x very, very big

With broken language the student tells something about the growth of the exponential function for big x . This broken language is an indicator for thinking aloud. Saying “sorry” the teacher interrupts the student's train of thought indicating that this question is important. However, the student does not answer the question. Instead, he defends the choice of the term “vertical straight line”. The student reacts to the so-called illocutionary level (telling something through saying something) of the teacher's question. Illocutionarily, the teacher's disruption is an indicator that there is something wrong with the vertical straight line while on the locutionary level (what is said) the teacher wants to know what vertical straight line G refers to.

During the following dialogue the student and the teacher talk about the function's growth, but, illocutionarily they negotiate about whose train of thought will be followed. The student begins to become involved repeatedly but is disrupted every time. In the end the teacher wins.

We now have a look at the last utterances.

- 14 T: eh, this is what seems to you by looking at; but you have here $x = 100$ billion, is this barrier overcome sooner or later, or not?
- 15 G: yes
- 16 T: in the moment it (the vertical straight line) is overcome, this $x 100$ billion, how many x do you have at your disposal, after 100 billion?
- 17 G: infinite

18 T: infinite... and how much can you go ahead after 100 billion?

19 G: infinite (points)

We see: The teacher is involved in arguing and the student's involvement is reduced to one (or two) word sentences (for a more detailed analysis of this episode see Arzarello, Bikner-Ahsbabs & Sabena, 2009).

A case of networking

Two theories were used to understand the episode above (for a short introduction: Arzarello et al., 2009b); a theory about the emergence of interest-dense situations and a theoretical approach about how a semiotic game between the teacher and the students shape the transition of mathematical knowledge.

The perspective of interest-dense situations

The first analysis is done from the view of the theory of the emergence of interest-dense situations. This theory – regarded as a triplet – is based on the following principles, methodology and questions:

- P1: Mathematical knowledge is socially constructed through interpretations of the others' utterances (see as well: Kidron et al., 2008).
- P2: The object of research is “meaning-making” within the process of social interaction.
- P3: In an interest-dense situation successful learning takes place as learners are deeply involved in the activity of social interactions constructing mathematical meanings in a deepening way. In these situations learning with interest is supported.
- P4: If the teacher focuses on the students' train of thought the emergence of an interest-dense situation is supported, if the teacher pushes the student to follow the teacher's train of thought the emergence of an interest-dense situation is hindered.
- M: Main part of the methodology is speech analysis on three levels. On the locutionary level an interlocutor says something; on the illocutionary level he tells something by saying something; on the perlocutionary level the intentions and the impact are taken into account.

The analysis is executed according to three questions:

- Q1: Did an interest-dense situation emerge?
- Q2: What conditions fostered or hindered it?
- Q3: How was mathematical knowledge constructed?

From the perspective of the emergence of an interest-dense situation the dialogues do not lead to increasing student involvement. Locutionarily (what is said) the student and the teacher negotiated the growth of the exponential function for very big x .

Illocutionarily (telling something through what was said) the student and the teacher struggle whose train of thought is followed. In some instances the teacher starts to focus on the student's thinking process but changes his argumentation immediately according to his own train of thought, namely to work out a "proof of contradiction": Given a vertical straight line –seen as a asymptote- this line would be passed by the graph of the exponential function. The degree of the student's involvement decreases while the teacher follows his own ideas, although the teacher tries to connect them with the student's utterances. Several times, an interest-dense situation is about to emerge, but this process is interrupted by the teacher's behaviour forcing the student to follow the teacher's train of thought. The construction of mathematical knowledge is carried out by the teacher; the contribution of the student is very low.

The semiotic bundle approach (Arzarello, 2006; Arzarello et al., 2009a)

- P1: Mathematics is transferred through a semiotic game with the help of the teacher.
- P2: The object of research is the semiotic game and its semiotic bundle.
- P3: Successful learning is interiorisation of mathematics by the help of the semiotic game.
- M: Analysis of the semiotic game according to the use of the semiotic bundle meaning the interplay of speech, gesture, representations and the transition of sign use.
- Q1: How was the mathematical content transferred through the semiotic game?
- Q2: Did the teacher tune speech and gestures with the student's ones?

From the semiotic bundle approach the semiotic game seems to be successful: The teacher takes over the student's words, using more precise explanations or following the students' ideas for a while. He points to the computer screen showing what is wrong in the way of the student's perception. He underpins his explanation and the proof of contradiction using gestures and tunes his words with those from the student. As far as the teacher is concerned, the semiotic game seems to be fruitful. From the perspective of the teacher's options to engage in the semiotic game he has done a lot of things to successfully transfer the mathematical content to the student. The student seems to be convinced, since, in the end, he correctly answers the teacher's questions.

The networking of the theories

At first glance, these results seem to be contradictory. Each theory serves as a resource for criticizing the other. After the networking process we found that the results are complementary since we could add an aspect that provided the integration of the different results: The teacher tries to tune his words with those from the student; but the gestures show that the epistemological views of the teacher and the student are different and they do not converge. The student uses his perception and

extrapolates the growth of the graph of the exponential function for very big x : the function seems to grow like a vertical straight line. The teacher's view is theoretical requiring potential infinity. Neither the teacher nor the student is able to bridge this gap.

Some methodological reflections

The contradictory results were a reason for us to meet and refresh our analysis. During this process five steps emerged:

1. *Re-analysis*: Analysing the data together again from both perspectives made our theories mutually more understandable.
2. *Comparing and contrasting*: As we contrasted and compared our theories we began to juxtapose some principles and methodologies. For example: our views on theory require different uses of the data.
3. *Establishing a common ground*: From the perspective of interest-dense-situations I could explain how the emergence of an interest-dense situation was hindered, but I could not explain why hindrance occurred. We agreed that the semiotic game was not successful as shown from the other theoretical perspective. The question was: why?
4. *Complementary analysis*: A hypothesis occurred as we looked at the semiotic game, the gestures and the speech complementarily: The student's epistemological resource was his perception of the computer screen: he extrapolated the growth of the exponential function for very big x . The teacher's epistemological resource was theoretical. This caused a gap that could not be bridged.
5. *Establishing an inclusive methodology*: We used the three levels of speech in a complementary way for the analysis of gestures and utterances and re-analysed the data carefully. Again we reconstructed the gap between the epistemological resources that could not be bridged through the semiotic game as it was executed.

Conclusions

Did we move forward? Well – yes, we did. The starting point was the contradiction of our results that served as a resource for critique and a challenge for the networking of our theoretical backgrounds. We developed a common methodology including gesture analysis and the levels of speech into one analysis. We have gained a methodological overlap but we do not know yet whether our views will converge. If we do not dig too deep we can say we followed the same question: How is mathematical knowledge gained? However, this question is still understood a bit differently because our principles and paradigmatic questions remained the same. In the end, we deepened our insights and widened our theoretical perspectives. This was possible because the grain sizes of analysis were similar and the theories' principles

were close enough to include the epistemological resource as a matter for explanation.

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PEOPLE AND THEORIES

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People are an essential consideration in networking theories. The dialectical relationship between people and theories is dynamic with regard to development. It is important to consider why people want to develop theories, their motive(s), and how they apply theories, their interpretation(s). Ascriptions of agency to theories, as ways of producing understandings or actions, need to be tempered by considerations of agency on the part of researchers applying theories. The appropriation of a theory by a person starts (and may end) with constructs from the theory.

I saw my role in this CERME plenary as that of reactor to Angelika's contribution. My first reaction is *I think what Angelika is doing is very interesting*. Indeed, "interesting" may be too weak a word as networking local theories is something that I expect to rise to prominence in mathematics education research in the near future. Angelika, Tommy, Ferdinando and I agreed at the outset that the CERME plenary should generate debate. With "debate" in mind I wanted a theme to my reaction that was honest (I did not want to generate debate by simply saying the converse of what Angelika was saying) but addressed issues that Angelika did not address and I focused on people because people network theories.

This paper follows my talk very closely and is in three parts. In the first part I argue that theories cannot be separated from the people theorising. In the second part I look at researchers' motives for adopting/creating theories and their interpretations of data. In the third part I argue that in practice researchers often appropriate parts of theories. I preface these three parts with some preliminary remarks.

PRELIMINARY REMARKS

I am not sure, in general terms, what a theory is. I am aware of discussions in mathematics education and in the social sciences of discussions of this issue. Prediger, Bikner-Ahsbahs & Arzarello (2008) consider the variety of theories in mathematics education research and conclude that 'We can distinguish theories according to the structure of their concepts and relationships' (ibid., p.168). A recent consideration of this issue in the wider social sciences is Ostrom (2005) who considers the difference between frameworks, theories and models with regard to her research interest, institutional analysis. Acknowledging that these terms 'are all used almost interchangeably by diverse social scientists' (ibid., p.27) she goes on to differentiate them according to their function in analysis: frameworks help to identify elements; theories help to specify relevant components for specific questions; models clarify assumptions regarding variables. These authors provide cogent considerations but I am still not sure what a theory is; but I know one when it is presented to me, e.g.

the Theory of Didactical Situations in Mathematics (TDS; Brousseau, 1997). So I will speak of theories with regard to the theories I am aware of that are referred to in mathematics education research.

Mathematics education researchers employ what might be called “out-of-mathematics-education theories” as well as those created within mathematics education. There are, I feel, problems for mathematics education researchers in both of these kinds of theories. The majority of us in mathematics education research are not experts in out-of-mathematics-education theories; most of us do not have a critical insight into all of their ramifications due to a lack of immersion in the academic literatures of philosophy, psychology, sociology etc. Out-of-mathematics-education theories can also miss fine mathematics detail (people interacting with mathematical relationships) that we are so very interested in. Mathematics education theories, on the other hand, can miss the big picture; the sites (classroom, workplace) of most mathematics education research are but a part of the lives of the participants.

THEORIES CANNOT BE SEPARATED FROM THE PEOPLE THEORISING

I present seven statements under this theme.

1 Theories do not exist without people

A theory without someone to interpret the theory is only words (and maybe symbols). A theory accordingly can be considered as a pair, (theory, person). For any given theory and n people there will be n such pairs. Some pairs will be almost identical, some will differ greatly; any given pair will depend on the interpretation of the theory by the person in the pair.

2 Theories develop and people develop them

(theory, person) pairs are dynamic, they change/develop. It is a bit sad if this does not happen! People develop in their understanding of a theory and through scholarships and research they develop theories. It can also be the case that a person appropriates particular development in the history of a theory, e.g. I am influenced by Davydov's (1990/72) mid 20th century use of activity theory but activity theory has developed in numerous ways since his time.

3 People hold implicit and explicit theories

I have heard it said that people can only see via a theory and that people adopt theories. I think both claims, without further explanation, are rubbish. We “see” via the artefacts (including implicit and explicit theories) available to us in our phylogenic and ontogenic development (Wartofsky, 1973). The word “adopt” is too passive. I think there is, to draw a close analogy with Guin & Trouche's (1999) ‘instrumental genesis’, a theoretical genesis in which people with initial ideas (I_I) interact with a theory (T), the person with I_I and T reviews experiences and, if T is convincing for that person, then (T', P) develops. NB This account is certainly too simple but suffices, for my purposes, as an initial hypothesis.

4 Many people subscribe to more than one theory

Theory_1 informs us on ... and Theory_2 informs us on ... With regard to person-theory pairs we do not just have (T1, P) and (T2, P) but (some combination of T1 and T2, P). Maybe this is where networking theories becomes really important.

5 A continuum with regard to theory expertise

At one extreme there are leading theorists; at the other extreme there are those who do not appear to understand a theory; and there are many intermediate positions. In France there is a maximal element in the pair (TDS, Brousseau) but I believe that it is intellectually dangerous to grant absolute authority to leading theorists.

6 Mathematics education researchers network and partially absorb others' ideas

We (mathematics education researchers) read but we also talk – to people. I was introduced to the anthropological theory of didactics (ATD; Chevallard, 1999) by talking to J-b Lagrange. I did eventually read the paper but my understanding of the theory was through my conversations with J-b Lagrange and his research.

1-7 Theories arise in communities and cultures

As academic we may aspire to objectivity but we cannot escape cultural and community influences in our work. The plenary debate took place in France and I have alluded to ATD and TDS above in homage to mathematics education theories from France. I referred to J-b Lagrange introducing me to ATD above but our relationships with this theory will be distinct simply because J-b Lagrange is a French mathematics educator and his, and not my, identity is partially shaped in relation to this French theory.

A different example is provided by Nkhoma (2002), a black South African mathematics educator. This paper comments on attempts to import learner-centred instruction from the USA into Black SA classrooms:

It is not beneficial to stereotype classrooms practices into, simply, teacher-centred therefore bad, and learner-centred therefore good ... rich experiences can be provided in practices that appear teacher-centred. (p.112)

In reading Nkhoma's paper it is difficult not to feel his anger at the importation of a "foreign" theory.

MOTIVES AND INTERPRETATIONS

I now look deeper into people and theories and examine researchers' motives for adopting/creating theories and their interpretations of data within theoretical frameworks.

To examine motives I consider a paper by Kieran & Drijvers (2006) and a response to this paper by Monaghan & Ozmantar (2007). Kieran & Drijvers worked in a form of ATD with they call "task-technique-theory" (TTT). It is a long and interesting paper

on the interplay between computer algebra systems (CAS) techniques and by-hand techniques. The students were working on factorisations of x^n-1 , and the CAS required specific values for n and did not give the classic factorisation every time. They state:

According to the TTT ... a student's mathematical theorizing is deemed to be intertwined with the techniques ... tasks ... we distinguish the following three theoretical elements.

1. Patterns in the factors of $x^n - 1$: Seeing a general form and expressing it symbolically
2. Complete factorization: Developing awareness of the role played by the exponent in $x^n - 1$...
3. Proving: Theorizing more deeply on the factorization of $x^n - 1$ (pp.242-243)

We viewed this with regard to HersHKowitz, Schwarz & Dreyfus's (2001) abstraction in context (AiC) recognising and building-with actions. Students' prior work had involved factorising binomial expressions with regard to the difference of squares and sums and differences of cubes. They *recognise* that expressions of the form x^5-1 can be factored and *build-with* this knowledge artefact to produce factorisations $x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$.

We also viewed it via Davydov's ascent to the concrete (an inspiration for AiC) whereby an abstraction progresses from an initial entity to a consistent final form. This progression depends on the

disclosure of *contradictions* between the aspects of a relationship that is established in an initial abstraction ... It is of theoretical importance to find and designate these contradictions. (p.291)

One student says, with regard to $x^{135} - 1$, 'how are we supposed to know if it's valid or not?' – the initial abstraction is fragile and limited to specific whole number exponents. The teacher introduced $x^n - 1$ and this required a *vertical* reorganisation of their knowledge. Student work shortly after includes *recognising* and *building-with* but on a higher vertical level than when the exponents were specific whole numbers.

Later in the Kieran & Drijvers paper we see students grappling with contradictions created from attempts to reconcile paper-and-pencil and CAS techniques and how this attempt at reconciliation led to synthesis and further insights.

The difference-of-squares 'proof', for example, with its accompanying treatment of the case $x^{n/2} + 1$, for odd values of $n/2$, helped to extend even further the thinking of students in the class. The $x^n - 1$ conjecture, which had issued from the earlier work of some students with the factoring of $x^{10} - 1$, helped others to integrate their ideas about odd, even, and prime exponents - theoretical ideas that had been generated in interaction with various CAS and paper-and-pencil techniques ... (p.253)

We also viewed Noss & Hoyles' (1996) situated abstraction and webbing as closely related to Davydov's ascent to the concrete

learning as the construction of a *web* of connections – between classes of problems, mathematical objects and relationships, 'real' entities and personal-specific experiences. (p. 105)

We are certainly networking theories (though in a different sense to how Angelika networks theories) but we also attend to differences arising from our ascribed personal motives of the theorist to theorise. Minimal ascriptions of motive are:

- Kieran & Drijvers – to understand the interplay of machine and by-hand techniques;
- Davydov – to develop theory to aid instructional design;
- Noss & Hoyles – to account for mathematical meaning making and the structuring of mathematical activities;
- Hershkowitz et al. - dissatisfaction with empirical theories of abstraction re students' actual development.

I now briefly consider interpretation. Angelika talks about a case where two theories are successfully networked. In CERME 6 Working Group 9 "Different theoretical perspectives and approaches in research: Strategies and difficulties when connecting theories" some papers focused on difficulties in networking theories. This is not new, six years ago Even & Schwarz (2003, p.283) commented 'We exemplify how analyses of a lesson by using two different theoretical perspectives lead to different interpretations ...' I have no problem with this but question whether different interpretations are only the result of different theoretical perspectives. Research is often a team effort. Have you ever disagreed with a colleague during data analysis? I have and the outcome is usually compromise or an impasse. I think this tends not to get reported in papers. This is a further refinement to my point 1 'any given pair will depend on the interpretation of the theory by the person in the pair' but with regard to the interpretation of data via the theoretical perspective. Is at least one interpretation wrong?

ISSUES, CONSTRUCTS AND CONSISTENCY

In this final section I consider the extent to which theories lead research, theories and constructs and consistency issues. This section expands on my "motive" considerations above in that people often turn to/develop theories in order to address issues that they regard as important. But often it is parts of theories that they appropriate and this can lead to potential consistency problems.

Radford (2008, p.320) states that 'a theory can be seen as a way of producing understandings and ways of actions based on' a system of basic principles, a methodology and a set of paradigmatic research questions. I take partial issue with this, as a generality. I consider issues and specific research projects.

In my experience there are fundamental issues that mathematics education researchers return many times over their working lives. In my case one of these is the link between school mathematics and out-of-school mathematics. I have grappled with this over many decades. Research questions, methodologies and principles have come and gone but the issue remains. The construct “transfer” often arises in discussions of this issue for something akin to transfer is central in linking school to out-of-school maths. Personally I hate the term and largely agree with the old Lave (1988) critique but the issue haunts me and I am prepared to consider any theory that will further my understanding of this issue.

I now consider research projects with regard to theories. These generally have a shorter time scale than “issues”. I, like most CERME delegates, write formal proposals with a theoretical framework, research questions and methodology. Almost every time, however, I develop in the process. I encounter unexpected phenomena (and revise the research questions) or experience problems in data analysis (and revise the methodology) or develop the theoretical framework. My point is that sometimes theories lead research, sometimes they do not; and, whatever the case, researcher development is in the dialectic mix. I now consider constructs.

A construct may be regarded as a proper part of a theory, e.g. *didactical contract* in TDS. I think people often appropriate a construct of a theory without appropriating the whole theory. I further think that if a person appropriates a theory, then they appropriate constructs of that theory prior to appropriating the theory. I included ‘I think’ in the previous two sentences because I base these remarks on my reflections of my own development; with regard to my point on “theoretical genesis” in (3) above I am not aware of research that traces the genesis of theory acquisition amongst academics but such research would be relevant to my reflections.

As an instance of construct appropriation in my own development I return to my comments above that J-b Lagrange introduced me to ATD. This is true, he did introduce me to ATD, but what I initially appropriated was the ‘task-technique’ part of ATD (and this focus as the only part of ATD I made sense of lasted several years). This focus was, I am sure, due to my prior experience. I had long experience of working with students and with teachers on using ICT-mathematics tools and the term “technique” in my country’s everyday mathematics-education-speech refers to value-free manipulation. To view, as Lagrange’s exposition of ATD does, techniques as not only being not value-free but techniques having both epistemic and pragmatic values and being viewed with respect to tasks was, quite frankly, a huge revelation and very relevant to my ICT work. Monaghan (2000) provides published evidence of this narrow focus. Perhaps it was due to the big impact this construct had on my thinking that appropriating other aspects of ATD took me a longer period of time.

I do not think the above (ATD, me) is an isolated example. I think the theory-person development is similar to that which I outline in (3) above: a person with a theoretical approach (T, P) interact with a construct C, the person with theoretical approach and

construct reviews experiences and, if C is convincing for that person, then (T+C, P) develops. As with my comments in (3) this is almost certainly simplistic.

As I prepared for the CERME plenary I kept returning to T and C, in (T+C, P), with the thought that T and C must, in some sense, be consistent. I tried to formulate consistency criteria but failed, my attempts to frame consistency criteria ended with grand but empty phrases. This failure may be a personal failure but it may be that there is not a suitable meta-language in which to couch consistency criteria for non-specific theories and if this is the case, then perhaps we just need to resolve consistency tensions in our own research in case and theory specific ways.

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DISCUSSION

The discussion was opened and guided by the following questions:

- How can we link our theories to the mathematical background? Is this necessary?
- Why should we care about theories as an object of research? Working with theories and constructing theories within mathematics education is our job. It is not our job to investigate the epistemological processes within ME themselves. We have enough problems to work on if we restrict ourselves to the teaching and learning of mathematics.
- What does consistency mean? Taking bits and pieces from different theories includes the danger to merge inconsistent parts. Does consistency depend on the grain size? How can we link different grain sizes?

The following comments, relating to the above questions, were made by members of the audience and the presenters:

Concerning the link to mathematics, on the one hand, the experience of participants in the *Advanced Mathematical Thinking* group of CERME is that linking the theories to mathematical content domains is crucial, especially crucial when trying to network with mathematicians. On the other hand, most mathematicians hesitate to go into didactical theories, and those who do sometimes point out difficulties of communication (Quinn, 2008).

Concerning the importance of reflecting at the meta-level how we work with more than one theory, other working groups than those mentioned above also reported that they were coming up against this issue; specifically, the working groups on *Affect*, on *Mathematics and Language*, on *Early Years Mathematics*, and on *Comparative Studies* were mentioned. In addition, two comments were made, namely that networking will never end since theories are dynamic entities, and that the important aspect of John Monaghan's presentation is the human one, never mind whether individual or social. However, the characteristic trait of this part of the discussion was that contributors tended to ask questions rather than make comments; these questions included:

- Isn't our research necessarily linked to what happens outside of the discipline since the research needs are defined by politicians, funding agencies, and teachers?
- How do we deal with theories that we adopt from other disciplines such as psychology, epistemology, or even medical science? In particular, how do we integrate mathematics (or at least a mathematical view) into these theories? How do we integrate theories from other disciplines into the area of mathematics education?

- To what extent do home-grown theories integrate, adopt or adapt elements from general (outside) theories?
- How would one distinguish local from global combining of theories?
- Shouldn't networking efforts also include cases where the researchers attempting to connect do not start from a specific phenomenon?
- Why not using the useful (but relative) distinction between background and foreground theories?
- Would it promote networking to start by comparing metaphors?

Concerning the issue of consistency, participants commented that criteria for consistency might better be found outside our community, that it might be preferable to use the term 'compatibility' rather than 'consistency', that the distinction between 'theories of' and 'theories for' could be useful, that looking at complementary phenomena could be a starting point for networking theories – a point already made by Steiner (1985), and that the main reason for connecting theories might be the complementarity of their aims, which is important by itself and might make their convergence rather less important.

While it is far from clear whether our community has already made substantial progress in its attempts to find ways of working with different theories in mathematics education research, this plenary session has made it amply clear that the issue of how to work with different theories is deep, that it occupies a central position for a large number of researchers and plays an important role in the discussions of a majority of the CERME working groups. It is therefore recommended that CERME continue its support of efforts to make progress on this issue and to discuss scientific ways of dealing with the diversity of theories in a manner that is comprehensive and includes researchers from different areas and backgrounds within mathematics education.

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CERME 6 – WORKING GROUP 1

Multimethod approaches to the multidimensional affect in mathematics education

TABLE OF CONTENTS

Introduction.....	28
<i>Markku S. Hannula, Marilena Pantziara, Kjersti Wæge, Wolfgang Schlöglmann</i>	
The effect of achievement, gender and classroom context on upper secondary students' mathematical beliefs	34
<i>Markku S. Hannula</i>	
Changing beliefs as changing perspective	44
<i>Peter Liljedahl</i>	
“Maths and me”: software analysis of narrative data about attitude towards math	54
<i>Pietro Di Martino</i>	
Students’ beliefs about the use of representations in the learning of fractions.....	64
<i>Athanasios Gagatsis, Areti Panaoura, Eleni Deliyianni, Iliada Elia</i>	
Efficacy beliefs and ability to solve volume measurement tasks in different representations	74
<i>Paraskevi Sophocleous, Athanasios Gagatsis</i>	
Students’ motivation for learning mathematics in terms of needs and goals.....	84
<i>Kjersti Wæge</i>	
Mathematical modeling, self-representation and self-regulation.....	94
<i>Areti Panaoura, Andreas Demetriou, Athanasios Gagatsis</i>	
Endorsing motivation: identification of instructional practices	104
<i>Marilena Pantziara, George Philippou</i>	
The effects of changes in the perceived classroom social culture on motivation in mathematics across transitions	114
<i>Chryso Athanasiou, George N. Philippou</i>	
“After I do more exercise, I won't feel scared anymore” Examples of personal meaning from Hong-Kong	124
<i>Maike Vollstedt</i>	

Emotional knowledge of mathematics teachers – retrospective perspectives of two case studies	134
<i>Ilana Lavy, Atara Shriki</i>	
Humour as a means to make mathematics enjoyable	144
<i>Pavel Shmakov, Markku S. Hannula</i>	
Beliefs: a theoretically unnecessary construct?	154
<i>Magnus Österholm</i>	
Categories of affect – some remarks	164
<i>Wolfgang Schlöglmann</i>	

INTRODUCTION

MULTIMETHOD APPROACHES TO THE MULTIDIMENSIONAL AFFECT IN MATHEMATICS EDUCATION

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The first working group on affect was organized in CERME 3 in 2003. This was the fourth affect working group and like the previous three, it was an energizing and inspiring event. We had 18 participants and 17 papers were submitted to our working group. One of the papers was cancelled, and the peer review process led to rejection of one paper before the conference. Several papers were revised and all except one of these were accepted for publication in the proceedings, leading to 14 published papers.

Early in the conference, Di Martino reminded us of why this field of study is important. He made reference to several mathematics education researchers who have emphasized the role of affect in our efforts to understand human behaviour in mathematical thinking and learning. One of the quotes he shared with us was the following:

“...researchers who are interested in human performance need to go beyond the purely cognitive if their theories and investigations are to be important for problem solving in classrooms” McLeod (1992).

Numerous research studies carried out more recently in mathematics education emphasize in similar fashion the importance, hence relevance of affective factors in interpreting students mathematics performance, behaviour and difficulties in mathematics (e.g. Philippou & Christou, 2002, Young, 1997). In the papers accepted for the proceedings you will find 14 interesting perspectives into the complex world of affect, emotions, motivation and humour in mathematical thinking and learning.

The participants in this Working Group considered it important to report also the way of organized our sessions. The dilemma is to focus discussion in a way that it relates to the papers that each participant is familiar with, but so that it also is able to go beyond presentation of papers. First of all, we were fortunate to have a more or less optimal group size that allowed rich discussions where each participant was able to contribute. The authors of the accepted papers were asked to prepare in advance one or two slides based on their paper on each of the following topics:

- Theoretical framework

- Methodology
- Key findings
- Implications for teaching
- Implications for further research

Slides were collected and organised according to themes at the beginning of the conference. In the sessions each slide was briefly presented by the respective author, which (usually) initiated a discussion. When the momentum of the discussion was used out, the next presenter took the stage.

This way of organizing allowed each participant to have his or her main ideas in the focus of attention. Moreover, this allowed discussion to focus on topics and supported referring to ideas from previous presentations.

THEORETICAL FRAMEWORKS

The group had very intensive discussion on the topic of theoretical frameworks. A helpful framework to structure discussion was the figure from CERME5 summary presentation (Figure 1).

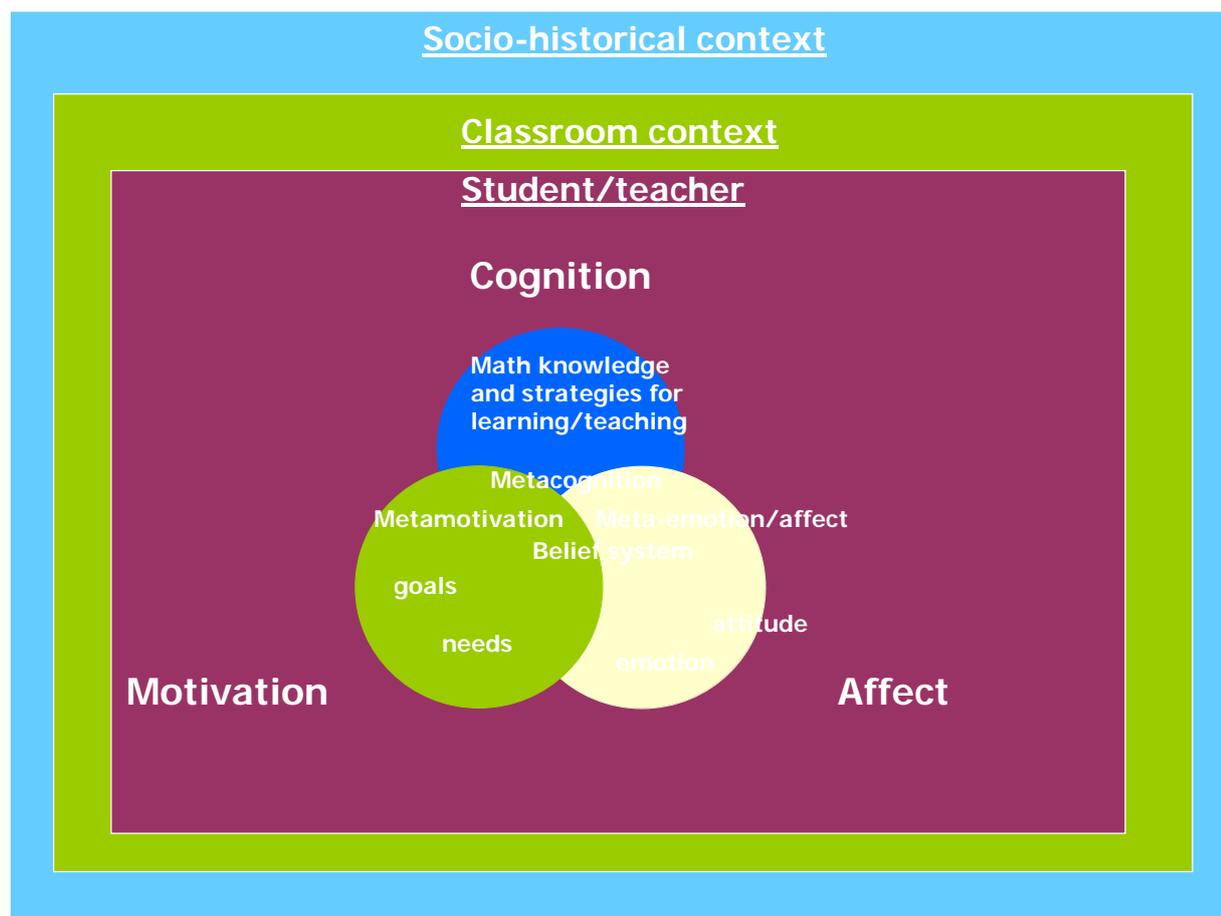


Figure 1. An overall framework for affective constructs within mathematics education research (Hannula, Op 't Eynde, Schlöglmann & Wedege, 2007, p. 204)

The proposed model is based on the socio-constructivist perspective on learning and it is characterized both by its focus on the situatedness of learning (classroom and socio-historical context) and by the recognition of the close interactions between (meta)cognitive, motivational and affective factors in students' learning (Op 't Eynde et al., 2006).

One of the issues that has been discussed in previous CERME-meetings and that was revisited again was the definition of beliefs (Di Martino; Liljedahl; Osterholm). This is an issue, where Furinghetti & Pehkonen (2002) concluded that there can not be a single definition for beliefs that is appropriate for all purposes.

We revisited the characterization by McLeod (1992), where affective domain is divided into emotions, attitudes and beliefs. There was an agreement that beliefs are different from the other concepts in that it is possible to consider their truth value, whereas emotions and attitudes are subjective by their nature. The paper by Österholm led us to discuss the distinction between beliefs and knowledge. Our preliminary conclusion at the conference was that the difference lies in knowledge being determined socially and beliefs being the individual aspect of knowledge (cf. Furinghetti & Pehkonen, 2002).

Self-efficacy issues were also presented in the group (Sofokleous and Gagatsis). We discussed Bandura's framework of self-efficacy, which has not been integrated into belief systems framework. Instead, self-efficacy beliefs seem to have remained a relatively independent framework with some connections to both belief theories and motivation theories.

Epistemological beliefs of mathematics was another framework of interest (Liljedahl). The differentiation between system, toolbox and process view of mathematics has long history from Dionne (1984); Ernest (1991); and Törner and Grigutsch (1994). Moreover, there was lively discussions about the generation of mathematical beliefs (Hannula).

Another concept which we discussed thoroughly was motivation. We recognized that motivation has two dimensions that require attention, namely the quality and the intensity of motivation. The different approaches used in the conference papers (Athanasidou, Pantziara, Wæge) include theory, personal Investment theory, Achievement goal theory and Self Determination Theory of needs and goals. Regarding the generation of motivation, needs, competence based variables, social, demographic and neurophysiological predispositions were recognized (Schlöglmann).

As new theoretical approaches to affect we were introduced to the concepts *personal meaning* (Vollstedt), *humor* (Shmakov & Hannula) and teachers' *emotional knowledge* (Lavy & Shiriki). In the discussion it was argued that it might be more appropriate to call the last of these *emotional skills*. It was reminded that one issue in earlier CERME affect groups had been the need to develop a more coherent language

and/or conceptual system for research on affect. Therefore the group concluded that these new concepts must be related to the existing ones in the domain.

RESEARCH QUESTIONS

The variety of the research questions presented in our group made the use of various research methods (qualitative and quantitative) necessary.

In particular three main themes of research questions were presented, with the first one referring to beliefs:

- ▶ The origin of the beliefs. Are all beliefs constructed in the same way or are some beliefs socially constructed while some others are mainly individual? (Hannula)
- ▶ Changing beliefs as changing perspective. (Liljedahl)
- ▶ “Maths and me”: software analysis of narrative data about attitude towards math. (Di Martino)
- ▶ Students’ beliefs about the use of representations in the learning of fractions. (Gagatsis, Panaoura, Deliyanni & Elia)
- ▶ The relation between self-efficacy beliefs and students’ achievement. (Sophocleous & Gagatsis)

The second theme referred to motivation aspects:

- ▶ Students’ motivation for learning mathematics in terms of needs and goals. (Wæge)
- ▶ Identification of students’ inner characteristics that may develop students’ motivation. (Panaoura, Demetriou & Gagatsis)
- ▶ Social variables (teachers’ practices) that may develop students’ motivation. (Pantziara & Philippou)
- ▶ The effects of changes in the perceived classroom social culture on motivation in mathematics across transitions. (Athanasiou & Philippou)

A third theme covered the new approaches to affect:

- ▶ The kind of personal meaning that students relate with mathematics education. Comparison between German and Hong Kong. (Vollstedt)
- ▶ Emotional knowledge of mathematics teachers. (Lavy & Shiriki)
- ▶ Humour as a means to make mathematics enjoyable. (Shmakov & Hannula)

The discussion on research methods showed several studies to have advanced beyond simple correlation and descriptive studies (Pantziara & Philippou). Some use a systemic approach and study several different aspects in connection with each other (e.g. Hannula; Panaoura et al.). There are also studies that use methods that allow examining changes in beliefs and motivation (Athanasiou and Philippou).

DISCUSSION AND CHALLENGES

One apparent main focus for research and practice in this domain has been to develop richer theoretical frameworks using aspects and develop better concepts and instruments, preferably combining qualitative and quantitative methods. The frameworks should recognize the close relation between beliefs, motivation and competence. Another, related focus has been the relations between different constructs in the affective domain and their connection to other areas in the realm of mathematics education. A third focus identified was change in beliefs and motivation; how it can happen and how to initiate change.

One specific issue we discussed was the different understandings of the stability of affective constructs. The first aspect here is to distinguish between affective state and affective trait. The second aspect to notice is affects resistance to change. The third aspect of stability is the robustness of affective constructs. The fourth aspect is the relative stability of affect, which means the tendency of people to keep the same order even if their affect might be changed.

When looking into the future, we recognized some promising approaches. In mathematics education affect has typically been approached through psychology. Looking at affect as biological or social phenomenon might open up new insight.

With regard to research on emotions, there is need to move beyond simplistic positive/negative view of emotions and distinguish different types of negative emotions (fear, dislike, sadness, anger) and positive emotions (joy, serenity). We also realized that most research on affective processes has focused on intensive emotions or non-routine mathematical activities. Therefore, it might be interesting to explore students' affect when they experience routine mathematics. Moreover, the research on affect could be extended to various contexts in mathematics, such as vocational education and mathematics at work.

CLOSING REMARKS

In each CERME the effort is denoted to identify some emerging or significant themes that might reflect the field in general, not restricted to the studies presented in the conference. The enrichment of the theoretical framework by clarifying specific constructs related to affect and by introducing new approaches has continued. Besides the illumination of relations among the various affective constructs (e.g. students' and teachers' beliefs, students' achievement goals, students' motivation) and other variables in the mathematics education domain (e.g. students' competence, teachers' practices, and teachers' knowledge) had been proceeded. The clarification of the terminology used in affect together with the new perspectives of stability of affective constructs develop this research domain. Due to the multidimensional face of the variables involved in the affective domain, the multi-method approach is becoming indispensable in the identification of relations among this area of research.

There is still much to be clarified and revealed in the realm of Affect in Mathematics Education. Therefore we go on and look ahead to the next affect working group at CERME 7.

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THE EFFECT OF ACHIEVEMENT, GENDER AND CLASSROOM CONTEXT ON UPPER SECONDARY STUDENTS' MATHEMATICAL BELIEFS

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The influence of achievement, gender and classroom context on students' mathematical beliefs were analysed from survey data from 1436 Finnish upper secondary school students. The results indicate that students of the same class tend to have similar effort, enjoyment of mathematics and evaluation of teacher. Students' mathematical confidence is influenced by gender while their perception of their competence mainly relates to their achievement in mathematics.

Keywords: beliefs, gender, secondary school, multilevel analysis

INTRODUCTION

Mathematical beliefs are on the one hand considered as individual constructs that are generated by individual experiences. On the other hand, beliefs are considered to be constructed socially, in a shared social context of a classroom. Which is more important? Are all beliefs constructed in the same way or are some beliefs socially constructed while some others are purely individual?

In Finnish research on affect in mathematics education the focus has clearly been on the level of human psychology, and only a few studies have explored also the social level (Hannula, 2007). One reason for this is most likely that differences between schools and geographic regions are low and the social variables have generally less pronounced effect on achievement in mathematics in Finland than in most other countries (OECD-PISA, 2004). Finland is also culturally rather homogeneous. Hence, it is not surprising that comparative studies between different groups of students within Finland have not been popular, gender being an exception to the rule. One study on regional effects indicated that students in capital province choose advanced syllabus more often than students in another province (Nevanlinna, 1998). This indicates that geographical differences in mathematics related beliefs may exist.

A general international trend has been that gender differences in mathematics achievement are disappearing. Gender differences in overall achievement of 15-year olds have disappeared also in Finland, but robust gender differences still exist in their affect towards mathematics (Hannula, Juuti & Ahtee, 2007). When attitude towards mathematics has been constructed as a single variable, studies generally have found boys to hold a more positive attitude towards mathematics (e.g. Saranen 1992). However, when different dimensions of attitude have been separated, interesting variations have been found. For example, all studies have not found gender differences in 'liking of mathematics' (Kangasniemi, 1989). Gender difference has

been clearer in how difficult mathematics is seen (Kangasniemi, 1989) and quite robust in students' self-confidence in mathematics (Hannula & Malmivuori, 1997; Kangasniemi, 1989; Hannula, Maijala, Pehkonen & Nurmi, 2005). Class-level factors are seen to influence students' self-confidence, and these seem to be more relevant to girls' than to boys' self-confidence (Hannula & Malmivuori 1997).

Although Finland scored to the top in PISA achievement scores, Finland was also characterised by less favourable results on the affective measures. Finnish students lack interest and enjoyment in mathematics, they have below average self-efficacy, and low level of control strategies. As a more positive result, levels of anxiety were also low. In Finland affect was an important predictor of achievement. Mathematical self-concept was the strongest predictor of mathematics performance, and this correlation was strongest among countries in the study. The study also revealed that gender differences favouring males in affect were larger in Finland than in OECD on average. (OECD-PISA, 2004)

In a study of elementary and secondary teachers' beliefs Pekka Kupari identified two types of mathematics teachers, traditional and innovative teachers. The traditional teacher emphasises basic teaching techniques and extensive drill, while the innovative teacher emphasises student thinking and deeper learning. (Kupari, 1996)

Moreover, Riitta Soro (2002) found out in her study that most mathematics teachers held different beliefs about students based on student's gender. Girls were seen to employ inferior cognitive skills and succeed because of their diligence, while boys were seen to be talented in mathematics but lacking in effort. However, there were also teachers who did not hold such gendered beliefs.

As there are quite different teachers, one would expect this to have an effect on beliefs of their students. If this is the case, then we are likely to find significant amount of variation of students' beliefs to be attributable to the class they study in. Moreover, this variation might be different for male and female students.

In this report we shall explore more deeply which aspects of mathematical beliefs are most affected by shared classroom context or gender, and which seem to be individual constructs, for which gender and class are poor predictors of the belief.

THEORETICAL FRAMEWORK

In the literature, beliefs have been described as a messy construct (Pajares, 1992). There are many variations for characterisations of belief concept (Furinghetti & Pehkonen, 2002). In this article we consider mathematical beliefs as "an individual's understandings and feelings that shape the ways that the individual conceptualizes and engages in mathematical behavior" (Schoenfeld 1992, 358). Op 't Eynde, De Corte and Verschaffel (2002) provide a framework of students' mathematics-related beliefs. Constitutive dimensions are object (mathematics education), self, and context (class), which further lead to several sub-categories:

- 1) Mathematics education (mathematics as subject, mathematical learning and problem solving, mathematics teaching in general),
- 2) Self (self-efficacy, control, task-value, goal-orientation), and
- 3) The social context (social and socio-mathematical norms in the class.). With regard to the social context, Op 't Eynde & DeCorte (2004) found out later that the role and functioning of one's teacher are an important subcategory of it.

In an earlier study (Rösken, Hannula, Pehkonen, Kaasila and Laine, 2007), we have explored the structure of mathematical beliefs among upper secondary school students. Our studies confirmed partially the aspects of mathematical beliefs that Op 't Eynde and his colleagues had suggested.

It is generally assumed that there is a link between teachers' and their students' affect towards mathematics (e.g. Cockroft, 1982). However, few studies seem to confirm this relationship. For example, the review of PME research on affect (Leder & Forgasz, 2006) does not mention any such study. As an example of research relating teacher and student beliefs we can take Crater and Norwood's (1997) study of seven teachers and their 138 students, where they found out that this group of teachers' beliefs influences their practices and what their students believed about mathematics

These different findings can be summarised on a model where there the three levels of gender, classroom context and individual are differentiated in the process of belief development (Figure 1).

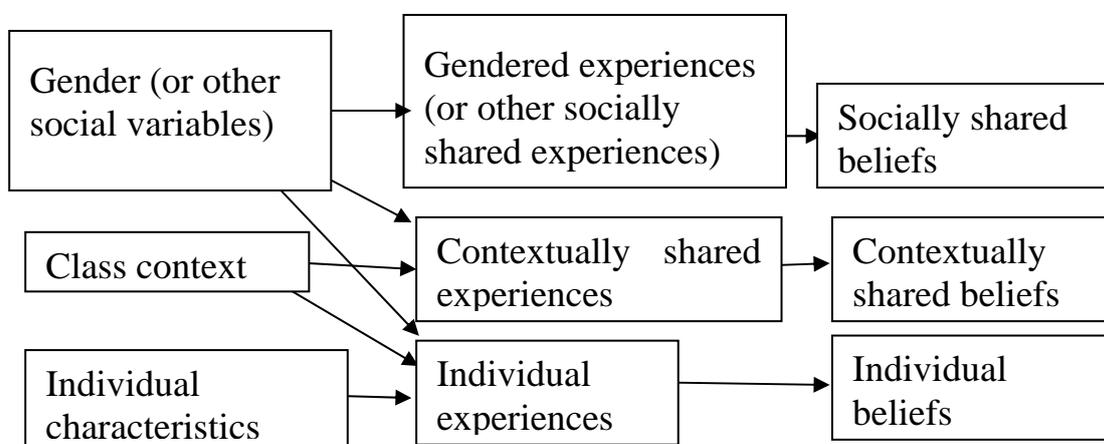


Figure 1. A model for generation of mathematical beliefs.

One origin of different student beliefs are the individual life histories that each student brings into the classroom. These life histories influence the way the students position themselves in the classroom, the way they engage with mathematics, teacher and peers and the way they interpret their experiences in the classroom. On the other hand, there are contextual factors that students of the same class share with each other. These are, for example, the personality of the teacher, the physical classroom and the implemented curriculum. These influence all students in a class and are the origin of shared experiences. Moreover, also students' individual experiences are

partly shaped by the shared events in the classroom. This is illustrated with an arrow from classroom context to individual experiences.

On the most general level there are experiences that people of the same social background (e.g. ethnicity, social class, hobbies, and social subcultures) share. One of such subsets is generated by students' gender. Gender is seen to play a significant part in the experiences in the classroom and in the beliefs that students develop (e.g. Hannula et. al, 2008). Also most teachers' have different beliefs about boys and girls as mathematics learners (Soro, 2002). Therefore it is reasonable to make the claim that individual experiences in mathematics classrooms are not the same for male and female students. Moreover, as teachers and classes are different, these gendered experiences may vary from one class context to another. Therefore, there are arrows from gender to both contextual and individual experiences.

METHODS

Instrument and Participants

The view of mathematics indicator has been developed in 2003 as part of the research project "Elementary teachers' mathematics" financed by the Academy of Finland (project #8201695). It has been applied to and tested on a sample of student teachers and was slightly modified for the present sample. That is, items addressing specifically aspects of teaching mathematics like View of oneself as mathematics teacher (D1-D6) and Experiences as teacher of mathematics (E1-E7) were removed. More information about the development of the instrument can be found e.g. in (Hannula Kaasila, Laine & Pehkonen, 2006).

The participants in our study came from fifty randomly chosen Finnish-speaking upper secondary schools from overall Finland, including classes for both, advanced and general mathematics. The respondents were in their second year course for mathematics in grade 11. Altogether 1436 students from 65 classes (26 general and 39 advanced) filled in the questionnaire and gave it back. The response rate was higher among advanced mathematics courses.

Through an exploratory factor analysis we obtained a seven-factor solution that counts for 59 % of variance and provides factors with excellent internal consistency reliability (Table 1). We related three factors to personal beliefs since a clear self-relation aspect regarding competence (F1), effort (F2) and confidence (F7) can be found. Two factors were related primarily to social context variables, namely teacher quality (F3) and family encouragement (F4), one to more emotional expressions concerning enjoyment of mathematics (F5) and one to mathematics as a subject; that is, difficulty of mathematics (F6). A description of factor analysis as well as all components and their loadings can be found in another report. (Rösken et. al, 2007)

A GLM univariate analysis was performed on SPSS. The seven belief factors were the dependent variables, gender was a fixed factor, and class a random factor.

Mathematics grade was a covariant. Students of advanced and general mathematics courses were analysed separately, and partial η^2 is used as a measure of effect size. It should be noted that although partial η^2 is a reliable estimate within a sample, it does not provide reliable estimate for the whole population. Because all variables did not confirm with the assumptions of normality, we made also a nonparametric Kruskal Wallis test to test the statistical significance of the grouping effect.

Name of the component	Sample item	Number of items	Cronbach's alpha
Competence	Math is hard for me	5	0.91
Effort	I am hard-working by nature	6	0.83
Teacher Quality	I would have needed a better teacher	8	0.81
Family Encouragement	My family has encouraged me to study mathematics	3	0.80
Enjoyment of Mathematics	Doing exercises has been pleasant	7	0.91
Difficulty of Mathematics	Mathematics is difficult	3	0.82
Confidence	I can get good grades in math	5	0.87

Table 1. The 7 principal components of students' view of mathematics.

RESULTS

The GLM univariate analysis indicated several statistically significant effects (Table 2 and Table 3). However, the assumption of equal variance did not hold true in all cases and nonparametric tests were necessary to confirm results (see below).

	General mathematics											
	Grade			Gender			Group			Gender x Group		
	F	Sig.	η^2	F	Sig.	η^2	F	Sig.	η^2	F	Sig.	η^2
Competence*	326,16	,000	,35	,12	,729	,00	1,58	,111	,61	,97	,507	,04
Effort	172,22	,000	,27	3,10	,087	,09	2,03	,041	,67	1,15	,278	,06
Teacher Quality	41,86	,000	,08	10,37	,003	,22	2,95	,004	,75	,92	,577	,05
Family Encouragement	,75	,388	,00	2,20	,147	,06	1,05	,456	,51	1,08	,359	,06
Enjoyment of Mathematics	196,65	,000	,30	2,94	,096	,08	1,65	,107	,62	1,00	,470	,05
Difficulty of Math*	194,80	,000	,30	4,73	,036	,12	1,90	,057	,65	,94	,550	,05
Confidence*	86,40	,000	,16	23,29	,000	,41	1,06	,444	,51	1,02	,433	,05

Table 2. GLM univariate analysis for general mathematics students (gender*group, grade as covariate). η^2 is partial η^2 . *) variance in groups was not equal (Levene's Test of Equality of Error Variance)

Most of the mathematical beliefs were related to the mathematics grade the student had. A simple correlation was calculated to determine the direction of the correlation

(correlation table is not reprinted here). All correlations were positive, except of correlation between grade and perceived difficulty of mathematics.

Regarding gender differences, the GLM Univariate analysis indicated that for both advanced and general syllabus female students were less confident and they perceived teacher quality lower and mathematics more difficult than male students. The effect was strongest in self-confidence.

The analysis indicated a strong group effect for teacher quality. In groups of general mathematics there was also a strong group effect on effort and in groups of advanced mathematics a strong group effect on enjoyment. Moreover, there was a gender and group interaction effect for enjoyment among advanced mathematics courses, indicating stronger group effect for female students.

	Advanced mathematics											
	Grade			Gender			Group			Gender x Group		
	F	Sig.	η^2	F	Sig.	η^2	F	Sig.	η^2	F	Sig.	η^2
Competence*	332,61	,000	,30	1,09	,301	,02	1,63	,077	,63	1,08	,355	,05
Effort*	254,72	,000	,25	,13	,717	,00	1,02	,479	,51	1,13	,278	,05
Teacher Quality*	53,34	,000	,07	5,83	,019	,10	7,26	,000	,88	1,14	,274	,05
Family Encouragement	1,20	,274	,00	,34	,561	,01	1,50	,116	,61	,73	,877	,03
Enjoyment of Mathematics	175,78	,000	,18	,30	,591	,01	2,41	,005	,71	1,49	,036	,06
Difficulty of Mathematics	254,08	,000	,24	34,27	,000	,40	1,67	,066	,63	1,24	,160	,05
Confidence	115,86	,000	,13	75,07	,000	,60	1,29	,228	,57	1,28	,132	,05

Table 3. GLM univariate analysis for advanced mathematics students (gender*group, grade as covariate). η^2 is partial η^2 . *) variance in groups was not equal (Levene's Test of Equality of Error Variance)

Because all variables did not confirm with the assumptions of normality, we made separate analysis to confirm some of the disputable results above (Table 4). Unfortunately this analysis did not allow a simple means to control for effect of achievement. The results confirmed the group effects partially. For students of general mathematics the statistically significant group effects were different for male and female students. For male students, groups had an effect on competence and effort, whereas for female students the group effect was found on teacher quality and confidence. This confirms the group effect on effort for male students and teacher quality for female students. The observed group effects on competence and confidence may actually be effects of grade.

For advanced mathematics a statistically significant group effect was found for teacher quality, effort, and enjoyment. This confirms the results of GLM Univariate analysis. Moreover, for female students only, a group effect on confidence was found.

Kruskal Wallis Test Statistics for group differences								
Course, Gender		Compe-	Effort	TQ	FE	Enjoy	Difficulty	Confidence
General, male	χ^2	36,39	46,10	27,053	21,96	25,16	26,56	20,38
	df	25	25	25	25	25	25	25
	Asymp. Sig.	,066	,006	,353	,638	,453	,378	,727
General, female	χ^2	30,64	24,70	66,369	47,61	31,12	23,41	43,72
	df	25	25	25	25	25	25	25
	Asymp. Sig.	,201	,479	,000	,004	,185	,554	,012
Advanced, male	χ^2	35,25	58,61	96,81	38,20	51,06	56,99	39,51
	df	36	36	36	36	36	36	36
	Asymp. Sig.	,504	,010	,000	,370	,049	,014	,316
Advanced, female	χ^2	40,71	52,04	140,12	33,8	99,700	47,43	54,14
	df	35	35	35	35	35	35	35
	Asymp. Sig.	,233	,032	,000	,523	,000	,078	,020

Table 4. Kruskal Wallis Nonparametric Test for the group effect on mathematical beliefs among male and female students in general and advanced mathematics courses. TQ = Teacher quality, FE = Family encouragement

CONCLUSIONS

The results of these analysis confirmed that there is a certain level of agreement in certain mathematical beliefs among students of same class. Most pronounced this was for perceived teacher quality. In our earlier studies on teacher education students (e.g. Hannula et. al, 2006) we were not sure whether the variation in respondents' beliefs about their teacher's quality was an effect of their own mathematical achievement or if it reflected actual differences in the teaching they had received. This study confirms that students' belief of their teacher's quality is shared among students of the same class and therefore it is likely to be generated by shared experiences in the classroom context. Yet, also student's gender and achievement had an effect on this evaluation of the teacher. This provides evidence for the suggested interaction between levels in the model (Figure 1).

Shared classroom context seemed to have an effect also in students' effort (general mathematics) and enjoyment (advanced mathematics). This is indicating that through choices in instruction, it is possible to create a 'culture' in the classroom that is

motivating or enjoyable. However, we can not rule out the possibility that these differences between classes be effect of geography or some other variable that differentiates these groups.

An interesting finding was that there was a gender and group interaction effect for enjoyment among advanced mathematics courses, indicating stronger group effect for female students. This might relate to the anecdotes that students still occasionally tell about chauvinistic mathematics teachers they have had. The small effect size (6%) indicates that this is not a major problem on the level of educational system. However, for those female students who have to suffer through these classes it may be a big problem. Alternatively, this might indicate that there are such teachers in Finnish upper secondary schools that are able to create lessons that female students find especially enjoyable.

It is worth to note that gender had a stronger influence on confidence in mathematics than mathematics grade. The same is true also for and perceiving mathematics difficult in advanced course. In this sense these beliefs are truly gendered beliefs.

The findings provide support for the presented model and give indication to the origin of the measured beliefs (Figure 2). The effects of context and gender were surprisingly strong and the results support the hypothesis of social origin of beliefs.

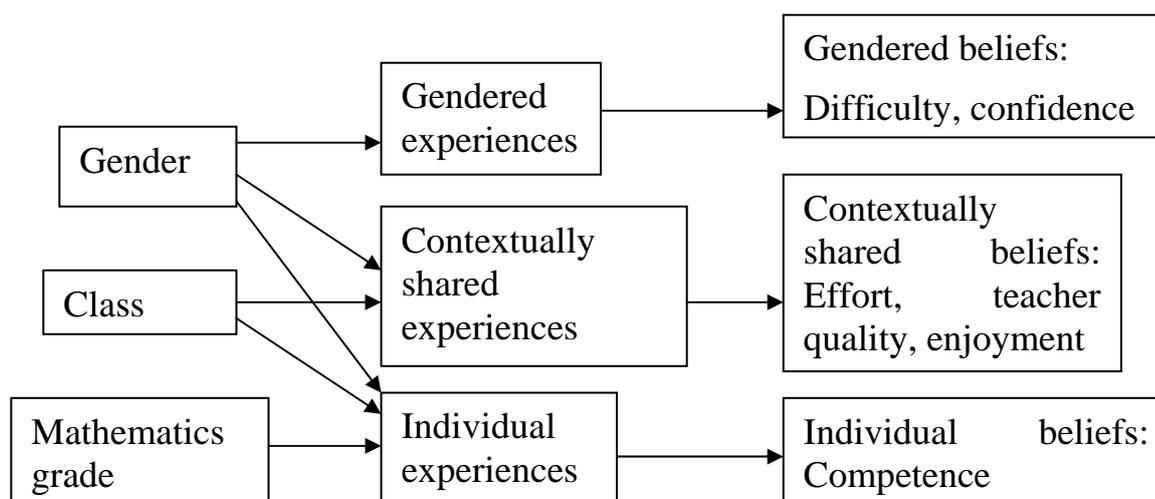


Figure 2. Empirically confirmed gendered, contextual and individual beliefs.

Enjoyment of mathematics, self-confidence in mathematics and self-efficacy beliefs are often considered as closely related aspects of attitude towards mathematics. This study highlights the different origin of these three aspects of attitude towards mathematics. Hence, it seems worthwhile to separate these different aspects also in future studies.

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CHANGING BELIEFS AS CHANGING PERSPECTIVE

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There is a phenomenon that has been observed in my work with inservice teachers. This phenomenon can be seen as embodying profound and drastic changes in the beliefs of the teachers participating in various projects. In this article I first describe this phenomenon and then more closely examine it using a framework of perspective. This framework allows for the articulation of the changes of beliefs as a foregrounding (or a reprioritization) of already existing beliefs. In doing so, I put forth a theory that allows for beliefs to be seen as both stable and dynamic – but always contextual.

INTRODUCTION

I work with inservice teachers. My reason for doing this is to affect change in these teachers' classroom practices, and ultimately, to affect change in the mathematical experiences of their students. In general, I try to accomplish this change through a focus on teachers' beliefs – beliefs about mathematics and beliefs about what it means to learn and teach mathematics. My assumption is that there is a link between teachers' beliefs and their practice (Liljedahl, 2008) and that meaningful¹ changes in practice cannot occur without corresponding changes in beliefs.

Recently, my main method of operating in this regard is to work with groups of teachers to co-construct some artefact of teaching – a definition, a task, an assessment rubric, a lesson, etc. This has proven to be a very effective method of reifying² the fleeting, and sometimes delicate, changes to beliefs that teachers experience within these settings (Liljedahl, in press, 2007). Within this context I am both a facilitator and a researcher. However, I am not a facilitator and a researcher in only the obvious sense. Although it is true that I facilitate the various activities that the teachers engage in – from discussions to the crafting of artefacts – it is also true that I facilitate the environment within which this all takes place. The sort of inservice work that I am involved in is more than simply the delivery of workshops, it is the provision and maintenance of a community of practice in which ideas are provisional, contextual, and tentative and are freely exchanged, discussed, and co-constructed. At the same time, while it is true that as a researcher I am interested in the down-stream effects of the work that I am engaged in (changes in teachers' practice in the classroom,

¹ Meaningful change is seen as a shift in teaching towards a more reform oriented practice. This change needs to be pervasive and robust.

² In this paper *reify* and *reification* is used in the tradition of Wenger (1998) rather than in the tradition of Sfard (1994). As such, reification means to make concrete – to turn some ephemeral aspect of teaching into *thingness*.

improvement in students' experiences and performance, etc.), it is equally true that I am interested in researching the inservice setting itself. There is much that happens within these settings. It is this later context which is the subject of this paper.

Working as both the facilitator and the researcher interested in the contextual and situational dynamics of the setting itself I find myself too embroiled in the situation to adopt the removed stance of observer. At the same time, my specific role as facilitator prevents me from adopting a stance of participant observer. As such, I have chosen to adopt a stance of *noticing* (Mason, 2006). This stance allows me to work within the inservice setting to achieve my inservice goals while at the same time being attuned to the experiences of the persons involved. I notice, first and foremost, myself. I attend to my choices of activities to engage in and the questions I choose to pose. I attend to my reactions to certain situations as well as my reflections on those reactions, both in the moment and after the session. More importantly, however, I attend to the actions and reactions of the teacher participants both as individuals and as members of a community. I observe intra-personal conflicts, interpersonal interactions, the dynamics of the group, as well as the interactions between individuals and the group. And in so doing, from time to time I notice phenomena that warrant further observation and/or investigation. Often these are phenomena that occur in more than one setting and speak to invariance in individual or group behaviour in certain contexts. Once identified these phenomena can be investigated using methodologies of practitioner inquiry that combine the role of educator with researcher – in this case teacher educator with researcher (Cochrane-Smyth & Lytle, 2004)³. Using a methodology of noticing I have observed rapid and profound changes in beliefs among individual teachers within a context of reification (Liljedahl, in press, 2007) and, more recently, among groups of teachers within this same context. It is this later phenomenon that I report on in this paper.

THEORETICAL BACKGROUND

Green (1971) classifies beliefs according to three dichotomies. He distinguishes between beliefs that are primary and derived. "Primary beliefs are so basic to a person's way of operating that she cannot give a reason for holding those beliefs: they are essentially self-evident to that person" (Mewborn, 2000). Derived beliefs, on the other hand, are identifiably related to other beliefs. Green (1971) also partitions beliefs according to the psychological conviction with which an individual adheres to them. Core beliefs are passionately held and are central to a person's personality, while less strongly held beliefs are referred to as peripheral. Finally, Green distinguishes between those beliefs held on the basis of evidence and those held non-evidentially. Evidence-based beliefs can change upon presentation of new evidence.

³ It should be noted that the main distinction between a methodology of noticing and a methodology of practitioner inquiry is that noticing doesn't presuppose a research question. It is a methodology of attending to the unfolding of the situation while being attuned to the occurrence of phenomena of interest.

Non-evidentiary beliefs are much harder to change being grounded neither in evidence nor logic. Instead they reside at a deeper and tacit level.

A person's belief system can, subsequently, be seen as a collection of beliefs competing for dominance in different contexts. Metaphorically, it is like a scene that is photographed from different perspective, with each perspective allowing something else to be foregrounded. Changes to learners' belief systems can then be seen as changes in perspectives⁴. Green argues that changing learners' belief systems is the main purpose of teaching. I argue that changing teachers' beliefs is the main purpose of inservice education.

METHODOLOGY

The data for the results presented here comes from three different, but similar, contexts in which I worked with groups of teachers in different schools and school districts. The first context (c1) involved a group of grade 5-8 mathematics teachers (n=10) working to design a task that could be used as district wide assessment of grade 8 numeracy skills in a school district in western Canada. This inservice project was comprised of 6 sessions (3 hours long, 3 weeks apart) during which we were to co-construct a working definition of numeracy (later adopted as the district definition) and design and pilot test a number of tasks that would reflect the qualities of our definition. The second context (c2) involved a group of grade 8 mathematics teachers (n=6) from a different district engaged in a very similar project. This time we were attempting to design a task that could measure the numeracy skills of their own students only. This project was comprised of 3 full day meetings 6 weeks apart. The third context (c3) involved all the mathematics teachers (n=18) in a middle school (grades 6-8). In this context we were working to design an assessment rubric that could capture some of the mathematical processes necessary for effective mathematical thinking. This involved a series of 12 one hour meeting held every two or three weeks.

As already mentioned, my method of operating within these inservice environments is through *noticing*. What this means from a more methodological perspective is that there is a great reliance on field notes taken both during the inservice sessions and more prolifically immediately after the inservice sessions. These field notes serve as a record of the things that I have noticed during individual sessions. Of course, they are limited in that they are only a record of that which has been attended to. However, these notes (or *noticings*) then form the basis of what is attended to in future sessions thereby creating an iterative process of refinement of attention. As this process continues phenomena that are deemed to be interesting receive more and more attention. This may simply mean a heightened awareness or anticipation of certain occurrences. Other times this means an adjustment in the facilitation practices in

⁴ This is not to say that changes in beliefs cannot also be seen as changes in beliefs, but for the purposes of this paper I stay with the metaphor of changing perspective.

order to more aggressively pursue the phenomenon. And sometimes it may mean stepping outside my role as a facilitator to investigate the phenomenon more directly as a researcher through methods such as interviews or questionnaires.

As such, the data for this study comes from a number of different sources. First and foremost, are the field notes from each of the aforementioned contexts. These notes increased in detail with each occurrence of the phenomenon. From c2 and c3 there are also transcriptions from interviews with different participants conducted at opportune times during or after certain sessions. These interviews were aimed at uncovering the participants own thoughts about the changes I was observing. The questions were of a semi-structured nature meant to preserve the conversational atmosphere that I had established with all of the participants while at the same time helping to illuminate the phenomenon itself.

THE PHENOMENON

The *exo/endo-spection* phenomenon, as I have come to call it, is comprised of a series of either three or four distinct phases, always in the same sequence, each having its own associated name. The names are an amalgamation – the prefix *exo-* and *endo-* comes from Greek meaning outer, outside, external and inner, inside, internal respectively; while *-spection* comes from the Latin *specere* which means 'to look at'.

Phase 1: exo-spection (x)

The teachers work on an activity which, at the time, occupies their focus. This could be a problem solving exercise or the designing of a lesson, task, or assessment rubric. Whether or not the activity is relevant to their own teaching practice is immaterial as the teachers' focus is on the completion of the task, rather than on the potential for the task to inform their own practice. That is, the teachers are looking at the activity as lying outside of themselves.

Phase 2: eXo-spection (X)

The teachers realize that the problem they have solved, or the lesson or task they have built, is not commensurate with their own classroom context. They see this as a large scale problem bemoaning the poor state of affairs of all students and the educational system in general. They look at the source of the problem as lying far outside of themselves – societal expectations, the curriculum, the evils of external examinations, deterioration of standards, etc. – and speak of systemic reform as the only solution. As such, they are not only pushing the problem further outside of themselves, but also broadening its scope.

Phase 3: eNdo-spection (N)

Suddenly there is a change in the teachers' disposition – the problem, regardless of where it lies, must be solved within their own practice in the scope of the classroom. Now the conversations are about what they can do

within their teaching in order to enable their students to be successful in solving a specific problem, completing a specified task, or performing well on a given assessment. The teachers' are no longer pushing the problem, and any subsequent solutions, away from themselves, but are rather bringing it back to their locus of control.

Phase 4: endo-spection (n)

For some teachers there is a final shift of attention to the plight of individual students. The conversations shift from the classroom to a particular student or subset of students, and with it comes a narrowing of focus on their influence as teachers. This final shift is also marked by a subtle shift in discourse from *teaching* to *learning*.

It should be noted that I have deliberately avoided using the term *introspection* which means to examine one's own thoughts and feelings. This is not what I am trying to capture here. *Endo-spection* is not about looking inside oneself, but about looking at something as lying inside of oneself or one's locus of control. Conversely, *exo-spection* is about looking at something as lying outside of oneself or one's locus of control.

In c1, x occurred in the first two sessions, X during the third session, N during the fourth session, and for two participants, n occurred in the last two sessions. In c2, x and X occurred in the first session, N in the second, and for one participant there was evidence of n in the third session. Finally, in c3, x occurred in the first 3 sessions, X in the fourth and fifth session, N in the sixth session, and for some of the participants, n occurred at various times during the last four sessions.

In general, the adoption of an exo-spection stance was uniformly a group position. That is, without prompting, every member of the group adopts an exo-spection stance and the group as a whole adopts an exo-spection stance. The discourse of the group did not deviate from this stance and there was a general sense that there was no need to do so – until there was a sudden transition to the eXo-spection stance. This transition, as well as the transition to eNdo-spection, was initiated by one or two members of the group, but then uniformly taken up by the group as a whole. It is almost as though the initiators were merely articulating what was already in the minds of the other members of the group, or the initiators merely precipitated an inevitable position. Conversely, the shift to an endo-spection stance, although articulated within the group context, was not taken up in the same way.

ANALYSIS

Because, for this paper, I am most concerned with changes in beliefs I will constrain my analysis to those points of greatest change – that is, the transitions between phases (x → X, X → N, and N → n). Further, I will look at these changes through a lens of changing perspectives.

exo-spection to eXo-spection ($x \rightarrow X$)

As already mentioned in the description of the exo-spection (x) phase, the teachers are initially contentedly working at completing the task at hand. In c1 and c2 this involved designing a numeracy task that conforms to a taken-as-shared definition of numeracy. In this case the teachers made extensive references to the published curriculum learning outcomes, the rationale that forms the underpinnings of the curriculum, as well as some ministry documents pertaining to the positioning of numeracy vis-a-vis the curriculum. In c3 the tasks that occupied the teachers in the first few sessions were increasingly challenging⁵ problem solving activities. Here the teachers were caught up in the excitement of doing mathematics that does not explicitly rely on mastery of specific learning outcomes. This can be seen in Barry's comments during one of the early sessions.

I love these problems. I mean, it's been a long time since I worked on problems myself, and I really like it. That card trick problem had me scratching my head all weekend. (Barry, c3, session II, field notes)

In either case, the teachers were focused on their own completion of these tasks, without much consideration for how they applied to their own practice.

The transition to X occurred in all three contexts when there was a sudden awakening to the fact that what the teachers were working on was not commensurate to their own classrooms contexts. This is nicely captured in the sudden change of tone in Barry's comments.

These problems are all fine and good. I mean, I enjoy doing them, but I don't have time for this with my kids. I have WAY too much stuff to get through to play around with these kinds of problems. Besides, my kids don't have enough patience for this kind of work. (Barry, c3, session V, interview transcripts)

It is also seen in the comments of Heidi and Charlotte working in c1.

I think we're getting it. The task is really starting to look like a numeracy task rather than just a word problem. It's not easy fitting all this stuff about communication, ambiguity, and multiple solutions into a task. But we're getting there. (Heidi, c1, session II, field notes)

I think these tasks are great, we've done a good job, but parents [of my students] are never going to go for this. The first time I send something like this home the phone will be ringing off the hook. We constantly have to work on drills to get the kids ready for the FSA's [Foundational Skills Assessment – an external high stakes exam, the results of which fold back onto the teacher]. And if we're not we're hearing about it from the parents and not because of the FSA's. They don't care about that, but these parents, a lot

⁵ This does not mean an increase in the mathematical complexity of the tasks. What is increasing is the demands on particular problem solving skills required (ability to organize work, communicate thinking, group work, deal with ambiguity, etc.).

of them are from Asia, and to them drills are important. (Charlotte, c1, session III, field notes)

The beliefs that these teachers are expressing (drills are important, learning outcome is curriculum, what parents want is important, kids are not capable) are not beliefs that have suddenly manifested themselves in latter sessions of the project. These are deep-seated beliefs (primary, core, evidential, tacit, or otherwise) that have been in the background during the teachers' initial encounters with their respective tasks. Working alone, or in a group, on something away from the multifaceted demands and expectations of their job less dominant beliefs (mathematics can be fun, numeracy is important, etc.) were able to come to the fore and inform their work in the initial sessions. But as the reality of their job rushed in on them the more dominant beliefs once again moved to the forefront, eventually paralysing their ability to see their initial work as being relevant to their own practice. However, there is still a wish that relevance could be found, but it is overwhelmed by the deep-seated belief that the problem is systemic AND can only be solved systemically. This can be seen in Adam's remarks.

Look, I agree that this is all very important. But there is just no way that we can make this work. There just isn't enough time, the kids aren't strong enough, we don't have administrative support, and, at the end of the day, the Ministry of Education just doesn't care. If they did, this is the kind of stuff we would see on the provincial exams. Until we can get them to change everything from the top down it just isn't going to work. I wish it were different, but it isn't. (Adam, c2, session I, interview transcripts)

eXo-spection to eNdo-spection ($X \rightarrow N$)

Initially, this transition is what drew my attention to the xXNn phenomenon. After commiserating about the negativity and hopelessness experienced in prior session of c1 there was a sudden rebirth of professional growth. This can be seen in Charlotte's comments in the fourth session of c1.

We have to keep pushing on in the direction we are going. If we don't design a task that shows what the kids can't do we're not ever going to be able to make any changes. We won't have anywhere to start. (Charlotte, c1, session IV, field notes)

Adam expressed a similar sentiment in the second session of c2.

In my opinion, these tasks aren't telling me enough. I'd like a task that really showed that these kids don't have a clue how to work together, for example. (Adam, c2, session II, field notes)

He adds details to these comments in a post-session interview.

I started to think about what we were doing here, with this whole project, and what it is we are trying to accomplish. I then started to think about how little I took away from my own math learning and what it is that is really important. We have an opportunity here to develop some really useful skills, stuff that these kids can use in grade 9, in grade 10, in

university, in life. They need to learn how to work together, how to deal with problems, how to tough it out, and stuff like that. But in order to do that we need to first show them that we are serious about this stuff. We can't just talk about it, we have to do it, and we have to mark them on it, and we have to start somewhere. (Adam, c2, session II, interview transcripts)

Tracey, also from c2, has a slightly different perspective.

They loved it. They asked me yesterday when we are going to do another numeracy task. I couldn't believe it. But you know what, they don't have a clue how to work together. So, now I'm working on that in my classroom. (Tracey, c2, session II, field notes)

As did Mary, who brought in samples of students' work.

As you can see there isn't much here – especially the boys. Like, you have to have a secret decoder ring to figure out what they are doing here. BUT, you know what, they did it. They worked on it and they got answers. Now we have to go forward with it. (Mary, c3, session VI, field notes)

The belief that assessments can be used formatively to inform both the teacher and the students is, again, not new. It has now moved into the forefront, however, buoyed by the realization of what it is that it is important, what the students can (or cannot) do, and what it is that the students enjoy doing. Whereas the transition from x to X can be seen as a regression to the norm (a return to a lower energy level, if you will) that is achieved almost subconsciously, the transition to N is almost wilful in nature. This re-prioritizing of beliefs is taxing and will require much effort and energy to sustain. It requires effort and motivation, and that motivation is found both in the successes of the students and the recapitulation of what is important. Or it can be found in the realization that what has come before isn't working, as is articulated by Phil.

I'm not sure if this is going to work. But I know for sure that what I've been doing before isn't working and I can continue to blame the system for all its faults or I can decide to do something about it. All I know is that I'm tired of both teaching my students AND learning for my students. Something has to change. (Phil, c3, session VI, interview transcripts)

eNdo-spection to endo-spection ($N \rightarrow n$)

As already mentioned, only some of the teachers moved to the final phase of the $xXNn$ phenomenon. Those who did, however, did so for seemingly the same reason – they were focusing on the learning of particular students or subsets of students. This was seen in their discourse about particular cases. Whereas some teachers spoke about cases as being exemplifications of the norm or the outliers within their classroom, these teachers spoke about the individual cases as standing for themselves. This can be seen in both Tracey's and Mary's comments.

So, I still have this one girl who is just toxic to anyone I put her with. No matter what I do she just will not work cooperatively. I've talked to the councillor and we think it has to do with self-esteem issues. So, I'm starting to think that this is where I should be putting my focus when it comes to her. (Tracey, c2, session 3, field notes)

In general, the students are doing much better. My work using graphic organizers has really helped. But, I still have a set of boys who just can't figure out which graphic organizer to use, or even that they have to use one. I'm not sure what to do about it, probably just keep working on it. But for now I'm still telling them which ones to use so that they can get through the task. (Mary, c3, session 11, interview transcript)

The belief that students are individuals and, thus, require differentiated instruction is likely not a new belief. However, with the use of formative assessment as an information gathering tool the teachers were giving this belief more and more prevalence.

CONCLUSION

Beliefs are stable patterns of thought, conscious or otherwise (Green, 1971). It is, therefore, unlikely that the teachers in this study changed their beliefs as drastically as the data may indicate. An alternative explanation is that the profound changes in beliefs are not a change at all, but rather a reprioritization of already existing beliefs - an affording of prevalence to less dominant beliefs. Such an explanation allows for both the robustness of beliefs and the possibility of profound change. This idea of reprioritization, or perspective, also allows for a more useful application of Green's organization of beliefs along three dimensions. A person's beliefs are hidden from us. Indeed, they may even be hidden from the person themselves. As such, knowing that beliefs may be central or peripheral, core or derived, evidential or tacit does us no good. Instead, recognizing that in different contexts different beliefs will be foregrounded, wilfully or otherwise, will allow us to think more holistically about belief systems as dynamic and contextual.

The xXNn phenomenon is such a context. Using a methodology of noticing and a framework of perspective I have described and analysed this phenomenon and concluded that the profound changes that are occurring within this context might just be due to a reprioritization of already existing beliefs. Further research into the phenomenon is necessary. There is great potential in analysing it using frameworks of psychology, group dynamics, as well as Gestalt. But it is early days, and this research is still in its exploratory phase. Now that the phenomenon has been identified, articulated, and even anticipated⁶, however, more detailed data can be gathered and more thorough analyses can be performed.

⁶ In fact, since gathering the data for the work presented here I have already identified the phenomenon, or subsets of it, within a master's course, a single session of a lesson study cycle, and a 90 minute workshop.

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**“MATHS AND ME”:
SOFTWARE ANALYSIS OF NARRATIVE DATA ABOUT
ATTITUDE TOWARDS MATH**

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Some years ago we undertook a research study aimed to obtain a ‘grounded’ characterization of attitude toward maths through the use of a narrative tool: we assigned to a large sample of Italian students the essay “Maths and me”, collecting more than 1600 texts. In this contribution we present some preliminary results, obtained using a piece of software for text analysis, regarding the way students of different grades describe their relationship with mathematics. In particular, we discuss the results from a comparative analysis between students of different school levels in order to find analogies and differences in the description of their own relationship with maths.

INTRODUCTION

Many research studies carried out in the last two decades in mathematics education highlight the relevance of affective factors to analyze and interpret students’ maths-related difficulties, and a specific field of research developed in recent years (for an overview see Zan R., Brown L., Evans J., Hannula M. 2006).

Among the affective factors, attitude toward mathematics is one of the most quoted constructs (by researchers in the field, teachers and educational institutions), but this “object” does not seem to have a well-defined and shared meaning. Among studies that explicitly give a definition, we can recognize three main different characterizations of attitude towards mathematics:

- a) a “*simple*” definition, that describes attitude as the positive or negative degree of affect associated with mathematics (Haladyna, Shaughnessy J. & Shaughnessy M., 1983; McLeod, 1992);
- b) a “*tridimensional*” definition, that recognizes three components in attitude: the degree of affect associated with mathematics, the beliefs regarding mathematics and the behaviour related to mathematics (Hart, 1989);
- c) a “*bidimensional*” definition, that includes only emotions and beliefs and does not consider behaviour (Daskalogianni & Simpson, 2000).

Some critical issues are linked to the choice of a definition for attitude (Di Martino & Zan, 2001), in particular: the consistency between the chosen definition of attitude and the instruments to observe/measure it, the definition of *positive/negative* attitude in the case of multidimensional characterizations. To characterize students’ attitude toward mathematics *from the bottom*, we carried out a narrative study investigating which dimensions students use to describe their relationship with mathematics. After

the characterization with the same data we could compare attitude of students belonging in different school levels.

In the field of mathematics education, narratives are more and more often used, especially in research about teachers' beliefs and teachers' practice (f.e. Da Ponte, 2001). Outside the field of teacher education, less numerous studies about *affect* make use of narratives: some have adults as their object (Karsenty & Vinner, 2000), others used narrative to report their own research (Hannula, 2003), others have students as their object (Ruffell et al., 1998). In this last case the studies are often carried out to criticize traditional instruments used to observe attitude rather than to characterize from the bottom the construct itself.

We used students' narratives (autobiographic essay), confident that in this way students could have the possibility to talk about the aspects *they* considered relevant in their own experience with mathematics. The chosen instrument is consistent with an interpretive approach and allows many typologies of data analysis.

From a qualitative analysis of students' description of their relationship with mathematics (Di Martino & Zan, submitted), a multidimensional model for attitude toward mathematics emerges, characterized by three strictly interconnected dimensions: the emotional disposition toward mathematics, the view of mathematics, the perceived competence in mathematics. That suggests the need to overcome the dichotomy between positive/negative attitude, and move to the identification of different profiles of negative attitude.

In this contribution, we present a quantitative analysis of the same data carried out with the help of T-Lab [1], a powerful software for text analysis, giving some preliminary interpretations of these results: in particular comparing the attitude of students from different educational levels.

METHODOLOGY

We proposed the essay "Me and mathematics: my relationship with maths up to now" to students from different school levels. For the administration of the essays we gave the following guidelines: essays had to be anonymous, assigned and collected in the class not by the mathematics teacher. At the end, we collected 1662 essays [2] ranging from grade 1 to grade 13: 874 from 51 classes of 14 primary schools (grade 1-5); 368 from 24 classes of 8 middle schools (grade 6-8); 420 from 29 classes of 10 high schools (grade 9-13).

In order to perform the statistical analysis with T-Lab we typed all data in a unique Corpus, respecting some specific guidelines, and we classified all essays with three control variables: identification number, grade and school level.

After this phase of data coding, we started to set the customized settings: selection of the lexical units to be included in the analysis, management of the lemmatization's phase, that is the reduction of the Corpus to their respective headwords called *lemmas* (for example general rules of lemmatization are: verbs' forms are taken back to the

infinite tense, nouns to the singular form, and so on).

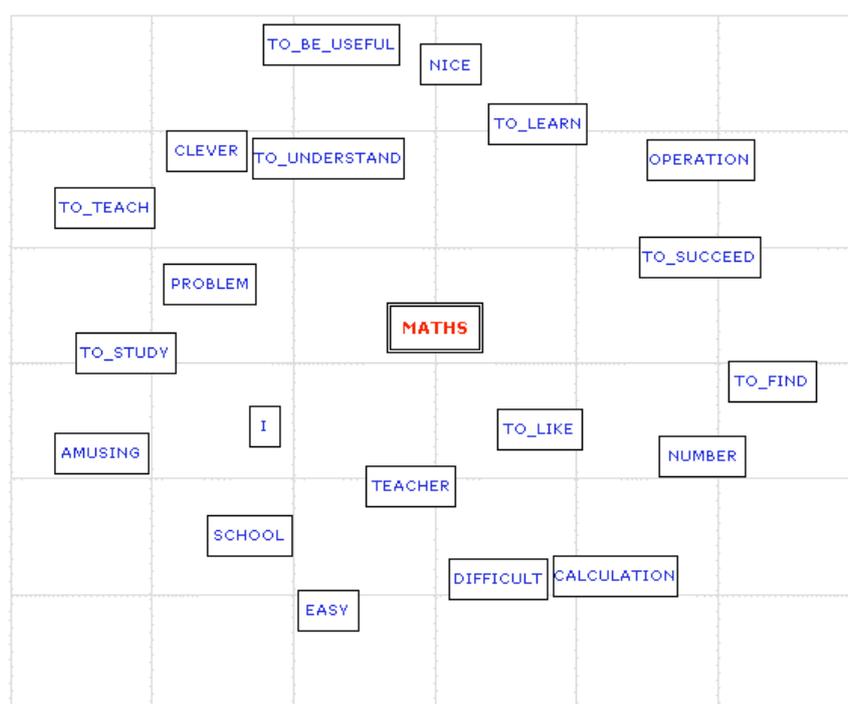
RESULTS AND DISCUSSION

Our attention will be focused on two typologies of analysis: co-occurrence and comparative analysis. The first one is finalized to find lexical units that more frequently are in co-occurrence [3] with some specific lemma, the latter is finalized to identify differences between texts from different subsets of the Corpus identified by some variables (in our case we selected the variable *school level*).

Co-occurrence analysis

Starting from the choice of the key-term '*maths*', the software calculates, in the whole Corpus, the lemmas with more co-occurrence with it through the association index of cosine [4]. This is a way to have a preliminary idea about the lexical units that students, in their autobiographical essays, more frequently associated with maths. Graph 1 is one of the outputs of the analysis: the nearness of each lemma to the central lemma '*maths*' is proportional to its degree of association.

Graph 1: Lemmas associated with *maths*

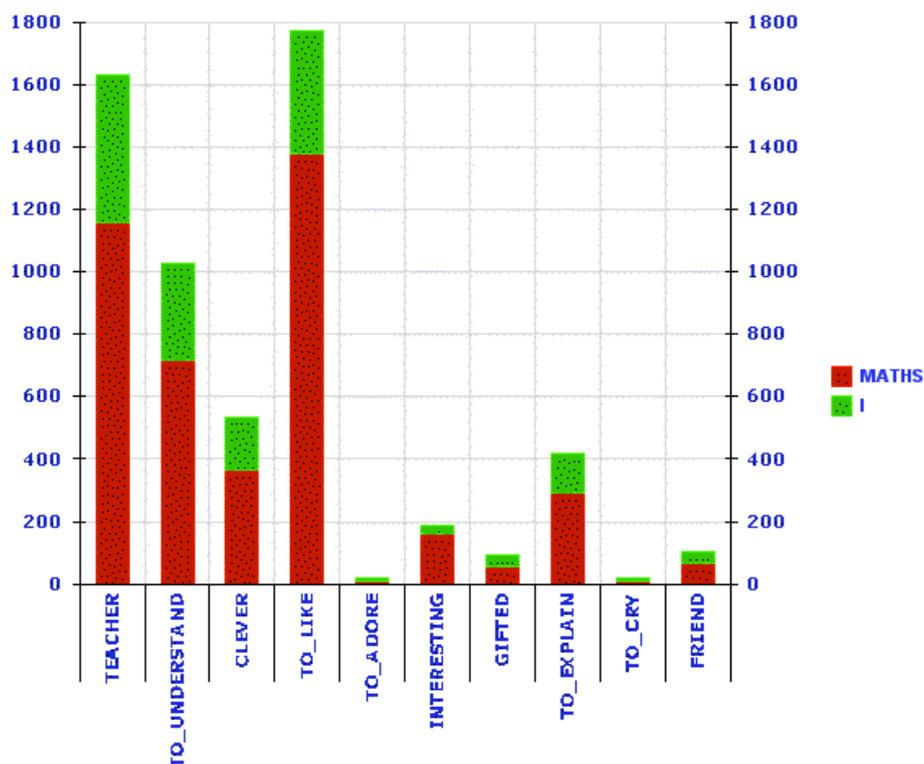


This representation strikingly shows that the emotional disposition (concisely expressed by “*I like/do not like maths*”) is very often in co-occurrence with maths: this is an indication that students tend to express their emotional disposition toward mathematics when they tell their relationship with mathematics itself. Moreover, the nearness of '*teacher*' can be interpreted in light of the fact that students recognize the teacher as a protagonist of their story with maths. For what concerns '*T*', it is obvious that, in an autobiographical essay regarding the writer's relationship with maths, the

lemmas *I* and *maths* are in co-occurrence.

Another analysis enables us to find the lemmas that are more correlated to both terms: *I* and *maths*. In graph 2 the co-occurrence with the two terms is shown in decreasing order with respect to the chi square test [5].

Graph 2: co-occurrences with *I* and *maths*



The relevance of the *teacher* in students' building of their own relationship with maths seems to be confirmed. But other two dimensions emerge heavily: an affective one (linked to lemmas as *to_like*, *to_adore*, *to_cry* and also *friend*) and one correlated with the idea of success in maths (associated to lemmas as *to_understand*, *clever*, *gifted*).

Comparative analysis

As we said earlier, with this typology of analysis we try to underline the differences between the three groups of students, as identified by the variable '*school level*'.

The first analysis regards the specificities of each group: T-Lab compares the subset A of the Corpus with the rest of the Corpus, individualizing which lexical units are typical (by the Chi-square test) or exclusive of the subset A. In table 1, for each group (Primary, Middle, High) the ten lemmas with the biggest chi-square value are reported.

Table 1: Specificities of three school levels

Primary school

Middle school

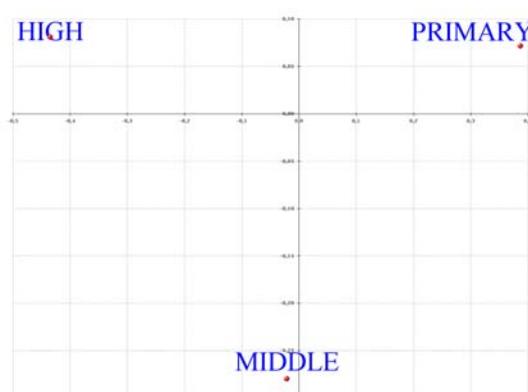
High School

WORD	CHI ²	SUB	TOT	WORD	CHI ²	SUB	TOT	WORD	CHI ²	SUB	TOT
to_like	459,7	1560	2488	expression	63,7	114	282	to_succeed	254,5	645	1078
operation	222,2	397	539	rule	38,3	50	111	school	185,8	399	638
to_learn	166,0	535	837	to_study	31,2	211	714	to_study	87,9	382	714
nice	149,8	302	423	algebra	27,2	48	118	exam	72,2	66	80
amusing	141,9	284	397	complex	25,9	39	91	time	69,1	156	252
examination	135,4	179	222	arithmetic	21,6	56	154	task	65,9	201	349
number	111,3	410	658	complicated	20,1	56	157	method	60,3	64	82
geometry	89,0	376	619	Easy	17,9	119	401	teacher	52,8	955	2168
calculation	67,6	250	401	important	17,3	84	267	to_apply	52,2	56	72
error	63,5	151	220	maths	14,0	1288	5603	insufficient	49,9	49	61

Sub = number of word's occurrences in subset, Tot = number of word's occurrences in Corpus

One interesting remark is about the strong characterization of the two extreme groups (Primary and High), testified by very high chi-square values. Moreover, looking at the first lemmas for each group, we can observe a shift from a *mastery-oriented* view (*to_learn*) of the relationship to a *performance-oriented* view (*to_succeed*, *exam*) and it is also interesting that the two first lemmas of the Middle group are related to an instrumental view of mathematics. According to our qualitative findings (Di Martino & Zan, submitted), this instrumental view is often combined with negative emotions towards mathematics and low perceived competence. This can be a possible explanation of the fact that in Italy the relationship with mathematics often becomes problematic just at middle school level. A factorial analysis allows us to characterize more precisely the specificity of the three groups: we can visualize their position in the factorial plane.

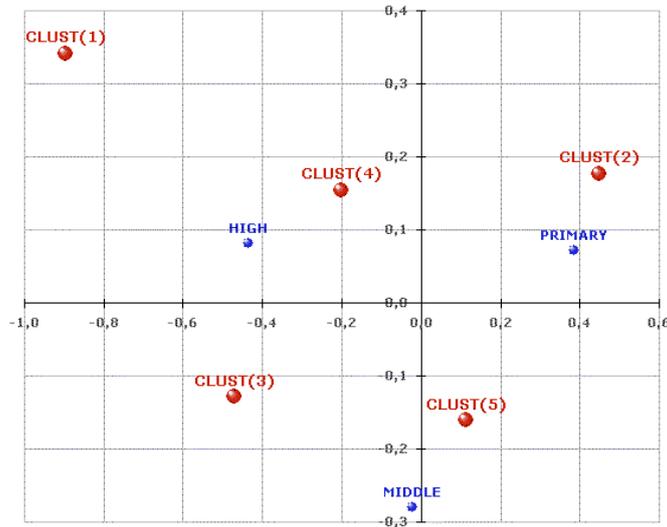
Graph 3: variables' position in factorial plane



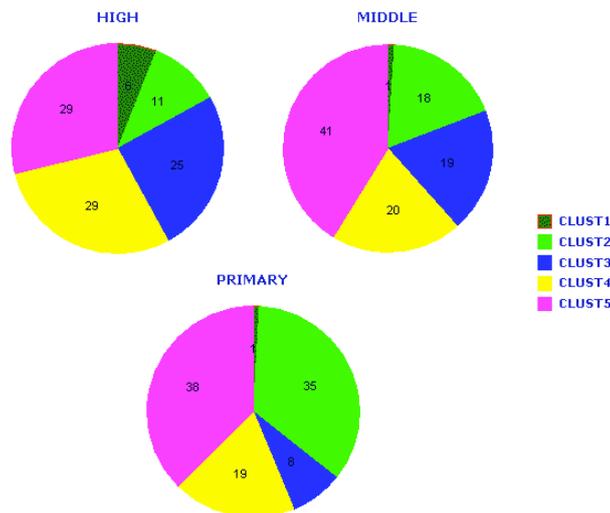
So we can observe that Primary and High groups are opposite poles in the X axis, while Middle group is characterized by its negative Y-component. In graph 4 all

their youngest colleagues. These interpretations are also reinforced by the cluster analysis that we performed with a partitioning method. We fixed to 5 the cluster numbers because with a smaller one we hadn't a clear distinction between groups identified by variables. We briefly report a table with the lemmas characterizing each cluster and the relationship between clusters and variables.

Graph 5: clusters and variables



Graph 6: Percentage groups subdivision in clusters



The percentages of cluster 1 are very small but it is present for any subdivision in clusters more than two. From an evolutionary point of view, we can observe that cluster 2 becomes less representative passing from 35% at Primary level to 11% at High level and cluster 5 is more or less stable from Primary to Middle level but becomes less representative at High level. While clusters 3 and 4 increase the number of their representatives. So it is very interesting to give a look to lemmas that characterize these four clusters in the following table (lemmas are in decreasing order

of relevance):

Table 2: description of clusters 2, 3, 4 and 5

Cluster 2

to_like, to_learn, number, geometry, operation, nice, calculation, amusing, error, examination, multiplication_table, to_write, to_make_a_mistake, fear, figure, logic, to_calculate, measure, drawing, correct, to_discover, wonderful, get_angry, question, to_play, to_draw, ability, exercise_book, brain, to_read, happy, to_worry, to_measure, anxiety, to_reproach, tidy, heart, to_sweat_blood, to_cry, gaiety, punishment, to_bore, mysterious, angry, test

Cluster 3

to_study, school, to_explain, mark, task, engagement, time, to_hate, to_hope, to_improve, to_carry_out, algebra, to_comprehend, rule, explanation, complex, oral_test, course_book, luck, best, future, to_love, worsening, resolution, cause, gifted, sincere, memory, reasoning, patience, to_overcome, positive, passion, to_forget, fundamental, serious, set_theory, possible, negative, genius, unpleasant, to_attract, to_fascinate, to_repeat_year, competition, to_give_up, theory, able, procedure, nightmare, frightened, torment, unlucky, serene, unbearable, tension, surprise, to_persecute, suffering

Cluster 4

teacher, to_understand, to_succeed, to_find, to_think, difficulty, interesting, to_know, to_believe, to_talk, formula, to_try, attention, will, ugly, to_memorize, immediately, friend, truth, effort, blackboard, sure, alone, strange, to_appreciate, idea, quiet, pleasant, clear, to_reflect, confuse, to_upset, experience, impossible, to_imagine, sense, thought, reality, stupid, to_resign, terrible, dream, terror, to_make_curious, hateful, slow, pride, success, disgusting, sadness, horrible, shame

Cluster 5

maths, I, problem, difficult, clever, to_teach, easy, boring, exercise, to_be_useful, certainty, expression, important, to_solve, simple, liking, arithmetic, useful, complicated, to_reason, game, quickly, severe, exciting, happiness, school_report, mathematician, to_implement, fascinating, tiring, to_support, challenging, to_listen, intelligence, shout, dubious, to_confuse, tremble

Cluster 2 is centred on the description of the *objects* of mathematics as well as on related activities (*to_learn, number, geometry, operation, calculation, multiplication_table, to_write, figure, logic, to_calculate, measure, drawing, to_discover, to_play, to_draw, exercise_book, to_read, to_measure*). Cluster 3 centres on theories of success (*to_study, engagement, time, to_comprehend, rule, cause, gifted, memory, reasoning, patience,...*) like cluster 4 (*to_understand, to_succeed, to_find, to_think, to_know, to_believe, formula, to_try, attention, will,*

effort,...), but whereas cluster 3 seems to be projected ahead (*to_hope, to_improve, future, to_overcome*), cluster 4 seems to be more *static* and centred on a definitive evaluation of what happened (*impossible, to_resign,...*), cluster 5 seems to be the cluster of balance between difficulties (*difficult, simple, complicated,...*) and usefulness (*to_be_useful, important,...*). Finally, all four clusters have some emotional components: surely clusters 3 and 4 are characterized by lemmas that evoke stronger emotions (*to_hate, to_love, nightmare, frightened, torment, tension, to_persecute, suffering* for cluster 3 and *terrible, terror, disgusting, hateful, pride, horrible, shame*) than cluster 2, which seems to be the one with the highest number of lemmas linked to *positive* emotions, and cluster 5.

CONCLUSIONS

An important aspect of the described research study is the combination of quantitative analysis with an interpretive approach. All the results we got led us to interpretive hypotheses, that become stronger if compared to, and interconnected with, the qualitative analysis performed on the same material (and partially described in Di Martino & Zan, submitted). We point out that if on the one hand, the obtained results offer extremely interesting stimuli, on the other hand they cannot provide certainties, due to the type of material we analyzed (*open* texts). In this case, we really ought to be cautious: the analysis of open texts based on lexical units only, without an analysis of the contexts within which these lexical units are used, might be problematic. To exemplify, the lemma *to_like* is not always referred to mathematics; the word *problem* might stand for a mathematical problem but also for a real life problem. Therefore, it was really important to compare results of this analysis with those of the qualitative one (described in Di Martino & Zan, *ibidem*): in particular, the results about the three dimensions characterizing attitude towards mathematics are confirmed.

The ‘evolutionary’ results that emerge from cluster analysis seem to be particularly interesting. A general deterioration of students’ relationship with mathematics can be clearly detected but, most of all, as the school level increases, the lemmas used to describe one’s relationship with mathematics suggest that the latter becomes more and more radical. Moreover, there seems to be a move from a phase of interest in the novelty of mathematics -the pleasure of discovery- to a phase in which succeeding prevails over the subject matter itself. One final remark: the fact that in this phase emotional aspects become more radical provides material for further reflection.

NOTES

1. The bibliography related to T-lab is available on-line: <http://www.tlab.it/en/presentazione.asp>
2. The collected essays constitute a convenient sample, obtained through a collaboration with teachers and heads of schools who accepted our requests. The schools are situated in six different area of Italy: from North to South.
3. Co-occurrence is when two or more lemmas are present together in the same text.

4. To calculate the cosine index between lemma X and lemma Y we have to consider $a = \#$ of essays with lemma X and Y, $b = \#$ of essays with lemma X and without lemma Y, $c = \#$ of essays with lemma Y and without lemma X. Cosine (lemma X, lemma Y) = $a / \sqrt{(a + b) \times (a + c)}$.

5. The Chi-square test is a well-known test used to check if the frequency values obtained by a survey are significantly different from the theoretical ones. T-Lab applies this test to 2x2 tables then the threshold values is 3.84 (df=1, p=0.05) or 6.64 (df=1, p=0.01).

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STUDENTS' BELIEFS ABOUT THE USE OF REPRESENTATIONS IN THE LEARNING OF FRACTIONS

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Cognitive development of any mathematical concept is related with affective development. The present study investigates students' beliefs about the use of different types of representations in understanding the concept of fractions and their self-efficacy beliefs about their ability to transfer information between different types of representations. The interest is concentrated on differences among students at primary and secondary education. Results indicated that students at secondary education have less positive beliefs for the use of representations at the learning of mathematics than at primary education. As a consequence they have less positive self-efficacy beliefs about their abilities to use them. Unexpected was their lower performance at solving tasks on fractions for which the information is represented in different forms.

Keywords: representations, beliefs, self-efficacy, fractions

Mathematics is a specialized language with its own contexts, metaphors, symbol systems and purposes (Pimm, 1995). From an epistemological point of view there is a basic difference between mathematics and other domains of scientific knowledge as the only way to access mathematical objects and deal with them is by using signs and semiotic representations (Duval, 2006). Cognitive development is related with metacognitive and affective development. One's behavior and choices, when confronted with a task, are determined by her/his beliefs and personal theories, rather than her/his knowledge of the specifics of the task. Thus, students' academic performance somehow depends on what they have come to believe about their capability, rather than on what they can actually accomplish.

The relationship between cognition and affect has the last decades attracted increased interest on the part of mathematics educators, particularly in the search for causal relationship between affect and achievement in mathematics (Young, 1997). This is due to the fact that the mathematical activity is marked out by a strong interaction between cognitive and emotional aspect. The affective domain is a complex structural system consisting of four main dimensions or components: emotions, attitudes, values and beliefs (Goldin, 2001). At the present study we focus on students' beliefs and mainly their self-efficacy beliefs in using different types of representations in mathematics learning and understanding. We concentrated our attention on the notion of fractions.

Fractions are among the most essential (Harrison & Greer, 1993), but complex mathematical concepts that children meet in school mathematics (Charalambous & Pitta-Pantazi, 2007). An important factor that may contribute to students' difficulties in learning fractions is the transition from primary in secondary school with all the changes that this encompasses in mathematical teaching and learning.

THEORETICAL BACKGROUND

Self-efficacy beliefs

Beliefs is a multifaceted construct, which can be described as one's subjective "understandings, premises, or propositions about the world" (Philipp, 2007, p. 259). According to Pehkonen and Pietila (2003) there are several difficulties in defining concepts related to beliefs. Some researchers consider beliefs to be part of knowledge (e.g. Pajares, 1992), some think beliefs are part of attitudes (e.g. Grigutsch, 1998), and some consider they are part of conceptions (e.g. Thompson, 1992).

The construct of self-efficacy beliefs constitutes a key component in Bandura's social cognitive theory; it signifies a person's perceived ability or capability to successfully perform a given task or behavior. Bandura (1997) defines self-efficacy as one's perceived ability to plan and execute tasks to achieve specific goals. He characterized self-efficacy as being both a product of students' interactions with the world and an influence on the nature and quality of those interactions. Self-efficacy beliefs have received increasing attention in educational research, primarily in studies for academic motivation and self-regulation (Pintrich & Schunk, 1995). It was found that self-efficacy is a major determinant of the choices that individuals make, the effort they expend, the perseverance they exert in the face of difficulties, and the thought patterns and emotional reactions they experience (Bandura, 1986). Furthermore, self-efficacy beliefs play an essential role in achievement motivation, interact with self-regulated learning processes, and mediate academic achievement (Pintrich, 1999).

Multiple representations in mathematics teaching and learning

The representational systems are fundamental for conceptual learning and determine, to a significant extent, what is learnt (Cheng, 2000). Learning involves information that is represented in different forms such as text, diagrams, practical demonstrations, abstract mathematical models, simulations etc (Schuyter & Dekeyser, 2007). Recognizing the same concept in multiple systems of representations, the ability to manipulate the concept within these representations as well as the ability to convert flexibly the concept from one system of representation to another are necessary for the acquisition of the concept (Lesh, Post, & Behr, 1987) and allow students to see rich relationships (Even, 1998). Recently the different types of external representations in teaching and learning

mathematics seem to become widely acknowledge by the mathematics education community (NCTM, 2000). The necessity of using a variety of representations or models in supporting and assessing students' constructions of fractions is stressed by a number of studies (Lamon, 2001). The geometric shapes used for introducing the continuous model of fractions are distinguished into two types: the circular model which is based on the use of circles and the linear model which is based on a rectangle divided into a number of equal part (Boulet, 1998).

An issue that has received major attention from the education community over the last years refers to the students' difficulties when moving from elementary to secondary school and to the discontinuities in the curriculum requirements, the use of teaching approaches, aids and methods. According to Schumacher (1998) the transition to secondary school is accompanied by intellectual, moral, social, emotional and physical changes. Pajares and Graham (1999) investigated the extent to which mathematics self-beliefs change during the first year of middle school. By the end of the academic year, students described mathematics as less valuable, and they reported decreased effort and persistence in mathematics. The findings of the Deliyianni, Elia, Panaoura and Gagatsis's (2007) study suggest that there is a noteworthy difference between elementary and secondary education in Cyprus concerning the representations used in mathematics textbooks on fractions. There are also differences in the functions the various representations in the school textbooks fulfil.

The present study investigated Grade 5 to Grade 8 students' beliefs about the use of different representations for the learning of the fractions and their self-efficacy beliefs about the use of those types of representations. That means that it explores the differences of students' beliefs at primary and secondary education concerning the use of different types of representations.

METHOD

The study was conducted among 1701 students of 10 to 14 year of age who were randomly selected from urban and rural schools in Cyprus. Specifically, students belonging to 83 classrooms of primary (Grade 5 and 6) and secondary (Grade 7 and 8) schools (414 in Grade 5, 415 in Grade 6, 406 in Grade 7, 466 in Grade 8) were tested.

A questionnaire was developed for measuring students' beliefs about the use of different types of representations for understanding the concept of fractions. The questionnaire comprised of 27 Likert type items of five points (1=strongly disagree, 5=strongly agree). The reliability of the whole questionnaire was very high (Cronbach's alpha was 0.88). The items of the questionnaire are presented at Table 1.

At the same time a test was developed for measuring students' ability on multiple representation flexibility as far as fraction addition is concerned. The test included 22

fraction addition tasks that examine multiple-representation flexibility and problem-solving ability. There were treatment, recognition, conversion, diagrammatic problem-solving and verbal problem-solving tasks (further details for the tasks can be found at the paper of Deliyanni et al. (2007). Indicative examples of the items are presented at Appendix. Cronbach's alpha for the test was 0.87.

The test and the questionnaire were administered to the students by their teachers at the end of the school year in usual classroom conditions. Right and wrong or no answers were scored as 1 and 0, respectively. Solutions in treatment, recognition and translation tasks were assessed as correct if the appropriate answer, diagram, equation or shading were given respectively, while a solution in the problems was assessed as correct if the right answer was given.

RESULTS

The analysis of students' responses to the items of the questionnaire resulted in six factors (KMO=0.933, $p < 0.001$) with eigenvalues greater than 1 (Table 1). The first factor corresponded to students' self-efficacy beliefs about conversion from one type of representation to another. The second factor was associated with their general self-efficacy beliefs in mathematics. The third factor represented their beliefs about the use of the number line, while the fourth factor represented their beliefs about the use of models, materials or representations. The fifth factor corresponded to students' beliefs about the use of diagrams in problem solving and the sixth factor to their self-efficacy beliefs about the use of verbal representations.

Item	F1	F2	F3	F4	F5	F6
I can easily find the diagram that corresponds to an equation of fractions.	.53					
I can easily solve tasks than ask to convert the part of a diagram into an equation.	.62					
I can easily find the diagram that corresponds to an equation of decimals.	.67					
I can easily find the equation of fraction addition that corresponds to a part of a surface of a rectangle.	.63					
I can easily find the equation of fraction addition which is presented with arrows in number line.	.58					
I am very good in solving tasks with decimals.		.70				
I am very good in problem solving fractions.		.78				
I can easily solve tasks with fractions.		.79				
I can easily solve equations of fraction addition.		.70				
I can easily solve equation of decimal addition.		.56				
Number line helps me in problem solving with fractions.			.68			
Number line helps me in solving equations with fractions.			.68			
My teacher usually uses number line in order to explain us the operations of fractions.			.64			
Number line helps me in solving equations with decimals.			.64			
A good student in mathematics can present the solution of a problem by many different ways.					.55	
For the problem solving the use of equation is necessary.					.65	
In mathematics the use of materials (fraction circles, dienes cubes etc) is						

useful mainly for students at primary education	.59
The diagrams (number line, rectangle etc) are useful for executing operations.	.42
If I have to explain how I have solved a problem with decimals, I prefer to use an equation.	.57
If I have to explain how I have solved a problem with fractions, I prefer to use a diagram.	.65
When I solve a problem with fractions, I use the number line for executing the operations.	.44
When I solve a problem with fractions by using a diagram, I then try to solve it by using an equation, as well.	.49
When I solve a problem with decimals I use a diagram.	.67
I can easily explain how I have solved a problem with decimals by using a diagram.	.47
I prefer solve problems with decimals which present the data verbally.	.79
I can easily explain verbally how I have solved a problem with decimals.	.69
Eigenvalues	7.87 2.48 1.92 1.58 1.25 1.17
Percentage of variance explained	24.6 9.76 6.77 5.01 4.20 3.34
Cumulative percentage of explained variance	24.6 34.3 41.1 46.1 50.3 53.6

Table 1: Factor loading of the six factors against the items associated with participants' beliefs

Analysis of variance (ANOVA) indicated that there were statistically significant differences in respect to grade for the factors F1, F2, F5 and F6. Specifically in the case of F1 there were differences at the means ($F_{3,1547}=9.09$, $p<0.001$) between students' self-efficacy beliefs to converse flexibly the concept of fraction addition from one representation to any other who were attending the Grade 8 with the students of the Grades 5, 6 and 7. In the case of the F2 the statistically significant differences ($F_{3,1574}=31.615$, $p<0.001$) were between the Grade 5 with Grades 7 and 8, the Grade 6 with the Grade 7 and 8. Students at the Grade 8 seemed to have less positive beliefs for the significance of using different types of representations (F5). There were statistically significant differences between Grade 5 and Grade 8, Grade 6 and Grade 8. In the case of their preference for using verbal explanations the differences were between Grade 5 with Grade 7 and 8 and Grade 6 with Grade 7 and 8. Therefore, most of the differences revealed were between the students at primary education and the students at secondary education. All the means are presented at Table 2.

	F	p	\bar{X}_5	\bar{X}_6	\bar{X}_7	\bar{X}_8
F1	9.09	<0.001	3.63	3.58	3.53	3.37
F2	31.615	<0.001	4.08	3.93	3.70	3.56
F5	6.209	<0.001	3.29	3.24	3.17	3.09
F6	21.036	<0.001	3.46	3.36	3.15	2.96

Table 2: The means for the factors F1,F2, F5 and F6 with the statistically significant differences in respect to grade.

Very impressive and unexpected were the descriptive results of the students' mathematical performance at the test. As it is obvious in Figure 1 students at the Grade 7 have lower performance than the students at the Grade 6.

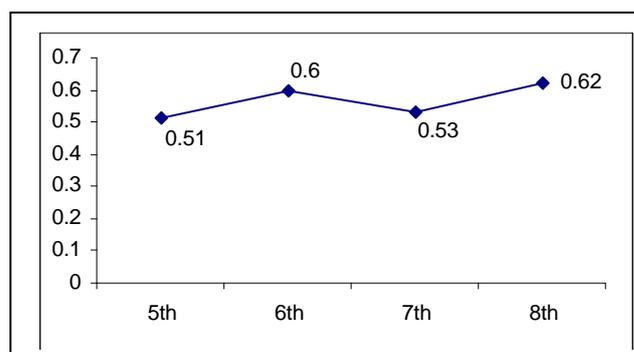


Figure 1: Students' of different grades performance on the mathematical test.

Students were cluster, by using cluster analysis, according to their performance at the test into three groups (Group1: 426 students with low performance, Group2: 788 students with medium performance, Group3: 487 students with high performance) Analysis of variance (ANOVA) with independent variable the three groups and dependent variables the six factors, which were comprised from the abovementioned factor analysis, indicated statistically significant differences in respect to F1 ($F_{2,1547}=51.819$), F2 ($F_{2,1474}=74.903$), F4 ($F_{2,1609}=12.057$) and F6 ($F_{2,1671}=8.844$). In all cases the first group had the most negative beliefs and self-efficacy beliefs and the third group had the most positive beliefs. That means that students with high mathematical performance had at the same time positive beliefs for the use of representations and high self-efficacy beliefs.

Finally students were clustered into two groups according to their general self-efficacy beliefs in mathematics (F2), by using cluster analysis. The group with higher self-efficacy beliefs consisted of 1047 students ($\bar{X}=4.31$) and the second group consisted of 528 students ($\bar{X}=2.82$). T-test analysis between the two groups in respect to the other five factors indicated that there were in all cases statistically significant ($p<0.01$) differences (Table 3). Students with higher general self-efficacy beliefs in mathematics had at the same time more positive beliefs for the use of different forms of representations and more positive self-efficacy beliefs for the use of those representations and their ability to transfer their knowledge.

	t	df	\bar{X}_1	\bar{X}_2
F1	22.82	1463	3.81	2.97
F3	6.508	1527	3.30	2.99
F4	13.897	1507	4.09	3.57
F5	9.151	1499	3.32	2.96
F6	14.616	1565	3.48	2.73

Table 3: Students' with high and low self-efficacy beliefs differences in respect to their beliefs about the use of representations

DISCUSSION

The main emphasis of the present study was on investigating students' self-efficacy beliefs for mathematics in relation to their beliefs about the use of representations for understanding the concept of fraction. The analysis of the data confirms earlier findings that young students have high self-efficacy beliefs (Bandura, 1986) and that they tend to overestimate their abilities. However those beliefs decreased at the secondary education. It seems that students' sense of efficacy diminishes somehow when they compare their abilities with classmates and even more in relation to their mathematical performance as it is revealed by their final grades at mathematics. The influence of those active experiences is too strong and with immediate results. Accepting that the most important step is getting individuals to become aware of their own processes, strengths and limitations in order to have an accurate self-representation, it seems that the specific result is important for the learning of the concept of fractions. Nevertheless it is not positive generally, because there are too many other concepts at the teaching of mathematics at secondary education for which students have to use flexibly different types of representations. For example the concept of function admits a variety of representations, each of which offers information about particular aspects of the concept without being able to describe it completely (Elia et al., 2008).

Interesting and unexpected was the differences between students' performance in the use of different forms of representations at primary and secondary education and mainly the lower performance at secondary education. A possible explanation for the lack of improvement regarding their mathematical performance observed are the differences regarding the representations and their functions in mathematics textbooks used in primary and secondary education in Cyprus (Deliyianni et al., 2007). Furthermore, the secondary school students may had not created referential connections between corresponding elements and related structures in a way that promotes understanding of this concept during their primary schooling. Their difficulties increased in secondary education since no emphasis is placed on learning with multiple representations.

Results confirmed that students with low performance in mathematics have at the same time negative beliefs for the use of different forms of representations because they cannot use them fluently and flexibly as a tool to overcome obstacles while solving tasks and handling the whole situation. It seems that there is a need for further investigation into the subject with the inclusion of a more extended qualitative and quantitative analysis. Most mathematics textbooks today make use of a variety of representations more extensively than ever before in order to promote understanding (Elia, Gagatsis & Demetriou, 2007). Much more research is needed for the students' beliefs about the role of those representations regarding different mathematical concepts in relation to their self-efficacy beliefs for using them as a tool for the better understanding of the concepts.

Appendix

1. Circle the diagram or the diagrams whose shaded part corresponds to the equation $\frac{2}{3} + \frac{1}{4}$.



(recognition)

2. Solve the following equation

$$\frac{1}{6} + \frac{2}{5} = \dots$$

(treatment)

3. Write the fraction equation that corresponds to the shaded part of the following diagram:



Equation:

(conversion)

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EFFICACY BELIEFS AND ABILITY TO SOLVE VOLUME MEASUREMENT TASKS IN DIFFERENT REPRESENTATIONS

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University of Cyprus

The aim of this study was to investigate the relationship between students' efficacy beliefs and their performance in volume measurement tasks which were given in different representations. A group of sixth grade students (N=173) completed a four-part self-report questionnaire and solved six volume measurement tasks in different representations format: text, diagram of 3-D cube array and net diagram. Perceived efficacy to solve volume measurement tasks was found to be a significant predictor of students' general performance. Furthermore, high-ability students had stronger and more accurate efficacy beliefs towards tasks with net diagram which were unfamiliar, whereas low-ability students had more accurate efficacy beliefs towards verbal tasks which were familiar.

Key words: efficacy beliefs, volume, 3-D cube arrays, net.

INTRODUCTION

The affective domain has in recent years attracted much attention from mathematics research community (Philippou & Christou, 2002). A number of researchers who have examined thoroughly the connections and the relationship among affect and mathematical learning found that affect plays a decisive role in the progress of cognitive development (Bandura, 1997; Ma & Kishor, 1997; Philippou & Christou, 2002). One of the components of affective domain are self-efficacy beliefs (Goldin, 2002), which were found to have significant correlations and direct effects on various math-related variables (Pajares, 1996). However, although much work has been done in this area, little attention has been given to the relationship between self-efficacy beliefs and the use multiple representations in mathematics (e.g. Patterson & Norwood, 2004).

In this paper we try to investigate the relationship between efficacy to solve volume measurement tasks and performance in volume measurement of cuboid tasks which are given in different modes of representations.

THEORETICAL BACKGROUND

Self-efficacy beliefs and mathematics performance

Self-beliefs, such as self-esteem, self-concept and self-efficacy, comprise components of the general beliefs system (Philippou & Christou, 2002). Students' perceived self-efficacy for a task, are defined as their judgments about their ability to complete a task successfully (Bandura, 1997).

A number of studies have found a positive relationship between students' self-efficacy beliefs and mathematics performance (Pajares, 1996). More specifically, Pajares and Miller (1994) reported that self-efficacy in solving math problems was more predictive of that performance than sex, math background, math anxiety, math self-concept and perceived usefulness of mathematics. Additionally to this, Pajares and Kranzler (1995) found that self-efficacy made as strong a contribution to the prediction of problem-solving as did general mental ability, an acknowledged powerful predictor and determinant of academic outcomes. In this line, Mayer (1998) stressed that students who improve their self-efficacy will improve their success in learning to solve problems.

Researchers have also indicated that high-ability students have stronger self-efficacy and have more accurate self-perceptions (e.g. Pajares & Kranzler, 1995; Zimmerman, Bandura, & Martinez-Pons, 1992). Schunk and Hanson (1985) found that students who expected to be able to learn how to solve the problems tended to learn more than students who expected to have difficulty.

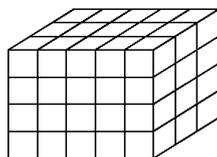
Self-efficacy beliefs have already been studied in relation to a lot of aspects of mathematics learning, such as arithmetical operations, problem solving and problem posing (e.g. Pajares & Miller, 1994; Pajares, 1996; Nicolaou & Philippou, 2007). However, these beliefs haven't been examined in relation to volume measurement tasks and this study tries to investigate this relationship.

Students' understanding of 3-D rectangular arrays of cubes

A number of researchers investigated students understanding of three dimensional rectangular arrays (3-D) of cubes, using interviews or tests (Ben – Chaim, Lappan & Houang, 1985; Battista & Clements, 1996). In particular, Ben – Chaim et al. (1985) indicated four types of errors that students in grades 5-8 made on the volume measurement tasks with three dimensional cube arrays. The first error was to count only the number of faces of cubes shown in a given diagram, while the second error was doubling that number. The third error was counting the number of cubes shown in the diagram and the fourth error was doubling that number (see for example figure 1). In this study, when researchers asked students to determine how many cubes it would take to build such prisms, they found that only 46% of the students gave the correct answer, while most of them made the errors of type 1 or 2 (Ben-Chaim et al., 1985). These results are in line with those from a recent work by Battista and Clements (1996) where they found that 64% of the third graders and 21% of the fifth graders double-counted cubes. These types of errors made by students are clearly related to some aspects of spatial visualization (Ben-Chaim et al., 1985). In addition to this explanation, Battista and Clements (1996) stressed that many students are unable to correctly enumerate the cubes in such an array, because their own spatial structuring of the array is incorrect. In particular, they found that for some students the root of such errant spatial structuring seemed to be attributed to their inability to coordinate and integrate the views of an array to form a single coherent mental model

of the array. However, Hirstein (1981) believes that these errors are caused by their confused notions of volume and surface area.

How many unit cubes does it take to make this rectangular solid? (Clements & Battista, 1996)



Four types of errors that students make on this problem:

Error type 1: Counting the cube faces shown in the diagram, e.g. $20+12+15=47$

Error type 2: Counting the cube faces shown in the diagram and doubling that number, e.g. $47 \times 2 = 94$

Error type 3: Counting the numbers of cubes showing in the diagram, e.g. $20+8+8=36$

Error type 4: Counting the numbers of cubes showing in the diagram and doubling that number, e.g. $36 \times 2 = 72$

Figure 1: Four types of errors that students make on volume measurement problems.

THE PRESENT STUDY

The purpose of the study

The purpose of this study was to explore the relationship between students' efficacy beliefs to solve volume measurement tasks and their ability to solve volume measurement cuboids tasks; these were given in different modes of representations, namely text, diagram of 3-D cube array and net diagram. More specifically, the present study addresses the following questions: (a) Are students' efficacy beliefs to solve volume measurement tasks strong predictor of their performance in these tasks? (b) What is the relationship between students' efficacy beliefs to solve volume measurement tasks and their errors in dealing with 3-D cube arrays and net diagrams? (c) Are there differences in the efficacy beliefs and the accuracy of these beliefs among students of varied abilities?

Participants and Test

In the present study data were collected from 173 sixth grade students (84 females and 89 males) ranging from 11 to 11.5 years of age. These students were from 10 primary schools in Cyprus from rural and urban areas.

All participants completed a five-part test which was developed on the basis of previous studies (e.g. Ben-Chaim et al., 1985; Battista & Clements, 1996; Nicolaou & Philippou, 2007). For the purpose of this paper, we did not use students' answers from the first part of the test. The first four-parts of the test measured efficacy beliefs towards mathematical problems and volume measurement tasks and the fifth part

measured students' ability to solve volume measurement tasks in different representations. Specifically, in the second part, students were asked to read each of the three volume measurement tasks: verbal task (SEiA), task with 3-D cube array (SEiB) and task with net diagram (SEiC) and state their sense of certainty to solve these tasks, without solving them. Responses were recorded on a 4 point Likert scale with 1 indicating not at all certain and 4 very much certain. In the third part, students were asked to state which one of the tasks from the second part was easy to solve (Es), was difficult to solve (Df), liked to solve (Lk) and did not find interesting to solve (Lint). The fourth part comprised of five cartoon-type pictures and statements explaining the situation presented by each picture; the students were requested to select the picture that best expressed their efficacy beliefs (very high-SEI, high-SEII, medium-SEIII, low-SEIV and very low-SEV) to solve volume measurement tasks. The fifth part of the test had six volume measurement cuboids tasks which were given in different modes of representations: text, diagram of 3-D cube array and net diagram (see figure 2).

Verbal tasks

1. Mary tries to put 28 unit-sided cubes (1 cm edge) in a rectangular box with dimensions 2 cm x 5 cm x 3 cm. Is this possible? Explain your answer. (VPr1)

4. Four friends went to the cinema. They decided to buy some bags of nuts during the movie. The vendor said to them that there were two size bags of nuts, where:

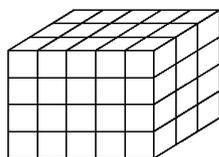
- The price of small bag was €1.
- The large bag's dimensions were two times the small bag's dimensions and its price was €6.

The dimensions of small bag were 20 cm, 10 cm and 5 cm.

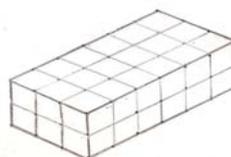
One child suggested to his friends that it was better to buy and share one large size bag, instead of buying four small bags. Do you agree? Explain your answer. (VPr4)

Tasks with diagram of three dimensional cube array

Find the volume (the number of cubes) of the following cuboids:



(SPr2a)

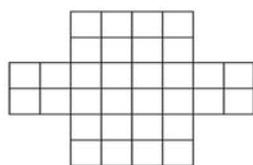


(SPr2b)

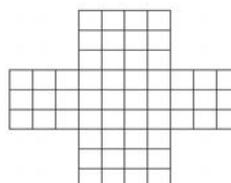
Which one of these cuboids has the greatest number of cubes? Explain your answer. (SPr2Ans)

Tasks with net diagram

The figures below show the nets of cuboids with one of its sides missing. Find the volume (number of cubes) of this net when folded:



(NPr3a)



(NPr3b)

Which one of these nets when folded can carry the least number of cubes? Explain your answer. (NPr3Ans)

Figure 2: Volume measurement tasks.

The coefficient of reliability Gronbach's Alpha of the five-part of test was very high ($\alpha=0.794$). Specifically, we found that the reliability of answers of students in the first four-part of questionnaire was $\alpha=0.782$ and the reliability of answers in volume measurement tasks was $\alpha=0.810$.

Data Analysis

Students correct responses in volume measurement tasks were marked with 1 and incorrect response with 0. However, the marks to responses of the questions: "Which one of these cuboids has the greatest number of cubes? Explain your answer." and "Which one of these nets when folded can carry the least number of cubes? Explain your answer." were: 1 for fully correct response, 0.5 for partly correct response (wrong explanation) and 0 for incorrect answer. We used the classification of errors made in previous studies (Ben Chaim et al., 1985; Battista & Clements, 1996) to code the students' errors while solving the volume tasks with 3-D cube array diagram and net diagram.

To answer the research questions of this study, four different analyses were conducted: a Regression Analysis, an Implicative Statistical Analysis with the use of the computer software CHIC (Bodin, Coutourier, & Gras, 2000), an Analysis of Variance one way and a Crosstabs Analysis. The implicative statistical analysis is a method of analysis that determines the similarity connections and the implicative relations of factors.

RESULTS

We used regression analysis with independent variable students' efficacy beliefs to solve volume measurement tasks (answers of students in forth part of test) and dependent variable their general volume measurement performance in the test. We found that students' efficacy beliefs to solve volume measurement tasks can be a statistically significant predictor of their performance in the test (10,1%). Furthermore, we examined the predictive role of students' efficacy to solve verbal

volume measurement tasks to their performance in these tasks and regression analysis confirmed that (6%). Additionally, students' efficacy to solve volume measurement tasks with 3-D diagram can be a statistically significant predictor of their performance in one of these tasks (3%). We also found that students' efficacy to solve volume measurement tasks in net diagram predicted only 4% of their performance in these tasks.

To examine the relationships between students' efficacy beliefs to solve volume measurement tasks, their performance in these tasks which were given in different representations and their errors in dealing with 3-D cube arrays and net diagrams, we employed the statistical implicative analysis for the data of this study and gave us the similarity diagram (see figure 3), which allowed for the grouping of the tasks and the statements based on the homogeneity by which they were handled by students.

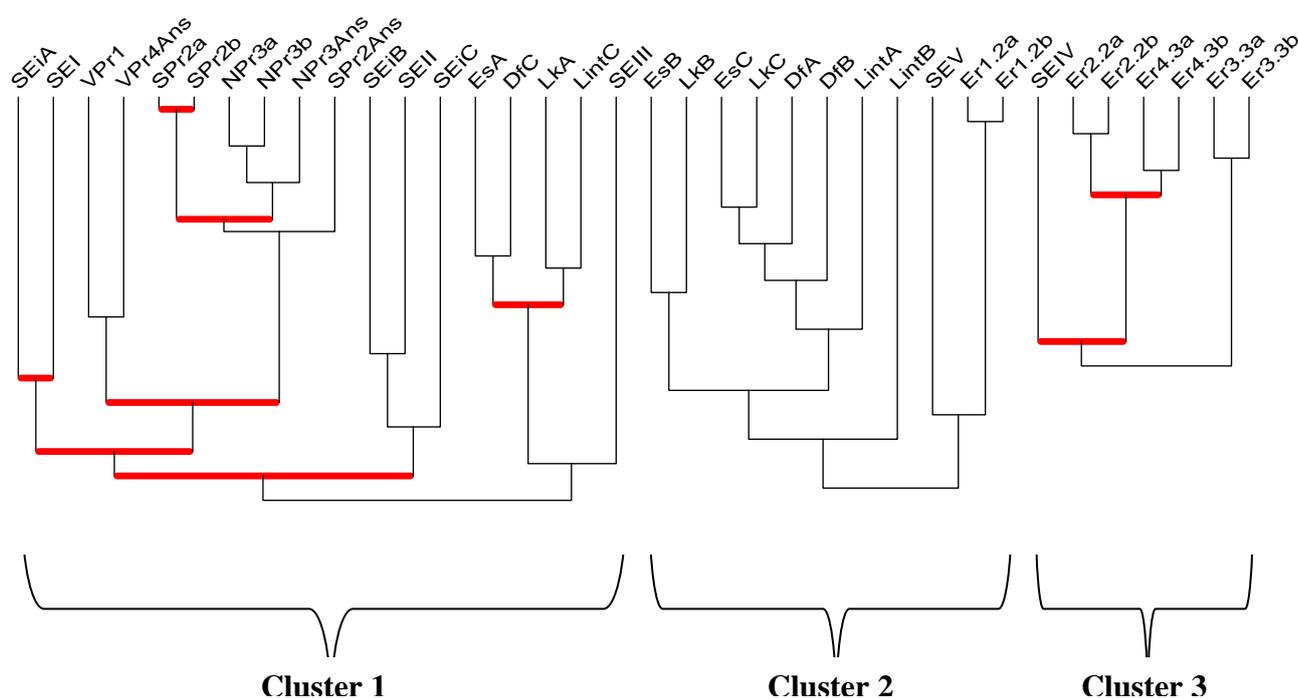


Figure 3: Similarity diagram of students' responses to the four-part of test.

Note: The similarities in bold color are important at level of significance 99%.

In figure 3, three distinct clusters of variables were formed. The first cluster consists of correct responses of students to volume measurement tasks and high efficacy beliefs, while the second and the third cluster consist students' errors and low efficacy beliefs. More specifically, the first cluster involved five similarity groups. The first group included the two statements of high efficacy beliefs to solve all volume measurement tasks and verbal tasks. The second group involved the verbal volume measurement tasks, while volume measurement tasks with 3-D cube array diagram and net diagram formed the third similarity group. These groups provided further support that different cognitive processes were required in order to solve verbal volume measurement tasks and volume measurement tasks with diagram.

However, their similarity connection indicated that equivalent content knowledge was needed to develop volume measurement ability in different representations. The fourth group included the three statements of high efficacy beliefs to solve all volume measurement tasks, tasks with 3-D cube array diagram and tasks with net diagram. Finally, the fifth group of the first cluster involved mainly four statements which referred to students' evaluation for verbal tasks as easy and interesting and for tasks with net diagram as difficult and less interesting. All above groups of similarity of the first cluster show that students with high efficacy beliefs to solve volume measurement tasks in different representations solved these tasks in a similar way. Furthermore, these students assessed the verbal tasks as easy and interesting, while the task with net diagram as difficult and less interesting. It is hypothesised that students solved mainly verbal volume measurement tasks in their textbooks and so they had more experiences to solve these tasks than tasks with net diagram. Therefore, they felt more certain to solve familiar tasks than unfamiliar ones.

The second cluster involved two similarity groups. The first group mainly included four statements which referred to students' evaluation for tasks with net diagram as easy and interesting and for verbal tasks as difficult and less interesting. The second group involved the statement of low efficacy beliefs to solve volume measurement tasks and the wrong strategy: count the number of faces of cubes shown in diagram, which used from students to solve tasks with 3-D cube array diagram. The third cluster involved the statement of lowest efficacy beliefs to solve volume measurement tasks and errors to tasks with diagram. From the second and third cluster indicated that different cognitive processes were required to calculate the number of faces of cubes shown in 3-D cube array diagram and in net diagram. However, in the case of errors: count the number of faces of cubes shown in diagram and double that number, similar cognitive processes were required to apply it in 3-D cube array diagram and in net diagram.

The sample of this study was clustered into three groups according to their volume measurement performance in the tasks of the fifth part of the test. The performance of the three clusters of students was examined in respect to their efficacy beliefs to solve volume measurement tasks. The comparison of the means by one way ANOVA indicated statistically significant differences between these groups ($F_{(2,169)}=6.240$, $p=0.002$) at efficacy beliefs towards volume measurement tasks. Using Bonferroni procedure, we found only statistical significant differences at efficacy beliefs between students with the lowest performance ($\bar{X} = 3.10$) and highest performance ($\bar{X} = 4.18$) in volume measurement tasks. Therefore, high-ability students have stronger efficacy beliefs towards volume measurement tasks than low-ability students.

However, at the same time, according to the results of the crosstabs analysis, students who solved the tasks of test correctly or wrongly indicated both very high efficacy beliefs and very low efficacy beliefs. We found that students who solved the tasks of the test correctly had more accurate self-efficacy than students who solved the tasks

of the test wrongly. More specifically, high-ability students were more accurate in their efficacy beliefs towards tasks with net diagram in relation to their performance in these tasks (73% of students who solved the tasks with net diagram correctly indicated very high and high efficacy beliefs and only 7.5% of them indicated very low and low efficacy beliefs). The tasks with net diagram considered as an unfamiliar form of the volume measurement tasks for the students, because they did not solve any similar tasks in their mathematics textbooks. Also, crosstabs analysis showed that low ability students were more accurate in their efficacy beliefs towards verbal tasks in relation to their performance in these tasks (37% of students who solved verbal tasks wrongly indicated very high and high efficacy beliefs and 35% of them indicated very low and low efficacy beliefs). The verbal tasks are more familiar to the students, since their mathematics textbooks have a number of these tasks.

Additionally, the sample of this study was clustered into five groups according to their efficacy beliefs towards volume measurement tasks. The efficacy beliefs to solve volume measurement tasks of the five clusters of students were examined in respect to their general volume measurement performance. The comparison of the means by one way ANOVA indicated statistically significant differences between these groups ($F_{(5,166)}=3.697$, $p=0.003$) on volume measurement performance. Using Bonferroni procedure, students with very high efficacy beliefs ($\bar{X}=2.43$) and students with very low efficacy beliefs ($\bar{X}=0.55$) differed significantly in their general volume measurement performance.

DISCUSSION

The purpose of the present study was to investigate the relationship between students' efficacy beliefs to solve volume measurement tasks in different representations and their performance in these tasks. We found that students' efficacy beliefs to solve volume measurement tasks was a statistically significant predictor of the general volume measurement performance of students. The predictive role of efficacy beliefs was indicated from various studies in different concepts of mathematics (Pajares & Miller, 1994; Pajares & Kranzler, 1995; Nicolaou & Philippou, 2007).

In the similarity diagram three distinct clusters of variables were formed. The first cluster included students who solved correctly the tasks of the test and indicated very high and high efficacy beliefs towards volume measurement tasks, whereas the second and the third group involved students who used wrong strategies to solve volume measurement tasks with 3-D cube array diagram and net diagram and indicated very low and low efficacy beliefs towards volume measurement tasks. Specifically, these different similarity groups which were formed show that the confidence with which students approached volume measurement problems connected and had direct effects on their volume measurement performance.

We found, also, that high-ability students had stronger and more accurate efficacy beliefs towards volume measurement tasks in comparison to low-ability students.

These findings confirm the earlier results by Pajares and Kranzler (1995) and Zimmerman et al. (1992). Furthermore, high ability students had more accurate efficacy beliefs towards volume measurement tasks with net diagram which were unfamiliar, whereas low-ability students had more accurate efficacy beliefs towards verbal volume measurement tasks which are more familiar to them.

Moreover, students who had high efficacy understand the volume measurement tasks better than the students who have low efficacy beliefs. This finding confirms the results of the study of Schunk and Hanson (1985). Also, students with high efficacy beliefs tend to assess the verbal tasks as easy and interesting, whereas the tasks with net diagram as difficult and less interesting. Therefore, these students' perceptions probably play an important role to their volume measurement performance and/or the development of their efficacy beliefs. This finding needs to be further explored.

In conclusion, the above findings about the predictive role of efficacy beliefs towards volume measurement tasks in different representations are very important in mathematics teaching and learning. Efficacy beliefs is an important component of motivation and behaviour (Pajares, 1996) and thus teachers need to develop ways to enhance efficacy beliefs of students of varied abilities. More specifically, high ability students need to solve "new" and creative tasks in which they will give the necessary attention and low ability students need to solve more easy and familiar tasks in which they can succeed.

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MOTIVATION FOR LEARNING MATHEMATICS IN TERMS OF NEEDS AND GOALS

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This article suggests a framework for analysing students' motivation for learning mathematics. In the present paper, motivation is defined as a potential to direct behaviour. This potential is structured through needs and goals. The author examines students' motivation in terms of needs and goals, and the emphasis is on the psychological needs for competence and autonomy. The proposed theoretical framework as an analytical tool is useful in describing the students' goals and changes in goals in details. It could also contribute to increased insight into relations between different aspects of instructional designs and the students' motivation for learning mathematics. The usefulness of the theoretical framework will be illustrated with some findings from the study.

INTRODUCTION

In mathematics education there has not been done much work on people's motivation to date (Evans & Wedege, 2004; Hannula, 2006). Only a few researchers have distinguished between intrinsic and extrinsic motivation in mathematics (Goodchild, 2001; Holden, 2003; Middleton & Spanias, 1999), or between task orientation and ego orientation (Nicholls, Cobb, Wood, Yackel, & Patashnick, 1990; Yates, 2000). Some mathematics educators have discussed students' motivation under the terms of motivational beliefs (Kloosterman, 1996; Op't Eynde, De Corte, & Verschaffel, 2002) and interest (Köller, Baumert, & Schnabel, 2001; Schiefele & Csikszentmihalyi, 1995). Evans and Wedege (2004; , 2006) consider people's motivation and resistance to learn mathematics as interrelated phenomena.

Hannula (2006) points out that many of the above approaches fail to describe the quality of the individual's motivation for learning mathematics in sufficient detail. He suggests that the reason for this is that the authors' approaches aim to measure predefined aspects of motivation, not to describe it (p. 166). Hannula developed a theoretical foundation of motivation as a structure of needs and goals, and his study shows that the students' goals vary a lot from person to person. The aim of this article is to present (develop) a theoretical framework for analysing the students' motivation for learning mathematics, in terms of needs and goals. The article reports on a particular aspect of a study where the focus is the development of Norwegian upper secondary school students' motivation for learning mathematics when they experience an inquiry mathematics teaching approach. The study followed a design-research approach in that it involved both instructional design and classroom based

research (Cobb, 2001). I collected a large and varied pool of data (participant observation, semi-structured interviews, videotapes of students working, conversations with the teacher, students' diaries, collection of material, assessment) on seven of the students. The focus of this article is the development of theory. Some findings from the study will be presented, mainly to illustrate the usefulness of the theoretical framework. Due to space constraint, the original data and analyses cannot be included. The interested reader should return to original papers.

MOTIVATION

Motivation is defined in different ways in the literature of (achievement) motivation, and I have chosen to use the following definition:

Motivation is a potential to direct behaviour that is built into the system that controls emotion. This potential may be manifested in cognition, emotion and/or behaviour. (Hannula, 2004, p. 3)

Motivation is considered as a potential to direct behaviour, and therefore, my focus is on the orientation of motivation. According to the definition, students' motivation may be manifested in cognition, emotion and/or behaviour. For example, a student's motivation to get a good grade in mathematics may be manifested in happiness (emotion) if he or she scores high on a test. It may also be manifested in studying for a test (behaviour) and in new conceptual learning (cognition) when studying for the test. Needs are specified instances of the potential to direct behaviour (Hannula, 2004). Psychological needs that are often emphasised in educational settings are competence, relatedness (or social belonging) and autonomy (e.g. Boekaerts, 1999; Ryan & Deci, 2000). I have chosen to define motivation as a potential to direct behaviour and therefore the orientation of motivation becomes central. Thus it is necessary to add a more fine grained conceptualization of motivation focusing on needs and goals.

Self Determination Theory and needs

Self Determination Theory (SDT) is a general theory of motivation that focuses on psychological needs, and I have chosen to use Ryan and Deci's (2002) definition of needs. Before presenting the definition, I will give a short presentation of the theory. Most contemporary theories of motivation assume that people engage in activities to the extent that they believe the behaviours will lead to desired goals or outcomes (Deci & Ryan, 2000). Within Self determination theory one is concerned about the goals of the behaviour and what energizes this behaviour. SDT is founded on three assumptions. The first assumption is that human beings have an innate tendency to integrate. *Integrating* means to forge interconnections among aspects of one own psyches as well as with other individuals and groups in his or her social world:

...all individuals have natural, innate and constructive tendencies to develop an even more elaborated and unified sense of self. (Ryan & Deci, 2002, p. 5)

This assumption of active, integrative tendencies in development is not unique to SDT. However, specific to this theory is that this evolved integrative tendency cannot be taken for granted. The second assumption in SDT is that social-contextual factors may facilitate and enable the integration tendency, or they may undermine this fundamental process of the human nature:

...SDT posits that there are clear and specifiable social-contextual factors that support this innate tendency, and that there are other specifiable factors that thwart or hinder this fundamental process of human nature. (Ryan & Deci, 2002, p. 5)

In other words, there is a dialectic between an active organism and a dynamic environment (social context) such that the environment act on the individual, and is shaped by the individual. To describe and organize the environment as supporting versus thwarting the integrative process, the concepts of needs are used. *Needs* are defined through optimal functioning (growth and well-being), and I have chosen to use the following definition:

There are necessary conditions for the growth and well-being of people's personalities and cognitive structures, just as there are for their physical development and functioning. These nutriments are referred to within SDT as basic psychological needs. (Ryan & Deci, 2002, p. 7)

Looking back at Hannula's definition, psychological needs are specified instances of the general potential to direct behaviour. The third assumption in SDT is that human beings have three basic psychological needs, the needs for competence, relatedness and autonomy (Deci & Ryan, 2000; Ryan & Deci, 2002). Within SDT, competence, relatedness and autonomy are defined in the following way:

Competence refers to feeling effective in one's ongoing interactions with the social environment and experiencing opportunities to exercise and express one's capacities (Ryan & Deci, 2002, p. 7). *Relatedness* refers to feeling connected to others, to caring for and being cared for by others, to having a sense of belongingness both with other individuals and with one's community (Ryan & Deci, 2002, p. 7). *Autonomy* refers to being the perceived origin or source of one's own behaviour (Ryan & Deci, 2002, p. 8). (My italics in the three quotations)

According to the definition, competence is not an attained skill or capacity, but it is a felt sense of confidence and effectiveness in action. The individual feels and experiences competence in the specific situation, it is not a product that shall be used (Wæge, 2007). In that case it is different from the way it is used by Hannula (2002). Hannula defines competence as the individual's functional understanding and skills. He considers competence to be a product, something the individual could use. Relatedness, in the definition above, refers to the psychological feeling of being together with other persons in a secure community or unity. In a similar way as for the construct of competence, Hannula considers social belonging (or relatedness) to be a target to attain. It also includes a goal of social status in the group. Within SDT

relatedness refers to the students' feelings of belongingness with others. When individuals are autonomous they experience themselves as volitional initiators of their own actions. Cobb and colleagues (Cobb, 2000; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; diSessa & Cobb, 2004) use the concept of intellectual autonomy as a characteristic of a student's way of participating in the practices of a classroom community. They speak of the students' awareness and willingness to draw on their own intellectually capabilities when making mathematical decisions and judgments as they participate in mathematics activities. Hannula define autonomy as "the need to have control over own actions and to feel self-determining" (Hannula, 2002, p. 74). His definition differs from Ryan and Deci's definition in that it adds an aspect of having control over own actions.

The concept of needs is useful because it allows the specification of the social-contextual conditions that will facilitate motivation. According to SDT, students' motivation will be maximized within social contexts that provide them with the opportunity to satisfy their basic psychological needs for competence, autonomy and relatedness. I have chosen to use Ryan and Deci's definitions of the three psychological needs [1]. The data in the study did not give a basis for detailed analyses of the student's needs for relatedness and the goals the students' have in relation to this need. Therefore, the need for relatedness was not a focus in my study. In my study I focused on the students' needs for competence and autonomy. In his study, Hannula focuses on the three psychological needs for competence, relatedness and autonomy, but as I pointed out above, his definitions of the constructs differ from Ryan and Deci's definitions, which are the ones I have chosen to use.

Needs and goals structures

Hannula's definition of motivation (above) purports the potential to direct behaviour is structured through needs and goals. Needs and goals are specified instances of the potential to direct behaviour. According to Hannula, goals are derived from needs, and the difference between needs and goals is their different level of specificity. A need may be directed toward a relatively large category of objects, while a goal is directed toward a specific object (Hannula, 2004). For example, in my study, Berit realised her need for competence as a more specific goal of gaining a good grade. She translated her need for autonomy into the more specific goal of developing her own ideas, independently of the teacher. Another student, David, realised his need for relatedness as a goal to gain the mathematics teacher's confidence and respect.

According to Boekaerts, the students' goal structures are complex, and they tend to pursue multiple goals. The goals are related to each other, and pursuing one goal might be necessary to attain another goal or different goals may be seen as contradictory (Boekaerts, 1999; Shah & Kruglanski, 2000). Learning goals and performance goals are usually considered as contradictory to each other (Lemos, 1999; Linnenbrink & Pintrich, 2000), but Hannula's (2004) and my own findings (Wæge, 2007) indicate that these goals should not be seen as mutually exclusive

goals in mathematics education. To exemplify this I present an utterance of a student [2]:

Berit: [...] I think it has been pretty enjoyable. In the beginning I thought it was a bit difficult (Interviewer: Mm) because I was not used to this kind of teaching approach. [...] I think this mathematical approach is much better. The full-day test [3] was pretty special this time, because usually I didn't quite understand what I was doing {inaudible}. Do this, follow rules and things like that. This time I thought that I understood everything and I thought the test went very well. And then I get a 4[4] and when I didn't understand it I used to get a 5. But I almost think it's better to try to understand a little more and nevertheless get a lower grade. Anyhow, I think it is possible to increase the grade. It's only a new way of thinking. It's quite interesting, I think {laughing} strange, yes.

My analysis of Berit shows that she has a specific goal of relational understanding in mathematics (Skemp, 1976). Her sense of mastery and her feeling of succeeding in mathematics are higher when she experiences that she understands the mathematics problems, than when she uses rules without understanding. Another important goal for Berit is to get good grades on the mathematics tests. Her goals of relational understanding in mathematics and good grades in mathematics support each other mutually. Getting good grades are important to Berit, but relational understanding in mathematics is the most important goal for her.

FIVE MOTIVATION VARIABLES

There is a serious methodological problem with research on a mental construct like motivation. Students' motivation cannot directly be observed, and thus measured, and it needs to be reconstructed through interpretation of the observable. I have developed an instrument to assess students' motivation for learning mathematics in terms of cognition, emotion and behaviour. In doing this I focus on the five sets of motivational variables that Stipek, Salmon, Givvin & Kazemi (1998) used in their study entitled: "The value (and convergence) of practices suggested by motivation research and promoted by mathematics education reformers" [2]. These are the students'

1. focus on learning and understanding mathematics concepts as well as on getting right answers;
2. enjoyment in engaging in mathematics activities;
3. related positive (or negative) feelings about mathematics.
4. willingness to take risks and to approach challenging tasks;

5. self-confidence as mathematics learners;

All these motivation variables figure prominently in the achievement motivation literature and in the mathematics reform literature. The five motivation variables are closely related to the needs for competence and autonomy. The first and the fourth variable, students' focus on learning and their willingness to take risks and approach challenging tasks, are closely related to the students' need for competence. Deci and Ryan (2002) claim that the students' need for competence leads them to seek adequately challenging mathematics tasks and to attempt to maintain and develop their mathematical understanding and skills. In my analysis I distinguish between students' learning orientation and performance orientation (Nicholls, Cobb, Wood, Yackel, & Patashnick, 1990). In addition, I also make a distinction between relational understanding and instrumental understanding (Skemp, 1976). The fifth variable, students' self-confidence, is related to students' willingness to approach tasks (Stipek, Salmon, Givvin, & Kazemi, 1998). The second and the third variable, students' enjoyment and their feelings about mathematics, are related to the students' intrinsic motivation in mathematics. According to Deci and Ryan (2002), intrinsic motivation represents a prototype of self-determined activity. They suggest that there is a strong connection between people's intrinsic motivation and their need for autonomy and competence. Mathematics classrooms that support the students' needs for autonomy and competence will enhance their intrinsic motivation in mathematics. Contextual events that students experience as thwarting satisfaction of these needs will undermine their intrinsic motivation.

In analysing the data, I assess these five motivation variables and I analyse the needs and goals of the students in relation to these specific motivational orientations. More specifically, the analysis is divided into two parts. First I analyse the data according to the five motivation variables. Although the variables can be seen as interrelated they are analysed separately in order to provide detailed insight into the students' motivation for learning mathematics. In the second part, I analyse the student's needs and goals in relation to these five specific motivational orientations. Furthermore, my emphasis is on the students' need for autonomy and competence.

TEACHING APPROACH

The teaching approach in the study was intended to give more space for the students to satisfy their needs for competence and autonomy, than teacher-centred and teacher-controlled teaching approaches. In the study attention was given to the development of students' mathematical thinking and reasoning. Our (the teacher and I) task was to create instructional activities that supported the development of both collective mathematical meanings evolving in the classroom community and the mathematical understanding of the individual student. We tried to support

...the collective learning of the classroom community, during which taken-as-shared mathematical meanings emerge as the teacher and students negotiate interpretations and solutions (Gravemeijer, Cobb, Bowers, & Whitenack, 2000, p. 226).

The teacher always asked the students “What did you think when you solved this problem? What strategies did you use?” In the written tasks we developed, the students were frequently asked to explain their solutions and strategies, and the students were invited to find several solution strategies to a problem. The teacher tried to promote a classroom microculture (Cobb, Boufi, McClain, & Whitenack, 1997) where active participation and encouragement to understand were emphasised. In some of the instructional activities the students had to develop their own ideas, apply the mathematics in realistic situations and draw their own conclusions. Collaboration was important in our teaching approach. When the student’s were given problems they were not familiar with, we wanted the students to collaborate. The students had an opportunity to experience themselves and their peers as active participants in creating mathematical insight. Every student brought a personal contribution at his or her level. These elements of our design study were suitable for meeting the students need for competence, autonomy and relatedness.

THE THEORETICAL FRAMEWORK – SOME KEY POINTS

The proposed theoretical framework for analysing students’ motivation is useful in describing students’ goals and changes in goals in detail. The framework is useful in clarifying students’ notion of what it might mean to understand in mathematics. For example, the analysis of Berit shows that for her, to understand means to know what to do and why. We may also understand the relations between different goals through the use of such a framework. The complete analysis of Berit shows that there was a strong connection between her goal of relational understanding and her goal of finding her own solutions. She believes that finding own strategies for solving problems helps her in learning and understanding mathematics. As I described above, her goal of getting a good grades in mathematics and mastery goal, in this case a goal of relational understanding in mathematics, mutually supported each other.

The study shows that students’ motivation for learning mathematics, although it is considered relatively stable, can be influenced by changes in the teaching approach. The case of Berit shows that students’ motivation for learning mathematics might change in a relatively short time. Within the first semester of the school year, Berit changed her goal of instrumental understanding (Skemp, 1976) to a goal of relational understanding in mathematics.

We may also understand the relations between different aspects of the instructional designs developed in the study and the students’ motivation for learning mathematics in terms of needs and goals through this framework. The analysis of Berit indicate that a combination of working with mathematics problems and routine tasks from the textbook, and the fact that the students were given opportunities to find their own

solutions and rules for solving the problems, in collaboration with peer students and with guidance from the teacher, contributed to a sense of understanding and mastery with Berit.

I perceive that the theoretical framework as an analytical tool captured the complexity and the richness of the students' motivation in detail, and the tool made it possible for me to present detailed descriptions of the students' motivation for learning mathematics.

NOTES

1. See Wæge (2007) for a detailed description of my interpretation of the definitions.
 2. Key to transcripts: [...] extracts edited out of transcript for sake of clarity; {inaudible} unclear words; {text} comments about context or emotional behaviour like laughing; {.} 1 sec pause, {...} 2 sec pause, and so on.
- The interviews took place in Norwegian. I have tried to translate from colloquial Norwegian to colloquial English, but it does not give an exact word for word translation. My analysis took place without any translation, that is, I analysed the transcripts in the original language.
3. At the end of each semester, the students have an all-day test in mathematics.
 4. 1 is the lowest grade and 6 is the highest.

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MATHEMATICAL MODELING, SELF-REPRESENTATION AND SELF-REGULATION

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The aim of the present study was to investigate the improvement of students' self-representation about their self-regulatory performance in mathematics by using mathematical modeling. Three materials were developed and administered at 255 11th years old students, for mathematical performance, self-representation and the use of self-regulatory strategies for problem solving. A web page with the proposed model (the model of Verschffel, Greer & De Corte, 2000) was constructed and used individually by students. Results indicated that the program created a powerful learning environment in which students were inspired in their own experiences. Although the program improved their cognitive and self-regulatory performance, it reproduced the differences among students in respect to their cognitive and metacognitive performance.

Keywords: self-regulation, self-representation, mathematical modeling

In the last decades, children's early understanding of their own as well as others mental states has been intensively investigated, reflecting growing interest for the concept of metacognition (Bartsch & Estes, 1996). In psychological literature, the term metacognition refers to two distinct areas of research: knowledge about cognition and self-regulation (Boekaerts, 1997). Self-regulation refers to the processes that coordinate cognition. It reflects the ability to use metacognitive knowledge strategically to achieve cognitive goals, especially in cases where someone has to overcome cognitive obstacles.

As regards the relationship between academic self-concept and academic achievement, extant literature supports both direct and indirect relationships between them; however, the range of correlations reported is a function of several factors (Guay, Marsh & Boivin, 2003). Age is a factor that affects this relationship since young students, academic self-concept is usually very positive and not highly correlated with external indicators, such as skills and achievement (Guay et al., 2003). Veenman and Spaans (2005) assumed that metacognitive skills initially develop on separate islands of tasks and domains. Beyond the age of 12, these skills will gradually merge into a more general repertoire that is applicable and transferable across tasks and domains. The present work is concentrated on the improvement of metacognitive performance on the domain of mathematics and more specifically on the improvement of self-regulatory behavior.

Learning mathematics, as an active and constructive process, implies that the learner assumes control and agency over his/her own learning and problem solving activities (De Corte, Verschaffel & Op't Eynde, 2000). Knowing when and how to use cognitive strategies is an important factor to successful word problem solving (Teong, 2002). Metacognitive behavior can be applied in every stage of the problem solving activity (Lerch, 2004). For example before starting solving a particular problem, students can ask themselves questions like what prior knowledge can help them develop a solution plan for the particular task; during the application of the solution plan the students monitor their cognitive activities and compare progress against expected goals. Finally, after reaching a solution, the students may need to look back, to check for the reasonableness of outcomes and integrate newly acquired knowledge to existing.

Problem solving procedure and the use of mathematical modeling

Studies on solving mathematical word problems refer to various conditions that cause transfer to occur, for example, providing solved examples (e.g. Bassok & Holyoak, 1989), having a scheme (Nesher & Hershkovitz, 1994), and providing feedback (Hoch & Loewenstein, 1992). The first step in solving a problem is to encode the given elements (Davidson & Sternberg, 1998). Encoding involves identifying the most informative features of a problem, storing them in working memory and retrieving from long-term memory the information that is relevant to these features. Incomplete or inaccurate metacognitive knowledge about problems often leads to inaccurate encoding and could generate learning obstacles.

A specific strategy frequently taught in math classes in order to enhance problem solving ability, is to use analogy in order to create a mental model of similar problems. In this regard, the students are expected to extract the relevant facts from the statement of the problem, compare it to their knowledge base, relevant to the problem domain, and recognize similarities between the new problem and problems they have previously encountered, and abstract the proper entities and principles. Empirical findings show that students fail to see the underlying principles unless they are explicitly pointed out (Panaoura & Philippou, 2005).

The modeling of open-ended problems have been of interest to mathematics educators for decades. Mathematical modeling of problem solving is a complicated procedure which is divided into different stages (Mason, 2001). When a mathematical modeling task is offered in a school the goal generally is not that students learn to tackle only that particular task. Rather, students are expected to recognize classes of situations that can be modeled by means of a certain mathematical concept, relation or formula, and to develop some degree of routine and fluency in mapping problem data to the underlying mathematical model and in working through this model to obtain a solution (Van Dooren, Verschaffel, Greer & De Bock, 2006).

A characteristic is that the modeling process is not a straightforwardly sequential activity consisting of several clearly distinguishable phases. Modellers do not move sequentially through the different phases of the modeling process, but rather run through several modeling cycles wherein they gradually refine, revise or even reject the original model. The present paper discusses the impact of the use of the mathematical model proposed by Verschaffel et al. (2000) on the development of students' self-representation about their self-regulatory behavior in mathematics. The main stages of the model are: 1) Understanding the phenomenon under investigation, leading to a model of the relevant elements, relations and conditions that are embedded in the situation (situation model), 2) Constructing a mathematical model of the relevant elements, relations and conditions available in the situation model, 3) Working through the mathematical model using disciplinary methods in order to derive some mathematical results, 4) Interpreting the outcome of the computational work to arrive at a solution to the real – word problem situation that gave rise to the mathematical model, 5) Evaluating the model by checking if the interpreted mathematical outcome is appropriate and reasonable for the original problem situation, and 6) Communicating the solution of the original real – word problem.

At the first phase of the problem solving procedure by the use of the mathematical model students have to consider and decide what elements are essential and what elements are less important to include in the situation model. In the next phase, the situation model needs to be mathematised i.e. translated into mathematical form by expressing mathematical equations involving the key quantities and relations. Students need to rely on another part of their knowledge base, namely mathematical concepts, formulas, techniques and heuristics. After the mathematical model is constructed and results are obtained by manipulating the model, numerical result needs to be interpreted in relation to the situation model. At this point, the results also need to be evaluated against the situation model to check for reasonableness. As a final step, the interpreted and validated result needs to be communicated in a way that is consistent with the goal or the circumstances in which the problem arose.

Nowadays problem solving skills have become a prominent instructional objective, but teachers often experience difficulties in teaching students how to approach problems and how to make use of proper mathematical tools. Many teachers of mathematics teach students to solve mathematical problems by having them copy standard solution methods. It comes as no surprise, therefore, that many students find it difficult to solve new problems, especially problems within a context (Harskamp & Suhre, 2006). Attempts to improve problem solving should focus on episodes students neglect when solving problems. The aim of the present study was to develop students' (5th grade) problem solving ability and to enhance their ability to self-regulate their cognitive performance in order to overcome cognitive obstacles when they encounter difficulties

while trying to solve mathematical problems. One of the main emphases was to oblige students reflect on their cognitive processes while trying to solve the problems and encounter difficulties in order to self-regulate their behavior. We hypothesized that the development of self-representation in order to be more accurate regarding the students' strengths and limitations would improve their self-regulatory behavior in mathematics. Especially for the problem solving procedure we hypothesized that the better distinction of problems and the clustering of those problems according to their similarities and differences would have as a consequence the better transfer of knowledge and strategies from the one domain to the others and from general situation to the specific ones.

METHODOLOGY

Participants: Data were collected from 255 children (107 experimental group and 148 control group), in Grade 5 (11 years old) from five different urban elementary schools. The participation at the program were voluntary because we had used the extra time students stayed at school for the program of the Ministry of Education, called "day-long school".

Procedure: The main emphasis was on the development of the program for the use of the proposed mathematical model, the training of students on the model and the evaluation of its results. At the first phase of the study three materials were constructed for pre and post test. The first one was about students' self-representation, the second for mathematical performance and the third one for their behavior while trying to solve mathematical problems. The first one comprised of 40 Likert type items of five points (1 = never, 2 = seldom, 3 = sometimes, 4 = often, 5 = always), reflecting students' self-representation about mathematical learning (e.g. "I can better explain my solution for a problem when I use a diagram", "I can easily compare two pictures in order to find their similarities"). The reliability was very high (Cronbach's alpha was .87).

The second questionnaire comprised of 20 mathematical tasks on counting, geometry, statistics and problem solving (e.g. "How the area of a square, side 4cm, will be changed if the side is doubled", "Construct the bigger four digit number with the digits 9 and 3", "In our neighborhood every year since 2000 we organize a celebration, For the three following years, after the first one it did not organize. At what date (chronology) did it start again?") All items in the mathematical performance questionnaire were scored on a pass-fail basis (0 and 1). The reliability was high (Cronbach's alpha was 0.85).

The third questionnaire comprised of ten couples of sentences and students had to choose which one expressed better their cognitive behavior while they were encountering a difficulty in problem solving (a. When I explain to my friend how to solve a problem, I prefer to use a diagram, b. When I explain to my friend how to solve a problem I prefer to do it verbally). All the questionnaires were first used at a pilot study in order to examine their construct validity.

Then an intervention program was developed in order to propose the use of the mathematical model (Figure 1) for problem solving, proposed by Verschaffel et al. (2000). The emphasis was on the understanding that different stages of problem solving would have as a consequence the use of different cognitive procedures and that the cognitive obstacles could be encountered by realizing the cognitive interruptions at one or more of those stages and mainly by self-regulating the cognitive performance. For example a self-regulatory strategy is the ability to recognize the “inner” mathematical similarities and differences of mathematical problems in order to transfer cognitive and metacognitive strategies among different domains. For the purpose of the project we had constructed a web page which was visited individually by each student of the experimental group (107 students) during 20 “meetings”. One of the main emphases was to oblige students rethink their cognitive processes while trying to solve the problems and encounter difficulties in order to monitor their performance.

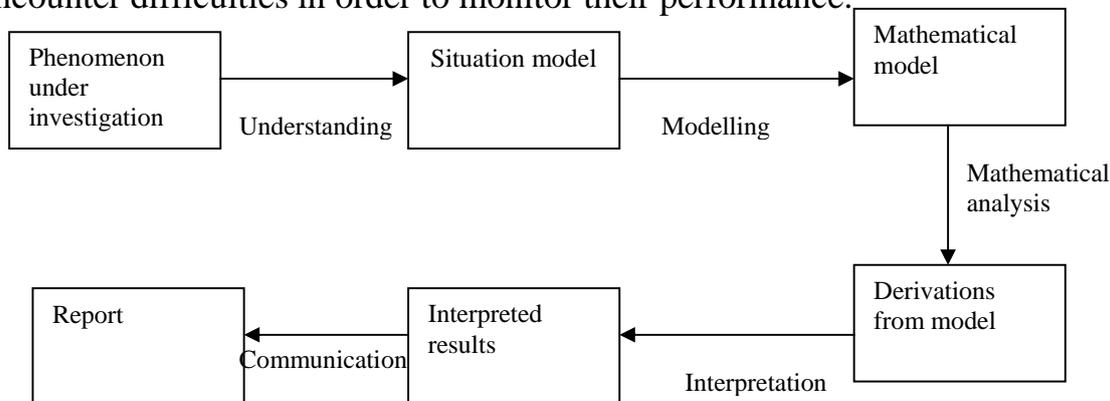


Figure 1: The mathematical model proposed by Verschaffel et al. (2000)

We had organized twenty “individual meetings” of the students with the webpage in order to work with the model (almost 20 minutes each meeting). Using the model used the first four “meetings” for the familiarization with the environment of the computer and for understanding the whole idea of the webpage for the problem solving procedure. The ten following “meetings” concentrated on different stages of the proposed mathematical model. For example at the stage of “understanding the problem” students had to solve problems with not enough data, or with more than the necessary data, they had to answer specific questions about the data of the problem, they had to explain in their own words the problem, to summarize it etc. At the stage of “modeling” they had to work on the classification of mathematical problems by explaining the criteria they used in order to classify the problems. There were problems with the same situational characteristics or the same context in order to oblige students to be concentrated on the structural mathematical characteristics. At the last six “meetings” students should solve mathematical problems by using all the stages of the mathematical model. In each stage the “cartoon” that was the hero of the web page asked questions such as “How did you get that? This isn’t a better solution? (for a proposed solution). Do you have any better

solution?”, in order to force students to self-regulate their cognitive performance. We wanted to have a reflection at all the stages of their work. The students’ responses were recorded automatically at a database with details such as when they had worked on the specific task and for how long. The whole procedure is presented at Figure 2.

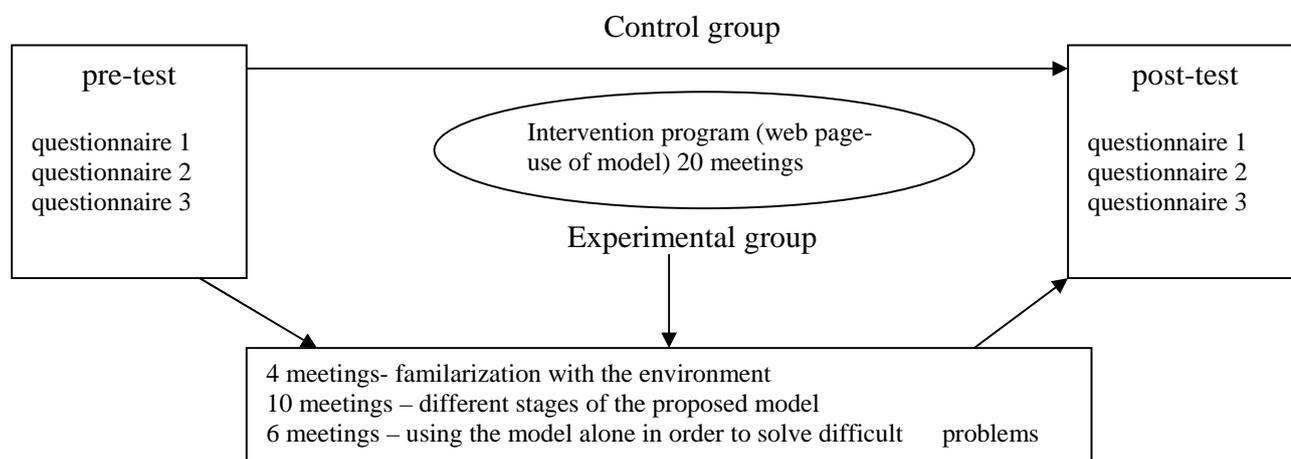


Figure 2: The development of the intervention program

RESULTS

The data about self-representation (1st questionnaire) were first subjected to exploratory factor analysis in order to examine whether the presupposed factors that guided the construction of the items of the first questionnaire were presented in the participants’ responses. This analysis resulted in 6 factors with eigenvalues greater than 1, explaining 65.56% of the total variance. After the content analysis, according to the results of the exploratory factor analysis items were classified in the following factors: F1: general self-image about mathematics, F2: self-representation about problem solving abilities, F3: self-representation about the strategies used in order to self-regulate the cognitive performance, F4: self-representation about students’ spatial abilities in mathematics, F5: self-representation about the degree of concentration on problem solving procedure, F6: the preference for different types of representations

We concentrated on the three factors which were related with self representation in respect to problem solving and self-regulation (F1, F2 and F3). The comparison of the means of the three factors between the pre and post tests for the experimental and the control group were statistically significant in all cases ($p < 0.001$). Nevertheless the improvement was highest for the experimental group in the case of the second and the third factors (Table 1). It is obvious the increase of the control group as well as a consequence of the age development and the impact of teaching and learning (those were factors that could not be controlled). However the improvement was in all cases higher in the case of the experimental group.

	pre - test		post - test	
	experimental	control	experimental	control
F1	3.92	4.00	4.00	4.07
F2	3.22	3.25	3.69	3.57
F3	2.76	2.78	3.35	3.20

Table 1: The means of the experimental and the control group for the three factors at the pre and post test.

At the same time for the experimental group the improvement was highest in the case of the general mathematical performance ($\bar{X}_{1\text{exp}}=0.27$, $\bar{X}_{2\text{exp}}=0.63$, $\bar{X}_{1\text{control}}=0.27$, $\bar{X}_{2\text{control}}=0.52$) and the problem solving performance ($\bar{X}_{1\text{exp}}=0.20$, $\bar{X}_{2\text{exp}}=0.47$, $\bar{X}_{1\text{control}}=0.20$, $\bar{X}_{2\text{control}}=0.39$). Specifically the highest differences were found in the domain of geometry ($\bar{X}_{1\text{exp}}=0.28$, $\bar{X}_{2\text{exp}}=0.47$, $\bar{X}_{1\text{control}}=0.29$, $\bar{X}_{2\text{control}}=0.44$) and statistics ($\bar{X}_{1\text{exp}}=0.38$, $\bar{X}_{2\text{exp}}=0.69$, $\bar{X}_{1\text{control}}=0.38$, $\bar{X}_{2\text{control}}=0.64$). This result reveals the positive impact of the use of the specific mathematical model on the mathematical performance.

The most important in the case of self-representation is the accuracy of this feature in relation to the real mathematical performance. We have clustered, depended on cluster analysis, the participants in respect to their general self-image about their mathematical performance into three groups. The first group was consisted of 42 students with low self-image ($\bar{X}=2.55$), the second one of 82 students with medium self-image ($\bar{X}=3.26$) and the third one of 99 students with high self image ($\bar{X}=3.94$). There were statistically significant differences between the first and the third group at the initial phase (pre – test) in respect to their real mathematical performance ($F=4.716$, $df=2$, $p=0.01$, $\bar{X}_1=0.466$, $\bar{X}_2=0.543$, $\bar{X}_3=0.605$). After the program the difference of the groups regarding their general self-image in relation to their mathematical performance (post test) was significant only in the case of the experimental group ($F=4.447$, $df=2$, $p=0.01$, $\bar{X}_1=0.557$, $\bar{X}_2=0.6059$, $\bar{X}_3=0.699$). Those results indicated that most students had accurate self-image in respect to their real mathematical performance and they did not seem to overestimate their abilities. At the same time students' means at the classification of similar mathematical problems according to the mathematical structure of the problems were highest at the post test. The development was statistically higher in the case of the experimental group ($\bar{X}_1=0.29$, $\bar{X}_2=0.49$, $t=12.79$, $p<0.001$) than the control group ($\bar{X}_1=0.29$, $\bar{X}_2=0.41$, $t=11.69$, $p<0.001$). The difference between the two groups was statistically significant ($t=3.32$, $df=228$, $p<0.01$).

A part of the couples of sentences at the third questionnaire were about the self-regulatory strategies they use in order to encounter difficulties and cognitive obstacles at the problem solving procedure. For the self-regulatory strategies the difference of the

means between the two measurements was statistically significant ($t=2.93$, $df=98$, $p<0.01$, $\bar{X}_1=0.65$, $\bar{X}_2=0.69$) only in the case of the experimental group. That means that students tended to develop more self-regulatory strategies or tended to believe that they have to develop those strategies. Even the second learning situation is an important step for the change of cognitive and metacognitive behavior, as well.

Students of the experimental group were clustered according to their self-representation about problem solving ability and their general mathematical ability into three groups (low self-representation: 24 students, medium: 36 students, and high self-representation: 34 students). Analysis of variance (ANOVA) indicated that there was a statistically significant difference concerning their self-representation about the use of self-regulatory strategies in mathematics ($F_{2,93}=6.094$, $p=0.003$). As it was expected the mean of the group with the high self-representation was higher (0.80) than the other two groups (medium: 0.63 and low: 0.58). The most interesting result was that the students' with medium and low mathematical performance was increased after the program (low: $\bar{X}_1=0.83$, $\bar{X}_2=0.87$, medium: $\bar{X}_1=0.90$, $\bar{X}_2=0.94$, high: $\bar{X}_1=0.94$, $\bar{X}_2=0.94$). In the case of the improvement on the self-representation about the use of self-regulatory strategies for the three groups the changes were similar (low self-representation: $\bar{X}_1=0.50$, $\bar{X}_2=0.53$, medium self-representation: $\bar{X}_1=0.64$, $\bar{X}_2=0.67$, high self-representation: $\bar{X}_1=0.80$, $\bar{X}_2=0.84$). This stability or low increase may indicate that students realized their difficulties and limitations and did not tend to overestimate their abilities in using strategies.

DISCUSSION

Results confirmed that providing students with the opportunity to self-monitor their learning behavior in the case of encountering obstacles in problem solving through the use of modeling is one possible way to enhance students' self-representation about the self-regulatory strategies they use in mathematics and consequently their mathematical performance. It seems that the program with the use of the model created a powerful learning environment in which students were inspired in their own experiences. Nevertheless it is obvious that students with high self-representation about their mathematical abilities in the initial phase were at the same time students with the most self-regulatory strategies after the impact of the intervention program, as well. That means that although the program improved the metacognitive performance and the mathematical performance of the experimental group, further research is needed in order to find ways to change the initial differences among students.

For the development of a more accurate self-representation about mathematical performance and self-regulation in problem solving teachers must create a powerful learning environment, in which children are allowed and inspired to, their own learning experiences. According to the self-regulated learning approach students are self-regulating when they are aware of their capabilities of the strategies and resources

required for effectively performing a task (Paris & Paris, 2001). Learners, who decide to ask a more competent person for assistance when faced with a task, indicate that they realize their difficulties and try to find out ways to overcome them. The accurate self-representation about the strengths and limitations is a presupposition for the development of self-regulation. Instruction should mainly lead students to self-questioning as a systematic strategy in helping them control their own learning and organize by themselves the different occasions they may encounter. In the area of mathematics, a number of important questions about metacognition remain unanswered. Much more research is needed to study the different aspects of metacognition in a more systematic and detailed way. We suggest specifically that further research could focus on interactive computer programs that may be designed to provide feedback and hints to assist students in becoming more aware of their cognitive and metacognitive processes. It would be optimistic and naïve to claim that such types of intervention programs would develop the self-regulatory strategies of all students. Possibly different models and programs are suitable for different groups of students.

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ENDORISING MOTIVATION: IDENTIFICATION OF INSTRUCTIONAL PRACTICES

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This paper presents some results of a larger study that investigates the relationship between instructional practices in the mathematics classroom and students' motivation and their achievement in mathematics. Data were collected from 321 sixth grade students through a questionnaire comprised of three Likert-type scales measuring motivational constructs, a test measuring students' understanding of the fraction concept and an observation protocol for teachers' instructional practices in the classroom. Findings revealed the importance of multi-level modelling in the analysis of instructional practices suggested by achievement goal theory and mathematics education research that promote both students' motivation and achievement in mathematics.

INTRODUCTION

Research on achievement motivation provides substantial evidences of instructional practices that foster students' motivation (Anderman et al., 2002; Turner et al., 2002). These instructional practices are alike the ones developed by mathematics educators to achieve both learning and motivational outcomes (Stipek et al., 1998). Motivation is treated in mathematics education as a desirable outcome and a means to enhance understanding (Stipek et al., 1998). In broad, the socio-constructivist perspective on learning (Op't Eyndne et al., 2006) underlines the interplay between cognitive, motivational and affective factors but also it highlights the influence of the specific classroom context in the whole process.

In this respect, the present study investigates variations in instructional practices and their impact on students' achievement motivation and outcome. Understanding the interplay between the characteristics of a particular instructional setting, and students' achievement-related goals and outcomes is an important direction for both motivational and mathematics education research (Anderman et al., 2002; Stipek et al., 1998). In the next section we consider the basic concepts and define the research questions.

THEORETICAL BACKGROUND AND AIMS

Motivation

Motivation cannot directly be observed but it can be noticeable only by its interaction with affect, cognition and behaviour. Hannula (2006) defines motivation as the preference to do certain things and to avoid doing some others. In regards to students' motivation four basic theories of social-cognitive constructs have so far been identified: achievement goal orientation, efficacy beliefs, personal interest in the task, and task value beliefs (Pintrich, 2003). In this study we conceptualise motivation

according to achievement goal theory because it has been developed within a social-cognitive framework and it has studied in depth many variables which are considered as antecedents of students' motivation constructs. Some of these variables are students' competence based variables, such as need of achievement or fear of failure, self-based variables, such as self efficacy beliefs, and demographic variables, e.g. gender (Elliot, 1999). In addition, one of the strengths of goal orientation theory in understanding students' motivation is that it explicitly considers the role of teachers and instructional contexts in shaping students' goal orientations. Thus a major tenet of goal theory is that students' adoption of personal goals is influenced even in part, by the goal structures promoted by the classroom and boarder school environments (Anderman et al., 2002).

Achievement goal theory is concerned with the purposes-goals students perceive for engaging in an achievement-related behaviour and the meaning they ascribe to that behaviour. A mastery goal orientation refers to one's will to gain understanding, or skill, whereby learning is valued as an end in itself. In contrast, a performance goal orientation refers to wanting to be seen as being able, whereby ability is demonstrated by outperforming others or by achieving success with little effort (Elliot & Church, 1997). Recently, there has been a theoretical and empirical differentiation between performance-approach goals, where students focus on how to outperform others, and performance-avoidance goals, where students aim to avoid looking inferior or incompetent in relation to others (Cury et al., 2006).

These goals have been related consistently to different patterns of achievement-related affect, cognition and behaviour. Being mastery focused has been related to adaptive perceptions including feelings of efficacy, achievement, and interest (Anderman et al., 2002; Elliot & Church, 1997; Cury et al., 2006). Although the research on performance goals is less consistent, this orientation has been associated with maladaptive achievements beliefs and behaviours like low achievement, fear of failure and superficial cognitive commitment, i.e. the use of 'surface' learning strategies such as copying, repeating and memorizing (e.g. Cury et al. 2006). Efficacy beliefs encountered as an antecedent variable in the achievement goal theory, refers to the beliefs in one's capabilities to organize and execute the courses of action required to manage prospective situations (Bandura, 1997).

Instructional practices

Environmental factors are presumed to play an important role in the goal adoption process and eventually in students' achievement (Anderman et al., 2002). Elliot & Church (1997) underline that if the achievement setting is strong enough it alone can establish situation-specific concerns that lead to goal preferences for the individual, either in the absence of a priori propensities or by overwhelming such propensities.

Earlier studies on achievement goals specify various classroom instructional practices as contributing to the development of different types of goals and consequently, eliciting different patterns of motivation and achievement outcomes (e.g. Ames,

1992). Goal orientation theorists relying on a large literature on classroom motivational environments focus on six categories that contribute to the classroom motivational environment. The categories, represented by the acronym TARGET refer to task, authority, recognition, grouping, evaluation and time. *Task* refers to specific activities, such as problem solving or routine algorithm, open or closed questions in which students are engaged in; *Authority* refers to students' level of autonomy in the classroom; *Recognition* refers to whether the teacher values the progress or the final outcome of students' performance and how the teacher treats students' mistakes (as a part of the learning process or as cause for punishment); *Grouping* refers to whether students work with different or similar ability peers; *Evaluation* refers to how the teacher treats assessment, giving publicly grades and test scores, or focusing on feedback as a means for improvement and mastery; *Time* refers to whether the schedule of the activities is rigid or flexible.

This framework has been adapted and developed by goal theory researchers working within classroom context (Anderman et al., 2002; Turner et al., 2002). Using classroom observations and qualitative analysis, they found that instructional practices in classrooms in where students adopted mastery goals differed from instructional practices in classroom characterized by students' low mastery goals or high performance goals. Specifically, according to the task variable, in mastery oriented classrooms teachers used an active instructional approach, ensuring that all students participated in classroom talk and adapted instruction to the developmental levels and personal interests of their students, while in low mastery oriented classrooms, learning was processed by students listening to information and following directions (Anderman et al., 2002; Turner et al., 2002). Regarding authority, in high mastery oriented classrooms teachers engaged the class in generating the rules, while in low mastery oriented classrooms the teachers presented their rules to the students (Anderman et al., 2002). In high mastery classrooms teachers emphasized the intrinsic value of learning, while recognition practices were characterized by warm praise, which was also task oriented, clear, consistent and credible (recognition). High levels of genuine enthusiasm, positive affect and enjoyment by these teachers with respect to engaging in academic tasks was also observed. In low mastery oriented classrooms teachers used punishment and threats with students who did not do what they were told (Anderman et al., 2002). In high mastery orientation classrooms students had considerable freedom within the classroom-e.g. talking to classmates (autonomy) and peer collaboration (grouping) (Anderman et al., 2002). Reversely, in high mastery classrooms teachers emphasized students' performance, relative performance and differential prestige (evaluation) while in low mastery classrooms teachers emphasized test scores and grades or students' differential performance on tasks (evaluation). Moreover teachers in high mastery classrooms valued the time during the lesson referring to time allocation for different activities (time) while students in the low mastery oriented classrooms were allowed to work on their paces (Anderman et al., 2002).

In mathematics education domain, Stipek et al. (1998) in a relevant study referring to instructional practices and their effect on learning and motivation found that affective climate was a powerful predictor of students' motivation and mastery orientation. Students in classrooms in which teachers emphasized effort, pressed students for understanding, treating students' misconception and in which autonomy was encouraged reported more positive emotions while doing math work and enjoying mathematics more than other students while they also scored higher in a fraction test. Teachers' provision of substantive feedback to students rather than scores on assignments was also associated with mastery orientation.

Despite the apparent utility of the list concerning the classroom practices both by achievement goal researchers and mathematics educators, very few studies have examined these practices in relation to students' perceptions of achievement goals and outcomes in the ecology of regular classroom. To the best of our knowledge none of these studies had employed multilevel statistical tools for the identification of teachers' practices that influence students' specific goals and vis-à-vis students' achievement. In this respect the purpose of this study was:

- To test the validity of the measures for the six factors: fear of failure, self-efficacy, interest, mastery goals, performance-approach goals and performance-avoidance goals, in a specific social context.
- To construct and test the validity of an observational protocol that includes convergent variables referring to instructional practices in the classroom from the mathematics education domain and the achievement motivation one.
- To identify instructional practices suggested by achievement motivation theory and mathematics education theory that affect students' motivation (mastery and performance goals) in the mathematics classroom applying multilevel analysis.

METHOD

Participants were 321 sixth grade students, 136 males and 185 females from 15 intact classes and their 15 teachers. All students-participants completed a questionnaire concerning their motivation in mathematics and a test for achievement in the mid of the second semester of the school year.

The motivation questionnaire comprised of six sub-scales measuring: a) mastery goals, b) performance goals, c) performance avoidance goals, d) self-efficacy, e) fear of failure, and f) interest. Specifically, the questionnaire comprised of 35 Likert-type 5-point items (1- strongly disagree, and 5 strongly agree). The five-item subscale measuring mastery goals, the five-item subscale measuring performance goals, the four-item subscale measuring performance-avoidance goals, as well as the five item subscale measuring efficacy beliefs were adopted from the Patterns of Adaptive Learning Scales (PALS) (Midgley et al., 2000); respective specimen items in each of these four subscales were, "one of my goals in mathematics is to learn as much as I can"

(Mastery goal), “one of my goals is to show other students that I’m good at mathematics” (Performance goal), “It’s important to me that I don’t look stupid in mathematics class” (Performance-avoidance goal), and “I’m certain I can master the skills taught in mathematics this year” (efficacy beliefs). Students’ fear of failure was assessed using nine items adopted from the Herman’s fear of failure scale (Elliot & Church, 1997); a specimen item was “I often avoid a task because I am afraid that I will make mistakes”. Finally, we used Elliot and Church (1997) seven-item scale to measure students’ interest in achievement tasks; a specimen item was, “I found mathematics interesting”. These 35 items were randomly spread through out the questionnaire, to avoid the formation of possible reaction patterns.

For students’ achievement we developed a test measuring students’ understanding of fractions. The tasks comprising the test were adopted from published research and specifically concerned students’ understanding of fraction as part of a whole, as measurement, equivalent fractions, fraction comparison and addition of fractions with common and non common denominators (Lamon, 1999).

For the analysis of teachers’ instructional practices we developed an observational protocol for the observation of teachers’ mathematics instruction in the 15 classes during two 40-minutes periods. The observational protocol was based on the convergence between instructional practices described by Achievement Goal Theory and the Mathematics education reform literature. Specifically, we developed a list of codes around six structures, based on previous literature (Ames, 1992; Anderman et al., 2002; Stipek et al., 1998), which were found to influence students’ motivation and achievement. These structures were: task, instructional aids, practices towards the task, affective sensitivity, messages to students, and recognition.

The structure *task* included algorithms, problem solving, teaching self-regulation strategies, open-ended questions, closed questions, constructing the new concept on an acquired one, generalizing and conjecturing. We checked whether teachers made use of instructional aids during their lesson. *Practices towards the task* included the teacher giving direct instructions to students, asking for justification, asking multiple ways for the solution of problems, pressing for understanding by asking questions, dealing with students’ misconceptions, or seeking only for the correct response, helping students and rewording the question posed. Behaviour referred to *affective sensitivity* included teachers’ possible anger, using sarcasm, being sensible to students, having high expectations for the students, teachers’ interest towards mathematics or fear for mathematics. *Messages to students* included learning as students’ active engagement, reference to the interest and value of the mathematics tasks, students’ mistakes being part of the learning process or being forbidden, and learning being receiving information and following directions. Finally, *recognition* referred to the reward for students’ achievement, effort, behavior and the use of external rewards by the teachers.

During the two classroom observations lasted for 40 minutes for each teacher, we identified the occurrence of each code in each structure.

RESULTS

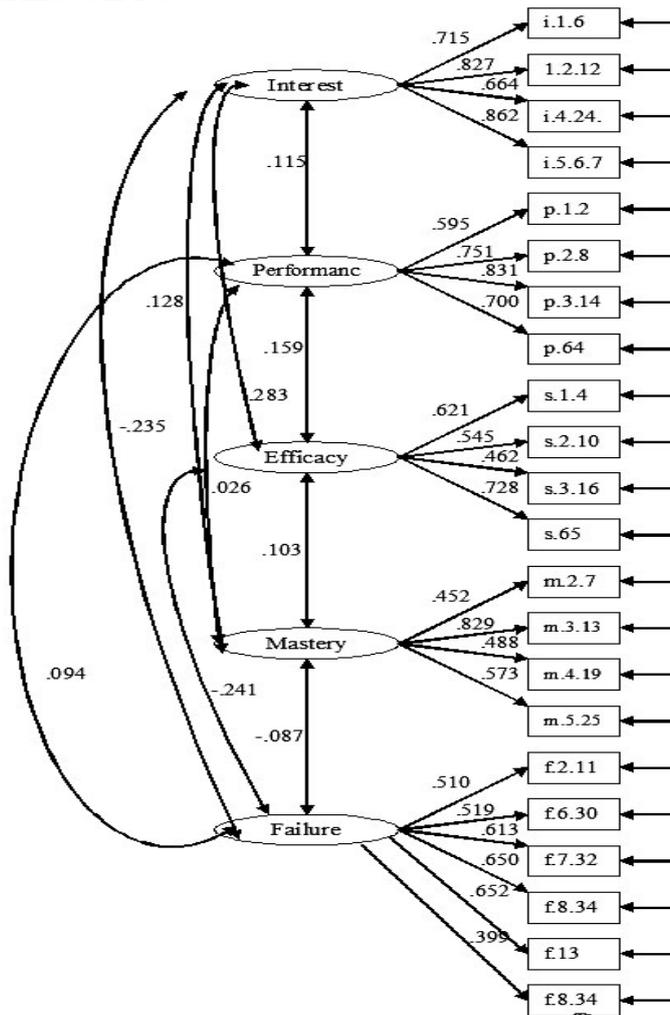


Fig 1: The factor model of students' motivation with factor parameter estimates.

With respect to the first aim of the study, confirmatory factor analysis was conducted using EQS (Hu & Bentler, 1999) in order to examine whether the factor structure yields the six motivational constructs expected by the theory.

In the analysis for the identification of the six factors, we followed a process including the reduction of raw scores to a limited number of representative scores, an approach suggested by proponents of Structural Equation Modelling (Hu & Bentler, 1999). Particularly, some items were deleted because their loadings on factors were very low (e.g. for the factor interest the item i.3.18. and for the factor fear of failure the item f.5.28) and some other items were grouped together because they had high correlation with each other (e.g. for the factor fear of failure the items f.1.5 and f.3.17). From the

analysis the factor performance-avoidance goals failed to be confirmed.

Then in line with the motivation theory,

a five-factor model was tested (fig. 1). To assess the overall fit of the model we used maximum likelihood estimation method and three types of fit indices: the chi-square index, the comparative fit index (CFI), and the root mean square error of approximation (RMSEA). The chi square index provides an asymptotically valid significance test of model fit. The CFI estimates the relative fit of the target model in comparison to a baseline model where all of the variable in the model are uncorrelated (Hu & Bentler, 1999). The values of the CFI range from 0 to 1, with values greater than .95 indicating an acceptable model fit. Finally, the RMSEA is an index that takes the model complexity into account; an RMSEA of .05 or less is considered to be as acceptable fit (Hu & Bentler, 1999).

Items from each scale are hypothesized to load only on their respective latent variables. The fit of this model was ($\chi^2 = 691.104$, $df = 208$, $p < 0.000$; $CFI = 0.770$ and $RMSEA = 0.086$). After the addition of correlations among the five factors the measuring model has been improved ($\chi^2 = 343.487$, $df = 198$, $p < 0.000$; $CFI = 0.931$ and $RMSEA = 0.049$).

Concerning the second aim of the study, analysis of the observations involved estimating the mean score of each code for the two 40 minutes observations using the SPSS and creating a matrix display of all the frequencies of the coded data from each classroom. Each cell of data corresponded to a coding structure. From a first glance, the observational protocol succeeded in detecting differences in teachers' practises during the mathematics lessons. Notably, teachers 4, 9, 13, 15 used more algorithmic tasks than the others, while teachers 2, 4, 7 used more problem solving activities than their other colleagues. Open-ended questions were used more by teachers 3, 5 while teachers 8 and 14 used more the closed type of questions. Very few teachers made use of the visual aids (4, 7, and 8). From the category practices towards the task justification of students' answers were asked from almost all teachers expect from teachers 2, 3, 10, 13. Press for understanding characterized teachers' 6 and 13 practices, while asking for multiple problem solutions was not popular to this sample of teachers. Teacher 5 was characterized by her willingness to help students. Regarding teachers' affective sensitivity, teacher 1 expressed anger while teacher 7 showed great sensitivity to students. Concerning the structure messages all teachers apart from teachers 1 and 15 treated students' erroneous responses as part of the learning process, while the other codes regarding this category were met rarely during these lessons. Regarding recognition, teachers 1 and 7 rewarded students for their performance.

According to the third aim of the study, the identification of instructional practices suggested by achievement motivation theory and mathematics education that affect students' mastery and performance goals, we applied Multilevel analysis using the program MLwin (Opdenakker & Van Damme, 2006). Multilevel analysis is a methodology for the analysis of data with complex patterns of variability, with a focus on nested sources of variability: e.g. students in classes, classes in schools, etc. The main statistical model of multilevel analysis is the hierarchical linear model, an extension of the multiple linear regression model to a model that includes nested random effects. Multilevel statistical models are always needed if a multi-stage sampling design has been employed (a sample of pupils and a sample of teachers) because the clustering of the data should be taken into consideration avoiding the drawing of wrong conclusions (Opdenakker & Van Damme, 2006). The simplest case of this model is the random effects analysis model (null model). The null model exhibits only random variation between groups and random variation within groups. (e.g. students and teachers). Estimating the variance at the distinguished level (e.g. students and teachers) it is possible to see which level is important for the estimation of the variance. For example if the estimation variance at student level (level one) is much higher than the estimation of the variance at the teacher level, then this means that differences between students with respect to the characteristics under study are largely related to individual students and not to the teachers. The null model can be expanded by the inclusion of explanatory variables. With the explanatory variables, we try to explain part of the variability of the dependent variable. It is possible to

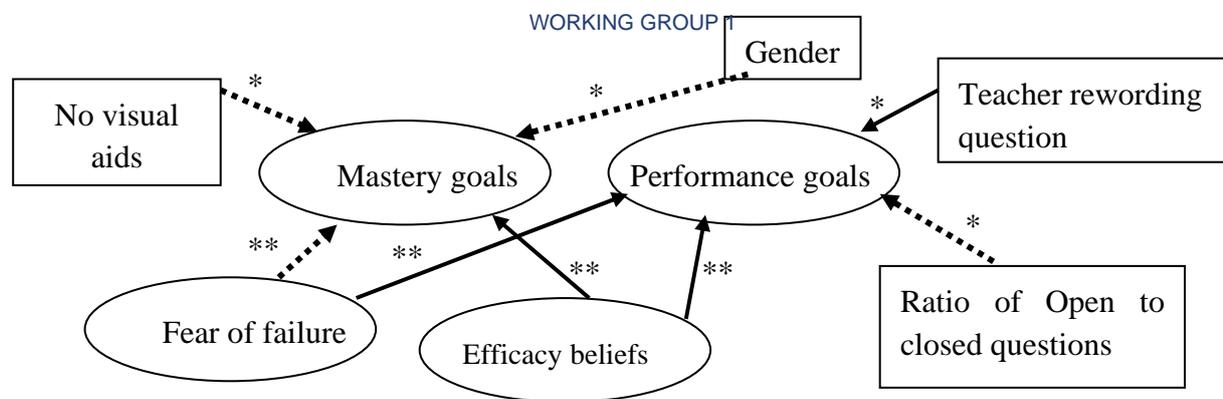
explain variability at level one as well as in a next-step at level two (Opdenakker & Van Damme, 2006).

In our case a two level model was employed with students' performance or mastery goals as the depended variable and students' motivational constructs and teachers' practices as the exploratory variables. The first test in the analysis regarding variables that influence the development of mastery goals was to determine the variance at the student level and teachers' level without explanatory variables (null model 0). The variance at each level reached statistical significance ($p < 0.05$) and this implied that MLwiN could be used to identify the variables which were associated with achievement in each subject. Regarding mastery goals, student effect was much higher than teachers effect (91% and 9% respectively). Following the procedure we added in model 1 student demographic variables. Model 1 explained 2% of the total variance. From the three variables (education mother-father and gender) only gender had statistically significant effect on students' mastery goals. The variance was explained solely to student level (2%). Explicitly, female students demonstrated higher mastery goals than male students. In model 2 all affective variables according to achievement goals theory were added to the model. Specifically the antecedent variables fear of failure and efficacy beliefs were added to the model and also performance goals. Model 2 explained 26% of the total variance. The antecedent variables had a statistically significant effect to the model, with fear of failure to have negative effect, while performance goals did not have any effect. From the 26% of the total variance 23% was at the student level and 3% at the teacher level. In Model 3 we added teachers' educational background but it turned out not to have any statistical significant effect on students' mastery goals. Then we added to the model teachers' practices concerning the structure Task and again they did not have any statistical significant effect to the model. We continue adding the other categories of teachers' practices. The only one that had negative statistical significant effect on students' mastery goals was the absence of visual aids. Model 3 explained 2% of the total variance and this variance was explained exclusively to teacher level.

We followed the same process to identify variables that had significant effect on students' performance goals. We ended that from student level, fear of failure and self efficacy had statistically significant effect on students' performance goals while from teacher level the practice, "teacher rewords the question asked" had statistically significant effect to students' performance goals.

Next, we followed Stipek et al. (1998) process grouping instructional practices in each of the six categories regarding the observational protocol together with the ratio of open-ended questions to closed questions. The ratio related to the questions had statistically significant negative effect on students' performance goals.

Figure 2 presents the results of the multilevel analysis in identifying exploratory variables that affect students' mastery and performance goals in mathematics. Dotted arrows represent negative effect.



* $p < 0.05$, ** $p < 0.001$

Fig 2: Results of the Multilevel analysis on mastery and performance goals.

CONCLUSION

Regarding the first aim of the study, data revealed that factors referred to the five of the six motivational constructs were confirmed in the Cypriot environment. The factor regarding performance-avoidance goals failed to be confirmed in contrast to the results of other studies (Cury et al., 2006). This may be due to students' age-usually this factor is confirmed in elderly students or to the different cultural context.

Regarding the second aim of the study, the data revealed important differences in the instructional practices used in the mathematics classrooms in line with other studies (Anderman et al., 2002; Pantziara & Philippou, 2007; Stipek et al., 1998). However the need for in-depth analysis of these practices born due to the study's evidence that while in some classrooms teachers applied the practices suggested by motivation and mathematics education to foster students' motivation, students' motivation was high while their mathematics performance was poor.

As far as the third aim is concerned, taking into consideration the clustering of the data in the multi-stage sampling (sample of pupils and sample of teachers) we applied the multilevel analysis to identify variables that have statistically significant effect on students' achievement goals. The results revealed that more effect on students' motivation had students' variables (gender, fear of failure, efficacy beliefs) while only few of the numerous instructional practices suggested by other studies (Anderman et al., 2002; Stipek et al., 1998) found to have statistically significant effect on students' motivation. This may be due to the new analytical tools used considering the variance between the different level of the depended variables or to the small number of teachers involved in the study. Whatever the case is, further research is needed using multilevel analysis in domains regarding achievement goals and mathematics education for the identification of instructional practices that endorse motivation and achievement in mathematics.

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THE EFFECTS OF CHANGES IN THE PERCEIVED CLASSROOM SOCIAL CULTURE ON MOTIVATION IN MATHEMATICS ACROSS TRANSITIONS

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This study investigates the effects of changes in the perceived classroom social environment on students' motivation in mathematics across the transition from primary to secondary school and during the transition from one grade level to the next within the same school (elementary or secondary school). The comparisons of students who perceived an increase, decrease or no change in the classroom social environment across the transition to middle school indicated that students' who reported a decline in their classroom social dimensions also reported a decline in social aspects of motivation and an incline in negative self-esteem. Furthermore, the effect of the changes in the classroom social dimensions on motivation were found to be larger across the transition to middle school than across the transition within elementary school, whereas they were mirrored in the secondary school transition.

BACKGROUND AND AIMS OF STUDY

The period surrounding the transition from primary to secondary school has been found to result in a decline in students' motivation in mathematics (e.g. Athanasίου & Philippou, 2007, MacCallum, 1997). This decline was found to be related to certain dimensions of the school and classroom culture (e.g. Eccles et al., 1993, Urđan & Midgley, 2003). It has been suggested that the two types of schools are very different organizations with respect to "ethos" as well as to practices and that this discrepancy influences students' motivation and performance. Most children move from a relatively small, more personalized and task-focused elementary school to a larger, more impersonal and performance-oriented middle school where they face differences in grading and teaching practices and expectations (Midgley et al, 1995).

The focus of the above studies has been on the academic aspect of motivation and of the school environment. However, students' social perceptions and goals were found to influence their motivation within a new school setting and thus are a significant part of motivation. The importance of attending to the social aspects of students' transition experiences in order to gain a fuller understanding of young adolescents' motivation in school was reinforced by the study of Anderman & Anderman (1999), in which students' social perceptions made significant, unique contributions to their achievement goal orientations. Furthermore, many longitudinal studies documented that the discontinuity in the social environment students' face across the transition to secondary school has an effect on motivation in mathematics (e.g. Eccles et al., 1993). Social discontinuities include changes in the diversity of the student population, relations with teachers and classmates and sense of school belonging.

In these studies middle school classrooms were characterized by less positive teacher-student relationships than elementary school classrooms (Midgley et al., 1995). The study of Eccles et al. (1993), revealed that the students who moved from the mathematics classroom of a high-support teacher (with respect to fairness and friendliness) to a classroom of a low-support teacher showed a decrease in their ratings of the intrinsic value and the perceived usefulness and importance of mathematics, whereas students who experienced a change from low-to-high-support teacher showed an increase in their ratings of intrinsic value. Furthermore, Anderman & Anderman (1999) found that the feeling of belonging in one's school and the endorsement of social responsibility goals were associated with an increased focus on academic tasks and predicted an increased task goal orientation, whereas endorsement of social goals for forming peer relationships and maintaining social status were associated with an increased focus on the self and predicted an increased ability goal orientation.

All the above longitudinal research shed some light on the nature of motivational change and the influence that social classroom and school environmental factors have on this process during the transition from primary to secondary school. These studies however examined motivational change for students as a whole group assuming and inferring that the transition affects all students the same way. This is not necessarily the case; recent research in the area of students' perceptions of their classroom environments supports the view that students perceive the same environment in variable ways at least on some of its dimensions (Urduan & Midgley, 2003). If there are differences in students' perceptions of their classroom environment across the transition which should really be expected, then it is possible that students perceive the transition differentially.

Despite the above theoretical considerations we are aware of only one study, by Urduan & Midgley (2003), which examined the effect of moving from a classroom perceived to emphasize a mastery goal in elementary school to a performance goal structure in secondary school (i.e. that the purpose of engaging in academic work is to develop competence or to demonstrate competence respectively). These researchers compared students who perceived an increase, decrease and no change in the mastery and performance goal structures of their classrooms during the transition to middle school and across two grades within middle school. The results of their study indicated that changes in the mastery goal structure were more strongly related to changes in cognition, affect and performance than were changes in the performance goal structure, whereas the most negative pattern of change was associated with a perceived decrease in the mastery goal structure of classrooms across the transition to middle school.

The aim of the present research is twofold. Firstly, to examine the effects of changes in the perceived classroom social environment on students' motivation in mathematics across the transition from primary to secondary school (grade 6 to 7). To

this end the classroom social environment was operationalized focusing on three dimensions: (a) teacher fairness and friendliness (**FAI/FRI**), (b) cooperation and interaction (**COOP/INTE**), and (c) competition (**COMPET**), whereas students' motivation was conceptualized involving social cognitive (orientations and goals) and affective dimensions (self-esteem). Secondly, to investigate whether the changes observed in students' perceptions of classroom social environment and the related motivation across the transition to middle school are mirrored across the transition from one grade level to the next within the same school context. More specifically, the research questions are formulated as follows:

- (1) What are the effects of the direction of change in the perceived classroom social environment on students' motivation in mathematics across the transition from primary to secondary school?
- (2) Are the changes observed in students' perceptions of the classroom social environment and the related changes in motivation across the transition from primary to secondary school mirrored across the transition from grade 5 to 6 in elementary school and across grade 7 to 8 in secondary school?

METHOD

Participants in this study were 331 students who were followed over a period of two consecutive school years. The students were divided in three Cohorts. The 220 students in Cohort T (CT) experienced the transition from primary to secondary school (grade 6 to 7); the 42 students in Cohort E (CE) were followed over the last two years of elementary school (grade 5 to 6), and the 69 students in Cohort S (CS) were followed over the first two years in secondary school (grade 7 to 8).

Data were collected through a self-report questionnaire in the spring semester of each school year, since by that time of the year students' motivation and their perceptions of the classroom social environment are well developed and established. The questionnaire was comprised of 42 items measuring four dimensions referring to students': (a) social motivational goals (students' social reasons for engaging in math work with 14 items tapping three specific motivational goals such as competition/social power, social concern and affiliation e.g. for affiliation "In mathematics I try to work with friends as much as possible"); (b) social motivational goal orientation (4 items tapping students' perceptions of how socially oriented they are e.g. "I am most motivated when I am showing concern for others in mathematics"); (c) self-esteem in mathematics (students' perceptions of their competence in doing mathematics with 8 items tapping two dimensions such as positive and negative self-esteem e.g. for negative self-esteem "I often make mistakes in mathematics"); and (d) classroom social dimensions (16 items measuring three dimensions referring to teacher fairness/friendliness, cooperation/interaction and competition e.g. for cooperation/interaction "We get to work with each other in small groups when we do math"). The items referring to the first three dimensions were

adapted from the Inventory of School Motivation Questionnaire (McInerney, Yeung & McInerney, 2000), whereas the items for the latter were adapted from the Student Classroom Environment Measure (Eccles et al., 1993). All statements were presented at a five-point Likert-type format (1=Strongly Disagree, 5=Strongly Agree). The reliability estimates were found to be quite high for all the scales ranging from $\alpha=.69$ to $\alpha=.88$.

Data processing was carried out using the SPSS software. The statistical procedure used in this study was Repeated Measures ANCOVA. Change group (CG-3 levels) was the independent, between-groups factor and time of measurement (TM-2 levels) was the within-groups repeated measures component. For all the analyses, gender was included as a covariate to control for any differences by gender.

In order to provide answers to the two research questions, three groups of students for each of the classroom environment variables were created. To create the three groups, students' classroom environment scores were firstly standardized. Next, the change score was calculated by subtracting students' scores on the first measurement from the respective scores on the second measurement, in each classroom dimension. The change scores for each dimension were then divided into three groups: (i) increase; (ii) no change; and (iii) decrease in classroom environment variable. The groups were created by using .50 standard deviations as the cut-off such that students in the "increase" groups scored at least half a standard deviation above the mean change score, those in the "decrease" groups scored at least half a standard deviation below the average change score, and those in the "no change" groups were within .50 standard deviations either above or below the mean change score. Half standard deviation was selected as the cut-off point to make sure that the groups created would be different from one another and yet maintain a large number of participants in order to allow comparisons across groups.

RESULTS

To answer the first research question, CT students' responses were analysed using Repeated Measures ANCOVAs. Table 1 presents the means, standard deviations and the F ratios for the Change Group x Time of Measurement interactions (CG x TM) for each of the three social dimensions change groups on each of the dependent variables. The alphabetical superscript "a" within each classroom social dimension change group indicates that the means in grades 6 and 7 are significantly different from one another. Similar numeric superscripts indicate non significant differences between group means on variables measured in 7th grade using univariate post hoc tests. The .05 level of significance was adopted for these comparisons.

The analyses indicated that the CG x TM effect was significant for social goal orientation, social concern and affiliation goals and negative self esteem for the **FAI/FRI** and the **COOP/INTE** change groups. Examining the results from the 6th to 7th grade transition, it appears that the most negative pattern of change in motivation

was associated with a perceived decline in **FAI/FRI** and **COOP/INTE** classroom social dimensions. Specifically, the tests of simple effects within groups indicated that students' social goal orientation, social concern and affiliation goals were significantly lower in 7th grade than in 6th grade within the group that perceived a decrease in **FAI/FRI** and in **COOP/INTE** across the transition to middle school. No significant differences were found between the 6th and 7th grade means for either the perceived "no change" or "increase" groups. The opposite pattern was observed for negative self-esteem, i.e., students' mean ratings were significantly higher in 7th grade than in 6th grade within the group that perceived a decrease in **FAI/FRI** and in **COOP/INTE** across the transition. The univariate post hoc tests of 7th grade means revealed that the mean ratings of students in the **FAI/FRI** and in the **COOP/INTE** "decrease" change groups on social goal orientation, social concern and affiliation goals were significantly lower than the mean ratings of students in the "no change" or "increase" groups, whereas their negative self-esteem was significantly higher. Also, the analysis of TM effect revealed a significant decline from 6th to 7th grade in social goal orientation ($F=3.341$, $p<0.05$), social concern ($F=8.656$, $p<0.01$) and affiliation goals ($F=2.946$, $p<0.05$) and a significant incline in negative self-esteem ($F=3.038$, $p<0.05$). Since no statistically significant differences were found between the means of students in the **FAI/FRI** and in the **COOP/INTE** "no change" or "increase" groups from primary to secondary school for social orientation, goals and negative self-esteem, these declines in orientation and goals and the incline in negative self-esteem were not evident for students who perceived no change or an increase in both the above classroom social dimensions.

The ANCOVA analyses for **COMPET** change groups indicated that the CG x TM effect was significant for social goal orientation, competition/social power, social concern and affiliation goals and negative self-esteem. The largest differences were associated with a perceived incline in **COMPET** classroom social dimension. Specifically, the tests of simple effects within groups indicated that students' social goal orientation, social concern and affiliation goals were significantly lower in 7th grade than in 6th grade within the group that perceived an increase in **COMPET** classroom environment across the transition from primary to secondary school. In both the perceived "no change" and "decrease" groups there weren't any significant differences between the 6th and 7th grade means. For competition/social power goal and negative self-esteem the opposite pattern was observed since students' mean ratings were significantly higher in 7th grade than in 6th grade within the group that perceived an incline in **COMPET** environment across the transition. The univariate post hoc analyses of 7th grade means revealed that the mean ratings of students in the **COMPET** "increase" change group on social goal orientation, social concern and affiliation goals were significantly lower than the mean ratings of students in the "no change" or "decrease" groups, whereas their competition/social power goal and negative self-esteem were significantly higher. Also, the analysis of TM effect revealed a significant decline in social goal orientation ($F=3.427$, $p<0.05$), social

concern ($F=9.507$, $p<0.01$) and affiliation goals ($F=3.105$, $p<0.05$) from 6th to 7th grade and a significant incline in competition/social power goal ($F=9.144$, $p<0.01$) and negative self-esteem ($F=3.247$, $p<0.05$). Since there were no statistically significant differences between the means of students in the **COMPET** “no change” or “decrease” groups from primary to secondary school for social orientation, competition/social power, social concern and affiliation goals and negative self-esteem, these declines in orientation and goals and the incline in competition/social power goal and negative self-esteem were not evident for students who perceived no change or a decrease in the **COMPET** classroom social environment.

Dependent Variables	Teacher fairness and friendliness change groups (FAI/FRI)						F Interaction: CG by TM
	Decrease (N = 62)		No change (N = 89)		Increase (N = 69)		
	6 th grade	7 th grade	6 th grade	7 th grade	6 th grade	7 th grade	
Social goal orientation	3.23 ^a (.81)	2.60 ¹ (.93)	3.09 (.97)	2.90 ² (.95)	3.14 (.84)	2.99 ² (1.03)	4.873***
Compet/social power	2.03 (.95)	2.47 (1.14)	2.24 (1.00)	2.72 (1.12)	2.25 (.89)	2.46 (1.12)	ns
Social concern goal	3.91 ^a (.87)	3.30 ¹ (.89)	3.95 (.91)	3.79 ² (.99)	3.90 (.93)	3.72 ² (1.10)	3.987***
Affiliation goal	3.17 ^a (.89)	2.73 ¹ (1.05)	3.40 (.96)	3.27 ² (.95)	3.32 (.93)	3.11 ² (.91)	4.268***
Positive self-esteem	3.70 (.75)	3.06 (1.07)	3.81 (.69)	3.42 (1.10)	3.79 (.74)	3.21 (1.19)	ns
Negative self-esteem	3.21 ^a (.88)	3.97 ¹ (.86)	3.27 (.86)	3.49 ² (.91)	3.39 (.77)	3.48 ² (.88)	5.488***
Dependent Variables	Classroom cooperation and interaction change groups (COOP/INTE)						F Interaction: CG by TM
	Decrease (N = 78)		No change (N = 63)		Increase (N = 79)		
	6 th grade	7 th grade	6 th grade	7 th grade	6 th grade	7 th grade	
Social goal orientation	3.27 ^a (.81)	2.56 ¹ (.84)	3.23 (.80)	2.98 ² (1.00)	2.96 (.99)	3.03 ² (1.05)	6.581*
Compet/social power	1.95 (.84)	2.43 (1.15)	2.24 (.95)	2.46 (.97)	2.37 (1.02)	2.79 (1.20)	ns
Social concern goal	4.02 ^a (.88)	3.33 ¹ (.97)	4.05 (.73)	3.83 ² (1.00)	3.73 (1.01)	3.73 ² (1.01)	5.912**
Affiliation goal	3.37 ^a (.98)	2.86 ¹ (.99)	3.35 (.83)	3.18 ² (.89)	3.22 (.95)	3.19 ² (.99)	4.259***
Positive self-esteem	3.75 (.76)	3.07 (1.15)	3.80 (.73)	3.18 (1.07)	3.77 (.69)	3.49 (1.12)	ns
Negative self-esteem	3.30 ^a (.86)	3.92 ¹ (.95)	3.32 (.72)	3.51 ² (.82)	3.26 (.90)	3.42 ² (.88)	5.018**
Dependent Variables	Classroom competition change groups (COMPET)						F Interaction: CG by TM
	Decrease (N = 76)		No change (N = 64)		Increase (N = 80)		
	6 th grade	7 th grade	6 th grade	7 th grade	6 th grade	7 th grade	
Social goal orientation	3.24 (.80)	3.08 ¹ (.96)	3.09 (.92)	2.92 ¹ (.91)	3.11 ^a (.93)	2.59 ² (1.03)	4.785***
Compet/social power	2.35 (1.06)	2.48 ¹ (1.16)	2.22 (.97)	2.40 ¹ (1.12)	2.00 ^a (.79)	2.79 ² (1.08)	4.955***
Social concern goal	3.88 (.94)	3.72 ¹ (.95)	3.92 (.88)	3.74 ¹ (.89)	3.97 ^a (.88)	3.42 ² (1.10)	3.877***
Affiliation goal	3.40 (1.00)	3.36 ¹ (1.04)	3.24 (.87)	3.17 ¹ (.88)	3.29 ^a (.91)	2.71 ² (.95)	3.744***
Positive self-esteem	3.91 (.70)	3.21 (1.07)	3.71 (.65)	3.25 (1.10)	3.69 (.78)	3.30 (1.21)	ns
Negative self-esteem	3.39 (.89)	3.55 ¹ (.90)	3.21 (.75)	3.41 ¹ (.87)	3.27 ^a (.86)	3.85 ² (.89)	4.057***

Table 1: Means, Standard Deviations and Summary of Repeated Measures ANCOVAs on motivational variables by changes in classroom social dimensions

* $p<0.001$ ** $p<0.01$ *** $p<0.05$

To answer the second research question, the same set of analyses were conducted as students moved from 5th to 6th grade in elementary school (CE) and from 7th to 8th grade in secondary school (CS). Table 2 presents the means and the F interaction (CG x TM) for all the classroom social dimension change groups for students in CE and CS. Standard deviations are not presented due to space limits.

Regarding the comparability of results involving the direction of changes in classroom social dimensions between the elementary to secondary school transition (grade 6 to 7) and the elementary school transition (grade 5 to 6), the patterns of results involving all the classroom social dimensions change groups across the

transition from primary to secondary school were not replicated during the elementary school transition. There were no significant interactions for **COMPET** change groups, whereas for **FAI/FRI** and **COOP/INTE** only one significant interaction was observed involving social goal orientation with students' perceptions across the transition within elementary school changing the same way as the perceptions of students across the transition from primary to secondary school.

	Teacher fairness/friendliness (FAI/FRI)				Cooperation/interaction(COOP/INTE)				Competition (COMPET)			
	CE		CS		CE		CS		CE		CS	
	5th	6th	7th	8th	5th	6th	7th	8th	5th	6th	7th	8th
Social orientation												
Decrease	3.44 ^a	3.12 ¹	3.36 ^a	2.62 ¹	3.47 ^a	3.02 ¹	3.50 ^a	2.57 ¹	3.33	3.37	3.39	3.25 ¹
No change	3.75	3.60 ²	3.12	2.95 ²	3.37	3.48 ²	3.25	3.17 ²	3.40	3.36	3.14	3.08 ¹
Increase	3.31	3.57 ²	3.12 ^a	3.43 ³	3.62	3.71 ²	2.85 ^a	3.09 ²	3.66	3.55	2.96 ^a	2.57 ²
F Interaction: CG by TM	3.181***		6.145**		3.560***		5.562**		ns		2.991***	
Compet/social power goal												
Decrease	2.15	2.17	2.63	2.27	2.43	2.25	2.73	2.51	2.20	2.23	2.63 ^a	2.24 ¹
No change	2.16	2.19	2.65	2.56	2.06	2.07	2.58	2.39	2.28	2.07	2.59	2.44 ¹
Increase	2.17	2.13	2.65	2.72	2.05	2.20	2.66	2.77	2.05	2.16	2.73 ^a	3.04 ²
F Interaction: CG by TM	ns		ns		ns		ns		ns		4.777***	
Social concern goal												
Decrease	3.86	3.71	4.00 ^a	3.66 ¹	3.18	2.89	3.59 ^a	3.05 ¹	3.66	3.30	4.01	3.89 ¹
No change	3.80	3.50	3.71	3.62 ¹	3.96	3.90	3.94	3.80 ²	3.79	3.93	3.73	3.77 ¹
Increase	3.78	3.75	3.72	3.73 ¹	4.17	4.03	3.71 ^a	3.94 ²	3.95	3.77	3.55 ^a	3.19 ²
F Interaction: CG by TM	ns		2.998***		ns		3.840***		ns		3.241***	
Affiliation goal												
Decrease	3.38	3.05	3.68 ^a	3.27 ¹	3.41	2.89	3.46 ^a	2.71 ¹	3.26	3.25	3.29	3.05 ¹
No change	3.61	3.26	3.12	2.92 ²	3.18	2.87	3.39	3.28 ²	3.29	2.88	3.11	3.16 ¹
Increase	3.00	2.70	2.75	2.89 ²	3.35	3.19	2.57 ^a	2.80 ¹	3.35	2.83	3.03 ^a	2.73 ²
F Interaction: CG by TM	ns		3.125***		ns		4.553***		ns		3.310***	
Positive self-esteem												
Decrease	4.01	3.97	3.34	3.23	3.31	3.49	3.34	3.21	3.51	3.44	3.47	3.27
No change	3.73	3.67	3.48	3.32	4.12	3.91	3.51	3.35	4.13	4.15	3.42	3.26
Increase	3.87	3.87	3.73	3.52	4.05	4.08	3.66	3.46	3.98	3.97	3.71	3.59
F Interaction: CG by TM	ns		ns		ns		ns		ns		ns	
Negative self-esteem												
Decrease	3.40	3.42	3.12 ^a	3.60 ¹	3.06	3.16	3.59 ^a	3.88 ¹	2.94	3.00	3.30	3.20 ¹
No change	3.34	3.36	3.29	3.16 ²	3.43	3.42	3.09	3.08 ²	3.72	3.76	3.26	3.19 ¹
Increase	3.01	3.01	3.41	3.34 ²	3.16	3.16	3.33	3.23 ²	3.16	3.14	3.27 ^a	3.63 ²
F Interaction: CG by TM	ns		3.565***		ns		3.243***		ns		2.987***	

Table 2: Means and Summary of Repeated Measures ANCOVAs on motivational variables by changes in classroom social dimensions for students in CE and CS

***p<0.001 **p<0.01 ***p<0.05**

On the contrary, the patterns of changes in classroom social dimensions change groups for students across the transition from primary to secondary school were mirrored for students across the transition within secondary school, with some notable exceptions. Firstly, social goal orientation increased significantly from 7th to 8th grade among those students who perceived an increase in **FAI/FRI** and **COOP/INTE** classroom social environment but decreased significantly for those students who perceived a decrease in **FAI/FRI** and **COOP/INTE** social environment over time. A similar pattern was observed for the analysis regarding social concern and affiliation goals as the dependent variable for the **COOP/INTE** social dimension. In addition, the comparison of the differences found across the transition to secondary school (6th to 7th grade) with those found during middle school (7th to 8th grade) among the **COMPET** social dimension change groups revealed similar directions of change for social orientation, social concern and affiliation goals and negative self-esteem. However, a significant difference over time was found for the competition/social power goal. The students who moved from 6th to 7th grade and perceived an increase in the **COMPET** social dimension of their classroom reported endorsing competition/social power goals significantly more, whereas students in the no change or decrease groups did not change significantly in their adoption of competition/social power goal. But when students moved from 7th to 8th grade, the endorsement of competition/social power goal decreased significantly among those students who perceived a decrease in the **COMPET** social environment over time.

DISCUSSION

The results of the study suggest that when students make the transition to middle level schools they are likely to move into classrooms that are characterized by less teacher-student relations, less cooperation and interaction whereas competitiveness is emphasized. Despite those general trends, there are students who perceive no difference in their classroom social environment before and after the transition and other students who perceive an increase in their classroom social orientation. Recent studies have contributed to our understanding of what occurs within classrooms, but nothing is known about the effects of moving from one classroom social environment to another. Thus, while it has been documented that the classroom social environment changes after the transition from primary to secondary school, it remains unclear what effects these differences might have on students' motivation in mathematics. The present study shed some light on these issues.

More specifically, the results of the study revealed that students who reported a decline in their classroom social environment across the transition to middle school also reported a decline in the social aspects of their motivation and an increase in negative self-esteem. Also, it was found that among students who reported an increase in the social environment of their classrooms after the transition, the general negative pattern of change in motivation was not evident. These results suggest that whereas a perceived increase in classroom social dimensions has advantages, the

disadvantages associated with a perceived decrease in the classroom social environment are even stronger. Perhaps social messages in the classroom are more evident to students when they are first removed than when they are perceived to be added. In other words, students may not notice the presence of social dimensions in the classroom as much as they notice their absence. This may be particularly true when students move from what has been described as the more nurturing elementary school environment to the more impersonal middle school classroom environment (Anderman & Anderman, 1999).

The changes in motivation associated with changes in the perceived classroom social dimensions were found to be larger during the transition to middle school than they were during the last two years in primary school. This finding is pretty logical taking into consideration the fact that the classroom environment in elementary school is almost the same across grades. On the contrary, the effect of changes in the perceived classroom social environment and changes in motivation that were found across the transition to middle school were replicated within the first two years of middle school. Therefore, the stress of moving to middle level schools does not enlarge the size of the effects of changes in the perceived classroom social dimensions on motivation, despite the fact that previous research has documented that the transition to middle level school can be a stressful time in students' lives (e.g. Eccles et al., 1993).

Although the size of the changes in motivation associated with changes in the perceived classroom social environment were quite similar across the transition to middle school and within the first two years in middle school, there were some interesting differences in the direction of the changes and in which change groups the largest differences were found. The changes in the means were largest among students in the decrease groups for **FAI/FRI** and **COOP/INTE** dimensions from 6th to 7th grade. For students in the 7th to 8th transition the differences within these groups remained whereas differences in the **FAI/FRI** and **COOP/INTE** increase groups were found since students' who perceived an increase in the above social dimensions reported higher social orientation and goals and lower negative self-esteem. It also appears that the pattern of change among the **COMPET** social dimension change groups differed across the two time periods of the study. For example, the **COMPET** increase group reported a decrease in motivation from 6th to 7th grade, whereas when students made the transition from one grade to the next within middle school the **COMPET** decrease group reported an increase in their motivation.

These shifting patterns of results are evident due to the fact that the transition to middle school influences the salience of the presence or absence of social messages in the classroom (Anderman & Anderman, 1999). When moving from a smaller and perhaps more social oriented elementary school environment to a middle school environment, students may be particularly aware of decreases in the emphasis on social orientations and goals in the classroom, creating stronger effects on motivation among those students who perceive a decrease in the classroom social environment.

Once familiar and comfortable with the middle school environment, however, increases in the classroom social environment become as salient as decreases and the effects of these two types of change become more even.

The findings of the present study highlight the effects of changes in the classroom social environment on students' motivation in mathematics during the transition from one school context to another or from one grade level to the next within the same context. Therefore, longitudinal studies examining these issues can assist in unravelling the complexity of motivational change across transitions. Such studies should examine different aspects of motivation (academic, social and affective) and various dimensions of the classroom or school environment. This multidimensional perspective is very important in order to understand not only the effects of what is more prevalent in classrooms but in determining what the most facilitative environments are, even if they are uncommon, in order to test the effects of these environments on the nature of change in students' motivation in mathematics. Such information will be useful for teachers, educators and policy makers in their planning to make systemic transitions easier so fewer students are lost.

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“AFTER I DO MORE EXERCISE, I WON'T FEEL SCARED ANYMORE” – EXAMPLES OF PERSONAL MEANING FROM HONG KONG

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What kind of meaning do students relate with mathematics education? To answer this question, the concept of personal meaning is developed and integrated in an interplay with context and culture. Personal meaning hereby denotes the personal relevance students relate with a certain action or object. Finally, the concept is illustrated with an example of personal meaning constructed by a 15-year-old student from Hong Kong. Along this example, the relation of personal meaning and (learning) culture is disclosed.

INTRODUCTION

The demand for meaning in the context of mathematics education and education in general has been noted for many years. Hurrelmann stated in the early 1980ies that students are in the need of meaning when dealing with learning contents at school (Hurrelmann, 1983). But what exactly is understood by the term *meaning* when thinking about school education? Do educators and students denote the same concept when using the term? To be more precise: What kinds of meaning are there? And which meaning do students see when dealing with mathematics in school context? To shed some light on the obscurity of this realm, this paper starts with briefly presenting different understandings of *meaning* before the focus is put on the perspective of the students. Then, the concept of *personal meaning* is related to the notions of context and culture. The discussion shows in what way personal experiences and perspectives are important for the student to construct meaning. Finally, examples of personal meaning constructed by a 15-year-old Hong Kong student are presented to illustrate the concept and to show its relations to the (learning) culture the student has been socialised in.

FROM MEANING TO PERSONAL MEANING

Meaning: A blurred concept

A review of the relevant literature shows that very different understandings of *meaning* are used. The notion may refer for instance to the act of leading the schema of an unconscious sensori-motor or mental activity to consciousness (Thom, 1973), to the development of a certain mathematical concept over time (Bartolini-Bussi, 2005), or to the collectively shared understanding and application of mathematical concepts (Biehler, 2005). These kinds of meaning deal primarily with mathematical concepts and develop a theory about its referents.

On the other hand, meaning can also be understood as a condition for students to engage in the action of learning (Alrø, Skovsmose & Valero, 2007), i.e. as an integrated aspect of acting (Lange, 2007) and the educational situation (Skovsmose, 2005), or as the personal relevance an object or action has for a certain student (Vollstedt, 2007). These interpretations move the focus from the meaning of concepts to the meaning of action, i.e. the educational process and the perspective of the students. The term *meaning* is therefore used here in a personal sense (Kilpatrick, Hoyles & Skovsmose, 2005).

Quite important differences between the understandings of *meaning* as described in the last two paragraphs can be detected. Howson therefore points out that

one must distinguish between two different aspects of meaning, namely, those relating to relevance and personal significance (e.g., ‘What is the point of this for me?’) and those referring to the objective sense intended (i.e., signification and referents). (Howson, 2005, p. 18)

To sharpen the terminology used, I will use the more specific terms *personal meaning* when denoting the personal relevance of an object or action for a certain person, and *objective* or *collective meaning* when denoting a collectively shared meaning of an object or action (Vollstedt, 2007; Vollstedt & Vorhölter, 2008 [1]).

Characteristics of personal meaning and its construction

As described in Vollstedt (2007), some assumptions can be made concerning personal meaning. It is characterized by the following traits:

- Personal meaning is subjective and individual. This means that every person constructs his/her own meaning with respect to a certain object or action. As the construction of meaning is not collective but individual, different students who attend the same lesson can also construct different meanings relating to the same object or action.
- The construction of personal meaning is also context bound. Here, context denotes on the one hand the subject context as well as the situation in the classroom. On the other hand, it also embraces the personal context of the students (see below).
- Personal meanings can be reflected on but normally do not have to. This means that the process of the construction of personal meaning can in some parts be dominant in the situation so that one is aware of it (e.g. in an Aha-experience); the meaning enters consciousness. On the other hand, meaning may remain latent and can be constructed implicitly.

The student's perspective

Bearing in mind that there are different understandings of meaning in relation with mathematics education, one has to decide which perspective to put the focus on: collective or personal meaning? This means one has to ask whether mathematical concepts or the students are in the centre of attention.

My dissertation project reported on in this paper (see below) evolves from the context of the Graduate Research Group of Educational Experience and Learner Development located at the University of Hamburg. In this research group we investigate processes of learning and *Bildung* from the learner's perspective. Special attention is paid to the individually experienced tensions resulting from societal or institutional demands on the one hand, and the learner's individual responses being rooted in his/her biography on the other hand. On the one hand, special emphasis is put on the way how students acquire knowledge and skills. On the other hand, research is done about how they develop the ability to come to decisions and to act responsibly in an increasingly complex and difficult world (Graduiertenkolleg Bildungsgangforschung, 2006).

Due to the connection to the field of Educational Experience and Learner Development, the focus of my study lies clearly on the learner's perspective. The study seeks to find out what kinds of personal meanings students construct in the context of mathematics education. Like Lange I therefore want to “look *with* children” (Lange, 2007, p. 271) instead of looking at them.

Personal meaning, context, and culture

Personal meaning cannot be constructed in a vacuum but is related to context. Context is here used as a cover term for both, situational context (i.e. context of the learning situation in terms of topic as well as classroom situation) and personal context. The personal context of a student then may consist of his/her personal traits (i.e. aspects which concern the student's self like his/her self-concept, motivation, or beliefs) and his/her personal background (i.e. aspects which concern the world around the student like his/her socio-economic status, migration background, or surrounding (learning) culture) (Vollstedt & Vorhölter, 2008).

Mercer describes context from the student's perspective in the following way:

What counts as context for learners [...] is *whatever they consider relevant*. Pupils accomplish educational activities by using what they know to make sense of what they are asked to do. As best they can, they create a meaningful context for an activity, and the context they create consists of whatever knowledge they invoke to make sense of the task situation. (Mercer, 1993, pp. 31–32, italics in original)

Therefore the student decides which information and experiences are relevant for him/her to deal with the given task. I interpret Mercer's description in a broad way as not only knowledge but also for instance beliefs, goals or other kinds of personal traits or background may be relevant for the student in a learning situation. These are, however, object to cultural influence as culture has a strong impact on the way how learning takes place in any learning situation (Leung et al., 2006).

This understanding goes along with Mercer, who states that learning in the classroom depends both on culture and context as learning is,

(a) culturally saturated in both its content and structure; and (b) accomplished through dialogue which is heavily dependent on an implicit context constructed by participants from current and past shared experience. (Mercer, 1993, p. 43).

When we take for instance the East Asian and the Western traditions, both, culture and context of a learning situation are very different as they are based on Chinese/Confucian and Greek/Latin/Christian traditions respectively (Leung, 2001). In how far culture also has an impact on the construction of personal meaning will be shown in the following section with the help of an example from Hong Kong.

PERSONAL MEANING CONSTRUCTED BY A HONG KONG STUDENT

To illustrate the concept of personal meaning, I will present some findings from a qualitative study which seeks to find out similarities and differences between the personal meanings constructed by students in two different learning cultures, namely Germany and Hong Kong. I will restrict myself here to Hong Kong data and results.

The study

In total, the study is based on 33 interviews with 15- and 16-year-old students in Germany (form 9 and 10) and Hong Kong (Secondary 2 and 3) [2]. In Germany I interviewed 16 students attending a grammar school; the 17 Hong Kong students attended band one EMI-schools (schools with the highest academic standards and English as medium of instruction [3]). The interviews began with a phase of stimulated recall (Gass & Mackey, 2000) based on a video-sequence of five to ten minutes from the last mathematics lesson the interviewee attended. The student was asked to utter and reflect on his/her thoughts he/she had when having attended the lesson. This was followed by a guided interview about various topics like the student's beliefs about and attitudes towards mathematics (lessons), his/her connotations of mathematics (lessons), or the feelings he/she associates with mathematics (lessons), i.e. personal traits. Aspects of personal background were not explicitly asked for [4]. In average, the interviews lasted for about 35 to 45 minutes. In the style of grounded theory (Strauss & Corbin, 1996), the theory of personal meaning was refined and deepened in the process of data evaluation. Data evaluation itself was a coding process following grounded theory with the aim to construct different types of personal meaning evolving from the data. These types are then reflected on from a cultural perspective.

Personal meanings constructed in the context of mathematics education in Hong Kong

Emma, a 15-year-old girl from Hong Kong, attends a highly selective band one school in which the classes are divided into academic achievement. She is a member of class *Secondary 3C*, which is the class of the top 40 students of her year. Although she attends this class, she explains that she has difficulties with mathematics and shows a low mathematical self-concept (Marsh, 1986). This low self-perceived ability in mathematics, being part of her personal traits (i.e. personal context), is an impor-

tant precondition for the personal meaning she constructs in relation with learning of mathematics at school. The following extract from the interview ([5]) may help to illustrate this point:

- 99 Interviewer: First of all, what comes to your mind when you hear the word *mathematics*?
- 100 Emma: First, at the beginning I feel, I'm afraid of mathematics. Because it is difficult for me to think. Think is the main problem for me. When I saw the mathematics sentence questions, I will feel scared. I think I don't understand, whether I understand that question or not, so that I feel scared. But after I do more exercise, I won't feel scared anymore and I feel I am safe.
- 101 Interviewer: So is it because of the language, the problem is given in or is it because it's something unknown, or do you know why you are scared?
- 102 Emma: I think it's not the language problem. I think is my problem because I think very slow. So I'm afraid I can't catch up with the other classmates.
- 103 Interviewer: But you are in C class and C class is the best, isn't it?
- 104 Emma: It is very difficult for me to go into this class because there is many pressure. There are many students are get high marks. So, there will be against students and students. So I need to study hard.

We can see that Emma comes to her low mathematics self-concept by means of internal and external references (Marsh, 1986). On the one hand she negates that her difficulties in mathematics are due to the fact that the mathematical problems and lessons are given in English (101-102), which is not her first language. The internal comparison of her self-perceived verbal ability with her self-perceived mathematics ability (Marsh, 1986) make her come to this conclusion. She also, on the other hand, compares her abilities in mathematics with those of her classmates (102, 104), i.e. significant others in her frame of reference (Marsh, 1986). Due to the selective process, there are lots of very good students in her class so that it is not astonishing that Emma experiences high pressure when she compares her own achievement with the ones of her classmates. Especially as she mentions that there is quite some competition going on between the students (104).

The reason Emma gives for her difficulties with mathematics is that she has problems to think fast enough (100, 102). Therefore she stresses that actively doing mathematics can help “train us our mind and the logic” (66). Also, practice can help her to overcome her difficulties (100) as well as meet the pressure experienced between the students (104). She also refers to this point in another sequence of the interview in which she explains the importance of good grades with relation to the pressure caused by the *Hong Kong Certificate of Education Examination* (HKCEE):

- 198 Interviewer: How important is it for you to achieve the mark you want to achieve in quizzes, or tests, or examinations, or whatever?

- 199 Emma: Do more exercise. And when you see the questions, you should not feel afraid of them. Just like homework or worksheets, not a quiz or exams. So that we can relax and we won't feel more pressure.
- 200 Interviewer: Is it important for you to get good marks?
- 201 Emma: Yes, because we need to study in form four. And when we study in form five, there is Hong Kong CEE. It is very important because if we got a pass in a Hong Kong CEE we can study in form six and form seven. And if we are not pass in a Hong Kong CEE, maybe we can't study in form six, form seven and so that at that time maybe we need to find a job. But it is very difficult to find a job with form five level because many companies needs a person who got a university level. So the competition is very big.

Emma describes how practice can help to overcome anxiety and pressure as quizzes and exams may lose their threatening power when having done enough exercises beforehand (199). Therefore she is of the opinion that “it is not enough for us to do the school work. We should do more, so we find more practice exercise” (230). Her aim is to “remember all the steps” (230) necessary to solve a question. As a consequence she can relax and does not feel more pressure (199). On the other hand, she explains that the results of the HKCEE are so important for Hong Kong students as their future depends on them (201). This means that Emma reflects here on her future opportunities or foreground (Skovsmose, 2005).

To meet this high pressure and competition, the warm and friendly atmosphere that relates her with the teacher is very important for her:

- 203 Interviewer: Which feelings do you relate with mathematics lessons?
- 204 Emma: Happy.
- 205 Interviewer: Why?
- 206 Emma: Because teacher is our friend and a friend teaches us things and it will be easy to remember a friend's words. So that we will more easily to understand mathematics and the explanation. So I think Ms. Wong's teaching method is good for us.

Describing her teacher as “friend” (206) shows Emma's strong need for relatedness (Ryan & Deci, 2004) with the teacher and its importance for her learning (206). This positive relation is the cause that Emma relates a happy feeling with mathematics lessons (204) in spite of great pressure and competition.

Taken together we can describe Emma as a girl with low mathematical self-concept who suffers from the high pressure experienced in her learning environment. Therefore she fears mathematics and examinations, especially the HKCEE. The situational context as well as personal traits are therefore highly influential for the personal meanings Emma constructs. The *positive atmosphere in the classroom* (resulting from the good relation with the teacher) *opposes high pressure*. In addition, *studying hard is soothing preparation for important exams* for Emma and works against her low mathematical self-concept.

Discussion from a cultural perspective

Emma's personal context as described in the last section can be explained with reference to the culture she was socialised in, i.e. the Chinese (a Confucian Heritage Culture (CHC) (Wong, 2004)). Leung shows that the CHC does have influence on how mathematics is taught in schools because “there exist distinctive features of mathematics education in East Asia and [...] those features are expressions of distinctive underlying cultural values” (Leung, 2001, p. 48). He identifies six features of mathematics education in East Asia and contrasts them with features in Western countries. To provoke discussion, he formulates these features in the form of the following six dichotomies (East Asia vs. West): product (content) vs. process; rote learning vs. meaningful learning; studying hard vs. pleasurable learning; extrinsic vs. intrinsic motivations; whole class teaching vs. individualised learning; and concerning the competence of teachers: subject matter vs. pedagogy (Leung, 2001). Leung, however, stresses the point that

[i]t does not mean that all East Asian societies are on one side of the dichotomies and all Western countries are on the other side. Very often, it is a matter of the relative positions of the two cultures on a continuum between two extremes rather than two incompatible standpoints. (Leung, 2001, p. 38)

Emma is certainly not the only student with a low mathematical self-concept who studies hard and practices as much as possible to pass the HKCEE. This behaviour is, as far as I can judge from observation and data evaluation, somehow typical for Hong Kong students. It seems to be culturally determined and can be related to the three features of East Asian mathematics education that refer to students' behaviour, namely rote learning, studying hard, and extrinsic motivation.

Emma's attitude to practice as many tasks as possible can be explained by the Chinese belief that practice makes perfect (Li, 2006). It is closely linked with the feature of rote learning which Leung describes to be rooted in the East Asian view on the nature of mathematics learning. In East Asia, rote learning or memorization are not negatively connoted but, on the contrary, accepted and necessary steps of learning (Leung, 2001). Also, memorization and understanding are not necessarily separated (as a Western view might presume) but may be intertwined to lead to higher quality outcomes (Dahlin & Watkins, 2000).

Closely linked to the belief that practice makes perfect is the belief that studying hard is necessary to gain deep knowledge of the subject. This belief comes from the East Asian view that learning is necessarily accompanied by hard work (Leung, 2001). How deeply rooted this belief is in China can be deduced from the Chinese characters denoting education: 教育. They consist of different parts which mean 'young people' (lower left part of the first character), 'hard burden' (upper part of the first character), and 'development' (second character). So taken together the characters of 'education' confer the idea that “young people grow and develop under the condition in which

they make every endeavor to tackle tough tasks” (Li, 2006, p. 131). Therefore, diligence and effort are needed to come to a deep level of pleasure and satisfaction as the outcome of study.

Finally, Emma studies hard to prepare herself for the HKCEE, which she has to sit in 2.5 years. Although the HKCEE is still fairly far in her future, it has already quite some power over Emma. This power comes, due to the large population, on the one hand from the serious competition between students for university admission. There is, however, also a historical argument of the big importance of exams in China or Hong Kong respectively. Throughout history, education has been a way for social advancement insofar as examinations had to be taken to be selected for important officer positions (Li, 2006). In addition, examinations are a warrantable source of motivation in the East Asian understanding. As Leung points out, “East Asians believe that, being human, we need some 'push' in our learning” (Leung, 2001, p. 43). Therefore, an optimal level of pressure is helpful to direct students' energy and attention to study and to learn.

From this illustration we can see that culture has an impact on the context of the individual in different ways: culture shapes the identity of mathematics education (see Leung (2001)) and with it the learning situation, and cultural beliefs seem to determine the individual's actions and beliefs about learning.

CONCLUSION

The discussion of personal meaning has shown in what way the personal context is important for constructing personal meaning in the context of mathematics education. It is of special importance that personal meaning may be explained with reference to culture (the Confucian Heritage Culture in Emma's case). Her personal meaning (practising mathematics soothes and prepares for important exams) could be related to the CHC on three levels. Some of her personal traits (being diligent) as well as some of the actions she carries out in line with her personal meaning (working hard, practising as much as possible) seem to be rooted in cultural beliefs which are part of the CHC culture. So – as culture seemingly does matter for the construction of personal meaning – it is at near hand to support Leung, Graf & Lopez-Real, who assume that “the impact of cultural tradition is highly relevant to mathematics learning” (Leung et al., 2006).

NOTES

1. The German term for *personal meaning* we use in our research is *Sinnkonstruktion*. *Objective* or *collective meaning* on the other hand are equivalents of *Bedeutung*.
2. In Hong Kong, compulsory schooling starts with primary school, which lasts for 6 years (*Primary 1* to *Primary 6*). Subsequently students attend up to 7 years of secondary school. After *Secondary 5*, the *Hong Kong Certificate of Education Examination* (HKCEE; similar to GCSE in the United Kingdom) has to be sit.

3. Secondary schools in Hong Kong are divided in band one to three. This division is based on the achievement of their students in the HKCEE. After finishing primary school, Hong Kong students are divided into different groups according to their achievement in relation to the standing of their school. Only high-achieving students are allowed to attend a band one school after primary school.
4. All students come from rather privileged and well-educated background. This can be argued by the kind of school they attend (private band one school/grammar school). For other aspects it was assumed that interviewees would give the information voluntarily or could be asked about it.
5. The transcripts of the interviews are simplified in language in the way that stuttering and break-ups are left out; grammatical mistakes are not corrected but left unchanged. As Emma is very fluent in English, it was not necessary to mark hesitation etc. in the quoted sequences.

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EMOTIONAL KNOWLEDGE OF MATHEMATICS TEACHERS – RETROSPECTIVE PERSPECTIVES OF TWO CASE STUDIES

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Abstract

In this paper we provide a partial description of certain facets and experiences that are central to the development of emotional knowledge from the retrospective perspectives of two highly experienced mathematics teachers in middle and high school. One of the study participants refers to the emotional knowledge she developed over the years regarding her interactions with her students, while the second participant also refers to the emotional knowledge she developed regarding her interaction with the school principal. Both indicate the differences in their emotional reactions between the first practice years and the years after. The differences are seen primarily in the type and in the intensity of their emotions. While negative feelings mostly accompanied the first years, later years were accompanied by more positive emotions.

1. Introduction

Teaching and emotions are inseparable. Emotions are dynamic parts of ourselves, and whether they are positive or negative, all organizations, including schools, are full of them (Hargreaves, 1998). In his literature review, Zembylas (2007) asserts that although "teacher knowledge" has become a major area of exploration in educational research, limited attention is given to the emotional aspects of teaching. While Shulman's (1987) work on pedagogical content knowledge (PCK) was further investigated and discussed by many researchers, teachers' understandings of emotional aspects of teaching and learning continued to be ignored. Zembylas argues that "any effort to expand current conceptions of PCK should include the connection between PCK and *emotional knowledge* (EK) in general – that is, a teacher's knowledge about/from his or her emotional experiences with respect to one's self, others (e.g. students, colleagues), and the wider social and political context in which teaching and learning takes place" (p. 356). Furthermore, Zembylas continues, in order to teach well, "teachers must be able to connect their emotional understanding with what they know about subject matter, pedagogy, school discourses, personal histories, and curriculum" (p. 364). In this paper we provide a partial description from a study we conducted that focused on themes identified by teachers as central to their

development of EK. We present two case-studies of mathematics teachers, each of whom has more than 30 years of teaching experience.

2. Theoretical background

In the process of determining mathematics teachers' qualifications, teacher educators focus on various types of knowledge identified as essential for good teaching: content knowledge, didactical knowledge, knowledge about students, and knowledge of class management (Shulman, 1987; Shulman, 2000). Often these types of knowledge are discussed, separately on the assumption that teachers are capable of integrating them into a coherent whole. However, issues concerning emotional aspects of teaching and their interrelations with the above knowledge types, are rarely discussed in mathematics teachers' training programs.

Planes and types of EK. Zembylas (2007) finds a reciprocal relationship between PCK and EK, and argues that the latter "occurs on different *planes* as there are different *types* of EK that are aspects of PCK" (p. 358). These planes are: individual, relational, and socio-political. The individual plane refers to how teachers experience and express their EK on the personal plane; the relational plane refers to how teachers use EK in their relationships with students; and the socio-political plane refers to EK of the institutional and cultural context of schooling and its influence on teachers' curricular decisions and actions. There is no hierarchical order between the three planes. Their boundaries are blurred, and mutual influence and interaction exist between them.

Positive vs. negative emotions. Smeltzer (2004) studied the emotions of beginning teachers, and discerned positive and negative emotions according to their characteristics and forcefulness, as they appeared in the teachers' reactions. The categories of positive emotions include: joy-happiness, fulfillment-reward-satisfaction, competence-confidence-motivation, and surprise-fun. The categories of negative emotions include: frustration-anger, incompetence-anxiety-fear-doubt, exhaustion-stress, and disappointment-discouragement-sadness. Smeltzer also found that the most dominant and intense category of emotion is frustration-anger. It comes as a result of the turmoil beginning teachers, experience as defeat, distress, or displeasure. The incompetence-anxiety-fear-doubt category represents low self-efficacy, expressed by feelings of inadequacy, uneasiness, apprehension, worry, hesitancy, or uncertainty. The exhaustion-stress category characterizes weariness, fatigue, and energy loss. The disappointment-discouragement-sadness category refers to the most desperate and desolate of emotions such as unfulfilled expectations, sorrow, low spirits, disheartenment, and dashed hopes.

The categories of positive emotion were found to be of less frequency and intensity. The joy-happiness category represents the delight, pleasure, and contentment experienced in the early years of teaching. The fulfillment-reward-satisfaction category extends the joy-happiness category, representing a deeper and more intense degree of gratification. The competence-confidence-motivation

category signifies teacher self-efficacy identified by assurance, certainty, and proficiency. The least dominant and intense of all the emotional classifications is the surprise-fun category that refers to unanticipated and spontaneous experiences in teaching. In the present study the research participants recounted various emotions that can be generally grouped into positive and negative headings. Moreover, these emotions can also be further categorized according to Smeltzer's types which were previously mentioned.

3. The study

Our study focuses on experienced mathematics teachers, each of whom who has more than 30 years of teaching experience. The aims of our study are to characterize: (i) facets and experiences that are central to the development of EK from retrospective perspectives; (ii) interrelations between EK and PCK; and (iii) the evolvement of teachers' EK during their years of practice from retrospective perspectives. In this paper we provide a partial description of the results from the first part of our study. We also present certain facets and experiences of the emotional component of teaching that are central to the development of EK, as shown in these two case-studies.

3.1 The study participants

Twelve mathematics teachers with more than 30 years of teaching experience each were interviewed. In this paper we will briefly present the narratives of only two of them: Betty (56) and Rose (55), both who teach mathematics in middle-high school. We chose to make use of their stories because more than the other participants, Betty and Rose were able to identify the "causes and effects" that impacted their emotions and the development of their EK. In section 4 we present excerpts from their actual narratives.

3.2 Method

Data collection. We asked the twelve teachers to tell us their stories, with deliberate attention given to emotional aspects of teaching and EK. The interviews were open. We asked the teachers several general questions (for example – why they chose to become teachers), and following their narratives we asked for further clarification. We were careful not to direct them, or to interfere in their associative train of thought. The interviews were tape-recorded. Each interview lasted between 3 to 4 hours and took place in an informal setting, such as the teacher's home or Cafeteria.

Data analysis. Scanning the transcripts of the recorded interviews, we first picked out all the excerpts which included expressions of emotion. Then we differentiated between various types of emotion according to the addressee of the emotional reaction, namely: emotional reactions towards students, the school principal or other colleagues.

Being aware of the small size of our sample, we cannot say that the data collected represents the general emotional profile of the teachers in our country.

However, it does shed light on some important aspects of the teaching experience that should be considered.

4. Results and discussion

In this section we make use of the narratives of Betty and Rose to characterize some of the important facets and experiences that emerged in relation to EK development. In the scope of this paper we focus merely on EK with respect to students and school principal.

Betty's story

Betty is 56 years old and has more than 29 years of teaching experience. Betty was born and raised in Lebanon. She remembers her classmates "standing tensely and quietly in their places until the teacher entered the class and gave us permission to sit down. All the students behaved politely and respected the teachers, and there were no disciplinary problems...When I came to Israel I knew it was a different country with a different culture but I could not anticipate the extreme differences."

Betty immigrated to Israel when she was 16 years old. When she was 18, she began to study computer science. After graduation she worked as a computer programmer for two years in a large commercial company, and then was offered a position as a mathematics teacher in a middle-high school. She accepted the offer. Betty chose to begin her story as a mathematics teacher with a description of her first lesson in the school:

"Although it happened many years ago I remember it as if it were yesterday. This was my first day at school and I had to teach mathematics in one of the 11th grade classes. I opened the door and I was shocked. All the students were half-sitting, half-lying on the tables and no one even bothered to turn his/her head toward me when I entered the classroom. I felt discouraged. I asked the students to sit properly so that we could start the lesson and they said: "This is how we behave!" I felt hopeless and speechless but after a few seconds I said: "If you do not follow my request, I will leave the classroom." One of the boys went to the door lay down on the floor and said: "Over my dead body!" The rest of the students laughed. I was very close to tears and felt very frustrated and hopeless. But I knew that if I showed any sign of weakness I would not be able to teach this class again. So with my remaining bit of strength I insisted that they follow my instructions which eventually they did. I must admit that from time to time I ask myself what I would have done had they had kept misbehaving..."

Unfortunately, I had to face similar situations several times during my first two years of teaching. I felt like the students were testing me, looking to see how consistent my behavior was...However the second time is never like the first. The first time you confront a certain situation which was not anticipated, the emotional effect is very powerful since it is accompanied by a sense of helplessness. The first time it happens to you, you do not know how to respond,

you feel a lack of proper communication skills, and your self-esteem plunges. However, when you face a similar situation again, knowing that you have already survived such an experience, your emotional reaction (ER) is less intense. You feel like you already know how to handle the situation successfully."

Betty claims that although the first years were difficult she chose not to quit her job: "I had many moments when I asked myself why keep on suffering? However, emotionally, I could not afford to give up. It was actually like admitting that I was not capable of handling a class. I could not bear this thought...It was my pride [smiling] that prevented me from quitting."

Betty's description of her first lesson is full of negative emotional expressions: shock, disrespect, hopelessness, and frustration. These emotions resulted in a sense of "being pushed to the corner," which affected her ER and her decision to use the threat of leaving the classroom against the students. After the students laughed, her emotions intensified to such an extent that Betty was close to tears. The fact that Betty chose to open her story with this lively and unpleasant memory demonstrates how powerful these emotional impressions were. Betty, however, quickly regained her composure and repressed her negative emotions. She chose to use an alternative ER, and then insisted that the students follow her instructions. Although this alternative reaction was successful, the pestering thought of "what would have happened if..." occupied her thoughts for years. It appears as if some sort of "emotional sequence" in Betty's mind remained unsolved.

According to Betty, ERs decrease in their intensity due to the building of EK. The second time she had to face such an episode in the teaching environment, she already knew what to do and how to react. Emotions can either paralyze one's actions or serve as a starting point for learning how to transform them into an actual response. This is the meaning of building EK. In Betty's case, EK that was translated into communication skills with students and knowledge about classroom management. In the ensuing years Betty asserts that she continued to suffer from negative emotional experiences and reactions within the classroom. Building her EK actually sustained her through the inner emotional struggle of whether to give up and thus lose her pride or whether to learn to confront her emotions and regulate and navigate her way through them. Gradually Betty built her self-image as a teacher:

"During the first few years of my teaching I remember that my students kept asking me personal questions. I believe this was their way to get to know me and to adjust their behavior to my expectations. At the beginning I was flattered and I cooperated with them. But then I realized that they interpreted this cooperative behavior of mine to mean I was their friend. When I had to be authoritative they were confused. So I realized that I had to operate differently - to be nice to them not as a friend but as a teacher. In fact, my image as a mathematics teacher was built during that period... I believe that after the first two years at the school my image as a mathematics teacher was solidified and the students conveyed that information about me to new incoming students."

Learning to reflect on her EK also enabled Betty to establish her image as well as her status as an appreciated teacher. Although she was tempted to cooperate with the students and to provide them with personal information, she chose to remain nice to them, but not too friendly. We might say that these were Betty's first steps in developing emotional understanding (Denzin, 1984). Betty concluded her story:

"The main difference between my functioning as a beginning and as an experienced teacher is that as a beginning teacher the types of knowledge I had were disconnected, isolated. I had no idea how to integrate my content knowledge, pedagogical knowledge, and EK. Moreover, I wasn't even aware of the fact that such integration was essential to my success as a teacher. I believe that my reflections on the complexity of class management and student-teacher relations was most dominant in developing my EK and in developing my ability to synthesize these types of knowledge. Only after I was able to balance between these types of knowledge did the intensity of my ERs significantly decrease, no longer being the dominant aspect of my teaching."

Betty's reflection on her evolution as a teacher focuses on the importance of merging academic content, pedagogical, and emotional knowledge. In the beginning her deficiencies in EK created a situation according to which her emotions governed and directed her actions, and they were highly intense. With time, her ability to regulate her emotions, reflect on them to generate EK, minimized their intensity and dominance, and enabled her to recognize EK as equally important as other types of knowledge. It was, however, only after she realized that all types of knowledge were interconnected that she felt she became a good teacher.

Rose's story

Rose is 55 and she has 32 years of teaching experience. Rose's parents were both teachers. Her father was a mathematics teacher. Rose claims that "since I was a child I knew I would never be a teacher. I saw my parents working very hard and I didn't want to be like them." When she was 18 she started studying statistics at the university. She recalls: "I hated every moment there. The teachers were bad. We were more than 100 students in a class, and the teachers didn't know us personally. I was shy, and in such a large class I was embarrassed to ask questions or provide answers." By the end of the year, after failing most exams, she started to wonder whether she had chosen the right profession. Before the beginning of the school year her father suggested that she work as a substitute teacher in his school until the beginning of the university's academic year. She accepted the suggestion "just to save some money." However, "the moment I entered the class I knew – this is what I wanted to do! It was something about the chemistry with the students." Rose left the university and started to study in a small college, where she graduated as a mathematics and physics teacher: "I loved the college. There were no more than 10 prospective teachers in a class, and our teachers knew each of us personally. They encouraged me to ask questions and listened to what I had to say." After her graduation she started to teach mathematics in a middle-high school:

"I was young and naïve, and at the beginning I didn't realize that I was sent to teach classes no other teacher wanted. There were many disciplinary problems, but it didn't bother me. The other teachers didn't understand how I managed to survive these students... When I reflected on my experience at the university and the college, I realized that the alienated attitude at the university as opposed to the close and warm relations between the teachers and students in the college had a tremendous influence on my ability to persist in my studies. So I guessed that if I treated each student warmly and personally, not as a problematic person but as an individual, I would be able to see beyond my immediate emotional difficulties that might stem from disciplinary problems. And it worked... I knew that many students hated mathematics and found it very difficult. It was very important for me to reduce their fears. I knew this was one of the keys to my success as a teacher... Nothing however prepared me for the struggle with the school management. I never realized why the principal of the school was hostile. He didn't speak nicely to me and didn't support me as a new teacher. I tried very hard not to let this affect my work with the students. For me, closing the door of the classroom was like entering an airplane and landing in a different country... As I said, I was naïve and I had nothing to do with intrigues. By the end of the year the principal told me that he didn't want me to teach high-school classes anymore, only middle-school classes. He didn't explain why. He said that because I didn't teach the high level classes he didn't consider me important for the school. I felt insulted and humiliated, and although I loved the students I couldn't bear this humiliation and decided to leave this school."

Rose left the school with "hard feelings. My self-esteem was harmed, and I was confused. I didn't realize what had been disrupted." She found a job in another school, but the supervisor of the former school pleaded to return. She acceded to his request on the condition that she continue to teach her students. Rose feels that "I returned to that school as a winner. I gained back my self-esteem. However, the principal couldn't accept the fact that he was forced to have me back against his will. Emotionally, it was very hard to arrive to school every day. I had no idea how to confront him." Three years later her father told her that there was a vacant position in his school and she "went back to where it all started." This new school was highly selective in those days, and she started to work with "totally different students."

From Rose's story it appears that she had a high emotional self-awareness when she started to teach. Reflecting on her emotional experiences as an undergraduate student, she realized that personal and attentive relations with students are essential for developing their readiness to learn. The fact that by the time she started to teach she had already gained some relevant EK helped her handle successfully problematic disciplinary situations, and not to consider them threatening. In fact, we might say that even if there were any conflicts with the students, Rose put them aside since she was emotionally more occupied by an unexpected front – the bad attitude of the school principal. As a new teacher in school she expected to receive supportive

attention from the school management in general and from the school principal in particular. The principal's attitude hurt her feelings and gave rise to feelings of humiliation and insult in her. Her lack of EK regarding relations with management prevented her from confronting her emotions and coping successfully with the situation she encountered. Rose was not able to resolve the situation, and therefore, with her damaged self-esteem, she chose to leave the school. Trying to recover her self-esteem Rose agreed to return to the school, but during the following three years she did not manage to further develop her EK with respect to teacher-management relations, and she decided to leave the school again, this time forever.

As regards to her relationships with students, Rose believes that she had "a breakthrough when my daughter entered middle-school":

"It happened fourteen years ago, and I realized that my approach to the students was too academic. I didn't really know their emotional world. I understood that when they were angry or in bad mood it wasn't because they wanted to struggle with me, but merely because they were teenagers with emotional distresses. I became more curious about their emotional lives. I wasn't angry when they didn't do their homework. I talked to them personally and tried to be more attentive to their emotions... I tried to develop awareness about what might insult them, to recognize those with whom I could be cynical with, those who needed my encouragement, and those who needed my embrace. I stopped punishing them, because I didn't want to insult them... This emotional approach turned out to be beneficial for them as well as for me. I started to enjoy teaching more... to emphasize values and emotions, and to treat them as equal partners... As I said before, many students are afraid of mathematics, and I became more sensitive to this emotion, and I kept looking for various didactical approaches to help them overcome their anxiety."

Rose's further development of her EK as a teacher occurred when she started to develop her EK with respect to her own daughter. From her, Rose became aware of the reasons that underlie her students' anger and dispositions and started to be more involved in their emotional lives. Her new EK directed her towards developing personal emotional relationships with the students on the basis of each student's personality. Although she was already aware of their fear of mathematics, it was only after she established her EK that she was able to successfully integrate her EK and her didactical knowledge as well as her knowledge about the curriculum.

Five years ago the principal of the school retired, and a new principal started to administrate Rose's school: "This principal is bad for school. Since his first day at school he gathered around him 'yes-men' and formed cliques...I refused to join the 'right' clique and, like other teachers in my condition, I have to deal with his harassment. However, unlike my first school, I don't let it ruin me emotionally. I believe I have learned how to control my emotions, to neutralize them when necessary. I don't take it personally. He has his own personal problems, and I can't be responsible for that."

Rose's last excerpt shows that throughout the years she developed her EK regarding teacher-management relationships. When she had to face hostile behavior for the second time, she was already prepared and her ER towards the situation was not as intense as it had been the first time.

5. Conclusions

Teaching is an emotional practice and the use of emotions can be helpful or harmful (Hargreaves, 2000). Thus there is a need to learn about teachers' EK in order to be able to redirect it in desirable directions.

EK is about developing emotional understanding. The last term is constituted from two words which come from totally different areas. *Emotional* refers to activities ruled by instincts and intuition, while *understanding* refers to activities ruled by logic and cognition. The combination of these two terms implies the need to control and lead the emotions by cognitive means, such as understanding. Moreover, while didactical and content knowledge can be acquired in teacher training programs, EK is dynamically built as a result of human interaction. Moreover, EK is subjective and varies from one person to another. Both Betty and Rose describe EK as a knowledge base that is gradually built and which comes as a result of human interaction. When Betty and Rose made their initial steps as teachers, they were well equipped with didactical and curricular knowledge. Their preliminary EK however was influenced by their previous experiences as learners: in Betty's case – her experience as a pupil in school and in Rose's case – her experience as an undergraduate. Both Betty and Rose refer to EK concerning their interaction with students while Rose refers in addition to EK concerning her interaction with the school's principal. Considering Zembylas' (2007) distinction between the three planes of EK, although Betty and Rose refer to the individual, relational and socio-political planes of emotion, in our paper we relate merely to personal relationships. EK that relates to inter-personal relationships develops as a result of what teachers encounter during their professional lives. Namely, when facing crises in teacher-student or teacher-management relationships, coping with the situation produces an ER which in turn produces a practical reaction that can affect the situation itself. Considering Betty's and Rose's narratives, it appears that ERs differ in their intensity and focal points. The intensity is heavily dependent on the rate of familiarity with the focal point, the teacher's personality, social-cultural background, and more.

That the interviews represent retrospective perspectives of events the teachers experienced many years ago, strengthens the feeling that after all these years they served as milestones in building their EK. It is harder to reflect on ER than on cognitive processes since the first action might involve the exposure of weaknesses and difficulties. It is therefore worthwhile to consider Betty's suggestion to create a kind of support group which can help teachers safely make it through the hard start is unusual, since people often tend to avoid the exposition of their feelings in public.

Both interviewees managed to develop a certain level of ability to reflect on their emotions during their teaching practice. This ability enabled them to develop their emotional understanding regarding their relations with students, the school principal, and other colleagues.

In most professions people face new situations, experience frustration and helplessness, joy and satisfaction, and difficult individuals, among other challenges. The inability to reflect on circumstances and ER, to grow and develop into the profession, can lead one to experience negative feelings such as frustration. These feelings, although essential to the process of growth and development, have a tremendous influence on other aspects of one's personal life (Yaffe-Yanai, 2000). It is therefore important that teachers be able to reflect on their experiences, design and develop their EK, and learn to integrate the different types of knowledge they possess. It would be interesting to listen to the stories of teachers who chose to quit teaching in various phases of their professional lives, and compare their EK to those who persisted.

Our focus is on middle- and high-school mathematics teachers. It is reasonable to assume that elementary school teachers have different stories. It would be also interesting to examine the differences between lower-elementary and upper-elementary school teachers to learn how the students' age influences teachers developing EK.

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HUMOUR AS MEANS TO MAKE MATHEMATICS ENJOYABLE

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The traditional educational system is constructed in such a manner that it excludes humour as a unique live process for promoting knowledge and understanding. Informational communications is the basis of logical thinking instead of vivid dialogue that has an informative purpose. Present work represents an intermediate stage of research of influence of CheCha math method. In particular, humour as the affective factor in mathematical reflection is being considered. With the use this method, the positive emotions that result can influence how teaching material is perceived, can facilitate creation of joyful atmosphere in the classroom, and can help maintain creative state of mind in students.

Key words: problem solving, emotions, humour, classroom climate, motivation

INTRODUCTION

"...the comic thought, which with contradiction, and is strengthened by imagination, is capable of delivering pleasure by training and induces the pupil to participate in dialogue... Game and laughter are higher expressions of living and rejoicing of life" (Muñiz, 1996).

Anyone who has paid attention to great speakers would know that humour is an excellent method for eliciting sympathy from the audience and opening them up to your message. Every teacher also knows that a sense of humour is necessary to winning the hearts of students. Research has established that one's affective state has an effect on cognitive processes (see e.g. Hannula, 2006). How should this inform teaching? Should the teachers focus on creating an entertaining show for their students? Or would the teachers change their lessons into therapy sessions?

This study presents a teaching approach that is built around math problems that are for the student at the same time **Cheerful** (entertaining, funny, cool) and **Challenging** (difficult). We call this CheCha mathematics.

THEORETICAL FRAMEWORK

CheCha math method is based on three educational approaches: acknowledging the role of affect in math learning (Hannula, 2006), using humour in teaching (Grecu, 2008) and use of open-ended problems in math teaching (Pehkonen, 2004).

Affect in mathematical thinking and learning

In order to study affect in math education in contexts of actual classrooms there are three main elements to pay attention to: cognition, emotion, and motivation. Achievement without motivation is not sustainable, and neither is motivation without

enjoyment. All three domains have a more rapidly changing state-aspect and more stable trait-aspect. (Hannula, 2006)

One "fundamental principle of human behavior is that emotions energize and organize perception, thinking and action" (Izard, 1991). Research has confirmed a positive relationship between positive affect and achievement. It seems that the affective outcomes are most important during the first school years, as they are less likely to be altered later on. Two key elements of a desired affective disposition are self-confidence and motivation to learn (Hannula, 2006).

Advances in our understanding of the neuropsychological basis of affect (e.g. Damasio 1995, LeDoux, 1998) have radically changed the old view of the relationship between emotion and cognition. Emotions are no longer seen as peripheral to cognitive processes or as 'noise' to impede rationality. Emotions have been accepted as necessary for rational behaviour. Moreover, research has also shown – although not yet fully understood – that certain emotions facilitate certain type of cognitive processing (Linnenbrink & Pintrich, 2004).

Focusing on motivation we may find ways to influence what the subjects want to do, not only how they try to achieve it. In the existing literature, psychological needs that are often emphasized in educational settings are autonomy, competence and social belonging (e.g. Boekaerts, 1999). These all can be met in a classroom that emphasises exploration, understanding and communication instead of rules, routines and rote learning. However, this requires that all feel safe and perceive that they can contribute to the process. A possible approach to meet all these conditions would be the open approach, and more generally focusing on mathematical processes rather than products (Hannula, 2006).

Humour

Already Kant (1952) considered the nature of humour. He stated "Laughter is the result of expectation which suddenly ends in nothing" (p. 199). His classical statement has started considering humour as a mental mechanism resulting in laughter. As another early scientific approach to humour, Freud (1991) divided comic into wit, humour and actually comic. Many kinds of activity, including wit, are directed on reception of pleasure from intellectual processes. A person feels pleasure from suddenly released energy, which is splashed out in the form of laughter. From this perspective already, we can perceive how a good joke can generate a joyful atmosphere and create a positive emotional background of activity.

The comic, humorous contents can be reached in various ways and techniques. For example, Veatch (1998) suggests a list of types that are funny: finishing to the point of irrationality, satire, literal understanding of metaphors, irony, ambiguity, word-play, contradiction, discrepancy, excessive rationality and a deviation from the usual.

Each of these types of the comic can be expressed as a joke or a problem in math context. As an example of a math contradiction we take a joke, here framed within the world of Winnie the Pooh:

Pooh and Piglet sit on a small bench and talk. Eeyore has sent them a box. In the box there are ten sweets and a note. In the note Eeyore tells them to divide them: seven for Pooh and seven for Piglet. Piglet: "How is that? I do not understand. What do you think of it?" Pooh: "I do not even want to think. But I have already eaten my seven sweets".

Humour can also act as means of a psychological discharge, and promote efficiency of pedagogical activity. Suhomlinsky (1975) wrote:

I would name laughter as a back side of thinking. To develop ability to laugh in the child, to enhance his sense of humour - means to strengthen his intellectual forces, abilities, to teach him to think and to see the world wisely.

Grecu (2008) has considered use of humour in teaching. She highlights seven basic functions of humour in pedagogical activity:

- 1) informatively-cognitive (Opens essential features and properties of subjects and the phenomena. Rejecting standard approaches, the humour bears in itself any discovery),
- 2) emotional (the Humour can act as means of creation of creative state of health and as means of emotional support)
- 3) motivational (The humour can serve as a stimulator of volitional processes)
- 4) communicative (the Person with humour is attractive for people)
- 5) developing (Humour promotes development of critical thinking, a sharpness of vision of the world, observation and consequently intellect)
- 6) diagnostic (by the laughter maintenance - at what the person laughs, it is possible to judge about his merits and demerits) and
- 7) regulative (the humour gives the chance to look at oneself from an unexpected angle, allowing self-evaluation).

In CheCha method most of these are relevant, the most important functions being on top of the list. Grecu suggest the following techniques for designing of humour for educational tasks. These pedagogical techniques are paradox, finishing to the point of irrationality, comparison by the remote or casual attribute, return comparison, wit of absurd, pseudo-contrast or false opposition, a hint, a self-exposure of own faults, intentional ignoring of things that might cause laughter, and exaggeration of the certain features of behaviour.

Grecu has offered also classification of means of the comic: 1) "word-play" based on violation of language norm (carrying of terminology over to a context unusual to it). Consider the following riddle: "*I am it while I do not know that I am. But I am not it when I know that I am. What am I?*" 2) Comparison, author's original neologisms, - based on artistic expressive means (double entendre, an ambiguity). Examples are

easy for finding in Carroll's books (2006, s. 50): “*Explain yourself!*” “*I can't explain myself.*” 3) Paradox, an example being the claim “*I am lying now*”.

Also Dzemidok (1993) distinguishes several humoristic methods: modification and deformation of the phenomena, unexpected effects and amazing comparisons, disproportion in attitudes and communications between the phenomena, imaginary association of absolutely diverse phenomena, creation of the phenomena which deviate from logic. As an example of the latter method consider the following:

There were only 3 students attending a professor's lecture in University. Suddenly 5 persons left the room. The professor said: "If 2 students enter this room, there is nobody attending."

Most types of humour and their techniques could be used at mathematics lessons. Thanks to entertaining tasks and comical contents of the problems the classroom climate promotes a positive interaction between the teacher and students. However, one must be aware that opportunities of humour as pedagogical means have their limits. Grecu (2008) gives several suggestions regarding these limits. She suggests that one should use humour gently and support humour of students. She also warns not to ridicule student's person, laugh at what the student is not able to correct or change or laugh at an involuntary mistake of the student. Rough joking would indicate lack of customs and disrespect of the student and hence is absolutely unacceptable for the teacher. Moreover, the teacher should avoid being the first to laugh at one's own joke, as it can cause the reaction opposite to expected.

Problem solving and open-ended problems

Problems are said to be open, if their starting or goal situation is not exactly given and they usually have several correct answers (cf. Pehkonen 2008). Open-ended problems emphasize understanding and creativity (e.g. Nohda, 2000, Stacey 1995). This would not mean lowering the expectations, quite the contrary. If an open task allows the solver to gain deeper and deeper insights (a "chain of discovery"; Liljedahl, 2005) it can facilitate a state of sustained engagement. This would also lead to more intensive working.

Research has shown that problem solving can be engaging and enjoyable for many students, but it does not attract everyone. Schoenfeld (1985) defined an individual's beliefs or "mathematical world view" as shaping how one engages in problem solving. For example, those who believe that math is no more than repetition of learned routines would be more likely to give up on a novel task than those who believe that inventing is an essential aspect of maths. Unfortunately, there are students who do not see the potential for engagement and enjoyment in a math problem. We see humour as a means to engage also those students who do not perceive math problems enjoyable to begin with.

THE FEATURES OF CHECHA MATH

This research is more about creating tools for teaching than about analysing the reality of classrooms. The work has been started based on the first author's pedagogical intuition as a teacher and his will to engage students with math. This research falls within didactical engineering (Artigue, 1994) or design research paradigm (Cobb, Confrey, diSessa, Lehrer, & Schauble 2003) and it has a clear practitioner approach: "How can the teacher use humour to engage students' interest in math?" Previous experience in teaching had shown that information, when presented in humoristic form, is more convincing and is more easily acquired. This approach has developed gradually over a few years into a teaching approach that assumes:

- * in the same assignment entertainment is combined with a set of difficulty levels;
- * during problem solving there are conditions for emotions to rise;
- * all students can participate actively in solving the assignment regardless of their abilities.

The educational space is constructed in such a manner that teamwork of the teacher and students accepts dialogue character and interest in mathematics is favored. While using CheCha method, we separate the following basic constructs: a) *entertainment in learning process*, b) *level of the problem's difficulty*, c) *plurality of problem solutions*. We refer to as *entertainment in learning process* the affective components which excite the interest, draw attention and/or create a joyful atmosphere. For example, as entertainment we assume appeal, extraordinary content, intriguing title and/or amusing formulations. *Level of the problem's difficulty* we define as the variable degree of solution's complexity, beginning from the "obvious", achievable for many children, proceeding to a more complicated. It is important that the simplest way not always guides to the right solution. *Plurality of problem solutions* is a construct that consists of variety of means and ways of solving problem on the same level of abstractness, understanding and complexity. Various approaches are possible in one problem and it is supposed to have both a set of ways of solving and sets of different solutions as a whole. For example, to create a problematic math situation such parameters, as incomplete condition, the overloaded contents, or introduction of "not existing in reality" factors are used.

RESEARCH METHODS

In this paper, we shall describe the method of creating mathematical assignments (CheCha problems) and evaluate the practice of CheCha math teaching. We explore

1. What mathematical problems are entertaining from the students' point of view?
2. How CheCha method influences the atmosphere in mathematics lessons?

The construction of CheCha problems

The technique of construction of such problems consists of certain stages. At the initial stage there is a search of "matrix" of a condition or its author's creation. Useful sources to find problems that can be developed into CheCha problems have been math jokes, E. Lear's (e.g., 2002) and L. Carroll's (e.g., 2006) books, collections of problems from math Olympiads. Chessboard has also been a good setting for such problems. The original problem is typically open or can be modified into an open problem, meaning that it has no unique and final solution.

The next principle is to consider age-typical interests of students, their specific personalities and personal preferences. Substantial richness of a context of a problem is carried out at a following stage. There is a transformation into a context that bears in it entertainment, extraordinary and comic flavour or lively situations. At the same time, level of difficulty and plurality of the solutions is considered, allowing a wide range of different levels of solutions and approaches.

Then the problem is introduced to students and there is the opportunity for feedback, which is stirring up cognitive activity through questions, solutions and discussions. The teacher observes and reflects upon students' thinking during problem solving, focussing on: the perception of a problem by students (acceptance or non-acceptance); questions asked by them (depth and breadth); a degree of understanding of the context. These help the teacher to find direction for task's development.

It is important to notice that for every area of math teaching and learning one can find or construct such CheCha problems. This may lead to creation of a new problem, or changing of the task. For example: *“Three tortoises go one after another along the road. The tortoise says, “Two tortoises follow my rear”. The second says, “One tortoise goes ahead”, “One goes back of me”. The third says, “Two are ahead”, “One creeps behind”. How can this be?”* One should note that this problem is more attractive than something about moving material points along a straight line, with particular coordinates. The most common answer here is that it is impossible. But, in fact, there can be the solutions. *“Three tortoises go...”*: the words of the third tortoise contradict each other. The solution might be that the last tortoise is lying! ...One tortoise is riding on another. ...There is a time lapse between the phrases, allowing one tortoise to run ahead. ...The fourth tortoise stays near or behind the last turtle, and begins moving after the first phrase of the third turtle... The road is circular... The road is triangular... There is a mirror behind the last turtle. When it looks at its back, it can see one more turtle. Progressing from considered examples, and, instead of tortoise, we turn to another object, e.g., cows. One more possible solution is the birth of a calf!

Using CheCha problems in teaching and feedback from students

Research was carried out in two Finnish schools (Espoo 2007-08 and Helsinki 2008-09), in 7th classes with different level of acquaintance with CheCha math method and various educational atmospheres. The first author was teaching in these schools.

1. In December, 2007 the first author surveyed students' preferences of entertaining features in maths. The questionnaire consisted of five questions of open and closed types, e.g. *"What in a math problem can be entertaining?"* Two questions were multiple choice questions concerning the respondents' view of entertaining maths. Respondents were 40 students from two seventh classes and one eighth class.

2. In February, 2008 a second questionnaire was given in the same school (Espoo) to the students, where they were asked *which kind of problems they preferred*. In this survey 40 seventh graders from the same three classes responded.

3. In September, 2008 another questionnaire was administered in a school in Helsinki. The data were collected in two 7th grade classes (40 students) within the first month of employment of first author as the teacher in this school. Students were asked to fill in a questionnaire and draw a picture of a topic "Me at a math lesson".

a. In the first class (19 respondents) there was a favourable educational atmosphere and teacher-student relations were built at dialogue level. The atmosphere was promoted by playing Chess, Go, Katamino and other intellectual games. This was a basis for the future introduction of the CheCha method.

b. A comparison group (for the same survey) was a seventh grade class (21 respondents) of another teacher, in which CheCha method was not applied.

RESULTS

1. When responding what can be entertaining in maths, the frequency of choices were humour (55 %), "something else" (27 %), "cutting and drawing" (25 %), "unusual names and properties" (13 %), "plurality of answers" (13 %), and "fabulousness of a plot" (10 %). Altogether 88 % of students mentioned reasons why maths can be entertaining, and 5 % of children had written "nothing" in their specification of what the 'something else' could be.

2. The students' task preferences has shown, that tasks of comic character were most popular (51 %), then were the tasks that could be solved using Lego or Chessboard (33 %), cutting and drawing (30 %), a fantastic plot (15 %) and unusual names and properties (12 %), (Figure 1).

3. a. In this second sample the preferences were slightly different (Figure 1). This time the most popular choice was cutting and drawing (58 %), then the comical character (47 %), the tasks solved with the help Lego and Chess of (26 %), further a fantastic plot (21 %) and unusual names and properties (11 %).

The results of this survey have shown that 74 % of the respondents mention reasons why mathematics can be entertaining. Half of the students mention chess, and 29 % the personality of the teacher as the defining factor.

In the drawing task, 63 % have drawn a joyful image of a math lesson, 11 % of respondents drew themselves thinking or pondering, 15 % represented subjects of maths presented in a positive light (e.g. a notebook with the tasks solved correctly).

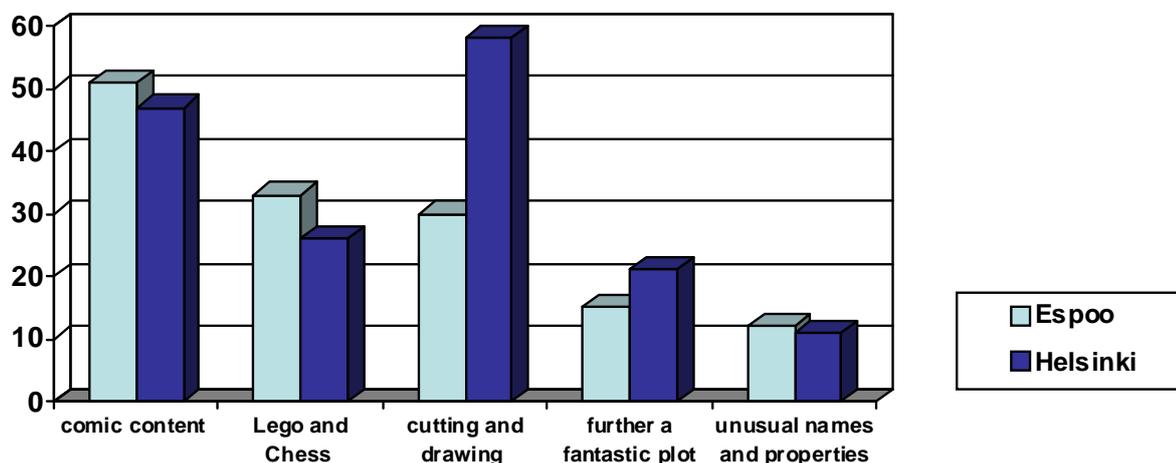


Figure 1. Students' responses to which types of tasks they prefer

When asked to continue the sentence "The CheCha-maths is ..." the most frequent answers were "Great!" (21 %) and "fun" (21 %). 16 % of students noted that it is simultaneously a game and study. There were also individual answers of such a character as "creative and interesting", "many-sided", "various" and "laughter".

3. b. The other survey in the class where CheCha maths was not applied produced somewhat different responses. For the question "It is possible to take pleasure at math lessons" only 26 % gave a positive answer, mostly responding utility of maths, instead any reference to its enjoyable nature. Also the drawing test did not show joyful atmosphere at a lesson. The priorities chosen by these respondents were cutting and drawing (67 %), the comical character (43 %), a fantastic plot (33 %), unusual names and properties (24 %) and the tasks solved with the help Lego and chessboard (10 %). On the offer to make definition "The entertaining maths is ..." the most frequent response was that such maths "is impossible" (29 %). Then was "drawing" (24 %) and there was a fair amount (29 %) of other positive characterisations (e.g. "games", "humour", "of a funny nature", "easy").

CONCLUSIONS

One growing branch in mass media is 'edutainment' where EDUcational purposes are combined with enterTAINING qualities and interaction possibilities (e.g. computer games). Could math education learn something from the edutainment business in order to deepen the students' engagement with maths? We strongly believe that it is

possible to develop suitable (open and multilevel) math tasks with attractive humorous flavouring, that make learning of maths very close to matter of laughter.

When this method was tried out and developed in different schools, the students' feedback points out how the teacher can use humour to engage students with maths:

1. From the students' point of view, entertaining tasks associated largely with humorous content. The longer students are working with humorous tasks, the higher percentage of students prefers such problems over other types of problems.

2. CheCha math method influences the atmosphere in the lesson. The use of intellectual games (or creating a favourable atmosphere in other ways) prepares the ground for the use of humour in the lesson. In an unfavourable atmosphere, comical assignments can lead to undesirable results. The importance of the overall receptive atmosphere was observed in fall 2008. In one of the 7th grade classes taught a part of students responded negatively to use of comic tasks, speaking about "irrelevance" of jokes. When math problems were not understood, the comic presentation of problems caused negative reaction in a part of children. However, tasks with fantasy characteristics did not cause negative reaction. Students were distracted into conversations among themselves, and they moaned about the inconvenient arrangement in a class (the uncomfortably big group was placed in a computer class, not suitable for math lessons). After a replacement into an ordinary classroom the atmosphere had changed into more positive. Playful statements and problems began to be perceived positively, increasing motivation to learn.

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BELIEFS: A THEORETICALLY UNNECESSARY CONSTRUCT?

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In this paper I analyze different existing definitions of the term beliefs, focusing on relations between beliefs and knowledge. Through this analysis I note several problems with different types of definitions. In particular, when defining beliefs through a distinction between belief and knowledge systems, this creates an idealized view of knowledge, seen as something more pure (less affective, less episodic, and more logical). In addition, attention is generally not given to from what point of perspective a definition is made; if the distinction between beliefs and knowledge is seen as being either individual/psychological or social. These two perspectives are also sometimes mixed, which results in a messy construct. Based on the performed analysis, a conceptualization of beliefs is suggested.

Key words: belief, definition, individual, knowledge, social

INTRODUCTION

There exists plenty of research in mathematics education focusing on aspects of beliefs, in recent years evident by books covering this specific topic (e.g., Leder, Pehkonen, & Törner, 2002b). However, Thompson (1992) points out that although the topic has been popular in educational research for many years, little attention has been given to theoretical aspects of the concept of beliefs. Specifically for mathematics education, Op't Eynde, De Corte, and Verschaffel (2002) note the same type of lack of theoretical studies about beliefs.

In many studies, the term 'belief' is not explicitly defined, but it is assumed that the reader knows what is meant (Thompson, 1992). For some purposes this might suffice, and in general different types of definitions, from informal to extended types, could be suitable depending on the situation (McLeod & McLeod, 2002). In addition, a theoretical perspective can focus on different aspects, for example by being more or less philosophically or psychologically oriented. When Schommer (1994) discusses different types of beliefs as key concerns in the conceptualization of epistemological beliefs, she argues that interesting results, perhaps of a more applied type, can be achieved also without explicit focus on the more philosophical aspects, but that the inclusion of such aspects would improve the conceptualization of beliefs. A philosophical perspective can include what McLeod and McLeod (2002) describe as part of a more elaborate definition, such as relations to nearby concepts. For beliefs, this elaboration could include relations between beliefs, knowledge, and different affective constructs.

When studying beliefs, instead of analyzing and arguing around different types of definitions of beliefs, it seems most common to describe different definitions found in the literature and then choose one of these or create your own for the study in question (if a definition is at all given). Even if it is perhaps impossible to create a general definition that is suitable for all types of research (as noted by Abelson, 1979; McLeod & McLeod, 2002), there is a need to discuss and analyze different types of definitions. In the present paper the focus is on such analyses.

Purpose

As the title of the present paper implies, I am taking a critical perspective regarding the concept of beliefs and suggestions of how this construct can be defined. This critical stance has evolved from informal, personal reflections when having read different types of studies of beliefs, and similarly as Pajares (1992), having noted a certain messiness regarding definitions and properties of beliefs. I have not only noted such messiness when looking at the breadth of different studies, where plenty of different types of definitions or properties are described, but also when trying to analyze the internal coherence of singular articles regarding definitions and properties of beliefs.

The main purpose of the present paper is to dig deeper into these reflections, in order to see what types of problems seem to exist when trying to define beliefs and also if and how these problems can be resolved. In particular, I will suggest a type of reconceptualization of beliefs, emerging from noted problems around (1) the point of perspective taken when defining and describing properties of beliefs, and (2) relationships between beliefs and knowledge.

It is important to note that I am not suggesting that the ideas presented here should be seen as final in some sense, but that they primarily constitute a starting point in my attempts to reconcile with some experienced problematic issues, for continued discussions and reflections and for continued work on a larger research project (see Österholm, in press). Also, I am not suggesting that I am presenting an entirely new perspective, regarding the mentioned reconceptualization, but as can be seen by references given throughout the present paper, others have presented similar suggestions, although sometimes done from other perspectives or focusing on somewhat different aspects of beliefs.

Research about beliefs

Historically, the interest in educational research in the study of beliefs seems to come from realizing that a focus on “purely cognitive” factors (in particular, content knowledge) is not sufficient when trying to describe and explain students’ problem solving activities (Pehkonen & Törner, 1996; Schoenfeld, 1983) or teachers’ classroom behavior (Speer, 2005). The relationship between (content) knowledge and beliefs is thus a central aspect. This relationship is also the most commonly referred to when discussing the definition of beliefs, and different views about this

relationship can also be seen as a major reason for experiencing beliefs as a messy construct (Pajares, 1992).

Since there can be different types of knowledge, such as procedural or conceptual, while beliefs are usually formulated as statements, the comparison between knowledge and beliefs can focus on factual, declarative knowledge.

BELIEFS – AS SEEN FROM DIFFERENT POINTS OF PERSPECTIVES

Abelson (1979) describes a cultural dimension of beliefs; that if all members of some type of group have a specific belief, then they might not label it as a belief but as knowledge. This cultural dimension corresponds to what other authors describe as a social property of knowledge (e.g., Op't Eynde et al., 2002; Thompson, 1992); that for something to be seen as knowledge it has to satisfy some type of truth condition – a condition that is negotiated and agreed upon within a community (of practice). Thus, depending on what social community you belong to, you can have different views on what is seen as knowledge and what is seen as belief. From this perspective, when focusing on social aspects, the difference between belief and knowledge can be defined by saying that knowledge fulfills the mentioned social criteria but that beliefs do not, or perhaps *cannot*, since there can exist statements that cannot be evaluated using existing criteria within a certain community.

This relative property of beliefs highlights the importance of taking into account from what perspective a labeling of something as a belief or as knowledge is being done. In addition, there is also the possibility of changing perspective when deciding on the definition of beliefs, from defining beliefs from a social perspective to defining beliefs from an individual perspective. For example, when Leatham defines beliefs he describes the relationship between belief and knowledge by seeing that

there are some things that we “just believe” and other things that we “more than believe – we know.” Those things we “more than believe” we refer to as knowledge and those things we “just believe” we refer to as beliefs. (Leatham, 2006, p. 92)

This type of definition describes the relationship between beliefs and knowledge as a psychological property. A somewhat different defining property of beliefs, but also from the individual perspective, is given by Abelson (1979); that the believer is aware that others may believe differently. This property includes a social dimension but the distinction between beliefs and knowledge is still being done from the individual perspective, and is psychological in nature. From this perspective, when focusing on the individual, the difference between belief and knowledge can be defined by seeing beliefs as something related to uncertainty, either in relation to other parts of an individual's beliefs/knowledge or in relation to what others claim to believe/know.

Sometimes an author describes some defining properties of beliefs that are from an individual perspective and some other properties that are from a social perspective. For example, I have mentioned Abelson (1979) when describing both these

perspectives, and Pehkonen and Pietilä (2003) also include both these perspectives when differentiating between beliefs and knowledge. The simultaneous use of these different perspectives when defining a concept could be a cause for creating a messy construct. However, it is often difficult to decide if all given properties should be seen as part of a homogenous definition or as something that can be inferred from a (sometimes implicit) definition or from empirical results.

From this analysis we can see that a central distinction in the discussion of beliefs and knowledge is from what perspective a definition or description is given, whether these concepts are construed as individual or social. This distinction deals with whether the decision regarding differences between belief and knowledge is located in the individual (i.e., that it is psychological in nature) or if it is located in the social community. Independently of which of these perspectives is used when defining beliefs, there is also another aspect of different perspectives; that different persons can have different views on what is regarded as knowledge and what should be labeled as belief, that is, there is a relative property of beliefs. This property is caused by taking the relationship between beliefs and knowledge as a starting point when defining beliefs and is also based on a general view of knowledge (which has previously not been stated explicitly in the present paper), that knowledge is “not a self-subsistent entity existing in some ideal realm” (Ernest, 1991, p. 48), but that knowledge is seen either as an individual construction (what Ernest labels as subjective knowledge) or as a social construction (what Ernest labels as objective knowledge).

TYPES OF DEFINITIONS OF BELIEFS

Sometimes it can be difficult to analyze some of the definitions and properties of beliefs since authors do not always motivate or describe these defining properties in detail. For example, it is sometimes mentioned, without further explanation, that beliefs can be conscious or subconscious (e.g., Leatham, 2006; Pehkonen & Törner, 1996), but since the concept of consciousness in itself is very complex (e.g., see Velmans, 1991) it is difficult to interpret such a suggested property of beliefs. In particular, the interpretation becomes more difficult if some definition of beliefs has not been given, or if no connection is made between a certain property of beliefs and a given definition.

One way to define beliefs is to focus on the claim that a person believes that (or has the belief that) a certain statement is true. The question of what you mean by such a claim deals with the definition of beliefs. For example, a belief can be seen as a type of knowledge that is “subjective, experience-based, often implicit” (Pehkonen & Pietilä, 2003, p. 2), or as a personal judgment formulated from experiences (Raymond, 1997, p. 552). However, many such definitions seem to be of an informal type (as labeled by McLeod & McLeod, 2002), since they most often do not explicitly describe what is meant by all words used in the definition and how these words/properties create a construct different from nearby concepts.

Another way to define beliefs, or at least to describe some properties of beliefs, is to focus on relationships between different beliefs, and thereby describe characteristic properties of so-called belief systems. Certain differences between belief systems and knowledge systems can then be taken as a characterization of beliefs. In the literature it seems common to refer to Abelson (1979) and Green (1971, as cited in for example Furinghetti & Pehkonen, 2002; Leatham, 2006; Op't Eynde et al., 2002; Pehkonen & Pietilä, 2003; Raymond, 1997) who both have proposed such differences between the two kind of systems. Since references to belief systems seem quite common in the mathematics education literature, I will in the next section analyze the notion of belief system regarding the view of knowledge that is implicitly, and sometimes explicitly, created through the separation of belief and knowledge systems.

While a definition that focuses on a singular belief/statement can be done from both an individual and a social perspective, implicit in the type of definition that focuses on belief systems seems to be a view that such systems are psychological constructs.

Properties of belief systems – creating an idealized view of knowledge

There is no consensus in the research community on the positioning of beliefs on a cognitive-affective scale (Furinghetti & Pehkonen, 2002), but it is sometimes claimed that a difference between belief and knowledge systems is that the former has, or at least has a relatively stronger, affective component (Abelson, 1979; Speer, 2005). However, it is unclear why, for example, a certain belief about mathematics teaching should have a greater affective component than the knowledge of the relationship between the diameter and the circumference of a circle. The situation (or the several situations) when the knowledge about the circle has been dealt with could very well have been strongly loaded with affect, for example from the joy of discovering this relationship or the dislike of having another fact to memorize. Such existing affective components of knowledge are also pointed out by Pajares (1992).

Also, it is seldom explained in detail how or why beliefs should be regarded as 'more affective' than knowledge, and when McLeod (1992) describes a framework for the study of affect, it is pointed out that beliefs are not emotional in themselves but that the role of beliefs is one (central) factor when attitudes and emotional reactions to mathematics are formed.

Some claim that belief systems are more episodic in nature than knowledge systems; that beliefs have a closer connection to specific situations or experiences (Abelson, 1979; Speer, 2005). This property seems to lie close to the clustering property described by Green (1971, as cited in Leatham, 2006), which permits the belief system to consist of clusters of beliefs that can be more or less isolated from each other. Leatham (2006) describes this property as a means to explain the contextualization of beliefs and that a person can hold different beliefs that can seem to contradict each other, if these beliefs belong to different clusters. However, learning and thereby knowledge is also always situated and context dependent, "resulting in clusters of situated knowledge" (Op't Eynde et al., 2002, p. 25).

Another suggested difference between belief and knowledge systems is that belief systems are built up using quasi-logical principles while knowledge systems are built up using logical principles (Green, 1971, as cited in Furinghetti & Pehkonen, 2002). For example, it is claimed that relationships between beliefs cannot be logical “since beliefs are arranged according to how the believer sees their connections” and also that “knowledge systems [...] cannot contain contradictions” (Furinghetti & Pehkonen, 2002, p. 44). If a person’s knowledge system is not built up around how this person sees the connections between different components of the system, it seems unclear exactly who or what is creating the structure within the system. In this case knowledge is perhaps not referred to as an individual, psychological construct but seen as a social construct. However, also when seeing knowledge from such a perspective it becomes difficult to reconcile with the statement that knowledge systems cannot contain contradictions, since the history of mathematics includes examples of such contradictions, for example regarding the connection between convergence of series and the limit of the general term (see Leder, Pehkonen, & Törner, 2002a, p. 9). You could explain this by viewing knowledge as something absolute and thus maintaining that knowledge systems cannot be contradictory, by seeing contradictions as stemming from beliefs and not from knowledge.

In summary, regarding the relationships between beliefs and knowledge based on existing suggested properties of belief systems, knowledge is described as less affective, less episodic, and more logical and consistent. These properties create an idealized picture of knowledge, as something pure and not ‘contaminated’ with affect or context.

A PROPOSED CONCEPTUALIZATION

Based on the analysis about different types of definitions of beliefs that can be made from different points of perspectives, I here discuss a conceptualization of beliefs that take into account the criticism that has been put forward. I am not suggesting that this conceptualization is necessarily suitable for all types of studies or situations, but that it is one way to relate to some of the problems that seem to exist when defining and describing beliefs.

Beliefs are seen as being related to uncertainty in some way. From some observer’s perspective a statement can be labeled as a belief for different reasons, but all related to some degree of uncertainty, as described in the following examples.

The first example is that if a statement cannot be included in, or directly related to, some (traditional) existing (scientific) content domains, such as mathematics or pedagogy, it can be labeled as a belief. For example, Ernest (1989) and Schoenfeld (1998), who do not explicitly discuss the definition of beliefs, describe beliefs and knowledge as two separate categories. Included in these categories are *knowledge* about teaching and learning, and *beliefs* about the nature of teaching and learning,

where the former can be included in the domain of pedagogy while the latter perhaps cannot (but perhaps can be included in the domain of philosophy).

A second example of a reason for labeling something as a belief is if a statement contradicts something that is part of some scientific domain. For example, this is done by Szydlik (2000) who discusses content beliefs, which for example include to see the existence of gaps in the real line.

Both these examples are from a social perspective since they relate to domains (i.e., communities of practice), but the property of uncertainty was also mentioned earlier when discussing beliefs defined from an individual perspective. What an individual regards as belief is something that is more uncertain than knowledge. The level of uncertainty refers to how confident a person is that a statement is true. That is, a person has some (implicit) criteria from which it can be decided if something is labeled as belief or knowledge. Törner (2002, p. 80) describes this as measuring certainty on a scale from 0 to 1, where knowledge can be seen as a special case in his framework of beliefs, possessing the certainty degree of 1.

Thus, uncertainty can be seen as a more general aspect of beliefs, regardless of from what perspective the concept is defined, either the social or the individual.

Unlike uncertainty, an aspect that can differ depending on from what perspective beliefs are defined is whether a belief, when compared to knowledge, is seen as a different type of psychological object. From a social perspective it becomes difficult to motivate that beliefs and knowledge refer to such different types of objects since the difference by definition is a social construction. Therefore, when studying the behavior of individual persons (such as teachers' activities in classrooms or students' problem solving activities) the social perspective does not seem suitable when defining beliefs. This has also been highlighted by other authors, for example by arguing that

individuals (for the most part) operate based on knowledge as an individual construct. That is, their actions are guided by what they believe to be true rather than what may actually be true. (Liljedahl, 2008, p. 2)

Others have also suggested that one should focus on the study of conceptions as a whole, which includes what some label as beliefs and knowledge (e.g., Thompson, 1992). However, there could be a reason to study beliefs as defined from an individual perspective, such that beliefs and knowledge from this perspective can be seen as psychologically different types of objects, since experienced differences in the degree of uncertainty could affect behavior differently. Empirical studies seem necessary for deciding if there is a reason to make such a distinction or if it is more reasonable to see the whole of a person's conceptions.

These presented perspectives on beliefs mainly focus on singular statements and not on properties of a system of beliefs compared to a system of knowledge. This type of conceptualization is chosen because of the problems noted about the systemic view

when defining and describing properties of beliefs, in particular the tendency to create an ideal and problematic view of knowledge. Also, the presented perspectives put an emphasis on the person making a claim about relationships between beliefs and knowledge, which some authors also have noted, but have not taken as a more fundamental aspect.

CONCLUSIONS

In the present paper, two main issues have been highlighted through the analysis of existing definitions and descriptions of properties of beliefs:

- (1) The important issue of explicitly focusing on the point of perspective taken when defining and describing properties of beliefs, in particular the difference between taking a social or individual perspective regarding where the difference between belief and knowledge is located.
- (2) The problematic issue of trying to define, in an objective manner and focusing on the individual, the difference between beliefs and knowledge through the separation of belief and knowledge systems.

Due to these issues one can question the necessity of the concept of beliefs, since the difference between beliefs and knowledge is not construed as so absolute, but that the meaning of the concept can be relative with respect to the person labeling something as a belief. In this way, beliefs are not seen as being used for making an important theoretical distinction between belief and knowledge, but more seen as a linguistic tool to signal what type of object/statement is in focus, as seen from the person making a claim about beliefs. Thus, the notions of belief and knowledge may say more about an observer than they do about some important theoretical distinction between two types of entities “within” the person being observed. In this sense, the concept might have lost some of its theoretical importance.

The most central point in my analysis and criticism is directed towards certain contradictory aspects in the existing literature, in particular that a common psychological perspective presented through the distinction between belief and knowledge systems implies a more idealized view of knowledge than what is existent in the social perspective of knowledge. Most often, when aspects of both these perspectives are mentioned, there is no in-depth analysis of possible relationships or contradictions between these aspects. Even when Op't Eynde et al. (2002) perform a more in-depth analysis of the social perspective, they also claim the existence of a psychological difference between beliefs and knowledge, by mentioning the quasi-logical property of beliefs. I see this use of a mixture of different perspectives as a central cause for the creation of beliefs as a messy construct. Thus, a main topic when defining beliefs is to decide, based on what is being studied, which perspective is the most suitable one when defining beliefs, the social or the individual, and then to be consistent within this one perspective.

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CATEGORIES OF AFFECT – SOME REMARKS

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Cognitive concepts were insufficient to explain some effects observed in mathematics learning, particularly differences in performance. So researchers began investigating the influence of affect on the learning process, using the concepts of beliefs, attitudes, emotions and values. This paper discusses questions connected with the theoretical status of these concepts.

Introduction

McLeod (1992) wrote in his survey paper, “Research on Affect in Mathematics Education: Reconceptualization”, that beliefs, attitudes and emotions are used in mathematics education research to describe a wide range of affective responses to mathematics. Although terms and concepts are often transferred from psychology to mathematics education, McLeod points out why such a transfer to the affective domain can be problematic:

Terms sometimes have different meanings in psychology than they do in mathematics education and even within a given field, studies that use the same terminology are often not studying the same phenomenon.... Clarification of terminology for the affective domain remains a major task for researchers in both psychology and mathematics education. (McLeod, 1992; 576)

There have been efforts to clarify the meanings of these concepts, particularly with respect to beliefs and attitudes. In a paper appearing in the collection, “Beliefs: A Hidden Variable in Mathematics Education”, Furinghetti and Pehkonen (2002) describe a process that clarifies some shared core elements commonly mentioned in characterizations of beliefs:

Using an international panel we looked for common background suitable in describing the characteristics of the concept of beliefs and the mutual relationship in the critical triad “beliefs – conceptions – knowledge”. (Furinghetti and Pehkonen, 2002; 46)

Even if it were not possible to reach a common shared definition of beliefs, the paper clarifies some of the common and contrasting meanings of this concept.

With respect to the problem of definition in the case of “attitude toward mathematics”, we find a situation analogous to the one described by Di Martino and Zan (Di Martino and Zan, 2001; Zan and Di Martino, 2008); namely, a

...lack of clarity that characterizes research on attitude and the inadequacy of most measurement. (Di Martino and Zan, 2008; 197)

In their analysis of academic papers, Di Martino and Zan found three types of definition of attitude toward mathematics: a “simple” definition where attitude toward mathematics is seen as being either a positive or negative emotional disposition toward mathematics; a multidimensional definition where three components constitute attitude – emotional response, beliefs regarding the subject

and behaviour related to the subject; and a bi-dimensional definition where attitude toward mathematics is seen as a pattern of beliefs and emotions associated with mathematics.

The lack of clarity in what “beliefs” or “attitude toward mathematics” means also has implications for research in the affective field. Thus Sfard writes:

Finally, the self-sustained “essences” implied in reifying terms such as *knowledge*, *beliefs*, and *attitudes* constitute a rather shaky ground for either empirical research or pedagogical practices – a fact of which neither research nor teachers seem fully aware. (Sfard, 2008; 56)

Hart, too, referred to this problem and wrote that

research on the affective domain in mathematics education is in need of a strong theoretical basis that will be developed only through sustained, systematic efforts over time. (Hart, 1989; 38)

All of this suggests we have to rethink the concepts used in research on affect, and, moreover, it seems necessary to consider the problem in a more general way: “Wherein lies the problem of defining concepts and, in relation to this, what is the status of research methods?” “Can results from other fields help us better understand the categories of affect?”

General aspects of concepts

In his paper, “Aspects of the Nature and State of Research in Mathematics Education”, Niss (1999) refers to a crucial fact permeating all research:

It is important to realise a peculiar but essential aspect of the didactics of mathematics: its *dual nature*. As in the case with any academic field, the didactics of mathematics addresses, not surprisingly, what we may call *descriptive/explanatory* issues, in which the generic questions are ‘what *is* (the case)?’ (aiming at description) and ‘*why* is this so?’ (aiming at explanation). Objective, neutral answers are sought to such questions by means of empirical and theoretical data collection and analysis without any explicit involvement of values (norms). (Niss, 1999; 5)

We use terms and concepts to describe and explain phenomena: therefore we have to see if this duality can be discerned in our terms and concepts.

In the literature on mathematics education numerous accounts exist of deep considerations of mathematical concepts (see, for instance, the Special Issue “Semiotic Perspectives in Mathematics Education” in *Educational Studies in Mathematics Education*, Saenz-Ludlow and Presmeg, 2006). In these papers, the focus is on the process of construction of the meaning of mathematical concepts. We therefore need to consider the process of constructing the meaning of concepts used in mathematics education research, with a special focus on affective concepts.

Let us discuss the meaning-construction-problem as encountered in the study of affect from a more general viewpoint; i.e. one that considers the ontological and other status of the concepts in the scientific research process, particularly in the way the latter’s relationship to a concept’s meaning.

In semiotics researchers analyse the relationship between symbols and referents. Frege discussed this in his important paper, “Zeichen, Sinn und Bedeutung (Sign, Sense and Meaning)”. Here, “meaning” represents the objective idea of a thing; “sense” contains the subjective interpretation made by a person relating to this thing; and “sign” designates the objective idea (Kilpatrick, Hoyles, Skovsmose, & Valero, 2005; Steinbring, 2005). In modelling the process of meaning construction, Steinbring (2005) uses the scheme of an “epistemological triangle”, in which sign/symbol, object/reference context and concept form the triangle’s corners:

Mathematics requires *certain sign or symbol systems* to record and codify knowledge... these signs do not immediately have a meaning of their own. The meaning has to be produced by the student or the teacher by establishing a mediation between *signs/symbols* and suitable *reference contexts*. (Steinbring, 2005; 22)

Sfard stresses the discourse aspect of a concept definition:

A concept is a symbol with its use. (Sfard, 2008; 111)

Within this concept definition, the term “symbol” includes more signifiers than words; and “use” refers to the use of a symbol in a discourse (Sfard, 2008; 236). This extension of the term “meaning of a symbol” to its use in a discourse process allows attention to be directed toward more perspectives (such as that of emotional reaction) than was possible in Frege’s classical concept of meaning. Otte refers to the important fact that all our perceptions include elements of interpretation as well as of generalization and therefore all knowledge is in a certain sense indirect knowledge and a function of symbols and representations (Otte, 2005; 231). Thus understanding concepts is a cognitive activity that is connected with intuition:

Thom, and Bruner as well, intend to draw attention to the fact that we cannot develop our cognitive activities if we do not believe in the reality of our intuitions, and that these intuitions or mental states nevertheless may be treacherous and without objective validity or reference. Subjective meaningfulness and objective validity may not coincide. (Otte, 2005; 231)

Reading this quotation, moreover, raises the question of how an individual acquires a concept. Two answers may be found in mathematics education research, depending on how the problem is viewed. Following the ideas of Piaget, intellectual growth results from a direct interaction between the individual and the world; on the other hand, according to social constructivism,

...whatever name is given to what is being learned by an individual – *knowledge*, *concept*, or *higher mental* function – all these terms refer to culturally produced and constantly modified outcomes of collective human efforts. (Sfard, 2008; 77)

We should probably accept that knowledge and concepts are outcomes of a cultural process and neither can be learned outside a discourse community. For instance, a learner needs help from an experienced person (Lave and Wenger describe this learning process as “legitimate peripheral participation” (Lave and Wenger, 1991)). Furthermore, we ought to consider the individual parts comprising the acquisition process. Lakoff and Nunez refer to the important role of metaphors:

One of the principal results in cognitive science is that abstract concepts are typically understood, via metaphor, in terms of more concrete concepts. This phenomenon has

been studied scientifically for more than two decades and is in general as well established as any result in cognitive science (although particular details of the analysis are open to further investigation). One of the major results is that metaphorical mappings are systematic and not arbitrary. (Lakoff and Nunez, 2000; 40 – 41)

This role of metaphors is important to keep in mind – especially if we transfer concepts, such as attitude, from other fields– because the borrowed concepts are combined with metaphors in our field to understand the concepts already present in our field. We must specify the metaphors required for using the concepts in our field, mathematics education.

A second crucial point is strongly connected to our use of language. We use words or symbols that are the endpoints of a process of objectification; and these words or symbols produce the illusion that they are in the same category as things, yet they can have no empirical manifestation:

After objectification, we often interpret metastatements, that is, statements about discourse, as statements about the extradiscursive world (...) This *ontological collapse* (a) may produce an *illusory dilemma*, (b) can result in *phony dichotomies* leading to tautologies disguised as causal explanations, and (c) is likely to lead us to *consequential omissions*; blinding us to potentially significant phenomena that cannot be described in ontologically “flattered” terms. (Sfard, 2008; 57)

In the light of this, we ought to keep in mind that concepts used in mathematics education research that are formulated in words have no empirical manifestation – and therefore no reference objects – and they get their meaning through the metaphors and associations that we imagine in connection with the symbol for the concept. In mathematics one can use a “realization tree” (Sfard, 2008; 165) to overcome, in a certain sense, the lack of a reference context; however, for concepts encountered in mathematics education we have no such realization tree.

The problem of meaning construction for affective categories

Research into affect was motivated by the fact that cognitive concepts were insufficient to explain some of the effects observed in mathematics learning (McLeod, 1992), such as differences in the outcomes of mathematics learning. To explain these differences, researchers used affective concepts such as attitudes and beliefs. Thus differences in mathematical performance were also viewed as a consequence of differences in attitudes or beliefs.

With reference to the general remarks on concepts in the previous chapter of the paper, in our context three components are important: the concept definition (independent of the formal state of this definition (see McLeod and McLeod (2002) for the case of beliefs); the associations and metaphors that combine with the concept definition; and the research methods that are used to investigate and measure the concept. It shall be argued below that with respect to the meaning-construction problem in mathematics education research, the components “concept definition” and “concept images” (or concept trees (Sfard, 2008)) are helpful, but the ontological

status of “research methods” is problematic, and the reason for this ought to be made widely understood.

Let us start with a definition of the affective categories, after Goldin (2002); also, in the following, we shall use the concept of beliefs to demonstrate the meaning-construction problem:

- (1) *emotions* (rapidly changing states of feeling, mild to very intense, that are usually local or embedded in context);
- (2) *attitudes* (moderately stable predispositions toward ways of feeling in classes of situations, involving a balance of affect and cognition);
- (3) *beliefs* (internal representations to which the holder attributes truth, validity, or applicability, usually stable and highly cognitive, may be highly structured);
- (4) *values, ethics, and morals* (deeply-held preferences, possibly characterized as “personal truth,” stable, highly affective as well as cognitive, may also be highly structured). (Goldin, 2002; 61)

In the following I also refer to the definitions of beliefs formulated by Op’t Eynde, De Corte and Verschaffel (2002) and Törner (2002; Goldin, Rösken and Törner, 2009):

Students’ mathematics-related beliefs are the implicitly or explicitly held subjective conceptions students hold to be true about mathematics education, about themselves as mathematicians, and about mathematics class context. These beliefs determine in close interaction with each other and with students’ prior knowledge their mathematical learning and problem solving in class. (Op’t Eynde, De Corte and Verschaffel, 2002; 27)

Törner uses constitutive elements (ontological, enumerative, normative and affective aspects) to define beliefs *B* as a quadruple $B = (O, C_0, \mu_i, e_j)$, whereby *O* is the belief object, C_0 the content set of mental associations, μ_i the membership degree function and e_j the evaluation map (Törner, 2002; Goldin, Rösken and Törner, 2009).

It is important to note that each of these definitions refers to descriptions of mental systems. These mental systems are activated in all situations in which mathematics is involved and these systems influence the thoughts and acts of a person in these situations (Furinghetti and Pehkonen, 2000; Hannula, 1998). The lack of reference objects for the concepts (all of which are discourse objects (Sfard, 2008)) leads to a problematic situation when attempting to give the concepts a meaning in the discourse process.

In the definitions we find certain keywords – “intensity”, “stability”, “structure” and “truth”. These keywords are supposed to lead to a meaning for the concepts: we therefore need to analyze them. Intensity is often described as “hot” or “cool” (McLeod, 1992), metaphors that are also used to describe affective states:

Affection, for example, is understood in terms of physical warmth. (Lakoff and Nunez, 2000; 41)

The terms “stability” and “balance” refer to a metaphor originating from physics and describing a state of equilibrium. In our case this term is used to evoke a twofold meaning. On the one hand, it is meant to capture the notion that some mental system always leads to the same endpoint that persists for an extended period; on the other

hand, it describes an equilibrium between the affective and cognitive systems. “Structure” refers to an ordering in the mental system that is clearly distinct from other systems. “Truth” is a metaphor borrowed from logic and used here in the singular sense that all utterances made by an individual are subjectively seen as true. However, all these keywords are also discourse objects and are therefore at the same level as the concepts that they are intended to give meaning to.

How do we proceed? Another opportunity to construct meaning for a concept is afforded by using insights from other scientific fields striving to understand the same phenomena. In our case we could use insights from neuroscience.

With respect to cognition and affect, neuroscience distinguishes two different systems: cognition and emotion. Both exist as a result of biological evolution, with the aim of aiding the individual’s survival (Wimmer and Ciompi, 1996; Damasio, 1999; LeDoux, 1998; Roth, 2001). Although located in different parts of the brain (Damasio, 1999; LeDoux, 1998; Roth, 2001), there are connections between the two systems that allow interactions. A very important consequence of the existence of these two systems is that we have to distinguish between “feeling” and “knowing that we have a feeling” (Damasio, 1999; 26); or “emotional reactions” and “conscious emotional experience” (LeDoux, 1998; 296).

For our problem we should note that although all processes on the neuronal level are not conscious, some of these processes lead to conscious results. We are aware only of these conscious parts of the processes. For remembrances, too, two memory systems exist with respect to emotions: an implicit emotional memory and an explicit memory of emotions (LeDoux, 1998). The implicit emotional memory operates unconsciously, is strongly connected to arousal systems and may often lead to bodily reactions. The explicit memory of emotional situations contains all the conscious knowledge of emotional situations, emotional reactions to objects, persons and ideas etc.. The most important consequence of this is that this memory system is part of the cognitive memory and there is no distinction between a remembrance of an emotion and a remembrance of a cognitive content (LeDoux, 1998). The fact that memory of emotions is cognitive has important consequences (Schlöglmann, 2002):

- 1) We have knowledge about our feelings, their origin and their effect. This knowledge is stored in memory systems as cognitive knowledge.
- 2) Memory of emotions is open to “rational” manipulation. That means we are able to think about our emotional remembrances, and that all verbal statements about emotional facts are controlled by cognition.
- 3) Knowledge of our affect with respect to objects and situations allows us to handle our affect at least in controlled situations (see Goldin’s example of the roller coaster experience (Goldin, 2002; 62)).
- 4) Humans are able to “construct” their remembrances in a way that they are able to live with this memory. Part of this process is forgetting unpleasant facts more easily than pleasant ones: our memory has suppression mechanisms to handle unpleasant remembrances (Roth, 2001).

Assimilation and accommodation processes lead to affective-cognitive schemata (Ciompi, 1999). The affective component is stored in two memories: in the implicit memory that works unconsciously but influences our actions and thoughts (Damasio developed the concept of “somatic marker” to explain this (Damasio, 2004; Brown and Reid, 2004)); and in the explicit memory that stores all the knowledge of affect with respect to people, objects and situations. Affective-cognitive schemata always contain both the unconscious and the conscious components. Repeated assimilation and accommodation processes in relation to a special problem leads to consolidation of the unconscious reactions, as well as to more and more conscious knowledge of feelings and emotional reactions. It provides information on the outbreak of emotional reactions and allows the development of strategies for handling such situations (Goldin, 2002; Schlöglmann, 2006)).

Neuroscientific research suggests that we ought to distinguish between reactions occurring within the two memory systems; however, according to neuroscience, we have no criteria to distinguish between knowledge and knowledge of our affective relationship to mathematics. This underscores the problem that a distinction is also difficult to formulate in philosophy (Österholm, 2009; Pehkonen and Pietilä, 2003), and helps us appreciate that the problem of defining affective categories, especially beliefs, must be considered at the discourse level. Yet we have seen that descriptions of affective categories as “discourse objects” themselves also use discourse objects (e.g. intensity, stability, structure, truth) together with some metaphors. We are in a circle situation: we are bound to define our concepts in terms that contain no reference objects.

On top of these considerations, in order to measure the categories, we need an operationalization of them, usually in terms of items of a questionnaire. The items are formulated by the researchers with the aim of grasping all of the important aspects of the definition, and are formulated as questions or simple statements. The attention of the responder is directed towards finding an appropriate answer or value on a scale. However, the items are more concrete than the definition, and we have a situation where the measurement methods are derived from theoretical concepts, while they themselves become an important part of the concept. This problem is inherent in all discourse objects.

Conclusion

The analysis of the problem of defining affective concepts shows that these concepts are objects of a discourse with no reference objects. To give these concepts a meaning we use discourse to clarify the meaning: in particular, by employing other terms and metaphors. However, these terms are often also objects of a discourse at the same level as the terms they are intended to give meaning to. In a discourse this obstacle can be successfully surmounted. In contrast, if we want to measure a concept, we must formulate the description of it mostly in the form of items of a questionnaire, and these items are a consequence of our definition – yet for the

purposes of the measurement they are the realization of the definition. The problem is that we cannot escape this situation. Therefore it is important to be aware of the problem. As a consequence of this state of affairs, researchers have developed numerous methods whose appropriateness depends on the complexity of the phenomenon at hand (for the case of beliefs research see (Leder and Forgasz, 2002)); indeed, in extending the basis of information about some phenomenon, more than one research method is often used to overcome, in a certain sense, the problem of defining a concept.

On the whole we can see three groups of methods: quantitative, qualitative and observational methods. The basis for quantitative methods is the questionnaire, together with the statistical methods used to handle the responses. Qualitative methods are mostly based on texts (protocols of interviews, essays, protocols of narratives and protocols of observations), and are used to look for keywords expressing affective or emotional reactions (see, for instance, Tsamir and Tirosh, 2009; Evans, 2002). Observations can also be used to look for keywords as well as other signs indicating emotional state, such as body language. (A small number of studies exist in which physiological facts are utilized.) All these efforts can help clarify the meaning of a concept, and, in a certain sense, overcome the theoretical obstacle in a discursive way.

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CERME 6 – WORKING GROUP 2

Argumentation and proof

TABLE OF CONTENTS

Introduction.....	176
<i>Maria Alessandra Mariotti, Leanor Camargo, Patrick Gibel, Kristina Reiss</i>	
Understanding, visualizability and mathematical explanation	181
<i>Daniele Molinini</i>	
Argumentation and proof: a discussion about Toulmin's and Duval's models	191
<i>Thomas Barrier, Anne-Cécile Mathé, Viviane Durand-Guerrier</i>	
Why do we need proof.....	201
<i>Kirsti Hemmi, Clas Löfwall</i>	
Proving as a rational behaviour: Habermas' construct of rationality as a comprehensive frame for research on the teaching and learning of proof.....	211
<i>Francesca Morselli, Paolo Boero</i>	
Experimental mathematics and the teaching and learning of proof.....	221
<i>Maria G. Bartolini Bussi</i>	
Conjecturing and proving in dynamic geometry: the elaboration of some research hypotheses.....	231
<i>Anna Baccaglioni-Frank, Maria alessandra Mariotti</i>	
The algebraic manipulator of alnuset: a tool to prove	241
<i>Bettina Pedemonte</i>	
Visual proofs: an experiment	251
<i>Cristina Bardelle</i>	
Teachers' views on the role of visualisation and didactical intentions regarding proof.....	261
<i>Irene Biza, Elena Nardi, Theodossios Zachariades</i>	
Modes of argument representation for proving – the case of general proof.....	271
<i>Ruthi Barkai, Michal Tabach, Dina Tirosh, Pessia Tsamir, Tommy Dreyfus</i>	
Mathematics teachers' reasoning for refuting students' invalid claims.....	281
<i>Despina Potari, Theodossios Zachariades, Orit Zaslavsky</i>	
Student justifications in high school mathematics.....	291
<i>Ralph-Johan Back, Linda Mannila, Solveig Wallin</i>	

“Is that a proof?”: an emerging explanation for why students don’t know they (just about) have one	301
<i>Manya Raman, Jim Sandefur, Geoffrey Birky, Connie Campbell, Kay Somers</i>	
“Can a proof and a counterexample coexist?” A study of students’ conceptions about proof	311
<i>Andreas J. Stylianides, Thabit Al-Murani</i>	
Abduction and the explanation of anomalies: the case of proof by contradiction	322
<i>Samuele Antonini, Maria Alessandra Mariotti</i>	
Approaching proof in school: from guided conjecturing and proving to a story of proof construction	332
<i>Nadia Douek</i>	

INTRODUCTION

ARGUMENTATION AND PROOF

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This chapter collects the contributions discussed during the working sessions of the WG2 at CERME6. The work of the participants of the Thematic Working Group on Argumentation and Proof was organized around the goals of

- Putting our research studies in relation to each other.
- Getting feedback for improving both our research work and our papers.

Each participant was expected to act as reactor to one of the other papers, presenting the key issues and posing questions to the author(s). Such intervention was aimed to trigger a collective discussion on the paper in focus as well on general issues.

Although they all share the issue of proof and argumentation, the contributions offer a quite varied spectrum of perspectives, both from the point of view of theoretical frameworks assumed and of issues in focus. The main themes that emerged from the papers were the frame according to which the working sessions of the group were organized, and it is the same frame we use to organize this introduction. These main themes were the following.

Historic and epistemological issues

Conjecturing and proving

Visual aspects in proving

Teachers and teaching of proof

Models to describe models to explain

HISTORIC AND EPISTEMOLOGICAL ISSUES

Historic and epistemological issues were specifically addressed in some of the papers presented. Molinini discusses mathematical explanation in Physics using the lens of history. His aim is to clarify how explaining a physical phenomenon via mathematics may foster its understanding and consequently may have a pedagogical value. As Avigad says: “We look to mathematics for understanding, we value theoretical developments for improving our understanding, and we design our pedagogy to convey understanding to students” (Avigad, 2008, p. 449).

The relationship between argumentation and proof is also addressed by Barrier, Mathé and Durrand-Guerrier. Taking a semantic approach the authors try to overcome the limits of previous discussions concerning the gap between argumentation and proof.

The function of proofs in the history of mathematics inspired the analysis presented by Hemmi and Lofwall that concerns the idea of *transfer*, that is the contiguity between proofs and methods for problem solving. The importance of proofs for the development of mathematics is compared with the opinion - shared by some of the mathematicians involved in the investigation - about the crucial role that certain proofs may have in the learning of mathematics.

Habermas' theory of rationality is proposed by Morselli and Boero as a research tool and a theoretical ground according to which new educational challenges can be pursued.

CONJECTURING AND PROVING: THE ROLE OF ARTEFACTS

The relationship between conjecturing and proving is addressed from the specific point of view of the contribution offered by artefacts, either in fostering the production of conjectures or in developing the sense of a theoretical approach. Our group's work in this area considers, in particular, three different artefacts, related to different mathematical domains: a linkage device to produce an ellipse – specifically the reconstruction of an ancient machine; a Dynamic Geometry environment – Cabri; a software for algebraic manipulation – Alnuset.

The papers present different potentialities offered by the use of such artefacts. The field of experience of linkages (mathematical machines) may be compared with that offered by a Dynamic Geometry System. Bartolini Bussi discusses direct manipulation, highlighting the potential of the exploration tasks, where a key request concerns the explanation of the functioning of the linkage. Exploration tasks are also discussed in the paper of Baccaglioni-Frank and Mariotti, where the authors present a model for describing and explaining the process of production of a conjecture, based on dragging strategies for grasping the relationship between geometrical invariant properties.

In her paper, Pedemonte discusses the use of a particular symbolic manipulator, Alnuset, with respect to enhancing the teaching and learning of proof in algebra.

VISUALIZATION

Some of our group's contributions address the issue of visualization in relation to proof and proving. Such issue is discussed from different perspectives, providing a good opportunity for reflecting on the diversity and the complexity of phenomena that are usually referred to as visualization. In fact, this issue was widely discussed in the working sessions, and the discussion provided a good opportunity to confront our

different epistemological assumptions as well as the different points of view about visualization. Exploring the use of visual reasoning is the goal of the paper of Bardelle. In her paper, she presents the results of a preliminary study concerning students' way of working with visual proofs. The difficulties in treating and accepting visual proofs described in Bardelle's study finds an echo in the paper of Biza, Nardi and Zachariades, where the authors elaborate on empirical results that clearly show the relationship between teachers' and students' beliefs. The instability of teachers' beliefs about the role of visual representation with respect to what counts as a valid proof has a counterpart in students' uncertainty on what counts as a proof. The role of visual reasoning was discussed not only with respect to the proving process but also with respect to the process of discovering and producing a conjecture. Difficulties emerge concerning the complexity of treating visual representation such as lack of basic geometrical knowledge or ambiguity of images from which it is difficult to extract useful information. However, the key issue concerns the uncertain status of images as argument for validating a statement. This issue brings to the forefront the role of the teacher in introducing students to a theoretical perspective in mathematics.

TEACHERS AND TEACHING PROOF

Several papers address the issue of teaching both in terms of teachers' mathematical competences and in terms of teachers' role in organizing and managing a learning environment that could (and should) enhance students' proving performances. In many countries – in Israel for instance – recent reform recommendations require that proof and proving become key components of classroom practice.

The paper of Barkai et al. reports on an empirical study showing how teachers are able to produce correct proofs of a given statement, but meet difficulties in understanding and evaluating the validity of students' arguments supporting the validity of the same statement. These results question the type of competences that teachers should have in order to face everyday practice with students' productions of proof. Along the same lines, the paper of Potari et al. discusses teachers' reaction to hypothetical classroom scenarios, specifically how teachers approach the refuting of students' claims. These results indicate teachers' misleading epistemological views about theorems and theory, as well about the role of counterexamples in mathematical reasoning.

These contributions enrich previous results concerning the relationship between teachers' beliefs and practices. At the same time they show the high complexity of treating visualization issues and the need of elaborating specific research questions that go beyond testing of teachers' ability of producing correct mathematical proofs.

Teachers' view of what constitutes a proof and its functions influences the choice of what is to be integrated into one's own teaching practices and consequently how students evaluate their own productions.

Shifting the attention from the teachers to the students, two papers address students' productions of proofs. The study presented by Back et al. aims at giving a clear picture of how students motivate their solutions and how these change throughout the course. The issue of evaluating students' productions of proofs is again the focus of this paper that discusses how students' justifications relate to both teachers' and textbooks' ways of justifying and explaining, focussing particularly on the opposition between verbal and symbolic expression. In this respect, the episode reported by Raman et al. is also significant. These researchers describe an episode in which students come very close to a proof (they reach something that a mathematician would have basically recognized as a proof), however they were not able to recognize their argument as a proof. That raises a natural but difficult question: why are students unable to recognize what they are saying as a proof? How to bridge the distance between students and experts in elaborating informal arguments into proofs?

More specific difficulties are described in the paper of Stylianides & Al-Murani and in the paper of Antonini & Mariotti. The first paper focuses on the possible coexistence of a proof and a counterexample for the same statement. Although the answers to a survey seemed to provide some evidence of such misconception, the interview data collected in the following suggest that students' responses originate from a particular interpretation of the given questions. The second paper focuses on difficulties related to indirect proof. Specifically, the paper discusses examples of abductive processes that are mobilized in order to produce explanatory hypotheses to establish what for the solver is a meaningful link between the contradiction produced in the indirect argument and the original statement to be proved.

No great discussion on didactic issues related to proof can be found in the contributions to the working group. The only exception is the specific example of a teaching intervention presented in the paper of Douek. In this paper, after a theoretical introduction, the author presents the outline of the didactic engineering, based on the notion of cognitive unity. The author highlights the crucial role of the situation for a student to engage him/herself in argumentative reasoning, nevertheless the difficulty of implementation clearly emerges from the reported results, raising many open questions.

CONCLUDING REMARKS

A considerable part of the discussion in the group was devoted to the illustration and the comparison of the different theoretical constructs that contributed to shape the different investigations, directing the researcher both in selecting the questions to be addressed and the ways to look for possible answers.

The opportunity of comparison that we had during the working sessions made us aware of the need and the usefulness of making theoretical assumptions explicit and clear. Similarly the comparison of different models and of their use in our investigations was very stimulating, suggesting possible integrations.

It is difficult to elaborate conclusions for a discussion group that spent a considerable amount of time exploiting the richness of diversity. In our discussion we were driven not only by the need of comparing but also by the curiosity of possible integration among different paradigms. This may constitute a program for our next up-coming meeting at CERME7.

UNDERSTANDING, VISUALIZABILITY AND MATHEMATICAL EXPLANATION

Daniele Molinini

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In this paper I focus on Mathematical Explanation in Physics and I analyse its interplay with the concepts of understanding and visualizability. Starting from a recent contextual approach to scientific understanding (De Regt & Dieks, 2005) I will try to see how an historical analysis of the formulation of a particular theorem could help to clarify the role of understanding and visualizability in mathematical explanation. My test case will be Euler's theorem for the existence of an instantaneous axis of rotation in rigid body kinematics. In particular, I will argue that the specific concept of vector space, defining a new standard of intelligibility, offers a good perspective in order to underline the dynamical character of mathematical explanation and its essential role in mathematical education.

1. INTRODUCTION

Different authors agree that the problematic of explanation is deeply connected to the debate about the nature of understanding in science. At the moment the major accounts of scientific explanation such as the Unificationistic (Friedman, 1974; Kitcher, 1981, 1989), the Causal (Salmon, 1984), the Pragmatic (Van Fraassen 1980; Achinstein, 1983) do not offer a satisfactory definition of understanding within their theories. While the authors and the supporters of those theories affirm that their particular accounts of explanation provide understanding, the notion of understanding remains still vague and is the cause of a series of controversies between philosophers of science. It seems quite plausible that a good explanation in science *must* provide understanding. But what is *understanding*? Is it really this “aha!” experience we are confronted with after some cognitive experience? And how can a good explanation provide understanding?

In this paper I will focus on the very specific notion of mathematical explanation, and in particular on the notion of mathematical explanation in physics. As clearly expressed by Mancosu in his studies on mathematical explanation (Mancosu 2005, 2008), we can have two different senses mathematical explanation:

In the first sense “mathematical explanation” refers to explanations in the natural or social sciences where various mathematical facts play an essential role in the explanation provided. The second sense is that of explanation within mathematics itself (Mancosu, 2008, p. 184).

Naturally, as pointed out by Shapiro (2000), mathematical explanation as intended in the first sense is connected to the more general problematic concerning the application of mathematics to reality and opens the mysterious problem of the “unreasonable effectiveness of mathematics in the natural sciences” (Wigner, 1967). However, leaving apart mysteries and ontological questions, many authors agree that it is possible to have a better comprehension of mathematical explanation of physical phenomena (MEPP) [1] starting from general discussions of scientific explanation and introducing an historical perspective (Tappenden, 2005; Kitcher, 1989). In this paper I will follow this line, getting my hands dirty *via* a bottom-up approach that starts from the mathematics itself. I will compare two different formulations of Euler’s theorem for the existence of an instantaneous axis of rotation in rigid body kinematics and I will try to discuss the concepts of understanding and visualizability under the light of dynamical MEPP. I assume as a starting point that in both the formulations the mathematical machinery has an essential role: they represent two mathematical explanations of the same physical fact. Naturally, in such a contextual analysis, the arena of mathematical education is of primary importance and I will offer a perspective in order to work in this direction.

In a recent series of papers De Regt and Trout have discussed the notion of understanding in science (De Regt, 2001, 2004, 2005; Trout, 2002, 2005). My point will be that, contrary to Trout’s idea that is impossible to give an objective epistemic role to understanding (Trout, 2002), some interesting ideas of De Regt’s account could be utilized in order to study the role of visualizability and understanding in mathematical explanations. I hope this study will make clear that MEPPs have a dynamical character, and in some case the role of understanding in them could be studied if we have at disposition conceptual tools like visualizability. After all, a number of new studies and a sort of “renaissance in visualization” (Mancosu, 2005, p. 13) have emerged during the last years in philosophy of mathematics and cognitive sciences. The impetus in this sense has been given for the most part by the rise of visualization techniques in computer science, from which has clearly emerged the heuristic and pedagogical value of visual thinking [2]. Naturally, I stress again, my analysis implicitly focus on the importance of mathematical activity and education. Explaining a physical fact via mathematics in order to make it understandable *is* a mathematical practice, and first of all a pedagogical practice. In particular, if I assume with De Regt and Dieks (2005) that understanding transcends the domain of individual psychology and is relative to scientific communities in a specific historical period (they call it the “meso-level in science”), the importance of the acquisition of skills should be take into account in a more complete analysis. As remarked by Jeremy Avigad (2008) in his discussion of the notion of understanding in mathematical proofs:

We look to mathematics for understanding, we value theoretical developments for improving our understanding, and *we design our pedagogy to convey understanding to students*. Our mathematical practices are routinely evaluated in such terms. It is therefore

reasonable to ask just what understanding amounts to (Avigad, 2008, p. 449. My emphasis).

So mathematical education is directly linked to the concepts of understanding, mathematical explanation and to the intelligibility standard of visualizability. In this direction: transitions in the formulation of Euler's theorem in mathematical (and physical) textbooks could be very helpful in order to study mathematical explanation in our sense and the variation of "what is considered more understandable" from a pedagogical point of view.

In the next Section I will briefly give an outline of the theorem and the two different mathematical explanations for the physical phenomenon. In Section 3 I will claim that MEPPs in this particular case have dynamical character, while in Section 4 I will focus on visualizability, understanding and on the particular role of vector space theory. I will defend the epistemic relevance of a contextual notion of understanding and I will put in evidence a shift in the notion of visualizability for this particular case of explanation. The final section will contain my conclusions and some epistemological and educational perspective.

2. EULER'S THEOREM

2.1 Euler's Original mathematical formulation in E177

Euler's contributions to mechanics are numerous and of primary importance. Between them, the remarkable fact that Euler was the first to prove the existence of an instantaneous axis of rotation in the kinematics of rigid body motion. He obtained the result of the instantaneous axis of rotation for the first time in his paper E177 *Decouverte d'un nouveau principe de Mecanique*. In this work Euler utilizes previous results in order to study the general motion of a rigid body with a fixed point and deduce the changes in the position and the velocity distribution from the given forces acting on the body [3]. His enterprise in the dynamics of rigid body motion in space was stimulated by the problem of the rotation of the Earth around its axis (as to explain the precession of equinoxes). The introduction of the perpendicular rectangular frame of reference permits Euler to apply Newton's second law separately with respect to each of the coordinates. This was brought about by a kinematical result: the instantaneous axis of rotation.

In the section *Détermination du mouvement en général, dont un corps solide est susceptible, pendent que son centre de gravité demeure en repos*, in order to study the velocity distribution, Euler introduces a cartesian system fixed in absolute space and assumes that a point Z of the body with coordinates x, y, z has velocities P, Q, R in the direction of the axis. The components of the velocity P, Q, R are functions of x, y, z . Euler's final purpose is to found those functions. He considers another point z "infiniment proche du précédent Z ", of coordinates $x + dx, y + dy, z + dz$ and velocities $P + dP, Q + dQ, R + dR$. After a mixed geometrical-analytical procedure

Euler is able to state that there are points, which have coordinates $(Cu, -Bu, Au)$, that do not move during time dt . In other words, those points are on a straight line through the origin, which is called the instantaneous axis of rotation [4].

... tous les points du corps, qui sont contenus dans ces formules $x=Cu$, $y=-Bu$, $z=Au$ demeureront en repos pendant le tems dt . Or tous ces points se trouvent dans une ligne droite, qui passe par le centre de gravité O ; donc cette ligne droite demeurant immobile sera l'axe de rotation, autour duquel le corps tourne dans le présent instant (Euler, 1750. p. 95).

Euler also added a geometrical proof of the existence of the instantaneous axis of rotation, discussing the infinitesimal motion of a spherical surface with a fixed point. The geometrical argument provided by Euler legitimates his analytical argument and holds not only for the instantaneous case but also for the discrete case.

2.2 A Modern formulation in Linear Algebra

As originally proved by Euler, the theorem for rigid body motion states that: “The general displacement of a rigid body with one point fixed is a rotation about some axis”. The motion of a rigid system in modern mechanics is described specifying at each instant the position of the points of the body with reference to a system of axis. To every point we associate a vector which belongs to an euclidean 3-dimensional space. The orientation of the rigid body in motion can be described at any instant by an orthogonal transformation, the elements of which may be expressed in terms of some suitable set of parameters. With the progression of time the orientation will change and the matrix of the transformation will evolve continuously from the identity transformation $\mathbf{A}(0)=\mathbf{1}$ to the general matrix $\mathbf{A}(t)$. Here we assume that at time $t = 0$ the body axes (the axes fixes in the rigid body) are chosen coincident with the space axes (a system of axes parallel to the coordinate axes of external space). The assumption that the operation implied in the matrix \mathbf{A} describing the physical motion of the rigid body is a rotation assures that one direction (the axis of rotation) remains unaffected in the operation and the same holds for the magnitude of the vectors. If we consider as the fixed point in the rotation the origin of the sets of axes (and not necessarily the center of mass of the object), the displacement of the rigid body involves no translation of the body axes and we can restate Euler's theorem in the following modern mathematical form: “Every matrix \mathbf{A} in $SO(3)$, with \mathbf{A} different from \mathbf{I}_3 , has an eigenvalue $+1$ with a 1-dimensional eigenspace” (Sernesi, 1993, p. 305).

A proof of the mathematical theorem in the form I have given involves the general concepts of matrix, vectors (in particular the more specialized concepts of eigenvalue and eigenvector), eigenspace, basis, orthogonality, bilinear forms (in particular the scalar product, which is a symmetric and non-degenerate bilinear form). All those concepts are included in linear algebra and their close interplay does not permit any

easy separate analysis of the elements which are found in the proof structure of such a theorem. Israel Klein pointed out this difficulty in his *History of Abstract Algebra*:

Among the elementary concepts of linear algebra are linear equations, matrices, determinants, linear transformations, linear independence, dimension, bilinear forms, quadratic forms, and vector spaces. Since these concepts are closely interconnected, several usually appear in a given context (e.g. linear equations and matrices) and it is often impossible to disengage them (Klein, 2007, p. 79).

The modern proof of the algebraical formulation is constructed into the general framework of linear algebra and the particular framework of euclidean 3-dimensional vector space \mathbf{R}^3 . Clearly, the proof's outcome is to show the existence of the eigenvalue $\lambda=1$ [5]. If we do not consider the concept of group, and we focus on the general concept of vector space, we could analyse the explanatory structure and make some relevant remarks.

3. SHIFT IN MATHEMATICAL EXPLANATION

It is clear that Euler did not have at disposition the modern concept of vector and vector space. But, as we can see from his papers, he did have the basic idea of geometrical transformation (point-to-point association in space and not transformation from physical magnitude to geometrical magnitude), which was central to his analysis. Differently from Euler's original argument, in which the mathematical explanation is given by a mixed geometrical-analytical argument by means of a geometrical space (and *via* a geometrical intuition [6]), the modern explanation of the existence of an instantaneous axis of rotation is given in the framework of linear algebra. Having the particular structure of euclidean 3-dimensional vector space is essential to Euler's theorem as formulated in modern terms because only the mathematical properties of a real vector space equipped with scalar product permit to “map” the properties of the kinematical system (angles, distances, orthogonality condition) into the algebraic structure.

In a recent paper Gingras (2001) has underlined how the shift in explanation and the “disparition of substances into the acid of mathematics” are an epistemic and an ontological effect of the process of mathematization started with Newton. As a consequence of an historical process concepts like determinant, matrix, orthogonality or transformation are today included in the mathematical apparatus of linear algebra and we could profit of their interplay without exit from this framework (i.e. the framework of abstract algebra). In other words: in the modern algebraical proof the geometrical part is already “included” in the structure of vector space and we do not need a geometrical argument [7]. It is very interesting to observe that Peano himself, in his *Analisi della teoria dei vettori*, remarked:

Thus the theory of vectors appears to be developed without presupposing any previous geometrical study. And since, by means of this theory, all of geometry can be treated,

there results thereby the theoretical possibility of substituting the theory of vectors for elementary geometry itself (Peano, 1898, p. 513).

After having proved the dynamical character of mathematical explanation (i.g. the mathematics is essential to both the two explanations but it *changes*), in the following Section I will use De Regt & Dieks's criteria for understanding and intelligibility in order to show how the theory of vector space offers a new conceptual tool of intelligibility and understanding.

4. UNDERSTANDING AND VISUALIZABILITY IN MEPP.

If I admit (and I do!) with De Regt & Dieks (2005) that visualizability constitutes a context-dependent standard of intelligibility, and only intelligible theories can provide an understanding of phenomena, then I can look at the shift between our two MEPPs in a more fruitful and interesting way. But, first of all, it is necessary to give a possible sense to the notions of visualizability, intelligibility and understanding.

As showed by De Regt (2001) being a spacetime theory is a necessary but not sufficient condition for visualizability. It might be objected here that I deal with mathematical entities and the term "spacetime" is very dangerous and misleading. Fortunately, I am referring to MEPPs and for my particular test case the conditions of necessity and sufficiency for visualizability are both fulfilled (Euler's geometrical framework and the framework of vector space theory both make the physical phenomenon *visualizable* in space -as a vector- at a particular time t , as could be seen from the diagrams we find in a common textbook of mechanics or mathematics). We can say that geometrical space in Euler and the modern concept of vector space *map* the physical space into a structure (a geometrical and a mathematical structure). In the case of vector space this mapping consists in an explicitly assumed isomorphism between the physical space and the 3-dimensional Euclidean space.

De Regt & Dieks (2005) propose two criteria for understanding and intelligibility: CUP (Criterion for Understanding Phenomena) and CIT (Criterion for the Intelligibility of Theories).

CUP: A Phenomenon P can be understood if a theory T of P exists that is intelligible (and meets the usual logical, methodological and empirical requirements).

The necessary connection between visualizability and understanding is made by De Regt through the Criterion for the Intelligibility:

CIT: A Scientific Theory T is intelligible for scientists (in context C) if they can recognize qualitatively characteristic consequences of T without performing exact calculations.

In the previous Criterion I substitute "Mathematical Theorem" for "Scientific Theory" and I assume the applicability of the CIT in both cases (with some differences that should be discussed). But how do we "recognize qualitatively

characteristic consequences of T without performing exact calculations”? A possible answer: through conceptual tools. In a particular historical or methodological context we have at disposition some conceptual tools and visualizability *could be* one of them [8]. In other words: visualizability is a conceptual context-dependent tool, i. g. a conceptual contingent tool which depends from the skill of the scientific-mathematical community and which is present during a precise historical period, and it *could* permit the intelligibility of a theory making possible the circumvention of the calculatory stage and the jump to the conclusion. So it is clear that also intelligibility is context-dependent. Naturally, as remarked by De Regt (2001), visualizability is not a necessary condition for intelligibility. Often other conceptual tools as abstract reasoning or familiarity could lead scientists and mathematicians to intelligibility as an immediate conclusion (see De Regt & Dieks, p. 156, for examples). Mathematical practice and theoretical physics are full of situations like this.

In Euler the tool of visualization is perfectly applicable in the classical geometrical framework (I call it Euclidean Geometrical Theory): point-to-point association and geometrical considerations offer the idea (a *visual idea*) of what is happening to the mechanical system in motion. The instantaneous axis of rotation could always be visualized in spacetime, and its existence could be established through a geometrical-intuitive reasoning [9]. In the modern explanation given in the framework of abstract algebra it might seem that this “chance” of intelligibility has been lost, but a deeper look shows that this is not completely true. The concept of 3-dimensional Euclidean vector space offers two new ways for obtaining the intelligibility (in line with CIT). Reading the modern formulation of Euler’s theorem a mathematician or a student could affirm “Yes, I see the eigenvalue +1”, just by looking at the formulation of the theorem in the matrix formalism. This is associated with the conceptual tool of familiarity, or abstract reasoning, and is related to a previous learning of matrix theory or other mathematical abilities. Instead of this approach, one can reach the same direct conclusion just by considering some general results in matrix theory and *visualizing* the eigenvector (the instantaneous axis) in the diagram [10]. The latter can be considered a new conceptual tool leading to the fulfilment of CIT. Naturally, the structure of $n \times n$ matrices with entries from \mathbf{R} and the structure of homomorphisms of a 3-dimensional space (over \mathbf{R}) into itself are isomorphic. From the last considerations is clear that visualizability still plays a very important role in understanding and in developing a fruitful strategy of mathematical education.

5. CONCLUSIONS AND PERSPECTIVES

MEPPs are context-dependent and have dynamical character. In particular, *via* a contextual approach to understanding, it is possible to recognize that the framework of linear algebra has defined new standards (or tools) for intelligibility which legitimate an explanation as “a good explanation” (an explanation which produces understanding). The understanding in this context is a payoff that directly comes from the availability of those conceptual tools. As I have showed, in the modern

formulation the understanding of the mathematical explanation for the existence of an instantaneous axis of rotation is obtained through *a double route* (visualization and abstract reasoning). I claim that this result might be very helpful in mathematical education and could offer a possible answer to Avigad's question "How do we design our pedagogy to convey understanding to students?" for the specific case discussed. A new interesting direction, as showed by Marcus Giaquinto in his studies on the epistemic function of visualization in mathematics (Giaquinto, 2005), could emerge from an analysis of visualization as a powerful educational tool in the context of discovery [11].

A better comprehension of mathematical explanation could profit from the historical study of the interplay between the proof structure of the theorem and the system of concepts that characterizes the explanatory structure. If a change in one of them influences the other, it could be interesting to study different formulations of Euler's theorem in textbooks in order to see how the mathematical explanation has been offered during this period and how it has changed in mathematical education. Naturally, the epistemological analysis of this paper opens the way to the more general question of how introduce proofs in classrooms and how concepts like explanation, understanding and visualizability should be taken into account in mathematical education.

NOTES

1. For shortness, from now on, I will refer to Mathematical Explanation of Physical Phenomena with the term MEPP.
2. For a panoramic of this field and the very interesting discussion of this point, including how computer graphics has helped to recognize mathematical structures such as Julia sets which would have been impossible to recognize analytically, see Mancosu (2005).
3. For a more precise reconstruction of Euler's argument in Euler (1750) see the paper "What we can learn about mathematical explanations from the history of mathematics" I've presented at *Novembertagung* Conference, in Denmark, 5-9 November 2008.
4. Euler does not use the word "instantaneous axis". He refers to it simply as "axe de rotation".
5. For a proof of the theorem see Sernesi (1993, p. 306).
6. The importance of the geometrical intuition in Euler emerges from the geometrical proofs he adds after his analytical arguments. The geometrical argument defines and legitimates the analytical procedure and is essential to the mathematical explanation of the existence of the axis.
7. Vector spaces firstly appear in their axiomatic form in Peano (1888).
8. Evidently, the intelligibility standard or tool of "casual connection" is of no interest in our discussion.
9. See Euler's geometrical argument or a modern geometrical argument (Whittaker, 1904, p. 2).

10. Here I am not claiming that the geometrical interpretation of matrices and eigenvectors is intrinsic in their definitions. I am assuming that under a particular “reading” (in our case Euler’s theorem in kinematics of rigid body motion), a subset of vectors of the vector space considered (the subset containing the instantaneous axis) has a geometrical representation in a diagram at time t (or a representation in a computer graphic simulation). A very good example of a case in which a precise situation is visualizable in the context of Vector Space Theory has been given by Artin (1957) and is discussed in Tappenden (2005).
11. For simple and interesting cases in which a case of visualization could provide the discovery of a theorem see Giaquinto (2005) or, in a different flavour, the famous Lakatos (1978).

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ARGUMENTATION AND PROOF: A DISCUSSION ABOUT TOULMIN'S AND DUVAL'S MODELS

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In this paper, we discuss the idea of a gap between argumentation and proof, an idea we think to be prevailing in the educational institution. Our claim is that the only use of propositional calculus is insufficient to the analysis of the validation process in mathematics and could artificially reinforce that idea of a gap. This claim can be understood as a criticism of Toulmin's and Duval's model, a criticism we hope to be a constructive one. We are then brought to the following proposal: taking explicitly into account the logical quantification and the mathematical objects in the models could help to explain mathematical creativity.

INTRODUCTION: THE PREGNANT IDEA OF A GAP

The issue at stake in this paper is the relationship between argumentation and proof. It seems to us that the assumption of a gap prevails in the educational institution. This prevalence could have major effects on mathematic education:

« Is it possible, yes or no, to shift from one to the other without too many efforts or misunderstandings?

[...]

If one answers No, one admits there is a gap between the cognitive processes of argumentation and the deductive reasoning at stake in a proof: the use of argumentation could not but maintain or even reinforce the obstacles and misunderstandings about what a proof is, because its discursive process acts against a valid reasoning process in ordinary language. » (Duval, 1992, p. 43, our translation)

Willing to take into account this gap between argumentation and proof, which is theorised in Duval's works, part of the teachers have been induced to put forward specifically the formal aspect of the proof (through structuring attempts like "I know that", "Now", "Therefore" for example) and to distinguish this aspect from the work on the content of statements. This phenomenon can be seen in Kouki's thesis (2008) through a survey carried out among six Tunisian teachers about learning and teaching of equations, inequalities and functions. Moreover, Kouki shows, through a more extended experimental study (involving 143 pupils and students in their transient period between secondary school and higher education) the consequences of these theoretical conceptions on the students' practices which tend to apply formal procedures as much as possible. In another context, Segal (2000) highlights the tendency of UK students to evaluate proof validity only from their formal aspect. There are a lot of examples of this phenomenon. We shall focus on two specific ones in order to point out the stakes of this issue.

Example 1

This example is taken from Barrier (2008) in which an extract of Battie (2003) is analysed. In this paper, a group of three students in scientific upper sixth form are asked to evaluate the following statement $\forall a \forall b (GCD(a,b) = 1) \Rightarrow (GCD(a^2, b^2) = 1)$. The group starts an argumentation built on the choice of some coprime natural numbers (3 and 2, 2 and 5, 9 and 17 then 4 and 15) and on the evaluation of the GCD of their respective square. Here is an extract of their dialogue (translated from French).

1. A : Or 125 and 16. They are relatively prime.
 2. I don't know.
- (Laughs)
3. You set 125 divided by 16 and you'll see... No, it is not the right way to do it. 16 by 16 is 4 2, 2 times 2/
 4. A : No, I think 16 and 125 are relatively prime.
 5. Yeah, when we square the things/
 6. A : Yeah, but we don't know, it's not written in the text book, but we can't prove it in the general case /
 7. Oh we make fun of it!
 8. A1 : We can't use it then. Well, I think, I really don't know, the teacher may have told it.

In (3), a student undertakes a prime factorization of 16. This method could be used for the emergence of a proof of the analysed statement. However, it seems to us that the students, influenced by their school culture regarding proof, disregard this possibility. They act as if the evaluation of a statement through an argumentation built on the manipulation of objects and the search for proof were two distinct and independent activities.

Example 2

Alcock & Weber (2005) analyse how thirteen student volunteers taken from first-term, first year introductory real analysis courses check the validity of the following proof (they were asked to determine whether or not the proof was a valid one):

Theorem. $\sqrt{n} \rightarrow \infty$ as $n \rightarrow \infty$

Proof. We know that $a < b \Rightarrow a^m < b^m$.

So $a < b \Rightarrow \sqrt{a} < \sqrt{b}$.

$n < n+1$ so $\sqrt{n} < \sqrt{n+1}$ for all n .

So $\sqrt{n} \rightarrow \infty$ as $n \rightarrow \infty$ as required.

The inference between the two last propositions is invalid. Exactly two students rejected the proof because they had familiar counter-examples. Their rejection was not founded on the recognition of a logical gap between the propositions. Three other students rejected the proof. They did it because they failed to recognize what they thought to be a proof structure. In particular, they argued that the definitions of the mathematical concepts involved in the argument were not used. Their decision seems to be grounded on exclusive formal considerations. From the point of view of mathematical activity, this is a misconception: definitions are not always employed in a mathematical proof and, above all, very few mathematical proofs are enough detailed so that their logical structure can be recognized without any work. To finish with this example, notice that while only two students refused the proof because of an invalid warrant, ten did it when the interviewer helped them to interpret " $n < n+1$ so $\sqrt{n} < \sqrt{n+1}$ for all n " as "the series is increasing" and " $\sqrt{n} \rightarrow \infty$ as $n \rightarrow \infty$ " as "the series is divergent". Our hypothesis is that this last intervention allowed the students to enter the semantic content of the proposition. Precisely, the translation into ordinary language could help them to go to a semantic interpretation in a familiar domain in which they know that there is some increasing and convergent series.

We shall now undertake a criticism of Duval and Toulmin's models which are often used in research in mathematical didactics about argumentation and proof (Mathé (2006), Tanguay (2005), Inglis & al. (2007), Pedemonte (2007, 2008)). Our main thesis is that using the proposition (in the sense of propositional calculus, as opposed to predicate calculus) as a basic element of modelling leads to overestimate the gap between argumentation and proof. In particular, we consider that taking into account mathematical objects and quantification in the didactical analysis allows a quite different approach to the validation process in mathematics.

BRIEF PRESENTATION OF DUVAL AND TOULMIN'S MODELS

We shall begin with a brief presentation of Duval's approach. Let us use Balacheff's presentation (2008, p. 509):

“Deductive reasoning holds two characteristics, which oppose it to argumentation. First, it is based on the operational value of statements and not on their epistemic value (the belief which may be attached to them). Second, the development of a deductive reasoning relies on the possibility of chaining the elementary deductive steps, whereas argumentation relies on the reinterpretation or the accumulation of arguments from different points of view. (Following Duval 1991, esp. p. 240–241).”

Duval often stands out that only argumentation lies on the content of propositions whereas what is important in a proof is the operating status of the proposition (in other words the way the proposition fits into the formal structure of the "modus ponens").

"This brings a first important difference between deductive reasoning and argumentative reasoning. The latter appeals to implicit rules which depend partly on the language

structure and partly on interlocutors' representations: therefore the semantic content of the propositions is essential. On the contrary, in a deductive step, the propositions do not intervene directly according to their **content** but according to their **operating status**, that is to the position previously assigned to them in the step process" (Duval, 1991, p. 235, our translation)

Duval especially focuses on this argument to support the idea that proof and argumentation involve very different cognitive processes. In this matter, Balacheff (2008, p. 509) points out that:

« One can imagine how this should raise question in our field considering that other researchers give a central role to “mathematical arguments” and “mathematical argumentation” in their consideration of what proof is.”

Recently, Toulmin's model has been used in many works focused on reasoning from a mathematics education viewpoint. The following example shows how Pedemonte (2008, p. 387) presents Toulmin's restricted model:

“In Toulmin’s model an argument consists of three elements (Toulmin, 1993):

C (claim): the statement of the speaker.

D (data): data justifying claim C.

W (warrant): the inference rule, which allows data to be connected to the claim.

In any argument the first step is expressed by a standpoint (an assertion, an opinion). In Toulmin’s terminology the standpoint is called the claim. The second step consists of the production of data supporting the claim. The warrant provides the justification for using the data conceived as a support for the data-claim relationships. The warrant, which can be expressed as a principle, or a rule, acts as a bridge between the data and the claim.”

This model has been used to analyse as well the production of arguments as the production of proof. In particular, Pedemonte uses this model to compare argumentation and proof relationships. Therefore the three elements (C, D, W) must be considered as more inclusive than the ternary structure of "modus ponens" (A, $A \rightarrow B$, B) used by Duval to analyse the proof in the sense where the Toulmin's model warrant is not necessarily a theorem. Nevertheless, these two models share a common point by both using the proposition in the sense of propositional calculus as a basic element of modelling. Mathematical objects and quantification are not explicitly taken into account in the model structure.

AN EXAMPLE OF USE OF A QUANTIFICATION THEORY

Several attempts have been made to use first-order theories in order to help analysing mathematical reasoning in our research team (natural deduction in Durand-Guerrier & Arzac (2005) and Durand-Guerrier (2005), Tarski's semantics in Durand-Guerrier (2008), Lorenzen's dialogic logic and Hintikka's game semantics in Barrier (2008)). The ambition of these theories is to allow for the relationships between the semantic

and syntactic aspects to be taken into account in the validation activities. On the contrary, Duval identifies a reasoning step when applying the "modus ponens" rule. He asserts for example:

«The deductive step process is well known. It is defined by the fundamental rule "modus ponens", also called Law of Detachment." (Duval, 1992, p. 43, our translation)

We also saw that Toulmin's model rested on the same type of ternary structure. Durand-Guerrier & Arsac (2005, p. 151-152) showed that this standpoint was insufficient for analysing proof, especially in the case of analysis. Furthermore, the only "modus ponens" rule cannot exhaust the propositional calculus insofar as other deductive rules are necessary (Vernant, 2006, Chapter 3). Nevertheless, the deductive step derived from the Law of Detachment prevails in proof learning at lower secondary school and certainly deserves special attention. Our contribution will rather focus on the theoretical effects of this restriction: we consider that restricting the model to the propositional calculus induces to overestimate the distinction between argumentation and proof. Let us consider how Duval (1992, p. 44-45, our translation) analyses the following text by Sartre:

« Jessica : Hugo ! You speak reluctantly. I watched you when you talked with Hoerderer :

0. He convinced you.

Hugo : 1. No, he didn't convince me.

2. Nobody can convince me that (one must lie to its friends).

3a. But if he had convinced me.

3b. It would be a reason more to shoot him.

4. Because it would prove that he would convince other guys. »

Duval asserts that this argumentation appeals to the following deductive step:

Premise: If he had convinced me

Warrant: Nobody can convince me that one must...

Conclusion: (it would prove that) he would convince other guys.

This modelling leads Duval to draw the fundamental differences between argumentation and proof. Indeed, the argumentation step as modelled by Duval is quite different from the proof step based on the "modus ponens". Our questioning on this model induces us to suggest an alternative interpretation of this argumentation step based on natural deduction (Durand-Guerrier & Arsac (2005)). We note that xCy is the assertion that « x has convinced y ». The first step of Hugo's reasoning may then be interpreted in the following way:

Data: $\forall x \neg (xCHugo)$ (2)

Inference rule: universal instantiation

Conclusion: $\neg(\text{HoedererCHugo})$ (1)

We shall go on with the analysis of the reasoning (setting apart the assertion (3b) and identifying (4) with «he would convince other guys », i.e. removing what seems to refer to metalanguage) in the following way:

Data : HoedererCHugo (3a)

Inference rule: existential generalisation

Conclusion: $\exists x\text{HoedererCx}$

Data: $\exists x\text{HoedererCx}$ (recycling)

$\exists x\text{HoedererCx} \rightarrow \exists x\exists y(x \neq y) \wedge \text{HoedererCx} \wedge \text{HodererCy}$ *

(implicit axiom)

Inference rule: modus ponens

Conclusion: $\exists x\exists y(x \neq y) \wedge \text{HoedererCx} \wedge \text{HodererCy}$ (4)

One shall notice that without the implicit axiom (*) (if Hoederer is able to convince one person, then he is able to convince two persons at least) the deduction from HoedererCHugo to $\exists x\exists y(x \neq y) \wedge \text{HoedererCx} \wedge \text{HodererCy}$ would be invalid. Therefore it is necessary, in a way, to complete the reasoning to make it valid. In this extract, one does not know whether the implicit theorem applied is part of a set of statements which are jointly accepted by Hugo and Jessica. However, this type of completion is not exclusive to argumentation, since in mathematics a fully explained proof would be much too long and therefore illegible. Weber (2008) puts forward an experimental study on how proofs are checked by mathematicians. This does not mean that the check is limited to the good practices of inference rules: proof checking, including validation, calls on not only a search for sub-proofs but also for informal or example-based arguments.

Now, an important question to be raised is the relationship between proof and proposition content. In the analysed example, we used an implicit axiom to complete the formal analysis of reasoning. This axiom is linked to a certain idea we have about the interpretation field objects (human beings in this example), what Duval calls the semantic content of propositions. In particular, the implicit axiom (*) is based on the idea that human beings are more or less homogeneous. The purpose of the following paragraph is to show that the content of propositions also intervenes in the proof construction.

« CONTENT » OF PROPOSITIONS AND PROOFS

We use here an experiment from Inglis & al. (2007). Andrew, an advanced mathematics student, is confronted with the conjecture « if n is a perfect, then kn is abundant, for any $k \in \mathbb{N}$ ». Notice that a perfect number is an integer n whose

divisors add up to exactly $2n$ and that an abundant number is an integer n whose divisors add up to more than $2n$.

ANDREW : Ok, so if n is perfect, then kn is abundant, for any k . Ok, so what does it, yeah it looks, so what does it mean ? Yeah so if n is perfect, and I take any p_i which divides this n , then afterwards the sum of these p_i s is $2n$. This is the definition. Yeah, ok, so actually we take kn , then obviously all kp_i divide kn , actually, we sum these and we get $2kn$. Plus, we've got also, for example, we've also got k dividing this, dividing kn . So we need to add this. As far, as basically, there is no disquiet, k would be the same as this. Yeah. And, how would this one go ? [LONG PAUSE]

INTERVIEWER : So we've got the same problem as up here but in general ? With a ... ?

ANDREW : Yeah. Umm, can we find one? Right, so I don't know. Some example.

INTERVIEWER : I've got some examples for you.

ANDREW : You've got examples of some perfect numbers ? OK, so 12, we've got $1 + 2 + 3 + 4 + 6$, then, ok, $+ 12$. [MUTTERS] But this is not ? Ok, perfect, I wanted perfect numbers. OK, so let's say six. Yeah, and we've got divisors 2, 4, 6, 12. Plus I claim we've got also divisors. Yeah actually it's simple because, err, because err, the argument is that we've also got 1 which is divisor, and this divisor is no longer here is we multiply.

At the beginning of the interview, Andrew manipulates the definition of the concepts involved in the conjecture but this strategy fails to construct a proof. Then, he asks for examples and begins to play a semantic game which involves several numbers. As Duval says, this game increases the belief of the students in the validity of the conjecture (the epistemic value of the conjecture). In this sense, those kind of games are cumulative. However that argumentation which is linked with the content of the conjecture seems to be the clue of the completion of Andrew's strategy in his former attempt of proof construction. This is the manipulation of the perfect number 6 which provides to Andrew the idea that for all n , 1 is a divisor of $2n$ which is not equal to any $2k$ (with k a divisor of n). Pedemonte (2008) provides several convergent examples concerning algebra. In particular, she stands for the need of an argumentation which would integrate what she calls abduction steps in the proof construction process. In our example, the purpose is to explain why 12 would be abundant, starting from the fact that it is abundant (this practice is sometimes called the *analysis of analysis/synthesis* dyade). The proof approach (the synthesis of the dyade) is based on this explanation (12 is abundant because 1 is a divisor of 12 which is not a double of any divisor of 6). Pedemonte (2007, p. 32-33) also gives an example of this type of approach in geometry. Besides, from experiments carried out in set theory and analysis, Weber & Alcock (2004) underline the weakness of syntactic proof procedures ("unwrap definitions" and "push symbols") compared with semantic procedures (which call on object instantiations).

ABOUT TOULMIN'S COMPLETE MODEL

In their above-mentioned paper, Inglis & al. (2007) advocate the use of Toulmin's complete model which includes three new categories: the backing, the modal qualifier and the rebuttal which are introduced as follows (p. 4) :

« The warrant is supported by the *backing* (B) which presents further evidence. *The modal qualifier* (Q) qualifies the conclusion by expressing degrees of confidence; and the *rebuttal* (R) potentially refutes the conclusion by stating the conditions under which it would not hold. »

The authors show that there are various types of *warrant* that the students (five students prepare a doctorate degree and one a master degree) connect with various *modal qualifiers*. They advocate the importance of inductive and intuitive justifications in the mathematical activity provided that these justifications are paired with the appropriate modal qualifier for the conclusion of the argument. They underline the interest for didactics researchers to use modal qualifiers specifically in the analysis process.

“The restricted form of Toulmin’s (1958) scheme used by earlier researchers to model mathematical argumentation constrains us to think only in terms of arguments with absolute conclusion.” (Inglis & al., 2007, p. 17)

In his *remark on Toulmin*, Jahnke (2008, p. 370) makes this argument his own and emphasises the role of open general statements in mathematics. It seems to us that the role assigned to modal qualifiers in Toulmin's model shows that it is very difficult, in the didactic of mathematics, to integrate mathematical objects and their manipulation into models which are basically built from the propositional logic and from a syntactic approach of the mathematical activity.

CONCLUSION

Barrier (2008) advocates the necessity to appeal to transactional and intra-world procedures (Vernant, 2007) in order to explain mathematical creativity, i.e. to take into account the students' specific interactions with mathematical objects and the following decisions. The quantification theories, in particular the theories which develop a semantic point of view, allow to explain the milieu's enrichment (Brousseau, 1997) along the proof processes. Durand-Guerrier (2008) also stresses the importance of the manipulation of objects in order to make mathematical practice fertile. This viewpoint seems to converge with Weber & Alcock's position:

“Just as most streets in a town intersect many other streets, at any given point in a proof, there are many valid inferences that can be drawn that might seem useful to an untrained eye [...]. Hence, writing a proof by syntactic means alone can be a formidable task. However, when writing a proof semantically, one can use instantiations of relevant objects to guide the formal inferences that one draws, just as one could use a map to suggest the directions that they should prescribe.” (Weber & Alcock, 2004, p. 232)

Obviously, every argumentation does not lead to proof, since the rules of the game are different in these two activities. In particular, in geometry, it is likely that the important gap between the different semiotic registers makes it more difficult to shift from an argumentation game to a proof game. As stated by Balacheff (2008, p. 509), it is necessary to bear this semiotic thinking in mind in order to understand Duval's approach. However, as we pointed it out in our examples, the assumption of an impassable gap between proof and argumentation is likely to hinder students' validation attempts. In particular, when validation is not immediate (we mean that it does not directly derive from the manipulation of the definitions of concepts involved in the statement of the proposition to be proven), it is often necessary to work on the content of the propositions. From a mathematical activity viewpoint, proof production seems to go with the familiarisation with mathematical objects.

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WHY DO WE NEED PROOF

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We explore teaching mathematicians' views on the benefits of studying proof in the basic university courses in Sweden. The data consists of ten mathematicians' written responses to our questions. We found a variety of ideas and views on the function of proof that we call transfer. All mathematicians in the study considered proofs valuable for students because they offer students new methods, important concepts and exercise in logical reasoning needed in problem solving. The study shows that some mathematicians consider proving and problem solving almost as the same kind of activities. We describe the function of transfer in mathematics, exemplify it with the data at a general level and present particular proofs illuminating transfer that were mentioned by the mathematicians in our study.

INTRODUCTION

The various functions of proof in mathematics and mathematics education have been discussed by researchers during many years and they have gained a wide consensus in the mathematics education research community (Bell, 1976; De Villiers, 1990; Hanna, 2000). Especially the functions of conviction and explanation have been in focus in the field (e.g. de Villiers, 1991; Hanna, 2000; Hersh, 1993). However, Weber (2002) states that besides proofs that convince or/and explain there are proofs that justify the use of definitions or an axiomatic structure and proofs that illustrate proving techniques useful in other proving situations. Lucast (2003) studied the relation between problem solving and proof and found support for the importance of proofs rather than theorems in mathematics and mathematics education for example from Rav's (1999) philosophical article. Lucast considers proof and methods for problem solving as in principal the same and states that proving is involved in the cognitive processes needed for problem solving.

According to the mathematicians in our earlier study, there are proofs that can introduce new techniques to attack other problems in mathematics or offer understanding for something different from the original context. For example, they mentioned the method of completing the square in deriving the formula for the solution of the second degree equation as useful in problem solving [1] (Hemmi, 2006). We decided to call this function of proof for *transfer* and we remarked that it had neither been in the focus in the research on proof in mathematics education nor involved in the earlier models about the functions of proof. It is close to and partly overlapping the aspect Weber (2002) describes but not exactly the same. Recently, Hanna and Barbeau (2008) have started to explore this function from a point of view of philosophy and mathematics education [2]. Also they stress that it has been overlooked in mathematics education research.

Extended information about various functions of proof communicates something about the meaning of proof in mathematical practice, and the consciousness of them should therefore be important for how newcomers experience the practice. Some students in our earlier study who had difficulties to follow and understand proofs that were presented in the lectures expressed for instance the lack of examples from mathematicians about connections between proofs and problem solving.

Most often you don't have to be able to know anything of the proofs in order to solve problems. (Student – Intermediate course, 2004 in Hemmi, 2006)

They also advocated working manners and tasks where they could use the proofs in some ways in order to enhance their own engagement with proofs.

I mean tasks in which you are supposed to calculate something using proofs. At least for me, it is easier to understand if I really use them for something. (Student – Intermediate course, 2004 in Hemmi, 2006)

Our recent study contributes to the field by exploring mathematicians' often tacit knowledge concerning the teaching and learning of proof in the practice of mathematics. In this paper, we first describe the function of *transfer* from the perspective of history of mathematics and then present an analysis of a pilot study with ten mathematicians concerning their views on proof, in particular with respect to the function of *transfer* in the basic courses [3] of mathematics in Sweden.

TRANSFER IN MATHEMATICS

Proof has not always been a natural part of mathematical activity. In the old cultures in Babylonia, Egypt and China, mathematicians seemed to be only interested in presenting results which could be used in different applications and not in the question of how these results were obtained. They might have done verifications of results also, but if so, they did not think it was worth while to write them down. With the Greeks, the deduction style of mathematics was born and the emphasis was put rather on the questions of truth, foundations, logic, and proving than on practical applications. Their work in geometry which we know from Euclid's *Elements* has since then been a model for scientific thinking. It was not until the 1900th century that proofs in algebra and analysis could be performed with the same kind of logical strength that was done in the *Elements*. Nowadays, proving has been almost a synonym for doing research in mathematics and an enormous amount of mathematical proofs are produced every year.

A natural question to ask is why the deductive style in mathematics has been so successful? Nobody can question the importance and usefulness of mathematics in the modern society, but do we need the proofs? It is only the very results in mathematics that are used in other sciences and, in the end, they are important for the production of all the facilities we see around us. We think the "market" should have forced mathematics to use the "handbook" style if this turned out to be as (or more)

efficient as the "deductive style". For the Greeks it might have been possible to study proofs just because they thought it was an intellectual challenge, but in our society we think this is impossible.

However, the deductive style in mathematics has survived and been successful. One important reason for this is indeed that the proofs contain information of how to get other results and also often contain methods of calculation used for example in applications. As an example, consider Archimedes result about the volume of the sphere. It is of course interesting for applications to be able to compute the volume of a sphere, and with the formula in hand also some other problems maybe solved, e.g., the volume of a half sphere. But without the proof it is hard to find formulas for the volume of other bodies. Archimedes described the method he used to find the formula, which may be seen as a form of integration and is interesting for other applications. It is a heuristic argument based on his law of the lever. The method contains a lot of information which may be used to reach far beyond the original problem. For other examples of theorems where the proofs are far more interesting than the results, see Rav (1999).

There is certainly a consensus among mathematicians that the proofs contain much more information than just the verification of the results, but how do they think about this function of proof in the teaching context?

METHODOLOGY AND THEORETICAL STANCES IN THE PILOT STUDY

In August 2008, we e-mailed to 16 mathematicians at various universities. We presented the aim of the study and invited them to share their thoughts with us concerning the following questions.

1. Why do you think that students in basic courses should become familiar with proofs and proving or do you think they do not need to do so and in this case why?
2. What specific proofs/derivations do you consider as central in basic courses which you have taught?
3. Are there specific proofs/derivations in the basic courses that teach students techniques, concepts, procedures, strategies or offer other tools that are useful in other contexts, for example in problem solving?
4. Are there proofs not filling the criteria in question 3 but which you in any way consider as central in the basic courses, in that case which proofs and why?

To encourage the mathematicians to response, we stressed that the answers would not need to be exhausting, it was enough to give some examples. Ten mathematicians from five different institutions e-mailed their answers. Although the responses varied both in length and in content we obtained very rich data. We had also the possibility to contact the mathematicians and ask for complementary information.

We consider the mathematicians as *old-timers* in their *communities of practice of mathematics* (see Wenger, 1998). All the mathematicians in the pilot study had at least ten years experience of teaching and all of them have somehow been engaged in the teaching of elementary courses. Learning is conceived as increasing participation in the mathematical practice where proof is a central *artefact* with many functions (see Hemmi, 2006). According to the theory of Lave and Wenger (1991) artefacts and their significance to the practice can be more or less visible for the newcomers. This is called the *condition of transparency* of proof in the teaching of mathematics, i.e. how and how much to focus on various aspects and functions of proof and how and how much to use proof in doing and presenting mathematics without a focus on it as proof (see also Hemmi, 2008).

This is one of the aspects in the conceptual frame that was created by combining the social practice approach with theories about proof obtained from the didactical studies in the field. The other aspects, relevant for this study, are the functions of proof of *conviction*, *explanation*, *communication*, *intellectual challenge*, *aesthetic* and *transfer*. All these aspects are intertwined and partly overlapping but have to be separated in order to be able to analyse the data.

We analysed the data with help of NVivo software by firstly relating the mathematicians' responses to the aspects in the conceptual frame. Then, we used an open approach and looked at the issues enlightening the function of *transfer* from various points of view and connected these issues to the themes described in the introduction (Weber, 2002; Lucast, 2003; Hemmi, 2006; Hanna & Barbeau, 2008).

We interpret the mathematicians' utterances as representative of views belonging to the community, utterances that are influenced by the social, cultural and historical context of the same mathematics environment but also from other possible environments they are members of. The aim of the pilot study is to investigate the diversity of ideas among mathematicians analysing a small sample in order to later explore a larger sample. This is why we cannot generalise the results and there is no use to give exact numbers of mathematicians talking about various themes. We make very little quantifications when reporting the results.

First, we sum up the main reasons mentioned by the mathematicians for why they wanted to include proof in the basic courses. Then we provide some examples about utterances concerning the function of *transfer* at a general level. Finally, we present some specific proofs that according to the mathematicians involved this function.

RESULTS

All the ten mathematicians stated that students in the basic courses should become familiar with proofs and proving. This is interesting because in our earlier study which concerned only one department, most of the mathematicians said they did not deal with proof so much in the basic courses for various reasons (Hemmi, 2006). Yet, some of the mathematicians in the present study pointed out that there was no use to

prove for example statements concerning limits of functions rigorously for the students studying engineering, chemistry or other sciences. One mathematician even stated that one should try to “serve up” mathematics for such students with so few proofs as possible and concentrate on applications.

The mathematicians gave various reasons for why proof is important to include in the curriculum for the basic courses. Some of them stated that proof helped to make visible the difference between school mathematics and university mathematics for the students and that inclusion of proof in the curriculum helped students to leave their preconceived interpretations about what mathematics actually is. Proof should be included in the basic courses because proof is the soul and the backbone of mathematics. It is the very idea of doing mathematics. According to one mathematician, working with some proofs also offered possibilities to discuss what proof is. This refers to the aspect of *transparency*.

In line with our earlier study many mathematicians consider school mathematics as teaching students to apply rules they get through examples from the teacher or a textbook. According to the mathematicians, this manner does not lead to understanding of what mathematics is, “i.e., concepts and intuitive and logical reasoning about these concepts and their relationships”. Proof explains how the concepts are related to each other. This view refers to the function of *explanation*.

Another reason the mathematicians gave was that proof connects all mathematics, without proof “everything will collapse”. You cannot proceed without a proof. This refers to the *verification* function of proof.

Some mathematicians stressed that it was important to present proofs (or convincing arguments) for statements which are not conceived as evident by the students. This refers to the attempts to create possibilities for the students to experience the function of *conviction* of proof.

One mathematician stated that proof enhanced students’ interest towards mathematics by giving aha-experiences and also that students were curious about proof. The latter was confirmed by our study among university entrants. It showed that about 80 percent of students were interested in proof and wanted to learn more about proof when they came to the university (Hemmi, 2006). This refers to the function of *intellectual challenge*.

One mathematician also pointed out that it was important to present some “beautiful proofs” even if he thought it was difficult to find such proofs suitable for the basic courses. This refers to the function of *aesthetic*.

Finally, one of the mathematicians talked about proof as useful in the learning of mathematical language. This refers to the function of *communication*.

All the functions mentioned above are interconnected and partly overlapping. Some of the reasons presented in this section that the mathematicians mentioned for why

they wanted the students to meet proof in the basic courses are already connected to the function of *transfer*, the main target of this article.

Transfer at a general level

All mathematicians considered proofs more or less important in a manner that they taught students concepts and techniques needed in problem solving even if one of them mostly saw benefits at this level for other proving tasks. Some of the mathematicians stated that all essential proofs in the basic courses carried this function whereas others had difficulties to find examples of proofs involving this function at the basic level.

At a general level, many mathematicians mentioned that proofs helped students to learn *mathematical and logical reasoning* valuable in problem solving.

If one becomes accustomed to study proofs one gets practiced with mathematical reasoning, something one can draw great advantages of in problem solving. Problem solving is an art of formulation. (M4)

But they (the proofs) should also contribute to demonstrate and develop students' skills of logical reasoning. This is useful in many situations. One of the function of mathematics in the engineering program is this. (M8)

Yet, not all mathematicians considered this function of proof so important for engineering students as the one in the citation above.

Also the *understanding of generalisations*, especially with respect of the *models for problem solving* within mathematics or in applied sciences could be enhanced by studying proof according to some mathematicians.

They have to start to argue for the solutions of the problems for example in applications that they present, show that they are correct, so they can work in a manner not just filling in numbers in given models but tackle new problems. (M10)

One mathematician talked about the value of proof for problem solving because they helped students to learn and *understand new mathematical concepts*.

Mathematics is about defining concepts and to study how these concepts are connected. To understand the concepts you have to understand how they are connected to each other. [...] From the proof one should learn something about the concepts involved in it. (M8)

Even *technical proofs* were considered as valuable by one mathematician as they helped students understanding of problem solving.

Also the technical proofs are useful to do: the technique leads to better understanding of problem solving. (M1)

Here, the mathematician might mean that the proof techniques could be explicitly used in problem solving.

Proving and problem solving involved in each other

Some of the mathematicians stated already in their responses to the first question that they considered proving and calculating/problem solving as in principle the same activity (compare with Lucast, 2003). By highlighting this in the teaching they wanted to “demystify proof”.

I don't consider “proof” as something different from other mathematical activities – obviously it is about reasoning, calculating, being ingenious/creative, using one's knowledge and experiences and then drawing conclusions. To prove the rule of squaring a binomial, to give an elementary example, is of course just to perform the calculation. (M9)

I would like to extend the meaning of “proof” to refer to logical reasoning in general. In proofs one meets such reasoning in a concentrated form. But it is present also in problem solving and in mathematical discussions in general. (M4)

There is no difference in principle between proving and calculation. When a student carries out a computation in several steps, then these steps is a proof of the statement that the final result is the answer to the question. It is important that students at all levels get the insight that it is always reasoning which is the core of mathematics. (M6)

Most of the mathematicians talked about *transfer* only at a general level but there were some examples of specific proofs that we found valuable to present in order to later explore their potentials for further studies.

Some examples of proofs that teach students concepts or techniques

The mathematicians mentioned a number of proofs and exercises as valuable for students in order to learn techniques applicable in *other proving tasks*. This refers to the function Weber (2002) writes about. We have gathered their suggestions in the following table.

The relation in Pascal's triangle can be proved by **induction**

There are an infinite number of primes enlightens **proof by contradiction**

The square root of 2 is irrational. The students can then surely find other results where the number 2 is replaced by another integer.

$n(n+1)$ is divisible by 2 , if n is a positive integer. The same proof techniques can be applied in other proving tasks concerning divisibility.

Is it true that the proposition $P(x)$ holds for all real numbers x ?” where $P(x)$ is for instance an inequality. This trains the ability to see what is required of a proof, and that a refutation just needs a counter example which is very important in many proving tasks.

Open tasks. They encourage the willingness to investigate and make hypotheses – which then are to be proved or disproved.

The next citation is an example about how studying proofs or proving statements concerning the derivatives is seen to help students to become familiar with and learn to understand *new concepts and definitions*, in this case the notion of the limit of a difference quotient as a derivative.

The derivative is defined as the limit of a difference quotient, and you get a geometric interpretation as the slope of the tangent, but you also have the technical interpretation as change of rapidity (in a broad sense). Next you derive (prove) the rules for the derivative of a sum, product, ... and you derive the derivatives of the elementary functions. All these you may of course find in a table of formulas and you should moreover know them by heart, they are so important for the applications. But through studying the proofs you get opportunity to many times consider limits of a difference quotient, and in that manner consolidate the definition of the important notion of derivative. (M8)

The last quotation below is about the proof of the factor theorem. The factor theorem states that $x - \alpha$ is a divisor of the polynomial $f(x)$ if and only if $f(\alpha)=0$. We find the proof of this theorem as a good example of such proofs at an elementary level that allow mathematicians to highlight importance of studying the methods and notions in proofs.

We can begin with the factor theorem. The theorem expresses for sure an equivalence and it is interesting to discuss that one implication is obvious while the other is deeper. If you look at the actual proof you then see that the proof gives a bit more than what the theorem states. Indeed, the proof gives us information about the remainder even in the case where the remainder is not zero. (M4)

As an example of a problem where the proof of the factor theorem could be useful, consider the following: Determine the remainder, without carrying out the division algorithm when $x^4 + x^3 + x^2 + x + 1$ is divided by $x - 1$.

DISCUSSION

The study shows that the function of *transfer* is a natural way of thinking about proof for many mathematicians and all mathematicians express the importance of teaching proofs also in the beginning courses at university. Yet, one of them states that the students studying applied sciences do not need any proofs and some others that they do not need all the rigorous proofs. Only one mathematician did not think that proofs could be useful in problem solving at the basic level.

Some mathematicians wanted to look at proving and calculation/ problem solving in a similar way. The resemblance between proving and problem solving has been studied and discussed by Lucast (2003). This is an interesting point of view as we can also think the other way around, i.e., students can learn concepts and techniques in problem solving that they can use in proving tasks.

We find it interesting to note that the connection between proving and problem solving is something fundamental in the area of *constructive mathematics*, where

these two activities are considered to be not just similar but in fact the same (see Nordström & Löfwall, 2006). It could be fruitful to study the notions of proving and problem solving from the perspective of constructive mathematics in order to get more insight in their connections.

In school mathematics and also in the beginning courses at university it has been a tendency to avoid the word “proof” in order to not frighten the students (Hemmi, 2006). However, students lack discussions about what proof is and why it is needed. An important didactical question is how to in the best way highlight the connections between proving and problem solving in the teaching of mathematics. Consider for example the following citation:

To prove the rule of squaring a binomial, to give an elementary example, is of course just to perform the calculation. (M9)

The mathematician expresses here a view that proving, in this case, is just calculating but we could also take it the other way around and consider this calculation as proving.

We have shed light on the function of proof that we call transfer from historical point of view and explored mathematicians’ pedagogical views on it. We have described transfer at a general level and exemplified some proofs where connections to problem solving can be made visible. It is clear that mathematical proofs are carriers of mathematical knowledge and there are various ways of enlightening this for students.

However, we do not want to look at the function of transfer mechanically, even if there are situations where it is possible to just copy a proof technique to another proving task. In this paper, we have described transfer from the perspective of teaching mathematicians. We have to acknowledge that what experts consider as evident connections may be difficult to see for a learner. When studying transfer we have to study the learners’ personal constructions of similarity across proving and problem solving from their perspective (Lobato, 2003). Our study shows that there is a lot to explore in university mathematics regarding the ideas from the mathematicians’ personal experiences of proof in the learning and doing mathematics.

NOTES

1. Consider for example the following problem: Determine the centre and the radius of a circle $x^2+2x+y^2-4y=0$. It should be easier to solve it if one is familiar with the method of completing the square.
2. However, Hanna and Barbeau (2008) do not use the word *transfer* for this function.
3. With basic and elementary courses, we refer to the courses taught during the first semester. With intermediate courses we refer to the courses taught during the second semester.

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PROVING AS A RATIONAL BEHAVIOUR: HABERMAS' CONSTRUCT OF RATIONALITY AS A COMPREHENSIVE FRAME FOR RESEARCH ON THE TEACHING AND LEARNING OF PROOF

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In this paper we draw from Habermas' construct of rational behaviour a construct for rationality in proving that we propose as suitable to investigate the teaching and learning of proof and generate new research developments. At first, we discuss our conception of the proving process, where cognitive and cultural aspects are shown to play a crucial role, and we present our adaptation of Habermas' construct as a way of taking into account both cognitive and cultural aspects. The adapted construct is shown to be useful in the discussion of some examples at tertiary level; finally, drawing from the analysis of the examples, we indicate some research questions (formulated in terms of the theoretical construct) that we feel worth to be explored.

Key-words: proof and proving, rational behaviour, Habermas, tertiary level

INTRODUCTION

The aim of our paper is to contribute to the debate on theoretical frameworks suitable to take into consideration the complex nature of the teaching and learning of proof.

When planning the teaching of theorems and mathematical proof and when analyzing students' difficulties in approaching them, we have at disposal several theoretical tools coming from epistemology, history of mathematics, psychology, and didactics of mathematics. In order to build a comprehensive framework for proof and its teaching and learning, encompassing the epistemological, psychological and didactical dimensions, we think that at first it is necessary to consider proof as a crucial component of mathematics and to look at mathematics from a cultural perspective. The definition of culture by Hatano & Wertsch (2001) suggests to consider mathematics as a multifaceted culture evolving through the history, which includes different kinds of activities and different levels of awareness, explicitness and voluntary use of notions, thus different levels of "scientific" mastery, according to the Vygotskian distinction between common knowledge and scientific knowledge (for further developments about mathematics as a culture, see Morselli, 2007). Within this cultural perspective we can situate the "culture of theorems" as the complex system of conscious systematic knowledge, activities and communication rules that refer to the processes of conjecturing and proving as well as to their final products. Consequently, we can describe the approach to theorems and proving as a process of scientific "enculturation" consisting in the development of a special kind of rational behaviour, characterized by the conscious mastery of the epistemic aspects of theorems (Mariotti et al., 1997; Balacheff, 1982) and by the intentional construction and control of the process that produces the proof, within a communication context

with its shared rules. From these considerations we can draw a link between the approach to theorems as a process of “scientific enculturation” and the three components of Habermas’ “rational behaviour” (the epistemic, the teleological and the communicative rationalities), as we will show in the subsequent section.

Another entry into the same line of thought derives from the process - product character of proving and proof. Balacheff (1982) points out that the teaching of proofs and theorems should have the double aim of making students understand what a proof is and learn to produce it. Accordingly, we think that, in mathematics education, proof should be treated considering both the *object* aspect (a product that must meet the epistemic and communicative requirements established in today mathematics - or in school mathematics) and the *process* aspect (a special case of problem solving: a process intentionally aimed at a proof as product). Here again we can identify potential links with Habermas’ elaboration about rationality.

PROVING AS A RATIONAL BEHAVIOUR

Habermas (2003, ch. 2) distinguishes three inter-related components of a rational behaviour: the *epistemic* component (inherent in the control of the propositions and their enchaining), the *teleological* component (inherent in the conscious choice of tools to achieve the goal of the activity) and the *communicative* component (inherent in the conscious choice of suitable means of communication within a given community). With an eye to Habermas’ elaboration, in the discursive practice of proving we can identify: an *epistemic* aspect, consisting in the conscious validation of statements according to shared premises and legitimate ways of reasoning (cf. the definition of “theorem” by Mariotti & al. (1997) as the system consisting of a statement, a proof which is derived according to shared inference rules from axioms and other theorems, and a reference theory); a *teleological* aspect, inherent in the problem solving character of proving, and the conscious choices to be made in order to obtain the aimed product; a *communicative* aspect, consisting in the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning, and the conformity of the products (proofs) to standards in a given mathematical culture.

Our point is that considering proof and proving according to Habermas’ construct may provide the researcher with a comprehensive frame, within which to situate a lot of research work performed in the last two decades, to analyze students’ difficulties concerning theorems and proofs (see the four examples in the next Section) and to discuss some related relevant issues and possible implications for the teaching of theorems and proof (see the last Section).

If we are interested in the epistemic rationality side, i.e. in the analysis of proofs and theorems as objects, mathematics education literature offers some historical analyses (like Arsac, 1988) and surveys of epistemological perspectives (like Arzarello, 2007): they help to understand how theorems and proofs have been originated and have been

considered in different historical periods and how, even in the last decades, there is no shared agreement about what makes proof a “mathematical proof” (cf. Habermas' comment about the historically and socially situated character of epistemic rationality). Concerning the ways mathematical proof and theorems are (or should be) introduced in school as “objects”, several results and perspectives have been produced, according to different epistemological perspectives and focus of analyses. In particular, De Villiers (1990), Hanna (1990), Hanna & Barbeau (2008) discuss the functions that mathematical proofs and theorems play within mathematics and advocate that the same functions should be highlighted when presenting proof in the classroom, in order to motivate students to proof and allow them to understand its importance. By referring proof to the model of formal derivation, Duval (2007) focuses on the distance between mathematical proof and ordinary argumentation; he also considers how to make students aware of that distance and able to manage the construction and control of a deductive chain. Harel (2008) uses the DNR construct to frame the classification of students' proof schemes (we may note that they concern proof as a final product). We note that, in terms of Habermas' components of rationality, Harel's ritual and non-referential symbolic proof schemes may be attributed to the dominance of the communicative aspect, with lacks inherent in the epistemic component (cf. Harel's N, “intellectual Necessity”).

Concerning the proving process, some analyses of its relationships with arguing and conjecturing suggest possible ways to enable students to manage the teleological rationality. In particular, Boero, Douek & Ferrari (2008) focus on the existence of common features (“cognitive unity”) between arguing, on one side, and proving processes on the other, and present some activities (from grade I on), based on those commonalities, that may prepare students to develop effective proving processes. Research on abductive processes in conjecturing and proving (Cifarelli, 1999; Pedemonte, 2007) and the construct of “abductive system” (Ferrando, 2006) take into account some aspects of the creative nature of conjecturing and proving processes and the need of suitable educational choices to promote creativity. Boero, Garuti & Lemut (2007) suggest the possibility of smoothing the school approach to mathematical proof through unified tasks of conjecturing and proving for suitable theorems (those for which the same arguments produced in the conjecturing phase can be used in the proving phase). However Pedemonte (2007) shows how in some cases of “cognitive unity”, students meet difficulties inherent in the lack of “structural continuity” (when they have to move from creative ways of finding good reasons for the validity of a statement, to their organization in a deductive chain and an acceptable proof): her study suggests to consider the relationships between teleological, epistemic and communicative rationality (see the last Section).

SOME EXAMPLES

Morselli (2007) investigated the conjecturing and proving processes carried out by different groups of university students (7 first year and 11 third year mathematics

students, 29 third year students preparing to become primary school teachers). The students were given the following problem: *What can you tell about the divisors of two consecutive numbers? Motivate your answer in general.* Different proofs can be carried out at different mathematical levels (by exploiting divisibility, or properties of the remainder, or algebraic tools). The students worked out the problem individually, writing down their process of solution (including all the attempts done); afterwards, students were asked to reconstruct their process and comment it. The *a posteriori* interviews were audio-recorded. In (Morselli, 2007) several examples of individual solutions and related interviews are provided, and in particular it is shown how students' failures or mistakes were due to lacks in some aspects of rationality and/or the dominance of one aspect over the others.

For the present paper, we selected four examples. At first, we present two very similar cases, concerning students that are preparing to teach in primary school, and we show how the theoretical construct of rationality in proving may help to single out important differences between the two students, as well as different needs in terms of intervention. Afterwards, we present two cases concerning university students in Mathematics: the first one is a case of success, while the second one is a case of failure. These two cases were analyzed in (Morselli, 2006) with a special focus on their use of examples. Here we discuss those proving processes by means of our adaptation of Habermas' construct.

The four examples have the double aim of illustrating how our adaptation of Habermas' construct works as a tool for in-depth analysis, and introducing a discussion that will suggest further research developments.

Example 1: Monica

Monica considers two couples of numbers: 14, 15 and 24, 25. By listing the divisors, she discovers that "Two consecutive numbers are odd and even, hence only the even number will be divided by 2". Afterwards, she lists the divisors of 6 and 7 and writes: "Even numbers may have both odd and even divisors". After a check on 19 and 20, she writes the discovered property, followed by its proof:

Property: two consecutive numbers have only one common divisor, the number 1. In order to prove it, I can start saying that two consecutive numbers cannot have common divisors that are even, since odd numbers certainly cannot be divided by an even number. They also cannot have common divisors different from 1, because between the two numbers there is only one unit; if a number is divisible by 3, the next number that is divisible by 3 will be greater by 3 units, and not by only one unit. Since 3 is the first odd number after 1, there are no other numbers that can work as divisors of two consecutive numbers.

Monica carries out a reasoning intentionally aimed (teleological rationality): first, at the production of a good conjecture; then, at its proof. Proof steps are justified one by one (epistemic aspect) and communicated with appropriate technical expressions

(communicative aspect). The only lack in terms of rationality concerns the short-cut in the last part of the proof: Monica realizes that something similar to what happens with 3 (the next multiple is “greater by three units”) shall happen a fortiori with the other odd numbers that are bigger than 3 (“Since 3 is the first odd number after 1”), but she does not make it explicit. Her awareness (cf. epistemic rationality) is not communicated in the due, explicit mathematical form (lack of communicative rationality). Monica’s *a posteriori* comments on her text confirm the analysis:

Monica: (...) and then I have thought that 3 was the first odd number after 1 and so if 3 does not enter there, also the bigger ones do not enter there [from the previous text, we know that “there” means: between two consecutive numbers on the number line].

Interviewer: to make more general what you said with 3, what would you write now?

Monica: ehm... I have tried to go beyond the specific case of 3, but I do not know if I have succeeded in it.

Example 2: Caterina

Starting from the fact that two consecutive numbers are always one odd and one even, we may conclude that the two numbers cannot be both divided by an even number. Afterwards, we focus on odd divisors; we start from 1, and we know that all numbers may be divided by 1; the second one is 3. We have two consecutive numbers, then the difference between them is 1, then they will not be multiples of 3, since it will be impossible to divide both of them by a number bigger than 1.

Caterina is able to justify all the explicit steps of her reasoning (epistemic rationality), she develops a goal-oriented reasoning (teleological rationality) and illustrates her process with appropriate technical expressions (communicative rationality). Differently from Monica, in spite of a good intuition there is a lack in her reasoning: divisors greater than 3 are not considered. A posteriori, after having seen also the production of her colleagues, Caterina comments:

My reasoning is not mistaken: indeed, I reach the conclusion giving a general explanation, saying that, since there is no more than one unit between the two numbers, the only common divisor is 1. Nevertheless, I can not create a mathematical rule. Observing the other solutions, I think that the correct rule is the following: along the number line we note that a multiple of 2 occurs every two numbers, a multiple of 3 occurs every three numbers, hence a multiple of N occurs every N numbers. Then, two consecutive numbers have only 1 as common divisor.

From the objective point of view of epistemic rationality, Caterina’s argument was not complete, and in her comment she reveals not to be aware of it. From her subjective point of view, Caterina is convinced to have found a cogent reason for the validity of the conjecture (“not mistaken reasoning”, “general explanation”), thus to have achieved her goal (teleological rationality). Some colleagues’ solutions induce her to reflect on the lack of a “mathematical rule”; however she doesn’t seem to

consider this lack as a lack in the reasoning, but as a lack in the mathematical communication.

Example 3: Sara

Sara (attending the third year of the university course in Mathematics), after having discovered the property by means of two numerical examples (1-2, 2-3), writes down:

“Two consecutive numbers are “made up” of an even number, divisible by 2 ($=2n$, $n \in \mathbb{N}$) and an odd number ($=2n+1$, $n \in \mathbb{N}$). Let’s suppose that 1 is not the only common divisor, that is $\exists k$ such that $k/2n$ and $k/2n+1$. $2n = ka$, $a \in \mathbb{N} \rightarrow$ also in ka there must be the factor 2 $\rightarrow k=2c$ or $a=2d$; $2n+1 = kb$, $b \in \mathbb{N} \rightarrow$ since k is common, $k=2c$, or $b=2e$. But only the product of two odd numbers is an odd number \rightarrow I could not finish for a matter of time.”

Sara seems to be aware of the way a proof should be presented (communicative rationality), of the importance of algebra as a proving tool and of the usefulness of the proof by contradiction in a case like this (two important strategic choices concerning teleological rationality). In particular, in the *a posteriori* interview she tells that she felt comfortable with the method of proof by contradiction, due to the fact that the uniqueness of 1 as a common divisor had to be proven.

Even epistemic rationality works till the last part of her algebraic work, where she derives the incorrect conclusion that “ $k=2c$, or $b=2e$ ”. However Sara gets lost after a few manipulations. Why did it happen? It is possible that in this case the arguments successfully used in the conjecturing phase (based on the distinction between odd and even, and thus on divisibility by 2) were misleading when applied in the proving phase. Incidentally, here we see that in some cases cognitive unity may act as a burden, if not controlled. Indeed, Sara could have reached the proof easily by substituting $2n=ka$ in the expression $2n+1=kb$, but she didn’t take into consideration this strategy, she just focused on divisibility by two. Substituting $2n=ka$ in the expression $2n+1=kb$ would have required to move from the odd/even semantic-based argument to a pure algebraic manipulation, with a break in the continuity of the conjecturing and proving process. Probably, Sara got lost because, when orienting her proving process, she did not fully concentrate on the meaning of the expression “1 is not the only common divisor”, being still focused on the odd-even dichotomy. Even her mistake (when she derived “ $k=2c$, or $b=2e$ ” from the previous step) might have depended on her intention to get the absurd conclusion that $2n+1$ would have been even (indeed she wrote: “But only the product of two odd numbers is an odd number”). Thus her failure might be interpreted in terms of one of her strategic choices not fitting with the aim of the proving process and not supported by a rigorous checking of inferences (i.e. in terms of a combined lack on the epistemic and teleological dimensions of rationality).

Example 4: Valentina

Valentina (attending the third year of the university course in Mathematics) chooses to carry out her exploration through an algebraic manipulation.

Given $n \in \mathbb{N}$, if it is divisible by $d \in \mathbb{N}$, then the remainder of the division of n by d is 0 , that is to say $n \bmod d$ is 0 , that is to say in \mathbb{Z}_d $n=0$. When I consider $n+1$, reasoning in the same way I realize that dividing by d I get remainder 1 , that is to say $n+1=1$ in $\mathbb{Z}_d \forall d \neq 1$. Then, the only common divisor for n and $n+1$ is 1 .

The exploration carried out by Valentina seems to be very useful: at the same time Valentina discovers the property and proves it, since the reasoning is already carried out in general terms. In the subsequent excerpt from the *a posteriori* interview, Valentina describes her process of conjecturing and proving. Valentina, being aware of the potentialities and limits of numerical examples, chooses to use algebra also in the exploration phase. We may say that the epistemic dimension (awareness of the limits of numerical examples) supports the teleological one (choice of algebra in the exploratory phase).

Interviewer: Try to explain to a secondary school student how to find the property.

Valentina: I think that... beh, I would start reasoning on data, on the hypotheses, and trying to see links between them, seeing what happens in various cases?

Interviewer: do you mean using numerical examples?

Valentina: maybe, even if this could be dangerous because induction does not always works, I mean, if we have limited cases, it is not a good method, it could even be absolutely wrong. But one could start from them; afterwards of course it is necessary to prove it in general... [...] and just consider the hypothesis and try and think about them, from a general point of view, just...non numerical, but n , $n+1$, what they mean, and try exactly to think about them, what this data mean.

Let us come back to Valentina's production. After the first phase, in which Valentina discovers and proves the property at the same time, Valentina writes down: "That were my fist ideas. Now I try to write them down in a better way". This sentence leads to a phase of systematization of the final product.

Given $n \in \mathbb{N}$, n and $n+1$ have only one common divisor, that is 1 . In fact, $\forall d \in \mathbb{N}$ such that $d|n$, $d \neq 1$, $(n) = (0)$ in \mathbb{Z}_d , while $(n+1) = (1)$ in \mathbb{Z}_d because $(n+1) = (n) + (1) = (0) + (1) = (1)$, hence $d \nmid n+1$. From the other side, $\forall p \in \mathbb{N}$ such that $p|n+1$ and $p \neq 1$ I have that $(n+1) = (0)$ in \mathbb{Z}_p and that $(n) = (n+1-1) = (n+1) - (1) = (0) - (1) = (-1)$, hence $p \nmid n$. On the contrary, $1|n$ and $1|(n+1)$ because 1 divides any natural number.

In the subsequent excerpt from the *a posteriori* interview, Valentina shows to put a great care both in the process and in the construction of the final product.

Interviewer: ok. May I ask you why did you do a second part, in which you systematized what you wrote in the first part?

Valentina: the first part was... I gave the idea, I started to write down, in a sort of draft, in order to make my ideas clear to myself, in order to formalize what I had in my mind. Afterwards, I tried to write in a more formal way, because the

first part was really... writing down ideas, while in the second part I tried to write in a more “mathematical” way, in clearer way.

Interviewer: what do you mean by “more mathematical way”?

Valentina: ehm... maybe using less words, trying to be more synthetic, and trying to use a mathematical language, then with more symbolic notation, rather than words.

Interviewer: ok. But actually, as concerns the mathematical content...

Valentina: it is the same. It is more or less the same. Yes, yes.

We may note that Valentina is able to describe the features that, according to her, a mathematical proof should have. Nevertheless, Valentina is aware that the first part of her production is already acceptable, even if written in a less appropriate way. We may say that Valentina is able to manage the crucial dialectic between epistemic and communicative dimension: the second part is an amendment from the communicative point of view, but Valentina is fully aware of the fact that the communication is subordinated to the epistemic dimension, that is to say to the validity of the produced arguments.

DISCUSSION: TOWARDS FURTHER DEVELOPMENTS

The analysis of some examples had the double aim of showing the viability and usefulness of our adaptation of Habermas’ construct in the special case of conjecturing and proving, and of suggesting new research questions, in terms of this construct.

As concerns the first aim, we have seen how success and failure may be read in terms of different intertwinings between the three components of rationality, or dominance, or lack on one of them. We may add that in the case of Valentina the communicative component is strictly depending on the epistemic one; furthermore, the teleological component intertwines with the epistemic one (choice and justification of the arguments) and with the communicative one (other readers will check the production). More generally the previous analyses suggest the opportunity of a closer investigation into the relationships between epistemic rationality, communicative rationality and teleological rationality in the case of proof and proving. Concerning this issue we note that in the historical development of mathematics, subjective evidence (or even mathematicians’ shared opinion of evidence) revealed to be fallacious in some cases, when new, more compelling communication rules obliged mathematicians to make some steps of reasoning (in particular, those concerning definitions: see Lakatos, 1976) fully explicit.

From the educational point of view, while it is easy (for instance, by comparison with other solutions) to help Monica to make her reasoning more explicit (according to her need, as emerged from her comments), the intervention on Caterina is much more delicate: how to make her aware that the “mathematical rule” is not only a matter of

conventional, more complete communication, but also a matter of objective, cogent arguing involving the goal to achieve (an exhaustive argument)? And how to exploit texts that are complete (communicative aspect) in order to develop the need of an exhaustive argument (epistemic aspect), but at the same how to avoid that the necessities inherent in the communicative aspect prevail over the epistemic aspect (cf. Harel's "ritual proof schemes")? A direction for productive educational developments might consist in the elaboration of a suitable meta-mathematical discourse (see Morselli, 2007) for students (including an appropriate vocabulary), as well as in the choice of suitable tasks that reveal how intuitive evidence not developed into an explicit, detailed justification sometimes results in fallacious conclusions.

These considerations raise another problem: Habermas' construct offers only the possibility to evaluate a production process and its written or oral products, while in mathematics education we need also to consider a long term "enculturation" process. We are working now on the articulation between a cultural perspective to frame this process (see Morselli, 2007) and tools of analysis derived from Habermas' elaboration on rationality. Indeed, it is within the cultural perspective outlined in the introduction that we think possible to deal with the approach to theorems and proving in school as a process of scientific "enculturation" consisting in the development of a special kind of rational behaviour, the one derived from Habermas, that is presented in this paper. We are trying to refine the Vygotskian common concepts - scientific concepts dialectics in the case of theorems and proofs in order to get a frame where to situate the long term planning of the school approach to the culture of theorems. Habermas' construct contributes to it by suggesting three interrelated dimensions along which to develop students' skills in proving and students' (and teachers') awareness about crucial features of proving and proofs. The educational challenge consists in leading students to move from the ordinary argumentative practices of validation of statements in different domains to the highly sophisticated and culturally situated management of the components of a rational behaviour in the specific case of proving.

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EXPERIMENTAL MATHEMATICS AND THE TEACHING AND LEARNING OF PROOF¹

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Aim of this paper is to discuss the role of experiments in mathematics for the teaching and learning of proof. I summarize some research findings from basic research studies and from teaching experiments. The examples comes from teaching experiments at all school levels on space and geometry by means of classical resources although some of the findings might be expanded to other subject areas and to ICT. They allow to frame the topic within the international literature on conjecture production and proof construction: they support the advantages of experimental approaches to the teaching and learning of proof and, at the same time, point at some critical points to be controlled in order to design appropriate teaching interventions.

INTRODUCTION

A growing interest is shown, at the international level, for the development of approaches to mathematics where the active participation of students is encouraged within a laboratory setting, with hands-on activities. The emphasis on experiments, manipulation and perception, measurement and examples is shared by the approaches developed within ICT environments (both DGE and CAS) and within classical technologies (straightedge, compass and ancient instruments). This experimental approach, where exploration plays a major role, seems appealing for students, who quite often find the evidence offered by a particular experiment much more convincing than a rigorous proof (Jahnke, 2007) and are bored by the request to produce also mathematical arguments. Hence, the appeal of experimental approach might be suspected of obstructing the development of mathematical styles of reasoning: some believe that hands-on activities are useful in either science centres or mathematical festivals, where popularization of mathematics is in the foreground, whilst are not useful and may be even risky in the mathematics classrooms, where the construction of mathematical meanings is at stake. In other words, many mathematics teachers are afraid that the need of mathematical proofs and of deductive arguments is put in a difficult position if experiments are given too much space in the mathematics classroom, at least in secondary schools. In the following, after a short review of literature, I present some effective experiments at all school levels where experiments and exploration have been combined with theoretical aims like conjecture production and proof construction.

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SOME STUDIES CONCERNING PROVING IN THE MATHEMATICS CLASSROOM.

The literature on proof and proving is large and encompass different aspects. In the recent book on "Theorems in School" edited by P. Boero (2007), the following aspects are highlighted: the historical and epistemological dimension; curricular choices, historical traditions and learning of proof (including two national case studies); the cognitive dimension of the relationships between argumentation and proof; the didactical dimension including both teacher education and classroom practices. In the chapter authored by Bartolini Bussi et al. (2007), a mathematical theorem – for didactical purposes - is conceived as a system of statement, proof and theory. All these three components are important: the theory as a system of shared principles (sometimes called postulates or axioms and definitions); the statement as the result of a conjecturing process, where exploration through experimental activity is in the foreground, the proof as a sophisticated argumentation that is, on the one hand, connected with the conjecturing process, and, on the other hand, consistent with the reasoning styles of mathematicians (e. g. deduction from the accepted principles). This approach is consistent with Jahnke (2007), who speaks about ‘local theories’, i. e. small networks of theorems based on empirical evidence and claims: “There is no easy definition of the very term ‘proof’ since this concept is dependent of the concept of a theory. If one speaks about proof one has to speak about theories, and most teachers are reluctant to speak with seventh graders about what a theory is”. And Arzarello (2007) adds: "A statement B can be a theorem only relative to some theory; it is senseless to say that it is a theorem in itself: even a proposition like $2+2=4$ is a theorem in a theory A (e. g. some fragments of arithmetic)".

In the above sense, it is possible to speak about theorems also within primary school, provided that the theories are “germ theories”, drawing on empirical evidence, with the expansive potential to capture more and more principles. Germ theories, with principles constructed on empirical evidence, are crucial up to 8th grade; later, accordingly to curriculum, the reference to more and more structured mathematical theories is possible. So, for instance, in the teaching experiments below, the reference theory from grade 11th on is expected to be elementary geometry (either 2D or 3D) with some additional parts concerning either isometries or conic sections.

The links between argumentation and proof from a cognitive perspective have been carefully analysed by Pedemonte (2007) who devoted her doctoral thesis to the development of the idea of cognitive unity, meant as a kind of continuity between the production of a conjecture and the construction of the proof. Experimental research shows that proof is more ‘accessible’ to students if an argumentation activity is developed for the production of a conjecture: in fact this argumentation can be used by the student in the construction of proof by organising in a logical chain some of the previously produced arguments. These studies may have important consequences

on the teaching and learning of proof: to explain why rote learning of ready made proofs is not successful for most students; to select suitable problems, which might foster conjecture production before proof construction; to understand why in some cases proving remains difficult in spite of the previous conjecturing process.

In the following sections I shall quote very quickly some experiments where conjecturing and proving were promoted, at different school levels and with different organization.

EXAMPLES FROM LONG TERM TEACHING EXPERIMENTS

In the attached table, some paradigmatic examples are quoted from long term teaching experiments developed as coordinated studies by different research teams. All the tasks concern a conjecture production before proving construction. They appear, however, different from each other.

Three tasks (tasks 1,2,3) concern individual activity, to be solved in paper and pencil setting; three tasks (tasks 4,5,6) concern small group activity, to be solved in writing after the exploration of a material object. The exploration is free in the case of sunshadows (task 4), whilst it is guided by sheets or by the teacher himself in the two cases from secondary school (tasks 5 and 6). The tasks 1 and 3 are construction problems: they require to produce a drawing and to justify the validity of the used method. The expressions "Explain" mean, in a language accessible for young learners, to justify the drawing process with reference to a shared (germ) theory. The task 2, on the contrary, seems to be given in a discursive way. Yet the explanation requirement with reference to a shared (germ) theory is implicit, as a part of the tacit rules shared within the classroom involved in these experiments. In the last three tasks proof is not explicitly required. Actually the focus is on the production of the conjecture. This is an intentional choice, because the problems are quite demanding. The tasks 4 and 6 concerns 3D geometry, that is usually not well mastered by secondary school students. The task 5 is difficult: the conjecture concerns a rotation around the lower point (O) in the Fig. 3. Actually to recognize it, it is necessary to "see" two line segments (OP and OP') that do not exist, to realize that they are always equal and, more generally, to be able to "see" invariants during the motion. The teachers, for the tasks 4, 5 and 6 had designed, according to the shared theoretical framework, an intermediate step where to collect and discuss the conjectures, before entering the proving process. In the task 4, students are explicitly requested to produce a general statement. This expression was used in those classrooms to foster the production of statements with universal quantifiers (all, always, and so on) and hopefully in conditional form (if ... then) to pave the way towards the construction of a proof with specified hypothesis and thesis.

Gr.	GERM THEORIES (REF)	CONJECTURES - PROBLEMS <i>THE TASK - TO BE SOLVED IN WRITING</i>	SETTING MATERIAL
1. Gr.2 - 8	The invariance of alignment in perspective drawing (Bartolini Bussi, 1996)	The centre of a table drawn in central perspective. <i>Draw the small ball in the centre of the table. You can use instruments. Explain your reasoning.</i>	Individual task (Fig. 1)
2. Gr.2 - 8	Motions of geared wheels (Bartolini Bussi et al., 1999)	The motion of trains of toothed wheels. <i>What about three wheels geared with each other?</i>	Individual task No material
3. Gr.4 - 8	The equality of the distance of the centres of two tangent circles to the sum of radii (Bartolini Bussi et al., 2007)	The drawing of a circle tangent to two given circles. <i>Draw a circle with a radius of 4 cm tangent to the given circles (radii 3 and 2). Explain carefully the method. Explain carefully why it works.</i>	Individual task (Fig. 2)
4. Gr.6 - 8	Mathematical model of sunshadows. Basic properties of lines, planes, parallelism and perpendicularity (3D geometry) (Boero et al., 2007)	The parallelism of sunshadows of sticks. <i>In recent years we observed that the shadow of two vertical sticks on the horizontal ground are always parallel. What can be said of the parallelism of shadows in the case of a vertical stick and of an oblique stick? Can shadows be parallel? At times? When? Always? Never? Formulate your conjecture as a general statement.</i>	Small group work. Pens, pencils, notebooks, rulers, to reify lines and planes
5. Gr.11	Elementary geometry (3D geometry). Definitions and properties of isometries. (Bartolini Bussi & Pergola, 1996)	The isometry (rotation) produced, as a correspondence, by a pantograph. After a guided exploration of the pantograph. <i>If P and P' are two writing points, draw two corresponding figures. Which are the common properties of the two figures? Can they be superimposed? Does it exist a simple motion which superimposes them? Describe it.</i>	Small group work. A pantograph with graphite leads in P and P' (Fig. 3).
6. Gr.12	Elementary geometry (3D geometry). Metric definition of conics. Equations of conics (Bartolini Bussi, 2005)	The conic obtained by cutting a cone in a suitable way. The task is given orally by the teacher. <i>You have to obtain an important property of parabola [...]. As you see, [the parabola] is in a 3D space, on the surface of the cone [...]. you have to discover the relationship between the green line segment [AE in the Fig. 4] and this line segment [EB in the Fig. 4].</i>	Small group work. A 3D model of a cone with a normal cutting plane (Fig. 4).

Table 1. Some paradigmatic examples.

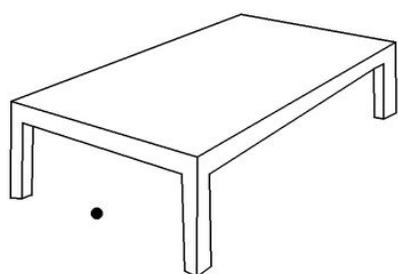


Figure 1. The small ball and the table

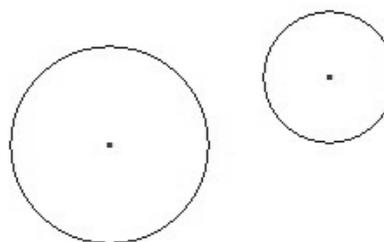


Figure 2. The two circles and the tangent circle

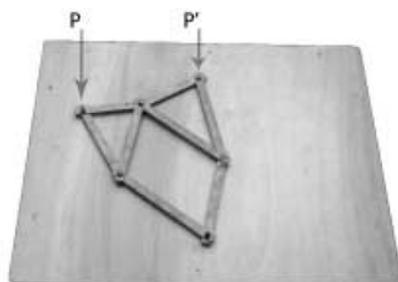


Figure 3. The pantograph

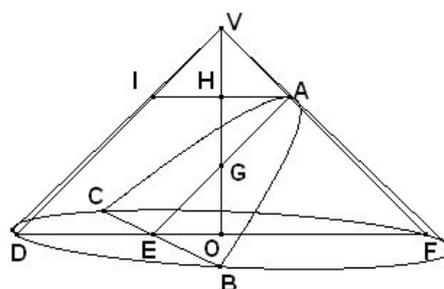


Figure 4. The parabola

At all ages, the dynamic exploration of a suitable problem situation has a crucial role both at the stage of conjecture production and during the proof construction. In particular, as to the conjecture production "the conditionality of the statement can be the product of a dynamic exploration of the problem situation during which the identification of a special regularity leads to a temporal section of the exploration process, which will be subsequently detached from it and then "crystal" from a logic point of view ('if ... then')"; and as to the proof construction, "for a statement expressing a sufficient condition ('if ... then'), proof can be the product of the dynamic exploration of the particular situation identified by the hypothesis" (Boero et al, 2007, p. 249 ff.). This phenomenon has been observed by Boero et al. (2007) for the task 4 about sunshadows, by Bartolini Bussi & Pergola for the task 5 about the pantograph (Bartolini Bussi & Pergola, 1996) and in other ongoing experiments on either transformation or curve drawing devices. As concrete manipulation of materials is not spontaneous and guaranteed with elder students, who had already spent years to learn (or better to be taught) that mathematics is just a mental activity, the teacher has to foster it in a very coercive way: concrete exploration in demanding tasks is quite often the only effective way to promote dynamic exploration. Younger pupils, on the contrary, were accustomed to explore and to evoke exploration when no concrete object was available.

THE PROCESSES

The six situations above, although in different modes, have been designed to foster cognitive unity between the conjecturing and the proving phases. I shall not try to summarize here the observed processes concerning them all: they are complex, long standing, different (also for students' age) and all available in the international literature. Rather I shall illustrate another simple case of conjecture production and proof construction at secondary school level (from grade 10 on, according on the curriculum), concerning a curve drawing device. I shall narrate the stories of dynamic exploration that show up when secondary school students are given this curve drawing device to foster reasoning, conjecturing and proving (another example is discussed by Bartolini Bussi, in press).

I shall collect some evidences from the field notes of the exploration sessions in both school classrooms and the Laboratory of mathematical Machines (www.mmlab.unimore.it), to highlight the patterns that emerge. The two parts of the fig. 5 show (on the left) a drawing from the XVII century treatise by van Schooten (1657, p. 339) and (on the right) a photo of the brass copy reconstructed on a wood platform (40 cm x 40 cm) by the team of the Laboratory of Mathematical Machines at the Department of Mathematics of Modena, to be used with secondary and university students. The students are supposed to know some early properties of conics, e.g. the string and pencil drawing of an ellipse (together with the ellipse metric definition).

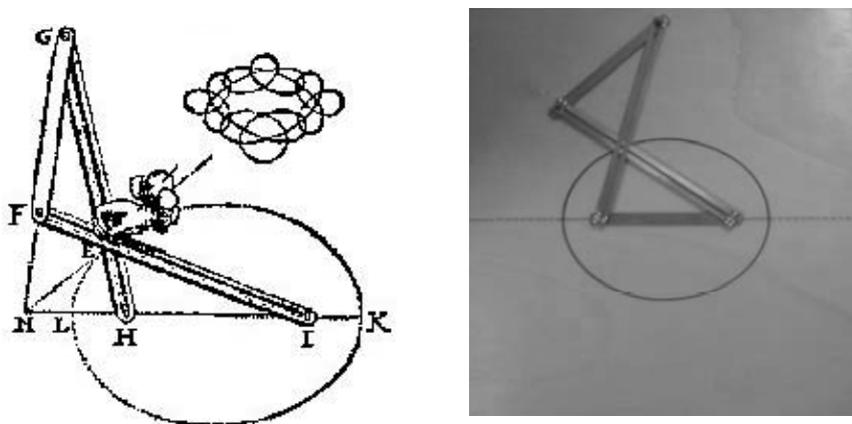


Fig. 5a and 5b: Van Schooten's Ellipsograph

There are several ways to explore the artefact (in order to produce a conjecture and to construct a proof of the conjecture) that span from strongly to weakly guided ones. In general, strongly guided exploration is suitable to the short term sessions (at most 2 hours, including the introduction and the conclusion of the hands on activity, Maschietto & Martignone, in press) which take place when a classroom come to the Laboratory, whilst weakly guided exploration is suitable to classroom activity, when the teacher plans to spend more time on the same topic. Actually with a weak guide, the time may expand, not matching the time constraints of a short visit to the Laboratory.

A) Strongly guided exploration. Students are given a worksheet where a layout of the artefact is drawn with coding letters (examples: <http://www.mmlab.unimore.it/online/Home/VisitealLaboratorio/Materiale/articolo10005163.html>) and are suggested to identify the fixed points, the trajectories of the moving points (e.g. G and F), the length of the bars, and so on. After this exploration, they are asked to conjecture the name (if any) of the trajectory of the point E (intersection of GH and FI in the fig. 5a) tracing it with a graphite lead on the wooden platform. The drawing is soon recognized as an arch of an ellipse and the conjecture is produced. Then the process of proof construction is to be started. We shall comment it later.

B) Weakly guided exploration: students are given the artefact and the information that it may draw curves; they are given the burden to produce conjectures and to prove them. A graphite lead to trace the trajectory of points is available with no special emphasis on this experiment: they can decide to use it or not. The artefact is without coding letters (Fig. 5b) and actually the need of coding may be one of the outcomes of the exploration to understand each other (Bartolini Bussi & pergola, 1996). When the students explore for some minutes the motion without drawing the arch, they may recognize a well known (although hidden) figure. HIGF (fig. 5a) is an isosceles trapezium with diagonals (HG and FI) and sides (FG and HI) given by brass bars, whilst the bases FH and GI have a variable length and are not reified by bars. The figure is not trivial to be noticed, as the two bases are not visible. Usually the students rotate G around H and observe the figure. Sometimes they seem fascinated by this rotation. They stay silent for minutes. They try to look at the artefact from different perspectives, also standing and moving around the table. They assume strange postures, twist their necks to follow the motion, point at the bars and follow the motion with the finger in the air, move the bars forward and backward to look for invariants and test them stopping the continuous process. In the small group work, sometime a conflict arises, when the speed of the motion controlled by the actor does not match the exploration planned by the observer. At one point they "see" the trapezium and notice that $EG = EI$ and $FE = FH$. When a student has "seen" the trapezium, this figure is immediately shared with others. When the trajectory of E is eventually drawn they have at disposal what they need to link the conjecture with the metric property of ellipse.

I have described two 'antipodal' exploration processes with a lot of mixed cases in between. The weakly guided one is enjoyed by experts. The strongly guided one suits novices' needs to avoid frustration: it aims at encouraging to handle the artefact and at scaffolding the process. In both cases the demanding part is not the conjecture production, especially when drawing by the graphite lead is encouraged. Actually, as soon as the user draws the curve, the conjecture springs up, because only a limited set of curves is known by students: it is neither a circle nor a parabola nor an hyperbola, hence it must be an ellipse. The demanding task in this case concerns proof construction. This situation is different from the one of the tasks 4 and 5 above, where also conjecturing is really demanding.

In the strongly guided exploration, the worksheet suggests some ways to explore the properties of the artefact. Yet, in order to notice the properties, measuring by rulers is suggested. Measuring requires to stop the motion and to transform the experience of continuous motion into the observation of a finite set of frames. The focus risks to be on measuring parts of still figures.

In the weakly guided exploration, the focus shifts on the observation of dynamically changing shapes and their invariants. The students have to move and observe. Their process seems time wasting and not effective and has to be monitored by a walking teacher who moves from one group to another showing how to explore the artefact, with changing speeds and, maybe, no word. The initial 'weak' guide seems to require a stronger teacher's control. The students do not need (and usually do not wish) to measure bars by a ruler. As soon as they notice some invariants, they use their hands: they pretend to pick up the line segment EG between forefinger and thumb and to rotate it until it matches EI. They repeat the action on the pair FE and FH. Silent gestures seem to be enough to convince them. Maybe words and deductive chains are missing. Writing and justifying (by symmetry, for instance) the equality:

$$HE + EI = HE + EG = HG$$

that represents the metric property of ellipse with foci H and I is the boring counterpart of a relationships discovered by making "infinitely many" experiments, during the continuous motion of G around H.

In both cases of exploration, if the drawing is produced too early, the attention is focused on the final result of drawing rather than on the dynamical process of drawing. I shall consider this later.

There is a difference between the strongly guided exploration, that foster the production of statements concerning pointwise construction of the trajectory and the weakly guided exploration, that foster the production of statements concerning the global construction of the trajectory by a continuous motion. This difference is epistemological and mirrors the ancient pointwise construction of curves and the modern (as from the 17th century) construction of curves by a continuous motion of a machine. In the pointwise construction, there is a gap between the statements concerning a particular point E obtained when the artefact is in a given position and the generalization to a whichever point of the trajectory. This gap might obstruct the proof construction, requiring additional arguments.

The situation is different, yet recalls the one analysed by Pedemonte (2007) and concerning the construction of proofs by mathematical induction. She analysed the sum of the interior angles of an n-sided convex polygon, but the reasoning might be applied to many cases of induction. The well known formula: $(n - 2)$ times 180° , may be conjectured in at least two ways, that draws on experimental activity and that are called: result pattern generalization (the cases of n-sided convex polygons are analysed separately, adding the measures of the interior angles, for $n=3$, $n=4$, $n=5$ and

so on); process pattern generalization (from an $(n-1)$ -sided convex polygon, for $n=3, 4, 5$ and so on, a new n -sided convex polygon is obtained by the juxtaposition of a triangle, whose sum is 180°).

The result pattern generalization does not help much to construct the proof by mathematical induction, because the argumentations used have no counterpart in the proof. On the contrary the process pattern generalisation paves the way towards the proof, showing how it is possible to shift from $n-1$ to n . Pedemonte (2007) says that in the second case there is a structural continuity between the conjecture production (by argumentation) and proof construction (by induction). Students may succeed in proving the conjecture also after a result pattern generalization, but they must reconstruct a suitable argumentation that links the conjecture to the proving process.

The shift to the analytic frame suggested in the Laboratory worksheets is an intentional break of the structural continuity, because the analytic frame is supposed to be the familiar context where conics are studied in secondary schools.

DISCUSSION

Some conclusions may be drawn from the quoted examples and research outcomes. First, there are good reasons to believe that conjecturing through exploration before proving might be very useful. Yet, when conjecture production is too fast, it might offer no element to be used in the proving process. Hence it is useful to look for strategies that slow down the conjecture production and encourage effective exploration of the problem. The time spent in conjecture production is not wasted and may be recovered in the proof construction. Second, it is not possible to give general rules about which exploration is effective in the conjecture production. In the last example, I have contrasted strongly guided and weakly guided explorations, which are only two examples of a very rich set of possibilities. What to choose in a classroom situation? The teacher's decision has to be contextualized and depends on a lot of issues: the time constraints, the curriculum, the students' qualifications and so on. This last issue is related to teacher education. The teacher's knowledge in order to design and to manage in the mathematics classroom this kind of activities is complex and does not fit in the space of this paper. A systemic approach to teacher education is now in the foreground in the literature on didactics of mathematics.

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CONJECTURING AND PROVING IN DYNAMIC GEOMETRY: THE ELABORATION OF SOME RESEARCH HYPOTHESES

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Research has shown that the tools provided by dynamic geometry systems impact students' approach to investigating open problems in Euclidean geometry. We particularly focus on types of processes that might be induced by certain uses of tools available in Cabri. Building on the work of Arzarello (Arzarello et al., 1998) and Olivero (1999, 2002), we have conceived a model describing some cognitive processes that may occur during the production of conjectures and proofs in a dynamic geometry environment and that might be related to the use of specific dragging schemes. Moreover, we hypothesize that such cognitive processes could be induced by introducing students to the use of dragging schemes.

Key words: conjecturing, dynamic geometry, dragging schemes, abductive processes, cognitive unity

INTRODUCTION

The contribution of a DGE to students' reasoning and proving is particularly evident during the investigation of open problems, since this process involves making conjectures (Mariotti, 2006). Instead of a static-conjecture built in a paper-and-pencil environment in a DGE a dynamic-conjecture [1] can be developed. Moreover, in a DGE, the invariant geometrical properties of a construction, which lead to conjectures, can easily be grasped. An interesting question is: what kind of support can a DGE provide first during the development of a conjecture and then during the production of a proof? The answer seems to depend on the nature of the problem. On one hand the ease to immediately grasp certain invariants seems to inhibit some argumentation processes that lead to finding useful elements for the construction of a proof. On the other hand, research has shown that a DGE can foster the learners' constructions and ways of thinking, and that it can help overcome some cognitive difficulties that students encounter with conjecturing and proving (e.g. Noss & Hoyles, 1996; Mariotti, 2002; De Villiers, 2004).

Building on the work of Olivero and Arzarello (Olivero, 1999; Arzarello et al., 1998), we have conceived a model of cognitive processes that can occur during the conjecturing stage of open problem investigations in a DGE. Through a qualitative study, our final goal is to give a detailed description of some cognitive processes related to conjecturing and proving, and of how a DGE might foster such processes, thus providing a base for further research and for the development of new curricular activities.

ORIGIN OF OUR HYPOTHESES

In the following paragraphs we will briefly outline the theoretical framework which the ideas are embedded in.

Semiotic Mediation and Semiotic Potential of an Artifact

A DGE like Cabri, which contains “objects” such as points, lines, circles, and ways to “manipulate” the objects, is a microworld (Papert, 1980; Balacheff & Kaput, 1996) built to resemble the mathematical world of Euclidean geometry. A key aspect of microworlds in mathematics education is that the “objects” included offer the opportunity for the user to experiment directly with the “mathematical objects” (Mariotti, 2005, 2006), because the logical reasoning behind the objects in the microworld is designed to be the same as that behind the real mathematical objects that they represent.

Recent research has developed the ideas of *tool of semiotic mediation* and of *semiotic potential of an artifact*:

“...any artifact will be referred to as a tool of semiotic mediation as long as it is (or it is conceived to be) intentionally used by the teacher to mediate a mathematical content through a designed didactical intervention” (Bartolini Bussi & Mariotti, 2008).

Computers in general, and a DGE in particular, are considered to be tools of semiotic mediation (Mariotti, 2006; Bartolini Bussi & Mariotti, 2008). However, the mediation can occur successfully only if their semiotic potential is exploited. Therefore it becomes necessary to study ways that foster exploitation of such potential. This was a main goal we had in mind when we started developing our hypotheses.

A First Theoretical Model and the Dragging Schemes

The dragging tool can be activated by the user, through the mouse. It can determine the motion of different objects in fundamentally two ways: direct motion, and indirect motion. The *direct motion* of a base-element (for instance a point), that is an element from which the construction originates, represents the variation of this element in the plane. The *indirect motion* of an element occurs when a construction has been accomplished. In this case dragging the base-points will determine the motion of the new elements obtained through the construction. The use of dragging allows one to feel “motion dependency”, which can be interpreted in terms of logical dependency within the geometrical context (Mariotti, 2002, p. 716). Starting from these phenomenological perspectives, a refined analysis of the dragging tool can highlight its semiotic potential that can be exploited by the teacher in school practice.

The use of Cabri in the generation of conjectures is based on the interpretation of the dragging function in terms of logical control. In other words, the subject has to be capable of transforming perceptual data into a conditional relationship between hypothesis and thesis. The consciousness of the fact that the dragging process may

reveal a relationship between geometric properties embedded in the Cabri figure directs the way of transforming and observing the screen image (Talmon & Yerushalmy, 2004). At the same time, that consciousness is needed to exploit some of the facilities offered by the software, like the ‘locus of points’ or ‘point on object’. Such a consciousness is strictly related to the possibility of exploiting the heuristic potential of a DGE (Mariotti, 2006).

The theoretical model presented by Olivero, Arzarello, Paola, and Robutti (Olivero, 2000; Arzarello, et al., 1998, 2002) addresses expert solvers’ production of conjectures, and how abduction marks the transition from the conjecturing to the proving phase, when a passage from “ascending control” to “descending control” occurs. Abduction guides the transition, in that it seems to be key in allowing solvers to write conjectures in a logical ‘if...then’ form, a statement which is now ready to be proved. Arzarello et al.’s analysis of subjects’ spontaneous development of dragging modalities led to the determination of a classification (Arzarello et al., 2002), which researchers have referred to as the “dragging schemes” (Olivero, 2002).

Abduction

In the previous section, the notion of abductive processes is mentioned. Peirce was the first to introduce the notion of abductive inference, and compare it with other inferences, such as deduction and induction. According to Peirce,

“abduction looks at facts and looks for a theory to explain them, but it can only say a “might be”, because it has a probabilistic nature. The general form of an abduction is: a fact A is observed; if C was true, then A would certainly be true; so, it is reasonable to assume C is true” (Peirce, 1960, p. 372).

Recently, researchers have renewed interest in abduction. In particular, Magnani defines abduction in a way that we find quite useful. According to him abduction is,

“the process of inferring certain facts and/or laws and hypotheses that render some sentences plausible, that explain or discover some (eventually new) phenomenon or observation; it is the process of reasoning in which explanatory hypotheses are formed and evaluated” (Magnani, 2001, pp. 17-18).

Moreover, the following distinction of direct abduction versus creative abduction will be useful for our study. *Direct abduction* is when the “rule” used in the abductive process consists of a theorem that is already known to the student; while *creative abduction* is when the “rule” of the abduction consists of something new, that is not previously known by the student (see also Magnani, 2001; Thagard, 2006). Other researchers have studied various uses of abduction in mathematics education (Reid, 2003), and abductive processes in relation to transformational reasoning (Simon, 1996; Cifarelli, 1999; Ferrando, 2006). The basic idea is that an abductive inference may serve to organize, reorganize and transform problem solvers’ actions (Cifarelli, 1999). Abductive processes have also been observed by Arzarello et al. (1998) during the development of conjectures when students were using the dragging schemes, as

mentioned above. In the next section we describe how our work builds on that of Arzarello et al., trying to study in detail the processes that occur during the conjecturing stage in open problem investigations, how these processes may be fostered, and what they might lead to during the phase of proof production.

OUR HYPOTHESES

While Olivero, Arzarello, Paola, and Robutti (Olivero, 2000; Arzarello, et al., 2002) focused their attention on the subjects' use of the dragging schemes during the development of a conjecture, we concentrate on the abductive processes that may be induced by certain dragging schemes. Arzarello et al. observed that abduction occurs during solvers' use of the dragging schemes. Moreover, they claim that the production of conjectures is based on abductive processes. Thus, it seems that the use of certain dragging schemes may foster abductive processes, and, consequently, the production of conjectures. To some extent, the dragging schemes can be seen as cognitive artefacts (Norman, 1991). We would like to investigate the relationship between the use of the dragging schemes and the development of abductive processes. In order to accomplish this investigation we need to induce solvers' use of dragging schemes, so we decided to introduce students to the specific dragging strategies. This way we seem to be able to induce the use of specific dragging schemes for the solution of open problems and, consequently, the appearance of abductive processes.

Below is a hypothesis of what might occur as a solver, who has been introduced to the dragging schemes, approaches an open problem in a DGE.

- Step 1: conscious use of different dragging strategies to investigate the situation – after *wandering dragging*, in particular *dummy locus dragging* (or *lieu muet dragging*) to maintain a geometrical property of the figure (*intentionally induced invariance*, or III), and use of the *trace tool*.
- Step 2: consciousness of the locus (*lieu*) that appears through *lieu muet dragging* – this marks a shift in control from ascending to descending – and description of a second invariance (*invariance observed during dragging*, or IOD).
- Step 3: hypothesis of a conditional link between the III and the IOD, to explain the situation.
- Other forms of dragging may be performed: *line dragging*, *linked dragging*, and the *dragging test*.
- Step 4: formulation of a conjecture of the form 'if IOD then III' (product of the abduction).
- Step 5: production of a mathematical proof of the conjecture (or attempt of it). Potential re-formulation of the conjecture.

The Notion of *Path* and an Example

Another hypothesis that we advance is that there is a key element, the *path*, that plays a fundamental role in the abductive process. In this section, we will try to introduce the concept of *path* and its significance for the model.

One of the dragging schemes, *lieu muet dragging*, involves dragging a point with the intention of maintaining a given property of the figure (which becomes the III). Some regularity may appear during this dragging stage, leading to the discovery of particular constraints that the dragged point has to respect (expressed in the IOD). Because of their origin from dragging, such constraints may be interpreted as the property of the point to belong to a particular figure. In mathematical terminology, that of course may not be consistent with students' way of thinking, we can speak of a hidden locus (*lieu muet*). Such locus can be made explicit by the trace tool, through which it appears on the screen (*lieu parlante*). During lieu muet dragging the solver notices regularities of the point's movement and conceptualizes them as leading to an explicit object. We refer to this object as a *path* when the solver gains consciousness of it, as generated through dragging, and consciousness of its property that if the dragged point is on it, a geometrical property of the Cabri figure is maintained. In this sense a *path* is the reification (Sfard, 1991) of a *lieu* that can now be used in a "descending control" mode (Arzarello et al., 2002). Zooming into Step 2, above, we observe that this is the point of the process in which the notion of *path* arises, and we can add a Step 2bis to indicate the (potential) geometric interpretation of the *path*, in order to (potentially, after Step 3) perform *line dragging*, *linked dragging*, and the *dragging test along such path*.

We believe that the *path* plays an important role in relation to the abductive processes that can be used to develop conjectures in a DGE. In particular, recognition of a *path* can act as a bridge, fostering the formulation of a conjecture. In fact, the *path* can be used during the abductive processes, but then it may no longer appear (or it may appear in a different form) in the formulation of the conjecture. Below, we zoom into a way in which abductive processes may take place and lead to a derived conjecture, and then we provide an example of the model in use during an activity.

- Intentionally Induced Invariance (III): the solver tries to maintain a certain geometrical property.
- Invariance Observed during Dragging (IOD): the solver notices that when he/she drags a certain basic point X along the *path*, the III seems to be maintained.
- Product of abductive process: it becomes reasonable for the solver to assume that if point X lies on the *path* (description of the IOD), the III is true.

If the path is recognized as a particular geometrical figure *F*, the derived conjecture may be: if X lies on F, the III is true.

Activity: Draw three points A, M, K, then construct point B as the symmetric image of A with respect to M, and point C as the symmetric image of A with respect to K.

Construct point D as the symmetric image of B with respect to K. Drag M and make conjectures about ABCD. Then try to prove your conjectures.

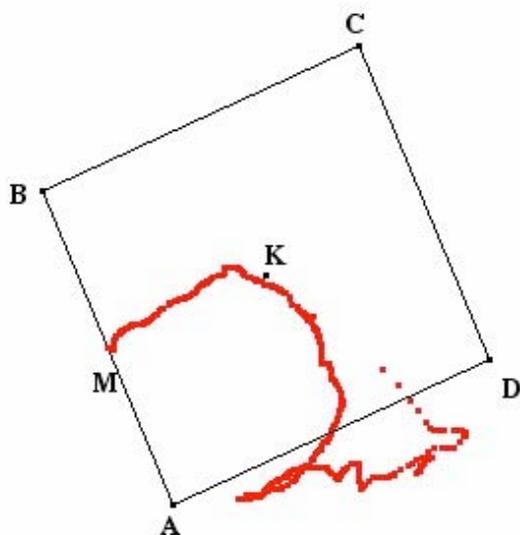


Fig 1: Dragging with the trace tool can help a student notice a *locus* (or *lieu*).

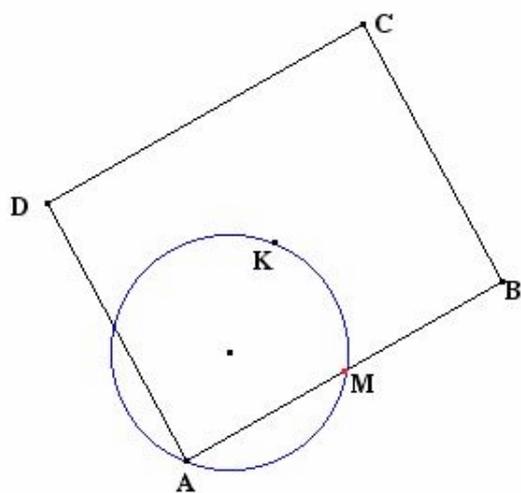


Fig 2: M is being dragged along the path (*line dragging*).

may hypothesize a conditional link between the III and IOD. At this point the abduction leads to a hypothesis of the form ‘if IOD then III’, leading to a conjecture like the following: “If M is on the circle of diameter AK, then ABCD is a rectangle,” or (if they discover or derive a property of the base-points which is equivalent to M lying on the circle): “If AKM is a right triangle, ABCD is a rectangle.”

In the case of the first conjecture, here is how we hypothesize the abduction (*creative abduction*) might go.

A Response [2]: Through *wandering dragging* solvers may notice that ABCD can become different types of parallelograms. In particular, they might notice that in some cases ABCD seems to be a rectangle (they can choose this as the III). With the intention of maintaining this property as an invariant, solvers might mark some configurations of M for which this seems to be true, and through the trace tool, try to drag maintaining the property, as shown in Fig 1. This can lead to noticing some regularity (IOD) in the movement of M, which might lead to awareness of an object along which to drag (the circle of diameter AK, potentially not yet recognized as “a circle”). At this point, when such awareness arises, we can speak of *path* with respect to the regularity of the movement of M.

If solvers recognize the path to be a familiar geometrical object, like in this case, they might be inclined to constructing it, as shown in Fig 2, and dragging along it (*line dragging*), or even linking the free point to it (*linked dragging*) and performing a *dragging test*. Through this abductive process, as an attempt at explaining the experienced situation, as Magnani describes (Magnani, 2001), solvers

- III: ABCD is a rectangle.
- IOD: when M dragged along the *path*, fact A seems to be true. The path is a known geometric figure: the circle of diameter AK.
- Product of the abduction: If point M lies on the circle of diameter AK, ABCD is a rectangle.

This product of the abduction coincides with a formulation of a conjecture. However, solvers might also perform a second abduction (this time a *direct abduction*) linking the property “M belongs to the circle” to a property of the base-points of the construction. In this case this may lead to a formulation of the conjecture like: “If the triangle AMK is a right triangle (with $\angle AMK$ as the right angle), ABCD is a rectangle.” In this case the further elaboration of the geometrical properties recognized in the path will have led to a key idea (Raman, 2003) of a possible proof. In particular, this idea together with that of triangles AMK and ABC being similar, should be enough for students to successfully provide a proof to their conjecture. In this sense, abductive processes involving the notion of *path* (as a reified concept the solver is aware of) might be a step towards the achievement of cognitive unity [3] (Boero, Garuti, & Mariotti, 1996; Pedemonte, 2003).

Some Research Questions

Given the hypotheses outlined above, we propose some general questions for a research study. First, it would be interesting to investigate what forms of reasoning (abductive, deductive, ...) are actually used (and how) during the conjecturing stage of an open problem in a DGE. In particular, if subjects use *lieu muet* dragging, what is the role of the *path*? Can our model be confirmed (even in a potentially modified version)? Second, how does a DGE contribute to the development of the proof of a conjecture? It would be interesting to compare the dragging schemes (if any) used during this stage to those used during the conjecturing stage. It might also be insightful to investigate the forms of reasoning used during the conjecturing stage in the cases in which subjects do produce a proof. Finally, it would be interesting to study whether it is possible to detect a relationship between the forms of reasoning analyzed, and, if possible, to describe such a relationship.

EXPERIMENTAL DESIGN AND POTENTIAL CONCLUSIONS

We propose to structure the study in the following general way: by a selection of the subjects, the introduction of the subjects to the dragging schemes, finally open-problem-activity-based interviews on pairs of students. We will use results from the pilot study to refine the model, the research questions, and the activities proposed during the interviews. In the results of this study we hope to be able to include: a description of some cognitive processes that occur during the conjecturing stage of the investigation of open problems in a DGE; and validation of the model (or of a revised version of it), or motivations for rejecting it as a useful descriptive model. Therefore, this study should help gain better comprehension of specific cognitive

processes. In particular, we hope to gain some insight into how abductive processes may occur, whether they can be fostered by preliminary introduction of the dragging schemes, and how the notion of *path* may foster the formulation of conjectures.

A secondary objective is to gain insight into how a DGE contributes to the development of proofs. The activities proposed during the interviews will all be open problems in which students are asked to make conjectures and then try to prove them. The *path* might also play a role in the generation of a proof, in that it may be a part of the “reorganization and transformation” that occurs with abductive reasoning (Cifarelli, 1999). This might very well be new powerful tool for the solver to use in a potential proof (or solution of the problem) as an aid to gain cognitive unity, as mentioned above. In this case, it would be reasonable to hypothesize that if the dragging schemes were to foster abductive processes, and abductive processes were to foster cognitive unity, then introducing the tool of the dragging schemes to the students a priori might accelerate and facilitate the entire process of making a conjecture and reaching a proof for it.

If our hypotheses are confirmed, and the dragging schemes and the notion of *path* do contribute positively to the formulation of conjectures (and potentially of proofs), we will recognize them as tools of semiotic mediation, with a semiotic potential that could be exploited by teachers. In this case, teaching experiments, which introduce the dragging schemes at a class-level, should be carried out, in order to further investigate how the teacher can exploit the semiotic potential of the dragging schemes in the classroom practice. Later, large-scale quantitative research on the induction of cognitive processes through introduction of the dragging schemes could be conducted, with the didactic objective of implementing the teaching of the dragging schemes in school curricula.

NOTES

1. With “static” and “dynamic” referred to conjecture, here we intend to emphasize the nature of the conjecture’s origin.
2. This is only one of the many possible responses leading to this specific conjecture. Of course different students might reach this conjecture in different ways. Moreover there are many different conjectures that students can formulate by focusing their attention on different geometric invariants (in this case, having ABCD be a kite, a rhombus, or a square).
3. Boero et al. introduce cognitive unity as follows: “During the production of the conjecture, the student progressively works out his/her statement through an intense argumentative activity functionally intermingled with the justification of the plausibility of his/her choices: during the subsequent proving stage, the student links up with his process in a coherent way, organizing some of the justifications (“arguments”) produced during the construction of the statement according to a logical chain” (Boero, Garuti, & Mariotti, 1996, p.113).

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THE ALGEBRAIC MANIPULATOR OF ALNUSET: A TOOL TO PROVE

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This report is devoted to analyzing the influence of an algebraic system, the Algebraic Manipulator of ALNUSET on students' construction of proof in proving equivalence among expressions. Results of an experiment, carried out with students at the second year of Upper Secondary school, are presented to show in which way this manipulator can be used in the educational practice to enhance the teaching and learning of algebraic proof.

INTRODUCTION

As underlined in the introduction to the special issue of ZDM on didactical and epistemological perspectives on mathematical proof (Mariotti and Balacheff, 2008), research work about mathematical proof has been growing in the last decade. Different perspectives (historical and epistemological issues, cognitive ones, didactical transposition of mathematical proof into the classroom) are taken into account framing the proof from different points of view. The actual invitation addressed to educational researchers is to find complementarities in this variety of approaches to make them converge (Balacheff, 2008). This required effort has double goal. On one hand, it could mean an acknowledged awareness of what connects and what separates our works, and on the other hand, it could strongly contribute to teaching and learning of proof in everyday classes. Finally, this effort could make possible the connection between educational research and the school context making our research work effective and fruitful.

Due to my concern for this aspect, I have been studying to find “effective supports” to the didactical transposition of mathematical proof into the classroom. Starting from evidence highlighted by existing research works about students' difficulties in approaching proof, I show a possible way to use technological artefact in the classroom to support the teaching and learning of proof effectively. In this report I present a part of this work in progress. In particular, some interesting results of an experiment carried out with students at the second year of Upper Secondary School are reported.

STUDENTS DIFFICULTIES IN LEARNING PROOF

Students' difficulties in learning mathematical proof have been pointed out by many different research works. In this report I am particularly interested in two of them: students do not see the usefulness of a mathematical proof and they do not understand its language and symbolism.

A new balance between the need to produce logical argument and the need to provide an argument that explains, communicates and convinces seems to be necessary (Healy and Hoyles, 2000). Various authors point out the importance of the explicative and justificative roles of proof (Hanna, 1989, 2000, Harel and Sowder, 1998) that often are not grasped by students. The importance of proof should go beyond the establishment of mathematical truth. A broader vision of proof is expected: proof should provide students with important mathematical strategies and methods for solving problems. (Hanna and Barbeau, 2008).

This new approach to proof could effectively support students in seeing the usefulness of a mathematical proof but other difficulties could come out and they have to be considered. For example, the deductive nature of proof and its symbolism should be explained and justified too. Research results highlighted the great difference between argumentation and proof both from a semantic point of view (Duval, 1995) and from a structural one (Pedemonte, 2007); it is important to distinguish between truth and validity from a logical point of view (Durand-Guerrier, 2008). Logical structure, language and symbolism are important aspects in the construction of proof but they remain often hidden for students. Proof can appear to students as a sub-minimal code with no vital information for understanding (Alibert and Thomas, 1991).

Furthermore, some studies highlight the role of the proof as theoretical organization. These studies focus on the importance of introducing students to the axiomatic structure of proof and to a theoretical perspective (Mariotti & al., 1997). Their aim is to help students access the meaning of theorem and support them in the transition from the need of justifying to the need of validating within a mathematical system (Mariotti & al., 1997).

In general, all these studies show that the role of proof in the educational practice is not well defined and very often difficulties emerge because some aspects of proof are not explicit for students and they are not well explained by teachers.

In teaching proof, certain often implicit aspects need to become part of explicit educational goals (Hemmi, 2008). Through the notion of “transparence”, Hemmi contributes to solve the dilemma to make more or less visible to students some important aspects concerning proof. The concept of transparency (Lave and Wenger, 1991) combines two characteristics: visibility and invisibility. Visibility concerns the ways that focus on the significance of proof (construction of the proof, logical structure of proof, its function, etc.). Invisibility is the form of “unproblematic interpretation” and integration to the activity (Hemmi, 2008, p. 414). It concerns the proof as a justification of the solution of a problem without considering it as a proof. It has been underlined that “*Proof as an artifact needs to be both seen (to be visible) and used and seen through (to be invisible) in order to provide access to mathematical learning*” (Hemmi, 2008, p. 425). The lack of transparency in the

teaching of proof regards the lack of knowledge about proof techniques, key ideas and proof strategies.

These considerations offer important insights to make the transposition of mathematical proof into the classroom effective.

In this context I intend to contribute through the Algebraic Manipulator of Alnuset. This system can be used in teaching and learning algebraic proofs to make rules and axioms used visible in proof processes and to make theoretical aspects usually implicit in algebraic manipulation emerge. The aim of this report is to show in which way the Algebraic Manipulator can be used in the educational practice to enhance the teaching and learning of algebraic proof.

ALNUSET

Alnuset is a system developed in the context of ReMath (IST - 4 - 26751) EC project for students of lower and upper secondary school (yrs 12-13 to 16/17). It is constituted by three integrated components: the Algebraic Line component, the Algebraic Manipulator component, and the Functions component. Even if the educational relevance of this system emerges better through the integrated use of these three components, in this paper I only consider the Algebraic manipulator component to show how it can be used to modify the approach to the algebraic proof.

To have a more complete idea about this system you can see the report presented in group 7 by Chiappini G., and Pedemonte B. of this edition of CERME.

The Algebraic Manipulator of Alnuset: a tool to prove

The Algebraic Manipulator component (AM) of Alnuset is a structured symbolic calculation environment for the manipulation of algebraic expressions and for the solution of equations and inequations.

Its operative features are based on pattern matching and rewriting rules techniques. In the AM these techniques are used in a different perspective with respect to the CAS where the basic rules (commutativity, associativity, etc.) are used internally in a sequence generally not controlled by the user, to produce a higher level result, like “factorize” or “combine”. As a consequence, the techniques of transformation involved in CAS can be obscure for a non expert user.

In the AM, pattern matching is based on a structured set of basic rules that correspond to the basic properties of operations, to the equality and inequality properties between algebraic expressions, to basic operations among propositions and sets. These rules are explicit for students. They appear as commands on the interface made active only if they can be applied to the part of expression currently selected.

An expression is transformed into another through this set of rules. Students can see the transformation of an expression as result of the application of a rule to it.

User Rules		Show	Import...	Export...	Clear
Addition					
$A+B \Leftrightarrow B+A$					
$A+(B+C) \Leftrightarrow (A+B)+C$					
$A \Leftrightarrow A+0$					
$A+-A \Leftrightarrow 0$					
$A-B \Leftrightarrow A+-B$					
$a_1+a_2+\dots \Rightarrow x$					
$n = a+b$					
Powers					
$A^n \Leftrightarrow A \cdot A \dots$					
$A^{n_1+n_2+\dots} \Leftrightarrow A^{n_1} \cdot A^{n_2} \dots$					
$(A_1 \cdot A_2 \dots)^n \Leftrightarrow A_1^n \cdot A_2^n \dots$					
$(A^m)^n \Leftrightarrow A^{m \cdot n}$					
$A^{-n} \Leftrightarrow \frac{1}{A^n}$					
$A^{\frac{1}{n}} \Leftrightarrow \sqrt[n]{A}$					
Computation					
$A = (A)$					
Remove extra ()					
Simplify numerical expression					
Expand					
Collect					
Eliminate variable					
Logic and Set					
Simplify boolean expression					
Simplify set					
$L \Leftrightarrow x \in S$					
$x \in S_1 \vee x \in S_2 \vee \dots \Leftrightarrow x \in S_1 \cup S_2 \cup \dots$					
$x \in S_1 \wedge x \in S_2 \wedge \dots \Leftrightarrow x \in S_1 \cap S_2 \cap \dots$					
$x \in S_2$					
\dots					
Multiplication					
$A \cdot B \Leftrightarrow B \cdot A$					
$A \cdot (B \cdot C) \Leftrightarrow (A \cdot B) \cdot C$					
$A \Leftrightarrow A \cdot 1$					
$A \cdot 0 \Leftrightarrow 0$					
$-A \Leftrightarrow -1 \cdot A$					
$1 \Leftrightarrow -1 \cdot -1$					
$A \cdot \frac{1}{A} \Leftrightarrow 1$					
$\frac{A}{B} \Leftrightarrow A \cdot \frac{1}{B}$					
$\frac{1}{A_1 \cdot A_2 \dots} \Leftrightarrow \frac{1}{A_1} \cdot \frac{1}{A_2} \dots$					
$a_1 \cdot a_2 \dots \Rightarrow x$					
$n = p_1 \cdot p_2 \dots$					
Distribute / Factor					
$A \cdot (B_1+B_2+\dots) \Leftrightarrow A \cdot B_1+A \cdot B_2+\dots$					
Solving					
$A \leq B \Leftrightarrow B \leq A$					
$A \leq B \Leftrightarrow A-B \leq 0$					
$A \leq B+T \Leftrightarrow A-T \leq B$					
$A+T \leq B \Leftrightarrow A \leq B-T$					
$T \cdot A \leq B \Leftrightarrow A \leq \frac{B}{T}$					
$A^p \leq B \Leftrightarrow A^q \leq B^q$					
$A^x \leq B \Leftrightarrow A \leq \sqrt[x]{B}$					
$T \cdot A \leq 0 \Leftrightarrow T \leq 0 \vee A \leq 0$					
$\frac{A}{B} \leq 0 \Leftrightarrow \begin{cases} A \leq 0 \\ B \leq 0 \end{cases}$					
Insert from Algebraic Line					
Factor roots					
Insert solution set					
Instantiate variable					

$(x-1) \cdot (x+1)$

$(x-1) \cdot x+(x-1) \cdot 1$

$x \cdot x-1 \cdot x+(x-1) \cdot 1$

$x \cdot x-1 \cdot x+x \cdot 1-1 \cdot 1$

$x \cdot x-1 \cdot x+x-1 \cdot 1$

$x \cdot x-1 \cdot x+x-1$

A sequence of rules (chosen from the left panel) are applied to the initial expression $(x-1)(x+1)$. At each step, the rule is applied to the green sub-expression, producing the expression on the next line. The last line shows the current selection ($x \cdot x$ in yellow), and one of the 7 rules highlighted in yellow can be applied to this sub-expression.

Moreover, the system allows the student to create new transformational rules (user rules) once these new rules have been previously derived. This feature also present in the L'Algebrista (Cerulli, Mariotti, 2003) is important because it can be used to construct an idea of structured theory.

In the following I show how the AM can be used to provide a good “transparency” (Hemmi, 2008) for the concept of proof. This system can be used to introduce proof in Algebra making visible the rules and procedures of manipulation supporting the comprehension of proof as part of a theoretical system. Moreover, the AM could be used to propose problems involving proof without a direct focus on it. For space reasons, in this report only the role of Alnuset as tool allowing the “visibility” of some important concepts about algebraic proof is analysed.

TEACHING EXPERIMENT

In this section, students’ resolution processes of some tasks involving the construction of proof in the AM of Alnuset are analysed. They are taken from a set of data collected from an experiment carried out in a class of 24 students of the second year of Upper Secondary School (15-16 years old) in the context of ReMath EC project.

The main aim of this experiment was to analyse the role of Alnuset in a teaching experiment centred on algebraic expressions and propositions. The experiment lasted ten weeks, with a 2-hour section each week. The first part of the teaching experiment

focused on algebraic expressions (equivalent expressions, opposite expressions, reciprocal expressions). In this part, a specific section was devoted to the manipulation of expressions. In this report I present results of this section.

During the previous weeks students had used the AM of Alnuset only twice.

Students worked in pairs with the AM of Alnuset under the supervision of the teacher and the researcher.

In the following, tasks proposed to students during the section are presented.

Tasks

- a) Use AM to prove that $(2+3)*5-25$ is equal to 0
Use AM to prove the same equality starting by 0.
Is this the only equivalence that it is possible to prove starting by 0?
- b) Use AM to prove that $(2/5+4/5)*5/6$ is equal to 1
Use AM to prove the same equality starting by 1.
Is this the only equivalence that it is possible to prove starting by 1?
- c) In solving tasks a) and b) you have used two specific commands, both in direct and indirect ways: the command to add two opposite expressions ($A+-A\leftrightarrow 0$) and the command to multiply two reciprocal expressions ($A*1/A\leftrightarrow 1$). Have you observed any difference in the direct and indirect use of these commands? If yes, what differences? In your opinion, is it more difficult to accomplish proofs based on the direct use or proofs based on the indirect use of these commands? Why?
- d) Try to prove that the expression $a/b+c/d$ is equivalent to the expression $(a*d + b*c)/bd$. If this proof is difficult for you, try to prove the equivalence between the two expressions starting from $(a*d + b*c)/bd$ and then to come back step by step in order to work out the more complex proof.
Use the accomplished proof to create a new manipulation rule.
- e) Try to prove that the expression a^2-b^2 is equivalent to the expression $(a+b)(a-b)$. If this proof is difficult for you, try to prove the equivalence between the two expressions starting from $(a+b)(a-b)$ and then to go backward, step by step, in order to work out the more complex proof. Use the accomplished proof to create a new manipulation rule.
- f) Use AM to transform the following expressions using, if necessary, the rules created in the previous tasks:

$$x^2-4; \quad x^2-1; \quad \frac{x+2}{x+1} + \frac{x+1}{x-2}$$

Tasks a) and b) introduce the two rules $A+-A\leftrightarrow 0$ and $A*1/A\leftrightarrow 1$ instantiated on specific examples. Task c) supports reflections about the direct and indirect use of these rules. Tasks d) and e) require to prove the rules $a/b+c/d = (a*d + b*c)/bd$ and $a^2-b^2 = (a+b)(a-b)$ using the two rules $A+-A\leftrightarrow 0$ and $A*1/A\leftrightarrow 1$. Task f) is useful to strengthen the use of the new proved rules.

Tasks a), b) and c)

The solution of task a) in the manipulator is reported in the following table. In the first part there is the manipulation from the numerical expression to 0 and in the second part there is the manipulation from 0 to the expression.

$(2+3) \cdot 5 - 25$	0	The second proof (right) is more difficult for students with respect to the first one (left). In the second proof, the equivalence needs a step that obliges the user to write 0 as addition of two opposite numbers (25-25). This is not obvious for students who in general are not able to manage it.
$(5) \cdot 5 - 25$	$25 + -25$	
$25 - 25$	$25 - 25$	
$25 + -25$	$5 \cdot 5 - 25$	
0	$(2+3) \cdot 5 - 25$	

The application of the rule $0 \Rightarrow A + -A$ requires to understand that 0 can be expressed as sum of two opposite expressions. The problem is that there are infinite possibilities that can be considered to replace 0.

In the same way, to apply the rule $1 \Rightarrow A \cdot 1/A$ students have to replace 1 with two reciprocal expressions.

$(\frac{2}{5} + \frac{4}{5}) \cdot \frac{5}{6}$	1	The second proof (right) is more difficult for students with respect to the first one (left). In the second proof, the equivalence needs 2 steps that oblige the user to write 1 as multiplication of two reciprocal numbers ($5 \cdot 1/5$ and $6 \cdot 1/6$). As in the previous case, this is not obvious for students who are not able to replace 1.
$(2 \cdot \frac{1}{5} + \frac{4}{5}) \cdot \frac{5}{6}$	$6 \cdot \frac{1}{6}$	
$(2 \cdot \frac{1}{5} + 4 \cdot \frac{1}{5}) \cdot \frac{5}{6}$	$6 \cdot 1 \cdot \frac{1}{6}$	
$((2+4) \cdot \frac{1}{5}) \cdot \frac{5}{6}$	$6 \cdot 5 \cdot \frac{1}{5} \cdot \frac{1}{6}$	
$((6) \cdot \frac{1}{5}) \cdot \frac{5}{6}$	$6 \cdot \frac{1}{5} \cdot 5 \cdot \frac{1}{6}$	
$6 \cdot \frac{1}{5} \cdot \frac{5}{6}$	$6 \cdot \frac{1}{5} \cdot \frac{5}{6}$	
$6 \cdot \frac{1}{5} \cdot 5 \cdot \frac{1}{6}$	$(2+4) \cdot \frac{1}{5} \cdot \frac{5}{6}$	
$6 \cdot 1 \cdot \frac{1}{6}$	$(2 \cdot \frac{1}{5} + 4 \cdot \frac{1}{5}) \cdot \frac{5}{6}$	
$6 \cdot \frac{1}{6}$	$(\frac{2}{5} + 4 \cdot \frac{1}{5}) \cdot \frac{5}{6}$	
1	$(\frac{2}{5} + \frac{4}{5}) \cdot \frac{5}{6}$	

As shown by the results of the experiment, in general these rules are used by students in their manipulations in paper and pen environment, in a completely implicit way. Most students are able to transform an expression into another one using these rules but they are not able to explicit them. In better cases they are able to use these rules as computational techniques but they are rarely able to justify them.

Analysis of results of tasks a), b) and c)

The results analysis of the experiment shows that most students constructed the direct proof in tasks a) and b) even if for task b) the intervention of the teacher was often necessary. Students knew the result of the sum $2/5 + 4/5$ but they were not able to make it in the AM because they didn't manage the properties and rules hidden in the

technique of addition of two fractions.

The construction of the inverse proofs (from 0 to the expression $(2+3)*5-25$ and from 1 to $(2/5+4/5)*5/6$) was not easy for them. As expected, difficulties emerged when students had to replace 0 as sum of two expressions and 1 as multiplication of two expressions.

Only observing the previously constructed direct proof some students (6 groups out of 12) were able to construct also the inverse proof, following step by step the direct proof and going backwards to the initial expression. Here is the dialog between two students while constructing the proof from 0 to the expression $(2+3)*5-25$.

I: But in which way can we prove this equivalence starting from 0?

F: perhaps...

I: wait a moment... if $a+-a$ is 0 it is also true that 0 is $a+-a$

F: yes, of course

I: then if $25-25$ is 0 it is also true that 0 is equal to $25-25$... then we can write in this way

F: following step by step the previous proof

The AM allowed students to make explicit rules $A+-A \Leftrightarrow 0$, $A*1/A \Leftrightarrow 1$ and to understand the intrinsic difference that characterises the two directions of the rules. Let's see the following example (answers reported in the copy of a group of student):

“a) Starting with 0 it is possible to prove whatever equivalence having 0 as result. So there are infinite equivalent expressions to 0. b) Starting with 1 it is possible to prove that 1 can be replaced by all reciprocal expressions having 1 as results. c) In our opinion it is easier to produce proofs based on the direct use of the command $A+-A \Leftrightarrow 0$, because in the inverse case it is necessary to look for the opposite expression, while the direct use of the command only requires the application of the correct axiom. For the rule $A*1/A \Leftrightarrow 1$ the principle is the same, but in this case consider reciprocal expressions and not opposite expressions”.

Answers given by these students to task c) show that they have developed awareness about the role of the two rules and the way they can be used in manipulation.

Tasks a), b), and c) allowed students to reflect deeply on these rules that are usually used in the algebraic manipulation in a completely “invisible” way. The AM of Alnuset allowed students to “make visible” these rules and their use in the construction of the proofs.

Tasks d), e) and f)

Task d) and task e) are very useful in approaching proof and in particular they are effective to understand the idea of theoretical systems. As a matter of fact, only when the rules $a^2-b^2 = (a+b)(a-b)$ and $a/b+c/d = (a*d + b*c)/bd$ are proved they can become new user rules and they can be used to prove expressions as those proposed in task f).

A possible solution of the task e) in the AM is reported in the following table.

$a^2 - b^2$	$(a+b) \cdot (a-b)$	<p>It is better to begin from the second proof (right) because in the first proof (left) it is necessary to insert 0 and replace it with the sum of the two opposite expressions ab and $-ab$.</p> <p>Once the proof is accomplished students can solve it as a new rule: the following one.</p> <div style="border: 1px solid black; padding: 5px; margin: 10px 0;"> <p style="text-align: center; background-color: #e06699; color: white; margin: 0;">User Rules</p> $a^2 - b^2 \Leftrightarrow (a+b) \cdot (a-b)$ </div> <p>This user rule can be used in the successive manipulations.</p>
$a^2 + 0 - b^2$	$(a+b) \cdot a - (a+b) \cdot b$	
$a^2 + a \cdot b + -(a \cdot b) - b^2$	$a \cdot a + b \cdot a - (a+b) \cdot b$	
$a \cdot a + a \cdot b + -(a \cdot b) - b^2$	$a \cdot a + b \cdot a - (a \cdot b + b \cdot b)$	
$a \cdot a + a \cdot b + -(a \cdot b) - b \cdot b$	$a \cdot a + b \cdot a + -(a \cdot b + b \cdot b)$	
$a \cdot (a+b) + -(a \cdot b) - b \cdot b$	$a \cdot a + b \cdot a + -1 \cdot a \cdot b + -1 \cdot b \cdot b$	
$a \cdot (a+b) + -1 \cdot (a \cdot b) - b \cdot b$	$a \cdot a + b \cdot a + -(a \cdot b) + -1 \cdot b \cdot b$	
$a \cdot (a+b) + -1 \cdot a \cdot b - b \cdot b$	$a \cdot a + a \cdot b + -(a \cdot b) + -1 \cdot b \cdot b$	
$a \cdot (a+b) + -1 \cdot a \cdot b + -(b \cdot b)$	$a \cdot a + 0 + -1 \cdot b \cdot b$	
$a \cdot (a+b) + -1 \cdot a \cdot b + -1 \cdot (b \cdot b)$	$a \cdot a + -1 \cdot b \cdot b$	
$a \cdot (a+b) + -1 \cdot a \cdot b + -1 \cdot b \cdot b$	$a^2 + -1 \cdot b \cdot b$	
$a \cdot (a+b) + -1 \cdot b \cdot a + -1 \cdot b \cdot b$	$a^2 + -1 \cdot b^2$	
$a \cdot (a+b) + -1 \cdot (b \cdot a + b \cdot b)$	$a^2 - 1 \cdot b^2$	
$a \cdot (a+b) + -1 \cdot (b \cdot (a+b))$	$a^2 - b^2$	
$a \cdot (a+b) + -1 \cdot b \cdot (a+b)$		
$(a + -1 \cdot b) \cdot (a+b)$		
$(a + -b) \cdot (a+b)$		
$(a-b) \cdot (a+b)$		
$(a+b) \cdot (a-b)$		

A lot of steps are necessary to prove the equivalence $a^2 - b^2 = (a+b)(a-b)$ in the AM of Alnuset, because manipulation requests students to make rules and axioms that are necessary to prove the equivalence explicit.

In the same way it is possible to produce the proof of the equivalence $a/b + c/d = (a \cdot d + b \cdot c)/bd$.

Analysis of results of tasks d), e) and f)

Tasks d) and e) required a lot of efforts by students. Nevertheless, these tasks were very fruitful to understand the meaning of proving a rule starting by a basic set of rules and axioms. Students who tried to prove the two equivalences $a/b + c/d = (a \cdot d + b \cdot c)/bd$ and $a^2 - b^2 = (a+b)(a-b)$ inserting the first expression ($a/b + c/d$ or $a^2 - b^2$) were not able to begin the manipulation. All students were forced to follow the suggestion given by the text of the tasks inserting the second expression and manipulating it. Also in this case the solution was not obvious. Some difficulties concerned denotative aspects: deletion of superfluous parentheses, application of properties in order to make the expression match with the rule to be applied, and so on. Nevertheless, in some cases, difficulties concerned “conceptual aspects” usually invisible in the ordinary manipulation in the paper and pen environment. For example, students were not confident with rules such as $a-b = a + -b$ and $-a = -1 \cdot a$. Thus steps concerning the

application of these rules were often introduced by the teacher. Let's see the dialogue of two students during the resolution of task d).

S: This is a specific product... Insert in Alnuset the expression a^2-b^2

L inserts the expression in AM

S: and then?

L: I really have no idea....

S tries to apply some rules without success.

L: Perhaps... it is better to start from the other side. Try to insert $(a-b)(a+b)$

S inserts the expression in AM and then she applies the distributive law.

She is not able to sum $-ab + ab$ because she was not able to transform the expression $aa+ba-(ab+bb)$ into the expression $aa+ba-ba-bb$.

L: What? We are not able to add these two expressions. We know that the solution is 0 but...

S: in which way can we find this result?

Teacher: You have to apply the rules $a-b=a+-b$ to transform $-(ab+bb)$ into $-1(ab+bb)$...

S: Ah ok! We try...

Students complete the proof and they try to perform the inverse proof.

Even if it was really hard for students to solve the tasks, the constructed proofs obliged them to make explicit axioms and rules that are used step by step during the transformation of an expression into another.

In general, students were very proud of their proofs and they liked a lot to save the proved rules as new rules that could be used in their successive proofs. Task f) was solved by most students without any difficulty. In this task they eventually realised that the previously proved rules were useful to prove other new rules.

CONCLUSIONS

The results of the experiment might show that the AM of Alnuset does not help students construct proofs and makes proofs more complicate for them. In a sense this is true - a lot of students are able to transform $(a+b)(a-b)$ into a^2-b^2 in paper and pen environment and perhaps it is not so important to be able to make the inverse transformation. The problem is that in school practice, algebra is usually considered as a body of rules and procedures for manipulating symbols. Students are usually able to develop calculus but they are not aware of the axioms and theorems they are using in performing it. Thus, algebra is taught and learned as a language and emphasis is put on its syntactical aspects. In this context, algebraic proof appears as a grammar structure made of a sequence of formulae connected by calculus rules. In this way, the meaning of proof is completely lost. Despite this, rigorous proof is generally considered as a sequence of formulae within a given system, each formula being either an axiom or derivable from an earlier formula by a rule of the system. The AM of Alnuset supports this kind of proof though in a different way. Each step in the manipulation is produced by the application of a rule that has to be chosen by the student from a set of rules. If the choice is not correct it could be very difficult for the student to construct the proof. During the experiment the intervention of the teacher often supported students that were unable to accomplish the task. Notwithstanding this, at the end of the experiment, students were able to explicit rules used during

their proofs spontaneously. Also during ordinary school practice, students justified their steps making the rule used in the transformation explicit. This kind of approach required a lot of effort but it supports the awareness of what it is an algebraic proof and in which way a mathematical theory can be constructed.

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VISUAL PROOFS: AN EXPERIMENT

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The main goal of this paper is to start a preliminary study of the basic features of visual proofs in mathematics and their use in mathematics teaching. The investigation, based on college mathematics students, shows a very poor use of visual reasoning in mathematical tasks involving figures. Moreover, students' use of visual semiotic systems is not spontaneous but seems to need some special training. Some of the ways of working students usually adopt when dealing with visual proofs have been identified, showing that most often diagrams are not seen as representations of complete processes, but rather as ready-made aids to solve problems.

INTRODUCTION

Many researchers have stressed the importance of visual reasoning in the learning of mathematics and have remarked that research in mathematics education has still a lot to develop about this topic (see e.g. Dreyfus 1991, Jones 1998, Presmeg 2006). In this perspective this paper focuses on visual proofs i.e. on proofs where the deductive steps are based on figures, diagrams or graphs. This means that the inferences are possible through just the reading of the figures. Although geometrical figures will be taken into account only, the expression ‘diagrammatic proof’ or ‘visual proof’ will be used in a more inclusive sense. At this regard, a number of works, such as Nelsen’s books (1993, 2001) have provided a wide selection of examples of visual proofs from different sources. In literature visual proofs are usually presented with no comments in verbal language (i.e. without words), but only based on diagrams, possibly equipped with numbers, letters, arrows, dots, or other signs and sometimes associated with symbolic expressions; the reconstruction of the proof is left to the reader. Nowadays visual arguments are far to be considered legitimate arguments for rigorous proofs probably due to the fact that they can easily misread and therefore lead to wrong inferences. Anyway their importance as an aid for the discovery of new results and the production of more formal proofs is widely recognized. In the last decades interest in visual proofs has grown up leading to both new mathematical investigations and applications to mathematics education. On the side of mathematical investigations above all we mention the work of Barwise and Etchemendy (1991) and further developments in the same line such as Jamnik’s study (2001). From the educational viewpoint the role of visual reasoning in mathematics teaching has been taken again into account and emphasized (see e.g. Dreyfus 1991, Dvora & Dreyfus 2004, Hanna 1989, Presmeg 1997, 2006).

The main goal of this paper is to identify the main difficulties in the use of diagrams in mathematics, in particular in the extraction of information. For this purpose some visual proofs have been taken into account. In the experiment I am describing some statements with the corresponding diagrammatic proofs have been given to

mathematics sophomore and third year students. Such proofs have been presented without any explanation on the inference steps implicit in the figures. The work is also aimed to compare the processes involved in visual proofs to those involved in the standard ones. Diagrams are not relevant only in relation to visual proofs, but they can also support either standard proof processes (i.e. proofs based on a verbal or symbolic text) or problem solving. Indeed the heuristic role of figures is widely recognized both by mathematicians and by mathematics educators. Therefore some features of diagrammatic proofs will be taken into account, which might be relevant from the educational viewpoint and to explore the opportunities that they can provide in order to improve the approach to mathematical theorems.

THEORETICAL FRAMEWORK

The production or the understanding of a diagrammatic proof involves constructing and treating (detaching, reversing, superposing, translating,...) figures and extracting information from them. All these operations will make evident the inferential steps that make up a visual proof of a statement. Moreover a diagrammatic proof is developed for a particular value of the domain of validity of the theorem but anyway it represents the proof for all values of the domain (character of generality, Barwise & Etchemendy 1991).

We did not find in literature a theoretical framework closely focused on visual proofs in mathematics education. Although here we are focusing on visual proofs that are based on geometrical figures, we take into account some different works about visual reasoning and visualization that could help us to interpret difficulties about this topic.

First of all, according to Fischbein (1993), geometrical figures are mental entities (named also ‘figural concepts’) which possess conceptual and figural characters at the same time. In this frame, as other studies in geometry, we refer to figures as the mental entities which possess properties imposed by, or derived from axiomatic systems and to drawings as their (external) representations. A major problem in the use of diagrams and figures is the potential conflict between conceptual and perceptual features of figures. Fischbein’s theory is very helpful at this regard. Fischbein argues that ‘...figural concepts constitute only the ideal limit of a process of fusion and integration between the logical and figural facets’ (Fischbein 1993, p.150). In particular visual proofs involve some logical questions concerning the nature of deductions based on diagrams and figures. Actually, it is to be considered that visual proofs are bound to correspond to some extent to proofs in the standard mathematical sense. In this work I do not mean to question the rigorousness of diagrammatic proofs (on this topic see Barwise & Etchemendy (1991), Jamnik (2001), Hanna & Sidoli (2007), Allwein, G. & Barwise J. – Eds. (1996) and references therein) but I assume that they can be regarded as legitimate mathematical processes.

Another main difficulty encountered by students is due to the lack of coordination of systems of semiotic representations (Duval 1993). Working with a visual proof requires a continuous interplay between the semiotic system of figures and the semiotic systems involved in the statement, usually verbal texts or symbolic expressions. Like Duval, I assume that semiotic systems are not neutral carriers of meanings but can contribute to the construction of meaning themselves. This explains the attention I am going to pay to semiotic systems through this paper.

AN EXPERIMENT

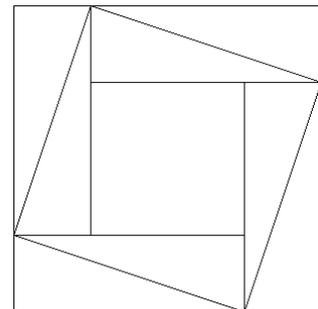
At the Università del Piemonte Orientale, in Italy, in the context of a course devoted to mathematical proof, we have given a group of 13 sophomore and third year undergraduate Mathematics students a number of tasks requiring to look at diagrammatic proof of some statement and to reconstruct such a proof (i.e. to describe how the proof could be extracted by the figure). The tasks have been administered as written tests and they were followed by interviews in order to better understand the arguments written by students.

The problems are the following:

Task 1.

The picture on the right represents a visual proof of the Pythagoras' theorem.

- Describe such a proof.
- Reconstruct the figure in the case that the legs of the right-angled triangle have the same measure.

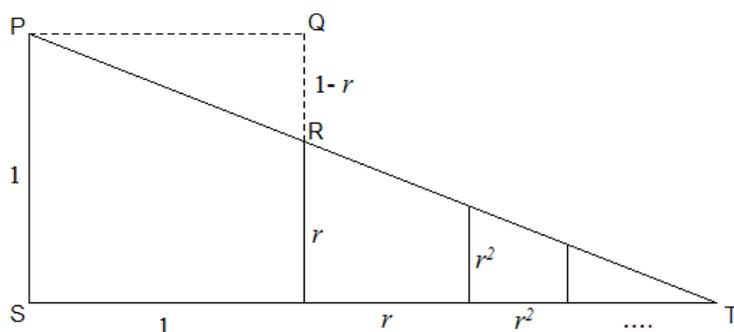


Task 2.

The picture on the right represents a visual proof of the theorem

$$\sum_{i=0}^{+\infty} r^i = \frac{1}{1-r} \quad \text{for } 0 < r < 1.$$

Describe such a proof.



The statements are in two different fields of mathematics: the Pythagoras' theorem and the geometric series. Pythagoras' theorem is customarily associated to visual representations, whereas the latter is less common (at least in Italy), as the convergence of the geometrical series is usually proven using a combination of algebraic and analytical arguments. So this visual proof is very unusual for Italian mathematics college students. The choice of theorems from different fields is aimed at finding common features and common difficulties related just to visual reasoning. As the results show, students find this kind of problems very difficult. The main difficulty is due to the fact that the drawing is a static object while a proof is made of an ordered sequence of inference steps. A drawing presents in a whole all written data and the reader has to choose the order of the construction and how to extract the information.

Analysis of task 1. Here the construction of the drawing may not present so many problems since it is not required a precise order of construction as far as one recognizes that there is a particular disposition of six right angle triangles. Troubles can arise when trying to find correspondences between the statement and the picture. This task is mainly based on visual arguments. Students could meet with difficulties in the identification of the area of the square built on the hypotenuse (Fig.1) and above all of the areas of squares whose sides are the legs of the right triangle (Fig.2) since they are not bounded with segments. Such a problem is related to the rearrangement of the figure.

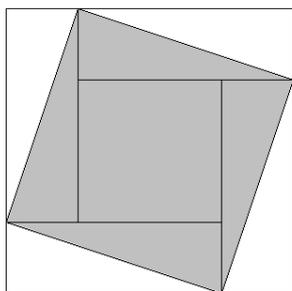


Fig.1

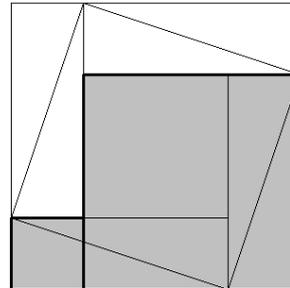


Fig.2

Therefore students could meet with difficulties from the perceptual side, as they might fail to spot the appropriate triangles or squares. In fact, as pointed out by Duval (1993) graphical sign can be either a help or a hindrance in understanding diagrams.

Analysis of task 2. In this case the reconstruction of the drawing itself is a difficulty. It requires the conceptualization that such a construction is made of infinitely many steps and that it proves that the series is convergent. All this requires a good conceptualization of the real numbers and their representation on the line. Moreover students could meet difficulties, not only with perceptual aspects, but above all with the lack of coordination of three different semiotic systems. One has to recognize that

$\sum_{i=0}^{+\infty} r^i = \frac{1}{1-r}$ is a proportion, then to translate it in the graphical system and finally to identify it in the given figure.

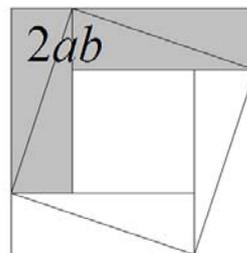
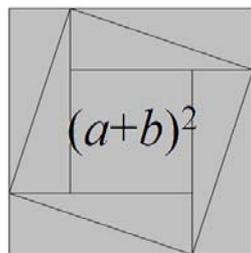
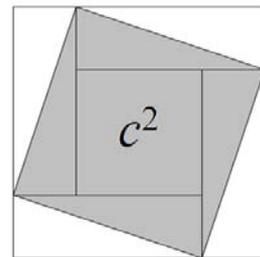
RESULTS

Task 1.

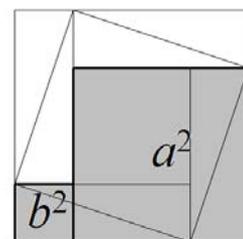
First of all in this task few students only provided an explicit description of the construction process of the figure. In this visual proof, the construction of the drawing is not related to the understanding of the proof since they succeed to achieve it even if with deductive arguments not based upon the whole picture but on some parts of it only. In particular, notice that some students do not feel the necessity to prove themselves that the tilted figure that looks like a square is indeed a square. In this case the perceptual facet is not controlled by the conceptual one. Second, all of them introduced letters a , b , c to indicate the measure of the sides of the triangle in order to find correspondence between the formula $a^2+b^2=c^2$ and the figure. Finally, students addressed the first task in three different but not necessarily separate ways:

1. *Modifying the formula in order to find correspondence with the figure*

Some students tried to connect the formula $a^2+b^2=c^2$ to the figure and to identify just c^2 in the picture to the right. They were not able to do the same for a^2 and b^2 . Then they wrote down $(a+b)^2-2ab=c^2$ most likely because they could find $(a+b)^2$ and $2ab$ in the picture too, as shown below:



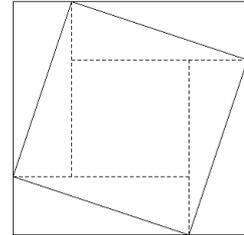
This way the students recognized the remaining area a^2+b^2 in the figure on the right. This kind of proof is mostly based on visual arguments except for the initial modification of the formula.



2. Area computation

This strategy is the most common in problems of this kind. It consists in calculating the area of the external figure in two different ways and then comparing the results to obtain the required relationship.

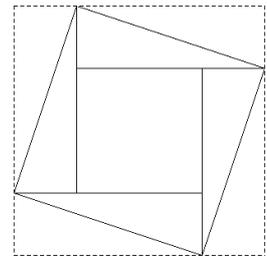
In the first problem they calculated the area of the square of side $a+b$ i.e. $(a+b)^2$ and then the same area as the sums of the five subfigures (four triangles and a square of side c) i.e. $4\frac{ab}{2}+c^2$. Comparing the two expressions they got the Pythagorean theorem through algebra. In this case they did not consider the dashed lines in the picture on the right.



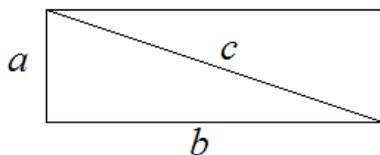
3. Figures as plain tools

The figure is not seen as a process embodying the proof of a statement but just as a tool that can be used to occasionally pick some piece of information useful to get a proof.

For example in this problem four students considered just the tilted square of side c and its five subfigures (four right-angled triangles and the square of side $a-b$). Actually they did not consider the dashed lines in the figure on the right. Comparing the area of the square of side c calculated as c^2 with the same area but regarded as the sum of the areas of the five subfigures one obtains the result as in point 2 (*Area computation*).



Another student just considered the rectangle



defining a the short side and b the long one. Then she used a so called “circular argument” or “begging the premise” (cf. Weston, 2000), i.e. she used the Pythagoras’ theorem to get $c = \sqrt{a^2 + b^2}$ and hence squaring both sides she got the Pythagoras’ theorem $c^2 = a^2 + b^2$.

Notice that also the answers in point 2. (*Area computation*) denote that the figure is not seen as an autonomous process of proof.

Task 2.

The Problem 2 proved the most difficult one. Nobody succeeded in understanding this visual proof. So a hint was given to them while they were solving the task. It was told them that a fundamental tool for its comprehension was the similitude of triangles and in particular the proportionalities between corresponding sides of the triangles. After that some of them succeeded to recognize that $\triangle PST$ and $\triangle PQR$ are

similar and they found the correspondence between the formula and the sides of triangles.

As a first result we have that students were able to match labels with the formula, and to understand the meaning of the dots ‘...’. As second finding we have that most students did not reconstruct the drawing. The reasons are three:

1. Students understood the need to reconstruct the drawing. Such construction is a necessary step in order to consider the visual proof as a process. Unfortunately they are not able to do such a reconstruction. One can see this outcome from the following excerpts:

A: Consider a square of side of length 1 ($l = r^0$) PQMS and construct a right-angled triangle PST such that the shorter leg is $\overline{PS} = r^0$ and one finds that the longer leg \overline{ST} is the sum of infinite segments having measure respectively r^0, r^1, r^2, \dots

(Student A understood that the measure of \overline{ST} is not an assumption but a finding of the construction but he could not prove that result, as it became clear from the interview) or

B: I can not understand how in the figure r^2 comes out from r .

2. Students considered figures just as plain tools. This is evident in task 2:

C: ...from figure I can see that \overline{PS} measures 1, \overline{ST} measures $\sum_{i=0}^{+\infty} r^i, \dots$

Student C did not see that $\overline{PS} = 1$ is an assumption while $\overline{ST} = \sum_{i=0}^{+\infty} r^i$ is the result of a deductive steps and in particular it means that the series converges.

3. Students understood the need to reconstruct the drawing but they failed to do it since they considered it trivial.

Finally some students could conclude the proof using the help given to them, but we distinguish

- students who were able to prove that the triangles PST and PQR are similar because they recalled this notion;
- students who did not recalled this notion or never learnt it.

In this case the problem is that even if students had a good knowledge of similitude of triangles they failed to introduce such “new” tool which could not be directly extracted by a simple manipulation of the objects already appearing in the proof.

General discussion

One of the main findings of this work is that visual proofs are not seen as processes but the figures are just plain tools which help to find results. The investigation of the protocols highlights that the unsuccessful results of this kind of tasks are due not only

to the semiotic system of figures or to the conflict between the conceptual and figural nature of visual proofs but it comes out that the concept of mathematical proof is not understood enough. This conclusion comes out above all from the fact that students do not feel the need for reconstructing the drawing. Moreover, in the first problem students used just some parts of the figure and not the whole of it, that is some students did not attribute values at every graphical sign, as it is explained in the analysis of the first task. Also this behaviour, in some cases, is due to a misunderstanding of the nature of the process of visual proofs. In fact the role of graphical signs and more in general of the perceptual learning of a figure is very important both in a positive and in a negative sense (Duval 1993). Perception can be a useful tool only if it is controlled by conceptual processes as pointed out by Fischbein.

Second it comes out that one of the main obstacles is the lack of geometrical knowledge: notions like similitude and congruence of triangles, Thales' theorem, etc. are hardly known, which severely prevents any attempt to work with the figure. This situation is found in the problem about the geometrical series. For example one of the fundamental steps for understanding this visual proof is to notice that the triangles $\triangle PQR$ and $\triangle PST$ are similar. No one spotted this geometrical fact. There might be two reasons of it. First, students have never learnt this or they have forgotten it. Second, they could not easily call to mind this notion, actually they knew something on similitude of triangles but they were not used to work with it. This means that students are not aware that there are some theorems, techniques, tools, which they can exploit when facing triangles. The fact is that Italian students work very little or do not work at all on the visualization of geometrical figures (for further details see Mariotti, 1998). Moreover, the time given for solving the task is not sufficient to remember or to reconstruct this notion. However, the necessity to use tools and constructions which are not directly related to objects at hand is a common feature in mathematical proofs, which do not refer to visual proofs only. Students could not overcome the difficulty of introducing such new elements in the visual proof we proposed them. Moreover, students were not even able to exploit the symbolic expression in the statement, since it would have required to represent it as a proportion, that in Italy is given prevalently by $\left(\sum_{i=0}^{+\infty} r^i\right):1=1:(1-r)$, and then into the figural system. The difficulty about the introduction of new elements, however, is not peculiar to visual proofs only. Indeed the first task does not present this problem. In this case all students succeeded in grasping the result even if in an improper way, for example using the figure just as a tool to extract information. Here one has just to manipulate the formula of the Pythagorean Theorem or manipulate its figure; there is no need to introduce new constructions, techniques, assumptions, tools, etc.

Finally the analysis of protocols shows that students prefer to work with algebra instead of using visual arguments coming from manipulation of figures. The visual

proof in task one is only of visual nature but no one addressed it with just visual arguments. Just one student used prevalently visual arguments (see strategy 1), but even in this case there was a preliminary modification of the formula.

CONCLUSION

This explorative research outlines the lack of skills in visual reasoning by a group of Italian mathematics college students. This lack is due to different reasons: poor knowledge of certain basic mathematical tools, poor acquaintance with the use of figural representations, conflict between the conceptual and perceptual nature of diagrammatic proofs and sometimes poor understanding of the concept of mathematical proof itself. Besides, the research points out that it is very difficult to learn proofs without being able to pick and use some basic pieces of mathematical knowledge. In this context tasks like those presented in this work might help students to develop a correct use of deductive method when working with figural representation and not only in the field of geometry but also in other context as in the second task presented. Obviously graphical representations in different mathematical settings can present different features related to different concepts. For example, in the case of the geometrical series one has to take into account the graphical representation of real numbers and of their properties. According to Duval, the coordination of at least two different semiotic systems of representation of a concept can improve its understanding. In particular I think that the passage from verbal and symbolic representations into the figural one and vice versa could be very fruitful. Moreover I think also that problems like the second one can help to overcome the trend to deal with mathematical subjects in isolation. Since, as our result confirms, the use of graphical representation presents a lot of difficulties, its use requires a particular training in order to exploit its potential. In this perspective tasks like those presented in this work could help students to develop some important tools to approach also other mathematical problems such as standard proofs.

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TEACHERS' VIEWS ON THE ROLE OF VISUALISATION AND DIDACTICAL INTENTIONS REGARDING PROOF

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In this paper we explore secondary teachers' views on the role of visualisation in the justification of a claim in the mathematics classroom and how these views could influence instruction. We engaged 91 teachers with tasks that invited them to: reflect on/solve a mathematical problem; examine flawed (fictional) student solutions; and, describe, in writing, feedback to students. Eleven teachers were also interviewed. Here we draw on the interviews and the responses to one Task (which involved recognising a line as a tangent to a curve at an inflection point) of two teachers. We do so in order to explore potential influences on the didactical contract regarding proof that these teachers are likely to offer their students. One such influence is the clarity and stability of their beliefs about the role of visualisation.

Key Words: teacher beliefs, proof, visualisation, tangents, didactical contract

INTRODUCTION

'The emphasis that teachers place on justification and proof no doubt plays an important role in shaping students' 'proof schemes'¹ (Harel & Sowder, 2007, p827). The not very extensive research in this area (p824) shows that this emphasis is insufficient both in terms of extent and in terms of quality. Internationally in most educational settings – even those with an official curricular emphasis on proof – little instructional time is dedicated to proof construction and appreciation (p828). Furthermore teachers' own proof schemes are often predominantly empirical and teachers do not always seem to understand important roles of proof other than verification (p836). For example, in Knuth's (e.g. 2002) study of practising secondary mathematics teachers, while all teachers acknowledged the verification role of proof, they rarely talked about its explanatory role. With regard to their proof schemes many of the interviewed teachers: felt compelled to check a statement on several examples even though they had just completed a formal proof; considered several of given non-proofs as proofs; and, accepted the proof of the converse of a statement as proof of the statement; and, found arguments based on examples or visual representations to be most convincing.

One of the aims of the study we report in this paper is to explore the relationship between teachers' pedagogical and epistemological beliefs about proof and their intended pedagogical practice (e.g. Cooney et al, 1998; Leder et al, 2002). Here we report some findings that relate to their beliefs about the role of visualisation.

¹ Harel & Sowder's (1998) term which describes an individual's and a community's perception of proof. They distinguish between external conviction (authoritarian, ritual, non-referential symbolic), empirical (inductive, perceptual) and deductive (transformational, axiomatic) proof schemes.

In the last twenty years or so the debate about the potential contribution of visual representations to mathematical proof has intensified (e.g. Mancosu et al, 2005), not least because developments in IT have expanded this potential so greatly. Central to this debate is whether, how and to what extent, visual representation can be used not only as evidence and means of insight for a mathematical statement but also as part of its justification (Hanna & Sidoli, 2007). For example, Giaquinto (2007) argues that visual means are much more than a mere aid to understanding and can be resources for discovery and justification, even proof. Whether visual representations need to be treated as adjuncts to proofs, as an integral part of proof or as proofs themselves remains a point of contention.

Visualisation has gained analogous visibility within mathematics education. Its richness, the many different roles it can play in the learning and teaching of mathematics – as well as its limitations – are increasingly being written about (e.g. Arcavi, 2003). These works address a diversity of issues, including: mathematicians' perceptions and use of visualisation; students' seeming reluctance to engage (and difficulty) with visualisation; etc. (Presmeg, 2006). Overall we still seem to be rather far from a consensus on the many roles visualisation can play in mathematical learning and teaching. So, while many works clearly recognise these roles, several (e.g. Arcavi, *ibid.*) also recommend caution with regard to 'the 'panacea' view that mental imagery only benefits the learning process' (Aspinwall et al, 1997, p315).

One of the aims of the study we report in this paper is to contribute to the above debate as outlined in the work of Presmeg, Arcavi and others through exploring secondary mathematics teachers' beliefs about the role of visualisation as evident in the reasoning and feedback they present to students. The specific part of the debate our study aims to contribute to concerns the relationship between these beliefs and teachers' intended pedagogical practice. Our particular interest is in the potential influences on the didactical contract (Brousseau, 1997) that teachers offer their students with regard to the role of visualisation. One such potential influence is the clarity and stability of teachers' belief systems (Leatham, 2006). Below we briefly introduce the study.

THE STUDY AND THE TANGENT TASK

The data we draw on in this paper originate in a study, currently in progress in Greece and in the UK, in which we invite teachers to engage with mathematically/pedagogically specific situations which have the following characteristics: they are hypothetical but likely to occur in practice and grounded on learning and teaching issues that previous research and experience have highlighted as seminal. The structure of the tasks we ask teachers to engage with is as follows – see a more elaborate description of the theoretical origins of this type of task in (Biza et al, 2007): reflecting upon the learning objectives within a mathematical problem (and solving it); interpreting flawed (fictional) student solution(s); and, describing, in writing, feedback to the student(s).

In what follows we focus on one of the tasks (Fig. 1) we have used in the course of the study. The Task was one of the questions in a written examination taken by candidates for a Masters in Mathematics Education programme. Ninety-one candidates (of a total 105) were mathematics graduates with teaching experience ranging from a few to many years. Most had attended in-service training of about 80 hours.

Year 12 students, specialising in mathematics, were given the following exercise:

‘Examine whether the line with equation $y = 2$ is tangent to the graph of function f , where $f(x) = 3x^3 + 2$.’

Two students responded as follows:

Student A

‘I will find the common points between the line and the graph solving the system:

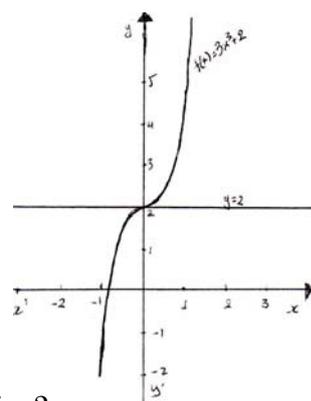
$$\begin{cases} y = 3x^3 + 2 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} 3x^3 + 2 = 2 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} 3x^3 = 0 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 2 \end{cases}$$

The common point is $A(0, 2)$.

The line is tangent of the graph at point A because they have only one common point (which is A).’

Student B

‘The line is not tangent to the graph because, even though they have one common point, the line cuts across the graph, as we can see in the figure.’



- In your view what is the aim of the above exercise?
- How do you interpret the choices made by each of the students in their responses above?
- What feedback would you give to each of the students above with regard to their response to the exercise?

Figure 1: The Task

The first level of analysis of the scripts consisted of entering in a spreadsheet summary descriptions of the teachers’ responses with regard to the following: perceptions of the *aims* of the mathematical exercise in the Task; mathematical *correctness*; *interpretation/evaluation* of the two student responses included in the Task; *feedback* to the two students. Adjacent to these columns there was a column for commenting on the approach the teacher used (verbal, algebraic, graphical) to convey their commentary and feedback to the students across the script. On the basis of this first-level analysis we selected 11 of the participating teachers for interview. Their individual interview schedules were tailored to the analysis of their written responses and, mostly, on questions we had noted in the last column of the spreadsheet. Interviews lasted approximately 45 minutes and were audio recorded.

The mathematical problem within the Task in Fig. 1 aims to investigate students' understanding of the tangent line at a point of a function graph and its relationship with the derivative of the function at this point, particularly with regard to two issues that previous research (e.g. Biza, Christou & Zachariades, 2008; Castela, 1995) has identified as critical:

- students often believe that having one common point is a necessary and sufficient condition for tangency; and,
- students often see a tangent as a line that keeps the entire curve in the same semi-plane.

The studies mentioned above attribute these beliefs partly to students' earlier experience with tangents in the context of the circle, and some conic sections. For example, the tangent at a point of a circle has only one common point with the circle and keeps the entire circle in the same semi-plane.

Since the line in the problem is a tangent of the curve at the inflection point A the problem provides an opportunity to investigate the two beliefs about tangency mentioned above – similarly to the way Tsamir et al (2006) explore teachers' images of derivative through asking them to evaluate the correctness of suggested solutions. Under the influence of the first belief Student A carries out the first step of a correct solution (finding the common point(s) between the line and the curve), accepts the line tangent to the curve and stops. The student thus misses the second, and crucial, step: calculating the derivative at the common point(s) and establishing whether the given line has slope equal to the value of the derivative at this/these point(s). Under the influence of both beliefs, and grounding their claim on the graphical representation of the situation, Student B rejects the line as tangent to the curve.

With regard to the Greek curricular context, in which the study is carried out, the Year 12 students (age 17/18) mentioned in the Task have encountered the tangent to the circle in Year 10 in Euclidean Geometry and the tangent lines of conics in Analytic Geometry in Year 11. In Year 12, they have been introduced to the tangent line to a function graph as a line with a slope equal to the derivative of the corresponding function at the point of tangency. Although in Years 11 and 12 the tangent is introduced as the limiting position of secant lines, this definition is rarely used in problems and applications. The students' mathematics 'specialisation' mentioned in the Task refers to the students' choice of mathematics as one of the curriculum subjects for more extensive study in Years 11 and 12.

The discussion we present in this paper is based on a theme that emerged from the first-level data analysis and was explored further in the interviews: the teachers' *beliefs about the role of visualisation in mathematics (epistemological) and in their students' learning (pedagogical)*. This theme emerged largely from our observation that, in their scripts, the majority of the teachers distinguished between (and often juxtaposed) Student A's *algebraic* approach and Student B's *graphical* approach. Most of these teachers included in their comments an evaluative statement regarding

the sufficiency/acceptability of one or both approaches. And often they referred explicitly to their beliefs about, for example, the **sufficiency/acceptability of the graphical approach**; or about the **role visual thinking may play in their students' learning**. The teachers' responses also appeared significantly influenced by the mathematical context of the problem within the Task; namely, by their **own perceptions of tangents** and their own views as to whether the line in the Task must be accepted as a tangent or not.

For example, with regard to the teachers' evaluation/interpretation of Student B's solution and feedback to Student B we scrutinised the scripts and designed the interviews with reference to questions such as: does the teacher turn the student away from the graphical approach (which may have led the student to an incorrect claim) and towards an algebraic solution in order to help the student change their mind about whether the line is a tangent or not? Does the teacher compare and contrast the algebraic solution to Student B's solution or do they proceed directly to the presentation of an algebraic solution? What types of examples/counterexamples, if any, do they employ in this process? What is the teacher's position towards Student B's grounding their claim on the graph and, generally, towards the validity of graphical argumentation as proof? Etc.. We presented a preliminary analysis of the above in (Biza, Nardi & Zachariades, 2008). This analysis suggested that there was substantial variation amongst the participating teachers in terms of the stability and clarity of their beliefs about the role of visualisation (epistemological and pedagogical). In what follows we present evidence from the scripts and interviews of two teachers, Spyros and Anna², whose cases exemplify this variation. Of particular interest in the accounts that follow is the interplay between the teachers' beliefs and their (stated) pedagogical practice. The data is translated from Greek.

SPYROS

Spyros has about fifteen years of teaching experience in secondary education. In his written response to the Task he described what led Student A and Student B to their respective answers. His feedback to the students was brief and stated rather generally. He emphasised the significance of mathematical definitions (in this case; the definition of tangent) and juxtaposed students' understanding and use of the definition with what he called 'intuitive' perception of the concept. He did not refer to any specific procedure through which the students could have determined whether the line is a tangent or not. At the same time he focused almost entirely, but rather generally, on the conceptual understanding of the definition and its 'history' in mathematics. We invited him to the interview in order to explore further his references to the 'history' of the concept and elaborate his feedback to the students.

² We note that Spyros is one of the 38 (out of 91) teachers who rejected Student B's claim that the line is not a tangent. Anna is one of the 25 teachers who agreed with Student B's claim. There was some evidence of support for Student B's claim in the scripts of another 18 teachers and there were also 9 blank or half-completed scripts.

During the interview he stated that he had not thought about the relationship between the circle tangent and the tangent to a curve. He recognised that Student A had regarded having a unique common point as a sufficient condition for tangency and stressed that this condition is neither sufficient nor necessary. He also described counter-examples that could help Student A reconstruct their image of a tangent line.

While discussing Student B's response we asked him to elaborate on whether he would accept an argument based on a graph. His answer was firm: 'No, first of all it is not an adequate answer in exams'. (We note that in the Year 12 examination, which is also a university admission exam, there is a requirement for formal proof). We asked him to let aside the examination requirements for a moment and consider whether an argument based on a graph would be adequate mathematically. He replied: 'Mathematically, in the classroom, I would welcome it at lesson-level and I would analyse it and praise it, but not in a test'. Asked to elaborate he says: 'Through [the graph-based argument] I would try to lead the discussion towards a normal proof...with the definition, the slope, the derivative etc.'. Asked to justify he says:

This is what we, mathematicians, have learnt so far. To ask for precision. For axiomatic... we have this axiomatic principle in our minds. Whatever I say I prove on the basis of axioms, on the basis of theorems, on the basis.... And this is what is required in the exams. And we are supposed to prepare the students for the exams.

In the above, Spyros's statement is clear: while he cannot accept a graph-based argument as proof, he recognises graph-based argumentation as part of the learning trajectory towards the construction of proof. He seems to approach visual argumentation from three different and interconnected perspectives: the *restrictions of the current educational setting*, in this case the Year 12 examination; the *epistemological constraints* with regard to what makes an argument a proof within the mathematical community; and, finally, the *pedagogical role* of visual argumentation as a means towards the construction of formal mathematical knowledge.

These three perspectives reflect three roles that a mathematics teacher needs to balance: educator (responsible for facilitating students' mathematical learning), mathematician (accountable for introducing the normal practices of the mathematical community) and professional (responsible for preparing candidates for one of the most important examinations of their student career). Spyros' awareness of these roles, and their delicate interplay, is evidence of the clear and stable didactical contract he appears to be able to offer to his students. Below we discuss a rather different case.

ANNA

Anna is a recent graduate with about four years of teaching experience in private tuition. In her written response to the Task she agreed with Student B's claim that the line is not a tangent. She interpreted Student A's answer as an implication of

accepting the uniqueness of the common point between the line and the curve without examining the ‘nature’ of this point (she pointed out that an infinite number of lines pass through one point). She attempted to reconstruct Student A’s views through reference to graphs and then to the definition. She did not elaborate on the use of the definition; she simply cited the related formula but did not apply it in the case of the function in the Task. She accepted Student B’s graphical approach. She stressed that students are rarely at ease with the graphical approach and are often reluctant to use it. She however wrote that she would draw Student B’s attention to the fact that a graphical approach is not always feasible. Therefore, she wrote, she would demonstrate the ‘analytical’ way through an appropriate worksheet in which she would use a function with a hard-to-construct graph. For a ‘more complete repertory’ she would encourage Student A to use graphs and Student B to use the analytical approach. We invited Anna to the interview because of her emphasis on the necessity of the algebraic approach in cases where the graphical approach is not possible – not because of her concern for its validity. Also because we wanted to explore further how this sat alongside her overt appreciation of Student B’s solution.

Anna, between writing the response and being interviewed, had realised that she should accept the line as a tangent. In the interviews, she attributed her, and the students’, ‘misunderstanding of tangents’ to earlier experience with circle tangents.

I thought that the tangent should be always like the circle tangent, but this is wrong. Because the student in question made the graph and saw it was horizontal and cuts the graph in half, he considered that this is not right, that’s why... he expected to see something like [*she gestures a line touching the graph without splitting it*].

When we asked her to describe the algebraic solution she managed only with extensive help on our part.

While discussing Student B’s response we asked Anna if she would accept Student B’s graphical solution as correct if the student had concluded with the acceptance of the line as a tangent. She said: ‘I think that we have to do all the procedure’ because ‘the line could be here, [*showing on the graph*] higher or lower, where it isn’t a tangent’ and ‘I cannot decline that it isn’t tangent but also I cannot say that it is. Don’t I have to do some...’. When we asked her why, in the light of these reservations, she accepted the graphical explanation in her *written* response, she replied: ‘I accepted it because he said that it wasn’t and I had in my mind that when I see the line splitting [*the graph*] there is no other choice, whatever it was’. So, would she accept a graphical solution, in general? ‘If it is correct, I would accept it’, she replied. Would she accept student B’s solution as correct if the student indicated on the graph that, although the line intersects the curve, the intersection point is an inflection point, as, for example, in the case of $f(x)=x^3$? She replied: ‘I would accept it [...] it is not necessary to use the algebraic method with formulas and all that, that’s what I believe. [*hesitating*] I am not sure this is correct [*awkward laugh*]’.

She then added:

Simply, I believe that students are not so familiar with graphical representations... and, for them, it is easier to use formulas...they see this as a methodology, as... I do not believe that they have gone into depth so that they know how to construct graphs perfectly and know how to interpret them well and this is why most of them usually use algebraic formulas. [...] Because to make a graph and analyse it you have to have understood something very, very well... to own it, completely, while for this [the algebraic formula] you learn how, somewhat blindly, and you solve it, that's what I believe. In any case if [the claim] was correct I would accept it because I would see that the student understood it better than someone who can follow the algebraic formulas... now I don't know, am I right? What do you think?! [to the interviewer]

Later on in the interview, we asked her what would happen if the inflection point wasn't at 2.00 but very close to it (e.g. at 2.02). That made her uncertain about the accuracy of the graph. She then reconsidered her previous statement and said: 'So I believe that the best is that the students do the algebra and then make the graph [*awkward laugh*]'. She elaborated her change of mind as follows:

I simply believe that after we solve through the algebraic formulas and find the result, then it is good to tell the students to make the graph because sometimes they reach the end and say 'ok, I found it' without having realised in their mind how it would look roughly and as soon as they see a graph they cannot answer immediately and I believe this is what happened to me... that is I was used to see circle tangents and it had crossed my mind... subconsciously that all of them must be like that ... all tangents have to be like that because I was not familiar with graphs.

In the above Anna's beliefs about the acceptability or not of a visual argument appear unstable. She appears ready to accept a visual argument without any algebraic justification if the information in the image constitutes, for her, clear and convincing support for a claim. She regarded the image in the Task as sufficient evidence for determining that the line is not a tangent – also drawing on her belief that a tangent cannot intersect the graph. However she stated clearly that to prove that the line *is* a tangent an algebraic argument was necessary. Later, she stated that she could accept a *correct* statement based on the graph. When we shook her faith in the graph she declared the algebraic solution necessary. While initially she did not speak of validation of the visual statement through reference to mathematical theory, she asked for such validation when she realised that the image could be misleading.

Many times in her interview she returned to her appreciation of visual representation and argumentation as evidence of a student's in-depth understanding and as an important means towards students' construction of mathematical knowledge. She did not specify whether she meant *formal* mathematical knowledge (for example, proof). Furthermore her views with regard to the sufficiency and acceptability of a visual argument appeared rather ambivalent and heavily dependent on the specific images involved in the discussion. In this sense the didactical contract she appears to be able to offer to her students seems less clear and stable than that of Spyros.

CONCLUDING REMARKS

Spyros' clear insistence on the class' collective arrival at a formal proof as closure to the lesson is distinctly different from Anna's fluctuation between cases where she would and would not accept a visual argument. Her willingness to rely, occasionally, on imagery in order to support a claim is 'a practice that may mislead students into thinking that such are acceptable mathematical 'proofs' and reinforcing the acceptability of their empirical proof schemes.' (Harel & Sowder, 2007, p829). Furthermore, her own criteria about what makes a visual argument acceptable appeared very personal and rather fluid. Within the unstable didactical contract that this vagueness might imply, how would her students distinguish between when a visual argument is acceptable and when not? In the already compounded didactical contract of school mathematics such vagueness can be detrimental.

A clearer contract could be as follows: in a classroom discussion where a visually-based (incorrect) claim is proposed, the class employs the algebraic, formal approach to convince the proposer about the incorrectness of their claim. Even when a visually-based (correct) claim is unequivocally accepted by the whole class, the class still employs the algebraic approach to establish the validity of the claim formally. In both cases visualisation emerges as a path to insight and proof as the way to collectively establish the validity of insight. In both cases there is a pedagogical opportunity for linking imagery with algebra and for embedding the algebra in the immediately graspable meaning in the image.

The above suggest a role for proof in the mathematics classroom that is not disjoint from the creative parts of visually-based classroom activity and that reflects an essential intellectual need. We conclude with quoting Harel & Sowder's (2007, p836) statement regarding this intellectual need:

The subjective notion of proof schemes is not in conflict with our insistence on unambiguous goals in the teaching of proof – namely, to gradually help students develop an understanding of proof that is consistent with that shared and practised by the mathematicians of today. The question of critical importance is: What instructional interventions can bring students to see an intellectual need to refine and alter their current proof schemes into deductive proof schemes.

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MODES OF ARGUMENT REPRESENTATION FOR PROVING – THE CASE OF GENERAL PROOF

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In light of recent reform recommendations, teachers are expected to turn proofs and proving into an ongoing component of their classroom practice. At least two questions emerge from this requirement. Is the mathematical knowledge of high school teachers sufficient to prove various kinds of statements? And does their knowledge allow the teachers to determine the validity of an argument made by their students? The results of the present study point to a positive answer to the first question in the framework of elementary number theory (ENT). However, the picture is much less positive with respect to the second one.

THEORETICAL BACKGROUND

The calls for enhancing students' abilities to prove and to refute mathematical statements appear prominently in various reform documents of different countries (e.g., Israeli Ministry of Education, 1994; National Council of Teachers of Mathematics [NCTM], 2000). In the NCTM document, reasoning and proof is one of five process standards for all grade levels. Still, there is a need to clarify what proof is in the classroom context. Stylianides (2007) made an attempt in this direction:

Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:

- It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justifications;
- It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; and
- It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community. (p. 107).

Stylianides' (2007) definition talks about the classroom community as the authority to determine the correctness of a proof. However, the teacher, as the representative of the mathematics community, has a special role in the endeavor. He needs to be attentive to both – the mode of argument for a

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given statement (such as general proof, counter example, supportive example), as well as the mode of argument representation (such as numerical, verbal or symbolic), to be able to determine the correctness of a justification.

To what extent are teachers prepared to implement proofs and proving as part of their classroom practice? Relatively little is known on teachers' subject matter knowledge in this area. Dreyfus (2000), following Healy and Hoyles' (1998) work with high school students, presented 44 secondary school teachers with nine justifications to the universal claim "The sum of any two even numbers is even". He found that most secondary school teachers easily recognized formal proofs, but had little or no appreciation for other types of justifications such as verbal, visual or generic ones. Knuth's (2002b) findings suggest that secondary school teachers recognized the variety of roles that proofs play in mathematics. Noticeably absent, however, was a view of proofs as tools for learning mathematics. Many of the teachers held limited views of the nature of proof in mathematics and demonstrated inadequate understandings of what constitutes proofs.

In a different study on in-service high school teachers' knowledge of elementary number theory (ENT), only a third of the 36 teachers provided counter examples to the (false) universal statement "All commutative actions are also associative" (Zaslavsky & Peled, 1996).

These studies focused solely either on universal or on existential statements. Tirosh (2002) presented the same group of elementary and middle school teachers with both universal and existential ENT statements. Tirosh and Vinner (2004) analyzed 38 prospective middle-school teachers' written answers to questionnaires on the issues of constructing and evaluating proofs and refutations in ENT. They found that about 20% of the prospective teachers incorrectly argued that some of the existence theorems in the questionnaires are false (e.g., "There exists a real number b so that $a + b < a$ "). Furthermore, about half of the prospective middle school teachers incorrectly argued that numerical examples that satisfy existential statements are just examples and could not be regarded as mathematical proofs. These responses suggest that some prospective teachers develop a general view that a mathematical statement is true only if it holds for "all cases", a view which is adequate for universal statements but not for existential ones.

The present study addresses a high school teacher's knowledge with respect to universal and existential statements in the area of ENT. It aims to give a preliminary answer to the following two questions. Is the mathematical

knowledge of high school teachers sufficient to prove ENT statements? And does their knowledge allow the teachers to determine the validity of an argument made by their students?

Note: the work of Tirosh (and Vinner) and Dreyfus differ from the work presented here in the population and the mathematical statements.

METHOD

Participants

A group of 50 high school teachers participated in the research. All teachers had some experience teaching in high school. Ms R was one of the teachers. Ms R was chosen as focus teacher for this study on the basis of her answers to the set of questionnaires below.

When participating in our project, Ms R had been teaching for five years in a high school, working with high-achieving students from a high socio-economic background. In parallel, she was studying for her Master's degree in mathematics education. The program included a number of mathematics courses and a number of psycho-didactical courses.

Tools

In one of these courses, the participants' mathematical knowledge was analyzed through their written reactions to two questionnaires that dealt with six ENT statements. No time-limit was imposed for the work on the questionnaires. In this section we briefly describe each of the questionnaires.

Predicate	Always true	Sometimes true	Never true
Quantifier			
Universal	S1. The sum of any five consecutive natural numbers is divisible by 5.	S2. The sum of any three consecutive natural numbers is divisible by 6.	S3. The sum of any four consecutive natural numbers is divisible by 4.
	<i>True/General proof</i>	<i>False/Counter example</i>	<i>False/Counter example</i>
Existential	S4. There exists a sum of five consecutive natural numbers that is divisible by 5.	S5. There exists a sum of three consecutive natural numbers that is divisible by 6.	S6. There exists a sum of four consecutive natural numbers that is divisible by 4.
	<i>True/Supportive example</i>	<i>True/Supportive example</i>	<i>False/General proof</i>

Table 1. Classification of the six statements

The Prove-Questionnaire was intended to identify the participants' production of proofs (validations and refutations) to various (true or false) statements. The questionnaire included six ENT statements (statements S1-S6 in Table 1). The statements were chosen to include one of three predicates (always true, sometimes true or never true), and one of two quantifiers (universal or existential). Clearly, the validity of a statement is determined by the combination of its predicate and its quantifier. Three of the statements are true (S1, S4, S5), and the other three are false (S2, S3, S6). Table 1 displays the six statements according to their quantifier and predicate; their truth value as well as a suitable proof method are also indicated. The participants were asked to examine each of the statements, to determine whether it is true or false, and to prove their claim.

The True or False-Questionnaire was intended to check the participants' identification of the correctness of 43 justifications for the six statements they had proven before, between six and nine justifications for each statement, using numerical, verbal or symbolic modes of arguments representations. For each justification, the participants were asked to determine whether it verifies (refutes) the statement, and to explain their evaluation. The justifications were presented as if they were written by students in various modes of argument representations.

In analyzing teachers' answers to the first and second questionnaire we related to the modes of argumentations as well as to the mode of argument representations.

RESULTS AND DISCUSSION

In this section we first present the participants' answers to the *Prove-Questionnaire*, with examples of Ms R's proofs. Then we discuss the participants' answers to the *True or False-Questionnaire*. Here we narrow the discussion to five justifications which relate to two statements – S1 and S6. We chose these two statements because they require general proofs. We present in detail the answers of Ms R to each justification, followed by a brief description of the results for all participants with regard to the same justifications.

Prove-Questionnaire

All the teachers produced correct proofs to each of the six statements. That is, the modes of argumentation the teachers chose for each statement were appropriate. Their proofs were presented in one of two modes of argument representation – symbolic or numeric (see Table 2).

All participants used the symbolic mode of argument representation for statements S1 and S6, which required a general mode of argumentation. About half of the participants produced numerical examples to refute the universal statements S2 and S3, and the majority of the participant provided a single numerical example to validate the existential statements S4 and S5. None of the participants provided several examples to prove or refute a statement. These findings indicate that the participants who used numerical examples knew when an example is sufficient for proving a statement.

	S1	S2	S3	S4	S5	S6
Numeric	---	50	44	72	80	---
Symbolic	100	50	56	28	20	100

Table 2: Percentages of modes of argument representation produced by the participants (N=50)

We present Ms R's proof for statement S1 which is a universal, always true:

Let's denote five consecutive numbers by $a, a+1, a+2, a+3, a+4$. Their sum is:
 $a+a+1+a+2+a+3+a+4 = 5a+10$.

$(5a+10):5 = a+2$. $a+2$ is a natural number for any a that is a natural number.
 Therefore the statement is true.

As we can see, the proof that Ms R provided related to all the cases in the domain, used correct inference rules, is concise, and thus exemplifies a sound proof.

Ms R's proof for statement S6, an existential, never true statement shows similar characteristics:

Let's check: a is a natural number. $(a+a+1+a+2+a+3):4=(4a+6):4$

We divide the last expression by 2, obtaining $(2a+3):2$. But, $2a+3$ is an odd number (the sum of even, $2a$ and odd, 3), and therefore is not divisible by 2.
 The statement is not true.

Again Ms R correctly identified the need for a general mode of argumentation, and used a symbolic mode of argument representation.

True or False-Questionnaire – Ms R's explanations.

We now focus on the two statements that required general proofs, meaning that the general mode of argumentation should be used. Yet, such an argument can be displayed in at least two modes of argument representation – verbal and symbolic. Five sets of justifications, Ms R's judgments, and her explanations are presented. A short discussion follows each set.

Example 1: Verbal justification to statement S1 and Ms R's explanation

The given correct justification:

Moshe claimed: I checked the sum of the first five consecutive numbers: $1+2+3+4+5=15$ is divisible by 5. The sum of the next five consecutive numbers is larger by 5 than this sum (each number is bigger by 1 and therefore the sum is bigger by 5), and therefore this sum is also divisible by 5. And so on, each time we add 5 to a sum that is divisible by 5, and therefore we always obtain sums that are divisible by 5. Therefore the statement is true.

Ms R's judgment: Moshe's argument is not correct.

Ms R's explanation

Moshe checked the case $1+2+3+4+5=15$, which can be accidentally true. In proving one needs to generalize, and therefore Moshe's justification is not correct.

From Ms R's explanation we can learn that she correctly identified the mode of argumentation needed for proving S1. Yet, she failed to notice the coverage aspect in Moshe's justification.

Example 2: Verbal justification to statement S1 and Ms R's explanation

The given correct justification

Mali claimed: I first tried the first ten examples of 5 consecutive numbers:

$$\begin{array}{lll} 1+2+3+4+5=15 & 2+3+4+5+6=20 & 3+4+5+6+7=25 \\ 4+5+6+7+8=30 & 5+6+7+8+9=35 & 6+7+8+9+10=40 \\ 7+8+9+10+11=45 & & 8+9+10+11+12=50 \\ 9+10+11+12+13=55 & & 10+11+12+13+14=60. \end{array}$$

I saw that the statement is true for the first ten. All other sums of five consecutive numbers are obtained by adding multiples of 10 to one of the listed sums (for instance, the sum $44+45+46+47+48$ is obtained by adding multiples of 10, 5 times 40, to the sequence: $4+5+6+7+8$ that I checked before). Since multiples of 10 are also divisible by 5, the statement is true.

Ms R's judgment: Mali's argument is not correct.

Ms R's explanation

Here also there is no generalization to all the natural numbers, and therefore this is incorrect. It is not a proof.

From Ms R's explanation in this case we can learn that Ms R is concerned with the mode of argumentation. She did not identify the cover aspect in Mali's correct verbal justification.

Example 3: Symbolic justification to statement S1 and Ms R's explanation

The given incorrect justification

Ayala claimed: Among any five consecutive numbers, there is one that is divisible by 5. Let's look at a sequence of five consecutive numbers : $5x$, $5x+1$, $5x+2$, $5x+3$, $5x+4$ ($5x$ is divisible by 5). The sum of this sequence is: $5x+(5x+1)+(5x+2)+(5x+3)+(5x+4)= 25x+10$, and $25x+10$ is divisible by 5 for any x . Therefore the statement is true.

Ms R's judgment: Ayala's argument is correct.

Ms R's explanation

x represents any number, and therefore the proof is general.

Ms R's explanation in this case relates to two important observations. x represents any number, and in this sense the justification is general. However, $5x$ represents a multiple of five, and thus the sequence 1, 2, 3, 4, 5, for instance, is not included. Hence, Ayala's justification is correct for only a subset of the cases that one needs to relate to in order to prove S1. Ms R failed to notice this flaw in Ayala's justification.

Example 4: Verbal justification to statement S6 and Ms R's explanation

The given correct justification

Moshe claimed: I checked the sum of the first four consecutive numbers: $1+2+3+4=10$, ten is not divisible by 4. The sum of the next four consecutive numbers is obtained by adding 4 to this sum (each of the four numbers in the sum grows by 1, so the sum grows by 4). It is known that adding 4 to a sum that is not divisible by 4 will yield a sum that is not divisible by 4 either. And so on, each time we add 4 to a sum that is not divisible by 4, and therefore we always obtain sums that are not divisible by 4. Therefore the statement is not true.

Ms R's judgment: Moshe's argument is not correct.

Ms R's explanation

Moshe chose an example, and on the basis of this example he concluded that there are no such four numbers. But maybe if he would have picked up four other numbers it could have been correct.

Once more, Ms R's reaction exemplifies her view that Moshe's verbal explanation is an example. Again she correctly determined that for this statement an example is not an appropriate mode of argumentation.

Example 5: symbolic justification to statement S6 and Ms R's explanation

The given incorrect justification

Ayala claimed: Among any four consecutive numbers, there is one that is divisible by 4. Let's look at a sequence of four consecutive numbers : $4x$, $4x+1$, $4x+2$, $4x+3$ ($4x$ is divisible by 4). The sum of this sequence is: $4x+(4x+1)+(4x+2)+(4x+3) = 16x+6$. $16x$ is divisible by 4 for any x , while 6 is not divisible by 4. So, the sum $16x+6$ is not divisible by 4. Therefore the statement is not true.

Ms R's judgment: Ayala's argument is correct.

Ms R's explanation

Ayala proved the claim for all four numbers, and hence it is not possible to show that there are four numbers, hence the justification is correct.

The same phenomenon as in example 3 is evident again in Ms R's reaction. On the one hand, it shows that she fully understands the mode of argumentation needed, but on the other hand she fails to recognize whether the given justification carries the general aspect needed.

It seems that for Ms R, the symbolic mode of argument representation, assures that the cover aspect of the proof is taken care of. Also, for Ms R, a verbal mode of argument representation is judged to be merely a numerical example.

One may wonder whether Ms R is unique in her judgments. Let's return to the entire population of 50 participants and check how many teachers made similar choices as Ms R.

For the first statement (S1), 34 percent of the participants rejected the correct verbal justifications (Examples 1 and 2), on the ground that they are not general, and at the same time accepted the incorrect symbolic justification (Example 3), on the ground that it is general. As Ms R, these teachers correctly identified the mode of argumentation needed for each statement.

For the last statement (S6), 26 percent of the participants rejected the correct verbal justification (Example 4), on the ground that it is not general, and at the same time accepted the incorrect symbolic justification (Example 5), on the ground that it is general. Also in this case, the teachers correctly identified the mode of argumentation needed for each statement.

Twenty percent of the participants were consistent in their answers, that is made the same choices as Ms R in the cases of the five justifications presented above.

SUMMING UP AND LOOKING AHEAD

The present study addressed the following two questions. Is the mathematical knowledge of high school teachers sufficient to prove mathematical statements from the field of elementary number theory? And does their knowledge allow the teachers to determine the validity of an argument made by their students?

Our findings indicate that the participants were able to produce correct proofs and refutations to the statements presented. While the teachers chose correct modes of argumentation for each statement, it was evident that they were concerned with this aspect in the second questionnaire.

The picture emerging from the *True or False-questionnaire* seems more complex. About a third of the teachers failed to identify as universal the general-cover aspects of the given arguments in verbal modes of representation. These findings substantiate similar findings reported by Dreyfus (2000), that teachers tend to perceive verbal proofs as deficient because they lack symbolic notations. However, Dreyfus (2000) found that teacher tended to reject verbal justifications. Our findings indicate that teachers had difficulties in understanding verbal justifications, but they did not reject them as such. Teachers' difficulties with verbal justifications are particularly worrying in light of the results reported by Healy & Hoyles (2000), namely that high school students not only preferred verbal proofs due to their explanatory power but also that their verbal arguments were more often deductively correct than their arguments in other modes of representation, yet at the same time they expected to get low grades for such proofs.

A quarter of the participants failed to identify when symbolic justifications did not cover all cases in the domain. These findings substantiate findings reported by Knuth (2002b): "In determining the argument's validity, these teachers seemed to focus solely on the correctness of the algebraic manipulations rather than on the mathematical validity of the argument" (p. 392). When being presented with an algebraic justification, the teachers' focus was on the examination of each step, ignoring the need to evaluate the validity of the argument as a whole.

The everyday practice of teachers involves a constant evaluation of students' justifications for statements. It is likely that verbal or symbolic justifications of the kinds presented in our study, will emerge during interactions with students. Therefore, it is important that teachers will be familiar with verbal justifications and able to judge their validity.

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MATHEMATICS TEACHERS' REASONING FOR REFUTING STUDENTS' INVALID CLAIMS

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This study investigates secondary school mathematics teachers' reasoning for refuting students' invalid claims in the context of hypothetical classroom scenarios. The data used in this paper comes from seventy six teachers' responses to a student's invalid claim about congruency of two given triangles and from interviews with a number of them. Some teachers responded to the claim by trying to refute it. Two main approaches to refuting the student's claim were identified: 1. by using known theorems; 2. by using counterexamples. Teachers' difficulties to generate correct counterexamples were traced. Moreover, a rather narrow meaning of the theorems and their use to refute invalid claims was manifested.

INTRODUCTION

Reasoning and proof are considered fundamental aspects of mathematical practice both in the practice of mathematicians and in the practice of students and teachers (Hanna, 2000). A large number of studies in mathematics education have explored students' justifications and proof strategies (e.g., Healy & Hoyles, 2000; Harel & Sowder, 1998). Refuting conjectures and justifying invalid claims requires reasoning that goes beyond the syntactic derivations of deductive proof which has been traditionally the focus of high school mathematics. It mainly involves the generation of counterexamples, the development of logical arguments that are grounded on exploration and experimentation, which are related to the construction of mathematical meaning and understanding. Balacheff (1991) discusses the diversity of ways of dealing with a refutation by referring to the epistemological work of Lakatos (1976) and to his own experimental study with high school students. Lin (2005) also demonstrates the complexity of the process by identifying the different types of arguments that secondary school pupils developed to refute false conjectures.

The process of evaluating and refuting students' claims is central to teacher practice. This often requires the teacher to give on the spot appropriate explanations that often involve the use of examples or counterexamples. Although the process of exemplification is highly demanding it has not been extensively investigated with regard to the teacher (Bills, Dreyfus, Mason, Tsamir, Watson & Zaslavsky, 2006). Desirable choice of examples depends on teacher's subject matter knowledge (Rowland, Thwaites & Huckstep, 2003) on her teaching experience (Peled & Zaslavsky, 1997) and on her awareness of students' prior experience (Tsamir & Dreyfus, 2002). The generation of examples and counterexamples in geometry gets a special meaning as the visual entailments of examples pose certain constraints (Zodik & Zaslavsky, 2008). In this paper, we investigate how teachers respond to students' invalid claims in the context of Euclidean geometry.

THEORETICAL BACKGROUND

We briefly present below the main theoretical constructs that framed our study. These include teacher knowledge, the process of refutation, and the nature and use of counterexamples.

The process of evaluating and refuting students' invalid claims strongly relates to mathematics teacher knowledge. Stylianidis and Ball (2008) studied the characteristics of teacher knowledge for reasoning and proof. Zodik and Zaslavsky (2008) also attempted to capture the dynamics of secondary mathematics teachers' choice and generation of examples in the course of their teaching. They offer an example-based teaching cycle with respect to teacher knowledge, the planning stage and the actual lesson.

The process of refutation has been mainly studied under the epistemological framework of Lakatos (1976) (e.g., Balacheff, 1991; Larsen & Zandieh, 2007). Lin (2005) developed a categorisation of students' refutation schemes. Accordingly, he distinguished between rhetorical arguments (reasons relative to the person spoken to), heuristic arguments (reasons taking into account the constraints of the situation), and mathematical proofs (the process of generating correct counterexamples).

Peled & Zaslavsky (1997) distinguished between three types of counterexamples suggested by mathematics teachers: specific, semi-general and general examples. Semi-general and general examples offer some explanation and ideas how to generate more counterexamples. Related to teachers' generation of counterexamples is the theory of personal *example spaces*, which encompasses examples that are accessible to an individual in response to a particular situation (Bill et al, 2006). Zazkis & Chernoff (2008) introduced the notions of *pivotal example* and *bridging example* and highlighted their role in creating and resolving cognitive conflict.

The study reported here is part of a larger study that investigates teachers' ways of responding to students' false claims. In this paper, we explore the different types of arguments that teachers use in dealing with an invalid claim in the context of geometry, an area where research is rather scarce.

METHODOLOGY

Seventy six teachers who were all candidates for a Masters in Mathematics Education programme participated in the study. Six of them were primary school teachers with an education degree, while the rest had a mathematics degree. Thirty of these were secondary school practicing mathematics teachers.

The teachers took a three hour exam as part of the selection process for the Masters programme. In this exam they had to respond in writing to five tasks in which they were asked to react to hypothetical teaching events. Four of these hypothetical events were related to the process of dealing with students' arguments and claims. Their written responses were analyzed from both mathematical and pedagogical perspectives. On the base of this analysis, 45 teachers were interviewed individually

in order to explore further their reactions and justifications. Each interview lasted about 15-30 minutes. One researcher interviewed the teachers while another one took notes of the conversation. Since these interviews were part of the selection process, we refrained from using any audio or video recordings, in order to avoid negative effects on the candidates.

In this paper we analyse the data based on the test and the first set of interviews concerning one of the tasks.

The task

The task was the following:

In a Geometry lesson, in grade 10, the teacher gave the following task:
 Two triangles $AB\Gamma$ and EZH have $B\Gamma=HZ=12$ and $AB=EH=7$ and the angles $A\Gamma B$ and EZH equal to 30 degrees. Examine if the two triangles are congruent.
 Two students discussed the above task and expressed the following opinions:
 Student A: The two triangles have two sides and an angle equal. Therefore they are congruent.
 Student B: We know from the theory that two triangles are congruent when they have two sides and a contained angle equal. Therefore, the given triangles are not congruent.
 If the above dialogue took place in your classroom, how would you react?

The task refers to a hypothetical classroom scenario which focuses on issues of learning and teaching mathematics. Further discussion about the importance of this type of tasks as a research tool for exploring teachers' thinking can be seen in Biza, Nardi & Zachariades (2007). This task was based on an example discussed and analysed by Zodik & Zaslavsky (2007). Its mathematical content, the properties of the triangles and their congruence, is part of the Euclidian Geometry course taught in grade 10 in Greek high schools. In the task, student A expresses his belief that if two triangles have two sides and one angle that are respectively equal then they are congruent. He seems to over-generalize the theorem "if two triangles have two sides and the contained angle that are respectively equal then they are congruent."

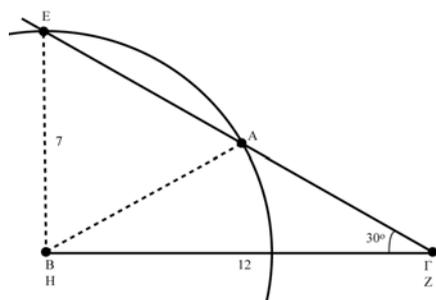


Figure 1: A geometric construction of a counterexample

There are at least three different approaches to refute the claim of this student. The first one is to provide a specific counterexample based on a geometric construction using a ruler and compass (Figure 1).

In this case we may continue and prove a general geometric theorem based on the geometrical construction, namely, that two sides (a and b , where $a > b$) and the angle (β) opposite the smaller side determine exactly two distinct triangles that are not congruent, except for a special case where $\sin(\beta) = b/a$. In the latter case the triangle is necessarily a right-triangle, therefore it is uniquely determined, that is, all triangles with these givens are congruent. The second approach is to prove this general theorem and apply it to the specific given case. The third approach is the use of the sine and cosine laws in trigonometry. By applying the cosine rule for the given angle, we determine the third side, and find that there are two possible values for its length. By applying the sine law we find that there are two possible angles opposite the larger side (a) – an acute one and its supplementary angle. An interpretation of this calculation and the verification of the existence of triangles with these sides or angles lead to the conclusion that there are two (and only two) distinct non-congruent triangles satisfying the givens.

RESULTS

Classifying teachers' justifications

In this section, we present a classification of teachers' justifications based on their written responses. Out of the seventy six mathematics teachers three did not reply while eight considered the given triangles congruent. The remaining sixty five teachers acknowledged that the given triangles were not necessarily congruent. Sixty-three of them gave an explicit justification to their assertion. These justifications were grouped in categories which are presented in the tree diagram in Figure 2. The numbers in brackets indicate the number of teachers' responses that fall in each category.

Out of 63 teachers, 18 justified their claim by drawing on mathematical theorems relevant to the problem and 45 asserted that a counterexample was needed to justify their claim.

Reasoning based on known theorems:

As mentioned above, this type of responses was manifested by 18 teachers. Only two gave a full valid proof. The rest gave invalid proofs that included proof-like arguments.

Valid proof. Interestingly, although the context is geometry, the two teachers who gave valid proofs based them on trigonometry. One of them (T64) used the sine rule, and the other (T32) used the cosine rule, as described earlier.

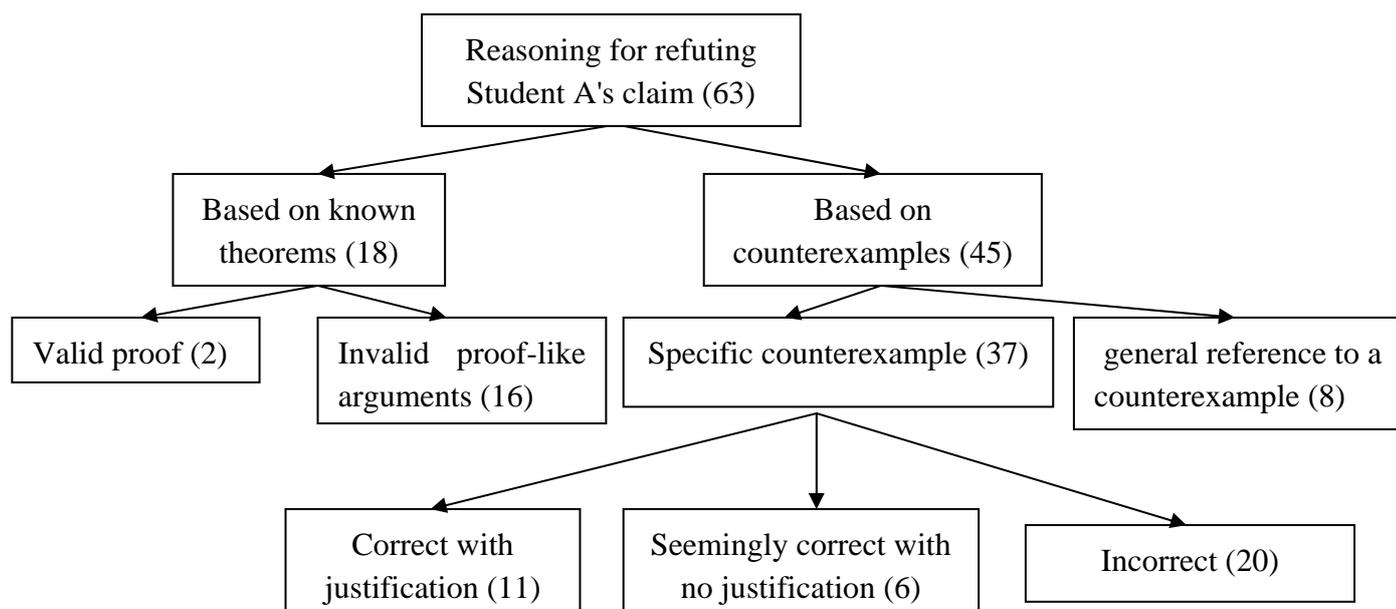


Figure 2: Mathematics teachers' justifications of the assertion that the two given triangles are not necessarily congruent

Invalid proof-like arguments. The remaining sixteen teachers provided invalid proof-like arguments to support their claim by maintaining that none of the known theorems about the congruence of two triangles applies in this case. The following example indicates the latter case: “Student A replied without considering the known criteria for congruence of triangles. I would encourage him to draw the two triangles so that to realise that these criteria cannot be applied” (T10). These teachers believed that this reasoning offers a valid proof for refuting student A’s claim.

Reasoning based on counterexamples:

This type of responses was manifested by 45 teachers. Only 11 gave a specific counterexample with correct justification.

General reference to a counterexample. Eight teachers only made reference to the need to give a counterexample by stating that they themselves or their students would give a counterexample. For example, T23 simply mentioned that “... to convince him (Student A) we could show him some triangles that have two sides and one angle equal but are not congruent” while T18 suggested asking the students: “... to experiment with the shapes and to make many different trials. So, Student A would see a good counterexample that would contradict his view”.

Specific counterexamples: The remaining 37 teachers in this category, constituting half of the participants, gave a specific counterexample. Twenty of them provided *incorrect counterexamples*. For example, some sketched two triangles that appeared to satisfy the given conditions and claimed that these triangles were not congruent although in their drawing these triangles seemed congruent. Thus, we consider this to be non-appropriate examples. Other examples had too many constraints - thus were non-existent. For example, T72 drew two triangles that seemed symmetrical in his

attempt to produce two triangles that were not congruent (Figure 3), however, they seemed congruent.

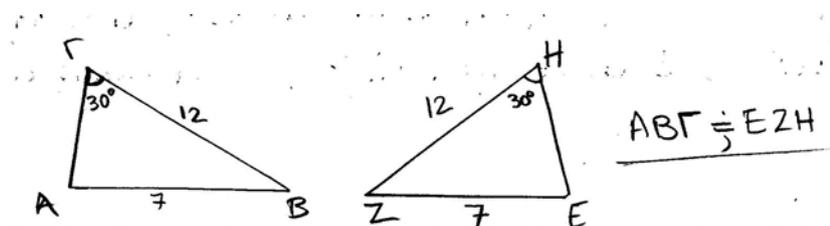


Figure 3: The drawing of T72

Some teachers considered the variation of a pair of angles but without giving specific measures. Others drew two triangles by attributing specific values to the angle contained between the two given sides. For example, T74 wrote: “I would ask the students to make two triangles $AB\Gamma$ and EZH with $B\Gamma=ZH=12$, $AB=EZ=7$, the angles $A\Gamma B$ and EZH equal to 30 degrees, the angle $AB\Gamma$ equals 90 degrees and the angle EZH equals 45 degrees”. In both cases, as it appeared also from the interviews and will be analysed further below, several teachers did not think about the existence of the suggested triangles and did not notice that they were suggesting non-existing cases.

Six teachers gave counterexamples that were *seemingly correct with no justification*. They drew two triangles which satisfied the given conditions for which the angles opposite to the sides of length 12 seemed supplementary, like in the appropriate counterexample. They claimed that this was a counterexample but did not give any justification for their claim. Finally, eleven teachers gave a *correct counterexample with justification* by constructing geometrically the two triangles that had the given elements and were not congruent. Some of them suggested to explore further with the students the situation and to formulate relevant theorems. For example, after the geometrical construction of a counterexample T33 wrote: “I would ask the students to try to prove that if one triangle has two sides and the angle opposite to one of these sides equal to the corresponding sides and angle of another triangle, then the corresponding angles that are not contained in the two sides are either equal or supplementary”.

Emerging epistemological issues

The issue of existence of a (counter)example

As shown in Figure 2, over one third of the teachers (26) had not considered the problem of existence in their initial responses. In the interviews, the teachers who had not justified the process of constructing a counterexample as well as those who gave an incorrect example, were asked about its existence: “How do you know that the triangles you have drawn exist?”. Some of them argued for the existence of their counterexample by inferring from a familiar theorem, recalling an image, or describing the drawing process. Following are some examples of their arguments:

“I have seen this counterexample in a textbook” (T32, recalling an image)

“If I remember well, there is a theorem that says that the non-contained angles are equal or supplementary”. (T40, inferring from a theorem)

“The sum of their angles is 180 degrees...They can be constructed...I can vary the angles” (T69, inferring from a theorem)

“I made them; I measured its sides with a ruler”. (T30, describing the drawing process)

When asked to consider the issue of existence, most of the teachers responded immediately that they had to check the existence of the suggested triangles. However, there were some who seemed to believe that the question about the existence of a triangle with specific properties had no meaning. Typical responses were:

“Yes, why can't we? Do we have to prove it?” (T46)

“Is it possible not to exist?” (T60)

“I thought that it is sure that there are two triangles (satisfying the given conditions) which are not congruent. So, I opened a bit the angle and I moved the side to that direction.” (T73)

Another issue that emerged and was related to the problem of existence was the number of possible counterexamples. There were teachers who believed that there was more than one counterexample and in some cases they described a process of generating an infinite number of triangles (for example, T69, T73 mentioned above). This finding concurs with the findings of Zodik & Zaslavsky (2007).

During the interviews, we observed that some of these teachers started to think about ways of constructing appropriate counterexamples. For example, T39 sketched two triangles and commented: “If we draw on the board two triangles $AB\Gamma$ and EZH with the given elements and the angles $AB\Gamma$ and EZH to be acute - one smaller than the other – it is easy to verify by using transparent paper that the two triangles are not congruent”. In the interview she formed a new hypothesis that: “if in both triangles (satisfying the given conditions) all angles are acute, they are congruent while they must be different if one triangle is acute-angled and the other obtuse.” Later in the interview, she used the sine-rule trying to prove her hypothesis. However, she did not manage to construct geometrically the suggested triangles. On the other hand, T32 had given as a counterexample two triangles, one right-angled and the other isosceles. In the interview she initially recalled a known theorem “that one pair of angles can be equal or supplementary” and finally she gave a correct geometrical construction of the counterexample.

The issue of over-reliance on familiar criteria

Another issue that emerged from our data was the use of theorems for justifying refutable (invalid) claims. A number of teachers believed that the non applicability of the known relevant theorems implied that the claim was wrong. In particular, some teachers concluded that the two triangles were not necessarily congruent as none of the three commonly used criteria about the congruence of triangles could be applied. For example, in his written response T11 reminded the (hypothetical) student these

three criteria, and added that “the problem statement does not satisfy the criterion S-A-S ... so from the given data we cannot conclude that the two triangles are congruent”. The above argument was the only one that the teacher gave for justification. T47 also expressed a similar view in his written response. In the interview, although this belief was challenged by one of the researchers, it seemed to be rather strong as demonstrated in the following extract:

- T47: The two triangles are not necessarily congruent
 R: How do you know this?
 T47: We cannot apply the criterion S-A-S.
 R: Ok, a known criterion cannot be applied. But how do you know that there is no other way to prove the congruence of the two triangles?
 T47: We cannot prove the congruence with the criteria we teach.

In the above cases, the teachers seem to base their reasoning on the principle that we can infer that two triangles with given properties are congruent only if these properties satisfy one of the three commonly used criteria (S-S-S, S-A-S and A-S-A). So, since these conditions were not explicitly given in the task, some teachers (falsely) inferred that these triangles cannot be congruent, while others claimed that they were not necessarily congruent. Although the latter claim may reflect legitimate logical inference, it may also have flaws and lead to wrong conclusions. For example, if the length of shorter side of the two triangles of our problem were 6 instead of 7, then the above kind of reasoning would lead to the conclusion that the two triangles are not necessarily congruent, while in fact, in this case the two triangles are right-angled and thus are indeed congruent.

It should be noted that even though the above way of refuting invalid claims is not a mathematical valid proof, in some cases it can be used as a tool for posing conjectures. For example, in his written response T21 initially stated that the known criteria could not be applied and then gave a geometrical construction of the counterexample.

CONCLUDING REMARKS

From the seventy six mathematics teachers of our study only thirteen refuted correctly the invalid claim of student A, eleven by constructing a counterexample and two by using theorems. Some of the characteristics of teachers’ reasoning that were identified in this study are similar to those reported by Lin (2005) in the case of students. For example, there were teachers who confirmed the invalid claim, others who suggested the possibility of a counterexample without generating it, and few who actually constructed a counterexample accompanied by a mathematical proof.

In our study, teachers seemed to draw on their personal example spaces in order to generate counterexamples. It should be noted that in the case of Student A's (false) claim, a carefully thought through construction of an appropriate counterexample is needed. Teachers who just randomly sketched two triangles were not able to come up

with an appropriate counterexample. In this problem, there is only one counterexample. This counterexample can be seen according to Mason and Pimm (1984) as a generic example, in the sense that it can reflect and lead to the general case (as illustrated in Figure 1). Similarly, in terms of Zaslavsky and Peled (1997) this counterexample has a high explanatory power. However, thinking of it for the first time turned out to be a strong demand on the teachers.

The two main phenomena that emerged from our study should be of great concern: overlooking the question whether an example exists, and over-relying on familiar theorems and criteria. Our findings illustrate how these two phenomena may lead to invalid inferences. Similar to Zodik and Zaslavsky's (2008) findings, there were several instances where teachers considered a non-existing example as if it existed, and did not seem to be aware of this issue at all. The second phenomenon reflects teachers' beliefs that a claim can be refuted if “all” the relevant theorems that they know (mostly those that are included in the school textbooks) cannot be applied. This conception indicates a misleading epistemological view of theorems and their status in mathematical reasoning.

In this paper, we focused mainly on teachers' mathematical knowledge as reflected in their responses. Pedagogical aspects of their knowledge that emerged from our data have not been discussed here. These aspects may provide a more comprehensive account of what is entailed in dealing with students' invalid claims.

ACKNOWLEDGMENT

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STUDENT JUSTIFICATIONS IN HIGH SCHOOL MATHEMATICS

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In this paper, we continue our previous work on evaluating the use of structured derivations in the mathematics classroom. We have studied student justifications in 132 exam solutions and described the types of justifications found. We also discuss the results in light of Skemp's (1976) framework for relational and instrumental understanding.

Keywords: student justifications, structured derivations, high school, instrumental and relational understanding

INTRODUCTION

The ability to justify a step in, for instance, a proof can be considered a skill that needs to be mastered, at least to some extent, before proof is introduced. In a wider sense, proof can even be regarded as justification (Ball and Bass, 2003). Unfortunately, students are not used to justify their solutions (Dreyfus, 1999). It is common for teachers to ask students to explain their reasoning only when they have made an error; the need to justify correctly solved problems is usually de-emphasized (Glass & Maher, 2004). Consequently, without the explanations, the reasoning that drives the solution forward remains implicit (Dreyfus, 1999; Leron, 1983).

A previous study (Mannila & Wallin, 2008) indicated that high school students can improve their justification skills in one single course. In this paper, we will present the results from a follow-up study, focusing on the types of justifications given by the students. We will first discuss some related work and also give a brief introduction to the approach used when teaching the course. The main research questions are the following: What types of justifications do students give in a solution? Do the types of justifications change as the course progresses, and in that case how?

RELATED WORK

Justifications as a condition for proof

The importance of proof and formal reasoning for the development of mathematical understanding is also recognized by the National Council of Teaching Mathematics (NCTM), which issues recommendations for school mathematics at different levels. According to the current document (NCTM, 2008), students at all levels should, for instance, be able to communicate their mathematical thinking, analyze the thinking of others, use mathematical language to express ideas precisely, and develop and evaluate mathematical arguments and proof. While discussing mathematical ideas is important, communicating mathematical thinking in writing can be even more efficient for developing understanding (Albert, 2000).

To think mathematically, students must learn how to justify their results; to explain why they think they are correct, and to convince their teacher and fellow students. “[M]athematical reasoning is as fundamental to knowing and using mathematics as comprehension of text is to reading. Readers who can only decode words can hardly be said to know how to read. ... Likewise, merely being

able to operate mathematically does not assure being able to do and use mathematics in useful ways.” (Ball & Bass, 2003; p. 29)

Justifications are not only important to the student but also to the teacher, as the explanations (not the final answer) make it possible for the teacher to study the growth of mathematical understanding. Using arguments such as “Because my teacher said so” or “I can see it” is insufficient to reveal their reasoning (Dreyfus, 1999). A brief answer such as “ $26/65=2/5$ ” does not tell the reader anything about the student’s understanding. What if he or she has “seen” that this is the result after simply removing the number six (6)?

Types of understanding and reasoning

A review of literature on mathematics education shows that there is an interest in studying the distinction between being able to apply a determined set of instruction in order to solve a mathematical problem and being able to explain the solution by basing it on mathematical foundations. Several frameworks have been presented for investigating types of learning and understanding.

Skemp (1976) discusses two types of understanding named by Mellin-Olsen: *relational* (“knowing both what to do and why”) and *instrumental* (“knowing what”, “rules without reasons”). People who exhibit an instrumental understanding know how to use a given rule and may think they understand when they really do not. For instance, getting the correct result when applying a given formula is an example of instrumental, not relational, understanding. One typical example can be found in equation solving, where students learn to “move terms to the other side and change the sign”, without necessarily knowing why they do it.

Sfard (1991) investigates the role of algorithms in mathematical thinking and discusses how mathematical concepts can be perceived in two ways: as objects and as processes. Pirie and Kieren (1999) present a theory of the growth of mathematical understanding and its different levels. More recently, Lithner (2008) has created a research framework for different types of mathematical reasoning, distinguishing between two main types: *imitative* and *creative*. Imitative reasoning is rote learnt and can be divided into two subtypes: memorised reasoning, where the student, for instance, solves a problem by recalling a full answer given in the text book or by the teacher, and algorithmic reasoning, where a problem is solved by recalling and applying a given algorithm. The other main type, creative reasoning, includes a novel reasoning sequence, which can be justified and is based on mathematical foundations. One of the main differences between imitative and creative reasoning is that the former does not necessarily involve analytical and conceptual thinking, whereas such thinking processes are essential to creative reasoning.

STRUCTURED DERIVATIONS

Structured derivations is a logic-based approach to teaching mathematics (Back & von Wright, 1998; Back & von Wright, 1999; Back et al, 2008a). The format is a further development of Dijkstra's calculational proof style, where Back and von Wright have added a mechanism for doing subderivations and for handling assumptions in proofs. Using this approach, each step in a solution/proof is explicitly justified.

In the following, we illustrate the format by briefly discussing an example where we want to prove that $x^2 > x$ when $x > 1$.

- Prove that $x^2 > x$: *task*
- $x > 1$ *assumption*
- ||- $x^2 > x$ *term*
- ≡ { Add $-x$ to both sides } *justification*
- $x^2 - x > 0$ *term*
- ≡ { Factorize } ...
- $x(x - 1) > 0$
- ≡ { Both x and $x-1$ are positive according to assumption. Hence, their product is also positive }
T

The derivation starts with a description of the task (“Prove that $x^2 > x$ ”), followed by a list of assumptions (here we have only one: $x > 1$). The turnstile (||-) indicates the beginning of the derivation and is followed by the start term ($x^2 > x$). In this example, the solution is reached by reducing the original term step by step. Each step in the derivation consists of two terms, a relation and an explicit justification for why the first term is transformed to the second one.

Another key feature of this format is the possibility to present derivations at different levels of detail using subderivations, but as these are not the focus of this paper, we have chosen not to present them here. For information on subderivations and a more detailed introduction to the format, please see the articles by Back et al. referred to above.

Why use in education?

As each step in the solution is justified, the final product contains a documentation of the thinking that the student was engaged in while completing the derivation, as opposed to the implicit reasoning mentioned by Dreyfus (1999) and Leron (1983). The explicated thinking facilitates reading and debugging both for students and teachers. According to a feedback analysis (Back et al., 2008b), students appreciate the need to justify each step of their solutions. They also find that the justifications makes solutions easier to follow and understand both during construction and afterwards.

Moreover, the defined format gives students a standardized model for how solutions and proofs are to be written. This can aid in removing the confusion that has commonly been the result of teachers and books presenting different formats for the same thing (Dreyfus, 1999). A clear and familiar format also has the potential to function as mental support, giving students belief in their own skills to solve the problem. Also, as solutions and proofs look the same way using structured derivations, the traditional “fear” of proof might be eased. Furthermore, the use of subderivations renders the format suitable for new types of assignments and self-study material, as examples can be made self-explanatory at different detail levels.

STUDY SETTINGS

Data collection

The data were collected during an elective advanced mathematics course on logic and number theory (about 30 hours in class) that was taught at two high schools in Turku, Finland, during fall 2007. All in all, twenty-two (22) students completed the course at either school and participated in the study (32 % girls, 68 % boys). The students were on their final study year.

The course included three exams held after 1/3, 2/3 and at the end of the course. The exams were of increasing difficulty level, i.e. the first was the easiest and the last the most difficult one. Two assignments from each were chosen for the analysis. Hence, we have in total analyzed 132 solutions (six solutions for each student) written as structured derivations.

The assignments analyzed were the following:

- A1: Determine the truth value of the expression $(x^2 + 3 \leq 7 \wedge y < x - 4) \vee x + y \leq 5$, when $x = 2$ and $y = 4$.
- A2: Solve the equation $|x - 4| = 2x - 1$.
- A3: Use de Morgan's law $(\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q)$ to determine if the expression $(\neg p \vee \neg q) \wedge (p \wedge q)$ is a tautology or a contradiction.
- A4: Prove that $b^2 - d^2 = ad + bc - ab - cd$ if $a + b = c - d$.
- A5: Prove or contradict the following: For any integers m and n , it is the case that if $m \cdot n$ is an even number, then both m and n are even.
- A6: Prove that $2 + 14^{30} \equiv_{13} 106 + 27^{30}$.

The topics covered in assignments A1 and A2 were familiar to the students from previous mathematics courses. The aim of these assignments was mainly to let students practice structured derivations and writing solutions using the new format.

The topics covered in the rest of the analyzed assignments (A3-A6) were new to the students. A3 and A4 focused on logical concepts and manipulation of logical expressions, whereas A5 and A6 covered number theory.

Method

The data collected, i.e. the justifications, were of qualitative nature. Qualitative data are highly descriptive, and in order to interpret the information, the data need to be reduced. In this study, a content-analytical approach was chosen for this purpose. The basic idea of content analysis is to take texts and analyze, reduce and summarize them using emergent themes. These themes can then be quantified, and as such, content analysis is suitable for transforming textual material into a form, which can be statistically analyzed (Cohen, 2007).

A first round of the content analysis was done by one of the authors, who analyzed 18 solutions from E1 and 24 solutions from E2. This initial coding resulted in a first view of the types of justifications. The authors discussed the results and agreed on how to combine the detailed justifications into higher-level categories. Next, all solutions were analyzed using the preliminary categories as the coding scheme. The second round analysis showed that the categories found in the

initial phase were sufficient for covering all justifications found in the 132 solutions. A quantitative approach was then taken in order to be able to illustrate the results graphically.

The use of both quantitative and qualitative methods has several benefits. Mixed methods avoid any potential bias originating from using one single method, as each method has its strengths and weaknesses. A mixed methods approach also allows the researcher to analyze and describe the same phenomenon from different perspectives and exploring diverse research questions. Whereas questions looking to describe a phenomenon ("How/What..?", our first research question) are best answered using a qualitative approach, quantitative methods are better at addressing more factual questions ("Do...", our second research question) (Cohen, 2007).

RESULTS

The content analysis revealed five main justification types:

- *Assumption*: Referral to an assumption given in the assignment directly or in a rewritten format.
- *Vague/broad statement*: A very brief and uninformative justification type: "logic" or "simplify".
- *Rule*: Referral to a name of a rule or a definition, e.g. the rule for absolute values, tautology, congruence etc. In some cases, the justification also included the rule explicitly written out in text.
- *Procedural description*: An explanation of *what* is done in the step, i.e. a description including a verb. E.g. "add $2x + 4$ to both sides", "move 3 to the other side and change the sign" and "calculate the sum".
- *Own explanation*: An explanation for *why* the step is valid in own words and/or with symbols, e.g. " $2k^2 + 2k$ is an integer if k is an integer. Therefore $2(2k^2 + 2k)$ is an even integer". In some justifications a mathematical definition was written out in own words, e.g. " $2 \equiv_{13} 106$ because $2 - 106 = -104$, $13 \mid -104$ ".

Figure 1 illustrates the proportion of different justification types found in the assignments respectively. The diagram also shows how the types of justification used varied depending on the assignment.

Some justification types are highly assignment specific. For instance, assumptions can naturally only be used in assignments where assumptions are present. In such assignments, it is common for the assumption to be used only once or twice, and the proportion of this type of justification will be rather low. The analysis showed that all students but one were able to handle assumptions correctly already in the first exam, i.e. after 1/3 of the course.

The use of rules can also be considered assignment specific. For instance, when manipulating logical expressions, rules become important as these make up the basis for the manipulation. When students gave a rule as a justification, most usually stated only the name of the rule, whereas only a few also wrote out the rule itself. In the final and most difficult assignment, where the rule was central to the solution, a larger proportion of students (46 %) had written it out explicitly, compared to those who had only provided the name of the rule (22 %).

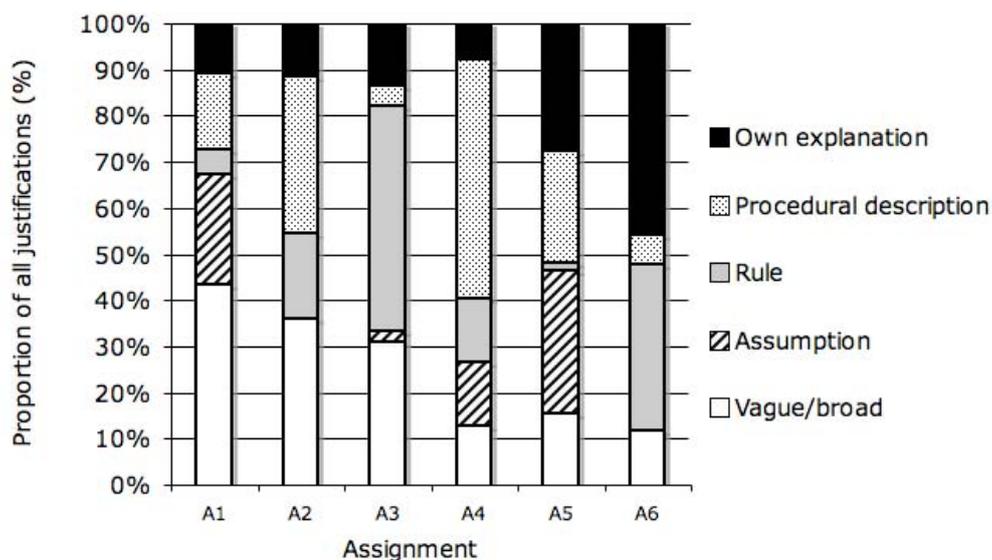


Figure 1: The proportion of justifications of different types in the six assignments

In addition to these specific dependencies, the analysis also revealed some other relationships. The assignments in the first exam (A1-A2) were not trivial but still familiar to the students (determine the value of an expression and solve an equation), who consequently mainly used short justifications (vague/broad, assumption, rule). Given the nature of equations, the solutions to A2 also contained a large proportion of procedural descriptions (“move 3 to the other side”).

In the second exam, students faced assignments (A3-A4) that were not as familiar anymore. In A3, students were to make explicit use of logical rules, which, as stated above, naturally has an impact on the types of justifications: almost half of all justifications referred to a given rule. The following assignment, A4, called for a formal proof (the Finnish high school curriculum does not include proofs in any other course than the elective one described in this paper). As the expression used in the proof was an equation, the main justification type used was, again, procedural descriptions.

The third exam (A5-A6) is probably the most interesting one from a research perspective. The assignments were in a completely new domain, with which students had no prior experience: constructing proofs in number theory. Thus, these assignments have potential to provide insight into how students use justifications when adventuring into a new terrain. As indicated in the diagram (figure 1), the proportion of own explanations increased, in particular at the expense of the less informative justification type “vague/broad”.

DISCUSSION

As seen above, the justification types changed throughout the course. Whereas some of the variation (e.g. the use of assumptions and rules) is a direct result of the nature of the task at hand, some seems to be more related to the perceived level of difficulty.

For instance, the most noticeable changes are found for “vague/broad” justifications and “own explanations”: whereas the former dominate the solutions early on, their frequency decreases towards the end as the number of the latter increases. The first exam was easier than the final one, and as easy assignments include more “straightforward” steps, students may not have seen the need

to justify those steps in any more detail. Rather, students seem to find the need to justify more carefully as the assignments become more difficult. Consequently, the occurrence of own explanations increase. Similarly, it is understandable that students are reluctant to write lengthy justifications when solving tasks similar to tasks they have solved many times before, whereas they may feel a need for writing more careful justifications in assignments that deal with new topics. This is supported by the results from our feedback study, where students found “extra writing” unnecessary for simple tasks (Back et al., 2008b).

Can justifications aid in assessing understanding?

Only two justifications types, “own explanations” and “procedural descriptions”, involve students writing in their own words. There is an important difference between these types. In a “procedural description”, students write *what* they do, but not why they have chosen or are allowed to do so. The “own explanation”, on the other hand, also gives information regarding *why* the step is valid.

This is closely related to Skemp’s instrumental and relational understanding (1976). Own explanations are clearly relational, but the remaining four types (vague/broad, assumption, rule, procedural descriptions) cannot easily be mapped to either type of understanding. We will therefore refer to own explanations as “relational justifications” and the other four types as “instrumental justifications”.

Although Skemp argued that instrumental justifications such as “move -3 to the other side” are examples of an instrumental approach to understanding, we do not think the situation is as black-or-white. For instance, a simple justification such as “logic” may be the result of complex thought processes. Knowing that students are not keen on writing, one can also assume that students may choose to write a short justification even in places where they could have been more expressive in order to indicate their understanding. An instrumental justification simply does not reveal enough information about whether the student has truly understood what he or she has done. Ruling out the possibility of relational understanding in such situations requires more than a mere justification.

To exemplify this, we now look at three different solutions to an assignment involving absolute values. The absolute value rule referred to below is the following: $|x| = c \Leftrightarrow (x = c \vee x = -c) \wedge c \geq 0$

- **Tom:** *instrumental justification, relational understanding*

Tom did not use the rule for absolute values learnt in class, but rewrote the expression in a way showing that he had really understood the absolute value concept. The solution was correct and indicated a relational understanding of absolute values.

$$|x - 4| = 2x - 1$$

\Leftrightarrow { rewrite the absolute value }

$$(x - 4 = 2x - 1 \wedge 2x - 1 \geq 0) \vee (-x + 4 = 2x - 1 \wedge 2x - 1 \geq 0)$$

- **Layla:** *instrumental justification, instrumental or relational understanding*

Layla used the absolute value rule and solved the problem correctly. Despite the correct solution, we cannot know whether Layla understood the concept or merely used a rule she had learnt that “should work” for this type of problems.

$$|x - 4| = 2x - 1$$

⇔ { rule for absolute values }

$$(x - 4 = 2x - 1 \vee x - 4 = -2x + 1) \wedge 2x - 1 \geq 0$$

- **Joe:** *instrumental justification, instrumental understanding*

Just like Layla, Joe also justified the initial step with “the rule for absolute values”. However, he used the rule incorrectly, as he “forgot” the second part of it (the requirement on x).

$$|x - 4| = 2x - 1$$

⇔ { rule for absolute values }

$$x - 4 = 2x - 1 \vee x - 4 = -2x + 1$$

This was a rather common error in our study (made by almost 36% of all students in assignment A2). Had Joe had a relational understanding for absolute values, the additional requirement would have been clear to him even if he had forgotten what the rule looked like.

Thus, it seems as if one can in fact conclude that a given instrumental justification is *not* an example of relational understanding – this is the case if the step is incorrect as for Joe above. However, doing the opposite, i.e. concluding that an instrumental justification to a correct step is relational, is not as straightforward.

Is a clearly relational approach always needed?

In high school mathematics, much time is spent on things like solving equations and simplifying expressions. Thus, to a large extent it boils down to using rules, and consequently a seemingly instrumental approach becomes dominant. However, this is foregone by the teacher explaining the theory behind the rules and the definitions. If the student later uses the rules in an instrumental or a relational way is up to how well he or she understood the theory. If the underlying concept is not clear to the students, the rules are most likely applied without reasons, i.e. instrumentally. One area of high school mathematics where relational understanding most likely becomes more evident is in textual problems, where students first need to formalize the problem specification. In order to correctly specify the problem, the student needs to understand the problem domain and the underlying concepts. Relational understanding is naturally also important when constructing proofs.

Furthermore, sometimes a justification with a seemingly instrumental approach is the best one that can be given. Take for example a complex trigonometric expression. Finnish high school students have a collection of rules that they can always have with them, even on exams. One can hardly require them to start explaining rules in order to be allowed to apply them. What is essential in such a situation is that they a) have an underlying understanding for trigonometry, b) know how to apply trigonometric rules correctly, and c) are able to manipulate the expression into a form where one of the many rules can be applied correctly.

As another example we can take equation solving and the “add -3 to both sides” type of instrumental justification mentioned above. Let us say we have two students: one who understands that whenever you have an expression of the form $a = b$, you can add the same value to both sides without changing the truth value of the full expression ($a + c = b + c$), and another who knows that

one should move “lonely numbers” to the other side while changing the sign. Both of these students would probably use similar justifications, but only one of them would have a relational understanding. This student would, however, hardly write out the rule ($a = b \Leftrightarrow a + c = b + c$), which would be needed in order for the teacher to be able to distinguish the justification from that given by the other student.

Justifications and validity of steps

As was described above, a seemingly “correct” justification can lead to an incorrect derivation step. This can happen for several reasons, one being the one exhibited by Joe above: not completely remembering a rule. Careless mistakes in a step do not seem to correlate with the type or the accuracy of the justification. Only a small number of this type of errors was found (in 9 % of the assignments throughout all three exams), which was also supported by students’ feedback as they pointed out that they made fewer careless mistakes using structured derivations than what they usually do (Back et al., 2008b).

CONCLUDING REMARKS

The type of justification chosen in a certain situation is closely related to the assignment and/or the step at hand. For example, assumptions or rules will not be used in problems where there are no assumptions or rules to apply. Our findings suggest that students choose the level of detail in their justifications mainly based on the difficulty level of the task at hand: in tasks that are familiar, students tend to opt for broad and vague justifications, whereas justifications which say more come into play as the topics covered are new and/or the assignments become more difficult. Especially justifications written in own words are of great importance to the teacher for understanding a solution and the student’s thinking; this is not necessarily the case for vague and broad justifications.

The study presented in this paper is a continuation on earlier qualitative studies on the use of structured derivations in education. Previous results indicate that students appreciate the approach (“it takes me longer, but I understand better”) and that it improves students’ justification skills as soon as during one single course (Mannila & Wallin, 2008). Furthermore, we have found that explicit justifications make students think more carefully when solving a problem (Back et al., 2008b). With this study, we now also have a rather clear picture of how students justify their solutions and how the justifications change throughout the course.

Getting students to clearly document their solutions step by step is a step forward, although “judging” the justifications is everything but straightforward. Thus, many questions still remain. Is it possible to teach a way of writing “good” justifications? And if we want to try, what characterizes such justifications? Another aspect, not considered so far, is related to teachers and course books. How do teachers justify their solutions when teaching using structured derivations? How are examples justified in texts? In order for students to develop relational understanding, we believe that it is essential that examples are explained freely (“using own words”) as often as possible.

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“IS THAT A PROOF?”: AN EMERGING EXPLANATION FOR WHY STUDENTS DON’T KNOW THEY (JUST ABOUT) HAVE ONE[†]

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This paper describes an episode taken from the third year of a design experiment aimed at improving the teaching and learning of proof at the university level. In the episode, students come enticingly close to having a proof, at least as judged by competent outsiders. However the students themselves, while satisfied with their result, abandon it when asked to write up a formal proof. We offer an analysis of this episode and offer questions for further study.

Key words: Proof, Tertiary Level, Key Ideas, Technical Handle, Design Experiment

INTRODUCTION

Design experiments, or “developmental research” as this work is often called in Europe, are becoming increasingly common, at elementary, secondary, and even tertiary education (e.g. Brown 1992, Collins 1999, van den Akker, Branch, Gustafson, Nieveen, & Plomp, 1999, Lesh 2002). The goal is to find theoretically grounded answers to practical questions of the classroom, done in as natural a setting as possible, with as Brown puts it, the “the blooming, buzzing confusion” that one can sometimes find in real classrooms, under real pressures, with real constraints and opportunities.

While the potential of merging theory and practice is quite alluring for many reasons, the practical and conceptual realities of doing so remain challenging. As Kelly (2002) suggests: if design experiments began in the early 1990’s as a sort of art, they are emerging in recent years as a type of science, guided by increasingly rigorous methodology and increasingly useful results. But specifying exactly what this science consists in, that is, how to merge research and practice in a mutually advantageous way, is still a matter of debate, discussion, and development.

This paper is an emerging product of a design experiment aimed at improving the teaching of proof at the university level. The research team, consisting of two

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mathematics educators and three mathematicians, came together with the aim of improving the teaching of “Introduction to Proof” courses, a type of course used frequently in American universities to help students prepare for the rigor of the theoretical courses like abstract algebra and analysis¹. The idea was to use videos of students struggling, and eventually succeeding, at proving claims that are known to be hard for students in this type of course, as a basis for discussion. These videos can be used both as a professional development tool for teachers who want to better understand student difficulties with proof and as a curriculum resource for class discussion to help students be more aware about their own mathematical thinking.

After three rounds of testing and piloting, we now have a fairly stable set of curricular materials, which include (1) carefully edited videos of students working on proofs that many other students find difficult, (2) materials to help teachers use these videos, both for their own understanding of student thinking and for classroom use. These materials have been tested in four colleges in the United States in the context of “Introduction to Proof” courses taught by members of the research team and their colleagues. We also are generating a number of research articles, this being one example, that probe questions of mathematical thinking that enhance and/or inhibit proof production.

We consider our particular marriage of theory and practice to be a happy one. The central questions which drive our research—how to reconcile student and faculty thinking about proof and proving—grew naturally from our experiences as teachers struggling to make the best of our own “Introduction to Proof” courses. While not eliminating common sense and experience as legitimate grounds for interpreting data, we felt a real need to move into theoretical territory to help make sense of some of the mysteries of mathematical thinking.

This paper describes one part of this theoretical journey. We begin by describing an episode that our team found particularly compelling. In the episode, students come enticingly close to finding a proof but do not seem to notice that they have done so. Rather than convert what outside observers recognize as a “key idea” of a proof into a formal proof, they abandon the idea and take a different, and ultimately unsuccessful path. This episode is useful for a starting point in understanding the nature of key idea in the process of proof production, but also points to some fuzziness about the notion of key idea, which a more theoretical analysis can help clarify.

FRAMING

The methodology for this project, for which this paper is one small part, follows the program set out by Cobb, et al (2003). The design is highly interactive and

¹ See Alcock (2007) for a similar project, focusing primarily on professional development, which has been successfully piloted in the US and UK.

interventionist, involving gathering and indexing of longitudinal data from a number of sources, including videos of classroom practice, individual and group interviews with teachers and students, journal and email records from the teachers, written records of student work, and audio and video records of behind-the-scenes discussion among the research team. Like Cobb, et al, we see this design experiment as a “crucible for the generation and testing of theory.” It is the tangible pressures of classroom realities that provide a needed spark for the theory to develop and crystallize, and one of the goals of this paper is to make part of that process visible to both research and practitioner communities.

The central research questions involve characterizing the trajectory of proof development in a way that both helps us see where students sometimes go wrong and also gives some guidance towards how to teach students to prove in a more effective way. In particular, as we traced one particular episode in which students struggled, came close, and eventually failed to find a proof we wanted to know (1) what were the critical “moments” when there was opportunity for the proof to move forward, and (2) what is the nature of these moments. In the end we found three such moments, which seem to play a critical role in proof production. These moments do not necessarily occur in every proof, nor do they necessarily occur in the order in which we present them, but they seem to be critical in the sense that if one is present, the proof can move forward in a fairly significant way, and if one is absent, it is quite possible that the proof will not move forward (or that a proof will be produced without a full sense of understanding).

The first moment is the getting of a key idea, an idea that gives a sense of “now I believe it”². The key idea is actually a property of the proof, but psychologically it appears as a property of an individual (we say that a particular person “has a key idea” if it appears that they grasp the key idea of a proof.) We refer to “a” key idea rather than “the” key idea, because it appears that some proofs have more than one key idea. While a key idea engenders a sense of understanding, it does not always provide a clue about how to write up a formal proof.

The second moment, is the discovery of some sort of technical handle, and gives a sense of “now I can prove it,” that is, some way to render the ideas behind a proof communicable³. The technical handle is sometimes used to communicate a particular

² More elaborated discussions of “key idea” can be found in Raman (2003), Raman & Weber (2006), and Raman & Zandieh (in progress). A key idea can be thought of as a certain kind of intuition that has both a public and private character: public in the sense it can be mapped to a formal proof, private in the sense that it is personally understandable as a sort of primary, or *prima facie*, experience. For a careful discussion about intuitions see Bealer (1992).

³ The term “technical handle” here is akin to the term “key insight” in Raman & Weber (2006). We have chosen to change the term in part because it sounded too similar to “key idea” which has a very different character, and in part because the technical aspect of this “moment” seemed central to

key idea, but it may be based on a different key idea than the one that gives an ‘aha’-feeling, or even on some sort of unformed thoughts or intuition (the feeling of ‘stumbling upon’.)

The third moment is a culmination of the argument into a standard form, which is a correct proof written with a level of rigor appropriate for the given audience. This task involves, in some sense, logically connecting given information to the conclusion. We assume that for mathematicians the conclusion is probably in mind for most of the proving process. But for students, the theorem might sometimes be lost from sight, adding a sense of confusion to their thinking processes.

In the data below we will illustrate how each of these moments occurs in the midst of proof production before turning more critically to trying to understand the nature of key idea.

THE EPISODE

The following example illustrates the presence and/or absence of these three moments as students work on the following task:

Let n be an integer. Prove that if $n \geq 3$ then $n^3 > (n+1)^2$.

Students were videotaped working on this task in the presence of the research team, and upon their completion, were asked questions about their thinking. Afterwards, the research team watched and discussed the videos. We were drawn to one part of the proof process that turned out to be a genuine mystery—an episode, near the beginning, in which the students generate what the faculty identify as a correct proof, but what the students, at least at some level, do not recognize as one.

Details: In the first two minutes of working on this task, the students made an observation that the professors identified as a key idea of the proof, namely that a cubic function grows faster than a quadratic. Rather than trying to formalize this idea, the students switched to an algebraic approach, what we label as a technical handle, to try to get to a proof. They wrote $n^3 > n^2 + 2n + 1$ which they manipulated into $n(n^2 - n - 2) > 1$ and then $(n-2)(n+1) > 1/n$.

The students then noticed that if $n \geq 3$ then the terms on the left are both positive integers so the product is a positive integer. And since n is an integer greater than 2, the right hand side is going to be between 0 and 1. They wrote these observations as

if $n \geq 3$ (line break) $n-2 > 0$ (line break) $n+1 > 0$ (line break) $0 \leq 1/n \leq 1$

its nature. The distinction between “key idea” and “technical handle” might appear at first sight to be similar to the distinction Steiner (1978) makes and Hanna (1989) builds on between proofs that demonstrate and proofs that explain. However, it is possible that a key idea gives rise to a proof that demonstrates or explains, and a technical handle can also lead to both kinds of proofs.

and seemed quite pleased with their reasoning, one student nodding and smiling as the other one wrote the last line.

S2: Yeah.

S1: This is if n is greater than 3, if n is greater than or equal to 3.

S2: Yeah.... Cool.

At this point in the live proof-writing, the three professors were convinced that the students had a proof. They believed that “all” the students needed was a reordering of their argument. To show $n^3 > n^2 + 2n + 1$, it suffices to show $(n-2)(n+1) > 1/n$, which one can establish by showing that the left-hand side is a positive integer while the right is between 0 and 1.

However, it turned out that the students, despite being pleased with their argument, were less than sure that they were near a formal proof. A professor asked the students “Is that a proof?” and S1 replied, “That’s what I’m trying to figure out.” As the students moved to now write up the proof, they switched to a new track, trying a proof by contraposition. This attempt ended up turning into a confusing case analysis in which they tried to prove the converse of the contrapositive and investigated many irrelevant cases.

AN EVOLVING EXPLANATION

That students can come so close to a proof without recognizing it is probably familiar to most experienced teachers⁴. *Why* the students are not able to recognize that they are so close is another, more difficult, question. Here we show how looking at the three “moments” of the proof, described above, allows us to compare what the students did in this problem with an idealized version of what faculty might have done.

The moments are represented graphically in Figure 1 below, with the blue line representing the “ideal” (professor-like) proving process, and the red line representing the students’ process⁵. The marks m_i indicate the points in the proof at

⁴ Another example can be found in Schoenfeld (1985) where two geometry students have what the researcher is convinced is a correct “proof” but when asked to write it up, they draw two columns and abandon all their previous work.

⁵ In creating this “idealized” version of a proof, we depict a continuity between the key idea and the technical handle, although we realize in practice that many proofs are made without the author being able to connect the two. The question about whether there exists such a connection, even if it has not been found, is an open one. We also realize that the process of proof development is not linear, even for an able mathematician, in many cases. This picture points out more the over-all trajectory of the proof, with minor false-paths ruled out. Further the heights of the peaks could vary.

which different moments are achieved: m_1 for the key idea, which both faculty and students achieved (though the students may not realize this), m_2 for the technical handle (which students in this case see as disconnected from their key idea), and m_3 for the organization of the key idea and/or technical handle into a clear, deductive argument (which in this case the students never reach.)

Specifically, m_1 is recognizing that cubic functions grow faster than quadratic ones. m_2 is choosing an algebraic approach, factoring the polynomials before and after the inequality sign. We label this as a technical handle even though the students do not know from the beginning where this might lead⁶. m_3 is connecting the assumption that $n \geq 3$ with the conclusion that $n^3 > (n+1)^2$. In this case, the students never reached m_3 , and in fact—during their attempt to write a formally accepted proof, they seem to lose sight of what they are proving.

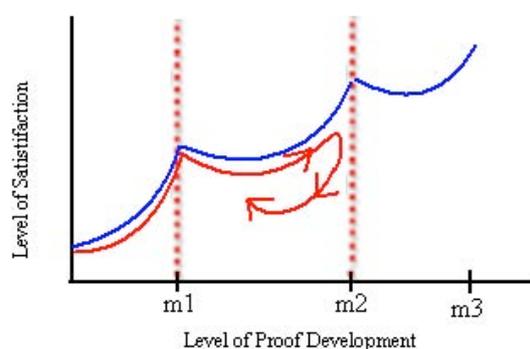


Figure 1: Comparing student (red) and faculty (blue) proof strategies

In the episode above, the students find two key ideas: one that cubics grow faster than quadratics, and another, after students have written $(n-2)(n+1) > 1/n$, that the right-hand term is trapped between 0 and 1 while the left grows indefinitely. Neither of these ideas gets developed into a formal proof. The curved line between m_1 and m_2 represents how students move towards a technical handle and end up at the second key idea.

⁶ The labeling of technical handle here is a bit tricky. If the students are not themselves aware of the way to link their algebraic manipulation to a proof, is it misleading to say they have found a technical handle since technically they do not seem to register that they “know” how to prove it. We have tentatively labeled this moment as a technical handle anyway, in part because as outside observers we can see that this algebraic manipulation could lead to a correct proof. In addition, while the students might not see exactly how to extract a formal proof from their algebraic arguments, they seem to take their arguments to be convincing and that they have grounds for making a formal argument.

The crucial distinctions between the “ideal” graph and the “student” graph are the breaks at m1 (students do not try to connect their key idea to a technical handle) and m2 (students lose sight of the conclusion and end up trying to prove a converse.) Our data indicate that these breaks are not merely cognitive—it isn’t that the students do not have the mathematical knowledge to write a proof, since they articulate the essence of the proof after three minutes. The problem is epistemological—they don’t seem to understand the geography of the terrain. Expecting discontinuity between a more intuitive argument and a more formal one, the students abandon their near-perfect proof for something that appears to them more acceptable as a formal proof.

Of course it is not always possible to connect key ideas to a technical handle, or to render a technical handle into a complete proof. But what distinguishes the faculty from the students is that the faculty are aware that this connection is possible, and might even be preferable given that sometimes it takes little work—in this case a simple reordering of the algebraic argument would suffice for a proof. As one professor in the study said:

“It became clear that to formalize meant something different to them and to us. To us, formalize seemed to mean ‘simply clean up the details’. To them, it seemed to mean ‘consider rules of logic and consciously use one’.”

Recognizing the difference between radical jumps that need to be made to move mathematical thinking forward and local jumps that allow one to delicately transform almost rigorous arguments into rigorous ones might be an essential difference that mathematics teachers can learn to recognize, diagnose, and communicate to their students.

FURTHER QUESTIONS

The episode and analysis described above, raise a number of questions which we would like to discuss briefly here.

1. Nature of key idea/technical handle

One nice feature of the episode above is that the identification of key idea and technical handle came fairly easy, with relatively little debate or discord among members of the research team. But are the notions of key idea and technical handle so clear that they can be picked out in any setting, for any proof? For this we need to continually refine the definitions (and in this paper we have actually backed away from a technical definition and given more general descriptions.) An ongoing research project of our team involves looking at a broad number of theorems, identifying key ideas and technical handles for different proofs, and refining the definitions based on that data.

2. Context of discovery vs. context of justification

The distinction between context of discovery and context of justification⁷, which has had a significant influence on epistemology and related fields, might be useful for understanding why students do not realize they have a proof. Taking the distinction to be psychological (which was not the original intent, but serves our purpose here), it seems natural to suggest that in the process of proving one has a phase of discovery and a phase of justification.

In the episode above, the students seem to be missing an important half of this combination. They sort of “discover” the key idea without seeing it as a justification⁸. Perhaps being able to toggle between the different contexts is a marker for mathematical maturity, and somehow central for being able to identify a proof as a proof. Specifically, the key idea might involve some combination of seeing the idea as a product of discovery and a grounds for justification (a thing to be justified). This is just a hypothesis, and a more careful analysis of the distinction between discovery/justification is needed to be able to substantiate it.

3. A Fregean telescope?

Another way of seeing the difference between student and faculty understandings in this episode might have to do with a deep connection (or lack of connection) between mathematical objects and the way they are grasped by the mind. This suggestion is highly tentative: to use Frege’s distinction between “sinn” (roughly, sense) and “bedeutung⁹” (roughly, reference) to better understand this relationship (Frege (1892/1997)).

Frege uses the following analogy to explain the difference between sinn and bedeutung: imagine a person looking at the moon through a telescope. The moon is a bedeutung, an object in the world, with a public status. The image on our retina is a sinn, the personal sense we have of that object, which has a private status. The telescope is sort of like a thought that connects the two—it has public status, in the sense that anyone can look through it, but it somehow makes an otherwise difficult to grasp object intelligible to the human mind.

Without going deeply into the way Frege extends this analogy to mathematics (in part because there are tricky moves, both going from the bedeutung of an *object* to the bedeutung of a *sentence*, and going from natural language to

⁷ For the original distinction see Reichenbach (1938), and for a critical discussion of this distinction in contemporary philosophy and history of science see Schickore & Steinle (2006).

⁸ Wright (2001) warns about misinterpreting the word “discover”. He points out that we would not say someone “discovered” the South Pole if they did not realize it was there. It is with this warning in mind that we use the term “discover” in quotation marks.

⁹ We retain the German names since the English translations are not completely accurate.

mathematical language) it might be useful to think if there is an analogy to the telescope in the episode described above.

Could it be that students stand facing some (to them) far away star, and with the aid of a telescope the public could be rendered private? If so, what would the telescope be, and is it something that we could better encourage students to develop and/or use as they learn to prove? Or is it possible that there is no telescope at all, just as when I look at the coffee mug on my desk, I feel I am simply getting sense data of the mug, without any mitigation. Perhaps the mind simply grasps key ideas. If so, then, what explains why some people grasp them and others don't?

This is perhaps merely a rephrasing of the central mystery found in the episode above. But by placing this mystery in a Fregean context (which also allows access to his critics), perhaps we gain some conceptual tools to try to better understand, not only the mystery, but also what we can do about it.

These questions mark a few of the places where we think it might be productive to push for a deeper analysis and where we see possibilities to connect to existing research. We are especially excited about the potential to use results from the field of epistemology where questions about the relation between mental representations and the external world (of which we consider mathematics to be a part) have been discussed extensively. In the next phase of our project, we plan to devote increasing time to developing and refining our theoretical ideas. We welcome any and all suggestions that can help us do so.

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“CAN A PROOF AND A COUNTEREXAMPLE COEXIST?”

A STUDY OF STUDENTS’ CONCEPTIONS ABOUT PROOF*

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Despite the importance of proof and refutation in students’ mathematical education, students’ conceptions about the relationship between proof and refutation have not been the explicit focus of research thus far. In this article, we investigate whether high-attaining secondary students have the misconception that it is possible to have a proof and a counterexample for the same mathematical statement. The data consisted of 57 student surveys augmented by follow-up interviews with 28 students. While analysis of the survey data alone offered considerable evidence for the existence of the misconception among several students in our sample, subsequent analysis with the inclusion of the interview data showed no evidence of the misconception. Implications for methodology and research are discussed in light of these findings.

INTRODUCTION

Despite the fundamental role that proof and refutation play in mathematical inquiry (e.g., Lakatos, 1976) and the growing appreciation of the importance of these concepts in students’ mathematical education (e.g., Lampert, 1992; Reid, 2002), students’ conceptions about the relationship between proof and refutation have not been the explicit focus of research thus far. The lack of research that aimed to investigate specifically students’ conceptions in this area creates a gap in the field’s understanding of how students perceive the standards of evidence in mathematics. Yet, existing research literature on proof and refutation allows us to make a hypothesis about students’ conceptions regarding the possible coexistence of a proof and a counterexample for the same statement.

Specifically, research studies identified two student conceptions whose combination gives rise to the hypothesis that some students believe that it is possible to have a proof and a counterexample for the statement. The first conception that some students have is that counterexamples do not really refute: students tend to treat valid counterexamples to general statements as exceptions that do not really affect the truth of the statements (Balacheff, 1988). The second conception that some students have is that proofs do not really prove: students have difficulties to understand that a valid proof confers universal truth of a general statement thus making further checks superfluous (Fischbein, 1982). However, we point out that the hypothesis that some

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students believe that a proof and a counterexample can coexist was derived by us considering findings from different studies that used different samples and methods and that were conducted in different cultural settings. So, the hypothesis is not attributed to any of those studies and should become the explicit focus of research.

In this article, we aim to contribute to this domain of research by reporting findings from an investigation of the possible existence of the aforementioned misconception among high-attaining secondary students. In this investigation, we used survey data from 57 students and follow-up interviews with 28 of them. With the interviews, we aimed to clarify some student responses to the survey and to test the tentative conclusions we had drawn from our analysis of the survey data.

BACKGROUND

The research was part of a *design experiment* (see, e.g., Schoenfeld, 2006) that was conducted in two Year 10 classes in a state school in England. The school had 165 Year 10 students (14 to 15 years old) who were set in seven classes according to their performance on a national assessment they took at the end of Year 9. A total of 61 students from the two highest attaining classes participated in the research.

Motivated in part by studies that showed that even high-attaining secondary students tend to have limited understanding of proof (Coe & Ruthven, 1994; Healy & Hoyles, 2000; Küchemann & Hoyles, 2001-03), the design experiment aimed to generate research knowledge about possible ways in which classroom instruction can help these students develop their understanding of proof. The design experiment involved development, implementation, and analysis of the effectiveness of a collection of lesson sequences that extended over one to three 45-minute periods. Each lesson sequence was intended to promote issues of proof in the context of mathematical topics and student learning goals that were consistent with the provisions of the English national curriculum, treating proof as a vehicle to mathematical sense making. As far as proof was concerned, the lesson sequences aimed to offer students opportunities to develop their understanding of the limitations of empirical arguments and of the importance of proof in mathematics, to construct proofs for true mathematical statements, and to formulate counterexamples for false mathematical statements. However, the issue of the possible coexistence of a proof and a counterexample for the same statement was not explicitly discussed in the classes.

The definition of proof that guided the work on proof within the two classes was an adapted version of the conceptualization of proof elaborated in Stylianides (2007, pp. 291-300). The following definition was used in the first lesson sequence in each of the two classes as part of students' introduction to the notion of proof.

An argument that counts as proof [in our class] should satisfy the following criteria:

1. It can be used to convince not only myself or a friend but also a sceptic. It should not require someone to make a leap of faith (e.g., "This is how it is" or "You need to believe me that this [pattern] will go on forever.")

2. It should help someone understand why a statement is true (e.g., why a pattern works the way it does).
3. It should use ideas that our class knows already or is able to understand (e.g., equations, pictures, diagrams).
4. It should contain no errors (e.g., in calculations).
5. It should be clearly presented.

The definition was discussed and referred to by both classes several times during the course of the design experiment, and it can be considered to reflect the classes' "idealized" shared understanding of the criteria for an argument to qualify as a proof.

METHOD

Data Sources

The data for the article are derived from: (1) 57 student responses to a survey that we administered to the two classes at the end of the third lesson sequence of the design experiment (some students were absent the day we administered the survey), and (2) follow-up interviews with 28 students. The students completed the survey part way through the design experiment, after they had been given learning opportunities to develop understanding of different issues related to proof as described previously.

Five cards have the odd numbers 1, 3, 5, 7 and 9 printed on one side, and the even numbers 2, 4, 6, 8 and 10 printed on the other side.

1	3	5	7	9
2	4	6	8	10
on the other side				

The cards are dropped on the floor and spread out.

Amina, Ben, Carol and Davor are discussing whether this statement is true:

When two of the visible numbers are even, the five visible numbers add up to 27.

Ben's answer

I tried all odd numbers first and got 25:

$$1 + 3 + 5 + 7 + 9 = 25.$$

If I change one odd number to an even number, the total will be 1 bigger.
So if I have two even numbers, the total will be 2 bigger.
So the total will be 27.

So Ben says it's true

Carol's answer

I wrote down these numbers:

1, 2, 3, 4, 9.

Two of the visible numbers are even but the total is 19. So you do not always get 27.

So Carol says it's not true

Figure 1: A mathematical problem and two sample solutions to the problem

Survey

The survey presented the students with a true statement contextualized in a mathematical problem, four sample solutions to the problem, and some open-ended and multiple-choice questions (figures 1 and 2); in this article we focus on students' evaluations of only two solutions (Ben's and Carol's).

Open-ended questions:			
1. Whose answer is closest to what you would do? Explain your answer.			
2. Whose answer would get the highest mark from your teacher? Explain your answer.			
3. Whose answer would get the lowest mark from your teacher? Explain your answer.			
Multiple-choice questions:			
For each of the following, circle whether you agree, don't know, or disagree.			
The statement is:			
When two of the visible numbers are even, the five visible numbers add up to 27.			
<i>Ben's answer ...</i>	agree	don't know	disagree
shows you that the statement is always true	1	2	3
only shows you that the statement is true for some examples	1	2	3
shows you why the statement is true	1	2	3
<i>Carol's answer ...</i>	agree	don't know	disagree
shows you that the statement is not true	1	2	3
shows you why the statement is not true	1	2	3

Figure 2: Open-ended and multiple-choice questions.

The survey derived from one used in the Longitudinal Proof Project (Küchemann & Hoyles, 2001-03; Technical Report for the Year 8 Survey, pp. 93-94). We added the third open-ended question and the probes inviting students to explain their answers. We hoped these additions would increase the survey's potential to reveal student thinking about the possible coexistence of a proof and a counterexample. While this issue does not seem to have been one that Küchemann and Hoyles aimed to explore (ibid, pp. 6-7), we thought the survey offered an excellent opportunity to do this: some students might not notice the (subtle) mistake in Carol's solution and consider it a valid counterexample to the statement, while at the same time recognize the value of Ben's deductive argument and consider it a proof for the statement.

Interviews

We interviewed 28 students based on their responses to the multiple-choice and open-ended questions in the survey. Most interview sessions began with us asking the

students to review their responses to the survey and then to explain which survey question they found the hardest. This general interview probe was followed by specific probes for the students to elaborate on particular responses in their scripts.

Procedure and Analysis

Patterns in students' responses were identified and used to formulate hypotheses about their conceptions. Interview data were then used to test/refine the hypotheses.

With regard to students' conceptions about the coexistence of a proof and a counterexample, our analysis of the survey data focused on those scripts that contained evidence to suggest the potential existence of the misconception. Specifically, we focused on the scripts that contained evidence of one or more of the following "inconsistencies": (1) the student found a mistake in Carol's solution and said that she would get the lowest mark from the teacher but agreed with the sentence that Carol's solution showed that the statement was not true; (2) the student said that the highest mark from the teacher would go to both Ben's and Carol's solutions; and (3) the student agreed both with the sentence that Ben's solution showed that the statement was always true and with the sentence that Carol's solution showed that the statement was not true.

We coded the type of evidence that was present in the scripts into two categories – strong or weak – depending on the degree of confidence that it gave us as researchers for the existence of the misconception. Specifically, we considered that strong evidence was offered by those scripts that had either "agree" or "don't know" in the first multiple-choice questions for both Ben's and Carol's solutions, and that included no relevant disconfirming evidence in the open-ended questions. The scripts that we considered offered weak evidence for the existence of the misconception had again either "agree" or "don't know" in the first multiple-choice questions for both Ben's and Carol's solutions, but included some relevant disconfirming evidence in the open-ended questions (e.g., they offered evidence that the student was aware that Carol's solution had a mistake in it). For each of the strong or weak evidence categories we used the interview data to examine the extent to which there was, overall, evidence to suggest that the students actually had the misconception. Also, we used the interview data to seek possible explanations (from the students' point of view) for the "inconsistencies" that we identified in their scripts.

RESULTS

General Findings

Our analysis of the survey scripts showed that 16 out of the 28 students interviewed exhibited some evidence to suggest the existence of the misconception that a proof and a counterexample can coexist. Of these, ten scripts showed strong evidence and six showed weak evidence for the misconception. Our subsequent analysis of the

interview data revealed that the students in each group (i.e., strong or weak evidence group) tended to offer similar justifications for their choices.

Regarding the strong evidence group, our interview data suggested that the inconsistencies in students' responses derived from them considering Ben's and Carol's solutions in isolation from one another when they were completing the survey. While discussing their responses with the interviewers, however, all the students in this group became aware of the potential inconsistency between their evaluations of Ben's and Carol's solutions, presumably because the interviewers' questions directed (implicitly or explicitly) students' attention to the relationship between their evaluations. Yet, the manner in which the students became aware of this inconsistency and how the awareness played out in the interviews varied.

On the one hand, some students realized the mistake in Carol's solution without any prompting from the interviewers and immediately dismissed her solution. As a result of this dismissal, there was no opportunity for the interviewers to explore further whether these students would experience any sense of conflict that a proof and a counterexample can coexist. On the other hand, some other students needed explicit prompting from the interviewers to reflect on whether or how their evaluations of Ben's and Carol's solutions fitted together before they appreciated the potential inconsistency between these evaluations. Believing that Carol had found a genuine counterexample, these students attempted to resolve the emerging conflict by assuming there was a flaw in Ben's argument, which however they were unable to identify. The interviewers then helped these students see the mistake in Carol's solution and realize it was not a genuine counterexample. As a result of this realization, the students subsequently rejected Carol's solution, but this rejection was not always accompanied with endorsement of Ben's solution as a proof.

Regarding the weak evidence group, our interview data suggested that the students in this group seemed to be aware of the following 'inconsistency' we identified in their scripts: the students pointed out the mistake in Carol's solution in their response to the open-ended questions, but in the first multiple-choice question for Carol's solution they agreed that the solution showed the statement was not true.

During the interviews, the students argued, with different degrees of clarity, that, in spite of the mistake in Carol's solution, her reasoning should be valued because her logic was correct and she had disproved a statement, albeit a different one from that in the problem. Consequently, none of these students changed their minds about their evaluations of Carol's solution during the interview. The issue of the misconception was not pursued further by the interviewers, as the students were already aware that Carol's argument was not a counterexample to the particular statement.

To sum up, there is no evidence from our interviews to suggest that any of the 16 students we originally identified as potentially having the misconception actually had it. Furthermore, the interview data showed that any potential conclusions that could be drawn from the survey data alone would be insecure, as students appeared to have

good reasons for ‘inconsistencies’ we identified in their scripts. For this reason we do not report findings with students we did not interview.

Illustrative Case 1: The Case of Emily

The first case is of a student we call Emily, whose responses to the survey showed strong evidence of the misconception. Emily’s script had “agree” in the first multiple-choice question for Carol’s solution and “don’t know” in the corresponding question for Ben’s solution. Furthermore, in response to the second open-ended question, Emily wrote that both Ben and Carol would receive the highest marks from the teacher and justified her thinking as follows:

Ben: It [Ben’s solution] is carefully thought out and written down in an understandable and clear manner.

Carol: She has shown when it [the statement] is not true.

During the interview Emily explained her thinking about Carol’s solution as follows:

The question was saying [that] when two of them [the visible numbers on the cards] were even that the answer is always 27, but she proved that it’s not, so she answered the question that was being asked.

In regard to Ben’s solution, Emily said:

It [Ben’s answer] was very, like, well set out and easy to understand and I think that was how I would have done it cause the other answers are like gabbling on a bit and they don’t really explain why it’s [the statement is] true or false.

She explained further that her “don’t know” response in the first multiple-choice question for Ben’s solution was because Ben “didn’t show that it’s always true, he only showed it for some numbers.” When asked whether she thought Ben had a proof, Emily said that Ben “needed to maybe expand it [his solution] a bit more to convince people that it was true” and noted that Ben could come up with a proof if he worked a bit harder on his solution.

After summarizing what Emily said about the two arguments, the interviewer asked Emily how her two evaluations fitted together. Realizing the inconsistency between the evaluations, Emily laughed and said: “they don’t [fit together] because Carol’s proved that it’s wrong and so it’s impossible to prove that it’s true... cause it’s not true!” Asked what she thought was going on with the two arguments, Emily asserted:

They [Ben and Carol] have both tried different ways and got different answers, so if they kept working at it, if Ben kept working on his [solution], he would eventually figure out that it’s not true.

The interviewer then helped Emily to see the mistake in Carol’s solution. Once Emily realized the mistake, she exclaimed: “Oh, so she [Carol] could be wrong... so hers is wrong then.” On reviewing her original responses to the multiple-choice questions for Carol’s solution, Emily decided to change her response to the first question from “agree” to “disagree,” because, as she said, Carol “hasn’t followed the instruction.”

Emily concluded that Ben's solution "might be true" but she decided not to change her responses to the multiple-choice questions for his solution.

Illustrative Case 2: The Case of Evans

The second case is of a student we call Evans, whose responses to the survey showed weak evidence of the misconception. Evans' script had "agree" in the first multiple-choice questions for both Ben's and Carol's solutions, an indication of the existence of the misconception. Furthermore, Evans' responses to the first two open-ended questions showed particular appreciation of Ben's solution: he wrote that Ben's solution would be close to what he would do and that the solution would get the highest mark from his teacher "[b]ecause [it] shows working and offers convincing proof." Yet Evans' response to the third open-ended question offered disconfirming evidence of the existence of the misconception as it indicated that he was aware of the mistake in Carol's solution and said that Carol's answer would get the lowest mark from the teacher. In a series of two interviews, we tried to understand the reasoning for the apparent contradiction in Evans' evaluation of Carol's solution.

Evans was aware that Carol's solution had a mistake in it, but on the basis that she applied a correct mathematical method and that this application warranted recognition, he consciously agreed that she had shown the statement was not true.

Well what she [Carol] has done is like impossible because 1 and 2 can't be seen at the same time, so then I would have disagreed because that can't be true. But seeing as though she has shown that she's thought it through and like, with her own reasoning she's come to an answer, then I would have put she technically has [shown the statement is not true] but she's got it wrong. [...] Carol tried to prove the statement wrong, so one counterexample was enough. She had the logic right but she didn't succeed to come up with a correct counterexample.

This interview excerpt shows that Evans evaluated Carol's solution from her own point of view and that he understood the fundamental idea that a single counterexample suffices to refute a general statement. Evans considered that Carol's solution embodied understanding of the latter idea, even though the counterexample she offered did not satisfy, as he observed, the problem's conditions.

When pressed by the interviewer to explain his thinking further, Evans described the different evaluation standards that he perceived existed in exams and in class work:

In an exam you don't get marks for the proof, do you? You get marks for showing your working and actually getting the answer in the end. But it [Carol's solution] does show the proof and everything. I don't know, it depends on what sort of question it is... if it's like what we're doing proof and stuff [referring to the proof work in class] then that [Carol's solution] would probably get the highest mark if that was what it was marked on... but in the exam it would be marked differently because it's not about how you are thinking, it's about getting the answer and getting the working and everything right.

The interviewer did not raise explicitly the issue of the possible coexistence of a proof and a counterexample, as Evans was clearly aware that Carol's argument was not a valid counterexample to the particular statement in the problem.

DISCUSSION

Although our analysis of the survey data alone offered considerable evidence (both weak and strong) for the existence of the misconception that a proof and a counterexample can coexist, our subsequent analysis with the inclusion of the interview data showed no evidence of the misconception. The size of the mismatch between the findings of the two analyses might have been influenced by what we considered as evidence for the possible existence of the misconception in our analysis of the survey data. Nevertheless, the existence of the mismatch reinforces and exemplifies the point that student responses to surveys may, by themselves, offer a rather limited insight into students' conceptions and that follow-up interviews with selected students are important for the construction of a more trustworthy picture of students' conceptions.

The latter statement is more than a reiteration of the well known methodological principle that triangulation of multiple data sources allows the examination of research questions in more nuanced ways than when using a single data source. The statement is also a cautionary remark that conclusions about students' conceptions that are based only on analysis of students' responses to surveys may be seriously misleading. This should not be taken as a criticism of the use of surveys in examining educational issues in general, but rather as a concern that the complexity that surrounds the particular issue of students' conceptions about multifaceted mathematical ideas may not be possible to be illuminated satisfactorily on the basis only of survey data. Of course this is not a black and white situation. The extent to which survey data alone can help illuminate complex issues depends on several factors: the methods that were used to validate a survey, the kinds of questions included in the survey, the conditions under which the survey was administered, the coding scheme used to analyse the survey data, etc.

In spite of the limitations in the conclusions that could be drawn based on the survey data alone, the survey offered a meaningful context in which we discussed during our interviews with students their ideas about the possible coexistence of a proof and a counterexample. This discussion was done with reference to Ben's deductive argument, which could be considered a proof, and Carol's purported counterexample. Carol's argument worked particularly well for the purposes of our research, as the subtle mistake in it passed unnoticed by several students, thereby helping us meet the challenge of presenting the students with a believable "counterexample" to a true statement. Ben's argument did not work as well as Carol's argument: students like Emily recognised the value of Ben's argument, but they did not accept it as a proof, primarily because they thought it needed "unpacking." The fact that some students did not consider that Ben's argument qualified as a proof gave them an "easy" way to

resolve the problematic situation regarding the possible coexistence of a proof and a counterexample: these students suspected a mistake in Ben's argument and thus felt less hesitant to endorse Emily's counterexample. Given that the statement in the problem was true, it would not be difficult to strengthen Ben's argument in the survey so that more students would accept it as a proof; this modification in the survey would increase its potential to elicit students' conceptions about the possible coexistence of a proof and a counterexample.

Future research on students' conceptions in this area can use this modified version of the survey. Also, it would be useful if future research used an additional problem that asked students to evaluate a valid counterexample and a believable "proof" for a *false* statement. This would complement our examination in this study, thus contributing to the development of a more comprehensive approach to eliciting students' conceptions about the possibility of having a counterexample and a proof for the same statement. The fact that our research did not reveal this misconception does not mean that there are no students who have it; less advanced students, younger students, or students with fewer experiences with proof are more likely to have the misconception than the students who participated in our research.

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ABDUCTION AND THE EXPLANATION OF ANOMALIES: THE CASE OF PROOF BY CONTRADICTION[^]

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Some difficulties with proof by contradiction seem to be overcome when students spontaneously produce indirect argumentation. In this paper, we explore this issue and discuss some differences between indirect argumentation and proof by contradiction. We will highlight how an abductive process, involved in generating some indirect argumentation, can have an important role in explaining the absurd proposition, in filling the gap between the final contradiction and the statement to be proved and in the treatment of impossible mathematical objects.

Key words: proof, argumentation, abduction, proof by contradiction, indirect argumentation.

INTRODUCTION

The relationship between argumentation and proof constitutes a main issue in mathematics education. Research studies have been based on different theoretical assumptions, proposing different approaches and consequently different didactical implications (Mariotti, 2006). In some studies (see, for example, Duval, 1992-93), a distance between argumentation and proof is claimed, while in others, without forgetting the differences, the focus is put on the analogies between the two processes and their possible didactical implications (Garuti, Boero & Lemut, 1998; Garuti & al., 1996). As a consequence, the authors hold the importance for students to deal with generating conjectures, and highlight that this activity can promote some processes that are relevant in developing students' competences in mathematical proof.

Elaborating on this first hypothesis, concerning the continuity between the argumentation supporting the formulation of a conjecture and the proof subsequently produced, Pedemonte (2002) developed the theoretical construct of *Cognitive Unity* in order to describe the relationship (continuity or break) between the argumentation process and the related mathematical proof in the activity of conjecture's production.

In this paper, we aim to investigate the relationships between argumentation and proof in the case of proof by contradiction. The reference to the framework of *Cognitive Unity* is of the interest for this study for the following reason. Although important difficulties have been identified in relation to this type of proof (see Antonini & Mariotti, 2008; 2007; Mariotti & Antonini, 2006; Antonini, 2004;

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Stylianides, Stylianides & Philippou, 2004; Wy Yu, Lin & Lee, 2003; Thompson, 1996; Leron 1985), in the literature we find evidence of arguments, spontaneously produced by students, that can be considered very close to proof by contradiction (see Antonini 2003; Reid & Dobbin, 1998; Thompson, 1996; Freudenthal, 1973; Polya, 1945). In fact, as reported by Freudenthal:

“The indirect proof is a very common activity (‘Peter is at home since otherwise the door would not be locked’). A child who is left to himself with a problem, starts to reason spontaneously ‘... if it were not so, it would happen that...’ “ (Freudenthal, 1973, p. 629)

We call *indirect arguments* the arguments of the form ‘*if it were not so, it would happen that...*’. Indirect arguments seem to be more like to appear in the solution of open-ended problems, as a natural way of thinking in generating conjectures, when one needs to convince oneself that a statement is true, or to understand because a statement is true.

Therefore, it seems important to study differences and analogies between proof by contradiction and indirect argumentation, and this is what we are going to do in the following sections.

DIFFICULTIES WITH PROOF BY CONTRADICTION

According with the terminology of the model presented in (Antonini & Mariotti, 2008, 2007), given a statement *S*, that we called a *principal statement*, a proof by contradiction consists in a couple of proofs: a direct proof of another statement *S**, that we call the *secondary statement*, in which the hypotheses contain the negation of *S* and the thesis is a contradiction (or a part of it); and a *meta-theorem* stating the logical equivalence between the two statements, the *principal* and the *secondary*. Here, we analyse two aspects and their relationships: the link between the *principal statement* and the contradiction achieved through the proof of the *secondary statement*; the treatment of impossible mathematical objects in both the argumentation and the proof.

The link between the contradiction and the *principal statement*

The link between the final contradiction and the principal statement is a source of difficulties for students (see Antonini & Mariotti, 2008). It can happen that such difficulties are openly shown when they appear astonished and disoriented after the deduction of an absurd proposition. This is the case for example of Fabio, a university student (last year of the degree in Physics), who explains very well this type of difficulty:

Fabio: Yes, there are two gaps, an initial gap and a final gap. Neither does the initial gap is comfortable: why do I have to start from something that is not? [...] However, the final gap is the worst, [...] it is a logical gap, an act of faith that I must do, a sacrifice I make. The gaps, the sacrifices, if they are small I can do them, when they all add up they are too big. My whole argument converges towards the sacrifice of the logical jump of exclusion, absurdity or exclusion... what is not, not the direct thing. Everything is fine,

but when I have to link back... [Italian: “Tutto il mio discorso converge verso il sacrificio del salto logico dell’esclusione, assurdo o esclusione... ciò che non è, non la cosa diretta. Va tutto bene, ma quando mi devo ricollegare...”]

Fabio identifies two gaps (he speaks also of a “*jump*”!) in a proof by contradiction: an initial gap and a final gap. According to our model, the initial gap corresponds to the transition from the statement S to the proof of S^* , and the final gap corresponds to the opposite move, from the proof of S^* to the conclusion that S is proved. The perception of these gaps makes Fabio feel unsatisfied, as if something were missing. In fact, he can accept the proof but he is not convinced, as he says it is “*an act of faith that must be done*”.

The treatment of impossible mathematical objects

It may happen that, at the beginning of a proof by contradiction, some of the mathematical objects have some characteristics that are absurd and strange, in an evident way. These mathematical objects are proved to be impossible in some theory. For this reasons, difficulties can emerge in the treatment of these absurd objects. As discussed in (Antonini & Mariotti, 2008; Mariotti & Antonini, 2006) difficulties may occur in the construction of the proof of S^* , but difficulties may also emerge after the proof of S^* is achieved, when absurd objects have to be discarded. In fact, at the end of a proof of S^* , once a contradiction is deduced, one has to realize that some of the objects involved do not exist; actually, they have never existed. As explained by Leron:

“In indirect proofs [...] something strange happens to the ‘reality’ of these objects. We begin the proof with a declaration that we are about to enter a false, impossible world, and all our subsequent efforts are directed towards ‘destroying’ this world, proving it is indeed false and impossible. We are thus involved in an act of mathematical destruction, not construction. Formally, we must be satisfied that the contradiction has indeed established the truth of the theorem (having falsified its negation), but psychologically, many questions remain unanswered. What have we really proved in the end? What about the beautiful constructions we built while living for a while in this false world? Are we to discard them completely? And what about the mental reality we have temporarily created? I think this is one source of frustration, of the feeling that we have been cheated, that nothing has been really proved, that it is merely some sort of a trick - a sorcery - that has been played on us.” (Leron, 1985, p. 323).

Our research interest is in exploring whether and how these difficulties may be overcome when students spontaneously produce indirect argumentation. Two elements seem important to take into account: on the one hand the indirect argumentation as a product and its differences with a proof by contradiction, on the other hand the processes involved in producing the argumentation (see also Antonini, 2008). In this paper we focus on the hypotheses that in many cases the students try to fill the gap between the contradiction and the statement in order to re-establish a link

and at the same time to give a new meaning to the “objects of the *impossible world*”, so that they can be modified without being discarded.

THE ABDUCTIVE PROCESS

Abduction is one of the main creative processes in scientific activities (Peirce, 1960). Magnani defines abduction as

“the process of inferring certain facts and/or laws and hypotheses that render some sentences plausible, that explain or discover some (eventually new) phenomenon or observation; it is the process of reasoning in which explanatory hypotheses are formed and evaluated” (Magnani, 2001, pp. 17-18).

The main characteristic of abduction is that of deriving a new statement that has the power of enlightening the relationship between the observation and what is known.

Many studies in mathematics education have dealt with abductive processes in students thinking: in problem-solving activities (Cifarelli, 1999), in generation of conjectures (Ferrando, 2005; Arzarello et al., 2002; Arzarello et al., 1998), argumentation and proofs (Pedemonte, 2007).

In this paper, through the analysis of a case study, we will show how an abductive process could assume a fundamental role in the production of indirect argumentation. Through an abduction a new statement is produced that has no logical need but allows one to make sense of the absurd and strange proposition and, in this way, to overcome the gap between the contradiction and the *principal statement*.

A CASE STUDY

The following open-ended problem was proposed to Paolo and Riccardo (grade 13), two students that, according to the evaluation of their teachers, are high achievers.

Problem: What can you say about the angle formed by two angle-bisectors in a triangle?

What follows is an excerpt of their interview. After a phase of exploration, the students generated the conjecture that the angle S (figure 1) is obtuse. Afterwards, the students started to explore the possibility that the angle S might be a right angle.

61 P: As far as 90, it would be necessary that both K and H are 90 degrees, then $K/2 = 45$, $H/2 = 45$...180 minus 90 and 90 degrees.

62 I: In fact, it is sufficient that the sum is 90 degrees, that $K/2 + H/2 = 90$.

63 R: Yes, but it cannot be.

64 P: Yes, but it would mean that $K+H$ is ... a square [...]

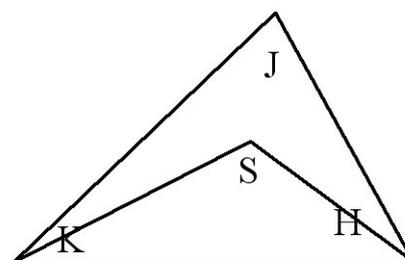


Figure 1: The angle between two angle bisectors in a triangle.

- 65 R: It surely should be a square, or a parallelogram
- 66 P: $(K-H)/2$ would mean that [...] $K+H$ is 180 degrees...
- 67 R: It would be impossible. Exactly, I would have with these two angles already 180, that surely it is not a triangle.
- [...]
- 80 R: [the angle] is not 90 degrees because I would have a quadrilateral, in fact the sum of the two angles would be already 180, without the third angle. Then the only possible case is that I have a quadrilateral, that is, the sum of the angles is 360.

The episode can be subdivided in three parts: the development of a first argumentation (61-63), the introduction of a new figure, the parallelogram (64-67), the production of the final argumentation (80). This last argumentation is expressed by Riccardo, after the students are explicitly asked to write a mathematical proof.

The argumentation developed in the first part (61-63) is indirect: assuming that the angle between two angle bisectors of a triangle is a right angle, the students deduce a proposition that contradicts a well known theorem of Euclidean Geometry. From the logical point of view, the deduction of the contradiction would be sufficient to prove that this triangle does not exist, or, equivalently, that the angle S is not right, thus concluding the argumentation. But, although convinced that the angle S cannot be a right angle, the students do not feel that the argumentation is concluded and they look for an explanation for the anomalous situation. In fact, the subsequent part (64-67) seems to have the goal to complete the argumentation; in particular, the students seem to look for an explanation to the false proposition " $K+H=180^\circ$ ". An explanation is found by formulating a new hypothesis: the figure is not a triangle, it is a parallelogram. In this case, it is true that the sum of two adjacent angles ($K+H$) is 180. In search of an explanation the original triangle fades becoming for the students an indeterminate figure that have to be determined in order to eliminate the anomalous consequences. In 67, Riccardo makes clear that the figure can be transformed during the argumentation. His expression "*surely it is not a triangle*" means "this figure is not a triangle" and it must be something else. Differently, in a proof by contradiction, as the proof that could arise from the first part of the argumentation (61-64), the figure is well determined, it is a triangle and it is not possible to modify it. Once a contradiction is deduced, it is proved that this figure does not exist. In this case, the triangle would be part of the "false, impossible world" and it would have had a temporarily role: at the end of the proof we know that it does not exist. Actually, it has never existed.

When the new case is selected and because this new case can solve the anomaly, Paolo and Riccardo seem to be satisfied. In 80, Riccardo summarizes the argumentation in what for him is a mathematical proof. The fact that the angle S is not right is not proved by contradiction but is based on the analysis of different cases: triangle, square, parallelogram. The figure is determined, and it is not a triangle, as

we have thought at the beginning of the argumentation. This argument seems very convincing for students, more than the argument based on deriving a contradiction.

The key point in the development of the argumentation is the generation of the new case that is the identification of the parallelogram. This process can be classified as an abduction, in fact an explanatory hypothesis is produced and evaluated, as Riccardo says “[the quadrilateral] it is the only possible case”.

The assumption of the parallelogram transforms a false into a true proposition. This argument allows students to overcome some of the difficulties that might be raised by a proof by contradiction (figure 2). In particular:

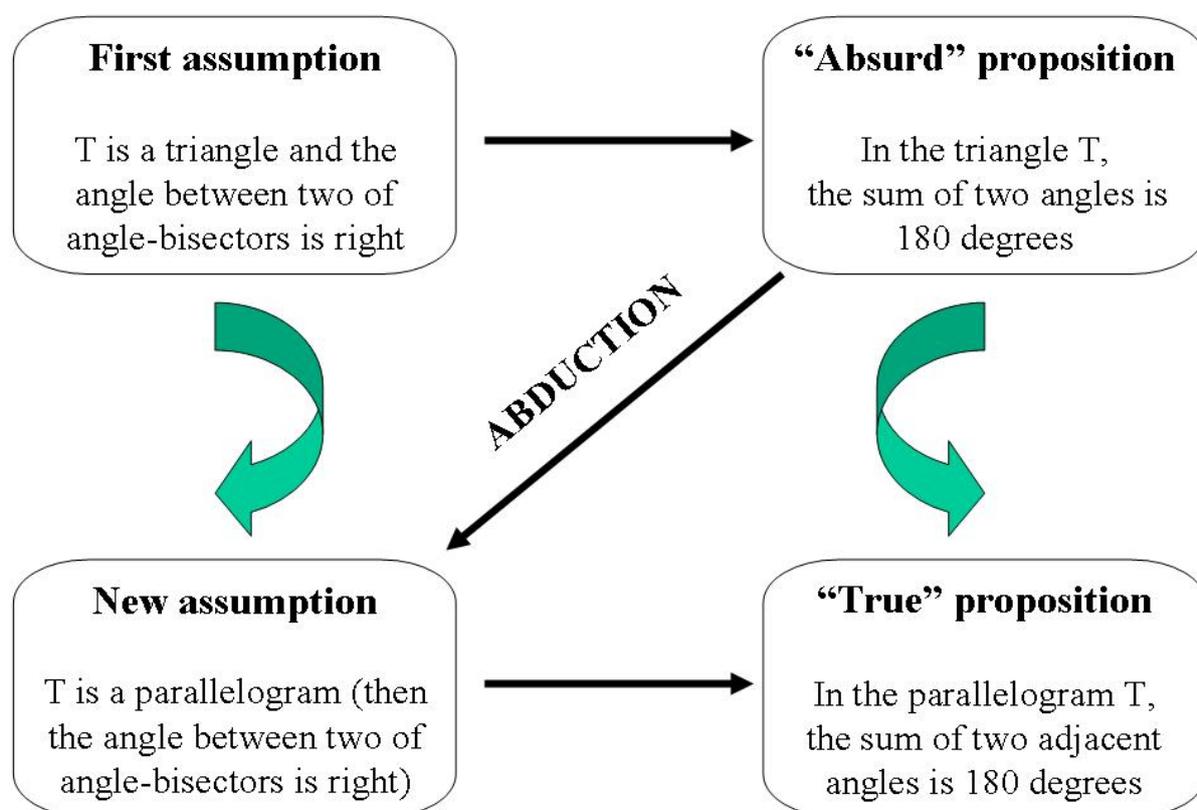


Figure 2: An abductive process in an indirect argumentation

- The false proposition - “in a triangle the sum of two angles is 180° ” – becomes a true proposition related to the new explanatory hypothesis (in a parallelogram the sum of adjacent angles is 180°);
- The mathematical object (the triangle) is considered an indeterminate object that is identified only through the abduction with the goal to explain the anomaly. Then the mathematical object is changed and not discarded as it happens in a proof by contradiction. The problem of treatment of mathematical object at the end of proof by contradiction highlighted by Leron (1985) is bypassed.

- Differently to what happens with proof by contradiction, a link, that is not only logical, between the *secondary statement* and the *principal statement*, is constructed: it is not possible that S is right because otherwise the triangle would become a quadrilateral.

As the previous example shows, in geometry, the identification of the case that can explain the anomaly and allow getting out of the “impossible world” seems to be related to the transformation of figures. Most of the students asked to solve the problem of angle bisectors provided arguments based on transforming the triangle in a quadrilateral or in two parallel lines.

Further researches are necessary to corroborate this hypothesis and investigate whether it can be extended to other context. In fact we hypothesize that also in contexts other than Geometry abduction can be for students the key to come out from the anomalous situation that occurs in proof by contradiction. In order to support this extension to other contexts, we report now a short episode concerning the algebra domain.

ABDUCTION AND PROOF BY CONTRADICTION IN ALGEBRA: AN EXAMPLE

In a questionnaire proposed to 68 secondary school students (grade 10, 11, 12) and 19 university students, a proof by contradiction of the incommensurability of the diagonal of a square with its side was presented. We aimed to investigate the recognition and the acceptability of this type of proof. In the presented proof, it is assumed that the ratio is a rational number, expressed by the fraction m/n where m and n are two natural numbers (with n different from 0). Then it is deduced that the number n is both odd and even. The students were asked to choose one of the following answers to explain what it is possible to conclude:

- This is not a proof*
- There is a mistake in some passages, but I can not identify it*
- There is a mistake, that is (specify the error):*
- We have not proved anything, because being even or odd has nothing to do with what we wanted to prove*
- We have proved what we wanted, in fact:.....*
- Other (specify):*

The 25 per cent of the students gave the correct answers and the 35 per cent chose the answer d). This expresses the feeling that something is missing and let us suppose the need to see a link between the contradiction and the statement. A hint in this direction comes from one of the answers. One student (grade 12) marked the correct answer and explained:

“we have proved what we wanted in fact one of the two numbers [the number n] is not a

natural number and then the ratio is not a ratio between two natural numbers”

The argument provided does not refer to what could be recognized as the *meta-theorem*, explaining the logical equivalence between the *principal statement* and the *secondary statement*, and thus rejecting the existence of the mathematical object m/n .

Differently, this student does not reject the initial assumption that the ratio is rational from the contradiction “ n is even and odd”, rather he changes the nature of the number n coherently (in his opinion) with the deduced proposition. If m/n is not a rational number, as we have believed before, everything is explained.

Inferring the explaining hypothesis that number n , odd and even at the same time, is not a natural number is the product of an abduction. The hypothesis that n is not a natural number can explain the anomaly “ n is odd and even” and, at the same time, it offers a link between the deduced proposition and the principal statement: n is not a natural number and then the ratio m/n is not a rational number. A link between the contradiction and the statement is now established and the proof can be accepted.

CONCLUSIONS

Main difficulties with proof by contradiction are related to the link between the contradiction and the statement to be proved, to the treatment of the impossible mathematical objects during the construction of the proof and at the end, to the need of discarding the mathematical objects involved in the proof of the *secondary statement*. The feeling of frustration that may emerge at the end of a proof by contradiction, as clearly expressed by Fabio’s words, is accompanied by the need of giving a meaning to the absurd proposition, the need of establishing a stronger link with the principal statement and adjusting the “false, impossible world”.

The analysis of the episodes proposed above shows how abductive processes may be mobilized to produce explanatory hypotheses. The system of relationships represented in the diagram of figure 2 shows the key role of the abductive process and highlights some differences between indirect argumentation and proof by contradiction.

Interpreting these results in terms of Cognitive Unity leads us to point out the distance between indirect argumentation as it is spontaneously developed and the scheme of a proof by contradiction. In particular, it clearly appears the distance between the *meta-theorem* - providing the equivalence between the *principal* and the *secondary statement* - and the abductive process that might emerge in an indirect argumentation. The question rises whether and how such distance can be filled through an appropriate didactical intervention.

Of course, further investigation is necessary to better understand the differences between argumentation and proof by contradiction and to identify and analyse other processes that could be important for the production and the development of indirect argumentation.

We think that the comprehension of these processes is fundamental for teachers to identify, explain and treat students' difficulties with proof. We also believe that indirect argumentation, even if it presents significant differences with proof by contradiction, should be promoted, in particular through open-ended tasks. As Thompson writes:

“If such indirect proofs are encouraged and handled informally, then when students study the topic more formally, teachers will be in a position to develop links between this informal language and the more formal indirect-proof structure.” (Thompson 1996, p.480)

As regards the transition from the argumentation to proof by contradiction, further researches are necessary to identify the tools to construct the didactical activity to face the gaps and promote the acceptability of method of proof by contradiction.

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APPROACHING PROOF IN SCHOOL: FROM GUIDED CONJECTURING AND PROVING TO A STORY OF PROOF CONSTRUCTION

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This paper presents some aspects of an ongoing research aimed at leading students (through activities of conjecturing, guided construction of proof and story making of the rationale of the proof) to become aware of some salient features of proving and theorems. Theoretical elaboration as well as an example of didactic engineering concerning Pythagoras' theorem will be outlined.

I INTRODUCTION

School approach to theorems has been a subject of major concern for mathematics education in the last two decades. Students' learning to produce proofs and their understanding of what does proof consist in (Balacheff, 1987) have been considered under different perspectives and with different aims: among them, how to make the students aware of the differences between proof and ordinary argumentation (Duval, 1991, 2007); how to favour students' access to the theoretical character of proof (M.A.Mariotti, 2000); how to exploit "cognitive unity" (which for some theorems allows students to exploit the arguments they produced in the conjecturing phase to construct the proof) in order to smooth the school approach to theorems (Boero, Garuti & Lemut, 2007); in what cases of cognitive unity do students meet difficulties in the passage from an inductive or abductive reasoning, to the deductive organization of arguments (lack of structural continuity: Pedemonte, 2007, 2008); what are the common aspects between ordinary argumentation and proving, and how to prepare students to proving by relying on those aspects (Douek, 1999a, 1999b; Boero, Douek & Ferrari, 2008). Previous research work helps us to formulate and situate some educational problems that arise in the school approach to theorems: how to tackle theorems for which cognitive unity does not work, or (if cognitive unity can work) when students meet important difficulties due to the lack of structural continuity? How to make the students aware of some salient characters of proving and proof? And how to lead them into some specific competencies of proving activity? In this paper we propose a possible way to tackle these problems in an integrated way. The idea is to guide students' constructive work on proving, then to help them focusing on the characteristics of the organisation of proof.

This paper presents a theoretical and pragmatic elaboration about how to deal with theorems for which cognitive unity does not work, and approach the rationale of a proof at first stages of proof teaching and learning. The theoretical elaboration also frames the accompaniment of students through two aspects of proving activity: exploration (in order either to find a statement, or to find reasons for validity of a statement); and organisation of reasons (or arguments), in the perspectives of

producing a proof, or of understanding the links between statements or arguments in a proof. Our hypothesis is that the rationale of a proof can be approached early in the school context through “story making” situations, preceded by suitable activities of conjecturing and guided proof construction, and related classroom discussions. We provide an example of such a didactical engineering concerning Pythagoras' theorem.

II FRAMING PROOF CONSTRUCTION

Inspired by Lolli's analysis of proof production (see Arzarello, 2007), we consider proving as a cognitive, culturally situated activity engaging four modes of reasoning:

1) Heuristic exploration. It occurs when one tries to interpret a proposition or to produce a proposition or an example. One has in mind a target but the main focus is not on attaining the target through an acceptable mathematical reasoning. Any accidental event, writing, metaphor, may move the exploration activity. This type of reasoning is typically open to divergent paths.

2) Organisation of reasoning, making explicit the threads of reasoning holding propositions together. When a proposition seems pertinent, a calculation promising, a writing efficient, one searches for a convincing coherent link to a local goal or to the global one. The links may be theoretical reasons of validity. The intentional and planning characters are typical of this mode, and abduction is a good example of it. Deductive reasoning is not yet a priority. Such organisational intention may concern partial arguments or the whole of the argumentation aimed at proof construction.

3) Production of a deductive text following mathematicians' norms. Once ideas of proof are brought to light, they must be organised in a deductive reasoning.

4) Formal structuring of the text, to approach a formal derivation. This mode will not at all be approached in the school context we are considering.

These four modes could be considered as successive phases of a proof construction, as different moments with different intentions. But in fact, as reasoning modes, they seldom do appear separately. Not only the succession of modes can vary and loop, but even two or more of them may intervene very closely in one phase aiming mostly at exploring or at writing a deductive text, for instance.

Methodologically, the consideration of these phases based on a cognitive analysis in terms of the four modes offers didactical tools to organise teaching-learning situations into sequences with clear didactical goals. As we refer to phases of predominant modes of reasoning, a didactical goal can be to lead the students to be aware of the processes they have to go through within a specific phase, essentially to favour students openness in exploration and their rational control in organising reasoning. But no exploration is blind nor any reasoning organising is totally controlled: when we analyse a phase of exploration activity, we ought to capture some reasoning organising activity, etc... (see sequence 1, in V).

The different modes and phases of reasoning involve several cultural rules of validity,

and they affect the delicate game of changes from what is allowed or even needed for one mode, to what is allowed or needed for another. For instance abductive reasoning is typical of mode 2 but is not allowed in modes 3 and 4, and student will have to move from it towards deductive reasoning, which is not easy. We can also consider the use of examples (pertinent in mode 1 and 2 but not acceptable in modes 3 and 4), and the conscious handling and conversion of different semiotic registers according to different modes of reasoning (Morselli, 2007; Boero, Douek & Ferrari, 2008).

This analysis leads us to give a special role to argumentation both as an intrinsic component of reasoning, and as a didactical tool to manage the different modes of reasoning and the relationships between them in a conscious way, keeping into account specific cultural rules (to be mediated by the teacher).

III ARGUMENTATION IN PROOF AND PROVING

In this paper, an "Argument" will be "A reason or reasons offered for or against a proposition, opinion or measure" (Webster), including verbal arguments, numerical data, drawings, etc. An "Argumentation" consists of one or more logically connected "arguments". Proof itself is an argumentation. But other argumentations play an important role in proving. Mode 2 is specially based on argumentative activity: discussing the use of a theory or a mathematical frame to produce a step of reasoning relies on a meta-mathematical argumentation (Morselli, 2007). It is not really part of a proof, but is needed to produce it. Analogies may implicitly affect mode 1 reasoning or be explicit arguments in mode 2 (Douek, 1999a, 1999b).

For teaching and learning purposes, argumentation is a fruitful means to control the validity of reasoning (as the legitimate use of examples and, or transitions from one mode of reasoning to another with their different cultural rules). We are therefore interested in two levels of argumentation: as part of the proving tasks, specially for producing and organising arguments (mode 2); and in discussing procedures, as a means to assimilate and master elements of proving processes.

Convergent structure of argumentation in a proof

In general, an argumentation is made of more elementary ones that may be organised in various ways (converging towards a conclusion, or being parallel as when producing different explanations, etc.). In a proof, the elementary argumentations may form a linear chain, each conclusion being input as an argument for the following argumentation, thus forming one whole "line of argumentation". But in many cases of proof, argumentation may contain parentheses "blocks", or side argumentation branches that meet the main line to input a supplementary data or argument. A parenthesis might be considered as a secondary line of argumentation. This description underlines the possible hierarchical relations between various argumentations involved in a proof (Knipping, 2008), which is a difficult matter for students who are being introduced to proof (see the Example for a suggestion).

IV EDUCATIONAL ASPECTS

In the early stages of proof teaching and learning, students can be smoothly introduced to theorems and proofs by conjecturing and proving activities provided that cognitive unity works (Boero, Garuti & Lemut, 2007). In particular exploratory activity (Mode 1) and justification (Mode 2) can be introduced at early stages. In a suitable educational environment, 7th and 8th graders are able to produce conjectures for non trivial arithmetic or geometric situations, and move (under a loose guidance by the teacher) towards constructing general justifications. Comparison of students' productions and classroom discussions about them, orchestrated by the teacher (Bartolini Bussi, 1996; Bartolini Bussi & al, 1999) allow students to appreciate some relevant cultural requirements of conjectures and proofs, like their generality and the conditionality of statements (Boero, Garuti & Lemut, 2007), and to become aware of processes favouring conjecturing and proving.

In the following we will focus on mode 2 reasoning, specially in the organisation of reasoning phases; then in the didactical engineering we will also consider mode 1 more specially related to conjecturing.

In spite of their usefulness to initiate students into conjecturing and proving, in those cases in which cognitive unity works well, with no difficulties due to the lack of structural continuity (Pedemonte, 2007, 2008), the peculiar argumentative structure of a proof does not emerge as an object of reflection for students. Indeed the fact that both easy-to-prove theorems must be proposed for a smooth approach to theorems, and that the students themselves are able to enchain the arguments in an autonomous way, make artificial and rather empty the discussion about the specific argumentative arrangement of those arguments. However students must be enabled to move from theorems for which cognitive unity works to theorems (like Pythagoras') for which proof cannot consist in the deductive arrangement of arguments produced by conjecturing. For other theorems students can meet difficulties in moving from creative ways of thinking (abduction, induction) typical of conjecturing to deductive arrangements of the produced arguments (Pedemonte, 2007). In both cases proving needs a strongly guided activity; and teachers' guidance can even initiate students into the mechanisms inherent in the Mode 2 reasoning, and open the perspective of Mode 3. Drawing from theoretical reflections, we make the hypothesis that the inherent argumentative activities could be promoted through debates (with real others) about arguments and their relations on one side, and story making on the other.

The debate

Classroom debates, if well oriented and guided, stimulate efforts of expression and explanation. These efforts, in turn, favour the consciousness of the logical rules and their range of validity. For instance, discussing a statement may bring students to methodological and meta mathematical reflections such as: producing an example to support the statement can be an efficient step in the exploratory phase, but is not a valid argument when organising a general mathematical justification; some semiotic

registers (like drawing) are crucial for exploration, and may be for organising reasoning, but insufficient to produce a suitable argument in a deductive reasoning. Such discussions question cultural rules of mathematical reasoning and mathematical knowledge too. Also the relation between arguments and the construction of lines of argumentation (mode 2) can be discussed in a debate, which draws students' attention to the goal of the line of argumentation in relation to its steps.

Making a story

Logic is concerned not with the manner of our inferring, or with questions of technique: its primary business is a retrospective, justificatory one - with the arguments we can put forward afterwards to make good our claim that the conclusions arrived at are acceptable because justifiable conclusions.

This quotation from Toulmin (1974, p. 6) inspires the hypothesis that in order to grasp the rationale of a proof, students may make an individual story from the ideas and calculations involved in a reasoning that validates the statement. We emphasise the story that connects steps and fragments with reasons, in order to serve the conclusion, and not particularly the story of how the steps occurred in one's mind (Bonaffé, 1993), nor of how learning has evolved through time (Assude & Paquelier 2005). The goal is that the students recognise the involved lines of argumentation, their possible hierarchical relations, and their role in the logical combination that produces the proof. At least at first stages of proof learning, these individual story makings need to be prepared by suitable tasks of guided construction of proof and by related debates putting into evidence some crucial "steps" of Mode 2 reasoning.

In our theoretical construction, debates and story makings should be considered together and arranged as a dynamic system of complementary situations. Individual story making involves students in an active personal reconstruction of the rationale of a proof, while a debate on the work done in individual tasks of conjecturing and guided proving (and story making as well) offers both openness to other possible combinations and regulation. We expect this system to draw students' attention to the "components" of the story. The deductive structure of the proof (through mode3) will consist of a particular relating of the pertinent components of a story.

Students need to be gradually initiated in both activities, possibly before the activities on theorems in order to establish a suitable didactical contract (Brousseau, 1986). However story making, in the case of theorems, shows particularities that need a careful mediation through sequencing suitable tasks.

Before illustrating the above theoretical reflection by an example, let us present the main activities we wish students to develop and their co-ordination, and give methodological precisions concerning the planned experiment:

- To associate exploration and conjecturing to enhance mode 1 (without excluding other modes).
- To stimulate proving the conjecture(s); either cognitive unity can work and thus the

students are able to produce a proof, or it cannot work and the teacher offers a task to guide them towards the proof.

- To engage the dynamic system of collective debate / individual story-making, starting from discussing some of students' productions, to enhance mode 2 (without excluding other modes). In case cognitive unity could not work, students would not be in a good condition to understand the proof nor to learn much of it, and this dynamic becomes particularly crucial.

The analysis of the experimentation should concern: Students' engagement in the proposed activities; and the evolution and the differences between the various individual productions. Observing the discussion (or its video registration), we need to track: How student's individual production reappears in the collective discussion; how the student hold his/her position in front of other's, and if some elements of consciousness awakened during discussion. However, some students may not take active part in the debate. The final individual production of story telling that follows should help completing the analysis. Comparing this individual production with the previous one (proof construction or proof reconstitution), one can see if the debate helped to bring to consciousness the necessity of some types of reasoning and the necessity of avoiding some others. The form of storytelling may reveal hierarchies of the types of reasoning and more particularly the linearity of the argumentation.

Another slightly different proving situation might follow to examine: transfer of the various reasoning competencies; the various methods; and the level of awareness of the variability of rules of validity.

V AN EXAMPLE CONCERNING PYTHAGORAS THEOREM

Pythagoras theorem was chosen for two reasons: it is an important and early met theorem in school mathematics; and it is not difficult to get the conjecture through a loosely guided path, while the construction of a proof needs a strong guidance by the teacher (cognitive unity cannot work, because the geometric constructions needed for the usual proofs are not suggested by the work done in conjecturing). Teachers' guidance, classroom discussions and story making will allow students to approach the rationale of the proof and offer occasions for learning about proof and proving.

First sequence: “Discovering” Pythagora's theorem, expressing the conjecture and making sense of it

Students have not only to grasp the theorem, but also to develop some proving skills (though no proving activity is demanded in this phase) and prepare for the further work; thus the activity on Pythagoras' theorem is prepared by Task 1 (an individual production on another theorem), followed by classroom discussion:

Task1: Consider the statement: "In a triangle of sides a , b and c , $a+b$ is always smaller than c ". Is it true? always? Why? Prepare yourself to explain how you checked it and why you think it is true, or it is not, or what makes you doubt.

No triangle is presented by the teacher; students are encouraged to draw some triangles for a check, if they did not do it spontaneously. This task aims at exploration through testing examples, and (specially in the discussion) at leading the students to express the rationale of the activity and to make visible the generality of the proposition they produce. An expression like “we wanted to see if it is true that... so we tried to verify it with four examples” is encouraged: such simple story making reflects an ability (and invites) to reconstruct the logical skeleton of the activity they went through. It bridges a Mode 1 reasoning with a Mode 2, and prepares Task 2.

Task 2 (individual): Now if we consider the squares of the lengths, instead of the lengths themselves, the situation is different. See if a relation between the squares of the lengths of the sides of a triangle exists. Once you think you produced a valid statement (a "conjecture"), put it clearly in words to explain it to other students.

Right angled, acute and obtuse triangles, are presented on the worksheet. Afterwards a collective discussion guided by the teacher is engaged to share and discuss the conjecture(s) produced, and the ways followed to produce them; and to attain and share acceptable expressions of the conjecture(s) (according to mathematical standards). An incomplete conjecture or an erroneous one may offer fine opportunities to make explicit the important elements of the theorem (in particular the condition of validity of Pythagoras' theorem, i.e. the angle being right) and their role.

Task 3 (individual): Write down the conjecture as now you think it should be. Explain it and illustrate it with some examples.

The teacher concludes with the standard formulation of Pythagoras' theorem.

Concerning proof learning, this first sequence aims at involving students in Modes 1 and 2: Exploring (drawing, measuring, calculating, induction when modelling and producing algebraic expressions, repeating procedures and modifying data) mostly in mode 1; and, mostly in mode 2, organising the exploitation of the gathered data, classifying them in order to find some rule, expressing results as general (everyday language being acceptable), etc; discussing and justifying propositions, and organising the steps of exploration in relation to a goal. Classifying and modelling are as much in mode 1 and mode 2. The explicit intentions of exploration and of organisation are satisfying sign in my opinion, as a main didactical goal is to enhance the processes students have to go through.

Second sequence: Guiding Pythagoras' theorem proof, and teaching/learning to organise the steps of reasoning into lines of argumentation

Given that cognitive unity cannot work, students are guided by means of individual and collective activities; then they reconstruct the lines of argumentation.

Task 4 (individual): Here we study the proof of the theorem we have conjectured, you will be guided towards this proof. Consider a right angled triangle with sides a , b , c . We use it to build the square A (see below). Its central square S is of area c^2 .



I) Can you describe how A can be obtained by using only our squared triangle? explain why S is a square (of area c^2)?

II) Try to write the area of A in two different ways (you may need to arrange the four identical square triangle differently). Find and explain the two ways.

III) How can this help us to validate our conjecture?

A geometrical reasoning is expected to intertwine with an algebraic reasoning in order to attain the equality between the areas. If needed, some supplementary tasks can be inserted either for the whole group or for some students.

After students' individual work, the teacher orchestrates a collective discussion (Bartolini Bussi, 1996) concerning the reasoning that allows to prove the steps of argumentation and the calculations and why they are needed, and in particular, the connection between geometrical arguments and algebraic arguments. The interactions must be based on their own reasoning productions, their insights and their shortcomings. Therefore the teacher selects elements of students' production to provoke fruitful interactions. Two complementary levers can help maturing students' awareness of the reasoning organization "rules", and their specificity in contrast to exploration reasoning: analyzing elements of reasoning, and rising direct methodological questions in the debate. The aim is to favor the elaboration of some satisfactory reasoning about the quality of which the student may agree, and, on another hand, to characterize some insufficiencies found in some produced reasoning. The parts of debate concerning specific algebraic or geometric steps and some sort of gap filling reasoning (directly concerned by the activity) need to be intertwined with methodological reflection about the validity of a reasoning, its communicability, the bases on which it can be accepted by another (indirect, implicit activity in student's individual production). Open "methodological" questions may be: how exploration and induction had been produced (algebraic induction); which different rules allow a reasoning to be valid (in exploration, measures and experiment are welcome, in proving deductive reasoning is needed, here based on elementary geometry and on algebraic calculus); and, in reasoning organization, how to come to such reasoning, and why (in particular how exploring the disposition of the four rectangles may favour algebraic exploration). This double level of discussion concerning the activity, on one hand, and the meaning and mathematical rules of the activity, on the other, is theoretically developed by M.A.Mariotti (2000) based on M.Bartolini Bussi's mathematical discussion theoretical frame.

Task 4 is formulated and organised in a way to approach a story making of the proof.

The subsequent discussion of the organisation of the lines of argumentation and the insertion of "blocks" of arguments/calculations in the main line is meant to prepare students to write a "story".

Third sequence: story making

Task 5 (individual): Write down how you organized your steps of reasoning to reach a general justification of the conjecture, and justify why those steps are important

This task is particularly important for the students who were not productive in the previous sequence. It should allow them (as well as the others) to grasp and reconstruct the rationale of the proof. Here is the kind of arguments we hope the students produce:

first (*block 1*) we calculate the area of A, then (*block 2*) we organised differently the calculation of the area (or we organised differently the disposition of the triangles) so that we found another algebraic expression of the area, because (*looking forward to the final goal*) surface measures of squares are written as algebraic squares. So we think that a^2 , b^2 and c^2 will appear and will be related (*possibility to rejoin the main line*). So, we can write the algebraic equality, and find the relation after transformations.

Mode 2 reasoning is needed for this task in block 2: students must go through an abductive reasoning ("how can I find a^2 and b^2 in this big square?") while deduction prevails in block 1 and will prevail afterwards, till the end.

It is important to notice with the students that the algebraic equality is the principal aim (and first to come to the mind, since it is near to the conclusion we want to reach) but that we have to begin with geometrical considerations, which are like parentheses besides the principal aim. Thus the reasoning is made of a principal line of argumentation and side parentheses involving geometrical reasoning and calculations, whose conclusions flow into the main argumentation line. Getting familiar with mathematical proof practices (like moving from a geometrical frame to an algebraic one, using geometry only for strategic purposes...) is a particular aspect of this work.

Difficulties inherent in the classroom implementation of the proposal

Comparing the proposal with the style of teaching of most teachers, and keeping into account my first experiences of work with teachers on this subject, I must say that teachers meet some difficulties in engaging in a coherent classroom implementation of the proposal. One difficulty consists in the fact that "To produce a conjecture" is a task that does not fit the most frequent didactical contract in our schools (statements are usually presented and illustrated by the teacher, and learnt by students who repeat and apply them afterwards; the same for proofs). Another is that teachers tend to identify student's task of reasoning and the task of explaining the rationale of a reasoning as bearing the same learning targets. And, finally, the presentation and management of the tasks in a way that guides students' work but does not prevents

creativity is not easy; however, if creativity is not practised, there would be no sense in making a story out of a series of calculations.

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CERME 6 – WORKING GROUP 3

ON “STOCHASTIC THINKING”

TABLE OF CONTENTS

Introduction.....	344
<i>Andreas Eichler, Maria Gabriella Ottaviani, Floriane Wozniak, Dave Pratt</i>	
Chance models: building blocks for sound statistical reasoning	348
<i>Herman Callaert</i>	
Recommended knowledge of probability for secondary mathematics teachers	358
<i>Irini Papaieronymou</i>	
Statistical graphs produced by prospective teachers in comparing two distributions.....	368
<i>Carmen Batanero, Pedro Arteaga, Blanca Ruiz</i>	
The role of context in stochastics instruction.....	378
<i>Andreas Eichler</i>	
Does the nature and amount of posterior information affect preschooler’s inferences	388
<i>Zoi Nikiforidou, Jenny Pange</i>	
Student’s Causal explanations for distribution	394
<i>Theodosia Prodromou, Dave Pratt</i>	
Greek students’ ability in probability problem solving	404
<i>Sofia Anastasiadou</i>	

INTRODUCTION

ON “STOCHASTIC THINKING”

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OVERVIEW

The Working Group 3 discussed 8 three aspects that reflect the diversity of the research approaches on stochastic thinking:

- theoretical issues of stochastic thinking,
- teachers' professional development, and
- students' learning in respect to their success in solving stochastic tasks.

The connective aim of all approaches was the students' learning of stochastic concepts, and the students' awareness that it is possible to use stochastics to cope with specific real situations. These aspects of the students' stochastic literacy (for the term statistical literacy see Gal, 2004), however, were discussed using three different perspectives, i.e. the stochastic content (C), the teaching of stochastics (T), and the students' learning about stochastics (S), that shape a didactical triangle referring to stochastics instruction.

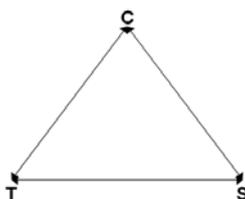


Figure 1: Didactical triangle involving three different perspectives on stochastics instruction, i.e. the content, the teachers, and the students

In the following we will introduce the papers that match one of the three perspectives, and we will sketch some results of our discussion.

STOCHASTICAL CONTENT

Stochastics is a cocktail of statistical ideas and probabilistic ideas. Although the latter thesis seems to be trivial, there is a lot of evidence that the emphasis on statistics and probability in curricula varies, often according to knowledge and feelings of the teachers. In the same way, the topics of interest to researchers vary over time.

Currently the focus of research concerning statistics is, for instance, on distributions, averages, variability (including informal inference, and co-variation and correlation), and graphs (Shaughnessy, 2007). Concerning probability the research focus is on random, sample space, and probability measurement (Jones, Langrall, & Mooney, 2007).

The research referring to these subjects has two aims:

- to clarify the notions, meanings or definitions of stochastic concepts. In our group, for instance, the talk of Hasan Akyuzulu deals with the undefined concept of risk highlighting the connection between risk and defined stochastic concepts.
- to develop and to evaluate teaching approaches that facilitate students' learning in respect to the different stochastic concepts. Matching this aspect, Herman Callaert discusses in his paper obstacles of the students' learning that emerge through ambiguous notations and explanations of stochastic concepts in widely-used text books and software.

Concerning the aspect of stochastic content, we, finally, discussed the recommendation of professional organisations regarding stochastics instruction. To this aspect, Irini Papaieronimou identifies in her paper many recommendations about the teaching of probability from four US professional organisations. We are concerned that there is insufficient support to effect a didactical transposition. Further, we noted an omission: such recommendations do not include the need for teachers to understand what it is that students know (as opposed to misconceptions).

TEACHING OF STOCHASTICS

A repeated claim towards the research on stochastic thinking is to increase the effort of investigating the teachers' knowledge and the teachers' beliefs concerning stochastic concept, and the learning and teaching of stochastics (Shaughnessy, 2007). According to this claim, we discussed two research approaches that concern both, the stochastics teachers' knowledge, and the stochastics teachers' beliefs.

- Carmen Batanero, Pedro Artega, and Blanca Ruiz discuss in their paper the knowledge of prospective primary Spanish teachers referring to statistical graphs based on the theoretical Framework of Curcio (1989). They found that some of the teachers were unable to use even basic statistical graphs, and that, in fact, only one third were able to draw a reasonable conclusion.
- the paper of Andreas Eichler refers to an analysis of "ordinary" upper secondary teachers' planning of stochastics instruction, the teachers' classroom practice and their students' learning. His report focus on teachers having differing orientations across two dimensions: seeing mathematics as: (i) emphasising applications or a formal subject; (ii) being dynamic or static.

The report gives some evidence about different modes of students' learning concerning their awareness of the benefit of stochastics in the real life.

We concluded on the one hand, that the teaching of stochastics needs to offer students experiences of statistics and probability before theoretical perspectives are introduced. On the other hand, we stated that there is much research to do to understand the teachers' knowledge and the teachers' beliefs about stochastics that both in some sense determine the students' learning of stochastics.

LEARNING ABOUT STOCHASTICS

Finally, we discussed three considerably different research approaches focusing students' learning in respect to their success in solving stochastic tasks.

- The paper of Zoi Nikiforidou and Jenny Page provides a psychological experiment on children (age 5 or 6 years), in which the children made decisions based on posterior information. The results of this research give some evidence that even such young children have some understanding of ideas that may be the roots of probability or inference. This result argues against the Piagetian framework.
- The paper of Theodosia Prodromou and Dave Pratt concerns students (15 years of age) using a computer simulation. This research yielded that it was possible to design a computer simulation such that students were able to make use of ideas about causality to make sense of distribution. In this sense, the deterministic and the stochastic worlds are not disconnected but connected through levels of complex causality.
- Finally, Sofia Anastasiadou provides in her paper a study referring to children's meaning-making with respect to set theory. She found that the students were not able to recognise the mathematical concept across differing representations. Perhaps the lack of transfer could be attributed to the students lack of preparation: time to discuss, interact and work on related tasks.

Although the papers focusing on the students' learning match some of the claims to the research into stochastics education, the three research approaches mentioned above showed the diversity of possible research questions in this field.

CONCLUSIONS

The papers of Working Group 3 highlighted the diversity of research approaches focusing on stochastic thinking. However, we concluded with three claims for future research that often combine several perspectives on the teaching and learning of stochastics that shape a didactical triangle (fig. 1):

- We need empirical results that give evidence, how we can support the implementation of recommendations from professional organisations.

- We need empirical based strategies we support teachers to be more connectionist in their approach.
- We need to research how students can transfer ideas from one domain to another. Reference could be made to connectionist theoretical frameworks.

One of the problems to achieve these claims is that it is sometimes not possible to transfer results yielded into mathematics education on stochastics education due to the fundamental difference of stochastics in contrast to other mathematical disciplines. For instance, the role of context is very different in statistics from in mathematics. Mathematics as a discipline aims to be decontextualised whereas statistics may draw on context. However, in both mathematics and stochastics learning, the students must experience the underlying ideas in meaningful contexts.

Another problem seems to be that stochastics instruction in Europe still emphasise probability, and, for this reason, studies in the field of stochastics education often focus on probability. Hence, we hope to see more research in statistics in future conferences of the ERME. Otherwise, we are afraid that statistics will be lost from CERME. But also, we as educationalists fear this might parallel a loss of statistics to mathematics education.

However, stochastics and, in particular, statistics are certainly useful to many subjects and to citizens in general but it is also important to mathematics education.

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CHANCE MODELS: BUILDING BLOCKS FOR SOUND STATISTICAL REASONING

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A good understanding of chance models is crucial for mastering basic ideas in statistical inference. Mature students should be introduced to the concepts of inference through a study of the underlying chance mechanisms. They should learn to think globally, in models. In an introductory course, these models should have their own clear and unambiguous notation. Fuzziness and flaws, as encountered by our students in textbooks and software, may hamper their learning process seriously. The above claims are based on my experience as an instructor for university students. They should be substantiated by systematic research on the potential advantage of “thinking in models”, possibly also for younger pupils.

INTRODUCTION

From my experience as a teacher of statistics, thinking in models is a stumbling block for many mature students when they are confronted with the basic concepts of statistical inference. As long as students do not master the connection between underlying chance mechanisms and statistical conclusions, procedures like the construction of confidence intervals remain “black boxes”. The main problems with confidence intervals have been discussed in a previous paper (Callaert 2007) where the ability of “thinking backwards” was shown to be essential. After seeing the data, the main question was: “how did those data come to me?” This is a question about an underlying probability model as an ideal mathematical construct for modelling outcomes in a physical world. Those models are the main theme of the current paper.

This paper has two parts. It first shows how mathematical mature students can be introduced to chance models at all places, from populations over samples to statistics. A simple example illustrates how the models are built. It points at the same time to the fact that a clear and unambiguous notation is crucial for acquiring clear and unambiguous insight. Students discover the need for distinguishing a population mean from a sample mean, or an “observable” chance model from an “unobservable” but fixed parameter. Many of the inaccuracies found in research papers, textbooks and software packages have their origin in a lack of insight in underlying chance models. Some examples are given in the second part of this paper.

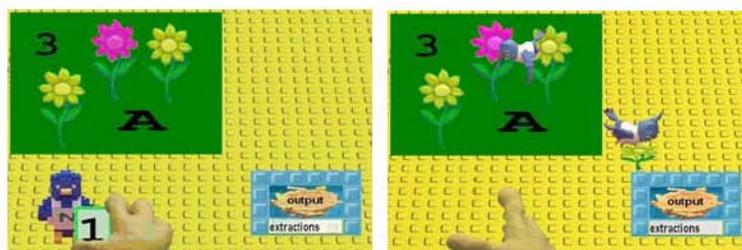
The current text is focused on mathematical mature students (using explicit mathematical notation). The underlying concepts however are very fundamental and it certainly is worthwhile finding out what can be done with younger pupils. Research

by Prodromou (2007) and Prodromou and Pratt (2006) is most interesting in this respect. They look at the connection between a *data-centric* and a *modelling* view on distributions, and write that: “*The modelling perspective reflects the mindset of statisticians when applying classical statistical inference*”. How and at what age can the connection with statistical inference be made?

THE POPULATION AS A CHANCE MODEL

From the very start, it is important that pupils not only are interested in “what” comes to them but also in “how” it comes to them. When they are allowed to build their own chance mechanisms, it is clear that (after some time and some experiments) they focus on both aspects. Nice examples can be found in a variety of research papers, such as in the study carried out by Pratt (1998) where children are able to manipulate “the underlying chance mechanism” (workings box). Another example is described in a paper by Cerulli et al. (2007) where they write:

The Random Garden is a microworld, for representing random extraction processes. The tool consists of a sample space (the *Garden*) a *Bird* and a *Nest*. When the user gives a number to the *Bird*, a corresponding number of objects is extracted (with repetitions) from the *Garden* and deposited in the *Nest*



In that study, one team of pupils creates not just a Garden but a *Random Garden*. This means that the pupils not only think about the composition of the garden (the flowers and trees) but they also know that the *Bird* will extract objects “at random and with replacement”. A competing team of pupils has to guess the *Random Garden* after they have inspected a *Nest*. That the objects in the *Nest* came “at random and with replacement” is key information and it is used (rather implicitly) by the competing team when they look at bar graphs and counters. One of the important consequences of the setup of this study is that pupils start discussing (and understanding) the concept of “equivalent chance mechanisms” (called equivalent gardens). If the study would have been set up differently, with the same flowers and trees but with a *Bird* that extracts objects not at random or without replacement, the “*Guess my Garden Game*” would have been completely different. This aspect might be stressed even more in such types of studies since it is important to find out at what age pupils are able to “think in models” and what kind of strategies can be used for enhancing (and evaluating) this type of thought-processes.

The above examples refer to studies with younger pupils (such as 11-12 years old). At a later stage the concrete objects in populations (such as flowers or colored

segments) are replaced by numbers. But the basic question about a population stays unaltered: “which numbers will come to me and with what probability?” For mathematical mature students, comfortable with abstract notation, it is helpful to make a distinction between a chance model and its outcomes. In line with the notation for random variables, a chance model can be denoted by a capital letter (such as X) and outcomes by the corresponding small letter x . An example of such a “population chance model” is what I call a *red die*. Physically, it is just a regular die (falling on each side with probability $1/6$) but the numbers on the faces have been changed. There are 3 faces with a 1, 2 faces with a 3, and one face with a 6. The way in which outcomes from this population appear is governed by a throw of this red die. Hence, one will never see a number 2 but, for example, one will get a number 3 with probability $2/6$, denoted as $P(X=3)=2/6$. The next table gives complete information about this population X .

x	1	3	6
$P(X=x)$	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

Table 1. The population X described by its chance model

Remark that also in the continuous case it is customary to describe a population by providing at the same time the range of the values and their chance behavior, as reflected by statements like: “we work with a normal $N(124;16)$ population”.

THE SAMPLE AS A CHANCE MODEL

Once students get used to look at populations from the perspective of chance models one would think that the step towards looking at a sample from the same perspective is straightforward. For most of my students, this was not evident. The following (simple) example became a real eye-opener for many of them.

What happens when one takes a sample of size $n=2$ from the population X described in table 1 (the *red die*)? The main point here is that students have to answer the question *before* they actually take the sample. Hence, the question: “What will be the result of the first draw?” is not answered by “How can I know?” (reasoning only about specific outcomes *after* an experiment has been carried out) but by “I can tell you, *beforehand*, every possible value together with its probability”. And then of course it is not difficult to come up with the chance model X_1 for the first draw. The second draw X_2 has the same behavior.

x_1	1	3	6
$P(X_1=x_1)$	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

Table 2.

x_2	1	3	6
$P(X_2=x_2)$	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

Table 3.

A model for a sample of size $n=2$ now follows easily from tables 2 and 3. The model is denoted by (X_1, X_2) and its outcomes by (x_1, x_2) . It is instructive for students to construct this model for themselves arriving at table A1 (appendix) or at an urn model with random draws from the urn (figure 1).

The insight that a sample result (x_1, x_2) is nothing but one of the possible outcomes of an underlying chance mechanism (X_1, X_2) is very important. It creates the appropriate context for a proper understanding of the behavior of the sample mean (or of any other statistic constructed from a sample).

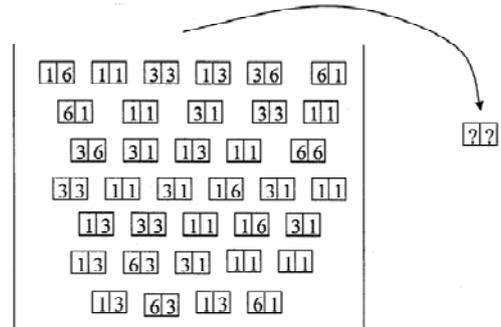


Figure 1.

THE SAMPLE MEAN AS A CHANCE MODEL

Continuing the above example, it takes just a few minutes to find all possible values of the sample mean together with their corresponding probabilities (see table A2 in the appendix). This leads to the following model:

\bar{x}	1	2	3	3.5	4.5	6
$P(\bar{X} = \bar{x})$	$\frac{9}{36}$	$\frac{12}{36}$	$\frac{4}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{1}{36}$

Table 4. The sample mean $\bar{X} = \frac{X_1 + X_2}{2}$ described by its chance model

Simulation tools might be extremely useful for learning statistical concepts but it is my experience that mature students (and secondary school mathematics teachers) also need an explicit confrontation with the more abstract tool of “thinking in models”. For many of them, the behavior of a sample mean is better understood in the context of chance models like table 4 than through the experience that a simulated bar chart or histogram is an approximation of a so-called sampling distribution. Properties like: “the mean of the sample mean is the population mean” can be discovered through simulations, but a clear view on underlying models surely can enrich insight in this discovery. In either case, an unambiguous notation is needed as a support to students for distinguishing populations from samples, and chance models from their outcomes. The next sections illustrate some problems.

EXAMPLES FROM TEXTBOOKS

During the past couple of decades reform in statistics education at the school level has been extensive in the United States. It has resulted in the production of new textbooks by authors such as: Yates, Moore and Starnes (2003) [YMS], Watkins, Scheaffer and Cobb (2004), Agresti and Franklin (2007), and many others. All these

books use capital letters (such as X) for random variables and small letters (such as x_1, x_2, \dots) for their outcomes. This is nice since this notation makes a clear distinction between an underlying chance process and a particular outcome. But once students start sampling, their attention is drawn to particular outcomes and the notation for underlying models, such as (capital) \bar{X} for the sample mean, is gone. Paul Velleman, author of *ActivStats*, says: “*Convention in the introductory course is to emphasize the observed values, which are usually not thought of as random. Every text I know uses a lower case \bar{x} to represent the sample mean. The r.v. version is a hypothetical construct of which the sample mean at hand is one realization. A bit sloppy at times, but, I think, less confusing for students*” [(1999) personal communication]. The experience I have with my students tells me the opposite. On p.525 of [YMS] one reads: “The sampling distribution of \bar{x} describes how the statistic \bar{x} varies in all possible samples from the population. The mean of the sampling distribution is μ , so that \bar{x} is an unbiased estimator of μ ”. The fact that \bar{x} stands for an outcome while at the same time it is said that \bar{x} is unbiased is confusing. The problem persists in the chapter on hypothesis testing where one reads on p.568 that $\bar{x}=0.3$ and that $P(\bar{x} \geq 0.3)$ is needed for computing the p-value. But probability statements are

statements
about chance
processes.
Hence, the p-
value is the
probability

In Example 10.9 the observations are an SRS of size $n = 10$ from a normal population with $\sigma = 1$. The observed mean sweetness loss for one cola was $\bar{x} = 0.3$. The P-value for testing	$H_0: \mu = 0$
	$H_a: \mu > 0$
is therefore	$P(\bar{x} \geq 0.3)$

that (under the null hypothesis) the chance process \bar{X} generates values which are at least as large as the observed outcome \bar{x} . Notation is crucial here and the above phrase should be written as $P(\bar{X} \geq \bar{x})$. If $\bar{x}=0.3$ in the sample of one student while $\bar{x}=0.4$ in the sample of another student, they now can start with the same notation $P(\bar{X} \geq \bar{x})$. Afterwards, they only have to plug in their \bar{x} -value for arriving at $P(\bar{X} \geq 0.3)$ [or at $P(\bar{X} \geq 0.4)$] as meaningful expressions.

EXAMPLES FROM SOFTWARE

Software can provide powerful educational tools and can create unique opportunities for gaining insight in statistical concepts. This is not only true for our students but also for adults who (sporadically) need to carry out a statistical analysis. At those instances, people often use their favorite package as a fast resource, both for ideas and for computations. From a “statistical literacy” point of view, one would hope that statistical information encountered in widespread packages is clear and accurate.

Excel

When your student says that, in a one-sided two-sample t-test, the null hypothesis assumes that the two means are equal and the alternative hypothesis says that one

mean is larger than the other, you might be willing to consider the answer as correct. But when he writes $H_0: \bar{x} = \bar{y}$ versus $H_1: \bar{x} > \bar{y}$ you can't believe your eyes. In his notation, he tries to find out whether the mean in his first sample is larger than the mean in his second sample $\bar{x} > \bar{y}$ instead of investigating whether the mean of the first population is larger than the mean of the second population $\mu_1 > \mu_2$. This type of confusion has been present in Excel for decades. Several versions in the nineties had in their "Data Analysis Toolpack" a help file called "Learn about the t-test: Two Sample Assuming Equal Variances Analyses". What you could learn was as follows. "This analysis tool performs a two-sample Student's t-test. This t-test form assumes that the means of both data sets are equal; it is referred to as a homoscedastic t-test. You can use t-tests to determine whether two sample means are equal". Apparently, when you have two datasets you can use the Data Analysis Toolpack in Excel for finding out whether \bar{x} equals \bar{y} . And you can do so at some alpha level, as follows. "Enter the confidence level for the test. This value must be in the range 0...1. The alpha level is a significance level related to the probability of having a type I error (rejecting a true hypothesis)". There is no clear distinction between a null and an alternative hypothesis (which is *the true hypothesis* to be rejected?) nor is there any reference to underlying populations. This type of fuzziness is disturbing. Attention to these problems has been drawn at several occasions, even in a publication (Callaert 1999). Change however is slow and confused. In Excel 2003 as well as in Excel 2007 it depends on the order in which you call for help. Press F1 (Help), type the phrase Data Analysis and click Search. Then click on Data Analysis and in the new window click on t-Test. The following text appears.

This analysis tool performs a two-sample student's t-test. This t-test form assumes that the two data sets came from distributions with the same variances. It is referred to as a homoscedastic t-test. You can use this t-test to determine whether the two samples are likely to have come from distributions with equal population means.
Hypothesized Mean Difference Enter the number that that you want for the shift in sample means. A value of 0 (zero) indicates that the sample means are hypothesized to be equal.
Alpha Enter the confidence level for the test. This value must be in the range 0...1. The alpha level is a significance level that is related to the probability of having a type I error (rejecting a true hypothesis).

But if you click on Formulas → More Functions → Statistical → TTEST → "Help on this function", then you can read about equality of *population means* together with a choice of using either a one-tailed or a two-tailed t-distribution.

Tails specifies the number of distribution tails. If tails = 1, TTEST uses the one-tailed distribution. If tails = 2, TTEST uses the two-tailed distribution.

Fathom

Never before I've worked with Fathom, so I only can give some first impressions by a novice (having downloaded a Fathom Evaluation Version 2.1). The fact that I was lost right from the start might be blamed on my inexperience. I think however that the rather abstract structure of Fathom working with "collections", "attributes",

“measures” and “statistical objects” is not obvious for beginning students. In contrast with this, Maxara and Biehler (2007) report on a study where Fathom was used systematically by their university students, apparently with success. I assume that those students’ first contact with Fathom was different from mine, since I clicked Help→Sample Documents→Statistics and started reading. I was quite amazed.

To start with, a clear notation could be helpful. The Fathom Documents use “mu”, “Mean”, “popMean”, “m”, “Avg”,... and “sigma”, “Std. dev.”, “popSD”, “s”, “sd”,... Why not stick to μ and σ for populations and to \bar{x} and s for sample results?

Furthermore, the notational distinction between a binomial model X (capital letter) and its x -values (small letter) should be applauded were it not that X is said to be a

binomialProbability (x , n , p , min, max)	This probability function computes the probability that $X = x$, where X is a random variable chosen from the set of possible values.
---	--

random variable chosen from the set of possible values.

The binomial model comes up several times but its discrete nature is seldom stressed, even in small samples. The “Polling Simulation” document wants to compare theory and experiment and uses `predictedProportion = binomialProbability (propYesForOnePoll*20, 20, 0.45)` resulting in a theoretical model where a lot of possible outcomes and their associated probabilities are missing. It is not because one has not seen 17 successes in a particular simulation (and hence not a proportion of $17/20=0.85$) that the predicted probability of a proportion of 0.85 doesn’t exist.

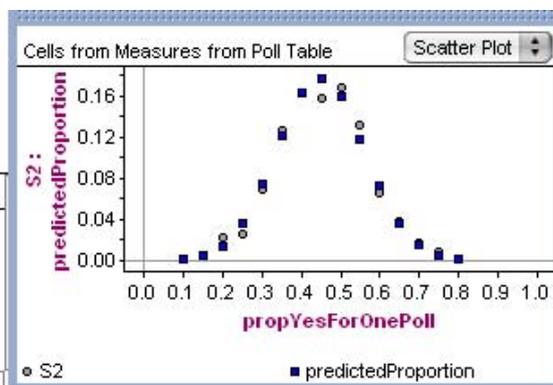
The Cells from Measures from Poll Table” enables us to compare theory and experiment numerically.



Cells from Measures from Poll Table

Cells from Measures from Poll Table

	propYesForOnePoll	S1	S2	predictedProportion
12	0.65	30	0.030	0.0000197
13	0.7	17	0.017	0.0149808
14	0.75	8	0.008	0.00490281
15	0.8	1	0.001	0.00125356

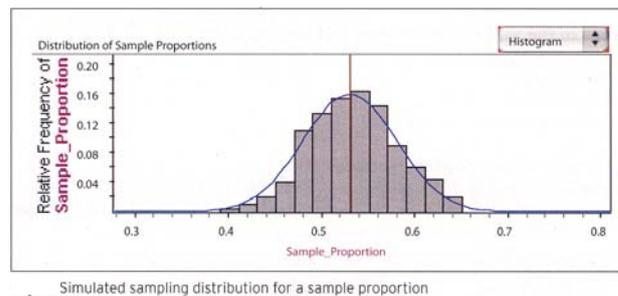
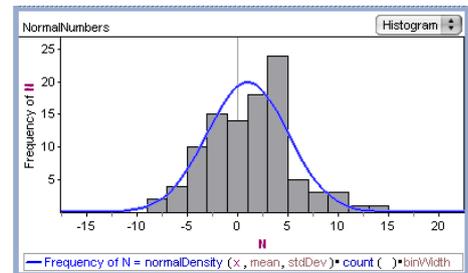


A further problem with this document lies in its histogram representation comparing the simulation results with the (also truncated of course) theoretical model. Repeating a poll of size 20 1000 times does not produce 1000 different outcomes. There still are only 21 different possible proportions. A bar graph comparing theoretical probabilities with experimental relative frequencies would make sense here since the chance model is discrete. By the way, try to let your students discover for themselves

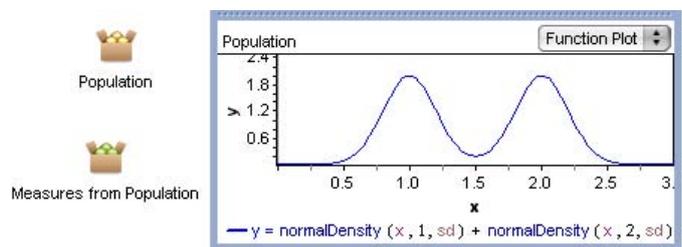
the formula `Density of propYesForOnePoll = binomialProbability (round (20x), 20, 0.45) *20` for drawing such a histogram. Of course, the problem is much deeper and relates to the obsession of making curves fit histograms who themselves have to represent experiments with discrete outcomes. The “Normal” document for example shows a histogram of 100

random numbers from a normal population together with [quote]: “a plotted curve of a normal distribution with the same mean, standard deviation, and area as the histogram”. Yes, with the same area! Fortunately the example uses a histogram on a density scale. But there is no problem if one would use a histogram with frequencies. In the same document under number 3 of the “To do” list attention is drawn to the fact that the density then has to be multiplied by both the count and the bin width. If you do this, you find the figure on the right.

But `normalDensity(x, mean, stdDev) * count () * binWidth` is a model for what? It is a curve fitting the “frequency histogram” but it certainly isn’t a model for an underlying chance mechanism. These problems are not uncommon. In Schaeffer and Tabor (2008) one finds a similar figure. This time, a histogram has been drawn on a Relative Frequency scale and the density has only been multiplied by the bin width. The authors write: “The figure shows a simulated sampling distribution of sample proportions. This sampling distribution has a mean of 0.53 and a standard deviation of 0.05 and is nicely represented by the normal distribution (overlaid smooth curve) with that same mean and standard deviation”. But the top of a normal density $N(0.53;0.05)$ is equal to 8, not to 0.16. So, what’s the name of a bell-shaped curve that (i) is nowhere negative and (ii) has an area under the curve equal to 0.02? Indeed, that’s the blue curve in that paper.



Fathom’s “Central limit Theorem” document has analogous problems. Wouldn’t it be nice to compare the histograms of the simulated sample means \bar{x} with the target model of \bar{X} ? That model is *normal* with *mean* $\mu=1.5$ and with *standard deviation* $\sigma/\sqrt{n}=\sqrt{0.5}/\sqrt{n}$. The document instead uses the mean and standard deviation of the randomly generated set of 200 \bar{x} -values. Moreover, the collection called “Population” is not the population but contains the sample values, while the population itself is represented by a bimodal curve integrating out to 2 (yes, two).



CONCLUSION

Thinking in chance models might be too abstract for the young learner but at some level in the developmental process the more mature student might need more than “approximations by simulation” in order to fully understand the underlying reasoning

of statistical inference. At this point one needs a careful identification of all the involved entities, together with a clear notation, both in textbooks and software. It might be interesting for further research to investigate the impact of an unambiguous notation on the effectiveness of student's learning and understanding of statistics.

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APPENDIX

first draw X_1		second draw X_2		sample (X_1, X_2)	
x_1	$P(X_1 = x_1)$	x_2	$P(X_2 = x_2)$	(x_1, x_2)	$P(X_1 = x_1, X_2 = x_2)$
1	$P(X_1 = 1) = \frac{3}{6}$	1	$P(X_2 = 1) = \frac{3}{6}$	(1, 1)	$P(X_1 = 1, X_2 = 1) = \frac{9}{36}$
1	$P(X_1 = 1) = \frac{3}{6}$	3	$P(X_2 = 3) = \frac{2}{6}$	(1, 3)	$P(X_1 = 1, X_2 = 3) = \frac{6}{36}$
1	$P(X_1 = 1) = \frac{3}{6}$	6	$P(X_2 = 6) = \frac{1}{6}$	(1, 6)	$P(X_1 = 1, X_2 = 6) = \frac{3}{36}$
3	$P(X_1 = 3) = \frac{2}{6}$	1	$P(X_2 = 1) = \frac{3}{6}$	(3, 1)	$P(X_1 = 3, X_2 = 1) = \frac{6}{36}$
3	$P(X_1 = 3) = \frac{2}{6}$	3	$P(X_2 = 3) = \frac{2}{6}$	(3, 3)	$P(X_1 = 3, X_2 = 3) = \frac{4}{36}$
3	$P(X_1 = 3) = \frac{2}{6}$	6	$P(X_2 = 6) = \frac{1}{6}$	(3, 6)	$P(X_1 = 3, X_2 = 6) = \frac{2}{36}$
6	$P(X_1 = 6) = \frac{1}{6}$	1	$P(X_2 = 1) = \frac{3}{6}$	(6, 1)	$P(X_1 = 6, X_2 = 1) = \frac{3}{36}$
6	$P(X_1 = 6) = \frac{1}{6}$	3	$P(X_2 = 3) = \frac{2}{6}$	(6, 3)	$P(X_1 = 6, X_2 = 3) = \frac{2}{36}$
6	$P(X_1 = 6) = \frac{1}{6}$	6	$P(X_2 = 6) = \frac{1}{6}$	(6, 6)	$P(X_1 = 6, X_2 = 6) = \frac{1}{36}$

Table A1. The sample (X_1, X_2) described by its chance model

sample result (x_1, x_2)	probability of this result $P(X_1 = x_1, X_2 = x_2)$	value of the sample mean $\bar{x} = \frac{x_1 + x_2}{2}$
(1, 1)	$P(X_1 = 1, X_2 = 1) = \frac{9}{36}$	$\bar{x} = \frac{x_1 + x_2}{2} = 1$
(1, 3)	$P(X_1 = 1, X_2 = 3) = \frac{6}{36}$	$\bar{x} = \frac{x_1 + x_2}{2} = 2$
(1, 6)	$P(X_1 = 1, X_2 = 6) = \frac{3}{36}$	$\bar{x} = \frac{x_1 + x_2}{2} = 3.5$
(3, 1)	$P(X_1 = 3, X_2 = 1) = \frac{6}{36}$	$\bar{x} = \frac{x_1 + x_2}{2} = 2$
(3, 3)	$P(X_1 = 3, X_2 = 3) = \frac{4}{36}$	$\bar{x} = \frac{x_1 + x_2}{2} = 3$
(3, 6)	$P(X_1 = 3, X_2 = 6) = \frac{2}{36}$	$\bar{x} = \frac{x_1 + x_2}{2} = 4.5$
(6, 1)	$P(X_1 = 6, X_2 = 1) = \frac{3}{36}$	$\bar{x} = \frac{x_1 + x_2}{2} = 3.5$
(6, 3)	$P(X_1 = 6, X_2 = 3) = \frac{2}{36}$	$\bar{x} = \frac{x_1 + x_2}{2} = 4.5$
(6, 6)	$P(X_1 = 6, X_2 = 6) = \frac{1}{36}$	$\bar{x} = \frac{x_1 + x_2}{2} = 6$

Table A2. Sample mean values $\bar{x} = \frac{x_1 + x_2}{2}$ for all possible sample outcomes (x_1, x_2) .

The arithmetic mean is computed for all outcomes (x_1, x_2) from table A1.

RECOMMENDED KNOWLEDGE OF PROBABILITY FOR SECONDARY MATHEMATICS TEACHERS

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Changes in school mathematics curricula in the last few decades have brought along an increase on the importance placed on probability (National Commission for Excellence in Education, 1983; National Council of Teachers of Mathematics, 2000). Since teachers' knowledge can have an impact on students' learning (Fennema & Franke, 1992), it is important that teachers have sufficient probability content and teaching knowledge. This paper identifies the suggested probability knowledge for secondary mathematics teachers through an examination of the recommendations from four professional organizations, namely the American Mathematical Society (AMS), the American Statistical Association (ASA), the Mathematical Association of America (MAA), and the National Council of Teachers of Mathematics (NCTM).

Keywords: teachers' knowledge, probability, professional recommendations

PROBABILITY CONTENT IN THE SECONDARY SCHOOL MATHEMATICS CURRICULUM

Since the late 1950s, there have been strong calls for an increase in the inclusion of probability in the US K-12 mathematics curriculum (NCSM, 1977; NCEE, 1983; NCTM, 2000). Probability has come to gain importance as a content area that students need to have experience with in order to be well-informed citizens since its study “can raise the level of sophistication at which a person interprets what he sees in ordinary life, in which theorems are scarce and uncertainty is everywhere” (Cambridge Conference on School Mathematics, 1963, p.70; as cited in Jones, 2004).

In 1963 a group of mathematicians and National Science Foundation (NSF) representatives published *Goals for School Mathematics* in which the importance of “some ‘feeling’ for probability” for all students was indicated (Jones, 1970, p. 291; as cited in Sorto, 2004). Following, the National Council of Supervisors of Mathematics (NCSM) defined probability as one of the basic skills that students should acquire (1977). In 1983, the National Commission for Excellence in Education (NCEE) published *A Nation at Risk*, a report aimed at pointing out the immediate need for reform in education, with the suggestion that high school graduates understand elementary probability and be able to apply it in everyday life.

More recently, the National Council for Teachers of Mathematics (NCTM) published the *Curriculum and Evaluation Standards for School Mathematics* (1989) in which it was recommended that in grades 5-8 students “explore situations by experimenting

and simulating probability models”, construct sample spaces in the attempt to determine probabilities of “realistic situations”, and appreciate the use of probability in the real world (1989, p. 109). Particular to grades 9-12, recommendations included the understanding of the difference between experimental and theoretical probabilities, theoretical and simulation techniques for computing probabilities, and interpreting discrete probability distributions (p. 171). In the mid to late 1990s the NCTM standards were revised resulting in the publication of *Principles and Standards for School Mathematics* (2000). Here, recommendations stated that

“middle-grades students should learn and use appropriate terminology and should be able to compute probabilities for simple compound events ... In high school, students should compute probabilities of compound events and understand conditional and independent events.” (NCTM, 2000, p. 51).

This increased attention on probability in school curricula is an indicator of how important it is that “teachers, mathematics educators, parents, and administrators, must provide their children and their students with alternative ways of approaching data and chance” (Shaughnessy, 2003, p. 223). Since “[T]here is perhaps no other branch of the mathematical sciences that is as important for *all* students, college bound or not, as probability and statistics” (Shaughnessy, 1992, p. 466, emphasis in original) and since misconceptions about probability are common among children, it is important that instruction allows students to confront their misconceptions and develop a deeper understanding of probability concepts (Garfield & Ahlgren, 1988; Konold, 1989; Shaughnessy, 2003). Since teachers’ knowledge can have an impact on students’ learning (Fennema & Franke, 1992), it is important that teachers be able to tackle these student difficulties and misconceptions on probability as they arise in mathematics classrooms. In order to be able to do so, teachers need to have sufficient probability content and teaching knowledge.

Teachers’ Knowledge of Probability

Although there have been calls for an increased attention on probability in the school curriculum, one of the problems encountered is the inadequate preparation of teachers in probability (Penas, 1987; CBMS, 2001). Many teachers have not encountered probability in their own K-12 mathematics courses and sometimes need convincing as to why they need to learn and teach probability topics (CBMS, 2001). Batanero et al. (2004) suggest that educators need to provide better initial training for teachers by offering courses at the college level specific to the didactics of probability. Such a course should include an introduction to the history of probability; information on statistics journals, associations, and conferences; the study of fundamental probability concepts; readings of literature on heuristics and biases in probability, as well as students’ difficulties and misconceptions in probability; identification of the educational theories and teaching approaches, assessment, teaching resources, and the use of technology; and examples of projects that can be used when teaching probability.

Teachers' Knowledge of Mathematics

Several scholars in the past three decades have provided insight into the definition of teachers' knowledge. In his work, Shulman (1986) provided a framework of teachers' knowledge which includes the following three categories: i) *subject matter content knowledge* which refers to “the amount and organization of knowledge per se in the mind of the teacher” as well as not only understanding *that* something is so but also *why* it is so and why it is important to the discipline (p. 9); ii) *pedagogical content knowledge* which refers to

“the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others” (p. 9).

This category also includes knowledge of common conceptions/preconceptions that students have; and iii) *curricular knowledge* which includes knowledge about the

“full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials ..., and the set of characteristics that serve as both the indications and contraindications for the use of a particular curriculum or program materials in particular circumstances” (p. 10).

The difficulty faced by educators is how to blend the components of teacher knowledge so as to effectively prepare teachers to help all students to learn meaningfully.

FOCUS OF THE PAPER AND QUESTION

With the above issues under consideration, a study was carried out by the author in which US state and national mathematics standards for grades 6-12, secondary mathematics textbooks, and recommendations from professional organizations were analyzed in order to identify the content and teaching knowledge that secondary mathematics teachers need to have relative to the domain of probability. A report of the results relating to the probability topics that secondary mathematics teachers should know and be able to teach was presented at a previous conference (Papaieronymou, 2008), whereas this paper focuses on the teaching aspects of these probability topics and more specifically on the following question:

What are the aspects of teaching knowledge of probability that secondary mathematics teachers need to have as suggested by professional organizations?

For the purposes of addressing this question, only the recommendations from professional organizations were analyzed. The data sources specific to students (i.e. national and state standards for grades 6-12 and secondary mathematics textbooks) were not very informative since they did not directly address teachers' knowledge.

METHODS

Data Sources

In particular, *A Call for Change: Recommendations for the Mathematical Preparation of Teachers of Mathematics* (1991) published by the MAA, the *Professional Standards for Teaching Mathematics* (1991) published by the NCTM, *The Mathematical Education for Teachers* (CBMS, 2001) published by the AMS, and the *Guidelines for Assessment and Instruction in Statistics Education (GAISE) College Report* (Aliaga et al., 2005) published by the ASA were analyzed. The ASA report presents a list of goals for college students – not specifically prospective mathematics teachers – and recommendations for the teaching of introductory statistics courses.

Data Analysis

The number of recommendations from each professional organization was as follows:

Data Source	Number of Recommendations before multi-coding
AMS (2001)	27
ASA (2005)	9
MAA(1991)	17
NCTM(1991)	6
Total	59

Table 1: Number of recommendations from each organization before multi-coding

These 59 recommendations were categorized according to Shulman's (1986) framework of teacher knowledge with 8 recommendations being placed under more than one category. In deciding under which knowledge category to place each recommendation, the verbs appearing in the recommendation and their use in association with the probability concepts mentioned in the respective recommendation were considered. Some examples of recommendations that were placed under each of Shulman's categories are:

Recommendation	Knowledge Category
Mathematics teachers should be able to use permutation and combinatorial computations in problems arising from several areas, including geometry, algebra, and graph theory. They should also understand how counting techniques apply in the calculation of the probability of events. (MAA report, p. 36)	Subject-matter content knowledge
The fact that, under random sampling, the empirical probabilities	Pedagogical

actually converge to the theoretical (the law of large numbers) can be illustrated by technology (computer or graphing calculator) so that an understanding of probability as a long-run relative frequency is clearly established. (AMS report, p.116)	content knowledge
Precede computer simulations with physical explorations (e.g. die rolling, card shuffling) (ASA report)	Curricular knowledge
Other topics that should be introduced include fair games and expected value, odds, elementary counting techniques, conditional probability, and the use of an area model to represent probability geometrically (NCTM, 1991, p. 138)	Subject-matter and pedagogical content knowledge

Table 2: Examples of recommendations under Shulman's (1986) knowledge categories

In the last recommendation provided in Table 2 above, the use of the area model to represent probability implies pedagogical content knowledge since this type of knowledge includes the ways of *representing the subject*. The reference to topics of probability that should be introduced implies subject matter content knowledge; the topics refer to the *amount* of knowledge that teachers should have with respect to probability so as to be able to introduce these topics in their mathematics classrooms.

RESULTS

Once the 59 recommendations were categorized under Shulman's framework for teacher knowledge, with 8 recommendations being placed under two of the knowledge categories, the results were:

Data Source	Subject-matter content knowledge	Pedagogical content knowledge	Curricular knowledge	Total
AMS (2001)	22	5	1	28
ASA (2005)	7	1	1	9
MAA (1991)	13	6	2	21
NCTM (1991)	2	4	3	9
Total	44	16	7	67

Table 3: Number of recommendations under each of Shulman's (1986) categories

As can be seen from Table 3, about 66% (44 out of 67) of the recommendations from the four professional organizations relate to subject matter content knowledge, 24% (16 out of 67) of the recommendations refer to aspects of pedagogical content knowledge and 10% of the recommendations specify aspects of curricular knowledge that should be included in the preparation of secondary mathematics teachers.

The analysis also showed that the following topics were recommended by at least two of the organizations:

Common Topic	Professional Organizations in agreement
Combinatorics	AMS, MAA, NCTM
Experimental and Theoretical Probability	AMS, MAA, NCTM
Simulations	ASA, MAA, NCTM
Probability Distributions	AMS, MAA, NCTM
Hypothesis Testing	AMS, ASA, MAA
Conditional Probability	AMS, NCTM
Expected Value	AMS, NCTM
Probabilistic Misconceptions	AMS, NCTM
Uses/Misuses of Probability	AMS, MAA

Table 5: Probability topics recommended by at least two of the organizations

DISCUSSION

Given the small number (59) of recommendations overall across all four organizations specific to the area of probability and that 66% of the recommendations relate to subject matter content knowledge whereas 24% refer to pedagogical content knowledge and only 10% refer to curricular knowledge, the results imply that it is still unclear what exactly the pedagogical content knowledge and curricular content knowledge that secondary mathematics teachers need to have in the area of probability is.

A closer examination of the recommendations indicates that with respect to pedagogical content knowledge specific to probability, teachers need to acquire an awareness and ability to confront common probabilistic misconceptions and student difficulties relative to probability concepts (as suggested by the ASA, the MAA, and the NCTM). In addition, teachers need to be able to use technology to carry out simulations in order to illustrate probabilistic concepts (as recommended by all four of the professional organizations) and should also be able to use concrete objects such as dice, cards, and spinners to demonstrate probability concepts to students in the mathematics classroom (as suggested by the ASA and the NCTM). Furthermore, secondary mathematics teachers should be able to represent probabilities using various models such as the area model (as suggested by the NCTM).

Specific to curricular knowledge, secondary mathematics teachers should be aware of the various materials and programs that they can use to help students understand probability concepts. That is, they should be aware that they can use various computer programs such as Fathom and DataScope in their mathematics classrooms

when working with probability concepts (as suggested by the AMS) and they should know the power of simulation as a technique that can be used to solve probability problems (as recommended by the MAA and the NCTM).

As can be seen from Table 5, the four professional organizations place considerable emphasis on experimental versus theoretical probability and simulations. Secondary mathematics teachers need to be able to plan and conduct experiments and simulations (Aliaga et al., 2005; CBMS, 2001; Committee of the Mathematical Education of Teachers, 1991; NCTM, 1991), distinguish between experimental and theoretical probability (Committee of the Mathematical Education of Teachers, 1991), determine experimental probabilities (CBMS, 2001; Committee of the Mathematical Education of Teachers, 1991), use experimental and theoretical probabilities to formulate and solve probability problems (Committee of the Mathematical Education of Teachers, 1991), and use simulations to estimate the solution to problems of chance (Committee of the Mathematical Education of Teachers, 1991; NCTM, 1991). Secondary mathematics teachers should be able to provide a model which gives a theoretical probability that can be compared to experimental results, which in turn is essential when studying the concept of relative frequency (CBMS, 2001). In order to help students develop an understanding of probability as a long-run relative frequency, secondary mathematics teachers need to understand the law of large numbers and be able to illustrate it using simulations (CBMS, 2001).

With regards to theoretical probability, teachers should know about and be able to use both discrete and continuous probability distributions (NCTM, 1991), understand probability distributions (CBMS, 2001) and especially the normal distribution (CBMS, 2001; Committee of the Mathematical Education of Teachers, 1991), as well as the binomial, poisson, and chi-square distributions (Committee of the Mathematical Education of Teachers, 1991). They should also be able to use simulations to study probability distributions (CBMS, 2001; Committee of the Mathematical Education of Teachers, 1991) and demonstrate their properties (CBMS, 2001). Moreover, they should be introduced to fair games (NCTM, 1991) and understand expected value (CBMS, 2001).

Another topic among the recommendations from three of the four professional organizations is that of hypothesis testing. Secondary mathematics teachers should understand the concept of statistical significance including significance level and p-values, and that of confidence interval (Aliaga et al., 2005; Committee of the Mathematical Education of Teachers, 1991) including confidence level and margin of error (Aliaga et al., 2005).

Returning to the idea of theoretical probability, secondary mathematics teachers should be able to use counting techniques (NCTM, 1991) such as permutations and combinations to determine such (theoretical) probabilities (Committee of the Mathematical Education of Teachers, 1991). In addition, they should be exposed to

the applications of combinatorics (CBMS, 2001) including their use in calculating the probability of events (Committee of the Mathematical Education of Teachers, 1991). Secondary mathematics teachers should also understand and be able to calculate the probabilities of independent and dependent events (CBMS, 2001), compound events made up of independent and dependent events (CBMS, 2001) and also understand conditional probability (CBMS, 2001; NCTM, 1991). Various representations such as area models and tree diagrams should be used by teachers to aid students in better understanding compound events (CBMS, 2001; NCTM, 1991).

In addition, teachers should know about the uses of probability in many fields and its misuses in such sources as newspapers and magazines (CBMS, 2001; Committee of the Mathematical Education of Teachers, 1991). Once experiments have been performed, teachers should be able to use probability to make decisions and predictions (CBMS, 2001; Committee of the Mathematical Education of Teachers, 1991).

An issue that arose as recommendations were being coded concerned the exact definition of the verbs that appeared in the documents. In many cases it was unclear as to what action or type of knowledge was expected of teachers based on the verb used since the meaning of the verb appearing in the report was unclear. Within the four documents of recommendations from the professional organizations, verbs appeared in different forms e.g. use, using, used or apply, applying, applied. Counting the different forms of a verb as one verb family gave rise to a total of 53 verb families being identified in the four reports. For example, consider the last recommendation on Table 2 which lists a set of probability topics that need to be ‘introduced’ in a mathematics classroom. The mere list of topics in this recommendation implies subject matter content knowledge. However, if the recommendation had established more clearly *how, in what order, what types of problems* should accompany these topics, and *how much emphasis* should be placed on each, the categorization might have been different. Let us also consider the verb family *understand* which had the highest frequency (29) in the four documents overall. In the mathematics education literature much has been written about the definition of this verb family. For example, Skemp (1976) makes a distinction between *relational understanding* (“knowing both what to do and why” (p.20)) and *instrumental understanding* (“rules without reasons” (p.20)). On the other hand, the National Research Council (2001) refers to *procedural understanding* and *conceptual understanding*. However, in the four reports examined in this study, it is not clearly indicated by the professional organizations which of these meanings the verb family *understand* carries when used in a recommendation. Such precise meanings are needed so as to accurately code the recommendations.

CONCLUSION

In recent decades, probability has come to gain importance as one of the content areas of school curricula in the United States. However, research on teachers’ knowledge in this content area is scarce. The identification of the knowledge of probability that

secondary mathematics teachers need to have in the form of content topics and their aspects of teaching is an essential tool that can be used in future research in this area. The analysis of recommendations on probability provided by professional organizations has revealed the importance of language in attempting to communicate to mathematics educators and teachers what is expected that they know and teach. As mentioned, 53 verb families were identified in the data sources. However, no clear definitions of these verbs, as related to the probability topics they accompanied, were provided by any of the sources leaving much to the interpretation of the researcher. Precise definitions of action verbs are needed in such documents to avoid possible errors in the coding of the recommendations and to help educators as they plan courses for prospective mathematics teachers.

Last, the analysis of the reports on recommendations for the preparation of secondary mathematics teachers by the AMS, ASA, MAA, and NCTM, revealed the inadequate number of such recommendations especially with regards to pedagogical content knowledge and curricular knowledge requirements specific to the area of probability at the secondary level. Given the increased attention of probability in school curricula, it is essential that professional organizations provide more extensive and detailed reports regarding the recommended skills in probability for future mathematics teachers. It would perhaps be most beneficial if professional organizations provide such a report collaboratively so that there is common agreement about the expectations of probabilistic knowledge of secondary mathematics teachers.

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STATISTICAL GRAPHS PRODUCED BY PROSPECTIVE TEACHERS IN COMPARING TWO DISTRIBUTIONS

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We analyse the graphs produced by 93 prospective primary school teachers in an open statistical project where they had to compare two statistical variables. We classify the graphs according its semiotic complexity and analyse the teachers' errors in selecting and building the graphs as well as their capacity for interpreting the graphs and getting a conclusion on the research question. Although about two thirds of participant produced a graph with enough semiotic complexity to get an adequate conclusion, half the graphs were either inadequate to the problem or incorrect. Only one third of participants were able to get a conclusion in relation to the research question.

Keywords: Statistical graphs, semiotic complexity, prospective teachers, assessment, competence.

INTRODUCTION

Graphical language is essential in organising and analysing data, since it is a tool for *transnumeration*, a basic component in statistical reasoning (Wild & Pfannkuch, 1999). Building and interpreting statistical graphs is also an important part of statistical literacy which is the union of two related competences: interpreting and critically evaluating statistically based information from a wide range of sources and formulating and communicating a reasoned opinion on such information. (Gal, 2002). Because recent curricular guidelines in Spain introduce statistics graph since the first year of primary school level and therefore, this research was oriented to assess prospective primary school teachers' graphical competence in order to use this information in improving the training of these teachers.

Understanding statistical graphs

In spite of its relevance, didactic research warn us that competence related to statistical graphs is not reached in compulsory education, since students make errors in scales (Li & Shen, 1992) or in building specific graphs (Pereira Mendoza & Mellor, 1990; Lee & Meletiou, 2003; Bakker, Biehler & Konold, 2004). Other authors define levels in graph understanding (Curcio, 1989; Gerber, Boulton-Lewis & Bruce, 1995; Friel, Curcio & Bright, 2001) that vary from a complete misunderstanding of the graph, going through reading isolated elements or being

able to compare elements to the ability to predict or expand to data that are not included in the graph. More recently, these levels were expanded to take into account the critical evaluation of information, once the student completely reads the graph (Aoyama, 2007):

1. *Rational/literal level*. Students correctly read the graph, interpolate, detect the tendencies and predict. They use the graph features to answer the question posed but they do neither criticise the information nor provide alternative explanations.
2. *Critical level*: Students read the graph, understand the context and evaluate the information reliability; but they are unable to think in alternative hypotheses that explain the disparity between a graph and a conclusion.
3. *Hypothetical level*: Students read the graphs, interpret and evaluate the information, and are able to create their own hypotheses and models.

Graphical Competence in Prospective Teachers

Recent research by Espinel, Bruno & Plasencia (2008) also highlight the scarce graphical competence in future primary school teachers, who make errors when building histograms or frequency polygons, or lack coherence between their building of a graph and their evaluation of tasks carried out by fictitious future students. When comparing the statistical literacy and reasoning of Spanish prospective teachers and American university students even when the tasks were hard for both groups, results were much poorer in the Spanish teachers, in particular when predicting the shape of a graph or reading histograms. Monteiro and Ainley (2007) studied the competence of Brazilian prospective teachers and found many of these teachers did not possess enough mathematical knowledge to read graphs taken from daily press. A possible explanation of all these difficulties is that the simplicity of graphical language is only apparent, since any graph is in fact a mathematical model. In producing a graph we summarize the data, going from the individual observations to the values of a statistical variable and the frequencies of these values. That is, we introduce the frequency distribution, a complex object that refers to the aggregate (population or sample) instead of referring to each particular individual and this object can be not grasped by the students.

THE STUDY

As stated in the introduction, the main goal in our research was to assess the graphical competence of prospective primary school teachers. A secondary aim was to classify the graphs produced by these teachers as regards its complexity. More specifically we analyse the graphs produced by 93 prospective teachers when

working in an open statistical project with the aim of providing information useful to teacher educators. These students had studied descriptive statistics (graphs, tables, averages, spread) the previous academic year (their first year of University) as well as in secondary school level. The data were collected along a classroom practice (Godino, Batanero, Roa & Wilhelmi, 2008) that was carried out in a Mathematics Education course (second year of University) directed to prospective teachers in the Faculty of Education, University of Granada. In this practice (2 hours long) we proposed prospective teachers a data analysis project. At the end of the session, participants were given a sheet with the data obtained in the classroom and were asked to individually produce a data analysis written report to answer the question set in the project. Participants were free to use any statistical graph or summary and work with computers if they wished. They were given a week to complete the reports that were collected and analysed.

The statistical project: “Check your intuitions about chance”

This project is part of a didactical unit designed to introduce the “information handling, chance and probability” content included in the upper level of primary education. Some aims are: a) showing the usefulness of statistics to check conjectures and analyse experimental data; b) checking intuitions about randomness and realising these intuitions are sometimes misleading. The sequence of activities in the project was as follows.

1. *Presenting the problem, initial instructions and collective discussion.* We started a discussion about intuitions and proposed that the future teachers carry out an experiment to decide whether they have good intuitions or not. The experiment consists of trying to write down apparent random results of flipping a coin 20 times (without really throwing the coin, just inventing the results) in such a way that other people would think the coin was flipped at random.
2. *Individual experiments and collecting data.* The future teachers tried the experiment themselves and invented an apparently random sequence (simulated throwing). They recorded their sequences using H for head and T for tail. Afterwards the future teachers were asked to flip a fair coin 20 times and write the results on the same recording sheet (real throwing).
3. *Classroom discussion, new questions and activities.* After the experiments were performed we started a discussion of possible strategies to compare the simulated and real random sequences. A first suggestion was to compare the number of heads and tails in the two sequences since we expect the average number of heads in a random sequence of 20 tosses to be about 10. The lecturer posed questions like: If the sequence is random, should we get exactly 10 heads and 10 tails? What if we get 11 heads and 9 tails? Do you think in this case the

sequence is not random? These questions introduced the idea of comparing the number of tails and heads in the real and simulated experiments for the whole class and then studying the similarities and differences.

4. At the end of the session the future teachers were given a copy of the data set for the whole group of students. This data set contained two statistical variables: number of heads for each of real and simulated sequences and for each student; n cases with these 2 variables each. As prospective teachers were divided in 3 groups, n varied (30-40 cases in each group). They were asked to complete the analysis at home and produce a report with a conclusion about the group intuitions concerning randomness. Students were able to use any statistical method or graph and should include the statistical analysis in the report.

RESULTS AND DISCUSSION

Once the students' written reports were collected, we made a qualitative analysis of these reports. By means of an inductive procedure we classified into different categories the graphs produced as a part of the analysis, the interpretations of graphs and the conclusions about the group intuitions. The classification of graphs took into account the type of graph, number of variables represented in the graph, and underlying mathematical objects as well as some theoretical ideas that we summarise below.

Font, Godino and D'Amore (2007) generalize the notion of *representation*, by taking from Eco the idea of semiotic function "*there is a semiotic function when an expression and a content are put in correspondence*" (Eco, 1979, p.83) and by taking into account an ontology of objects that intervene in mathematical practices: *problems, actions, concepts-definition, language properties and arguments*, any of which could be used as either expression or content in a semiotic function. In our project we propose a *problem* (comparing two distributions to decide about the intuitions in the set of students) and analyse the students' practices when solving the problem. More specifically we study the graphs produced by the students; these graphs involve a series of *actions, concepts-definitions* and *properties* that vary in different graphs. Consequently the semiotic functions underlying the building and interpretation of graphs, including putting in relation the graphs with the initial question by an argument also vary. We therefore should not consider the different graphs as equivalent representations of a same mathematical concept (the data distribution) but as different configurations of interrelated objects that interact with that distribution. Five students only computed some statistical summaries (mean, median or range) and did not produce graphs; we are not taking into account these students in our report. Using the ideas above we performed a semiotic analysis of

the different graphs produced by the other 88 students and defined different levels of semiotic complexity as follow:

L1. Representing only his/her individual results. Some students produced a graph to represent the data they obtained in his/her particular experiment, without considering their classmates' data. These graphs (e.g. a bar chart) represent the frequencies of heads and tails in the 20 throwing. Students in this level tried to answer the project question for only his /her own case (tried to assess whether his/her intuition was good); part of these students manifested a wrong conception of chance, in assuming a good intuition would imply that the simulated sequence would be identical to the real sequence in some characteristic, for example the number of heads. Since they represented the frequency of results in the individual experiment, in fact these students showed an intuitive idea of statistical variable and distribution; although they only considered the Bernoulli variable "result of throwing a coin" with two possible values: "1= head", 0= tail" and 20 repetitions of the experiment, instead of considering a Binomial distribution "number of heads in the 20 throwing" that have a wider range of values (1-20 with average equal to 10) and r repetitions of the experiments (r = number of students in the classroom).

L2. Representing the individual values for the number of heads. These students did neither group the similar values of the number of heads in the real nor in the simulated sequences. Instead, they represented the value (or values) obtained by each student in the classroom in the order the data were collected, so they did neither compute the frequency of the different values nor explicitly used the idea of distribution. The order of data in the X-axis was artificial, since it only indicated the arbitrary order in which the students were located in the classroom. In this category we got horizontal and vertical bar graphs, line graphs of one or the two variables that, even when did not solve the problem of comparison, at least showed the data variability. Other students produced graphs such as pie chart, or stocked bar charts, that were clearly inappropriate, since they did not allow visualizing the data variability.

L3. Producing graphs separate for each distribution. The student produced a frequency table for each of the two variables and from it constructed a graph or else directly represented the graph with each of the different values of the variable with its frequency. This mean that the students went from the data set to the statistical variable "number of heads in each sequence" and its distribution and used the ideas of frequencies and distribution. The order in the X-axis was the natural order in the real line. In case the students did not use the same scale in both graphs or used different graphs for the two distributions the comparison was harder. Examples of correct graphs in this category were bar graphs and frequency polygons. Students also produced incorrect graphs in this category such as histograms with incorrect

representation of intervals, bar graphs with axes exchanged (confusing the independent and dependent variable in the frequency distribution), representing the frequencies and variable values in an attached bar graph or representing variables that were not related.

L4. Producing a joint graph for the two distributions. The students formed the distributions for the two variables and represented them in a joint graph, which facilitated the comparison; the graph was more complex, since it represented two different variables. We found the following variety of correct graphs: attached bar chart; representing some common statistics (e.g. the mean or the mode) for the two variables in the same graph; line graphs or dot plots in the same framework. Example of incorrect graphs in this category were graphs presenting statistics that were not comparable (e.g. mean and variance in the same graphs) or the same statistics for variables that cannot be compared.

In Figure 1 we present an example of graphs produced in each category. Even when within each of these categories we observe a variety of graphs and configurations of mathematical objects it is evident a qualitative gap between each of the different levels. In Table 1 we present the distribution of students according the semiotic complexity of the graph, its correctness, the interpretation of the graph and the conclusion about intuitions.

Table 1. Results

	Correctness of the graph			Interpretation of graph			Conclusion on the intuitions			Total in the level
	1	2	3	1	2	3	1	2	3	
L1. Representing only the student data	1		1	1	1				2	2
L2. Representing individual results	10	1	4	4	10	1		3	12	15
L3. Separate graphs	15	17	14	15	15	16	1	12	33	46
L4. Joint graphs	14	6	5	9	11	5	1	7	17	25
Total	40	24	24	29	37	22	2	22	64	88

(1) Correct; (2) Partially correct; (3) Incorrect or no interpretation / conclusion

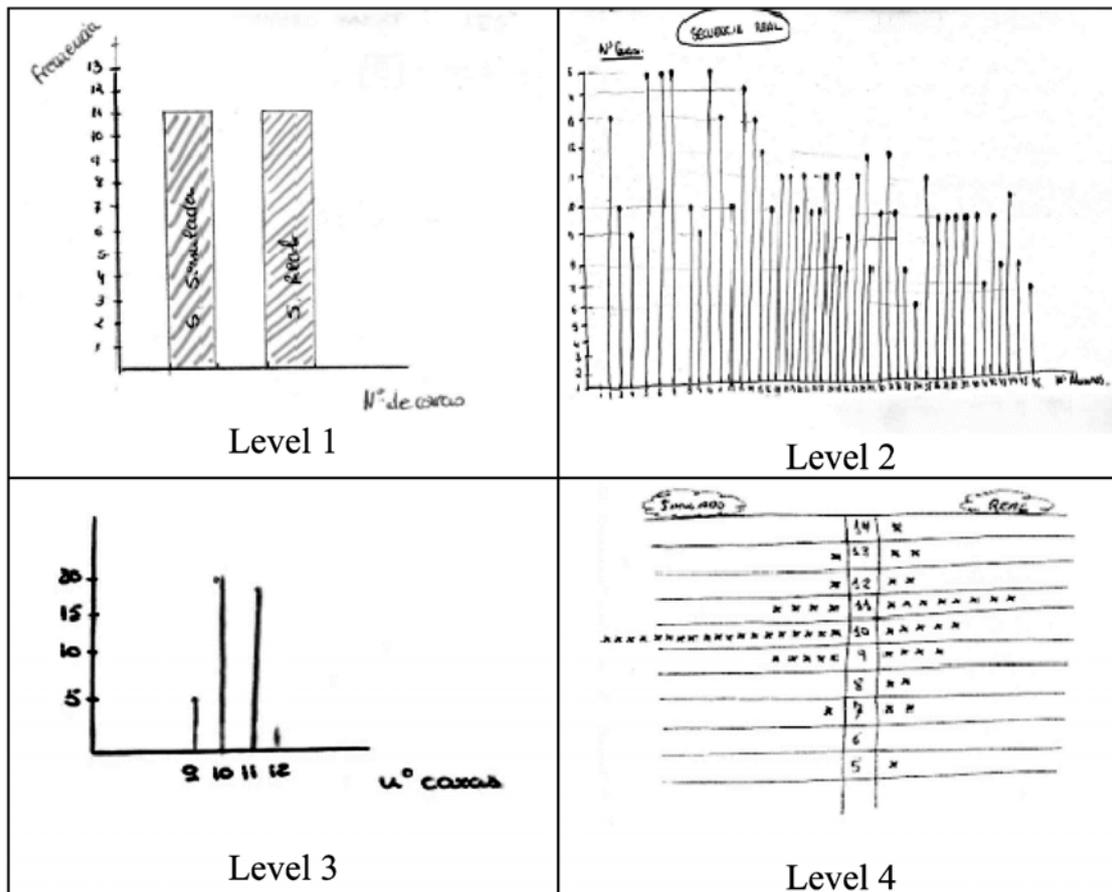


Figure 1. Examples of graphs in each different level of semiotic complexity

From a total of 93 students 88 (94,6%) produced some graphs when analysing the data, even if the instructions given to the student did not explicitly require that they constructed a graph. This fact suggests that students felt the need of building a graph and reached, by a transnumeration process some information that was not available in the raw data. Most students (52,2%) produced separate graphs for each variable (level 3), that were generally correct o partly correct (correct graph with different scales or different graph in each sample; not centring the rectangles in the histogram, or missing labels).

14 students in this level constructed a non-meaningful graph since they represented the product of values by frequencies, exchanged the frequencies and values of variables in the axes thus confusing the independent and dependent variable in the frequency distribution. 28,4% students worked at level 4, and produced only a joint graph for the two variables, although 6 of these graphs were partly correct and 5 incorrect (same reasons than those described in level 3). Few students only analysed their own data (level 1) and only 17% of participants studied the value got

by each student without forming the distribution. Consequently the concept of distribution seemed natural for the majority of students who used it to solve the task, although the instructions did not require this explicitly.

In general, these prospective teachers interpreted correctly or partially correctly the graphs in all the levels, reaching the Curcio's (1989) intermediate level (reading between the data) and the difficulty of interpretation of graphs increased with its semiotic complexity. However, an important part of students in our levels 3 and 4, even when they built correct graphs did not reached the "reading between the data" level, because either they did not interpret the graph either made only a partial interpretation. As regards the Aoyama's (2007) levels, the majority of prospective teachers only read the graphs produced at a rational/literal level, without being able of read the graphs at a critical or a hypothetical level. The teachers performed a mathematical comparison of the graphs but did not get a conclusion about the intuitions in the classroom (e.g. they correctly compared averages but did not comment what were the implications in relation to the students' intuitions). Only two students in the group reached the hypothetical level in reading the graphs, as they got the correct conclusion about group's intuition. These two students realised that the group have correct intuitions about the average number of heads but poor intuitions about the spread. Students were supposed to get this conclusion from comparing the averages and range in the variables in the simulated and real sequences distributions. At higher level statistical tests could also be used to support this conclusion that have been observed in previous research about people perception of randomness. 22 participants got a partial conclusion that the intuition as regards averages was good, as they were able to perceive difference or similitude in the averages, but they did not considered the results obtained in comparing spread of the variable (number of heads) in the two sequences. These students also work at the Aoyama's (2007) hypothetical level, although they did not considered spread in comparing the two distributions. Those working at levels 1 and 2 got few partly correct conclusions and none correct conclusion, so that these levels of complexity in the graph were not adequate to get a complete conclusion.

CONCLUSIONS

In the project posed the prospective teachers went through the different steps in the statistics method as described by Wild and Pfannkunch (1999) in their PPCAI cycle: setting a problem, refining the research questions, collecting and analysing data and obtaining some conclusions. They also practiced the process of modelling, since, beyond working with the statistics and random variables, they should interpret the results of working with the mathematical model in the problem context (whether the students' intuitions was good or not). This last step (relating the result

with the research question) was the most difficult for the students, who lacked familiarity with statistical projects and modelling activities. Since these activities are today recommended in the teaching of statistics since primary school level in Spain and are particularly adequate to carry out group and individual work as recommended in the Higher European Education Space we suggest they are particularly suitable for the training of teachers. Our research also suggest that building and interpreting graphs is a complex activity and confirm some of the difficulties described by Espinel, Bruno and Plasencia (2008) in the future teachers, in spite that they should transmit graphical language to their students and use it as a tool in their professional life. Improving the teaching of statistics in schools should start from the education of teachers that should take into account statistical graphs.

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THE ROLE OF CONTEXT IN STOCHASTICS INSTRUCTION

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This report focuses on a research project that combines two aspects of a stochastics curriculum related to teachers' classroom practice, and their students' stochastic knowledge and beliefs. Data were collected with questionnaires. The development of the questionnaires derived from results of a qualitative research project will be sketched. Afterwards, some results concerning the role of the context will be discussed.

Keywords: stochastics teachers, students' learning, beliefs, role of the context

INTRODUCTION

One central aim of the teaching of stochastics in school is to prepare students to deal with real stochastic situations in their lives (Jones, Langrall, & Mooney, 2007). This aim involves two goals, the students' comprehension of stochastic concepts, and the students' awareness that it is possible to use stochastics to cope with specific real situations. There is a wide consensus between researchers into stochastic education that to achieve these two goals, students must explore stochastic concepts on the basis of realistic situations instead of exploring solely pseudo realistic situations (cards, urns, dices) or learning stochastics in a formal and abstract way (e.g. Jones et al., 2007). While there is a consensus about the *role of the context* for the teaching and learning of stochastics, there is, however, still little insight into the daily teaching practice of "conventional" stochastics teachers. In this report, the results of a research project involving a quantitative survey concerning the classroom practice of German stochastics teachers will be discussed. The main focus is the role of the context based on the following aspect:

1. The teachers' beliefs about the goals of teaching stochastics,
2. the students' beliefs about the usefulness of stochastics, and
3. the impact of the teachers' beliefs on the students' beliefs.

The research project discussed in this report is part of a larger research project involving a qualitative designed investigation of stochastics teachers' classroom practices and the impact of the latter on students' learning (Eichler, 2008a; Eichler, 2007). The results of the qualitative part of the research that provides the basis for the quantitative survey will be sketched in the following.

RESULTS OF THE QUALITATIVE RESEARCH

The first step of the qualitative research comprised an interview study with eight stochastic teachers (Eichler, 2007a). This study yielded four types of (individual) statistics curricula that are similar concerning the content, but considerably differ

with regard to the teachers' objectives or beliefs. The distinction between the four types is characterised by differences of the teachers concerning two dimensions. The first dimension can be described with the dichotomous pairs of a static versus a dynamic view of mathematics or stochastics. The second dimension can be described with the orientation on formal mathematics versus mathematical applications. The four types of statistics teachers were characterised with reference to their main objectives as follows (Eichler, 2007a).

(dimension 2)	application	<i>Application preparers</i> : their central goal is to have students grasp the interplay between theory and applications. Students firstly must learn stochastic theory in order to cope with mathematical applications later.	<i>Every-day-life preparers</i> : their central goal is to develop stochastical methods in a process, the result of which will be both the ability to cope with real stochastic problems and the ability to criticise.
	formal	<i>Traditionalists</i> : their central goal is to establish a theoretical basis for stochastics. This involves algorithmic skills and insights into the abstract structure of mathematics, but it does not involve applications.	<i>Structuralists</i> : their central goal is to encourage students' understanding of the abstract system of mathematics (or stochastics) in a process of abstraction which begins with mathematical applications.
		static view of mathematics	dynamic view of mathematics

(dimension 1)

Figure 1: Four types of stochastics teachers

The second step of the qualitative research comprised the observation of the classroom practice of four teachers (Eichler, 2008a). One central result of this step of observation was that the instructional practice of the teachers provides strong evidence that they pursue their main objectives. Concerning the role of the context, the traditionalists and the every-day-life-preparers represent the extreme positions. The students of the traditionalists predominantly explore stochastical concepts on the basis of formal or pseudo realistic situations (cards, urns, dices). They seldom explore realistic situations. In contrast, realistic situations are crucial in the classroom practice of the every-day-life-preparers. Their students predominantly explore stochastical concepts on the basis of realistic situations or real problems, which arise, for instance, from articles of newspapers.

The third step of the qualitative research comprised an interview study with five students of each of the four teachers who were observed before. In this step the construct of statistical knowledge (Broers, 2006) and the distinction of declarative knowledge, procedural knowledge, and conceptual knowledge (Hiebert, & Carpenter, 1992) was used to describe the students' knowledge (Eichler 2008a). A central result of the third step of the qualitative research was that the students differ in their knowledge and beliefs. The differences consist between the students of one teacher, and between sets of students of different teachers. The students also differ concerning the role of the context. Thus, the students differ in the use of stochastic situations (formal, pseudo realistic or realistic) to explain stochastical concepts. Further, the

students differ considerably concerning their beliefs about stochastics and mathematics referring to their relevance for society and their relevance for the own life (Eichler, 2008a).

METHOD

In regard to the characterisation of the four types of teachers (figure 1), a questionnaire including four parts was developed. The first part concerns the instructional contents of stochastics courses. The other three parts of the questionnaire concern the teachers' objectives of statistics and mathematics instruction. In each of the latter three parts of the questionnaire the teachers were asked to rate typical statements of the teachers who represent one of the four types (from full agreement to no agreement, coded with 1 to 5). In these three parts respectively two statements of every type have to be rated.

The questionnaire for the students involves items concerning declarative knowledge and conceptual knowledge. Concerning their *declarative knowledge*, the students were asked to rate a list of 28 statistical concepts whether they: are not able to remember the statistical concept (coded with 0), are able to remember the statistical concept (coded with 1), are able to approximately explain a statistical concept (coded with 2), are able to exactly explain a statistical concept (coded with 3).

Concerning the conceptual knowledge, the students were asked to indicate interconnections into the consecutively numbered concepts (category *declarative knowledge*)

Four parts of the questionnaire comprise the role of the context. Thus, the students were asked to indicate

- stochastic situations of the classroom (category *application*).
- statistical applications along with related statistical concept (category *connections*).
- real situations (outside of the classroom), for which stochastics may be useful (category *benefit*).
- the benefit of stochastics for students' future life, the benefit of stochastics for the students' professional career. These two categories were linked with a single item, in which the students are asked to rate the relevance of stochastics for their lives from high relevance (coded with 5) to no relevance (coded with 1, category *relevance-life*, and category *relevance-profession*).

A random sample of 240 German secondary high schools was selected. These schools were asked to conduct the survey. 166 of these agreed. Two teachers' of each of these schools and three students per teacher with different statistical performance were asked to fill out the questionnaire (January to July 2007). The completed

questionnaires of 107 teachers and 315 students were analysed. The stochastics courses last between three and six month with three to five hours a week.

RESULTS CONCERNING THE TEACHERS

The statistics curriculum is dominated by the so called classical block of probability (see table 1).

Block	Topics and percent of teachers teaching the topic (n=107)
Classical block of probability	Frequencies (98%), Laplacean probability (97%), statistical probability (72%), probability tree (100%), Bernoulli experiment (99%), binomial distribution (100%), expected value (95%), standard deviation (95%)
Inferential statistics	Hypothesis testing (89%), confidence intervals (51%), Bayesian statistics (27%)
Conditional probability	Conditional probability (81%), (in)dependence (80%), theorem of Bayes (74%)
Distributions	Normal distribution (79%), hypergeometrical distribution (49%) Poisson distribution (49%)
Descriptive statistics	Frequencies (98%), mean (87%), spread (74%), median (52%), regression and correlation (16%)

Table 1: Percentage of teachers teaching different instructional content

Factor analysis concerning the objectives of the teachers' statistics curricula in the responses to questionnaires yield three interpretable factors (table 2) which include 15 of the 24 items referring to the objectives of the statistics curriculum. For each factor the number of items and the Cronbach's Alpha is shown in table 2.

Factor	Factor 1 (5 items, $\alpha = 0.689$)	Factor 2 (6 items, $\alpha = 0.725$)	Factor 3 (4 items, $\alpha = 0.779$)
Interpretation	Traditional curriculum, little reference to real data	Curriculum with high reference to real data	Curriculum with high reference to process

Table 2: Factors concerning the objectives the statistics curriculum

In the following the main focus is on the first two factors or rather on the teachers with a high acceptance to the items of one of these two factors. These items are shown in the following table. The items involve a statement of a teacher who represents one of the four types of stochastic teachers (figure 2). The type is indicated in the brackets (T: traditionalists; S: structuralists; A-P: application-preparers; E-P: every-day-life-preparers).

Factor 1	Factor 2
<ul style="list-style-type: none"> - The objective of teaching stochastics is to establish a theoretical foundation of stochastics (T). - Students must learn to deal successfully with abstract and formal systems (S). - Algorithmic skills constitute the basis of learning statistics or mathematics 	<ul style="list-style-type: none"> - The main goal of the teaching of stochastics is the students ability to understand decision-making processes in our society (E-P) - Students must explore stochastic concepts solely on the basis of real stochastic situations (E-P). - Students must learn to use stochastic or mathematical theory to be able to argue referring to real problems (A-P).

<p>(T).</p> <ul style="list-style-type: none"> - Students must be well prepared concerning final exams and studies (T). - Students must learn a precision in reasoning in order to deal successfully with abstract and formal mathematics (S). 	<ul style="list-style-type: none"> - Students must understand that stochastics or mathematics is part of the general ability of problem solving (E-P). - Students must learn to solve real problems either for their own or in a team (E-P). - Students solely will be motivated if they understand that stochastics or mathematics is applicable in the reality (A-P).
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Table 3: List of the items included in factor 1 and factor 2.

The correlation coefficient between factor 1 and factor 2 is - 0,1. For the distinction between teachers with high acceptance to the items of one factor and low acceptance to the other, two clusters were defined by the medians concerning the value of the two factors. Cluster 1 includes those teachers with high acceptance to factor 1 and low acceptance to factor 2. Cluster 2 includes those teachers with high acceptance to factor 2 and low acceptance to factor 1. Cluster 1 includes 39 teachers, cluster 2 34 teachers.

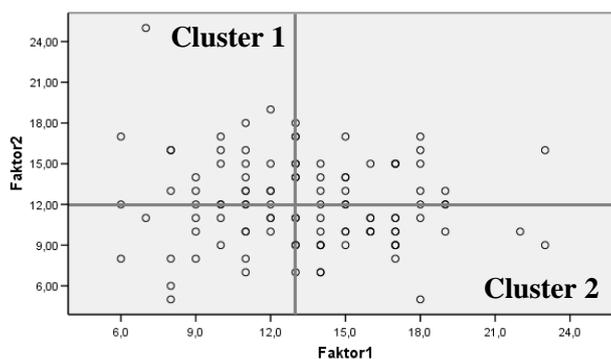


Figure 2: Clusters of teachers concerning factor 1 and factor 2

RESULTS CONCERNING THE STUDENTS

Figure 3 shows the results concerning five categories:

1. the students' self estimated ability to explain the 28 different stochastic concepts (the students' declarative knowledge),
2. the number of connections between two different stochastic concepts as part of the students conceptual knowledge (for instance: if a student indicated the connection between the three concepts of expected value, variance and standard deviation, the number of possible connection is 3 over 2 or rather 3)
3. the number of stochastic situations of the classroom (*application*).
4. the number of pairs of applications and statistical concept (*connections*).
5. the number of real stochastic situations (*benefit*).

Due to the fact that different teachers indicated different numbers of stochastic concepts taught in the classes, figure 3 shows the results concerning the category *knowledge weighted*. For this category the students' self estimated knowledge is

divided by the number of concepts taught by the teachers. This category alludes to a restricted sample, which involves the set of completed questionnaires of one class (some of the completed questionnaires allude only to the teachers or only to the students).

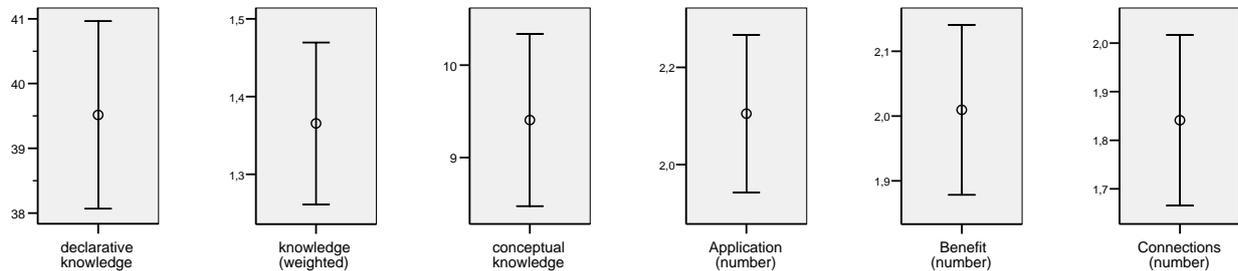


Figure 3: Results concerning the students knowledge and beliefs (average and 95%-interval)

The interpretation (only for the averages) is as follows: The sum of the students' self-estimations concerning the 28 given stochastic concepts is in average about 39. In average, the students rate their knowledge about the stochastic concepts taught by their teachers with about 1,4. The students indicate more than 9 connections between different stochastic concepts, they indicate about 2,1 stochastic situations of the classroom and about 2 stochastic situations outside of the classroom. Finally, the students indicate in average about 1,9 connections of a stochastic situation and a specific stochastic concept.

Concerning the role of the context it is important whether the indicated stochastic situations to the categories application, benefit, and connections refer to realistic situations or pseudo realistic situations (the pseudo realistic situations include games of chance). Table 4 shows the distribution of the indicated stochastic situations (with the number of indications in brackets) for the first two categories:

Application		Benefit	
realistic situations (255)	pseudo realistic situations (385)	realistic situations (359)	pseudo realistic situations (270)
quality control (48)	game of chance (100)	economy (63)	game of chance (100)
forecasts (30)	lottery (91)	quality control (45)	lottery (78)
elections (28)	dice (66)	elections (39)	poker (13)
statistics (24)	urns (33)	statistics (37)	bets (18)
clinical diagnostic (23)	coins (23)	polls (32)	dice (14)
polls (16)	cards (15)	clinical diagnostic (26)	bingo (13)
economy (16)	poker (13)	further education (26)	
weather (11)	lots (10)	weather (17)	
		stock market (16)	
		insurance (12)	
other situations with less than 10 indications		other situations with less than 10 indications	

Table 4: Distribution of stochastic situations and number of indications in brackets

The stochastic situations are topics: the situation *economy* includes, for instance, market research, promotion and some more specific situations. Although some of the

stochastic situations were indicated for both categories, application and benefit, it is obvious that

- concerning the category benefit, the pseudo realistic situations are restricted to existing games of chance, and
- concerning the category application, the majority of situations refers to pseudo realistic situations.

Some of the indicated situations stem from typical tasks in German textbooks, in particular quality control, elections, and clinical diagnostic. Students predominantly use these three different situations connecting a stochastic situation with a specific stochastic concept. The students, however, more often use pseudo realistic situations for connecting a stochastic situation with a specific stochastic concept, and, in this case, predominantly dice, urns and lottery (see table 4).

Realistic situations (157)		Pseudo realistic situations (341)	
Situation	Connected stochastic concepts	Situation	Connected stochastic concepts
Quality control (85)	hypothesis testing (17), binomial distribution (6), confidence interval (5), Bernoulli experiment (4), conditional probability (4) normal distribution (3), expected value (2), spread (2), probability tree (1), combinatorics (1) 2 x 2 table (1)	Dice	Laplacean probability (36), Bernoulli experiment (14), probability tree (9) random experiment (7), expected value (5), binomial distribution (4) probability (2), statistical probability (2), normal distribution (2), hypothesis testing (1), variance (1) simulation (1), combinatorics (1)
Clinical diagnostic (33), elections (9)		Urns (79), lottery (53)	

Table 5: stochastic situations and related stochastic concepts

Obviously, students remember predominantly connections between pseudo realistic situations and specific stochastic concepts. Further, the variation of indicated stochastic situations concerning the category connections is much lesser than the variation of indicated situations concerning the categories application and benefit.

Although the students estimated their declarative knowledge by themselves, these estimations give evidence of the students' factual knowledge. Thus, the correlations between the students' declarative knowledge and other categories discussed above are shown in table 6:

	conceptual knowledge	Application		benefit		connections	
		realistic situations	pseudo realistic situations	realistic situations	pseudo realistic situations	realistic situations	pseudo realistic situations
declarative knowledge	0,418**	0,172**	-0,233**	0,277**	-0,181**	0,269**	-0,177**

Table 6: Correlations between students' declarative knowledge and 5 other categories

The correlations are predominately weak, although they are significant different from zero. However, the correlations as a whole give evidence that the students' self estimated declarative knowledge measure in some sense the students' flexibility of

dealing with statistical concepts. Further, there is evidence that the higher the students' flexibility of dealing with statistical concepts is the higher their reference to realistic statistical situations is, and the lower the reference to pseudo realistic situations is.

TEACHERS – STUDENTS

To prove possible interrelations between the teachers' orientation concerning the goals of the stochastics instruction and the students' knowledge and beliefs, the sample must be restricted. This was necessary, because sometimes a teacher sends his completed questionnaire back but his students not, sometimes the students send their completed questionnaires back, but the teacher not. Two strategies were used for the following analysis. Firstly, the correlations between the factors, i.e. factor 1 and factor 2 (or rather the sum of ratings the teachers given to the items of the two factors), and the categories concerning the students (knowledge weighted, application, benefit, and connections). Secondly, the clusters of teachers defined above (figure 2) were used to split up the sample of the students. The averages of the two new samples concerning the categories knowledge weighted, application, benefit, and connections were compared by a t-test.



Figure 4: Students' weighted knowledge and students' procedural knowledge. f1F2: teachers, who have low acceptance to factor 1 and high acceptance to factor 2, F1f2: teachers, who have high acceptance to factor 1 and low acceptance to factor 2

Most parts of the analysis give no evidence of an interrelation between the teachers' orientation and the students' knowledge and beliefs. For instance, concerning the clusters of teachers, who have low acceptance to factor 1 (traditional curriculum) and high acceptance to factor 2 (curriculum with high reference to real data) or who have low acceptance to factor 2 and high acceptance to factor 1 (see figure 2), the distribution of the students' weighted knowledge and the students' ability to indicate connections between stochastical concepts (figure 4).

Although there are differences in detail, these differences are statistically not relevant. Thus, there is little or no evidence that a teacher's orientation towards a traditional curriculum (factor 1) or a curriculum that includes real data (factor 2) promote (or impede) students' learning in reference to the students' declarative knowledge, the students' conceptual knowledge, and the students' beliefs concerning the relevance of statistics except the category benefit. For this category t-test give some evidence that the students of teachers with high acceptance to factor 2 and low

acceptance to factor 1 use more often realistic situations than pseudo realistic situations to explain the relevance of stochastics for the society. However, the differences are not significant (table 7).

Benefit	Realistic situations (F1f2)	$\bar{x} = 1,14$	Pseudo realistic situations (F1f2)	$\bar{x} = 0,66$
	Realistic situations (F1f2)	$\bar{x} = 0,83$	Pseudo realistic situations (F1f2)	$\bar{x} = 1,00$
		$\alpha = 0,121$		$\alpha = 0,063$

Table 7: Difference of the students concerning the category benefit

In contrast to the low interrelations between the teachers' objectives concerning the statistics curriculum and their students' knowledge and the students' beliefs, there is stronger evidence that the amount of contents has an impact on the students' knowledge. So, the greater the number of statistical concepts taught by the teachers is, the lower the declarative knowledge of the students seems to be (Pearson's correlation coefficient $r = -0,43^{**}$).

CONCLUSION

The results of the quantitative survey concerning the curriculum of statistics teachers and the learning of students give evidence that:

- "The traditional way of teaching statistics, with its heavy emphasis on formal probability" (Broers, 2006, p.4) is still existent in German secondary high schools;
- the teachers' instructional contents are similar, but the teachers' objectives differ considerably;
- the quality of students' declarative knowledge affects their conceptual knowledge and their beliefs concerning the relevance of statistics;
- the students predominately indicate few realistic situations to explain both the relevance of stochastics for the society and connections between stochastic situations and specific stochastic concepts;
- the teachers' orientation towards a curriculum with high reference to real data seems to affect the students' ability to use realistic stochastic situations to explain the relevance for the society.

However, the latter interrelation between the teachers' orientation and the students' beliefs is weak. Above all, there is no evidence for the impact of the teachers' orientation and the students' knowledge and beliefs. The lack of statistical relevant interrelations between the teachers teaching and the students learning may be caused by the fact, that there are only small differences of the teachers' stochastics teaching with the emphasis on probability. It is possible that a stronger orientation to a data driven curriculum has a stronger impact of the students' knowledge and beliefs concerning the role of the context. Further it is possible, that the quantitative survey discussed in this report is not able to measure possible differences concerning the

students' knowledge and beliefs. There is some evidence that qualitative research can show differences in detail between students' of teachers who have different goals concerning the role of the context (see Eichler, 2008a).

However, the stochastics teachers' teaching is determined by the teachers' fundamental orientation, i.e. the teachers' system of objectives (or beliefs) concerning stochastics teaching. Pajares (1992) stated that it could be difficult to change the teachers' central beliefs. One approach to change these central beliefs may start by the teachers' conviction that a changed curriculum actually will promote students' stochastical knowledge. For this reason it would be desirable to have more research into the stochastics teachers' curricula, the students' stochastical knowledge and beliefs, and, in particular, the relations between stochastics teachers' curricula and the students' stochastical knowledge or beliefs.

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DOES THE NATURE AND AMOUNT OF POSTERIOR INFORMATION AFFECT PRESCHOOLER'S INFERENCES

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Children as young as 5 have been found to possess basic notions of probability, in contradiction to the piagetian perspective. In the current pilot study, preschoolers (N=25) participated in a probability task of single events, with alterations in the given posterior information. Children took into account the new sets of information and responded differently in each condition, depending on the nature and the amount of information. Such findings stress the importance of designing probability tasks in accordance to the children's cognitive capacities and probabilistic understanding.

Key words: preschoolers, posterior probability, design of probability tasks.

INTRODUCTION

The development of probabilistic thinking is a topic of much interest during the last decades from many perspectives, i.e. mathematical, cognitive, and educational.

Early research carried out mainly by Piaget and Inhelder (1951) supported that children undergoing the pre-operational developmental stage (4-7 years old) have no intuitions of randomness and no conceptions of chance and probability. Under this traditional perspective, probabilistic concepts develop as complementary to logical operational structures which emerge in relation to age (Kreitler & Kreitler, 1986). At the age of 5, children cannot differentiate certain from random events.

On the other hand, Fischbein (1975) suggested that young children possess a particular intuition of chance and probability in the sense that they possess 'primary intuitions' which are 'cognitive acquisitions derived from the experience of the individual, without the need for any systematic instruction' (Fishcbein et al, 1971).

Based on this intuitive perspective, young children show a minimal understanding of randomness and can identify the most/least likely outcomes (Way, 2003). Preschoolers have been found to understand the probability of an event (Jones et al, 1997; Falk& Wilkening, 1998), to make use of random sampling and base rate information (Denison et al, 2007), to realize part-part comparisons in order to estimate probability (Spinillo, 2002), to make use of probabilistic evidence in order to infer about causal strength (Kushnir& Gopnik, 2005). Preschoolers are able to compute prior probabilities in order to predict an uncertain event.

In the current study preschoolers were tested onto whether they can take into account and manipulate posterior probability. Posterior probability is a revised probability that integrates new available information. What happens when children are asked to consider new specific information in order to make judgments about the outcome of a probabilistic task? According to a study carried out by Girotto & Gonzalez (2008), even kindergartners were found to be able to use posterior information in order to update their evaluations about random outcomes. Young children made optimal decisions while integrating new information into prior information of single events.

The general hypothesis is that preschoolers are expected to take into consideration the extra-posterior information while building-up their inferences. The nature and amount of information that characterizes each condition (base rate vs category) is expected to affect children's responses: the more precise information (condition 2 vs condition 1), the more accurate judgments.

METHODOLOGY

This pilot study took place in a public kindergarten in a town of Western Greece, in 2008. The random sample consisted of both girls and boys. In this study we did not consider age and gender effects due to the small sample. Participants (N=25), aged 5 to 6, were asked to make predictions in a two-stage procedure: at a first point they were asked to infer given *prior* information and then they were asked to infer again by taking into account new, available *posterior* information.

The probabilistic task consisted of animal cards that depicted ducks and mice. In every condition the sample space was invariably 8 and cards were distributed unequally in 2 identical boxes. Among the 8 cards there was one lucky-card that had a sticker on it. Once children found that particular card in the correct box, they gained a sticker themselves. The lucky animal in all cases was a duck -participants were aware of that from the beginning of the task- and consequently mice were used as 'noise'.

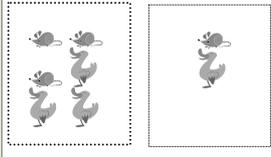
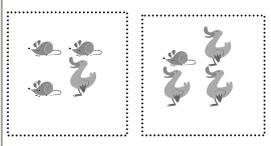
		1st stage of choice (based on prior information)	2nd stage of choice (based on posterior information)
1st condition: base rate		No info provided about the content.	Aware that one box has 6 animal-cards vs the other box with 2.
2nd condition: category		Aware that both boxes have 4 cards each.	Aware that the distributions are 3:1 and 1:3

Table 1: Design of the probabilistic task.

The design of the task (Table 1) comprised 2 conditions with differences in the nature and amount of information and 2 stages of provided information that affected participants' choice. In both conditions, participants began with information that didn't favor any box; both boxes had equal chances to carry the lucky-animal (level of probability, 50:50). Then, posterior information would provide additional evidence about in which box the lucky-duck might be.

In precise, in the 1st condition, children were given as prior information nothing, they were just asked to choose one box at random. As posterior information, they were informed that one particular box contained 6 whereas the other 2 cards.

In the 2nd condition, information was more detailed both in the prior and the posterior stages. In the beginning, preschoolers were aware that both boxes had 4 cards each, and after, they were given as posterior information each box's distributions of the sample spaces (3:1 vs 1:3).

Children participated in pairs in a separate room of the school. They were given instructions about the task and were motivated by the fact that they would win stickers. During the game, cards remained on the table reminding them the given information. At a 1st level, participants were asked to select orally the box they believed contained the lucky animal-card. As soon as they pointed to a box and before drawing a card of their choice, they were given new information orally by the experimenter about where the lucky card might be. Based on this new information, children either reconsidered their prior choice and switched box or made new predictions in order to succeed the desired outcome, i.e. the lucky –card. All participants carried out the 2 conditions in the same order.

Children recorded by themselves their final choices on specially designed sheets, independent of the actual outcome. These recorded sheets were used for further analysis.

RESULTS

Overall, children made correct predictions; they gave in total 36 correct answers out of 50. For the purposes of the current study, 'correct' is the answer that relates to the box with the higher probability of hiding the lucky animal. The predictions that related to the less probable box were scored as 'incorrect'. Such coding is used just for the analysis of the current results, as there is no such 'correct- incorrect' in probability tasks.

From the descriptive analysis (Table 2) it can be seen that in condition 1, children predicted the correct box by 60% and in condition 2 they responded correctly by 84%, in terms of selecting the more probable box.

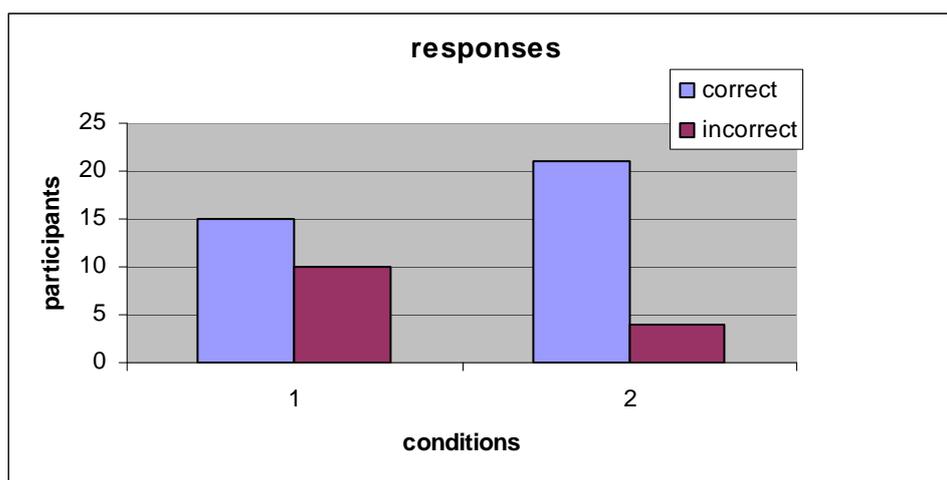


Table 2: Overall responses in cases 1&2.

The differences in the available information of each condition affected children's responses. Concerning the nature and the amount of information, it was found by the paired-sample t-test analysis concerning proportions, that there is a significant difference between conditions 1 and 2, $t(25) = 2.295$, $p < 0.05$. There is a significant difference between the means of the two conditions. This implies that children's inferences in tasks that relate to posterior probability get affected by the kind and the range of information provided as new.

DISCUSSION

The results of this pilot study support that preschoolers may participate in probabilistic tasks successfully and integrate any available information, while forming their inferences in more than one stage. These results comply with the findings of Girotto & Gonzalez (2008). Among these lines, young children correctly revise their decisions when given new sets of information about single, non-repeatable events.

The baseline for both conditions was that the sample space was 8 and the lucky animal was a duck. The amount of given information was more complex and detailed in condition 2 and was not of equivalent difficulty as in condition 1. Thus, in this 2nd condition preschoolers were found to be able to make more correct predictions in terms of choosing the more probable set of given information. Overall, children showed the capacity to consider and handle information while participating in a probabilistic task.

However, the limited sample considers an issue for further research. Another limitation that could be taken into account refers to the children's participation in pairs. If children conducted the task individually would they make the same predictions? Or do they get influenced by their classmates? In addition, more

conditions, randomization of the boxes, more variations in the given information (i.e. qualitative) and other stimuli such as cards with different themes or pictures could lead to different interpretations.

In this game, children made more correct predictions when given more detailed and precise information about the sample space (i.e. condition 2 vs condition 1). This has a methodological significance that should be considered while designing probabilistic tasks. Children express and develop probabilistic ideas, depending on the design of the given activity (Papaparistodemou & Noss 2004; Pratt, 2000). The nature and the amount of information are important factors that affect children's probabilistic thinking.

Opposed to the piagetian perspective, young children before the age of 7 can make inferences and handle more than 2 combinations in order to participate in probability tasks. Recent studies have shown that children as young as 4 demonstrate an understanding of probabilities and expected value, adjust preferences based upon probability, understand basic notions of probabilistic thinking (Acredolo et al, 1989; Schlottmann, 2001; Way, 2003; Nikiforidou & Pange, 2007) and possess specific concepts and skills associated with probabilistic reasoning (Langrall & Mooney, 2005).

Furthermore, preschoolers make use of additional information and reveal a capacity to proceed in posterior probabilities (Giroto & Gonzalez, 2008) or in a two-stage choice task. Future research has to focus in this direction; in setting all the factors that are cognitively equivalent to young children's probabilistic thinking. The types of random generators, the mathematical structure of sample space, the type of responses, the nature of comparison or estimation (Way, 2003), the sort and amount of given information should be taken into consideration while designing probability tasks for preschoolers, who are characterized by intuitive and non-formal thinking.

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STUDENT'S CAUSAL EXPLANATIONS FOR DISTRIBUTION

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This paper presents a case study of two students aged 14-15, as they attempt to make sense of distribution, adopting a range of causal meanings for the variation observed in the animated computer display and in the graphs generated by the simulation. The students' activity is analysed through dimensions of complex causality. The results indicate support for our conjecture that carefully designed computer simulations can offer new ways for harnessing causality to facilitate students' meaning-making for variation in distributions of data. In order to bridge the deterministic and the stochastic, the students transfer agency to specially designed active representations of distributional parameters, such as average and speed.

Keywords: causality, agency, stochastic thinking, variation, randomness, probability

VARIATION AND CAUSALITY

This research study builds on ideas which emerged from two research studies: 1) the seminal work of Piaget (1975, translated from original in 1951) and 2) Pratt's work (1998; 2000) as it attempts to clarify how students let go of determinism whilst at the same time re-apply such ideas in new ways to account for variation (Prodromou, 2008; Prodromou & Pratt, 2008).

Piaget and Inhelder (1951) reported how the organism fails in the first place to apply operational thinking to the task of constructing meanings for random mixtures, which were therefore unfathomable. Only much later, according to Piaget, the organism succeeds in inventing probability as a means of operationalising the stochastic. In contrast, students soon gain mastery over the deterministic, appreciating cause and effect at least in a basic manner, apparently lending itself more easily to operational thinking. Instead of interpreting Piaget's work as presenting an impregnable divide between the stochastic and the deterministic, at least until a late stage of development, we began to wonder whether the divide was a manifestation of conventional technologies and whether digital technology might provide a means by which the deterministic might be harnessed to support new ways of thinking about the stochastic.

In Pratt's work (for example, 2000, 2002), students aged 11 years explored computer-based mini-simulations of everyday random generators, such as coins, spinners and dice. These simulations provided functionality beyond that which would be experienced in everyday life. For example, the students were able to change the workings of the simulation and so explore their ways of thinking about randomness. Gradually, the students articulated the heuristic that "the more times you throw the dice, the more even is its pie chart". We detect in this statement a sense that the number of throws *determined* the appearance of the pie chart. Similar causal

statements were made about other aspects of the system, such as the effect of changing the workings of the simulation.

Pratt referred to these causal heuristics as *situated abstractions* (Noss and Hoyles, 1996), internal meanings for making sense of phenomena that capture the abstracted nature of the meaning, expressed in language tied to the situation. Pratt and Noss (2002) have further elaborated on the nature of situated abstractions as part of a model for the micro-evolution of mathematical knowledge.

We believe Pratt has made a *prima facie* case that, in certain conditions, possibly deeply connected to the potential of technologically-based environments, students can construct stochastic meanings out of causality. In this study, we examine this possibility further by building a digital simulation to provide a *window* on students' *thinking-in-change* (Noss & Hoyles, 1996) about average and spread as parameters within a distribution.

First though, we must be more specific about what we mean by causality. In fact, causality can be seen at a variety of levels (Grotzer and Perkins, 2000; Perkins and Grotzer, 2000). Grotzer and Perkins have proposed a taxonomy or a classification scheme that attempts to organise increasing complexity of causal explanation. The taxonomy comprises causal explanations organised in four dimensions along which causal complexity is characterized:

Mechanism includes the most superficial causal explanations, appealing to the most general of phenomena, or to token agents, perhaps “luck”, “destiny” or “god’s will” in the case of stochastic. Within this dimension we begin also to see inferences of underlying mechanisms.

Interaction pattern begins with simple cause and effect explanations but extends to complex relational causality, involving the co-existence of two or more interdependent factors, possibly with feedback mechanisms. For example, agent A affects agent B but feedback from agent B then affects agent A.

Probabilistic Causality relates to the use of uncertainty in modelling causal relationships. Often apparently deterministic systems hide uncertainty in a chaotic complexity. Thus, does the cup which rests on the table express the equilibrium of underlying static forces? Or should we seek explanation by reference to the chaotic dynamic motion of the sub-atomic particles that constitute the table and the cup? Conversely, we choose to explain phenomena in terms of probability to avoid reference to deep layers of underlying causality. Thus, we might choose to model the outcome from the throw of a dice in terms of probability, rather than by reference to multiple and interacting forces, such as the strength of the throw, the weight of the dice and the friction at the surface.

Agency describes those explanations that recognise that causality is distributed across many elements. Such explanations might use ideas of emergence. For example, we

might consider a theoretical distribution as a pattern that emerges from the many pieces of data.

We wished to explore what sorts of computer-based tools might provide us with a window on the use of these differing levels of causal complexity to make sense of distribution, as generated within a computer simulation. We set out to design a virtual environment that supported students in attributing agency to the emergent shape of the distribution while they were discriminating and moving smoothly between data as a series of random outcomes at the micro level, and the shape of distribution as an emergent phenomenon at the macro level.

In that respect, we conjectured that the computer simulation environment could enable students:

- at the micro level to use their understanding of causality whilst at the same time begin to recognise its limitations in explaining local variation, and
- at the macro level to see parameters such as average and spread as causal agents, impacting on the shape of distribution, whilst nevertheless not completely defining the distribution.

METHOD

Approach and tasks. The approach of this research study falls into the design research methodology (Cobb et al., 2003) resulting in the *BasketBall* simulation as depicted below (Fig 1). The animation of the basketball player was controlled by

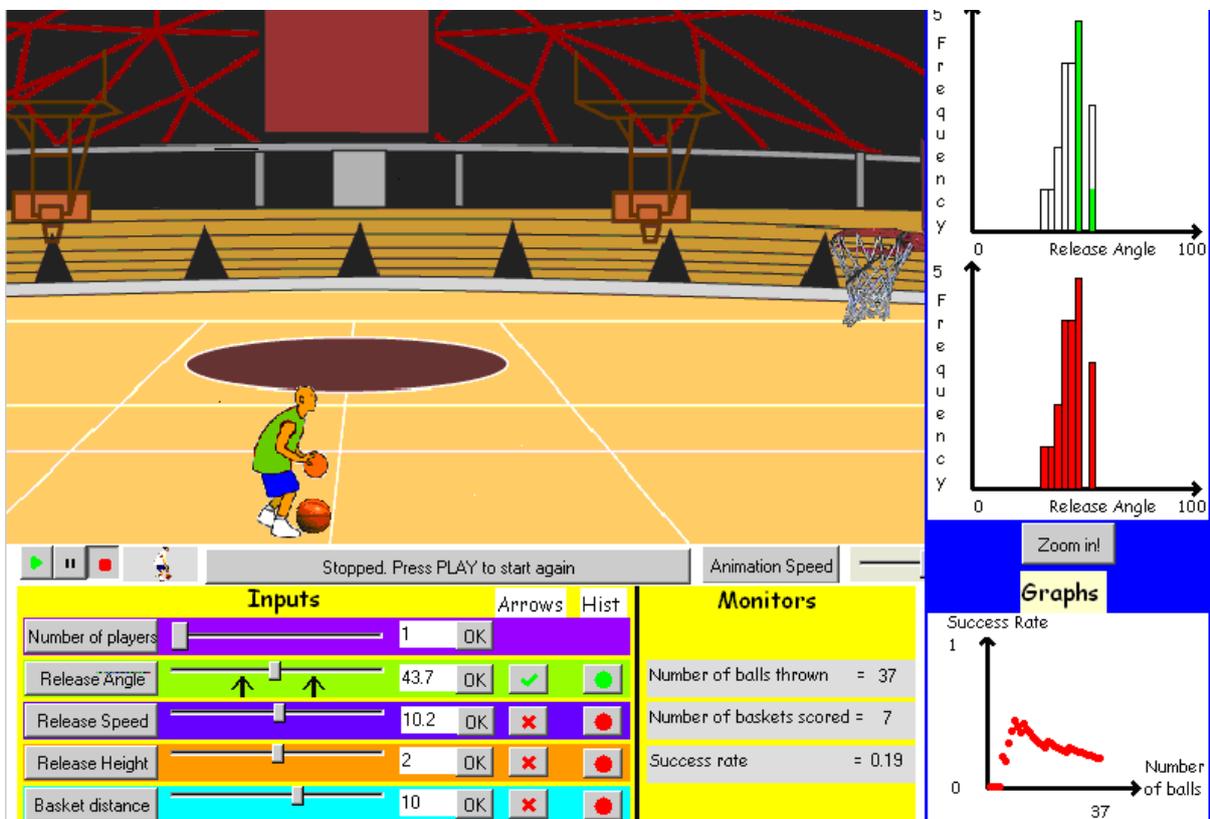


Fig 1: The interface of the *BasketBall* simulation.

varying the handles on the sliders of the release angle, speed, height and distance or by entering the data directly. Once the play button has been pressed, the player continues to throw with the given parameters until the pause or stop button is pressed. The trace of the ball can be switched off. Feedback is made available from the Monitors and Graphs panes. When the arrows button has been switched on, two arrows appear from both sides of the handle on the slider (Fig 2), in which case the value of the parameter is chosen from a distribution of values, centred on the handle of the slider. The students are able to vary these arrows to increase or decrease the spread of the values of the parameter around that centre. The microworld also allowed the students to explore various types of graphs relating the values of the parameters to frequencies and frequencies of success. The students have access to a linegraph of the success rate as well as a histogram of the frequency of successful throws or throws in general against release angle (or release speed, or height, or distance). Initially, the students were challenged to throw successfully the ball into the basket. When the parameters were determined, the histograms of the frequency of successful throws against release angle (or release speed, or height, or distance) appeared as a single bar columns.

Once the preliminary task was completed, some discussion about the realism of the simulation followed, which normally introduced notions such as skill-level, the use of the ‘arrows’ buttons and the appearance of the histograms. When bias had been introduced to the throws, the graphs appeared as histograms. The subsequent task for the students was to model a real but not perfect basketball player (one who was not successful on *every* throw).

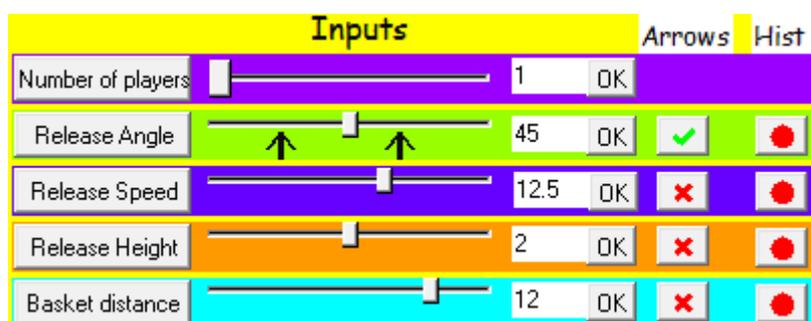


Fig 2: The value of the parameter was selected from a distribution of values, centred on the position of a slider.

Participants. The simulation was used by eight pairs of students in a UK secondary school. It was assumed that the simulation would be used only by students ranging in age from fourteen to fifteen years because a tight focus on the students’ intuitions of the distributions indicated that the age of 14-15 years old was mainly ripe for conceptual change in this domain. Another important advantage of working with students of this age was curriculum-based. In the UK National curriculum (DfES, 2000) students of this age are expected to know how to graph data using histograms, dotplots and boxplots, and compare distributions and make inferences, using the shapes of distributions and measures of average and range. Students of this age,

therefore, encounter distribution as a collection of data, either given or generated through experiments and surveys.

In this paper, we concentrate on the work carried out by two students, Ethan and Emma (aged 14-15 years), as they engaged with modelling a real but not perfect basketball player. These students had already experienced moving either or both of the arrows, generating values that corresponded to distributions with different spread and bias. The first author was a participant observer during this process. She frequently intervened in order to probe the reasons or intuitions that might lie behind participants' actions.

Data collection and analysis. The data collected included audio recording of the students' voices, video recording of the screen output on the computer, and the first author's[2] field notes. The analysis was one of progressive focussing (Robson, 1993). At the first stage, the recordings were simply transcribed and screenshots were incorporated as necessary to make sense of the transcription. Subsequently, the first author turned the transcript into a plain account. At the third stage, an interpretative account was written by the first author and discussions about the validity of those interpretations with the second author followed, making therefore an account of the data before accounting *for* the activity (Mason, 1994).

FINDINGS

The case of Ethan and Emma provides an illustration of students' typical causal explanations for the observed variation. The two hour session with Ethan and Emma demonstrates how the two students mobilized combinations of different tools to create explanations of variation.

Having already found how to make a successful basket, in the following extract, Ethan and Emma were first introduced to the arrows and they had spent a little time looking at the effect on the animation:

- 1 Re[1]: What do you think these arrows do?
- 2 Et: ...Do they change the angle and the height?
- 3 Em: It's just changed the angle, so we will get better results, so we can see.
- 4 Re: What do you mean by 'better'?
- 5 Em: Because each result is different on the graph (Fig 3).
- 6 Re: Why are they better?
- 7 Em: Because they much more like realistic.

By looking at the animation, Ethan had recognized that the arrows were causing changes in the throws made by the Basketball player (line 2). Emma refers to the changes in the graph (line 5), and seems to acknowledge that it is more realistic for the basketball player to throw at varying angles (line 7).

A few minutes later however, Emma deliberated upon the role of the arrows in determining the choice of angle:

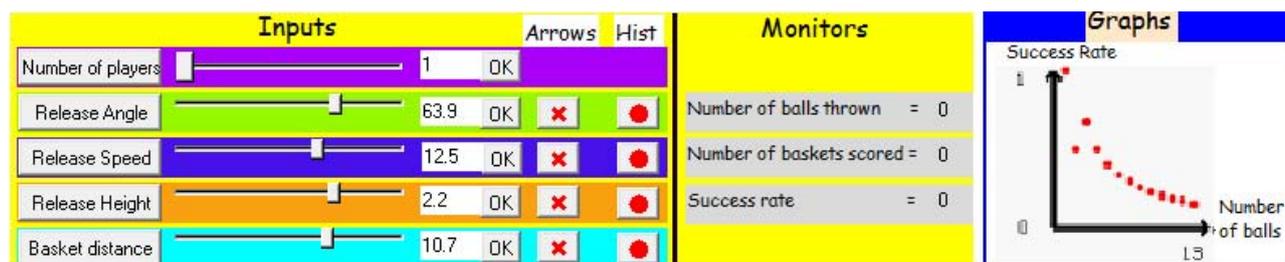


Fig 3: Emma seems to be referring not only to the different values of the angles which were chosen by the basketball player, but also appears to refer to the graph of success rate.

- 8 Re: What do you think the arrows are for?
- 9 Em: Is it... where the two arrows are, every time he throws is going to be the distance between that arrow (the arrow to the left of the vertical bar on the slider) and that arrow (the arrow to the right of the vertical bar on the slider)...
- 10 Re: Do you mean the angle?
- 11 Em: Yeah ... the angle ... You can only throw from here to there (pointing to the two arrows). You cannot go any place outside the two arrows.

Emma seemed to be conjecturing that the angle was chosen from between the two arrows (lines 9 and 11), though she still had offered no sense for the mechanism by which the choice was made.

For several minutes, the students experimented with the arrows, at which point their attention was re-focused on the variation which could be perceived through the histograms:

- 12 Re: Tell me what do you think your graphs will look like. Do you expect these graphs to have one bar, two bars, three bars, or four bars?
- 13 Em: ...about three bars.
- 14 Re: So, it will not be only one bar? Why?
- 15 Em: Because he is throwing at different angles... so... he is not throwing at the same angle all the times, so there would be more than one bar.

Emma asserted that variation in the throwing angles would result in additional bars in the histogram (line 15), and soon went further to predict that “the wider apart the arrows around the handle, the more bars there would be in the histogram”. Although, as can be seen, Emma tended to lead the discussion, Ethan was also comfortable at this point that variation could be perceived in the player’s throws and through the frequency histograms.

Their thinking about the relationship between the gap in the arrows and the number of bars was tested further a few minutes later when the bars were moved very far apart:

- 16 Re: Would there be more or less bars on the histograms?
- 17 Em: Because he can throw any distance between those two arrows... We haven’t given him a fixed angle to throw it at, so they would not be

the same every time. It will be different... because the arrows give him more of a choice... because the computer like assigns any angle at random between those two arrows... it records it in the graph.

For the first time, Emma referred to a random mechanism operating to make the choice from the gap between the arrows (line 16). She referred also to the interactions between a group of agents (arrows, basketball player, computer), which somehow cooperated to accomplish variation in the distribution.

So far, the discussion had centred on the connection between the gap in the arrows and the variation as seen in the animation or in the graphs. Later, the discussion switched to whether the score was successfully made or not. In the following extract, the handle is positioned on an angle which would successfully throw the ball into the basket and Emma and Ethan know this to be the case. They considered the effect of the arrows on success:

- 18 Em: Yeah... because when we put the arrows closer together, so it doesn't have enough choice, like... He can only pick between those two arrows for the release angle... so, he gets a better chance of... to score.
- 19 Et: As he's got the release angle inside... that space so... so got to choose that release angle that is scored...
- 20 Re: Which is inside ...?
- 21 Em: 63.3... and 76.3... he can only choose... a release angle between those two numbers.

Emma and Ethan both seemed to grasp that a small gap reduced the possibilities for failing to throw a successful basket (lines 17 and 18).

DISCUSSION

As an expert observing Emma and Ethan's activity, it is not difficult to recognise the connection between the arrows and the statistical notion of spread. Such an expert might see the distance between the arrows as a measure of spread. In fact, the data that is actually generated might portray spreads greater or less than that predicted by the gap between the arrows. In this sense the gap between the arrows operationalises the spread parameter of an underlying theoretical distribution, whereas what the students observe is a set of data generated randomly from that distribution.

The above protocol illustrates, through the case of Emma and Ethan, the use of causal explanations, at differing levels of causal complexity, to make sense of variation as it is depicted in the simulated animation of a basketball player and in graphical feedback. These explanations do not take the form of formal robust theory-oriented statements but rather they emerge more as tentative, situated, conjectural utterances, though as the exploration continues the utterances carry more authority and assurance and begin to sound more like conclusions than conjectures.

In Table 1, we list seven observed situated abstractions, based on the body of evidence, which the above protocol typifies:

Ref	Characterisation of situated abstraction	Line
SA1	“the arrows affect the angle that the basketball player throws the ball”	2
SA2	“the arrows affect the graph”	5
SA3	“angles are chosen from between the arrows”	15
SA4	“the wider apart the arrows around the handle, the more bars there would be in the histogram”	15
SA5	“the computer assigns any angle at random between the arrows and records it in the graph”	16
SA6	“The computer assigns a random value from the gap between the arrows for the basketball player to throw the ball”	17
SA7	“the closer together the arrows, the more is that chance to score”	18

Table 1: Examples of situated abstractions

The situated abstraction, SA1, reflects an awareness that the arrows have a causal affect on the variation in throws by the animated basketball player. SA2 similarly recognises a causal effect on the graph. Both these situated abstractions seem to operate at the mechanism level in the Grotzer and Perkins taxonomy. There appears at this stage to be little appreciation of further underlying levels of causal complexity though these begin to emerge later. Situated abstractions, SA3 and SA4, show an increased focus on mechanism as Emma and Ethan strive to make sense of how the arrows affect the player’s actions and the appearance of the graphs.

Situated abstraction, SA5, portrays the relationship not as purely deterministic but as including a random element. This introduction of uncertainty seems to represent a move from the mechanism level to probabilistic causality in the terminology of Grotzer and Perkins. Emma and Ethan do not have a sophisticated understanding of probability and so they do not progress deeply into this level but they do seek out, as articulated in both SA5 and SA6, explanations that accept a probabilistic language as a means of coping with a possible multitude of unknown factors. Of course, this move may have been all the easier to make because randomness is something they perhaps regularly experience on computers through, for example, playing computer games. In SA7, Emma and Ethan recognise, even with their ongoing probabilistic language, combinations of agents, as predicted in the interaction pattern level in the Grotzer and Perkins taxonomy. Emma and Ethan envisage a transference of agency from the computer to the arrows and then to the Basketball player. We note that we have previously reported a similar transference of agency from the student itself to the arrows (Prodromou, 2008; Prodromou & Pratt, 2006).

CONCLUSION

The facility to transfer agency seems to be a crucial move in making connections between the causal and the stochastic (from our perspective on the student's psychological state) and in harnessing the deterministic (from the perspective of designing for the student's abstraction). Indeed, by providing handles, arrows and a basketball player, together with feedback on "their actions" (and here we intentionally give these things agency), we set up the possibility that distribution might be seen as generated by the agents. Technological tools, therefore, may have been especially significant in supporting the construction of stochastic meanings out of causality and that in this sense they may provide a route towards operationalising the stochastic in the absence of formal operations.

We believe that such a view of distribution is consistent with the expert position in which a theoretical distribution is sometimes viewed as a generative model, for example sending out a signal *determined* by the average parameter and noise *determined* by the spread parameter. Such a position accepts that the deterministic view of distribution is useful within limitations. Simulations such as basketball might provide opportunities for students to begin to appreciate that expert position.

Even though we have referred regularly to agents, the reader may have noticed that nothing has actually been said about the final level in the Grotzer and Perkins taxonomy, that of agency in which causality is distributed across many agents. In fact, we intend to report elsewhere on students' attempts to make connections from the distribution of data to the theoretical distribution, a direction which demanded an emergent perspective from the students.

When students view variation as an accomplishment of a combination of agents, they think about distribution in terms of a relational model. Their expressions move along the underlying causality dimension towards considering that the simulated *BasketBall* is a context perturbed by a random mechanism. Students' accounts began gradually to address dimensions of probabilistic causality, such as noisy systems, chancy systems. Students were able to view the activities in the *BasketBall* context as noisy processes dependent on a variety of intervening variables. Those accounts were themselves preceded by students' understanding of mediating causality, where predominant causal agents, such as the arrows, and neglected agents of lower saliency in the context, such as the basketball player and the computer, mediate the effect of one agent to another in order to cause variation in the setting (Interaction pattern).

NOTES

1. 'Re' refers to the first named author (Dr. Theodosia Prodromou).
2. The data were collected for the first author's doctoral thesis.

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GREEK STUDENTS' ABILITY IN PROBABILITY PROBLEM SOLVING

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This study aims to contribute to the understanding of the approaches students develop and use in solving probabilistic tasks and to examine which approach is more correlated with students' ability in probability problem solving. Participants were students from the 12th grade. Implicative statistical analysis was performed to evaluate the relation between students' approach and their ability to solve problems. Results provided support for students' intention to use the algebraic approach and avoid Venn's diagrams. Students who were able to use the coordinated approach by using multiple representations had better results in problem solving. In addition the results suggest the flexibility in multiple representations is a trivial predictor of probabilistic problem solving.

Keywords: Probability, problem solving, 12th grade students, representations.

INTRODUCTION

There is an increasing recognition that statistical and probabilistic concepts are among the most important unifying ideas in mathematics. Statistical concepts form the single most important idea in all mathematics, in terms of understanding the subject as well as for using it for exploring other topics. The reasons to include probability and statistics teaching refer to the usefulness of statistics and probability for daily life, its instrumental role in other disciplines, the need for a basic stochastic knowledge in many professions and its role in developing a critical reasoning (Gal, 2002).

The understanding of probabilistic and statistical concept does not appear to be easy, given the diversity of representations associated with this concept, and the difficulties presented in the processes of articulating the appropriate systems of representation involved in probabilistic and statistical problem solving (SPS) (Anastasiadou, 2007).

Probability is difficult to teach for various reasons, including disparity between intuition and conceptual development even as regards apparently elementary concepts (Chadjipadelis and Gastaris, 1995). Since an education that only focuses on technical skills is unlikely to help teachers overcome their erroneous beliefs, it is important to find new ways to teach probability to them, while at the same time bridging their content knowledge and their pedagogical content knowledge (Batanero et al, 2005).

There is general consensus in the mathematics education community that teachers need a deep and meaningful understanding of any mathematical content they teach (Chadjipadelis, 2003). Biehler (1990) suggests that teachers require meta-knowledge

about probabilities and statistics, including a historical, philosophical, cultural and epistemological perspective on statistics and its relations to other domains of science.

In primary and secondary school levels, probability and statistics is part of the mathematics curriculum and primary school teachers and mathematics teachers frequently lack specific preparation in statistics education (Anastasiadou and Gagatsis, 2007; Chadjipadelis, 2003). According to Batanero et al., (Batanero et al, 2005) probability is increasingly taking part in the school mathematics curriculum; yet most teachers have little experience with probability and share with their students a variety of probabilistic misconceptions. The understanding of probabilistic concepts has been a main concern of statistics education that is an important focus of interest for the International Statistical Institute and of the International Association for Statistical Education.

In the field of statistics learning and instruction, representations play an important role as an aid for supporting reflection and as a means in communicating statistical ideas. Furthermore the NCTM's Principles and Standards for School Mathematics (2000) document include a new process standard that addresses representations and stress the importance of the use of multiple representations in statistical learning. In addition, an important educational objective in statistics is for pupils to identify and use efficiently various forms of representation of the same mathematical concept and move flexibly from one system of representation of the concept to another.

A representation is defined as any configuration of characters, images, concrete objects etc., that can symbolize or "represent" something else (Confrey & Smith, 1991, Goldin, 1998). Representations have been classified into two interrelated classes: external and internal (Goldin, 1998). Internal representations refer to mental images corresponding to internal formulations that we construct of reality. External representations concern the external symbolic organizations representing externally a certain mathematical reality. In this study the term "representations" is interpreted as the "external" tools used for representing statistical ideas such as tables and graphs (Confrey & Smith, 1991). The need for a variety of semiotic representations in the teaching and learning of probabilities is usually explained through reference to the cost of processing, the limited representation affordances for each domain of symbolism and the ability to transfer knowledge from one representation to another (Duval, 1987). By a translation process, we mean the psychological processes involving the moving from one mode of representation to another (Janvier, 1987). Several researchers in the last two decades addressed the critical problem of translation between and within representations, and emphasized the importance of moving among multiple representations and connecting them (Gagatsis & Elia, 2004; Goldin, 1998; Yerushalmy, 1997). Different representations referring to the same concept complement each other and all these together contribute to a global understanding of it (Gagatsis & Siakalli, 2004). Duval

(2002) claimed that the conversion of a mathematical concept from one representation to another is a presupposition for successful problem solving. A person who can easily transfer this knowledge from one structural system of the mind to another is more likely to be successful in problem solving by using a plurality of solution strategies and regulation processes of the system for handling cognitive difficulties. Kaput (1987) suggest that the concept of representation involves the following five components: a representational entity, the entity that it represents, particular aspects of the representation entity, the particular aspects of the entity that it represents that form the representation and finally the correspondence between the two entities. According to the above definition, the representation is considered a mental symbol or concept, which represents a concrete material symbol. It takes the place of another element and obtains more capabilities than the object itself. Many studies identified the difficulties that arise in the conversion from one mode of representation of a mathematical concept to another. They revealed students inconsistencies when dealing with relative tasks that differ in a certain feature, i.e. mode of representation. This incoherent behavior was addressed as one of the basic features of the phenomenon of compartmentalization, which may affect mathematics learning in a negative way (Gagatsis & Elia & Mousoulidis, 2006). According to Duval (Duval, 2002), the phenomenon of compartmentalization reveals a cognitive difficulty that arises from the need to accomplish flexible and competent conversion back and forth between different kinds of mathematical representations.

In Greece, the introduction of Statistics in the mathematics textbook of primary schools took place at the end of nineties. The teaching of fundamental statistical concepts was assigned to primary school teachers who are responsible for teaching all the curriculum subjects in the primary level. (Anastasiadou, 2007). The emphasis on statistics and probability in curricula varies, often according to knowledge and feelings of the teacher.

Although that many researches have been done in relation to study of the of the representations role in mathematical understanding and learning, there only a few that explore students' performance in using multiple representations of statistical and probability concepts with emphasis on the effects exerted on performance and on the relations among the various conversion abilities from one representation to another.

The purpose in this study is to contribute to the statistics education research community understands of approach students build up and use in solving statistical tasks and to examine which approach is more associated with students' ability in solving statistical concepts. A main question of this study referred to the approach primary school students use in order to solve simple probability tasks. It is important to know whether students are flexible in using algebraic, graphical and verbal representations in probabilistic problems. Most of the students used an algebraic approach in order to solve the simple probabilistic tasks. This study intends to shed light on the role of different modes of representation on the understanding of some basic probabilistic concepts. This study

investigated pre-service teachers' performance, in two aspects of probabilistic understanding: the flexibility in multiple representations and the problem solving ability.

METHOD

Participants- Data analysis-Tasks

The sample of the study involved 132 12th grade students from secondary schools in different regions of Thessaloniki (Western Thessaloniki, Eastern Thessaloniki, Central Thessaloniki) in Greece. These regions were selected because of their diversity in size and population. In Greek secondary education only students of the 12th grade are taught basic concepts of probability theory.

For the analysis of the collected data the similarity statistical method (Lerman, 1981) was conducted using a computer software called C.H.I.C. (Classification Hiérarchique, Implicative et Cohésitive) (Bodin, Coutourier & Gras, 2000). This method of analysis determines the similarity connections of the variables. In particular, the similarity analysis is a classification method which aims to identify in a set V of variables, thicker and thicker partitions of V , established in an ascending manner. These partitions, when fit together, are represented in a hierarchically constructed diagram (tree) using a similarity statistical criterion among the variables. The similarity is defined by the cross-comparison between a group V of the variables and a group E of the individuals (or objects). This kind of analysis allows for the researcher to study and interpret in terms of typology and decreasing similarity, clusters of variables which are established at particular levels of the diagram and can be opposed to others, in the same levels. It should be noted that statistical similarities do not necessarily imply logical or cognitive similarities. The red horizontal lines represent significant relations of similarity.

The test consisted of 12 tasks of two "equivalent" problems in difficulty from the mathematical point of view. In particular, the tasks concerned concepts of the probability theory such as probability, Venn's diagrams, events and probability problems.

Right and wrong or no answers were scored as 1 and 0, respectively. Students' responses to the tasks comprise the variables of the study which were codified by an uppercase V (variable concerns Venn's diagrams) or P (probability problem), η R (concept definition, e.g. event), followed by the number indicating the exercise number. Following is the letter that signifies the type of initial representation (e.g. r =representation, t =table, g =graphic, v =verbal) and, lastly, comes the letter that signifies the type of final representation.

For example the first and second tasks are the following ones: Task 1. Given two events A and B of a chance experiment and with the help of set theory we have the following event $A' \cap B'$. Present with a Venn diagram this event ($V1sg$). Task 2. Given two events A and B of a chance experiment and with the help of set theory we have the following

event $(A \cap B) \cup (A \cap B')$. Express the verbal representation of this event (V2sv).

RESULTS

Descriptive results

Table 1 presents the success rates of third, fifth and sixth grade indigenous students and immigrants in all types of conversions.

Tasks	Type of translation	12 th grade success rate of students (%)	Tasks	Type of translation	12 th grade success rate of students (%)
V1sg	Symbolic - Graphic	52.8%	P7va	Verbal - Algebraic	32.6%
V2sv	Symbolic - Verbal	51.6%	P8vg	Verbal - Graphic	28.3%
V3gs	Graphic - Symbolic	34.5%	P9vs	Verbal - Symbolic	22.6%
V4gv	Graphic - Verbal	30.7%	P10vv	Verbal - Verbal	27.5%
V5vg	Verbal - Graphic	46.2%	R11vv	Verbal - Verbal	23.1%
V6vs	Verbal - Symbolic	48.6%	R12vs	Verbal - Symbolic	22.9%

Table 1: Success rates of indigenous students and immigrants in the tasks

Similarity diagram of students' responses to the two tests

The similarity diagram in this study concern the data 11th grade and allow for the arrangement of students' responses ((V1sg), (V2sv), (V3gs), (V4gv), (V5vg), (V6vs), (P7va), (P8vg), (P9vs), (P10vv), (R11vv), (R12vs), to the tasks into groups according to their homogeneity.

Two clusters (Cluster A and B) of variables are identified in the similarity diagram of 11th grade students' responses as shown in Figure 1. Cluster A involves three pairs of variables V1sg-V2sv, V3gs-V4gv, V5vg-V6vs in Cluster A and concerns events representations with the aid of Venn diagrams. Cluster B involves three pairs of variables R11vv- R12vs, P7va-P8vg, P9vs-P10vv and involves variables relating to probability problem solving. This grouping suggests that students dealt similarly with the conversions involving probability problems.

The structure of the diagram reveals a cognitive difficulty that arises from the need to accomplish flexible and competent conversion back and forth between different kinds of probabilistic representations. Thus, this particular structure of the diagram indicates a

compartmentalization of the tasks of the tests. Students approached in a completely distinct way the tasks which involved the use of Venn's diagrams and the probability problems. Therefore, possible instructive activities would focus on the identification of the two different groups. The strongest similarity (almost 1) occurs between variables (V3gs-V4gv) (Figure 1) that were the most difficult for the students of 12th grade (Table 1). Furthermore the similarity (V1sg-V2sv, V3gs-V4gv) is also important (0.923).

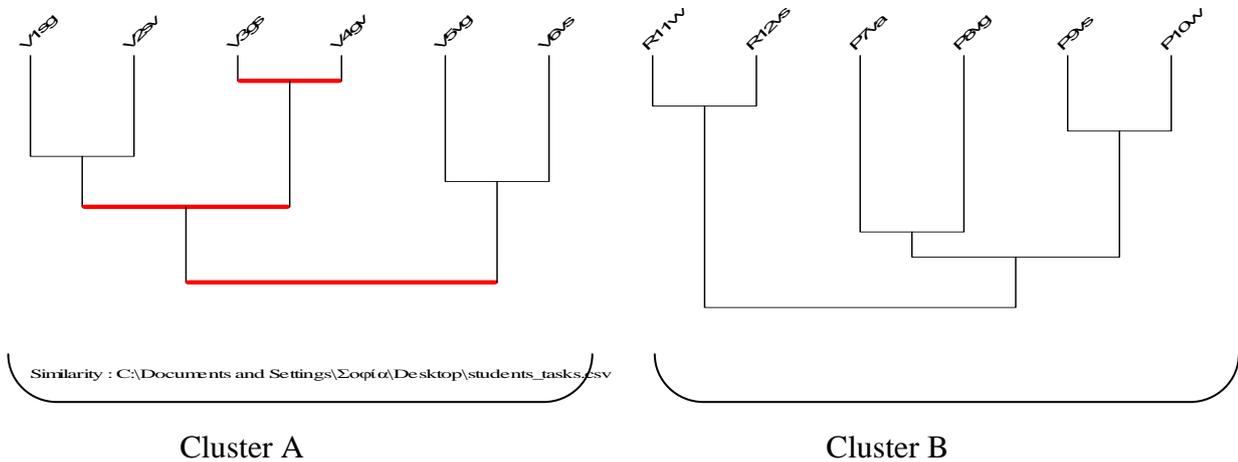


Figure 1: **Similarity Diagram**

CONCLUSIONS RESULTS

Representations enable students to interpret situations and to comprehend the relations embedded in probabilistic problems. Thus, we consider representations to be extremely important with respect to cognitive processes in developing probabilistic concepts. The main contribution of the present study is the identification of secondary students' abilities to handle various representations and to translate among representations related to the same probabilistic relationship. Our findings provide a strong case for the role of different modes of representation on 12th grade students' performance to tasks on basic statistical concepts such as frequency. At the same time they enable a developmental interpretation of students' difficulties in relation to representations of Venn diagrams. Lack of connections among different modes of representations in the similarity diagram indicates the difficulty in handling two or more representations in probabilistic tasks. This incompetence is the main feature of the phenomenon of compartmentalization in representations, which was detected in students of both grades. This inconsistent behavior can be seen as an indication of students' conception that different representations of the same concept are completely distinct and autonomous mathematical objects and not just different ways of expressing the meaning of a particular notion. An alternative explanation for the difficulty in transferring knowledge

could be the emphasis on stating with representations and defining transfer as connecting those representations. Perhaps links that were more powerful and meaningful for the students would have led to a space of the utility of the statistical and probability construct (Ainley and Pratt, 2002). Transfer might then be achieved by recognizing new situations which are consistent with the same meaning. In addition the lack of transfer may be attributed to the students' lack of preparation: time to discuss, interact and work on related tasks.

Probability instruction needs to encourage pupils' involvement in activities including translations between different modes of representation. Even more educators should focus on reasons that we use a specific representation or another of the same probability concept. As a result, students will be able to overcome the compartmentalization difficulties and develop their flexibility in understanding and using a concept within various contexts or modes of representation and in moving from one mode of representation to another. Moreover there is a strong need for teachers to understand what it is that students know about stochastic and offer them experiences of probability before theoretical perspectives are introduced.

It seems that there is a need for further investigation into the subject with the inclusion of a more extended qualitative and quantitative analysis. In the future, it is interesting to compare the strategies and modes of representations students used in order to solve the problems. Besides, longitudinal performance investigation in the multiple representation flexibility tasks for secondary students should be carried out.

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CERME 6 – WORKING GROUP 4

ALGEBRAIC THINKING AND MATHEMATICS EDUCATION

TABLE OF CONTENTS

Introduction.....	415
<i>Janet Ainley, Giorgio T. Bagni, Lisa Hefendehl-Hebeker, Jean-Baptiste Lagrange</i>	
The effects of multiple representations-based instruction on seventh grade students’ algebra performance	420
<i>Oylum Akkus, Erdinc Cakiroglu</i>	
Offering proof ideas in an algebra lesson in different classes and by different teachers.....	430
<i>Michal Ayalon, Ruhama Even</i>	
Rafael Bombelli’s Algebra (1572) and a new mathematical “object”: a semiotic analysis	440
<i>Giorgio T. Bagni</i>	
Cognitive configurations of pre-service teachers when solving an arithmetic-algebraic problem.....	449
<i>Walter F. Castro, Juan D. Godino</i>	
Transformation rules: a cross-domain difficulty.....	459
<i>Marie-Caroline Croset</i>	
Interrelation between anticipating thought and interpretative aspects in the use of algebraic language for the construction of proofs in elementary number theory.....	469
<i>Annalisa Cusi</i>	
Epistemography and algebra.....	479
<i>Jean-Philippe Drouhard</i>	
Sámi culture and algebra in the curriculum	489
<i>Anne Birgitte Fyhn</i>	
Problem solving without numbers – An early approach to algebra.....	499
<i>Sandra Gerhard</i>	
The ambiguity of the sign $\sqrt{\quad}$	509
<i>Bernardo Gómez, Carmen Buhlea</i>	
Behind students’ spreadsheet competencies: their achievement in algebra? A study in a French vocational school.....	519
<i>Mariam Haspekian, Eric Bruillard</i>	

Developing Katy's algebraic structure sense	529
<i>Maureen Hoch, Tommy Dreyfus</i>	
Children's understandings of algebra 30 years on: what has changed?.....	539
<i>Jeremy Hodgen, Dietmar Kuchemann, Margaret Brown, Robert Coe</i>	
Presenting equality statements as diagrams	549
<i>Ian Jones</i>	
Approaching functions via multiple representations: a teaching experiment with Casyopee	559
<i>Jean-Baptiste Lagrange, Tran Kiem Minh</i>	
Equality relation and structural properties	569
<i>Carlo Marchini, Anne Cockburn, Paul Parslow-Williams, Paola Vighi</i>	
Structure of algebraic competencies	579
<i>Reinhard Oldenburg</i>	
Generalization and control in algebra	589
<i>Mabel Panizza</i>	
From area to number theory: a case study	599
<i>Maria Iatridou, Ioannis Papadopoulos</i>	
Allegories in the teaching and learning of mathematics	609
<i>Reinert A. Rinvold, Andreas Lorange</i>	
Role of an artefact of dynamic algebra in the conceptualisation of the algebraic equality	619
<i>Giampaolo Chiappini, Elisabetta Robotti, Jana Trgalova</i>	
Communicating a sense of elementary algebra to preservice primary teachers	629
<i>Franziska Siebel, Astrid Fischer</i>	
Conception of variance and invariance as a possible passage from early school mathematics to algebra.....	639
<i>Ilya Sinitsky, Bat-Sheva Ilany</i>	
Growing patterns as examples for developing a new view onto algebra and arithmetic.....	649
<i>Claudia Böttinger, Elke Söbbeke</i>	
Steps towards a structural conception of the notion of variable	659
<i>Annika M. Wille</i>	

ALGEBRAIC THINKING AND MATHEMATICS EDUCATION

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In CERME–6 Working Group “Algebraic Thinking” we continued the work done in previous CERME conferences, both by following the discussions raised and by pointing out unanswered questions (Puig, Ainley, Arcavi & Bagni, 2007).

More particularly, in CERME–6, Working Group “Algebraic Thinking” was concerned with further discussion on historical, epistemological, and semiotic perspectives in research in the teaching and learning of algebra. The role of artifacts, technological or not, was also considered in this perspective. In general, Working Group “Algebraic Thinking” was interested in proposing to address the issue of the actual impact of research on curriculum design and development, and on practice.

In order to allow a detailed discussion of the contributions, we decided to split the working group into two subgroups:

- the subgroup A (co–ordinated by Lisa Hefendehl–Hebeker) included some contributions mainly focused on cognitive aspects. The Authors were W.F. Castro and J.D. Godino; M.–C. Croset; A. Cusi; J.–P. Drouhard; C. Fernández and S. Llinares; B. Gómez and C. Buhlea; M. Hoch and T. Dreyfus; J. Hodgen, D. Kuchemann, M. Brown and R. Coe; R. Oldenburg; I. Papadopoulos and M. Iatridou; F. Siebel and A. Fischer; I. Sinitsky and B.–S. Ilany; E. Söbbeke and C. Böttinger; A.M. Wille.
- the subgroup B (co–ordinated by Janet Ainley) included some contributions mainly focused on pedagogical aspects. The Authors were O. Akkus and E. Cakiroglu; M. Ayalon and R. Even; G.T. Bagni; A.B. Fyhn; S. Gerhard; M. Haspekian and E. Bruillard; I. Jones; J.–B. Lagrange and T.K. Minh; C. Marchini, A. Cockburn, P. Parslow–Williams and P. Vighi; M. Panizza; R.A. Rinvold, and A. Lorange; E. Robotti, G. Chiappini and J. Trgalova.

Posters presentations by R. Berrincha and J. Saraiva, Ç. Kiliç and A. Özdaş, B.M. Kinach, A. Matos, C. Monteiro and H. Pinto, A.I. Silvestre, I. Vale, T. Pimentel produced important contributions to our discussion.

In the following file, contributions are organised according to the alphabetic order of the corresponding authors.

GENERAL REFLECTIONS

The invention of the symbolic language of algebra influenced the development of mathematics in all domains. Symbolic language is used throughout all mathematics:

for instance, there is no possible calculus or analysis without solving inequalities, structures (groups, rings, ...) are used to describe all parts of mathematics (Drouhard, 2009). This must be taken into account when considering early algebra.

Historically, algebra results from what evolution scientists call co-evolution. This co-evolution involves: first an art, then a science of resolution of numerical problems; first informal representation systems, then formal registers (semiotic representation systems); first a science of numbers, then a science of structures (Drouhard, 2009). So today algebra is a science of resolution of numerical problems, a family of semiotic systems (linguistic or not), and a science of numbers and structures.

In a passage of his *Questions Concerning Certain Faculties Claimed for Man*, Charles S. Peirce (1839–1914) suggests that it is impossible to “think without signs” (Peirce, 1868/1991, p. 49). In a Peircean perspective, algebraic language is based upon iconicity. Let us quote Peirce (1931–1958, 2.279, *MS 787*):

Particularly deserving of notice are icons in which the likeness is aided by conventional rules. Thus, an algebraic formula is an icon, rendered such by the rules of commutation, association, and distribution of the symbols [...]. For a great distinguishing property of the icon is that by direct observation of it other truths concerning its object can be discovered than those which suffice to determine its construction.

Two remarks must be taken into account. Firstly, every sign “contains” all the components of Peircean classification, although one of them (e.g. iconicity) is predominant. For instance, algebra is not characterised by the presence or absence of letters: algebra is characterised by the existence of a semiotic representational system, a system which allows us to solve numerical problems and to express number properties. So algebra is not but has got a language (Drouhard, Panizza, Puig, & Radford, 2006).

Secondly, Peirce’s semiotics hardly explains the complexity of sign-based human thought processes and the manner in which they relate to their corresponding historical settings (Douek, forthcoming). The historical dimension of cognition and its cultural subbasement (see Bradford & Brown, 2005; D’Ambrosio, 2006) are a fundamental theme in recent sociocultural perspectives where cognition is conceptualized as “a cultural and historically constituted form of reflection and action embedded in social praxes and mediated by language, interaction, signs and artifacts” (Radford, 2008, p. 11). Sociocultural perspectives lead to both new conceptions of cognition and new views about knowledge and the cognizing subject: algebraic thinking can be framed into the mentioned perspective.

Algebraic language must be described by linguistic terms (“syntax”, “semantics”). In terms of semantics, the power of algebra lies in the capability to judiciously “forget the meaning”. From an educational viewpoint, it is worth noting that students must at the same time master the languages (natural and symbolic), their respective syntax and semantics and the semiotic aspects of these languages, and be flexible, so be able

to work both with meaningless and meaningful expressions (see remarks in Puig, Ainley, Arcavi & Bagni, 2007).

COGNITIVE ASPECTS

As regards cognitive aspects (subgroup A), it is worth noting that the tension between the possibility of formal manipulation and the necessity of semantic understanding, which is typical for algebraic activities, causes particular cognitive demands for the learners. There are many partial abilities which should be learned and grow together to an interrelated system. Mental acts and ways of thinking (Harel, 2008) which are essential for algebraic thinking have to be activated on different layers:

- *Structuring*: The symbolic language of algebra is a tool to conceive arithmetical structures, and as a semiotic system it has a structure of its own. Comprehensive learning of algebra and successful manipulation of its language deserves “structure sense” in different respects.
- *Generalizing*: Generalizing belongs to the essence of algebra. It means to grasp something typical, which all cases under consideration have in common. Variables are tools to express indeterminacy and generality. To describe a sequence of geometrical patterns by a formula and to find a common form of a set of formulas (for example quadratic equations) are activities on different stages of generalization.
- *Representing*: The representation system of algebra in its final stage is symbolic and formal, that means, it allows context-free manipulation. This makes it difficult to grasp for learners, but for experts it gains a new kind of meaning and richness in itself.

Many contributions showed that there are previous stages in the development of these ways of thinking, which should be cultivated in the learning process. Such activities might help to reduce the “cognitive gap” between arithmetic and algebra:

- *Structuring and generalizing*: For example pre-service primary teachers experience structuring and thus develop “algebraic awareness” when they analyze, describe and continue patterns and structures in geometric and algebraic contexts. A fruitful interplay between arithmetic and geometric visual approaches can also be experienced on later stages.
- *Representing*: L. Radford demonstrated in his plenary address that alphanumeric symbolism is not the only way to express algebraic thinking. He pointed out that there is a conceptual zone before, where algebraic thinking is contextual and embodied in the corporeality of actions, gestures, signs and artefacts.

Nevertheless such approaches to teaching algebra have their own problems.

PEDAGOGICAL ASPECTS

In considering pedagogical approaches to teaching algebra (subgroup B) there is a potential tension between the need to focus on structure independently of context (for

example to develop understandings of equality, equivalence), and the uses of context as ways to make structure visible (for example by means of metaphor, metonymy, allegory, artefacts, narratives, ...). Teachers and pupils may be attending to different aspects of the activity: while the teacher is looking through a context such as a visual pattern in order to see generality, pupils may be looking at the stages of construction of the particular pattern.

Different perceptions of the nature of algebraic activity may become apparent when considering the role of, and need for, proof. Similarly, alternative perceptions of the nature of tools, artefacts and representations emerge from close study of the conversations in classrooms. This presents real challenges for teachers in their interactions with learners, and of their interventions in activities.

A continuing challenge is the design of tasks which may motivate a real need for algebraic thinking. There is clearly no single ‘best’ approach to algebra; many good approaches can support each other. It is important to interrogate each approach to identify what it may offer and for whom. The design of such tasks must take account of the rich variety which may be covered by the phrase ‘algebraic thinking’ and the ways in which such thinking may be expressed. Rather than focussing on differences between arithmetic and algebraic thinking, it may be powerful to see this as a continuum, or parallel development, rather than as a dichotomy. Generalisation may be embodied through gesture, including virtual gestures on a computer screen, or expressed through natural language as well as through symbolism. Variable is an algebraic idea that children must understand on their way to learning symbolic generalisation because it allows thinking about change, generalisation and structure. It is an idea which may be introduced and expressed in many ways: the design challenge is to find ways to engage learners in the real need for, and power of, algebra.

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THE EFFECTS OF MULTIPLE REPRESENTATIONS-BASED INSTRUCTION ON SEVENTH GRADE STUDENTS' ALGEBRA PERFORMANCE

Oylum Akkus¹ and Erdinc Cakiroglu²

The purpose of this study was to investigate the effects of multiple representation-based instruction on seventh grade students' algebra performance. The study was conducted on four seventh grade classes from two public schools lasting eight weeks. For assessing algebra performance, three instruments called translations among representations skill test, objective based achievement test, and Chelsea diagnostic algebra test were used. The analyses were conducted by using multivariate covariance statistical model. The results pointed out that multiple representation-based instruction had a significant effect on students' algebra performance compared to the conventional teaching. In addition to this, students from experimental groups found this way of teaching fruitful.

INTRODUCTION

Various meanings can be given to the concept of “representation” in connection with the teaching and learning of mathematics. Seeger, Voight, & Werschescio (1998) summarized some of those definitions in very general terms as follows: “...representation is any kind of mental state with a specific content, a mental reproduction of a former mental state, a picture, symbol, or sign, symbolic tool one has to learn their language, a something “in place of” something else”.

Multiple representations can be generally defined as providing the same information in more than one form of external mathematical representation by Goldin and Shteingold (2001). The usage of multiple representations in mathematical learning was investigated in depth by Janvier who defined it “understanding” as a cumulative process mainly based upon the capacity of dealing with an “ever-enriching” set of representations (Janvier, 1987, p. 67). There are two important key terms in a theory of representation that are; “to mean or to signify, as they are used to express the link existing between external representation (signifier) and internal representation (signified)” (Janvier, Girardon, & Morand, 1993, p. 81). External representations were defined as “acts stimuli on the senses or embodiments of ideas and concepts”, whereas internal representations are regarded as “cognitive or mental models, schemas, concepts, conceptions, and mental objects” which are illusive and not directly observed (Janvier, et. al., 1993, p. 81).

Another approach to the theory of multiple representations which is called Lesh Multiple Representations Translations Model (LMRTM) has been suggested by Lesh (1979). His theory draws the theoretical framework of this study since he improved a

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model involving translations among representational modes and transformation within one representational mode. According to Lesh, Post and Behr (1987), representations are crucial for understanding mathematical concepts. They defined representation as “external (and therefore observable) embodiments of students’ internal conceptualizations” (Lesh, et al. 1987, p. 34). This model suggests that if a student understands a mathematical idea she or he should have the ability of making translations between and within modes of representations. According to this view, a good problem solver should be able to “sufficiently flexible” in using variety of representational systems. He claimed further, “As a student’s concept of a given idea evolves, the related underlying transformation/translation networks become more complex; and teachers who are successful at teaching these ideas often do so by reversing this evolutionary process; that is, teachers simplify, concretize, particularize, illustrate, and paraphrase these ideas, and imbed them in familiar situations” (p. 36).

A MULTIPLE REPRESENTATIONS TRANSLATION MODEL

After reviewing a number of theories about multiple representations, this study emphasizes investigating particularly students’ ability to use the given representational mode for solving problems, and to make translations among the representational modes. A multiple representational translations model combined from the models belonging to Lesh and Janvier would seem to be perfect modeling for this research study. The five distinct representational modes; namely, manipulatives, real-world situations, written symbols, spoken symbols, and pictures or diagrams in LMRTM were directly included in the model of this study. Some of those representational modes were named differently referring the Janvier Representational Translation Model (JRTM). Instead of “written symbols” from LMRTM, wording of “formulas” from JRTM was included in this study. Besides in lieu of the combination of “situations, pictures, and verbal descriptions”, the researcher decided to use those representational modes separately. Therefore instead of “situations, pictures, and verbal descriptions” in JRTM, “manipulatives,” “pictures or diagrams,” and “spoken symbols” were taken from LMRTM. “Tables” and “graphs” were taken separately from JRTM. Janvier’s Representation Translation Process was revised in light of the Lesh (1979) ideas as appeared in Table 1.

Table 1: The combined model of Lesh and Janvier for translations among representation modes

From \ To	Spoken Symbols	Tables	Graphs	Formulas (Equations)	Manipulatives	Real Life Situations	Pictures
Spoken Symbols	–	Measuring	Sketching	Abstracting	Acting out	Acting out	Drawing
Tables	Reading	–	Plotting	Fitting	Modeling	Modeling	Visualizing
Graphs	Interpre-	Reading Off	–	Fitting	Modeling		

	tation						
Formulas (Equations)	Reading	Computing	Sketching	–	Concretizing	Exemplifying	Localizing
Manipulatives	Describing	Exemplifying	Concretizing	Symbolizing	–	Simplifying, Generalizing	Drawing
Real Life Situations	Describing	Exemplifying	Plotting	Modeling, Abstracting	Particularizing	–	Modeling
Pictures or Diagrams	Describing	Describing	Sketching	Abstracting	Constructing	Situationalizing	–

SIGNIFICANCE OF THE STUDY

The issue of what instructional approaches should be used in algebra classes does not have a single and clear answer. No matter which instructional approach is used, the primary goal of mathematics instruction should be to help students in forming conceptual understanding. Janvier (1987) mentioned that if teachers enrich their algebra classrooms by placing multiple representations, the students can more efficiently make connections between the meaning of algebraic concepts and the way of representing them, therefore they simply “go for the meaning, beware of the syntax” which results in conceptual understanding.

The improvement of mathematical understanding and representational thinking of students require flexible use of multiple representations and the interaction of external and internal representations (Pape & Tchoshanov, 2001). Since making meaningful translations in representational modes plays a crucial role in acquisition of mathematical concepts and there are still unanswered questions about the instructional outcomes of using multiple representations, we believe that it would be worth to investigate the multiple representations in this respect.

Since this study focuses on the effects of multiple representation-based environments in mathematics classroom, its results should help mathematics educators who seek alternative pedagogical instructions in classroom settings. Furthermore, if a teacher is aware of his/her students’ understanding of the multiple representations and what kind of learning is supported by multiple representation-based environments, s/he can better choose and utilize appropriate type of methods, manipulatives, or activities to meet the needs of students. Moreover, providing students with a multiple representation-based algebra instruction would promote a conceptual shift to thinking algebraically. Therefore, receiving such kind of instruction makes students more competent in the area of algebra.

RESEARCH QUESTION

The purpose of this study is to examine the effects of a treatment based on multiple representations on seventh grade students’ performance in algebra, and this study attempted to answer the following research question;

“What are the effects of the multiple representations-based instruction compared to conventional teaching method on seventh grade students’ algebra performance when students’ gender, mathematics grade of previous semester (MGPS), age, prior algebra level are controlled?”

METHOD

The research question was examined through a quasi-experimental research design since this study did not include the use of random assignment of participants to both experimental and control groups. The target population of this study consists of all seventh grade students from public schools in Çankaya district in Turkey. There were 103 public schools in this region. However, two schools from this district were determined as the accessible population. There were 2 seventh grade classes in School A, and 7 in School B. One experimental and one control group were selected from both schools. There were 15 girls and 13 boys in experimental group and 16 girls and 13 boys in control group taken from School A. On the other hand, the experimental group from School B consists of 17 girls and 21 boys and in the control group the number of girls and boys were equal, that is 18. The participants in this study ranged in age from 11 years to 14 years old.

INSTRUMENTS

To assess algebra performance, three distinct instruments namely Algebra Achievement Test (AAT), Translations among Representations Skill Test (TRST), and Chelsea Diagnostic Algebra Test (CDAT) were used. The rationale of using combined instruments is to perceive algebraic learning in a multi dimensional way. It includes procedural, conceptual, and translational knowledge and skills in its nature (Lesh, Landau & Hamilton, 1983). By utilizing three instrument it was aimed to assess algebraic learning within its all dimensions and each instrument was tried to assess different aspect of algebra learnings. It can be claimed that when a student gets higher scores from three instruments s/he can be called as successful in algebra since getting high score means that s/he can use procedural algebra knowledge in problem solving, understand algebra conceptually, and also make simple translations among representations.

Among three instruments Algebra Achievement Test (AAT) was administered to analyze students’ computation skills in algebra intensively. 10 essay type questions were used in this instrument which combines traditional school algebra test items including symbolic manipulations and computations in algebra. The items which are related with the procedural skills in school algebra are criterion-referenced tasks addressing key learning goals specified in the Mathematics Curriculum for Elementary Schools, published by Turkish Ministry of National Education (MEB, 2002). The required time for this instrument was 30 minutes. The internal reliability value of Cronbach alpha was calculated as .90. To score the students’ responses to each question in AAT four-point rubric was used. The highest point of 4 indicated a complete understanding of underlying mathematical concepts and procedures while the lowest point of 0 was

given for irrelevant or no responses. The minimum and maximum possible scores from the test items are 0 and 40 points, respectively. Students who got scores above mean score of the group was accounted as high achievers.

Another instrument for assessing students' algebra performance was the Translations among Representations Skill Test (TRST). The purpose of this test was to obtain data about students' abilities in making translations among different representational modes. TRST contains 15 open-ended items which were designed to measure skills of translation among representations, use of certain representations, and creating new representations. The items in TRST required a translation from one representation type to another, such as from tabular representation to graphical one. In the last two items all type of representations were required to solve the problem. Duration of the test was 40 minutes. It was scored by using a three-point holistic scoring rubric. The highest point of 3 was awarded for responses showing that the problem was solved correctly and that the appropriate translations among representations were used. The lowest point of 0 indicated if the response is completely wrong or immaterial to the. The possible minimum score was 0, and the possible maximum score was 36. The internal reliability estimate of TRST was found to be .79 by calculating the Cronbach alpha coefficient.

The last instrument was Chelsea Diagnostic Algebra Test (CDAT) which was developed by the Concepts in Secondary Mathematics and Science Team (Hart, Brown, Kerslake, Küchemann, & Ruddock, 1985) to determine 13-15 years old children's algebraic thinking levels. This test was designed to measure the conceptual knowledge of elementary algebra. In CDAT there are six different categories of interpreting and using the "letter". Apart from these six categories, four levels of algebra understanding were developed with respect to the children's responses and the items themselves. In Level 4, children can deal with the items that require specific unknowns and which have a complex structure (Hart, et al., 1989) and they can be accounted as successful in algebra. The students answered the items in this test approximately in 60 minutes. The discrimination power of the items ranged from 0.20 to 0.60. Reliability measure as based on KR-20 coefficient was found to be 0.93. There were 53 items in the adapted version of CDAT. The possible minimum and maximum scores were 0 and 53 respectively. Besides, CDAT was used as a pretest to find out experimental and control group students' conceptual algebraic knowledge before the intervention. It was considered that seventh grade students' algebra knowledge coming from their previous mathematics background might affect the experiment therefore CDAT as pretest was also taken to MANCOVA statistical model as a profounding variable.

TREATMENT BASED ON MULTIPLE REPRESENTATIONS

For this study, the instructional design for experimental groups consists of daily lesson plans in which several activities took place. There were 21 activities which were involved in the lesson plans of the instructional unit in order to aid in teaching of a unit of algebra. All 21 lesson plans which had distinct contexts and problem situa-

tions were developed in order to reflect the procedure of translations among representations, transformations within a specified representation, usage of any representational mode in dealing with algebraic situation. In particular, students were required to learn constructing the multiple representations of algebraic situations, including expressing them in tables, graphs, and symbols. Instead of teaching these representation skills in isolation, it was anchored within meaningful thematic situations. Instead of direct instruction in how to construct and use mathematical representations in algebra, students were only guided in the activities to explore different representations and to develop their understanding of each one. In experimental groups students were frequently given tasks that require them to make translations among different representations. This approach was used to present and develop concepts from verbal, algebraic, graphical, and tabular standpoints. To illustrate, for instance, a concept first introduced a numerically intuitive approach in which tables were used to collect and work on data. Then a verbal representation was used to verbally complement the relationship among numbers in the tabular representation. Finally, a transition was made to the algebraic representation. The usage of multiple representations varied for each activity presented in this treatment. For instance, for the topic of equations, first the tabular representation then the verbal representation were constructed; however, for conceptualizing the concept of graph, first, the algebraic representation, and then the other representations were used.

The actualization of treatment can be illustrated in one activity namely; “*Inequalities*”. In this activity students were responsible to find out the main characteristics of inequalities using the tabular representation. At the beginning the activity sheets were given to the students, and then they examined the activity. They filled the given table by required numbers, and then the translation from one representation to another came. For this, the daily life situations and the algebraic representational modes were selected. Students were required to give one daily life example to the inequality of “ $x-3 < 7$ ”. Students’ examples were like;

“There are x number of teachers in one school, then 3 of them are appointed to another school, and the number of the remaining teachers was less than 7”.

“Let us say that the number of the desks in our class was x , we get rid of 3 of them, then there are less than 7 desks in our class”.

After getting students translations among representations, all of them were discussed in class. It is compulsory for the students to keep the activity sheets in the folder that the researcher gave them, since they did all the works on those papers. They were also responsible to bring their folder to the class every mathematics lesson.

In the treatment, particularly the translations among representational modes were stressed and valued by the researcher. In conventional algebra teaching, however, translation among representations might occur only when the students are required to draw a graph. In this case, instead of constructing a table to represent the given equation, they only identified two points where the line pass through. Then, by the help

of this information, a graph could be drawn. However, the multiple representations-based instruction emphasizes the translations from variety of representational modes to the other modes. Therefore, students could have the opportunity to notice that one mathematical concept can be represented in several ways and these ways can be complementary to understand this concept. The same task of drawing the graph of a linear equation is taken in a way that, students analyze the equation through daily life situations, plain language, tables, and graphs. In that sense, drawing the graph of an equation is not an end but it is a means of interpreting the existing mathematical situation. The treatment lasted eight-weeks. Each week experimental groups received four lesson hours, with each session lasting 40 minutes.

RESULTS

To test the null hypothesis related to the research question, the statistical technique of Multiple Analysis of Covariance (MANCOVA) was used for comparing the mean scores of control and experimental groups separately on the AAT, TRST, and CDAT. MANCOVA was carried out by putting experimental groups together as a one experimental group and control groups as one control group as well (Cohen & Cohen, 1983).

Initial descriptive analysis revealed that the experimental groups had the higher scores on all the instruments compared to the control groups. Before conducting MANCOVA the assumptions called normality, multicollinearity, homogeneity of regression, equality of variances, and independency of observations were verified (Green, Salkind, & Akey, 2003).

The MANCOVA results revealed that, there was a significant effect of two methods of teaching on the population means of the collective dependent variables of seventh grade students' scores on the AAT, TRST, and CDAT after controlling their age, the MGPS, and PRECDAT scores. 37% of the total variance of MANCOVA model for the collective dependent variables of the AAT, TRST, and CDAT was explained by group membership of the participants. Using the Wilks' Lambda test, significant main effects were detected between the groups experimental group and control group ($\lambda = .63$, $p = .000$). Therefore, the results of this study were of practical significance. The significant finding of a group effect from MANCOVA, allowed further statistical analysis to be done in order to determine the exact nature of significant differences found in main effect. Therefore univariate analyses of covariance (ANCOVA) were carried out on each dependent variable in order to test the effect of the group membership. From the analyses, it can be stated that, multiple representation based instruction has a significant effect on the dependent variable scores of CDAT [$F(1,125) = 38.005$, $p = .000$], TRST [$F(1,125) = 25.942$, $p = .000$], and AAT [$F(1,125) = 18.271$, $p = .000$]. Furthermore, for the observed treatment effects, it was obvious that the values of eta squared for the scores of the CDAT, TRST, and AAT were .233, .172, and .128 respectively which are equal to the medium effect size. This explains 23% of the variance in CDAT, 17% of the variance in TRST, and 13% of the variance in AAT related with the treatment. Power for the scores of the CDAT, TRST, and AAT

were found as 1.00, .92, and .78 respectively. Step-down analysis was carried out as significant MANCOVA follow up analysis. By the help of this analysis, the unique importance of dependent variables which were found as significant in the MANCOVA analysis was investigated. Since there are three significant dependent variables namely, CDAT, TRST, and AAT, three step-down analyses were conducted. By doing so, any possible variance overlap among the dependent variables was planned to be detected. According to these results, the effect of multiple representation-based instruction had still significant effect on each dependent variable.

DISCUSSION

This research study has documented that, compared to conventional instruction, multiple representations-based instruction did make a significant influence on the algebra performance of seventh grade students. There might be various reasons to this result. Visualization of algebraic objects, connections among algebraic ideas, and the improvement of translational abilities in algebra problem solving (Lesh, Post, & Behr, 1987) can be counted as what multiple representations-based instruction provide for students. By the help of this instruction, students avoid memorization in algebra learning, and understand concepts meaningfully. As suggested in Swafford and Langrall's (2000) study; multiple representations-based instruction promotes conceptual understanding of algebra and makes students conceptualize algebraic objects. The results of this study are supported in the literature by numerous studies. One of them is Brenner's (1995) and her colleagues study. They conducted only 20 days multiple representations unit including variables and algebraic problem solving. After treatment they implemented four instruments related to algebra learning to the seventh and eight graders. Significant difference was found between experimental and control group of students in favor of the students in experimental groups. The findings of this study are also consistent with the findings of previous studies (Ozgun-Koca, 2001; Pitts, 2003) that provided evidence for the effectiveness of multiple representations-based instruction in engaging students in meaningful algebra learning. Additionally, in Herman's (2002) study similar results were found. It was stated that after multiple representation based instruction in college algebra course, students were better able to establish connections between varieties of representational modes.

This study confirmed the need for considering other kinds of representations, such as; representations used in graphic calculator and computer programs or representations that students create and unique for them. As it was suggested by Ozgun-Koca (2001), computer-based applications can be used to provide linked and semi-linked representations, and graphical form of representations. These applications can make students to abstract mathematical concepts from virtual world. Besides, allowing students to create their own representations for solving algebra problems makes them more creative and flexible in mathematics (Piez & Voxman, 1997). In this study it was observed that, students were mainly restricted by four types of representations which are tabular, graphical, algebraic, and verbal. This can be due to the activities or re-

researcher's emphasize on those representation types. However, students should be given an opportunity that they can use representations that they invent or create. Moreover, it can be suggested that future research can focus on teachers and teaching strategies in algebra classrooms. All of the data for this study was collected from students. Future research could combine data from students and their teachers, because teachers have also impact on shaping students' representation preferences. What teaching strategies and representation types are used within algebra classrooms by teachers and how those representations are conceptualized by the students seems to be worthwhile to investigate.

According to the researcher, mathematics educators ought to recognize making establishment between concepts for the mathematics instruction for all students. Nowadays, many attempts can be observed to improve mathematics instruction. Multiple representation-based instruction for conceptual algebra understanding is just the one that the researcher implemented and appreciated the benefits of using this method. Giving opportunity to new instructional methods like multiple representation based instruction in mathematics classrooms enables students better mathematics learner. As Klein (2003) implied; *'Learning to create and interpret representations using specific media such as texts, graphics, and even videotapes are themselves curricular goals for many teachers and students'* (p. 49). As a three-year experienced mathematics teacher before, the researcher could say that in traditional mathematics classroom, there is a need to encourage students to think more deeply on mathematical concepts, to intrinsically motivate for learning, to make students appreciate the nature of mathematics by getting rid of rote memorization, and to avoid overemphasizing mathematical rules and algorithms. In fact, new instructional methodologies like multiple representation-based instruction can address this need.

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OFFERING PROOF IDEAS IN AN ALGEBRA LESSON IN DIFFERENT CLASSES AND BY DIFFERENT TEACHERS

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This paper analyzes the ways proof ideas in an algebra lesson were offered to students (1) by two different teachers, and (2) in two different classes taught by the same teacher. The findings show differences between the two teachers, and between the two classes taught by the same teacher, regarding the proof ideas made available to learn in the lesson.

Keywords: Proof ideas, algebra, classroom, curriculum, teacher.

INTRODUCTION

Research suggests that getting students to understand what a mathematical proof is and the role that proofs play in mathematics is not an easy task (de Villiers, 1990; Dreyfus & Hadas, 1996; Harel & Sowder, 2007). However, most of the research on proof focuses on the individual student's cognition and knowledge. There is an absence of studies that focus on the complexity of teaching and learning proof in the classroom (Mariotti, 2006), and on the role of the content and sequencing of the curriculum on the quality of teaching proof (Holyes, 1997; Stylianides, 2007). Moreover, research related to proof is commonly conducted in the context of geometry, and examination of proof in algebra is sparse. This study addresses this shortcoming of current research. Its aim is to examine the enactment of a written algebra lesson, which centers on determining and justifying equivalence and non-equivalence of algebraic expressions. The study focuses on ways important proof ideas were offered to students, the extent to which they were explicit in the lessons, and the contributions of the teacher and the students to their development. Two of these ideas are general: refutation by a counter example as mathematically valid, and supportive examples for a universal statement as mathematically invalid – two ideas that are difficult for students (e.g., Balacheff, 1991; Fischbein & Kedem, 1982; Jahnke, 2008). Another idea is algebra specific: the use of properties and axioms in proving that two algebraic expressions are equivalent as mathematically valid.

Recent research suggests that different teachers enact the same curriculum materials in different ways (Manouchehri & Goodman, 2000), and that the same curriculum materials may be enacted differently in different classes taught by the same teacher (Eisenmann & Even, in press). Thus, we chose to focus here on the ways the proof ideas in the algebra lesson were offered to students (1) by different teachers, and (2) in different classes taught by the same teacher. This study is part of the research program *Same Teacher – Different Classes* (Even, 2008) that compares teaching and learning mathematics in different classes taught by the same teacher as well as classes taught by different teachers. Various aspects are examined, with the aim of gaining

insights about the interactions between mathematics teachers, curriculum and classrooms.

PROOF IDEAS IN THE WRITTEN LESSON

The lesson appears in a 7th grade mathematics curriculum program developed in Israel in the 1990s (Robinson & Taizi, 1997). The curriculum program used by the teachers in this study is intended for heterogeneous classes and includes many of the characteristics common nowadays in contemporary curricula. One of its main characteristics is that students are to work co-operatively in small groups for much of the class time, investigating algebraic problems situations. Following small group work, the curriculum materials suggest a structured whole class discussion aimed at advancing students' mathematical understanding and conceptual knowledge. The curriculum materials include suggestions on enactment, including detailed plans for 45-minute lessons.

The lesson "Are they equivalent?", which is the focus of this paper, is the 6th lesson in the written materials. Prior to this lesson, equivalent expressions were introduced as representing "the same story", e.g., the number of matches needed to construct a train of r wagons. The use of properties of real numbers (e.g., the distributive property) was mentioned briefly as a tool for moving from one expression to an equivalent one, but it was not yet presented explicitly as a tool for proving the equivalency of two given expressions.

Based on an analysis of the textbook and the teacher guiding, three proof-related ideas were found as being explicit in this lesson:

Idea 1: Substitution that results in different values proves that two expressions are not equivalent (a specific case of refutation by a counter example as mathematically valid).

Idea 2: Substitution cannot be used to prove that two given algebraic expressions are equivalent³ (a specific case of supportive examples for a universal statement as mathematically invalid).

Idea 3. It addresses the problem that emerges from idea 2: the use of properties in the manipulative processes is a mathematically valid method for proving that two expressions are equivalent.

The lesson is planned to start with small group work aiming at an initial construction of Ideas 1 and 2. Students are given several pairs of expressions; some equivalent and some not. They are asked to substitute in them different numbers and to cross out pairs of expressions that are not equivalent. After each substitution they are asked whether they can tell for certain that the remaining pairs of expressions are equiva-

³ Students were not familiar at that stage with the properties of linear expressions.

lent. Finally, students are instructed to write pairs of expressions, so that for each number substituted, they will get the same result.

Then small group work continues, asking students to write equivalent expressions for given expressions. The aim is to direct students' attention to the use of properties in relation to equivalence of algebraic expressions, which is relevant to idea 3.

The whole class work returns to idea 1, and moves, through idea 2, to idea 3, aiming at consolidating these ideas, by discussing questions, such as: How can one determine that expressions are not equivalent? that expressions are equivalent? By substituting numbers? If so, how many numbers are sufficient to substitute? If not, what method is suitable? Finally, the teacher guide recommends that the teacher demonstrate the use of properties for checking equivalence, and together with the students implement this method on several pairs of expressions in order to check their equivalency.

Ideas 1, 2, and 3 are connected to three other ideas, none of which appears explicitly in the first six lessons in the written materials:

Idea 4 justifies Idea 2: There may exist a number that was not substituted yet, but its substitution in the two given expressions would result in different values, thus showing non-equivalence.

Idea 5 justifies Idea 3: The use of properties of real numbers in the manipulative processes guarantees that any substitution in two expressions will result in the same value, thus showing equivalence.

Idea 6 is the underpinning for Ideas 1, 2, and 3, as well as for Ideas 4 and 5. It defines equivalent algebraic expressions: Two algebraic expressions are equivalent if the substitution of any number in the two expressions results in the same value.

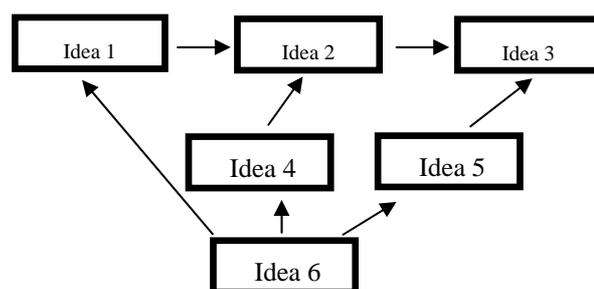


Figure 1: Connections among the proof-related ideas in the lesson

Ideas 4 and 6 are implicit in the written lesson, and Idea 5 does not exist.

METHODOLOGY

The primary data source include video and audio tapes of the enactment of the written lesson in four classes, each from a different school (i.e., four different schools). One teacher, Sarah, taught two of the classes, S1 and S2; another teacher, Rebecca, taught the other two classes, R1 and R2 (pseudonyms). The talk during the entire class work

was transcribed. The transcripts were segmented according to focus on the six ideas, yielding 3-4 more or less chronological parts in each class. Next, the collective discourse in the classroom was analyzed by examining the contributions of the teacher and the students to the development of the proof ideas in each enacted lesson. We compared how the teachers structured and handled the proof ideas in each lesson, and what was available to learn in different classes of the same teacher and in the classes of the two teachers.

PROOF IDEAS IN THE ENACTED LESSONS

Idea 1

In line with the written curriculum materials, the whole class work in all four classes included an overt treatment of Idea 1. However, contrary to the recommendations in the written materials, in none of the classes did the whole class work begin with the question, how can one determine whether algebraic expressions are not equivalent. Instead, the students performed substitutions in pairs of algebraic expressions from Problem 1 because the teacher requested them to do so, and not as a way of addressing a problem. When the substitutions resulted in different values, the classes concluded that the two expressions were not equivalent. In all four classes, it was the teacher who eventually presented Idea 1 explicitly, attending only to the specific context of non-equivalence of expressions, with no reference to the general idea of refutation by using a counter example as mathematically valid.

Idea 2

After working on non-equivalence, the four classes proceeded to work on equivalence of algebraic expressions. In both of her classes Sarah presented Idea 2, that substitution cannot be used to prove that two given algebraic expressions are equivalent. She explicitly incorporated in the presentation of this idea its underlying justification (which does not appear explicitly in the written materials) that possibly there exists a number that was not yet substituted, but its substitution in the two given expressions would result in different values (idea 4). For example, Sarah said in class S1:

We saw that with substitution, it is always possible that there is a number that I will substitute, and it will not fit. We can substitute ten numbers that would fit, and suddenly we will substitute one number that will not fit, and then the expressions are not equivalent... We have to find some way other than substitution, which will help us determine whether expressions are equivalent.

Contrary to the recommendations in the written materials, the students in Sarah's classes did not participate in constructing Idea 2 in class. Sarah merely presented it as motivation for finding a method to show equivalence, and immediately proceeded to work on using properties in the manipulative processes as a means to prove equivalence (Idea 3).

The idea that substitution cannot be used to prove that two given algebraic expressions are equivalent was dealt with differently in Rebecca's classes. In general, in both classes Rebecca pressed on finding a method that works, rather than evaluating the method of substitution, which does not work. However, the issue of substitution continued to be raised. In class R1, following the students' suggestion, the initial focus was on rejecting substitution because of the inability to perform substitution of all required numbers (an infinite number), as the following excerpt illustrates:

Rebecca: When will I be sure that these three [points to the pairs on the board] are indeed equivalent? That each pair is equivalent? When will I be sure?

S: When you check all the numbers.

...

S: There is an infinite number of numbers so you will never finish.

Rebecca: So I am not going to substitute infinite numbers. I need to find some other trick.

Idea 2, that supportive examples (i.e., substitution) could not be used to prove a universal statement (i.e., that two given algebraic expressions are equivalent), was not dealt with in class R1. Rather, it seemed to be taken as shared. Repeatedly, after substituting numbers in pairs of expressions and receiving the same value, the class concluded that the pairs appeared to be equivalent but that it was impossible to know for certain. For example,

Rebecca: OK, we are told to check another number, four.

S: Right.

Rebecca: You checked four. What did you get?

S: That they are equivalent.

Rebecca: I got the same result, right?

S: Yes, right.

S: All is well so far.

By stating, "I got the same result" following the statement "they are equivalent" Rebecca signaled that they did not yet know whether the latter claim was correct. Students then agreed, "All is well *so far* (emphasis added)". Later in the lesson, a similar conversation took place,

Rebecca: So, does it mean that they are equivalent?

S: Yes.

S: Yes. Ah, no, not necessarily.

Rebecca: Why? Do you have a counter example?

S: We don't know that they are equivalent.

Still, there was no explicit rejection of substitution for proving equivalence, as a specific case of supportive examples for a universal statement as mathematically invalid. Instead, Rebecca changed the focus of the activity to looking for a connection be-

tween the two algebraic expressions in each pair, as a transitional move towards Idea 3.

In contrast with class R1, class R2 embraced the idea that substitution is a valid means of determining equivalence of algebraic expressions. Unlike R1, where after several substitutions that resulted in the same value, students claimed that they still could not conclude that the two expressions were equivalent, in similar situations R2 students claimed that the expressions were equivalent because all the numbers they substituted resulted in identical numerical answers. This happened even after Rebecca offered idea 4, that there may be a number, which was not yet substituted, but its substitution in the two given expressions would result in different values. For example,

Rebecca: So, what do you say, what should I do, check all the numbers; maybe there is a number that won't fit here?

S: No [interrupts the teacher]

Rebecca: Or will it always fit?

S: Always.

...

Rebecca: Why are they equivalent? Why do I say that these are equivalent...?

S: Because we checked at least thirty.

Rebecca: We didn't check thirty, but I am asking: Why are these equivalent, in your opinion?

...

S: Because we checked.

Rebecca: Because you checked, but we said that maybe there is one number that you did not check.

S: But we checked almost all the [inaudible].

Eventually, Rebecca changed the focus of the activity to looking for algebraic expressions that are equivalent to given expressions, aiming at Idea 3. Thus, unlike Sarah, who used the brief mention of Idea 2 (and 4) as a motivational transition from Idea 1 to Idea 3, in R2, Rebecca did not motivate the search for a method different from substitution.

Idea 3

Led by Sarah, in line with the written materials, S1 and S2 searched for properties that show that the expressions produced when working on Problem 1 (S1), or given in Problem 3 (S2), were equivalent. Sarah then stated that the use of properties is the way to show equivalence, not substitution. When introducing Idea 3 in S2, Sarah explicitly connected with Ideas 5 and 6, which underpin and justify Idea 3. However, no such connections were made then in S1. Only later on, in her concluding remarks in S1, when summarizing both ways of proving equivalence and non-equivalence of expressions, did Sarah explicitly propose Idea 6.

Class R1 started to work on Idea 3 by searching for connections between pairs of expressions from Problem 1 that remained as potentially consisting of equivalent ex-

pressions. The class then quickly embraced the discovery that by using properties, it was possible to move from one expression to another, by indicating equivalence. Rebecca then introduced explicitly Idea 3. However, in R1, like in S1, no connections were made then to Ideas 5 and 6. Nevertheless, Idea 6 was introduced explicitly at the beginning of the lesson, when a student asked for the meaning of equivalence expressions.

Class R2 had a different starting point than R1 for treating Idea 3 because the students were confident that based on the substitutions they performed they could infer that the remaining pairs of expressions from Problem 1 were equivalent. Rebecca then slightly deviated from the written materials' suggestions and asked the students to find new expressions that would be equivalent to the given ones. Eventually, R2 embraced the idea that equivalence can be determined by manipulating the form of expressions, using properties. In R2, too, no connections were made with Ideas 5 and 6. Moreover, Idea 6 was not proposed at all.

Figure 2 depicts the teaching sequences of the proof-related ideas as offered during the whole class work, in the written materials, as well as in the four classes.

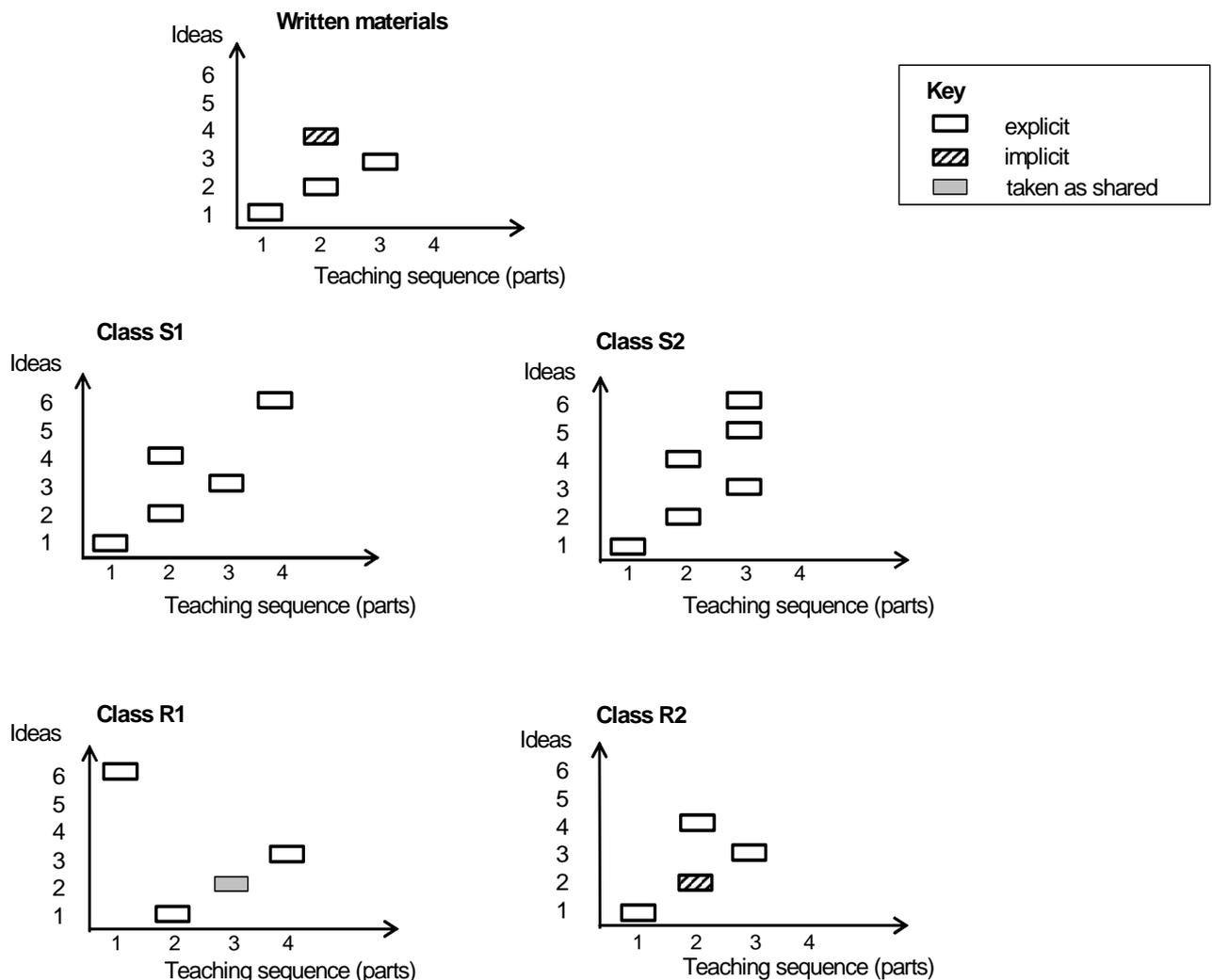


Figure 2: Teaching sequences of the proof-related ideas, as offered in the whole class work, in the written materials, as well as in the classes

The figure clearly demonstrates that Sarah was the only one who explicitly proposed the sequence of the three proof-related ideas (1, 2, and 3) that were explicit in the written lesson, whereas Rebecca explicitly proposed only Ideas 1 and 3. Moreover, any connections between these three ideas and the other three ideas (4, 5, and 6), which did not appear explicitly in the written lesson, were made only in Sarah's classes: Idea 2 was connected to its underlying justification, Idea 4 in both of Sarah's classes, whereas Idea 3 was connected to its underlying support by Ideas 5 and 6 in S2 only. Nevertheless, Idea 4 was offered by Rebecca in R2 with no explicit connection to Idea 2, and Idea 6 was offered in S1 (at the end of the lesson) and in R1 (at the beginning of the lesson), with no explicit connections to the other ideas.

FINAL REMARKS

Sarah and Rebecca taught the written lesson “Are they equivalent?” using the same written materials, which included a detailed lesson plan. Thus, it is not surprising that the mathematical problems enacted in class were similar in all four classes. However, the ways the proof ideas in the lesson were offered to students differed to some degree from what was recommended in the written materials. There were also differences between the two teachers, and between the two classes of the same teacher, in what was available to learn in the lesson. One of the main differences is related to offering Idea 2. This idea is central in the written materials. However, Sarah only briefly mentioned it in her classes, just as a transition to Idea 3. In R1 this idea was taken as shared, never made explicit, as was the case in R2, which strongly embraced the opposite idea. Another central idea in the written materials is Idea 3. The way that the written materials deal with Idea 3, without making Ideas 5 and 6 explicit, seemed to make teaching it a challenge. Eventually, each teacher handled this idea somewhat differently in each of her two classes.

These differences seem to be related to differences in teaching approaches. Sarah tended to make clear presentations of important ideas. Rebecca hardly made presentations, but instead, attempted to probe students, expecting them to explicate these ideas. Thus, some ideas were never made explicit, in one class more than the other, because of differences in students' mathematical behaviour and performance.

These initial findings illustrate the complexity of the interactions among teachers, curriculum and classrooms (Even, 2008). Rebecca faced serious challenges in her attempts to make students genuine participants in the construction of mathematical ideas, as was recommended in the written materials – more so in one of her classes – challenges that lie at the meeting point of the specific teacher, specific curriculum and

specific class. Sarah, who chose to make clear presentations of the mathematical ideas, faced different challenges, even though she used the same materials.

The mere fact that different teachers offer mathematics to learners in different ways, even when using the same written materials, is not entirely surprising, and has been documented by empirical research (e.g., Manouchehri & Goodman, 2000). Nonetheless, the nature of the differences is important because what people know is defined by ways of learning, teaching, and classroom interactions, as documented by Boaler (1997). Consequently, Sara'h and Rebecca's students were offered somewhat different proof-related ideas that are central in algebra and in mathematics in general, and that are known as not being easy for students. Furthermore, when instead of focusing solely on the comparison between teachers, different classes taught by the same teacher were also compared, important information was revealed about the interactions among curriculum, teachers and classrooms.

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RAFAEL BOMBELLI'S ALGEBRA (1572) AND A NEW MATHEMATICAL "OBJECT": A SEMIOTIC ANALYSIS

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In the theoretical framework based upon the ontosemiotic approach to representations, some reflections by Radford, and taking into account Peirce's semiotic perspective, I proposed to a group of 15–18 years–old pupils an example from the treatise entitled Algebra (1572) by Rafael Bombelli. I conclude that the historical analysis can provide insights in how to approach some mathematical concepts and to comprehend some features of the semiotic chain.

INTRODUCTION

In this paper I shall examine a traditional topic of the curriculum of High School and of undergraduate Mathematics that can be approached by historical references. The introduction of imaginary numbers is an important step of the mathematical curriculum. It is interesting to note that, in the Middle School, pupils are frequently reminded of the impossibility of calculating the square root of negative numbers. Then pupils themselves are asked to accept the presence of a new mathematical object, " $\sqrt{-1}$ ", named i , and of course this can cause confusion in students' minds. This situation can be a source of discomfort for some students, who use mathematical objects previously considered illicit and "wrong". The habit (forced by previous educational experiences) of using only real numbers and the (new) possibility of using complex numbers are conflicting elements.

Although the focus of this paper is not primarily on the analysis of empirical data, I shall consider an educational approach based upon an historical reference that can help us to overcome these difficulties. More particularly, I shall consider the semiotic aspects of the development of the new mathematical objects introduced (imaginary numbers) and I shall ask: can we find an element from which the semiotic chain is originated? Can we relate the early development of the semiotic chain to the objectualization of the solving procedure of an equation?

THEORETICAL FRAMEWORK

Radford describes "an approach based on artefacts, that is, concrete objects out of which the algebraic *tekhnē* and the conceptualization of its theoretical objects arose. [...] They were taken as *signs* in a Vygotskian sense" (Radford, 2002, § 2.2). In this paper I shall not consider concrete objects. Nevertheless Radford's remark about the importance of "signs in a Vygotskian sense" can be considered as a starting point of my research.

When we consider a sign, we make reference to an object, and in the case of mathematical objects, to a concept. However my approach does not deal only with "con-

cepts”. Font, Godino and D’Amore (2007, p. 14) state that although “to understand representation in terms of semiotic function, as a relation between an expression and a content established by ‘someone’, has the advantage of not segregating the object from its representation, [...] in the onto–semiotic approach [...] the type of relations between expression and content can be varied, not only be representational, *e.g.*, ‘is associated with’; ‘is part of’; ‘is the cause of/reason for’. This way of understanding the semiotic function enables us great flexibility, not to restrict ourselves to understanding ‘representation’ as being only an object (generally linguistic) that is in place of another, which is usually the way in which representation seems to us mainly to be understood in mathematics education”.

In my research I shall consider the ontosemiotic approach to mathematics cognition. It “assumes socio-epistemic relativity for mathematical knowledge since knowledge is considered to be indissolubly linked to the activity in which the subject is involved and is dependent on the cultural institution and the social context of which it forms part” (Font, Godino & D’Amore, 2007, p. 9, Radford, 1997).

My framework is also linked with some considerations about semiotic aspects, based upon a Peircean approach (although, for instance, the relationship between Vygotsky and Peirce is not trivial: Seeger, 2005). According to Peirce we cannot “think without signs”, and signs consist of three inter–related parts: an object, a proper sign (*representamen*), and an interpretant (in Peirce’s theory sign is used for both the triad “object, sign, interpretant” and the representamen, in late works). Peirce considered either the immediate object represented by a sign, or the dynamic object, progressively originated in the semiotic process. As a matter of fact, an interpretant can be considered as a new sign (*unlimited semiosis*). The limit of this process is the *ultimate logical interpretant* and it is not a real sign, which would induce a new interpretant. It is an *habit–change* (“meaning by a habit–change a modification of a person’s tendencies toward action, resulting from previous experiences or from previous exertions of his will or acts, or from a complexus of both kinds of cause”: Peirce, 1931–1958, § 5.475. I shall cite paragraphs in Peirce’s work).

The sign determines an interpretant by using some features of the way the sign signifies its object to generate and shape our understanding. Peirce associates signs with cognition, and objects (“mathematical objects” will be considered as “objectualized procedures”: Sfard, 1991, Giusti, 1999) “determine” their signs, so the cognitive nature of the object influences the nature of the sign. If the constraints of successful signification require that the sign reflects some qualitative features of the object, then the sign is an *icon*; if they require that the sign utilizes some physical connection between it and its object, then the sign is an *index*; if they require that the sign utilizes conventions or laws that connect it with its object, then it is a *symbol*.

According to Peirce, the formulas of our modern algebra are icons, i.e. signs which are mappings of that which they represent (Peirce, 1931–1958, § 2.279). Nevertheless pure icons, according to Peirce himself (1931–1958, § 1.157), only appear in think-

ing, if ever. Pure icons, pure indexes, and pure symbols are not actual signs. In fact, every sign “contains” all the components of Peircean classification, although one of them is predominant. So our algebraic expressions are *complex icons* (Bakker & Hoffmann, 2005). Moreover, it is worth noting that a sign in itself is not an icon, index or symbol. From the educational viewpoint, the identification of signs is not just a question of classifying a sign as *e.g.* an icon, but it is a question of showing their cognitive import (Bagni, 2006).

Frequently Peirce underlined the importance of iconicity. He argued (1931–1958, § 3.363) that “deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts”. (Peirce distinguished three kinds of icons: images, metaphor, and diagrams). According to Radford (forthcoming), since the epistemological role of “diagrammatic thinking” rests in making apparent some hidden relations, it relates to actions of *objectification*, and a diagram can be considered a *semiotic means of objectification*.

HISTORY OF MATHEMATICS AND IMAGINARY NUMBERS

History of mathematics can inform the didactical presentation of topics (although the very different social and cultural contexts do not allow us to state that ontogenesis recapitulates phylogenesis: Radford, 1997). Let us consider the resolution of cubic equations according to G. Cardan (1501–1576) and to N. Fontana (Tartaglia, 1500–1557). R. Bombelli (1526–1573), too, is one of the protagonists of history of algebra. His masterwork is *Algebra* (1572), where we find some cubic equations, and sometimes their resolution makes it necessary to consider imaginary numbers.

The resolution of the equation $x^3 = 15x+4$ leads to the sum of radicals $x = \sqrt[3]{2+11i} + \sqrt[3]{2-11i}$ where $2+11i = (2+i)^3$ and $2-11i = (2-i)^3$. So a (real) solution of the equation is $x = (2+i)+(2-i) = 4$. In the following image (Fig. 1) I propose the original resolution on p. 294 of Bombelli’s *Algebra*.

$$x^3 = 15x+4$$

$$[x^3 = px+q]$$

$$(4/2)^2 - (15/3)^3 = -121$$

$$[(q/2)^2 - (p/3)^3 = -121]$$

$$x = \sqrt[3]{2+11i} + \sqrt[3]{2-11i}$$

$$x = (2+i) + (2-i) = 4$$

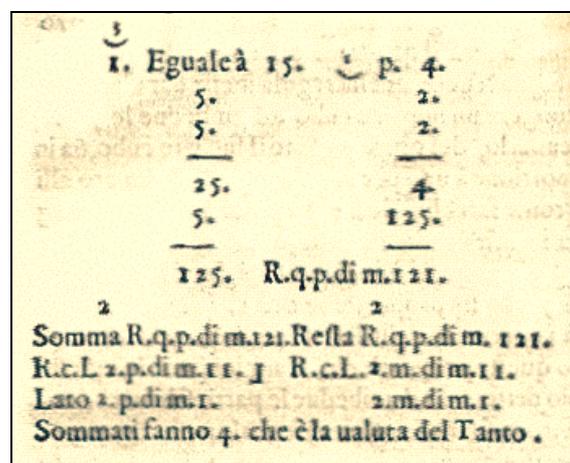


Fig.1

Bombelli justified his procedure using the two-dimensional and three-dimensional geometrical constructions (1966, pp. 296 and 298, Fig. 2 and Fig. 3 respectively). (Space limitations prevent a detailed discussion of these. The reader is referred to Bombelli: Bombelli, 1966).

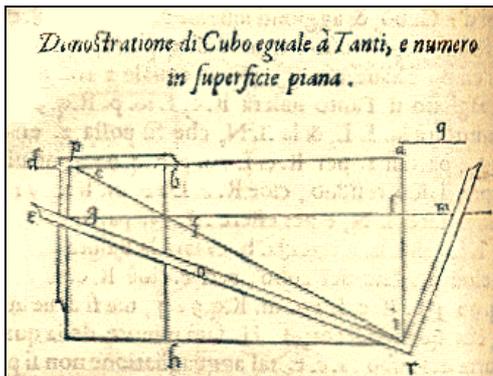


Fig. 1

From the educational point of view, Bombelli's resolution can help our pupils to accept imaginary numbers. As a matter of fact, its effectivity supports Bombelli's rules for *pdm* and *mdm* ("più di meno" and "meno di meno" respectively, today written as i and $-i$). In the image see the original "rules" as listed on p. 169 of Bombelli's *Algebra*, Fig. 4).

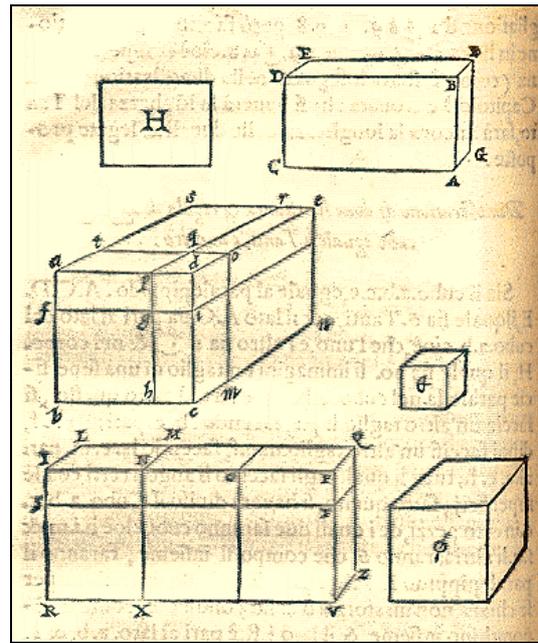


Fig. 2

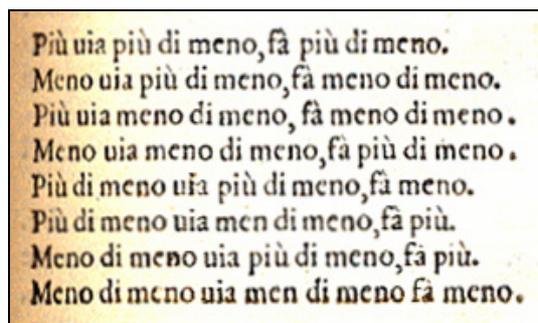


Fig. 3

IMAGINARY NUMBERS FROM HISTORY TO DIDACTICS

It is worth noting that the introduction of imaginary numbers, historically, did not take place in the context of *quadratic* equations, as in $x^2 = -1$. It took place by the resolution of *cubic* equations, whose consideration can be advantageous. Their resolution, sometimes, does not take place entirely in the set of real numbers, but one of their results is always real. A substitution of $x = 4$ in the equation above ($4^3 = 15 \cdot 4 + 4$) is possible in the set of real numbers. In the quadratic equation, the role of i and of $-i$ seems very important. As a matter of fact results themselves are not real, so their acceptance needs the knowledge of imaginary numbers.

Let us briefly summarize the results of an empirical research. In a first stage I examined 97 3rd and 4th year High School students (Italian *Liceo scientifico*, pupils aged 16–17 and 17–18 years, respectively). In all the classes, at the time of the test, pupils knew the resolution of quadratic and of biquadratic equations, but they did *not* know

imaginary numbers. Responding to a question about the statement $x^2+1 = 0 \Rightarrow x = \pm i$ only 2% accepted the resolution (92% refused it; 6% did not answer). A subsequent question proposed the following as a resolution of the cubic equation $x^3-15x-4 = 0 \Rightarrow x = \sqrt[3]{2+11i} + \sqrt[3]{2-11i} \Rightarrow x = (2+i) + (2-i) = 4$. This resolution was accepted by 54% of the pupils (35% refused it; 11% did not answer).

So imaginary numbers *in the passages* of the resolution of an equation, but *not* in its result, are frequently accepted by pupils (the *didactical contract* ascribes great importance to the result). Under the same conditions, a similar test was then administered to 52 students of the same age group, where the equations were presented in the reverse order (Bagni, 2000): 41% accepted the solution of the cubic equation (25% rejected it and 34% did not answer). Immediately after that, the solution of the quadratic equation was accepted by 18% of the students, with only 66% rejecting it (16% did not answer).

These data suggest that teaching a subject using insights from its historical development may help students to acquire a better understanding of it.

THE SEMIOSIC CHAIN

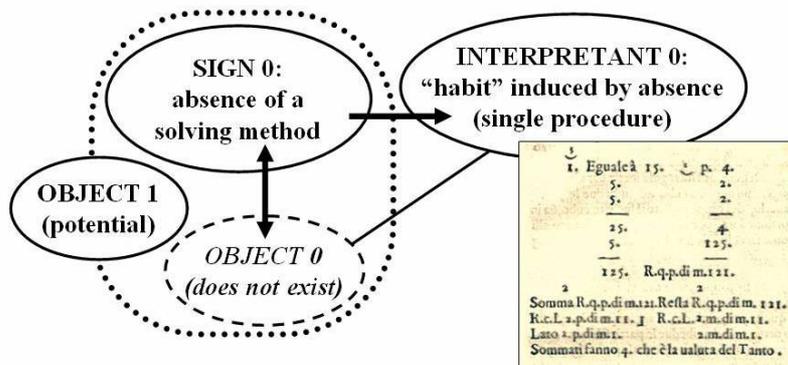
As previously noticed, this focus of this paper is not the detailed presentation of this experimental data (see, Bagni, 2000). Rather I shall consider some features of students' approach, making reference, in doing so, to Peirce's *unlimited semiosis*. As highlighted in section 2, every step of the interpretative process produces a new "interpretant n " that can be considered the "sign $n+1$ " linked with the object (considered in the sense of an objectualized procedure, following Sfard, 1991, and Giusti, 1999, p. 26). However we must ask ourselves: what about the very first sign to be associated to our object?

Our mathematical object (in this case, a procedure to solve an equation) would be represented by a first "sign". In fact, "absence" itself can be considered as a sign. Peirce (1931–1958, § 5.480) made reference to "a strong, but more or less vague, sense of need" leading to «the *first logical interpretants* of the phenomena that suggest them, and which, as suggesting them, are signs, of which they are the (really conjectural) interpretants». So I suppose that this kind of absence can be the starting point of the semiotic process.

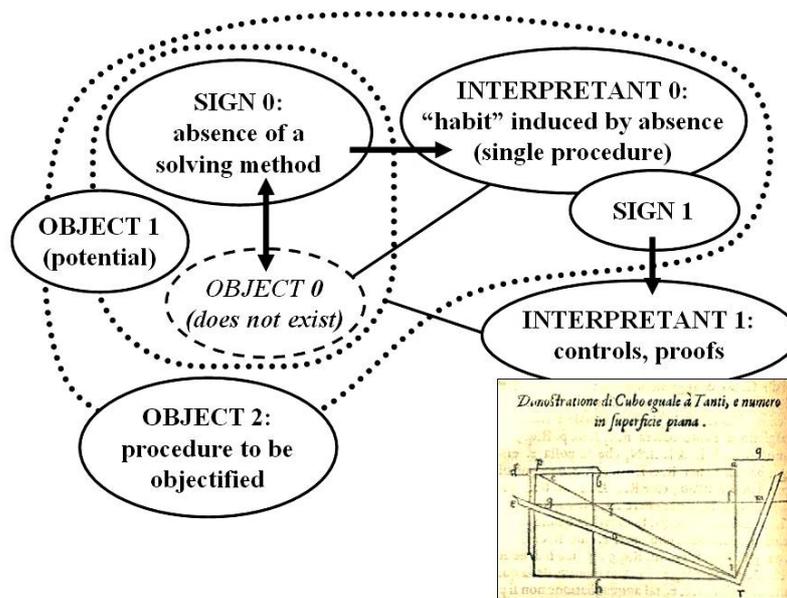
From an educational viewpoint this is influenced by important elements, e.g. the theory in which we are working, the persons (students, teacher), the social and cultural context. Of course by that I do not mean that there is a unique historical trajectory for every "mathematical object". Nevertheless this starting point can be described as a complexus of "object–sign–interpretant" without a particular "chronological" order. It can be considered a *habit* linked to the absence of a procedure, or, better, *a procedure to be objectualized*. So the situation is characterized by some intuitive sensations, and by the influence of social, cultural, traditional elements. Later, with the emergence of formal aspects, our object will become more "rigorous" (making refer-

ence, of course, to the conception of rigor in an historical and cultural context – the rigor for Bombelli and the rigor for modern mathematicians are different). These stages are educationally important.

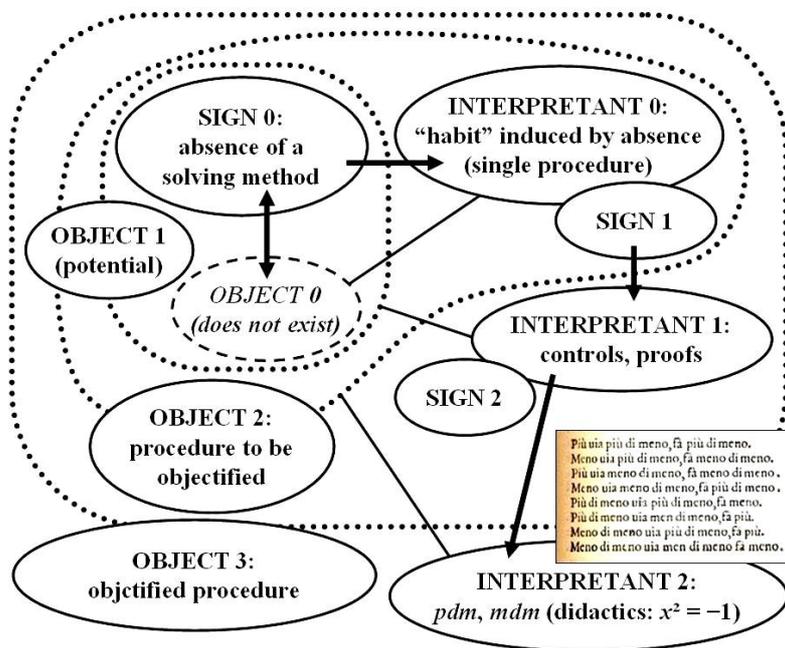
According to an ontosemiotic approach, knowledge is linked to the activity in which the subject is involved and it depends on the cultural institution and the context (Font, Godino & D'Amore, 2007, Radford, 1997). In the case considered, pupils have the perception of an absence, referred to the strategy to be followed, namely the procedure to be objectualized. Historical references gave them the opportunity to consider a situation, and the context is characterized by the “game to be played” (the resolution of an equation) at the very beginning of our experience. We cannot make reference to a semiotic function related to an object to be represented. The “object” will be considered just later, on the basis of the solving strategy. A real strategy is actually absent, and only a “potential object” is connected to the possibility to find out an effective procedure in order to play the (single) game considered.



In Bombelli's work the iconicity has a major role, and this aspect can be relevant to students approach (further research can be devoted to this issue). Educationally speaking, in this stage the effectiveness of the procedure is fundamental. There is not a real mathematical object to be considered, nevertheless pupils have a “game to be played”, and this can be considered as a sign (sign 1). Now controls and proofs are needed, and geometrical constructions can be considered as an interpretant (interpretant 1). So the possibility to provide a first “structure” to the strategy (e.g. the consideration of standard actions) makes it to become a procedure to be objectualized.



Both from the historical viewpoint (let us remember the aforementioned Bombelli's geometrical constructions) and from an educational viewpoint (with reference to the substitution of the result, $x = 4$, in the given equation, $x^3 - 15x - 4 = 0$ so $4^3 - 15 \cdot 4 - 4 = 0$), a first objectualization can be pointed out. The experience considered do not allow to state that pupils reach a complete objectualization. In the following picture, the interpretant 2 is related to an objectualized procedure and it is referred to the "rules" listed by Bombelli (as noticed, only some students accepted them).



Later, the strategy will become an autonomous object and its transparency (in the sense of Meira, 1998) will be important from the educational point of view. It will not be linked to a single situation and it will be applied to different cases (Sfard, 1991). This stage can be characterized by the emergence of a *schema* of action (Rabardel, 1995).

According to Font, Godino and D'Amore (2007, p. 14), “what there is, is a complex system of practices in which each one of the different object/representation pairs (without segregation) permits a subset of practices of the set of practices that are considered as the meaning of the object”. The starting point of the semiotic chain can hardly be considered in the sense of semiotic function. It can be considered as a first practice that will be followed by other practices in order to constitute the meaning of the object.

FINAL REFLECTIONS

In my opinion the importance of an ontosemiotic approach to representations can be highlighted by a Peircean (or post-Peircean) perspective giving sense to the starting point of the semiotic chain. The analysis of this stage of the semiotic chain can help us to comprehend both our pupils' modes of learning and the essence of mathematical objects themselves.

Nevertheless, from a cognitive viewpoint, the question is not only to show how a process becomes an object. The main problem is to understand how signs become meaningfully manipulated by the students, through social semiotic processes. It is also important to notice that Peircean semiotics seems not completely suited to account for the complexity of human processes in problem-solving procedures. In fact, we do not go always from sign to sign, but more properly from complexes of signs to complexes of signs (and usually they are signs of different sort: gestures, speech, written languages, diagrams, artifacts, and so on).

According to L. Radford and H. Empey, «mathematical objects are not pre-existing entities but rather conceptual objects generated in the course of human activity». It is worth noting that “that mathematics is much more than just a form of knowledge production – an exercise in theorization. If it is true that individuals create mathematics, it is no less true that, in turn, mathematics affects the way individuals are, live and think about themselves and others” (p. 250). As a matter of fact, a strategy to be objectualized can influence pupils' approaches both to mathematical tasks and to different (non-mathematical) activities: “within this line of thought, in the most general terms, mathematical objects are intellectual or cognitive tools that allow us to reflect upon and act in the world” (p. 250). These remarks lead us to reflect about the importance of “mathematical objects” and of their representations. They were conceived by mathematicians in the history, they are reprised and re-invented by our pupils today. So they affected – and, nowadays, affect – “all of society and not only those who practice it in a professional way” (p. 251).

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COGNITIVE CONFIGURATIONS OF PRE-SERVICE TEACHERS WHEN SOLVING AN ARITHMETIC-ALGEBRAIC PROBLEM

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The objective of this paper is to describe the cognitive configurations exhibited by the students when solving word problems which could be solved using arithmetic-algebraic methods. The configurations will be described in terms of theoretic elements provided by the onto-semiotic approach to mathematics knowledge and instruction.

Key words: elementary algebraic reasoning, cognitive configurations, primary teachers, didactic reflection.

INTRODUCTION

A number of researchers recommend the incorporation of elementary algebraic reasoning at different levels of primary education (e.g., Booth, 1988). Carraher and Schliemann (2007) state that algebra at the primary school is not simply a subset of the high school syllabus; rather, it is a rich sub-domain of mathematics education with its own approaches and problems to research.

The introduction of student primary teachers to elementary algebraic reasoning is a long and complex process (Van Dooren, Verschaffel and Onghema, 2003). It is considered that primary teachers should be able to recognize and to foster the algebraic reasoning manifested spontaneously by their students (Carraher and Schlieman, 2007). Therefore, research about fostering elementary algebraic reasoning in student teachers is of great relevance to initial teacher education (Borko et al, 2005).

On this research domain there are two questions posed by Carraher and Schliemann (2007, p.675): ‘can young students really deal with algebra?’ and, ‘can elementary school teachers teach algebra?’. Some researchers have tackled the second question. For example, Schmidt and Bernarz (1997) detail student teachers’ resistance and conflicts in the passage from arithmetic reasoning to algebraic reasoning. Similar findings are reported by Van Dooren et al. (2003).

Our purpose is to present the initial findings of a student teachers educational proposal on mathematics reasoning. The proposal offers opportunities to student teachers to develop didactic analysis knowledge (Godino, J. D., Rivas, M., Castro, W. F. y Konic, P, 2008) that could aid student teachers to recognize and to foster elementary algebraic reasoning in their pupils.

We focus the attention on the notion of cognitive configuration introduced by the “onto-semiotic approach”, OSA, (Godino, Batanero, and Roa, 2005; Godino, Batanero, and Font, 2007) to characterize the mathematic knowledge that is mobi-

lized in order to solve an arithmetic-algebraic problem. We consider that this notion offers a wider view of the construct of strategy by considering the conceptual, propositional, argumentative, representational and situational aspects of knowledge alongside the traditional procedural approach.

INSTITUTIONAL CONTEXT AND METHODOLOGY

The research has been carried out with a sample of 94 primary student teachers enrolled in a mathematics method course at University of Granada, Spain. The course aims to develop mathematical knowledge as well as didactical reflection. It is to mention that algebra as such was not studied in the course. During the course several mathematical problems that could be solved using elementary algebraic reasoning were given to students. In this paper we analyze the students' solutions to one of these problems which were given during a test.

A ball is thrown from an unknown altitude; it bounces up to one fifth of the altitude it was thrown from. If after three rebounds the ball reaches an altitude of 6 cm, a) What is the altitude it fell from the first time?, b) Explains the resolution using algebraic notation.

The problem belongs to a category of very well studied word problems. However, within the framework of this course, we are specifically interested in the arithmetic and algebraic solutions provided spontaneously by students.

EPISTEMIC ANALYSIS OF THE PROBLEM⁴

The OSA focuses on five dimensions in analysing the objects and meanings used in solving a mathematical problem: linguistic objects, concepts, properties, procedures and arguments. In what follows we analyse the problem using OSA⁵. This analysis has two purposes for the teacher educator: to explore the objects and meanings put into effect during the solution of the problem, and to identify eventual meaning conflicts and to foresee difficulties and errors that could emerge in students' solutions to similar problems.

The word problem is stated in terms of linguistic elements, which refer to quantities, magnitudes and relationships between them. These can be expressed in arithmetic or algebraic terms.

The statement "*A ball is thrown from an unknown altitude*" refers both to a real experience and to the unknown value of a quantity. Next it enounces a condition "*it bounces up to one fifth of the altitude it was thrown from*" that establishes the numeric relationship, invariant during the bouncing, between the altitude the ball falls from and the altitude to which it bounces, expressed by the fraction $1/5$.

⁴ To see an example of such analysis, we refer the readers to the work of Godino et al. (2008).

⁵ A priori analysis of the solution to the problem done by an expert.

The statement “*If after three rebounds the ball reaches an altitude of 6 cm*” establishes that the numeric relationship is compounded three times with itself, fraction of fraction. Additionally it assigns a value to the last altitude.

Finally the statement, “*What is the altitude it fell from the first time?*” establishes the quantity that must be identified in terms of the given information in the problem wording.

The linguistic terms refer to mathematic concepts (e.g., fraction, equality, unknown, operation), whose meanings, properties and procedures are related argumentatively in a complex way and favors or inhibits the solution to the problem.

It is worth to mention that both the eventual arithmetic and algebraic solutions place the primary entities in different configurations. For instance, in an arithmetic solution, if it is assumed that 6 is the fifth of an unknown quantity, then we can find the unknown quantity by multiplying for five, inverting the fractioning operation used initially. However, in an algebraic solution, it is not necessary to use either this property or the associated concept. The unknown quantity is multiplied, three times, by $1/5$ and this is equated to 6. Subsequently the unknown is isolated using a procedure that frames the solution in terms of multiplication/division.

COGNITIVE ANALYSIS OF THE STUDENTS’ SOLUTIONS

In what follows we will describe our typology of cognitive configurations evident in the solutions produced by the students. In each case, we identify the mathematical objects and meanings used by the students in representing their solutions.

*Algebraic configurations*⁶

Algebraic solutions are those where the use of unknowns is clearly manifested. The types of algebraic solutions are: use of unknown, assigning tags to equations, use of three unknowns, and additive relationships.

ALC1⁷: Use of unknown. On this type of procedure the unknown appears explicitly written and it is isolated. The students have attributed meaning to the linguistic objects “a bounce” and “*If after three rebounds*”, and have represented such linguistic elements in procedural objects, this can be deduced from the actions carried out on fractions, on relationships established and expressed by the equal sign and, finally, on isolating the unknown.

ALC2: Assigning tags. Students explicitly associate each rebound with an equation. They use a process made of three steps: initially identify the unknown “*altitude the ball fell from*” which is named x , later name the equation corresponding the first bounce as “*first rebound*”, and so two times more, up to the point where they write the equation that corresponds to the third bounce, and name it “*third rebound*”, equate to six and obtain the sought value.

⁶ See Godino et al. 2008.)

⁷ The code ALC and ARC stands for algebraic and arithmetic configurations, respectively.

Every solution on this category is correct. It seems that students control the alleged difficulty that rises when dealing with unknowns by assigning a tag that lets them to isolate each rebound, represented linguistically, and at the same time allocated it in the problem context. On this type of solution the students have isolated the linguistic object “*it bounces up to one fifth of the altitude it was thrown from*”, and have identified it as an operative invariant in the whole process and have given it a procedural role expressed by multiplying by one fifth.

The procedural and linguistic objects are materialized argumentatively through the appropriate use of the equality in its relational meaning and by means of numerical operations and properties that are carried out on the equation with the purpose of isolating the unknown.

ALC3: Use of three unknowns. Students use three unknowns, each one of them associated to the unknown’s numerical values corresponding to each bounce. Then they propose an equation and they execute a nested replacement of variables, from the expression corresponding to the last one up to the expression corresponding to the first bounce, and they proceed to isolate the unknown.

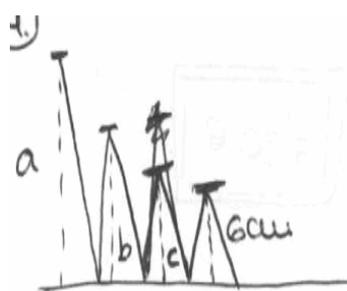
The problem is tackled by means of a procedure that breaks up it in three moments; the first and the second are represented by an equation with two unknowns, and the third, by an equation with one unknown. The mastering of linguistic elements that describe the relationships is predominant on this procedure.

The possible meaning conflicts on the description of the problem are overcome by assigning a semiotic function, whose antecedent corresponds to each and every bounce, and the consequent is a relationship, expressed as an equation.

On this procedure the students operate “with” and “on” the unknown (Tall, 20001) and spontaneously use the transitive property of equality (Filloy, Rojano and Solares, 2004).

It is observed, on this solution strategy, the use of procedures on two levels, the first that involves the “process” of dividing the problem in three parts, and the second, the use of properties and procedures, in the usual manner as mathematical procedures are used. This type of solution is illustrated on Figure 1.⁸

⁸ A translation is provided



a is the initial height from which the ball is thrown. Each bounce a , b , 6 cm is $\frac{1}{5}$ of the previous bounce. We isolated the first equation in order to substitute it in the others.

$$6 = \frac{1}{5} \text{ of } c = \frac{c}{5}; \quad c = 6.5 = 30; \quad 30 = \frac{1}{5} \text{ of } b = \frac{b}{5}; \quad b = 30.5 = 150$$

$$150 = \frac{1}{5} \text{ of } a = \frac{a}{5}; \quad a = 150.5 = 750, \text{ the initial height } 750 \text{ cm}$$

Figure 1. Use of three unknowns (ALC3)

CAL4: Additive relationships. On this type of solution, the students use an unknown and produce expressions and equations that relate arithmetic data by means of additive expressions. Some students wrote expressions (not equations) to represent the problem. The operative invariant “one fifth” appears multiplying the unknown that is operated, additively with the numbers three and six but without establishing a relationship expressed by an equation. In some cases the fragility of knowledge about properties of rational numbers is manifested.

In some other solutions it can be seen that some relationships are proposed among the numerical values “three” and “six”, where “one fifth” multiplies the unknown, the students identify the presence of an unknown and recover the numbers out of the problem wording, however they do not related them in any way.

Arithmetic configurations

Arithmetic solutions were classified as those where only arithmetic operations are used without any reference to unknowns. The types of arithmetic solutions identified are: Reverse multiplication, multiplicative relationship, additive relationship, and rule of three.

ARC1: Reverse multiplication. The solution procedure consists of inverting the operation: it is known that the altitude to which the ball bounces is one fifth of the altitude it was thrown from, as 6 is the last altitude, therefore the previous altitude is 6×5 and the previous altitude to the last one is $6 \times 5 \times 5$. Finally the altitude the ball was thrown from is: $6 \times 5 \times 5 \times 5$.

Students using ARC1 exhibit competence and fluency in the use of the multiplication operation in the context of known quantities. It is of note that this aspect of “operation sense” underlies algebraic thinking Slavit (1999, p.256).

On this category are located the right arithmetic answers given by the students. The only meaning conflict found on some answers is considering four bounces instead of three. Figure 2 illustrates this type of solution.

1er bote - 150 cm
 2º bote - 30 cm
 3er bote - 6 cm x 5

$$\begin{array}{r} 150 \\ \times 5 \\ \hline 750 \text{ cm} \end{array}$$

La pelota se lanzó desde 750 cm

③ ↑ 6 cm = 1/5 de ② ↑ 30 cm = 1/3 de ① ↑ 150 cm = 1/5 de (PI) ↓ 750 cm

Figure 2. Reverse multiplication (ARC1)⁹

ARC2: Arbitrary use of multiplication. Students focus their attention simply on the numbers contained in the problem: 6, 3 and 5, and the solution they offer is an arbitrary combination of multiplicative operations among these three numbers. The students appear to construct their solution without paying any attention either to the conditions on numbers or to relationships among them. According to Garolafo (1992), these students do not exhibit a “numeric approach”, because they do not display strategies neither to decide which operations to use nor to assess a plan to solve the problem.

It is deduced from the students’ solutions that they have not comprehended the meaning, in operative terms, of the linguistic objects “*first*”, “*second*” and “*third*” bounce, nor in relational terms of “*If after three rebounds the ball reaches an altitude of 6 cm*”. The students are incapable of expressing numerically the relationships present in the problem.

The two approaches to rational numbers duplicator/partition and stretcher/shrinker (Behr, et. al. 1997) are stressed on this strategy due to the fact that 6 cm is not identified as the last bounce, corresponding to one fifth of a quantity that can be found by multiplying for five, inverting the operation initially implemented, fractioning by five. The operative actions corresponding to adding up fractions are carried out correctly even though it seems to be a lack of meaning that students attach to the numbers and operations between them.

ARC3: Arbitrary use of addition. As with ARC2, the students only pay attention to numeric data, and simply add up the numbers, in some cases, without appearing to establish *any* relationship among them. It seems that students have assumed that the problem has an additive structure, where the length of the bounces are added up and the data 6cm, corresponds to the sum of the altitudes of the three bounces.

The meaning conflicts are located in the linguistic elements corresponding to “*first*”, “*second*” and “*third*” bounce, as well as, to the statement “*one fifth of*”, which is interpreted only in its numeric dimension. It seems that the relationships among the numbers and expressed linguistically in the problem wording are superfluous to students.

⁹ The translation for the Spanish in the graph is: 1) Ball was thrown from 750 cm; 2) Bote stands for bounce

ARC4: Rule of three. The students establish a proportionality relation between the number three, that corresponds to the bounces and 6cm, then formulate the question: what is the altitude corresponding to one bounce? The meaning conflicts on this category are much more profound. It seems that students have associated the data format presentation and the problem wording to the archetypal format of proportionality problems that are solved through the so called “rule of three”.

On this type of solution the students carry out the change of type of register procedure that lets them to produce meaning in numerical terms but with no link to the problem. It seems that problem complexity compels students to veer towards more familiar grounds and to perform arithmetic operations (Herscovics & Linchevski, 1994).

A discussion of results

The last three types of arithmetic solutions (ARC2, ARC3 and ARC4) are characterized by a wrong meaning assignment to linguistic objects. Understanding the statement of a word problem requires recognition of the existence of dependence among meaning corresponding to elementary entities. Anghileri (1995) suggests that the close relationship between real settings and the procedures used to solve problems characterized the initial states in learning mathematics. The students have not succeeded in writing a numerical “argument” that links different objects appearing during the resolution process.

The difficulties in representing the problem arithmetically or algebraically are evident from the analogy between ALC4 and ARC3. Nonetheless the meanings and the ways they are related differ essentially. Along with each type of resolution it has been shown that the problem structure raises a number of interpretative challenges, and how the solutions correspond to particular configurations of primary entities, where these facilitate or hinder the arithmetic or algebraic problem representations. The mathematic objects invoked in the problem are the same but the meanings, the relationships among them and the meaning conflicts are diverse to students.

To Filloy, Rojano and Puig (2007), “the mode of thought- be arithmetic or algebraic- appears to be determined by the type of ‘ relational calculation’ that underlies the problem structure” (p.216). We consider that the relational calculation can be expressed and objectified in terms of primary entities, which could be useful for the teachers to recognize both the mathematic tasks complexity and the variety of mathematical reasoning leading to the solution.

RESULTS SUMMARY

Table 1 gives a detailed breakdown of the number and proportion of each type of algebraic and arithmetic solution.

Types of algebraic solutions	ALC1	ALC2	ALC3	ALC4	Correct/incorrect
Number of students	25	17	5	11	37/58
Types of arithmetic solutions	ARC1	ARC2	ARC3	ARC4	10/23
Number of Students	4	14	3	2	
Do not answer	13				

Table 1: Type of configuration and number of students in each one

It can be seen that the number of algebraic solutions as the number of right solutions outnumbered the corresponding arithmetic solutions. The proportion between right solutions and solutions of each type is bigger for the case of algebraic solutions.

Even though students are asked to provide an algebraic solution in the second problem's item, they could have provided an arithmetic solution in the first problem item as well. Given that algebra was not studied during the course, it is worth noting the students' algebraic preference.

IMPLICATIONS FOR STUDENT TEACHER TRAINING

A finding of this research is that the algebraic methods used by the students to solve the problem outnumber in quantity and in effectivity the arithmetic strategies. Just a small number of students choose to solve the problem by means of a right arithmetic strategy in contrast to the findings reported by Nathan and Koedinger (2000). Another finding is the apparent disarticulation among the linguistic, conceptual and procedural elements in the cognitive configurations exhibited by the students, who do not manage to elaborate an "argument" leading to a problem solution.

We consider that teacher's activity not only concerns with planning mathematic tasks but also with the promotion and recognition of the meaning present in the students' solutions, where the primary entities are articulated. Recognizing the entities involved students' solutions could help teachers guide their didactic actions.

Therefore it is important to make teachers conscious of the network of objects, meanings and configurations that are put into effect during the mathematics problems solutions to help identifying the meaning conflicts manifested by pupils and therefore, to give answers to those conflicts in the classroom context. As a consequence, it is convenient to use the cognitive-epistemic analysis (Godino et al. 2008) in initial teacher training programs.

Some researchers have contended that teacher's competence to understand and to use the mathematic knowledge adapting it to students' achievements is important (Ball, 1990; Wilson, Shulman and Richert, 1987). More recently Hill, Rowan and Ball

(2005) found that content knowledge is related meaningfully to students' achievements.

We conclude with the observation about the arithmetic strategies that we have discussed above. Our study suggests that algebraic thinking underlies successful problem solutions. We believe that a focus on elementary algebraic reasoning can aid teachers in enabling their pupils to more fully understand the arithmetic domain.

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TRANSFORMATION RULES: A CROSS-DOMAIN DIFFICULTY

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The learning of a symbolic system such as algebra relies on the learning of the use of transformation rules. The implementation of rules in a CAS (Computer Algebra System) for students' modelling has pointed out some questions that are at the junction of three research fields: informatics, mathematics and didactics. Each of these communities has its own perception of algebraic objects, founded on models or practices. The implementation of objects that live in school has questioned object reliability. In this paper, a parallel is proposed between difficulties of informatics implementation of transformation rules and novices' difficulties.

Keywords: algebraic calculations, rules, informatics implementation, students' difficulties.

An important part of school algebra rests on algebraic calculations, what Kieran calls the “transformational activity”, which she distinguishes from the generational and global activities (Kieran, 2001). This activity focuses on changing the form of an expression or an equation in order to *maintain equivalence*. This includes, for instance, collecting like terms, factoring and expanding expressions. These are algorithmic tasks like the transformation of $(5 + x)x + 10 + 2x$ into $(5 + x)(x + 2)$. The conservation of equivalence relies on *correct rules* that allow substituting expressions by others. These rules will be called “transformation rules” in this paper. They are supported by the laws of the polynomial ring –commutative law, distributive law and so on. Rules produce objects of a particularly interesting form. Their use is guided by what the desired expression has to look like: reduced polynomial expression or factored polynomial expression. Bellard et al. (2005) call them the constituent rules of mathematic theory: “these rules constitute the base of the [transformational] activity, govern the motion and predetermine the permitted actions”. Such mathematical rules are supposed to be accurate and self-sufficient.

Nevertheless, Durand-Guerrier and Herault (2006) stress the fact that rules are objects the usage of which is not so obvious: “the rule is not only a way to learn but it is also an object which has to be learned”. It is, in fact, impossible to present a rule alone to students. Rules have to be transposed, adapted and as such lose a part of their accuracy. The implicit notions of rules are compensated by a necessary didactical contract (Brousseau, 1997): “it is an illusion to believe that one can produce the meaning in the mind of someone by indirect ways through the rule and examples” (Wittgenstein, Ambrose, & Macdonald, 1979). Durand-Guerrier and Herault (op.cit.) also point out the illusion to think that the use of a rule is plain, such as “rails that would guide un-failingly and in advance the way to be followed”. Actually, it is an interpretation that allows these implicit details between the rule and its application to be overcome. But what are exactly the notions underlying the learning of a transformation rule?

Our research is in line with the identification of systematic errors that students commit when solving transformational exercises. A library of correct and incorrect transformation rules has been built for that purpose and an automatic diagnosis mechanism has been implanted in order to associate a sequence of applied rules to student's transformation (Chaachoua, Croset, Bouhineau, Bittar, Nicaud, 2008). The implementation of these rules has raised questions about the kind of representation of a transformation rule. Automating the process forces the researcher to clarify some implicit mechanisms for the expert: how does a rule work? In which way does it work? How is it matched? It has led to three crucial points about implantation difficulties:

- The reading direction of a rule;
- The notion of sub-expression;
- The generic status of a rule.

Each of these points is discussed in the next sections. We propose, in addition, to link these three points to three classical difficulties which novices may experience when doing transformational activities: the difficulty of understanding the symmetric aspect of the equal sign (see e.g. (Kieran, 1981)); the difficulty of the structural aspect of an algebraic expression (see e.g. (Sfard, 1991)) and the difficulty of applying a general rule to a particular case (see e.g. (Durand-Guerrier, Herault, 2006, p. 144)).

The choices made to raise difficulties in programming may shed light an improvement of the teaching of algebraic rules and may overcome students' problems. Indeed, the reading direction of a rule is essential for a deductive reasoning, the notion of sub-expression allows matching correctly a rule and the generic status of a rule is the power of algebra.

1. REPRESENTATION, READING DIRECTIONS AND REASONING PROCESS

Transformation rules can be represented by two kinds of writings: equality or implication. Both present advantages and have good reasons to be used. Yet, we will see that rules as implication form are interesting in that it highlights the reasoning process in the transformation activity.

Rules as equality, used in school

The first representation –a rule as an equality– is the usual one used in school. Rules can be called by different names in the textbooks: proposition, property, identity, equality and sometimes even theorem (Bellard et al., 2005). Whatever their name, rules are often coming in the form of equality. For example, the distributive law is presented as:

$$k(a + b) = ka + kb, \text{ where } k, a \text{ and } b \text{ are real.} \quad (Eq1)$$

There is a double meaning of the equal sign: that of “identity” or that of “relation”. In transformation rules, the equal sign is of course used as “identity”, whereas in equations, the equal sign is used rather as “relation”. This well-known duality is a real dif-

difficulty for students. Presenting rules as identity can, on the one hand, be interesting to get students used to and, on the other hand, provoke confusion.

Such representations are declarative rather than procedural: this form of identity has no explicit reading direction since the equal sign has a double way: from left to right and vice versa. A learning of the way to use such a rule has to be taught. Whereas the process-product has been many times denounced (Davis, 1975) and that special exercises are proposed to students in order to grasp the equivalence notion, here is a case where the equality has to be used in one of the two ways. In fact, textbooks sense that, most of the time, it is necessary to distinguish the two ways by proposing two identities: not only (*Eq1*), which is used to expand expressions but also “ $ka + kb = k(a + b)$ ” to factor. This kind of presentation requires a specific work to become operative: associate a reading direction to the equality for application, according to the aim.

Rules as implication, used in informatics

The second representation –a rule as an implication– is the one used in informatics. One calls “implication” what Durand-Guerrier, Le Berre, Pontille, & Reynaud-Feurly (2000) call “formal implicative”, representation used in geometry:

$$\forall x \in \mathfrak{R}, P(x) \Rightarrow Q(x).$$

Implemented rules are represented as *oriented* mechanisms, also called “rewrite rules” (Dershowitz & Jouannaud, 1990): $A \rightarrow B$, where A is *rewritten* in B . The object A produces the object B and B can not produce A unless an other rule $B \rightarrow A$ is considered. For example, the rules:

$$k(a + b) \rightarrow ka + kb \text{ (R1) is used to expand,}$$

$$ka + kb \rightarrow k(a + b) \text{ (R2) is used to factor.}$$

It is rather a necessity in computing modeling to represent rules as oriented ones than a choice. Indeed, it is not really possible to implement rules as identity. If a single rule is implemented both for expanding and for factoring, there will be some loop and ending problems. For example, with the single rule (*Eq1*), the expression “ $3(x + 4)$ ” would be transformed into “ $3x + 12$ ”, then into “ $3(x + 4)$ ” and so on.

Even if it is a necessity, this kind of representation is interesting because its reading direction is explicit: given a real or a polynomial expression under the form “ $ab + ac$ ”, where “ a ”, “ b ” and “ c ” are reals or polynomials, it can be rewritten into “ $a(b + c)$ ”. One can suppose that the use of rules as implication is easier because of its procedural aspect. The kind of representation has an impact on its use easiness, as we will show in the next section.

Impact of the reasoning process

Although geometry is a special introduction field for proof, the latter is not a prerogative of geometry. The “deep structure” (Duval, 1995) of the transformational activities can be presented as a ternary organisation proposed by Duval. A premise (here, a certain expression), a proposition (a transformation rule) and a conclusion (an other expression), as shown in Figure 1, constitute a deduction step. These steps follow on, the conclusion of the current step becoming the premise of the next one. Using Duval’s classification (Duval, 1990), the algebraic calculation is formed by *deductive reasoning of steps explicitly concatenated in reference to a transformation rule*. Thereof, this activity can be viewed as a process of demonstration:

“Demonstration would be defined to be, a method of showing the agreement of remote ideas by a train of intermediate ideas, each agreeing with that next it; or, in other words, a method of tracing the connection between certain principles and a conclusion, by a series of intermediate and identical propositions, each proposition being converted into its next, by changing the combination of signs that represent it, into another shown to be equivalent to it” (Woodhouse, 1801)

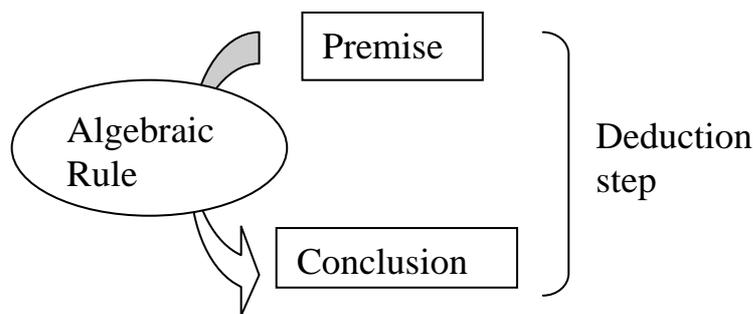


Figure 1: Deduction steps.

Representing rules as implications could allow the user to follow this reasoning process explicitly, as shown in Figure 2.

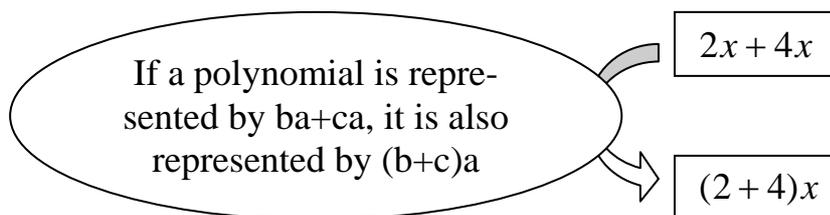


Figure 2: Example of the reasoning process in algebra. The level of making explicit a demonstration and the granularity of a deductive step evolves with the level of the student. Here, for example, we have omitted to explicit the commutative law. As Arsac notes: “any demonstration is shortened from another demonstration” (Arsac, 2004).

Splitting an identity into two implications conceals the fact that rules are equivalent but clarifies the *way of application* and, above all, it allows following the Duval’s structure of a deduction step. This is the *modus ponens* mode: “if p , then q , now p ,

then q ". The representation form of a rule has an impact on its use easiness but it lets the difficulty to know to which object the rule can be applied.

2. MATCHING AND SUB-EXPRESSIONS NOTION

An unrefined syntactic unification between the premise of a rule and a part of an expression does not produce an algebraic behaviour. With an unrefined unification, a rule as $x + x \rightarrow 2x$ would transform " $5x + x$ " into " $5 2x$ ", which has no sense (what is the operator between "5" and "2"?) nor the expected result. This is a well-known mistake committed by students: substitute an expression by another by working only on a syntax level and taking no account of semantics. Mastering substitution needs knowing the notion of what a *sub-expression of an expression* is.

The definition of an expression from the rewrite rule theory of Dershowitz (1990), in which rules are applied on *sub-objects*, underlines the notion of sub-expression, thanks to its recursive definition. Let us consider a set of symbols of terminal objects (e.g., integers), a set of symbols of variables (e.g., $\{x, y, z\}$), and a set of symbols of operators (e.g., $+$, $-$, \times , \wedge , sqrt , $=$, $<$, and, or, not). An *algebraic expression* is a finite construction obtained from the following recursive definition:

- a symbol of terminal object
- or a symbol of variable
- or a symbol of operator applied to arguments which:
 - are algebraic expressions,
 - are in the correct number (correct arity [1]),
 - and have correct types [2].

With this definition, matching a rule R to an expression E would consist of finding a sub-expression E_1 of E , replacing E_1 in E by the expression that produces R . For example, in " $5x + x$ ", the algebraic (sub) expressions are " $5x$ ", " x " (two times), " 5 " and " $5x + x$ ". The expression " $x + x$ " is not a sub-expression of " $5x + x$ ". To deal with this problem, the internal representation of expressions in computer algebra systems (CAS) is a tree representation, in which the structure of the expression is explicit, as shown in Figure 3.

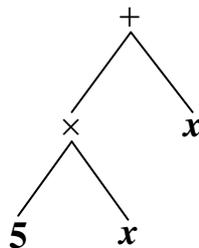


Figure 3: Tree representation of the expression $5x + x$.

The necessity of the tree representation appears also in school curricula. Although school approach of expressions is foremost syntactic -algebraic expressions are defined as “writings including one or more letters”-, new French curricula encourage making students work on tree representations. As they claim, tree representation allows pointing out the structural aspect of an expression as defined by Kieran:

“The term *structural* refers, on the other hand, to a different set of operations that are carried out, not on numbers, but on algebraic expressions. [...] the objects that are operated on are the algebraic expressions, not some numerical instantiations. The operations that are carried out are not computational. Furthermore, the results are yet algebraic expressions.” (Kieran, 1991)

This structure notion is essential to deal with matching difficulties. It enables understanding why such rule like $k(a+b) \rightarrow ka+kb$ (*RI*) can be applied on sub-expressions of expressions such as $3+4(x+1)$. Nevertheless, is it sufficient to understand that this rule can be applied also on expressions such as $4x^2(x+1)$ or $4x^2(x+1+x^2)$? Either in informatics or at school, we will see that most of the time, one needs to precise as many rules as there are cases.

3. GENERIC STATUS OF RULES

The third idea which emerges of rules implementation turns on the generic status of a rule: how a rule such as (*Eq1*) or (*RI*) can be sufficient to apply to the expressions “ $7(3+x)$ ”, “ $-7(3+x)$ ”, or even “ $7(3+x+x^2)$ ”? How to deal with the matching of “ $a+b$ ” with “ $3+x+x^2$ ”? It is, with no doubt, the principal difficulty for novice users of rules: the application of a general rule to a particular case. It is, in fact, the same in informatics. Although the two first points –reading direction & matching problems– have been easily resolved in informatics, it has not been the same for this third problem.

The entry by rewriting rules –and thus a syntactic presentation– leads to some new problems. Let us study again the case of (*RI*). For experts, it is not really this rule that is used but much more the single distributive law. With this last one, experts can expand any product of polynomials. In informatics, one needs rules to be implemented and so, the exact structure of an expression has to be specified. For (*RI*) implementation, “ k ”, for example, has to be defined: is “ k ” a real, a product such as a monomial or a sum? It is not possible to just say “given a polynomial k ”. Indeed, to transform “ k ”, its structure has to be specified. For example, if “ k ” is negative, the sign of the entire expression is changed. The main operator of the expression becomes “minus” and not “times”: the entire internal tree representation is changed, as shown in Figure 4. The same difficulty is found when “ $a+b$ ” is a sum of three terms: it can change the mechanism of the implementation of the rule. Without genericity, one needs to distinguish cases like “ $k(a+b)$ ” from “ $k(a+b+c)$ ”. To deal with that, the concept

of distributive law has been implemented. Let us consider two lists and an operator, the distributive rule can be written as:

$$(a_1, a_2, \dots, a_n) \Delta (b_1, b_2, \dots, b_n) \rightarrow (a_1 \Delta b_1, a_1 \Delta b_2, \dots, a_1 \Delta b_n, a_2 \Delta b_1, a_2 \Delta b_2, \dots, a_n \Delta b_n).$$

We do not have to specify the length “ n ” of the lists.

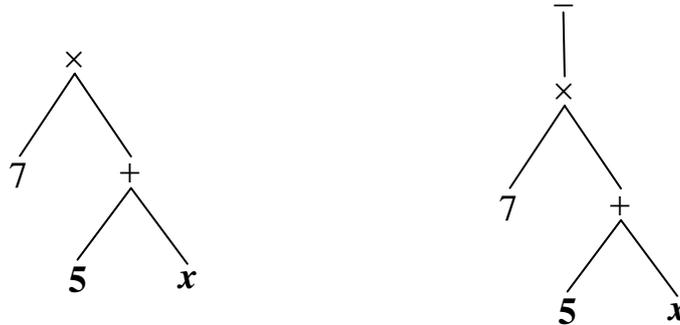


Figure 4: The single change of the real 7 into -7 changes the entire structure of the tree representation of the expression. On the left, the expression $7(5+x)$; on the right, the expression $-7(5+x)$.

Another example is very representative of this problem: the rule of monomials addition, which can be written as $ax + bx \rightarrow (a \oplus b)x$, where \oplus is the calculated sum operator. Such rule is not so easy to implement. If we ask the premise to be a sum of two products, this rule will not apply to expressions such as “ $ax + x$ ” because “ x ” is a single argument and not a product: an automatic mechanism does not recognize “ x ” as the product of “1” and “ x ”. To deal with this problem, some concepts have been implemented like the *monomial concept*. We have implemented the added fact that a monomial can be either a product of a real and a variable –of explicit degree or not– or a single variable –of explicit degree or not. Thus, expressions such as “ $4x^5$ ”, “ $4x$ ”, “ x^1 ” or “ x ” are read as monomials, and the rule $ax + bx \rightarrow (a \oplus b)x$ can be easily implemented: one needs just to specify that the premise has to be a sum of two monomials.

The same problem occurs at school: the polynomial notion is not taught in France [3]. The variable “ k ” from the rule (*Eq1*) is then defined as a real, so are “ a ” and “ b ”. Understanding that “ a ” can be itself a sum, or even a sum with variables, requires a real work. How do French textbooks deal with this problem?

To answer this question, we have used the concept of praxeologies from the Chevallard’s anthropological theory of didactics. Let us remain that Chevallard proposes to describe any human activity by a quadruplet which enables an activity to be cut in types of task, which can be solved by techniques –a way of doing–, which can be explained by a rational discourse, “logos” (Chevallard, 2007) [4]. Our work in progress (Croset, 2009) shows that French textbooks distinguish three types of task for expanding expressions:

$$“k(a + b)”, “k(a - b)” \text{ and } “(a + b)(c + d)”.$$

Cases like “ $-(a + b)$ ” or “ $+(a + b)$ ” are discussed in another part of textbooks (“how to remove brackets?”). Some textbooks propose even more distinctions: they discern also “ $(a + b)k$ ” and “ $(a - b)k$ ”.

On the one hand, it seems that textbooks decide to specify many cases although all these tasks are explained by a single “logos”: the distributive law. The fact that textbooks need to precise many cases points out the well-known difficulties of students to apply a general rule to particular cases. On the other hand, all possible cases cannot be specified. Textbooks do not specify types of tasks as “ $k(a + b + c)$ ” or “ $k(a + b)(c + d)$ ”. Understanding the structure of the expression is supposed to be sufficient to deal with all these forms. Nevertheless, we have not found such work and reflection about the generality of rules. Only a few textbooks precise links between the three types of task described above. Explanations such as using $k(a + b) \rightarrow ka + kb$ to expand “ $(a + b)(c + d)$ ” are not common. Neither are presented the iteration concept to expand “ $k(a + b)(c + d)$ ” whereas our work (ibid.) shows that students’ mistakes occur specially in this sub-type of task.

The second problematic example about monomials revealed by the computing implementation occurs also in students’ difficulties: recognizing “ x ” as a monomial is not an easy task for a novice. A novice’s common mistake is precisely to transform “ $ax + x$ ” into “ ax ” because of the lack of the explicit coefficient “1” ahead of the “ x ”: when “ a ” is added to “nothing”, it remains “ a ” [5]. The concept of monomial is not taught currently in French curriculum. We speak about “like terms” but few textbooks precise that “ x ”, “ $1x$ ”, and “ x^1 ” are “like terms” which can be collected.

The force of algebra lies in the writings generic status. Its interest is lost if all cases are presented. To avoid that, a specific work on concepts such as distributive law or monomial could be proposed to novices, just like it has been done for the computing implementation.

4. CONCLUSION

The learning of the transformational activity cannot be restricted to memorizing rules. This requires a specific work about the application of rules. Our research focusing on automatic student modelling has brought to light three important difficulties concerning the application of transformation rules, which have been compared with similar novices’ difficulties: knowing that a rule has a reading direction allows students to follow a reasoning process when they transform algebraic expressions; knowing the structure of an expression permits a correct matching; finally, having a good perception of the generic status of rules allows students to apply a general rule to a particular case. All these elements are necessary conditions for learning the algebraic symbolic system. Our paper has described the parallel between informatics implementation difficulties and the ones met by novices. One can wonder if the way to deal with the first ones could be used to deal with the second ones.

Regarding these three points, rules have been looked at from a technical point of view. Another point of view would be considering what experts' criteria are to control their transformations: substitute numerical values to equivalent expressions in order to verify the equivalence; in other words, being aware that equivalent expressions denote the same object. Similarly, another interesting point of view is to explore how to *choose* the appropriate rule. We have seen that a rule is general but the choice of a rule is crucial to obtain the form that one needs. The *raison d'être* of a rule, the strategic process and elements that guide an expert in choosing this particular rule, and not another one, have not been discussed here, despite the fact that informatics is also interested in such questions. We can expect that a parallel would be again possible between novices' strategic difficulties and the implementation ones.

NOTES

1. The arity of an operation is the number of arguments or operands that the operation takes. For example, addition is an operation of arity 2, sqrt is an operation of arity 1.
2. For example the expression " $\sqrt{5x=3}$ " has not a correct type.
3. A recent study has compared the algebra learning in France and in Vietnam (Nguyen, 2006). It shows that algebraic expressions found in French textbooks rely on the notion of polynomial function whereas the ones that can be found in Vietnamese textbooks rely on the polynomial notion.
4. The reference (Chevallard, 2007) is not the best one for the notion of praxeology but it presents the advantage of being written in English. French reader can see also (Bosch & Chevallard, 1999).
5. Haspekian (2005) proposes another explanation to this mistake: the difficult notion of neutrality of the multiplicative law. We think that, in our context, the mistake is more relative to a visual lack.

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INTERRELATION BETWEEN ANTICIPATING THOUGHT AND INTERPRETATIVE ASPECTS IN THE USE OF ALGEBRAIC LANGUAGE FOR THE CONSTRUCTION OF PROOFS

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Abstract. *This work is part of a wide-ranging long-term project aimed at fostering students' acquisition of symbol sense (Arcavi, 1994) through teaching experiments on proof in elementary number theory (ENT). In this paper I present some excerpts of students discussions while working in small groups on activities of proof construction. My analysis of these transcripts is aimed at highlighting the incidence of anticipating thoughts and of the flexibility in the coordination between different conceptual frames and different registers of representation in the development of proof in ENT. In particular, I singled out four main sources of interpretative blocks, highlighting the strict interrelation between anticipating thought and students' difficulties in the interpretation of the algebraic expressions they produce.*

1. INTRODUCTION

Many research studies support an approach to algebraic language related to the development of reasoning. Arcavi (1994), for example, claims that, in addition to stimulating students' abilities in the manipulation of algebraic expressions, teachers should make them see the value of algebra as an instrument for understanding, introducing them to algebraic symbolism from the beginning of their studies through specific activities that encourage an appreciation of the value and power of symbols. A central aspect in Arcavi's approach to algebraic language is, in fact, the concept of symbol sense. The author chooses to characterize symbol sense highlighting, through meaningful examples, the attitudes to stimulate in students to promote an appropriate vision of algebra. Particular attitudes that he names include: the ability to know when to use symbols in the process of finding a solution to a problem and, conversely, when to abandon the use of symbols and to use alternative (better) tools; the ability to see symbols as sense holders (in particular to regard equivalent symbolic expressions not as mere results, but as possible sources of new meanings); the ability to appreciate the elegance, the conciseness, the communicability and the power of symbols to represent and prove relationships. Many researchers share a similar vision of the approach to the teaching of algebra. Among them, Bell (1996), states, in particular, that it is necessary to favour the use of algebraic language as a tool for representing relationships, and to explore aspects of these relationships by developing those manipulative abilities that could help in the transformation of symbolic expressions into different forms. This idea is strictly connected with Bell's description of "the essential algebraic cycle" as an alternation of three main typologies of algebraic activity: representing, manipulating and interpreting. Similar observations are also found in Wheeler (1996), who asserts the importance of ensuring that students acquire the fundamental awareness that algebraic tools "open the way" to the discovery and (sometimes) crea-

tion of new objects. Kieran (2004) also stresses the importance of devoting much more time to those activities for which algebra is used as a tool but which are not exclusively to algebra (global/meta-level activities according to Kieran's distinctions) because they could help students developing transformational skills in a natural way since meaning supports manipulations. Proof is certainly one of the main activities through which helping students develop a mature conception of algebra. I adopt Wheeler's idea that activities of proof construction through algebraic language could constitute "a counterbalance to all the automating and routinizing that tends to dominate the scene". I believe that activities of proof in ENT would both provide students with the opportunities they need to progress gradually from argumentation to proof (Selden and Selden, 2002)) and help them to appreciate the value of algebraic language as a tool for the codification and solving of situations that are difficult to manage through natural language only (Malara, 2002).

I agree with Zazkis, Campbell (2006) who state that "the idea of introducing learners to a formal proof via number theoretical statements awaits implementation and the pros and cons of such implementation await detailed investigations" (p.10). In order both to investigate these aspects and to foster the diffusion of activities of proof in ENT in school, aiming at making student appreciate the value and power of algebraic language, I am working with upper secondary school students (10th grade) [1]. I planned and experimented a path for the introduction of proofs in ENT. The path was articulated through small-groups activities (some groups were audio-recorded), followed by collective discussions (audio-recorded) on the results of the small-group activities. In order to foster a widespread participation during group activities, I decided to work with homogeneous (according to competencies and motivations) small groups. In this work I will dwell on a central moment in the path: the small-groups' work aimed at constructing the proof of some conjectures they produced starting from numerical explorations. In particular I will present the main results of the analysis of group discussions when students were trying to prove one of the conjectures.

2. THEORETICAL FRAMEWORK WHICH SUPPORT MY ANALYSIS OF STUDENTS' DISCUSSIONS

Many different competencies are required of a student who has to face proof problems in ENT. In particular, he/she has to: (a) know the meaning of the mathematical terms in the problem text and interpret them correctly by reference to it; (b) translate correctly from verbal to algebraic language; (c) be able to interpret the results of the transformations operated on the algebraic expressions in relation to the examined situation; and (d) control the consequences of his/her assumptions. I identified a set of theoretical references that are both appropriate to the analysis of the transcripts of group discussions dealing with proofs and in tune with the view of algebra that I am trying to promote. The main reference in my research is the work by Arzarello, Bazzini and Chiappini (2001). The authors propose a model for teaching algebra as a *game of interpretation* and highlight the need for the promotion of algebra as an efficient

tool for thinking. An awareness of the power of the algebraic language can be developed only once the student has mastered the handling of some key-aspects that arise in the development of algebraic reasoning. In particular, the authors highlight the use of *conceptual frames* defined as an “organized set of notions, which suggests how to reason, manipulate formulas, anticipate results while coping with a problem”, and *changes from a frame to another* and from a knowledge domain to another as fundamental steps in the activation of the interpretative processes. According to the authors, a good command in symbolic manipulation is related to the quality and the quantity of anticipating thoughts which the subject is able to carry out in relation to the effects produced by a certain syntactic transformation on the initial form of the expression. Boero (2001) also argues that *anticipation* is a key-element in producing thought through processes of transformation. The author defines anticipating as “imagining the consequences of some choices operated on algebraic expressions and/or on the variables, and/or through the formalization process”. In order to operate an efficient transformation, the subject needs to be able to foresee some aspects of the final shape of the object to be transformed in relation to the target. Arzarello *et Al.* stress that the ability to produce anticipations strictly depends on changes in the frame considered in order to interpret the shape of the expression.

Another theoretical reference that I take as fundamental for analyzing students’ management of meaning in algebra is the concept of *representation register* proposed by Duval (2006). The author defines representation registers those semiotic systems “that permit a transformation of representations”. Among them, he includes both natural and algebraic language. Duval asserts that a critical aspect in the development of learning in mathematics is the ability to change from one representation register to another because such a change both allows for the modification of transformations that can be applied to the object’s representation, and makes other properties of the object more explicit. According to the author, real comprehension in mathematics occurs only through the coordination of at least two different representation registers. He analyzes the functions performed by different possible typologies of transformations, distinguishing between *treatments* (“transformations of representations that happen within the same register”) and *conversions* (“transformations of representation that consist of changing a register without changing the objects being denoted”) and highlighting both the fundamental role of each of these typologies of transformations and the intertwining between them.

In order to clarify how this set of theoretical references could help in analysing the role played by algebraic language in the construction of proofs (or attempts of proof) in ENT, the next paragraph will be devoted to the *a priori* analysis of the problem on which the working group activities, examined in this paper, were focused.

3. A PROBLEM AND ITS A PRIORI ANALYSIS

The problem, on which this paper is centred, is the following: “Write down a two digit number. Write down the number that you get when you invert the digits. Write

down the difference between the two numbers (the greater minus the lesser). Repeat this procedure with other two digit numbers. What kind of regularity can you observe? Try to prove what you state”.

The regularity to be observed is that the difference between the two numbers is always a multiple of 9; precisely it is the product between 9 and the difference between the digits of the chosen number. The proof requires the polynomial representation of each number: since a number of two digits m and n can be written as $10m+n$, where $m>n$, the difference can be represented as $10m+n-(10n+m)$. Through simple syntactical transformations it is possible to turn the initial expression into a form that makes the required property explicit: $10m+n-(10n+m)=9m-9n=9(m-n)$. The initial conceptual frames to which the statement of the problem refers are ‘difference between numbers’ and ‘two digits numbers’. It can be assumed, therefore, that the student will not automatically choose the ‘polynomial notation’ frame to represent the problem (some students might apply the ‘positional representation of a number’ frame and then get stuck). The reference to the ‘divisibility’ frame, which allows them to foresee the desired final shape of the expression after correct treatments (i.e. $9 \cdot k$, where k is a natural number), seems to be less problematic but possible blocks in the treatments to perform on the initially constructed polynomial expression can be ascribed to interpretative difficulties, which are strictly related to students' inability to correctly anticipate the final shape of the considered expression (it is necessary to recognize the transformation that leads to an expression that can be easily interpreted in the final frame ‘divisibility’). Finally, some observations about possible students' behaviours could be proposed. Many students could end their numerical explorations after having observed that the difference between the two numbers is always a multiple of 9, without recognizing the relationship that exists between the two digits of the first number and the difference between the two numbers (i.e. the considered difference is the product between 9 and the difference between the digits of the chosen number). Consequently, the analysis of the final expression could provide another index of students' interpretative abilities, in that access to the new meanings it embodies depends on those abilities.

4. RESEARCH HYPOTHESIS AND AIMS

My hypothesis is that the production of good proofs in ENT depends upon the management of three main components: (a) the appropriate application of frames and coordination between different frames; (b) the application of appropriate anticipating thoughts; and (c) the coordination between algebraic and verbal registers (on both translational and interpretative levels).

The aim of this paper is to investigate the role played, in students' proving processes, by the three components I singled out and the mutual relationships between them. In this work I will present a sample of prototype-productions [2] helpful to highlight that the lack or unsuccessfully application of one of these components leads to failure

and/or blocks of various types. In particular, I will highlight the interrelation between anticipating thought and interpretative blocks.

5. RESEARCH METHODOLOGY

Theoretical models I used helped us identify some interpretative keys for the analysis of protocols of students' discussion while working in small groups. My analysis focused on the following: (1) The conceptual frames chosen to interpret and transform algebraic expressions and the coordination between different frames; (2) The application of anticipating thoughts; and (3) The conversions and treatments applied and the coordination between verbal and algebraic registers.

My choice of analyzing small groups' discussions is motivated by the conviction that only when students are involved in a communication it is really possible for us to produce an in-depth analysis of the coordination between verbal and algebraic register. Moreover I believe that the analysis of the sole written protocols is not enough to highlight students' actual interpretations of algebraic expressions they construct. The need to communicate their reasoning to others forces students not only to verbally make what they are writing explicit, but also to explain both the objectives of the transformations they carry out and their interpretation of results.

6. THE ANALYSIS OF PROTOTYPE-PRODUCTIONS

In this paragraph I will present two examples of prototype-protocols of discussions, chosen because they highlight how students' interaction allows to identify the reasons of erroneous conversions and the difficulties in the interpretation of expressions.

6.1 Example 1:

The following example is characterized by the application of an initial suitable frame, not associated to an adequate conversion and a correct interpretation of the produced expressions.

After having considered many numerical examples, students A, C and N conclude that the considered difference is always a multiple of 9. The following dialog represents the proving phase.

27 C: Let us do with letters.

28 N: It is more complicated.

29 C: It will be $10x$... plus ...

30 A: ...plus y (*they write $10x+y$*) [3]

31 C: If we invert the digits, it will be $y+10x$

32 A: and now ... we have to do the difference

33 C: (*She writes and reads*) $10x+y$... minus ... (*she writes $y+10x$*) it becomes $10x+y-y-10x$

34 N: I think there is a mistake because the result is zero ... they cancel each other out.

We are not able to prove it.

35 C: We have $10x+y$ and it represents the number ... Then we have to ...

36 A: (*She reads*) 'when you invert the digits' ...

37 C: It is the same of having 1 and ... It is as if we take it on this side, so y should be take on the other side... however, if we take 10 on this side, it will be left a ...

38 A-N-C: one!

39 C: So it is not $10x$. I think it is x ... So it would become $10x+y-(y+x)$. The two y cancel each other out, so they will be left $10x-x$. Exactly $9x$! We were able to prove it! ...

40 C: ... (*C. is looking to the numerical examples*) But here I can see something more, I think. I can see that, practically, this is ... Look what I noticed (*she is looking at the differences 86-68, 92-29, 76-67, 52-25*) ... if you subtract the two tens, 8-6, you have only to consider the product between 9 and the difference between the two tens: 9 times 2 is 18; 7-6 is 1, 9 times 1 is 9; 5-2 is 3, 9 times 3 is 27.

41 A: We have to write it down. I would have never noticed it!

42 C: (*she dictates*) It is always a multiple of 9 and we can observe that the result of the subtraction ... you have to subtract the two tens and to multiply the result by 9... Do you know how I thought of it? Because I saw $9x$ and I said "it is a multiple" because there is 9 times x . Then I said "but ... what is x ? x is the tens!". Then I tried to do x minus x .

43 A+N: Good!

This protocol can be subdivided in three key-moments: (1) *Initial conversion and first treatments* (lines 27-33); (2) *Identification of a problem, modification of the conversion and new treatments* (lines 34-39); (3) *Attempt of interpretation of the obtained expression and refinement of the conjecture* (lines 40-43).

Initially C carries out a *first erroneous conversion* (line 31), translating this concept through the expression $y+10x$. While students correctly interpret the natural language term "invert" when they work on numerical examples in order to formulate the conjecture, when they have to carry out a conversion into algebraic register, the concept "exchanging the place" is translated through the pure exchange of the order of the monomials which constitute the polynomial $10x+y$, dispelling serious difficulties in coordinating the 'positional notation' and 'polynomial notation' frames and lack in the internalization of the last. The difference (zero) they obtain starting from this erroneous conversion lead them to detect the inaccuracy of their initial conversion and to look for a new correct one. They detect a mistake in having supposed that $10x$ should represent the units digit, so they decide to correct this mistake, substituting x instead of $10x$, but they do not consequently modify the representation of y as tens-digit. Therefore, writing the polynomial as $y+x$, they carry out again an incorrect conversion. Probably because of the prevailing of the anticipating thought they carry out (expecting a multiple of 9, they only concentrate on the factor 9 when they look at the expression $9x$), once they obtain $9x$ as the difference between the two numbers, they do not immediately subject the new result to a careful interpretation. Only afterwards C interpret x as the tens-digit of the initial number and decide to investigate the considered examples in order to refine their conjecture. C concentrates on the tens-digits of the two numbers (x and y in the correct representation) and observes, starting from examples, that the result is obtained multiplying 9 by the difference between those digits. This observation, however, does not help her in critically interpreting the ex-

pression $9x$. In her final intervention, she even tries to translate into algebraic language, through the expression $x-x$, the difference between the two tens, but she is not able to ‘grasp’ the gap between the algebraic representation she proposes and her verbal considerations.

6.2 Example 2

In the following transcripts we can highlight what kind of difficulties students meet when appropriate application of the initial conceptual frame and conversions are not supported by anticipating thoughts and by a semantic control.

The three students G, B and A decide to work separately on the conjecture: while A and G analyze numerical examples only, B works on the algebraic formalization of the difference to be considered. Without speaking with her friends, B is able to perform the correct conversion, representing the considered difference with $10x+y-(10y+x)$. Afterwards she performs correct treatments on this expression, obtaining $9(x-y)$, and she decides to illustrate her result to A and G.

19 B: I obtained this thing ... Why 9? 9 is 9! 9 is odd! Is it possible that the result is always an odd number?

20 A: No. Consider 20! The difference is 18!

21 G: I sincerely can't find a regularity ...

22 B: I could only find that the result is 9 multiplied by $x-y$, but ... why is 9 here? There is 9 only because there is 10!

23 G: Let's try with 28 ... $82-28$... the result is 54! So ... What have these numbers in common???

24 B: I found it!! I found it!! If I choose 65 and 56, the difference is 9. In the algebraic case the result is 9 multiplied by $(x-y)$!

25 G: Please, explain it!

26 B: Because, independently from the initial number, the difference is always 9.

27 G: No! Consider 82 and 28!

28 B: What a pity! I liked this observation! ... Wait a moment ... here (*she refers to the examples she chose*) we pass from a ten to the next ten. I found it! Only if we start from a number whose digits are consecutive, the difference is 9!!! 34 and 43 ... All the numbers have consecutive digits!

29 G: It is true! 54 e 45!

30 B: 12, 23, ... Do you understand? 1 and 2 are consecutive numbers.

32 A: 14 and 41? 15 and 51?

33 B: No! The two digits must be consecutive! When they are consecutive, the difference is always 9!

34 A: So ... what does it happen?

35 B: I don't know ... It happens that the difference between the numbers is 9. If you look at the algebraic case ... Can you see that it is always 9 multiplied by something?

36 A: Only if the digits are consecutive the difference is 9?

37 B: I don't know why ...

38 G: But ... I think that the distance between the numbers is not the only reason ...

(silence)

39 B: ... It is always a multiple of 9!!!

40 A: In what sense?

41 B: Let's try! 52-25! The result is 27!

42 A: Also if we choose 15 and 51 ...the result is 36!

43 B: They are all multiple of 9! Can you see that every case is the same?! Tell me other numerical examples!

44 A: 51-15 is 36

45 G: 52-25 is 27

46 B: 21-12 is 9, which is a multiple of 9!

47 G: So we can observe that the result is always a multiple of 9.

This excerpt could be subdivided in two key-moments: (1) *Attempt to interpret the expression produced during an 'algebraic exploration' of the problem situation* (lines 19-38); and (2) *Formulation of the conjecture* (lines 39-47).

Students' choice to proceed separately turns out to be not effective. In fact, while the analysis of numerical examples does not help A and G in formulating a conjecture, the total absence of anticipating thoughts about the objective of the algebraic manipulations B operates blocks her interpretation of the obtained expression $9(x-y)$. In fact, B initially tries to guess the correct interpretation of the expression as the representation of an odd number (line 19). When this interpretation is refuted by a counterexample proposed by A (line 20), B decides to refer to numerical examples in order to meaningfully look at the obtained expression. The choice of the numerical examples she considers (only numbers whose digits are consecutive) suggests her that the difference is always 9 (line 24). Now the presence of an anticipating thought (the difference is 9) negatively influences B's interpretation of the expression $9(x-y)$. When, again, G proposes a counterexample against B's conjecture (line 27), she does not try to re-interpret the expression and limits herself to look at numerical examples to understand what are the conditions under which the regularity she first observed (the difference is 9) is valid (lines 28 and 30). Although her correct observation about the interrelation between the digits of the initial number and the difference between the two numbers, again B is not able to correctly re-interpret the expression $9(x-y)$, focusing on the role assumed by the factor $(x-y)$ (lines 35 and 37). B's troubled conquest of an only partial interpretation of the expression $9(x-y)$ and her necessity to refer to numerical examples to understand what she obtained testify that, if algebraic manipulations are not guided by an objective, significant interpretations are blocked. An evidence of this problematical aspect is the fact that, paradoxically, the working group activity ends with the formulation of a conjecture.

7. CONCLUSIONS

The analysis I presented in the previous paragraph allows to offer some conclusions with respect to the role played by the three components I identified and the mutual relationships between them. The first protocol highlights the strict correlation between lack of flexibility in coordinating different frames, difficulties in carrying out

conversions from verbal to algebraic register and lack of interpretative games in the analysis of the expressions produced. Moreover, it testifies how such correlation causes failures in the production of proofs in ENT. In fact, the three students display rigidity in their use of frames and an incapability of simultaneously manage different frames. Such rigidity makes them produce partial or incomplete interpretations of the constructed expressions, so they are not alerted about the non-acceptability of their proof. The second protocol testifies the strict interrelation between anticipating thoughts, the activation of conceptual frames and the subsequent interpretations of the produced expressions: since the conversion and the treatments operated by B are not oriented by an anticipating thought, the activation of a proper conceptual frame and a correct interpretation of the final expression are blocked. Moreover, this protocol represents a good example of results produced by the strict interrelation between *blind manipulations* (i.e. produced without an objective) and blocks in the interpretative processes. The rigidities highlighted in the analyzed protocols are shared by other protocols (not presented here because of space limitations), to which different problems could be add, such as: (a) blocks related to the activation of an incorrect initial frame of reference; (b) blocks in the treatments and in the interpretative processes due to an inability to foresee the expression to be attained by the activation of the correct final frame; (c) difficulties in the choice of the treatments to be operated caused by the absence of anticipating thoughts.

These observations helped us in singling out an initial classification of interpretative blocks in relation to causes that have produced them. Summarizing, I identified *interpretative blocks associated to*: a) *difficulties in simultaneously managing different frames* (example 1, line 42); b) *total absence of anticipating thoughts* (example 2, line 19); c) *activation of erroneous anticipating thoughts* (example 2, lines 24-26); d) *activation of a predominant (partial) anticipating thought* (example 1, line 39; example 2, lines 39-43). This classification let us highlight, in particular, the fundamental role played by anticipating thoughts during these kind of activities, thanks to the strict interrelation between them and students' difficulties in the interpretation of the algebraic expressions they produce.

In conclusion, my analysis of students' discussions during small group activities turned out to be an effective methodological instrument to verify my hypothesis on the importance of the key-components I singled out for the analysis of proof productions in ENT.

The results of this analysis will be a starting point for the next step of my research. I am convinced that the only way to make this approach to algebraic language really effective is to help teachers act as fundamental models in guiding their students toward the acquisition of the essential competencies that can help them overcoming difficulties and blocks identified in this work and developing awareness of the central role played by algebraic language as a reasoning tool. Therefore I will focus my research on the role played by the teacher during class activities in order to highlight the attitudes of an aware teacher, the choices he makes and the effects of his/her ap-

proach on students, from the point of view of both awareness shown and competencies acquired.

NOTES

1. The study was conducted in some classes (10th grade) of a *Liceo Socio-Psico-Pedagogico*, which is an upper secondary school originally aimed at educating future primary school teachers.
2. The term “prototype-production” is here used with the meaning of “representative of a category of productions of the same kind”.
3. The difficulties I hypothesised in the identification of the initial frame are not highlighted by this protocol because students have faced the problem of the representation of two and three-digit numbers in a previous activity.

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EPISTEMOGRAPHY AND ALGEBRA

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We propose to address the problem of how to know students’ knowledge in an entirely new approach called “epistemography” which is, roughly, an attempt to describe the structure of this knowledge. We claim that what is to be known is made of five tightly interrelated organised systems: the mathematical universe, the system of semio-linguistic representations, the instruments, the rules of the mathematical game, and the identifiers.

Keywords: epistemography, algebra, semiotics, language, subparadigm.

One of the most commonly shared principle of didactics of mathematics is that teaching must ground on students' previous knowledge. Therefore we researchers (and teachers too!) need to know what students know and what they are supposed to know.

But the point is that knowing what students are supposed to know is less easy to do than it appears at a first glance, particularly when they shift from primary studies to secondary studies and when there are frequent curricular changes in the primary studies. In this case, secondary teachers cannot rely on remembering their primary school time; reading curricular documents is not very helpful, neither discussing with primary teachers. The problem is the lack of a common language, or better said, that the common language is not accurate enough. Saying that “students know the sense of operations” or that they are able to solve “simple word problems” is far too fuzzy and superficial.

We propose to address this problem (how to know students’ knowledge) in an entirely new approach called “epistemography” which is, roughly, an attempt to describe the structure of this knowledge.

Epistemography is based on an attempt to generalise and conceptualise findings about knowledge we made mainly during previous researches on algebraic thinking. According with many authors we found that semiotic and linguistic knowledge plays a central role in Algebraic Thinking. And we faced the following question: to what extent is this knowledge, *mathematical*? Letters and symbols are not mathematical objects in the same way that numbers or sets or functions are¹¹; but on the other hand they are equally necessary to do mathematics.

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¹¹ More precisely, digits, letters, symbols and expressions made with them form a “language”. Languages are mathematically described by the “Language Theory” (a part of Mathematical Logic, shared with computer science).

tionality of $\sqrt{2}$ does not depend on how integers, square roots or fractions are written. Actually the Greeks' notations of the first proof had nothing in common with ours (in particular they did not use any symbolic writing). Semiotic properties, on the contrary, rely on representational conventions. The property that in order to write $1/3$ you need an infinite number of (decimal) digits is true – in base ten only; it is false in base three (“ $0,1$ ”: zero unit and one third) or, as in the Babylonian system, in base sixty: \llcorner : two times ten sixtieths.

Mathematical language

What are the characteristics of the semio-linguistic system? First of all, the “mathematical language” (in a loose sense) is a written one¹³. Mathematical semio-linguistic *units* are written texts. Following and extending Laborde's ideas (1990), written mathematical texts are heterogeneous, made of natural language sentences, symbolic writings, diagrams and tables, graphs and illustrations. Their organisation follows what we call the fruit cake analogy, the natural language being the dough and the symbolic writings, diagrams, graphs and illustrations being the fruit pieces. To describe rigorously such a complex structure is far from easy.

Linguistic system

Students' ability to understand natural language mathematical texts (the “dough”) is linguistic by nature. Mathematical natural language (we call it the “mathematicians jargon”) is mostly the natural language itself; but Laborde (1982) showed there are some differences (unusual syntactic constructions like “Let x be a number...”) between the jargon and the mother-tongue, difficult to interpret by students.

Symbolic writings (like “ $b^2 - 4ac > 0$ ”) make up a language, too (Brown & Drouhard, 2004, Drouhard et al, 2006), which is far more complex and different from mother-tongue than it appears at first sight; detailed and accurate descriptions of this language can be found in Kirshner (1987) and Drouhard (1992). Students must learn this language and its syntax¹⁴ – which allows symbolic manipulation (Bell, 1996): the actual mathematic language, ruled by a rigid syntax, permits to perform operations on the symbolic expressions rather than on (mental or graphic) representations.

The present mathematical language is also characterised by a complex but precise semantics. Semantics (the science of the meaning) is the set of rules and procedures which allows interpreting expressions, in other words which allows relating expressions to mathematical objects.

The most accurate description of this semantics (how symbolic writings refer to mathematical objects and properties) is based on G. Frege's ideas (Drouhard, 1995).

13 which puts upside down the usual relationship between oral speech and written texts

14 the syntax is the part of the grammar which deals with the rules that relate one to another the elements of a language. (Syntax says that a parenthesis must be close once opened...

G. Frege's key concepts are "denotation" (which can be a numerical value (in the case of "20"), a numerical function (in the case of " $x+1$ "), a truth value (in the case of " $1 > 20$ ") or a boolean function (" $x+1 > 20$ "), according to the type of symbolic writing)¹⁵, and "sense" (the way denotation is given). The linguistic nature of students' difficulties with symbolic writings is often underestimated, or confused with conceptual difficulties.

Semiotic system

Let's give an example of a semiotic problem in algebra. How to represent an infinite series of decimals? Imagine I ask you what the properties of the number 0,666... are. When multiplied by 3 it gives 2? No. Actually I had in mind the number 1999/3000. And yes, I cheated: I broke the representational rule of decimals, which is a semiotic rule (on how to interpret elements like "...") about linguistic objects (the numeric expressions).

There are more than one approach to mathematics semiotics, which were fully presented in the special issue N° 134 (2003) of *Educational Studies in Mathematics*. Duval dedicated his lifelong work to an extensive and coherent theory of semiotics of mathematics education. Three key concepts are the semiotic representation registers, the treatments (within a register) and the conversions (between different registers). Other researchers (see amongst others Otte, 2006) are investigating how to interpret mathematics education using the terms of the founder of semiotics, Charles S. Peirce (1991): the three types of signs –index, icon, symbol– and, maybe more interesting, the three types of inferences –induction, abduction, deduction).

An entire communication paper would not suffice to present even a small part of the outcomes of semiotics for the study of algebraic thinking. Hence we called "semio-linguistic" the mathematics representation system. Therefore students must handle both aspects of this representation system, the linguistic as well as the semiotic one, and the complex interaction between them.

INSTRUMENTS

Up to now we have seen that to do mathematics, students must not only know objects and how to represent them: now we will see that they need also to know how to use instruments (Rabardel & VÉrillon, 1995) to operate on the representations of objects.

However, unlike object/representation opposition, instruments are not characterised by their nature (mathematical objects can also be tools, as noted by Douady, 1986) but instead by their use. Students, then, must learn what these instruments are and how to use them. Given that instruments are only characterised by their use, it is possible to propose a typology, based on their nature: material instruments (like rulers or

15 The AlNuSet software, developed by Giampaolo Chiappini allows (in a totally original way) a dynamic view of the denotation of algebraic expressions.

compasses, see Bagni, 2007), conceptual instruments (mathematical properties, like theorems), semiotic instruments (manipulations on semiotic representations) – this idea appears in L. S. Vygotsky, 1986); eventually one may consider “meta” instruments like strategies and, more generally, meta-rules.

THE RULES OF THE MATHEMATICAL GAME

We have seen that students must know what mathematical objects are and their properties, how to represent them and how to use instruments. Is this sufficient to do mathematics? Not at all: using a given instrument to operate on a given representation may be, or not, legitimate (even if done properly). For instance, to solve some numerical problems, some procedures are arithmetical (and are not legitimate in algebra) and other are algebraic (and are not legitimate in arithmetics).

Therefore algebra is not just a question of objects, representations and tools, but also of rules, which are saying what the actions are that we *may* or *may not* do amongst the actions we *can* do. Algebra is not a game in the same sense that chess is a game, but, like chess, algebra does have rules. These rules, moreover, are changing with passing times: the present way of doing differs from, say, the Renaissance Italian way of doing algebra. L. Wittgenstein (the “second Wittgenstein”, the author of the *Philosophical Remarks*, or *On Certainty*, 1986) is an invaluable guide to clarify the extremely complex relationship between objects, signs, practices and rules. (Ernest, 1994, Bagni, 2006).

SUBPARADIGMS

Some rules (in particular logic) are universal for all mathematics. But other rules are related to a certain domain of mathematics. A square number is always positive, except when studying complex numbers. We call these domains “subparadigms”, which are analogous Kuhn’s paradigms, but less vast, and commensurable between them). This notion of subparadigm allows us to understand the shift from arithmetics to algebra. Semantics (and instrumental value) of the “=” sign change, thus objects (the equalities, the expression with letters) also change. The semiotic systems, although looking quite the same (“ $2+1 = 3$ ” and “ $2+x = 3$ ”), are different in fact.

IDENTIFYING KNOWLEDGE

A last type of knowledge allows us to identify (or recognise) if what we do is mathematical or not, and to identify to what domain of mathematics it belongs. When a student writes something that superficially looks like algebra but actually is wrong or meaningless, the teacher might say: “This is *not* algebra”; and if later the student succeeds in writing a meaningful and correct algebraic text, the teacher might comment: “*This is* algebra”. With these statements, the teacher speaks about the student’s text but also about algebra; he is actually teaching the student *what* is algebra – and what

is not¹⁶ (Sackur et al., 2005). We call this Identifying Knowledge; it is also that which allows us to recognise whether a mathematical problem is arithmetical or algebraic, and to choose the appropriate instruments to solve it (without certainty: this kind of knowledge is more abductive than deductive, see Panizza, 2005).

THE LAYERED DESCRIPTION

As said above, epistemography is not the theory of everything (or, better said, of every kind of knowledge)! Firstly, we only consider here the part of knowledge which is specific to mathematics; this leaves aside nonspecific knowledge, related with the use of (oral and written) natural language or with general reasoning capabilities. “Mathematical activities”, however, remains too vague to allow a precise description. Then, by analogy with the Internet reference model, which is a layered abstract description for the very complex communications and computer network protocol design, we propose a layered description of students’ mathematical activities.

The five descriptive layers of students’ mathematical activities are:

1. the School Layer (what are the students’ rights and duties, why and how to work in the classrooms and at home, what kind of participation is expected by the teacher etc.). This is what french-speaking researchers like Sirota (1993) or Perrenoud (1994) call ““being a student” as a job”¹⁷. A great number of students’ difficulties may be analysed in terms of the school layer: when they don’t want to learn, or don’t know how to, for instance.
2. The Maths Classroom Layer (how to do maths in the classrooms and at home, what kind of participation is expected by the maths teacher and what is the math teacher supposed to do, etc.). This part of the students’ activities is ruled by what Brousseau (1997) calls the *didactical contract* (see also Sarrazy, 1995, for an extensive survey of this notion). Many students’ difficulties can be analysed in terms of didactical contract, as it was brilliantly done by Brousseau (*ibid*) and followers.
3. The Modelling Layer, which is the description of, for instance, how students change a word problem into a mathematical problem, or even how they change a mathematical problem (i. e. expressed in mathematical terms) into an other problem which they can solve with their mathematical tools. A whole field of mathematics education is devoted to the modelling part of the students’ mathematical activities (see for instance Lesh and Doerr, 2002).

16 which would be almost impossible to do with an explicit discourse within this context: definition or characterization of mathematics are epistemological statements, not mathematical statements

17 unfortunately, according to Dessus (2004) this concept is almost non-existent in English-speaking sociology of education studies.

4. The Discursive Layer, which is the description of students' reasoning on mathematical objects. This reasoning may be expressed by a discourse (like "if x is greater than -3 then $x+3$ is positive and therefore..."), hence the name of this description layer¹⁸. In France, Duval (2006) is a main contributor in this domain, which is closely related to researches on argumentation (see for instance Yackel and Cobb, 1997) and on proofs (see for instance Gila Hanna, 2000).
5. The deepest, Symbolic Manipulation Layer, describes how students operate on symbolic forms to yield other symbolic forms which represent the solutions of the problem. In the case of algebraic thinking, not too many authors (see for instance Bell, 1996 or Brown & Drouhard, 2003) stress on that – mainly because on the contrary it is often overemphasized by textbooks and teachers.

It is important to notice that what is layered is the description, not the student's activity. It is very similar to what happens in linguistics: the language's description is split in phonetics, syntax, semantics, pragmatics etc. but the subject's act of speech, on the contrary, is of a whole.

CONCLUSION

A way to cope with the problem of identifying students' mathematical knowledge has long been to focus on students' solving abilities and this can explain the prominent role which has been given to assessment throughout the world. However, many mathematics educators remain reluctant to reduce assessment criteria to solving abilities. Our point is that solving abilities are not so relevant clues on what students know and what they are supposed to know. On the one hand, the student's failure in achieving a task does not give much information on what his or her deficiencies or misconceptions are. On the other hand, the student's success may just show his or her technical abilities, but we cannot be sure that s/he understood conceptually.

Then, how can we determine what students know and are supposed to know? We claim that epistemography can provide accurate answers to this question.

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¹⁸ It is not called "reasoning layer" since that could lead to the erroneous idea that there is no reasoning outside this level.

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SÁMI CULTURE AND ALGEBRA IN THE CURRICULUM

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Abstract: The Sámi culture's richness of patterns and structures give rise to the question whether an implementation of Sámi culture in the teaching of algebra might improve this teaching for the Sámi pupils. The Sámi have their curriculum but Sámi culture does not seem to be implemented in its algebra syllabus. Mathematical archaeology with respect to metonymy upon the Sámi cultural elements duodji and joik indicate possibilities for the teaching of algebra. But a remaining question is the Sámi mathematics teachers' view of the situation and of the suggested possibilities. The paper aims to prepare for empirical studies which focus on the Sámi mathematics teachers' mathematical archaeology upon their own cultural elements, as a basis for the teaching of algebra.

Key words: algebra; curriculum; mathematical archaeology; patterns; Sámi

BACKGROUND AND RESEARCH QUESTIONS

The Sámi are an indigenous people of the arctic who live in the northern part of Norway, Sweden and Finland, and in the Kola Peninsula of Russia (Kuhmunen, 2006). In 1990 Norway ratified the ILO Convention No. 169 concerning indigenous and tribal peoples in independent countries, and after this the Sámi in Norway got their curriculum (KUF, 1997). In the three latest national curricula, the Norwegian Ministry of Education has worked out special Sámi syllabuses for several subjects, but not for mathematics. One quite common interpretation of the curriculum is that the teaching of algebra should be the same for pupils in the Sámi core area in Northern Norway as for any pupil in our capital Oslo in the south. A quite different interpretation is that the Sámi should have their syllabus in mathematics.

This paper constitutes parts of a basis for a project which intends to research the possibilities of a Sámi algebra syllabus. The idea is that one researcher and one group of Sámi mathematics teachers together design and develop a teaching of algebra based upon Sámi cultural expressions. One lower secondary school in the Sámi core area wants to join a meeting where this project is introduced. The aim of this paper is to obtain important basis material for this important meeting. The basis material includes a) an analysis of the present situation regarding the teaching of algebra for Sámi pupils, and b) an analysis of some Sámi cultural expressions with respect to possibilities for a teaching of algebra. This leads to the two research questions of this study: 1: How is Sámi culture implemented in the algebra part of the national mathematics syllabus for lower secondary school? 2: If there are any (algebraic) structures to be found in Sámi cultural expressions, then how may these structures emerge?

THEORETICAL FRAMEWORK

Algebra

According to Lakoff & Núñez' (2000, p. 110), "Algebra is the study of mathematical form or "structure"". According to the latest TIMSS framework (Mullis et. al., 2007) algebra consists of patterns, algebraic expressions, equations/formulas and functions. Barton (1999) describes mathematics as a system of quantities, relations and space. His term "relations" is interpreted to be wider than just algebra. Fyhn (2000) uses the metaphor "pattern" similar to Lakoff & Núñez' (2000) "structure". Lakoff & Núñez (ibid.) focus on the terms "essence" and "structure" in their approach to algebra,

Algebra is about essence. It makes use of the same metaphor for essence that Plato did – namely, Essence is form. ...Algebra is the study of mathematical form or "structure". Since form (as the Greek philosophers assumed) is taken to be abstract, algebra is about abstract structure. (ibid., p. 110)

The analyses in this paper use the term algebra as by Lakoff & Núñez (ibid.).

Aesthetical Expressions as Basis for the Teaching of Algebra

Fyhn (2000) searched for and analysed relations between pupils' participation in different leisure time activities and their score in some TIMSS mathematics tasks from 1995 and 1998. The pupils were categorised according to their participation in different leisure time activities, activities which they performed at least once a week. The results pointed out some common features for the categories "creative-crafts-girls", girls who participate in activities that concern drawing or handicraft, and the "musicians", pupils who play an instrument. The creative-crafts-girls' mean test score was below the mean score, while the musicians scored high above the mean. Geometry was expected to be a domain where the creative-crafts-girls had their highest score, but their score in geometry turned out to be rather low. Actually these girls' highest scores were on tasks which tested the pupils' understanding of patterns. The musicians turned out to have a test score profile that to a large extent was parallel to the creative-crafts-girls' (ibid.). This gave raise to the idea of a teaching of algebra that is based on the pupils' understanding of patterns.

Symmetry is an important part of the two latest Norwegian mathematics syllabuses for primary school (KUF, 1996; KD, 2006b). But the approach to symmetry is limited to be via geometry. Norway give less priority to algebra in school, and algebra is the domain where the Norwegian pupils have their lowest score in the TIMSS (Trends in International Mathematics and Science Studies) (Grønmo, Bergem, Nyléhn & Onstad, 2008). This opens for new ways of teaching of algebra. Due to the Sámi culture's apparently richness of patterns and structures, a good implementation of Sámi culture in the mathematics subject syllabus might lead to an improved teaching of algebra for Sámi pupils. Before any approaches can be done towards the design of new approaches to school algebra, there is a need for investigating how and to what extent structures and patterns from Sámi culture are integrated in the mathematics syllabus.

Parts of this investigation will take place in cooperation with the teachers; the rest will take place in this paper. In addition the apparently richness of structures and patterns in Sámi cultural expressions need to be confirmed and described before they can be treated as a basis for the teaching of algebra.

Mathematical Archaeology

Mathematics can be integrated into an activity to such a degree that it disappears for both the pupils and the teachers. According to Skovsmose (1994, p. 94) “Mathematics has to be recognised and named, that is the task of a mathematical archaeology.” It makes a difference whether the teaching is built upon situations that contain possibilities for application of mathematics or just for descriptive purposes. Many sorts of descriptive uses of mathematics can be possible as well as appropriate through mathematical archaeology; mathematics can be treated as an emerging subject (ibid., p. 90). It is important to a project, which contains mathematics as an implicit element, to spend some time on mathematical archaeology. The reason is: “If it is important to draw attention to the fact that mathematics is part of our daily life, then it also becomes important to provide children with a means for identifying and expressing this phenomenon” (ibid., p. 95). If there exists any algebra in the Sámi culture, it has to be implicit and hidden. A result of a mathematical archaeology may be that such algebra is recognised, named and described. A description of such algebraic structures may lead to an increased consciousness about possibilities for the teaching of algebra.

METHOD

The first research question will be answered by a) a survey of the development of the Sámi Curriculum in general and analyses of the treatment of algebra in it, b) a survey of the mathematics textbooks for Sámi pupils and analyses of their treatment of algebra, and c) analyses of the treatment of algebra in the national tests for Sámi pupils in mathematics and Sámi language. The second research question concerns the emergence of mathematics from elements in the Sámi culture. The research question will be enlightened by performing mathematical archaeology (Skovsmose, 1994) upon *duodji* and *joik*. *Duodji* is the name of Sámi craft, handicraft and art (KD, 2006a), while *joik* is the old Sámi folk music (Graff, 2001). The emergence of mathematics is categorised into three different levels; 1: recognition, 2: naming and 3: description.

ANALYSES

The Sámi Curriculum

The Sámi’s right to take care of and develop their language and culture has not always been accepted in Norway. The *norwegianisation* (assimilation) of the Sámi has been extensive and long-lasting (Minde, 2005). The norwegianisation also has led to a disparagement of Sámi culture, and this gives reasons to believe that there are few tracks of Sámi culture to be found in the Norwegian curriculum. In 1989 the Ministry of Education published the Sámi syllabuses (KUD, 1989) as a special supplementary

booklet to the national curriculum for the compulsory school. The intention was to adapt the traditional syllabuses to the Sámi culture and the Sámi surroundings. Some subjects got their own syllabus, but mathematics did not. The 1997 national curriculum (KUF, 1996a; KUF, 1996b) included a special Sámi curriculum (KUF, 1997). The mathematics syllabus was identical with the Norwegian one except for the illustrations.

The national curriculum of 2007 (KD, 2006a) includes a special Sámi syllabus for seven subjects, but not for mathematics. Reasons for a particular Sámi mathematics syllabus are that the Sámi and the Norwegian numerals are structured differently (Nickel, 1994), and that the traditional Sámi measuring units are based on body measures and not on the SI-system (Jannok-Nutti, 2007). For the pupils who learn Sámi as their first language, Sámi units of measurement and mathematical methods are treated as Basic Skills in mathematics, as an integrated part of the subject Sámi language (KD, 2006c). And “skills in mathematics require understanding of form, system and composition” (ibid., p. 3). According to Lakoff & Núñez (2000), this is algebra. But according to the curriculum, this is part of the syllabus in Sámi language. For the pupils who learn Sámi as their second or third language, basic skills in mathematics mean general concept development, reasoning and problem solving as well as the understanding of quantities, amounts, calculations and measurements (KD, 2006a). For these pupils the syllabus has no aims regarding their understanding of form, system and composition.

In the Norwegian national curriculum (KD, 2006b), the subject area “numbers and algebra” for the lower secondary school is presented this way

The main subject area numbers and algebra focuses on developing an understanding of numbers and insight into how numbers and processing numbers are part of systems and patterns... Algebra in school generalises calculation with numbers by representing numbers with letters or other symbols. This makes it possible to describe and analyse patterns and relationships. Algebra is also used in connection with the main subject areas geometry and functions. (ibid., p. 2)

As for the pupils who learn Sámi as their first language at school, the understanding of form, system and composition may be integrated with descriptions and analyses of patterns and relationships in the mathematics lessons. But this message is only implicit in the curriculum. Thus an interesting question is whether the Sámi culture is integrated in the teaching of algebra for the Sámi pupils.

Textbooks

The Sámi mathematics textbooks are Norwegian textbooks translated into Sámi language. The Norwegian Directorate for Education and Training (Udir, 2004) presents two mathematics textbooks in Sámi language for lower secondary school; one of them is approved for the curriculum of 1997, and the other one is Finnish. For economic reasons Norway offer the lower secondary school pupils no Sámi mathematics

textbooks which are approved for our latest curriculum. However, these pupils have their right to get appropriate books: The United Nations' Declaration on the Rights of Indigenous Peoples Article 14 claims that "Indigenous peoples have the right to establish and control their educational systems and institutions providing education in their own languages, in a manner appropriate to their cultural methods of teaching and learning" (UN, 2007, p. 6). The Sámi parliament, the Sámediggi, is an elected representative assembly for the Sámi in Norway (Kuhmunen, 2006). The Sámediggi's Youth Committee underlines the importance of getting Sámi textbooks. They sent an open letter to the Norwegian Minister of Education where they demand that the Ministry carry out necessary actions in order to improve the school-days for Sámi children (Nystø Ráhka, 2008). The textbook situation for Sámi pupils is far from satisfactory. Thus it is not any surprise that no attention is paid towards including Sámi culture in the algebra paragraphs in the existing textbooks.

National Tests

From 2003 Norwegian pupils have taken part in national tests as part of a national system for quality assessment (KD, 2003). From 2007 the mathematics tests were replaced by tests in mathematics as a basic skill in every subject. One result of this is that algebra is no longer part of the tests. The tests are translated from Norwegian to Sámi language; the Sámi pupils are offered no special tasks. The Ministry of Education and Research have decided that pupils who have Sámi as their first or second language will be tested in mathematics as a basic skill in this subject (KD, 2007c; KD, 2008). The Norwegian Directorate for Education and Training will carry out the translations of the mathematics tests into the three Sámi languages (ibid.). The national tests in mathematics as a basic skill do not reflect the pupils' achievement of any goals which are particular for the Sámi curriculum, and the algebra goals for the Norwegian pupils are neither reflected in these tests.

Duodji and Joik

The ornamentations in Sámi handicraft, *duodji*, and the Sámi folk music, *joik*, are both rich on structures and patterns. This claim is based upon the doctoral dissertations of Dunfjeld (2001) and Graff (2001). According to Dunfjeld (2001) the Sámi people's understanding of their own ornamentation differs from the pure formal understanding of ornamentation that we find in Western Europe. Thus she introduced the term "Tjaalehtjimmie" which has a meaning beyond pure decoration; "it is composed by signs, ornamentals and symbols which together may give meaning" (ibid., p. 102, my translation). For example may the meaning of the triangular engraving be decided from its localisation and orientation related to other symbols in a composition like in figure 1.

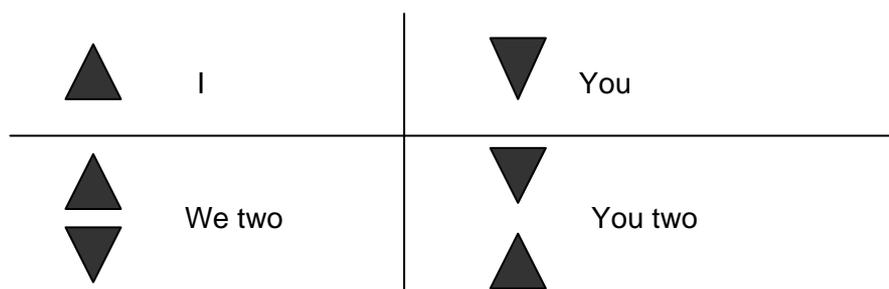


Figure 1. How the triangular engraving may symbolise personal pronouns in first and second person, and in singular and dual (Parts of figure from Dunfjeld, 2001, p. 109)

In duodji there are several more or less advanced plaited patterns. Fyhn (2006) describes hair plaiting by first splitting the hair into three equal parts. Plaiting can be further described by numerous repetitions of “take the right part and cross it over the mid-part. Then take the left part and cross it over the mid part”. The right part, whichever it is, can refer to all of the three parts of the hair, and so is for the mid part and the left part as well. This is what we understand with conceptual metonymy (Lakoff & Núñez, 2000), and this exists outside mathematics.

This everyday conceptual metonymy ...plays a major role in mathematical thinking: It allows us to go from concrete (case by case) arithmetic to general algebraic thinking... This everyday cognitive mechanism allows us to state general laws like “ $x + y = y + x$ ”, which says that adding a number y to another number x yields the same result as adding x to y . It is this metonymic mechanism that makes the discipline of algebra possible, by allowing us to reason about numbers or other entities without knowing which particular entities we are talking about. (ibid, p. 74-75)

According to the curriculum the Sámi ornamentations are geometry (KD, 2006a). Dunfjeld (2001) denotes these structures as geometry, too, and she refers definite to figures as triangles, rhomboids, squares and rectangles. Her mathematical archaeology is at level two; naming. When she refers to the organisation of the geometrical figures and the patterns they shape, she does not denote it as mathematics anymore. Fyhn’s (2006) description of ornamentation as metonymy is mathematical archaeology at level three, description.

Graff (2001) claims that researchers have focused on joik from different perspectives: as text, as melodies and rhythms, and as communication. To “joik” a person means to perform a particular joik which is dedicated to this person; the joik is an expression with a meaning (ibid.). The pitch constitutes an analogy for conceptual metonymy in music, when two or more people sing together. The structure of the song is given on beforehand; independent of what particular pitch to use. Graff (ibid.) points out, among other things, that the melodic motive in joik is based upon melodic patterns which in turn might have different shapes. The structuring of the joiks which he investigated, show that a rhythmic motive might be repeated throughout the complete joik (ibid.). Algebra is the study of mathematical form or “structure” (Lakoff & Núñez, 2000), and joik is just a way of expression that like other music is built up by

given structures. According to the curriculum, the understanding of how different patterns and structures influence artistic and musical expressions is part of mathematical skills in the subject music (KD, 2006a). Graff's (2001) term "rhythmic motive" is the name of a structure and could be denoted as mathematical archaeology at level two. He gives thorough descriptions of the structures as well, and he uses words like "ascending –descending melody line (inverted U-form)" (ibid., p. 210, my translation) and "transposition" (ibid., p. 214, my translation). But there is no mathematics connected to the names and the descriptions of these structures. The structures that constitute a basis for duodji ornamentations and for joiks may be identified and described by mathematical terms. The process in which algebra is emerging from these aesthetic expressions can be carried out as mathematical archaeology (Skovsmose, 1994) at three levels. But because joik as well as duodji express more than just aesthetics and structure, the meaning aspect need to be focused and enlightened.

CONCLUSION

The Sámi curriculum (KD, 2006a) offers a special Sámi syllabus for several subjects, but not for mathematics. "The understanding of form, system and composition" is part of the syllabus for Sámi as first language. Together with "descriptions and analyses of patterns and relationships" from the algebra syllabus, this opens for an integration of elements from the Sámi culture in the mathematics lessons. But that depends on whether the Sámi mathematics teachers are aware of and agree to these possibilities, and how the Sámi language teachers approach "form, system and composition" in their lessons. Due to the norwegianisation (Minde, 2005) there are reasons to believe that the teachers are not aware of the possibilities of integrating elements from their culture in the teaching of algebra.

The United Nations' Declaration on the Rights of Indigenous Peoples (UN, 2007), states that the Sámi lower secondary school pupils have their right to get appropriate mathematics textbooks in their own language. There are Sámi versions of Norwegian textbooks for primary school and for some of the grades in lower secondary school, but many of these books are based on a lapsed curriculum. And no special attention is paid towards including Sámi culture in the algebra parts of these textbooks. The lack of Sámi mathematics textbooks results in extra work for the teachers. Sámi pupils are offered translated versions of the Norwegian national tests in mathematics as a basic skill in every subject. The fact that these tests do not concern any algebra is an example of how Norway gives less priority to algebra in school. The national tests neither reflect the pupils' achievement of any goal in the Sámi curriculum. Aesthetic expressions may become a resource in the teaching of algebra: According to the Sámi curriculum (KD, 2006a) the relations between aesthetics and geometry are elements in the work with duodji decorations, while the music syllabus focus on the understanding of different patterns and structures. No connection between aesthetics and algebra is found in the Sámi curriculum.

One question for the further research is whether and to what extent the Sámi mathematics teachers find the project relevant and worthwhile taking part in. One more question is how metonymies might function in bridging the gap between Sámi cultural discourses and the algebra teaching discourse. The term “discourse“, is here used as by Foucault (2004, p. 53), “...discursive practice is a place in which... objects is formed and deformed.” These questions are closely interwoven and the further development of the project depends on the meeting between the researcher and the teachers. Maybe the teachers really want to join the project. But one other outcome is that the teachers dislike the ideas of creating an algebra teaching based upon Sámi cultural expressions. Another outcome might be that the teachers give priority to other parts of mathematics than algebra at the moment. A third possible outcome is that the teachers want to take part in the project, but that metonymies turn out to be less useful than they seem at the moment.

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PROBLEM SOLVING WITHOUT NUMBERS AN EARLY APPROACH TO ALGEBRA

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Abstract: *This paper reports a research project that aims at finding a good approach to school algebra using magnitudes and measurement. Thereby we not only focus on the way algebra can be taught effectively but also at on when in student's mathematical education a geometric and measuring approach can be successful. For this purpose we provide a theoretical framework and modify an early algebra program developed for first-graders to implement it in different age-levels.*

Key Words: *Algebraic Symbolizing, Early Algebra, Cognitive Gap, Measurement*

INTRODUCTION

In Germany, as in many other countries, algebra is taught as generalized arithmetic (see e.g. Lins & Kaput, 2004) after a long term arithmetical education. Reasons can be found on the one hand in the historical development of algebra as a medium for solving advanced arithmetical problems, on the other hand in the Piagetian stages of cognitive development. According to Piaget's theory children achieve the formal operational stage – and therewith the capability for abstract reasoning - not before the age of eleven (Piaget & Inhelder, 1972). It is however not self-evident that all aspects of algebraic thinking require achievement of the full formal operational stage.

Linchevski (2001) talks about a “cognitive gap”, which characterizes “these steps in the pupil's learning experience where without a teaching intervention [...] he or she would not make a certain step” (Linchevski, 2001, p. 144), and this is independent of the Piagetian stages.

If one reinterprets the cognitive gap in terms of Vygotski's zone of proximal development (Vygotski, 1987), the cognitive gap marks not only the difference between what a learner can achieve without help and what a learner *cannot* achieve *without* help, but what a learner *can* achieve *with* help: in this case developing algebraic skills.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

Early Algebra

The idea of teaching algebra in earlier grades beyond a preparatory pre-algebraic way is most welcome as one can see in several early algebra projects (see Carraher & Schliemann, 2007). A reason for the popularity of early algebra is that the problems that students have with school algebra is likely to be based mostly on long experience of arithmetic classes without algebraic contents (see McNeil, 2004). This leads us to a first question:

1. Are there coherences between students' arithmetical skills and their effective approach to algebra?

From Carraher and Schliemann's (2007) review of the seven most common difficulties middle and high school students have with algebra (Carraher & Schliemann, 2007, p. 670) we can extract at least two main ideas that are demanded in arithmetic but are no longer desired while dealing with algebra. These are on the one hand the belief that the equal sign only represents an unidirectional operator that produces an output on the right side from the input on the left side, and on the other hand a focus on finding particular answers.

Algebraic symbolizing

Regardless of whether it is taught as regular school algebra in grade 7 or as early algebra in an earlier grade, if algebra is to be taught at school we have to think about what school algebra is meant to be. School algebra is taught as dealing with algebraic symbols, terms and equations, but often without context. This is accompanied by the problem, that students do not see the point in algebraic symbolizing.

“The lesson from history has implications for teaching in the sense that the potential of dominating algebraic syntax will not be appreciated by students until they have experienced the limits of the scope of their previous knowledge and skills and start using the basic elements of algebraic syntax.” (Rojano, 1996, p. 62)

Van Amerom proposes that “algebra learning and teaching should be based on problem situations leading to symbolizing instead of starting with a ready-made symbolic language.” (van Amerom, 2002, p. 10)

An alternative to conventional algebraic symbolizing is to allow the students to develop their own sign system when solving algebraic problems. But the algebraic syntax, as we know it and the way it is used worldwide, is a sophisticated tool for communicating about algebraic problems, and thus the understanding of and the ability to use and manipulate conventional algebraic symbolism is an important goal of algebra education (see Dörfler, 2008).

Summarizing, on one hand there is a negative correlation between students' advanced arithmetical skills and their effective approach to algebra. On the other hand there is the need to teach algebraic syntax in an environment that brings students to the limit of their mathematical abilities. This leads us to the conclusion that if algebra and algebraic syntax can in fact be taught in early grades successfully then it should indeed be taught in these early grades for the following reasons.

First of all, an earlier approach to algebra offers a lot more mathematical exercises that children can understand but cannot solve with the mathematical knowledge they've achieved up to then. At the same time the emphasis on arithmetic is reduced, which may decrease a habituation effect to arithmetic. Apart from that, lower achiev-

ers in arithmetic may profit from an early approach to algebra and algebraic syntax can support their algebraic thinking strategies.

The MeasureUp- Program

An unconventional way of teaching school algebra is taken by the MeasureUp-Program (Dougherty & Slovin, 2004) which combines early algebra with a fast introduction to common algebraic symbolization, at an early stage in primary school even before numbers are introduced. MeasureUp is based on a teaching experiment from the 60s implemented by Davydov (1975), a Wygotskian student. Within this teaching experiment the students develop abstract algebraic thinking by comparing magnitudes, like length, area, volume, etc. of concrete objects. The comparison of magnitudes is written down firstly with the help of signs of different sizes and finally with letter inequations and equations. The teaching of numbers follows only when the students can handle the algebraic syntax of elementary linear equations properly.

Our main concern is with the idea of introducing the abstract use and manipulation of the algebraic symbol system by concrete comparison of the magnitudes excluding numbers. We want to find out if this concept, which we will call the MeasureUp-Concept, will work for primary school children even though they have already have been introduced to numbers and arithmetic. This leads us to the following question.

2. Does the MeasureUp-Concept give German primary school-children a “good” approach to algebra and algebraic symbolism?

To answer this question we concentrate on two basic ideas of algebra, expressing magnitudes and their relations in letters and detaching the thinking from the concrete context.

The various aspects of letter variables range from letters as specific unknown over letters as generalized numbers to letters as changing quantity (see e.g. Küchemann, 1978). In our very first approach we have not seen it as important which of these aspects the children were working with. We are primarily interested in the question of whether the children are really seeing the letters as numbers and not developing the misconception of seeing letters as objects. As it is not intended to focus the children on magnitudes as numbers we have to differentiate the two categories letter as magnitude and letter as object. Bertalan (2008) claims, that a geometric approach supports the (mis)conception of letters as objects.

Within the intervention the children are working with concrete objects whose different magnitudes are compared. We want to know if the children are able to detach their thinking from the concrete material and if they are able to deal with word problems that do not refer to concrete material.

When to teach algebra and algebraic syntax?

Our focus of interest lies in the Measure Up-Concept, the introduction of abstract use and manipulation of the algebraic symbol system by concrete comparison of the

magnitudes excluding numbers, which is only a small but important part of the MeasureUp-Program. Because the MeasureUp-Program starts with the first grade it is reasonable to arrange our first observations at this age-level.

However, there are several widespread reasons, why algebraic syntax without numbers should *not* be taught in primary school, including curricular issues and the argument that this is too far away from a primary school students' everyday use of mathematics and thus should not be subject of mathematic lessons. With these reasons in mind, we come to another question of interest:

3. Does the Measure Up-Concept work in high school grades lower than grade 7 in the sense that none of the difficulties named above appear.

METHODOLOGY

Our research is based on the paradigm of design based research (DBR), which “blends empirical educational research with the theory-driven design of learning environment” (The Design-Based Research Collective, 2003, p. 1). It contains two main goals which have to be well-connected. These are on the one hand designing learning environments, on the other hand developing theories of learning. DBR happens in multiple cycles of design, implementation, analysis and redesign. The following investigation marks the first completed cycle of design, implementation and analysis. Later we will state conclusions for redesign.

As the starting point for the intervention we chose the MeasureUp-Program which we modified for our purpose. As variables are not part of primary school curricula, we have been looking for a school that enables us to teach the MeasureUp-Concept. We found that a Montessori primary school class with mixed age-groups would fit best for our first investigation. The self directed activity of children in a Montessori class allows us a flexible intervention alongside the regular class.

Implementing the MeasureUp-concept in a Montessori class made it necessary to develop material that children can work with on their own. So we developed exercise books which contain the introduction and comparison of magnitudes not only of Montessori but also other concrete materials, the setting up of equations and inequations, the so-called statements, and transforming inequations in equations, including transitivity and commutativity.

Example 1:

Compare

1. Take boxes I, II and III
2. Name the volumes of the boxes.
3. Compare the volumes of boxes I and II, write a line-segment and a statement.
4. Compare the volumes of boxes II and III, write a line-segment and a

statement.

5. Which statement can you write down for the volumes of boxes I and III without comparing the volumes?

The last exercise book contains word problems that do not refer to concrete material and word problems that contain numbers.

Example 2:

Word problems

A street has length A. Julia has already walked length B. How far does she still have to go?

A street has length L. Tim has already walked 200 m. How far does he still have to go?

A street has length 845 m. Hans has already walked 220 m. How far does he still have to go?

To address the question of whether there are coherences between students' arithmetical skills and their effective approach to algebra, we had to collect data about the arithmetical knowledge of the children. Thus every student attended the half-standardized interview ElementarMathematisches BasisInterview (EMBI, basis interview on elementary mathematics,) before the intervention (Peter-Koop et al, 2007). Thus we are able to compare high achievers with low achievers.

Then we introduced the exercise books to the children and allowed them to work with them during their free activity time. With some students or student groups we made appointments which gave us the opportunity to videotape the students while they were working with their exercise books and explaining their work to an interviewer. This happened within the principles of the Montessori school which means: students join voluntarily, the intervention will take part in an individual atmosphere and mistakes are not to be corrected. The work will consider the individual stage of development and, if required, the exercises will be extended or modified. So the interviewer held a double role as interviewer and teacher. Then we transcribed the videos and conducted a series of qualitative content analyses. To answer our first question

1. Are there coherences between students' arithmetical skills and their effective approach to algebra?

we have been coding in regard to the following topics:

- The students' possible belief that the equal sign only represents a unidirectional operator that produces an output on the right side from the input on the left side.
- The students' focus on finding particular (i.e. numerical) answers.

These we used as categories for our content analysis. Then we compared the findings of a, according to the EMBI, lower achiever with findings of a higher achiever.

To answer our second question

2. Does the MeasureUp-Concept give German primary school-children a “good” approach to algebra and algebraic symbolism?

we concentrated on the ideas of expressing magnitudes in letters and detaching the thinking from the concrete context. We did a qualitative content analysis with the two categories letter as number and letter as object. Also we did a qualitative content analysis on the children’s work with concrete material and also on the situations where children are solving word problem which does not refer on material (Example 2). For the latter we did not use pre-set categories, but generated them inductively.

For answering the third question,

3. Does the MeasureUp-Concept work in lower high school grades than grade 7 in the sense that none of the difficulties named above appear.

we are planning further cycles of design, implementation, analysis and redesign in a 5th grade of a German high school.

OBSERVATIONS ON STUDENTS’ ACTIVITIES

The design of the study only allowed us a focus on a small number of students. So our following interpretations are based on two case studies, Jay and Elli, which have been chosen for following reasons. Both students, a boy and a girl, are 3rd graders and will leave the class in the following year to join grade 4-6.

As showed by the EMBI, Jay is good at counting and handles interpreting and sorting of numbers beyond 1000 easily. He shows multiple strategies in addition, subtraction and multiplication and is able to solve division problems in an abstract way. Elli is also good at counting, but not as good as Jay and she is able to interpret and sort three-digit numbers. She is solving addition and multiplication problems through counting and needs proper material for solving multiplication and division problems. So we can call Jay a higher achiever and Elli a lower achiever. This is important for our first question, whether success in algebra class depends on arithmetic skills.

The analyses of both the transcripts and the exercise books showed that there is no dominance of the belief that the equal sign only represents a unidirectional operator that produces an output on the right side from the input on the left side. Jay and Elli both wrote and completed several equations of the form $D+B=A$ and $D=A-B$, without accounting for the direction of the equation. The transcripts also did not show any sign of preference or confusion about writing the equations the one or other way.

We had a different result when analyzing the focus on particular answers. We take a look at how Elli and Jay dealt with Example 2 (see above).

Jay: ...how far does she still have to go?

- Teacher: Right, you have said...
- Jay: D . *J writes down D , but the teacher stops him.*
- Teacher: Wait, can you write an equation?
- Jay: What's that?
- Teacher: A statement, with equal signs and plus and minus.
- Jay: Err, D plus B equals A .
- Teacher: Yes, right, you can write that down. *J writes it down.*
- Jay: Yes, but first of all I can write down D . *J writes down D and underlines it.*

Here we can see that Jay is looking for a particular answer. He names the length that still has to be travelled with D and wants to write it down as answer. The intervention of the teacher reminds him, that he can find a statement that shows how he can get length D with length A and B . Certainly, because of the early intervention of the teacher, we do not know if Jay would have written a statement without prompting. As we can see, he has no difficulties in finding the equation $D+B=A$ and later on he will have no problems with transforming the equation into $D=A-B$. But for him, both equations do not belong to the solution. In his exercise book we can find both equations in a subsidiary position. By contrast he insists in writing down and underlining D “first of all” right behind the word problem. The underlining is an indicator that for Jay D is the particular answer of the word problem but the equations are not.

Elli handles the word problem differently. At first she has problems with understanding the question and after the encouragement of the teacher she draws the street and attaches the given information. Then she suggests different statements that are however not solution-orientated. With some help by the teacher she finally writes down the statement $S=A-B$.

The following transcript shows that generally Elli feels comfortable with using letters.

- Elli: A street has length 845 meters.
- Teacher: Hm.
- Elli: Is the length M . Hans already walked 200 meters. How far does he still have to go?
- Teacher: Hm.
- Elli: I want to do that with letters.
- Teacher: You want to do that with letters? Ok. Which letters do you want?
- Elli: N and M .

By contrast Jay again is eager to calculate the solution and notes “that’s easier”.

If we interpret the observed situation, while keeping the research question in our mind, we explicitly have to differentiate algebraic thinking from using algebraic syntax. Elli’s difficulty with the last word problem that prompts the wish to use letters is a sign of her low achievement in arithmetic. We can also see her difficulties with algebraic thinking and algebraic syntax, but nevertheless Elli is expecting benefit from using algebraic syntax. Jay on the contrary has no difficulties with solving the word problems because he realizes their algebraic structure. He does not use the algebraic

syntax, but this is not because he cannot use it. We have seen that he can easily find a proper statement and is able to manipulate the equation. We conjecture he does not use algebraic syntax because the word problems are easy for him and he is focussing upon an answer where the approach is a minor matter.

We do not suspect that lower achievers in arithmetic will be likely to have fewer difficulties with algebraic thinking and using and manipulating algebraic syntax than higher achievers. But they may be more open for the use of algebraic syntax while working on word problems, because they expect a benefit for solving word problems and therewith are more accessible for the use of algebraic syntax.

As we have seen students at that age-level can work easily with letters as denotation.

For a “good” approach to algebra we need to know whether they name the object or the magnitude. By viewing the transcripts we found evidence for both letter as object and letter as magnitude. But we also observed a third category as is seen in the following transcript.

Teacher: Which letter stands for example for this length? *The teacher shows a grey stick.*

Jay: Err, the lowest, the lowest letter of all, which...ah...which is the lowest one?
Jay is sorting the letter-cards

Jay: So we call the small grey ones U. This is an U.

Teacher: So, then you can name all.

Jay: A is always the biggest one.

Jay is naming “the small grey ones”. Thus he is naming not only one object, but a class of objects with the same attributes. But he is naming the objects and not the magnitudes. Although the letter U names an object, the size of the object is still contained in the letter, because it is “the lowest” letter and the grey sticks have the lowest length. There is no lower letter than U because the letters V - Z are not available on letter-cards. Furthermore we can see that there is also a highest letter, the letter A which names “always the biggest one”. Elli shows a different but similar performance when she has to compare the width of two stripes which have same width but different length.

Elli: Do you have an U?

Teacher: I do.

Elli: Like Urs? And a D like Donatella?

Teacher: A D like Donatella? Ok.

Elli: My mother. An U and a D like my mum.

Teacher: There’s the D, look. So, you can already write that down. Here is.... which has the width U?

Elli: Dad is bigger.

Like Jay, Elli includes the size of the object in the letter. For that purpose she refers to the size of family members. But Elli is focusing on what differentiates the objects and not on what is being compared. So she is choosing the letters while focusing on

the length and not the width. Therefore she picks two letters that refer to two family members which different length, U for her “bigger” dad and D for her smaller mum.

Beside the categories letter as object and letter as magnitude we can summarize the above observation under the category letter as object with a certain size. This leads to new questions of interest. Does a geometric approach to algebra support the idea of a letter as object with a certain size instead of letter as object and letter as magnitude? And if so, is it to be seen as positive or negative for a “good” approach to algebraic thinking and/or algebraic syntax?

Finally we take a look at the word problems of Example 2 again, to find out how Jay and Elli handle problems that do not refer to concrete materials but to imagined objects, in this case a street. Both were offered the opportunity to use paper strips or sticks to represent the street or to draw the street. For solving the second word problem, which mixes letters with numbers, Eli drew a street, while Jay used paper strips. The following observation was made as Elli was working on the word problem.

Teacher: So, a street has length N , Tim already walked 200 meters.

Elli: Then he still has to go 400 meters.

With Jay we can make a similar observation.

Jay: ...that is length L . *J displays a different paper strip.*

Teacher: That is length L ? Ok.

Jay: 200 meters, how big is a man, that big, then, I think, these are about 200 meters.

Teacher: Ok.

Jay: And this small edge here, that goes here, are the remaining...?

Teacher: Meters. How do you call the remaining meters?

Jay: 50 meters?

Both understand the offered material not as aid for visualising the real street but as a scaled down model version of the street. They can't detach themselves from the concrete material thus they are not able to solve this word problem without assistance.

PERSPECTIVE

In regard to our questions the evaluation of the exercise books and the transcripts did not provide as conclusive results as we had hoped for. In particular looking at how the students perceive the letters brought up new questions. These questions have to be considered in our redesign. We also have to work more closely on the abilities of the children. We have seen that Jay did not use the algebraic syntax in some cases because he did not require it. As a main goal of the intervention is to adapt the use of algebraic syntax, we have to modify these particular exercises so that we can adapt them easily and flexibly at the abilities of the students. Furthermore we decided to move the question of the ability to detach the thinking from the concrete context to the projected intervention in grade 5. There we also will try to gain more clarity if a long term arithmetic education gets in the way of an effective approach to algebra.

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THE AMBIGUITY OF THE SIGN $\sqrt{\quad}$ ¹⁹

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In this paper an educational problem is discussed deriving from the ambiguity of the radical sign, $\sqrt{\quad}$, produced by its shift in meaning when passing from arithmetic to algebra. This problem is concerned with understanding difficulties that are linked to a particular tradition of teaching in which the radical sign is introduced by means of the square root notion. As a conclusion it indicates that any teaching proposal should take into account the distinction between root and radical.

Key words: roots, radicals, meaning, textbook

INTRODUCTION: The problem under investigation

The ambiguity of the sign $\sqrt{\quad}$ as a consequence of the change in its meaning when passing from arithmetic to algebra often goes unnoticed by teachers and textbook authors. This lack of perception may be the cause of certain cognitive conflicts experienced by teachers and students.

This work takes its cue from one of these conflicts. It is a conflict expressed by a Spanish secondary school mathematics teacher called Patricia, on attempting to understand the definition of equivalent radicals. She states that the equality $\sqrt[4]{3^2} = \sqrt[3]{3}$ cannot be true, since in the expression on the left the index of the root is even, so that it has two opposing roots, two solutions, whereas in the expression on the right the index is odd so it only has one root, which means that the two expressions have a different number of roots.

The conflict expressed by Patricia leads to the difficulties and controversies related to the values, properties and rules of radicals, which are the ultimate aim of this work.

Examples of that, are the students opinion about the statement $\sqrt{25} = \pm 25$. (Roach, Gibson and Weber, 2004), the value of $(-8)^{1/3} = -2$ (Even and Tirosh, 1995; Goel and Robillard, 1997; Tirosh and Even, 1997), and the rule for multiplying imaginary numbers (Martínez, 2007).

THEORETICAL FOUNDATIONS

To support the work carried out, a theoretical approach has been adopted that has three fundamental references.

1. One of these looks at the cognitive side, taking into account the need to re-conceptualise signs that change meaning when passing from arithmetic to algebra (Kieran, 2006, p. 13).

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This happens with the sign $\sqrt{\quad}$ which changes meaning, since it either indicates an operation, as happens in $\sqrt{4}$, or indicates the main root of this operation, as happens in the solution to the equation $x^2 - a = 0 \rightarrow x = \pm\sqrt{a}$.

There are examples of this double meaning to be found in the teaching tradition that appears in textbooks by such influential authors as Euler.

Euler considered that:

150. (...) the square root of any number always has two values, one positive and the other negative; that $\sqrt{4}$, for example, is both +2 y -2, and that, in general, we may take $-\sqrt{a}$ as well as $+\sqrt{a}$ for the square root of a (...). (Euler, 1770, p. 44)

In Euler's text the $\sqrt{\quad}$ sign is used ambiguously. In $\sqrt{4}$ it is perceived as an indicated operation (finding the square root of 4) and it is associated to the set of two results, in this case +2 and - 2. In $+\sqrt{a}$ it is perceived as a result of the aforementioned process and designates one of the two roots of a .

This duality of meaning starts in arithmetic when introducing the $\sqrt{\quad}$ sign in order to indicate an operation in an abbreviated way, *the fifth elementary operation*²⁰. In arithmetic this number can be found and it is unique. Thus, for example, the square root of 4 is 2, which is written $\sqrt{4} = 2$.

Things change in algebra, since the square root of a ($a > 0$) cannot be calculated, so that to indicate its value the expression \sqrt{a} is introduced, which no longer represents an indicated operation but a result.

3. The second reference looks at the formal component. The mathematicians have decided to assign to the radical expression, \sqrt{x} , $x \geq 0$, only one value, one of the roots of x , the root no negative, the one that they name principal root. With this restriction, the right thing is to write $\sqrt{4} = 2$, not $\sqrt{4} = \pm 2$.

We agree to denote by \sqrt{a} the positive square root and call it simply the square root of a . Thus $\sqrt{4}$ is equal to and not -2, even though $(-2)^2=4$ (Lang, 1974. p. 10).

With this decision, the mathematical problem of the ambiguity of the radical sign disappears, but not the didactic problem. Students do not learn only what they are told; much of students' learning occurs when they attempt to make sense of the mathematical situations that they encounter (Roach, et al. 2004). To help students to make sense of the formal definition there are several options:

A) To avoid contradictions. If $\sqrt{4} = \pm 2$, then $\sqrt{4} + \sqrt{4} = (\pm 2) + (\pm 2) = \{-4, 0, +4\}$; $\sqrt{4} - \sqrt{4} = (\pm 2) - (\pm 2) = \{-4, 0, +4\}$ and $\sqrt{4} + \sqrt{4} = \sqrt{-4} + \sqrt{-4}$

²⁰ This consists of given a number, find another which when multiplied by itself gives the first.

B) To satisfy the requirements for the definition of operation of exponentiation to rational exponents. This definition should not depend on the representatives of numbers involved in the operation. We want $a^r = a^{\frac{m}{n}} = \sqrt[n]{a^m} = a^{\frac{km}{kn}} = \sqrt[kn]{a^{km}}$ (see Tirosh, & Even, 1997, p. 327). Nevertheless, if $\sqrt{4} = \pm 2$, then $\sqrt[6]{3^2} \neq \sqrt[3]{3}$. And, in general, $\sqrt[kn]{a^{km}} \neq \sqrt[n]{a^m}$, when kn is even and n is odd,

C) To satisfy the requirements for functions. The basic arithmetic operations addition and multiplication by a number different from zero establish bijective functions: $x \rightarrow x+a$, $x \rightarrow x \cdot a$, $a \neq 0$. These functions have unique inverse functions corresponding to the inverse operations. But, an operation like: $x \rightarrow x^2$ does not establish an injective function; because $x^2 = (-x)^2$. Consequently, the function $x \rightarrow x^2$ has to be confined to one of its branches to be inverted, $x \geq 0$. In the same way the inverse operation, $x \rightarrow \sqrt{x}$, has to be confined to positive domain, and range, in order to be unique.

2. The third reference takes on a psychological point of view, taking into account the dual operational/structural nature of mathematical conceptions and their role in the formation of concepts, indicated by Sfard (1991).

Sfard (1991) supports this theory with the fact that a mathematical entity can be seen as an object and a process. Treating a mathematical notion as an object leads to a type of conception called structural, whereas interpreting a notion as a process implies a conception called operational.

For Sfard, the ability to see a mathematical entity as an object and a process is indispensable for a deep understanding of mathematics, such that the “concept formation implies that certain mathematical notions should be regarded as fully developed only if they can be conceived both operationally and structurally” (p. 23).

It is worth pointing out that when referring to the role of operational and structural conceptions, Sfard conjectures that when a person gets acquainted with a new mathematical notion, the operational conception is usually the first to develop, whereas the structural conception follows a long and difficult process that needs external interventions (of a teacher, of a textbook), and may therefore be highly dependent on a kind of stimulus (of teaching method) which has been used (p. 17).

Pointing out that, the investigation on the conceptualization of the radical sign should be held in a revision of manuals and textbooks.

OBJECTIVES

Once the general problem to be studied has been pointed out, as well as the theoretical references, it is necessary to specify the general aims that are to guide the investigation's design and methodology:

1. To determine the characteristic aspects of teaching the radical sign, just as they are shown in textbooks today.

2. To diagnose mathematical knowledge with respect to the radical sign that some secondary school teachers have.
3. To explain teachers' possible conceptual and operational difficulties.

PATRICIA'S CONFLICT

The aims are linked to Patricia's conflict. Patricia is a high school mathematics teacher (in Spanish public education) and a student in a post-graduate programme. She presented the following conflict to her professor:

In the textbook, the concept of equivalent radicals is defined as follows: "Two radicals are equivalent if they have the same roots" (and so I had learned). On the other hand, simplifying a radical by dividing the index of the radical and the exponent of the radicand by the same number, results (in theory) in a radical equivalent to the first. However, in a case like the sixth root of three squared, the cube root of three is obtained. As the index of the first radicand is an even number, two solutions exist (one being the opposite of the other) but in the second case, the index is an odd number and therefore there is a single root. Therefore, it cannot be said that these two radicals have strictly the same roots. So, are they equivalent?

Patricia says:

(A) Two radicals are equivalent if they have the same roots.

Also Patricia makes reference to the following equivalency:

$$(E) \quad \sqrt[nk]{a^k} = \sqrt[n]{a}, k, n \in \mathbb{N}^*, n \geq 2, a \geq 0.$$

Applying the equivalency (E), Patricia obtains that: $\sqrt[6]{3^2} = \sqrt[3]{3}$. However, to her the sixth root of three squared has two opposed roots, "two solutions", as the index is an even number and the cube root of three has a single root as the index is an odd number, which means that the two expressions do not have the same number of roots and so according to (A) they would not be equivalent.

Hypothesis in relation to this conflict

In order to try to explain the causes of conceptual and operative difficulties that give rise to Patricia's conflict, the following hypothesis has been formulated:

(H₁) The lack of perception of the difference between the operational and structural conceptions of the radical sign that Patricia expresses is the cause of her conflict.

(H₂) This lack of perception is a product of a traditional teaching proposal, which does not pay attention to the need to re-conceptualise the $\sqrt{\quad}$ sign when passing from arithmetic to algebra.

(H₃) In an alternative teaching proposal, where the meanings of root and radical are formulated, the conflict expressed by Patricia is not expected.

METHODOLOGY

To verify the solidity of the hypotheses an exploratory study was carried out, as a step prior to a more rigorous inquiry in terms of methodology, still to be carried out.

This exploration is based on a revision of current and representative textbooks of two alternative proposed ways of teaching: the Spanish one, which introduces the radical sign in arithmetic, and the Rumanian one, which introduces it in algebra. The revision of textbooks is has been complemented by a questionnaire followed by an interview with two representative individuals, Patricia (Spanish) and Iulian (Rumanian), two typical high school mathematics teachers.

With the revision of textbooks an attempt has been made to identify characteristic features in the teaching of roots and radicals in Spanish and Rumanian textbooks, and to identify comments that may favour the ambiguity of the $\sqrt{\quad}$ sign, and Patricia's conflict.

The questionnaire

The questionnaire consists of a paper and pencil test which included four tasks. The first one is based on the teaching proposal given in the Spanish textbooks. In the task it is considered, as in Euler's text, that the square root of any positive number has two solutions, one positive and another negative. However, to represent this set of results the symbolic form $\sqrt{4} = \pm 2$ is used as well as the rhetorical form: "the solution is double, positive and negative". The intention of this task was to know if the difference is perceived between the structural and operational conception. The task is:

In the class of 9th grade, after introducing the theme of the roots and radicals, the students were asked to calculate the square root of four.

One student wrote $\sqrt{4} = \pm 2$, justifying thus:

"As the radicand is positive and the root's index is even, then the solution is double, positive and negative".

Is this correct?

Task 1

The interview's design took into account the answers produced by Patricia and Iulian to task 1. If the answer was "No", then the interviewee was asked to justify why and if it was "Yes", then they were given the second task with the aim of bringing in a cognitive conflict, in order to study the students' reaction

The second task is based on substituting $\sqrt{4}$ for ± 2 in a context of calculation. With this the aim was to put the affirmative answer to the task 1 into conflict.

If $\sqrt{4} = \pm 2$ then complete:

$$\sqrt{4} + \sqrt{4} = (\pm 2) + (\pm 2) = \dots$$

$$\sqrt{4} - \sqrt{4} = (\pm 2) - (\pm 2) = \dots$$

Explain the answer.

Task 2

A third task is based on the restriction of the property of radicals in the case where k is an even number and $a < 0$, which requires the intervention of the module.

$$(P) \quad \sqrt[n]{a^k} = \begin{cases} \sqrt[n]{a}, & k, n \in \mathbb{N}, n \geq 2, a \geq 0 \\ \sqrt[n]{|a|}, & k, n \in \mathbb{N}, k - \text{even}, n \geq 2, a < 0 \\ \sqrt[n]{a}, & k, n \in \mathbb{N}, k - \text{odd}, n \geq 2, n - \text{odd}, a < 0. \end{cases}$$

Here, the intention was to confirm that the interviewee was taking into account the radical's formal definition, in a traditional problematic case. The hypothetic situation that is present is the following:

In a class of 10th grade, after introducing the radicals theme, the students were asked to simplify:

$$\sqrt[6]{(-8)^2}$$

One student wrote: $\sqrt[6]{(-8)^2} = \sqrt[2 \cdot 3]{(-8)^2} = \sqrt[3]{-8} = -2$

and said: "I have applied the following rule: $\sqrt[n]{a^{mk}} = \sqrt[n]{a^m}$. Is this correct?"

Task 3

If the answer to the task was "No", then the interviewee was asked to justify why and if it was "Yes", then the fourth task was given with the aim of introducing a cognitive conflict, in order to study the student's reaction.

Task 4 imposes the strategy for calculating $\sqrt[6]{(-8)^2}$ that leads to a different result from -2. With this, the intention was to put the affirmative answer given previously to the task 3 into conflict, in order to again study the reaction of the interviewee.

If you consider:

$$\sqrt[6]{(-8)^2} = \sqrt[2 \cdot 3]{(-8)^2} = \sqrt[3]{-8} = -2$$

then complete:

$$\sqrt[6]{(-8)^2} = \sqrt[6]{64} = \dots$$

Task 4

RESULTS OF TEXTBOOKS REVIEW

1. In the Spanish textbooks reviewed the sign $\sqrt{\quad}$ is used to express the reverse operation of taking a number to the power of two (Figure 1):

Calculating the square root is the reverse operation of calculating the power of a square:

$$b^2 = a \leftrightarrow \sqrt{a} = b.$$

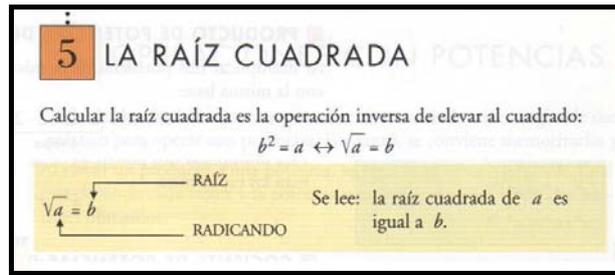


Figure 1. 1^o Secondary (7th grade), Anaya 2006, p. 52

The expression that has the $\sqrt{\quad}$ sign is called a radical, that is to say the operation shown, and not the main root of said operation (Figure 2).

It is called the n^{th} root of a number a , and is written $\sqrt[n]{a}$, where a number b meets the following condition: $\sqrt[n]{a} = b$ and $b^n = a$

$\sqrt[n]{a}$ is called radical; a , radicand, and n , the root's index.

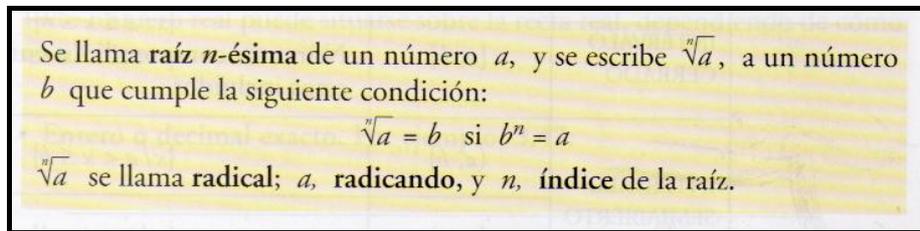


Figure 2. 4^o Secondary (10th grade), Anaya, 2006 b, p. 32

As a consequence it is considered that a radical has roots and that its number depends on the index of the radicand's sign (Figure 3).

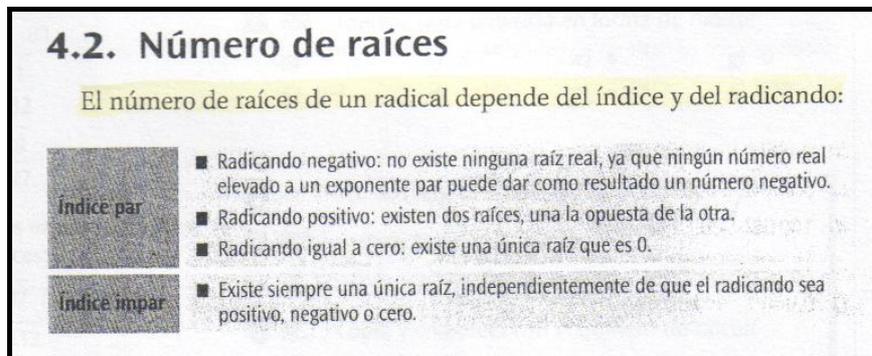


Figure 3. 3rd Secondary (9th grade), Oxford, U. P., 2007, p.32

So, equalities appear written as $\sqrt{36} = \pm 6$ (Figure 4).

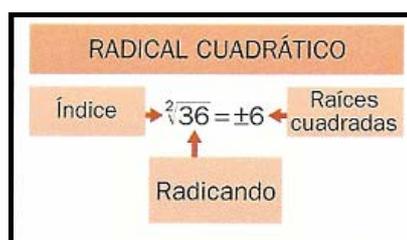


Figure 4. 3rd Secondary (9th grade), S. M., 2003, P. 36

The properties of the radicals are stated without mentioning their field of validity. So it is not taken into account that $\sqrt{a^2} = |a|, \forall a \in \mathbf{R}$ (Figure 5).

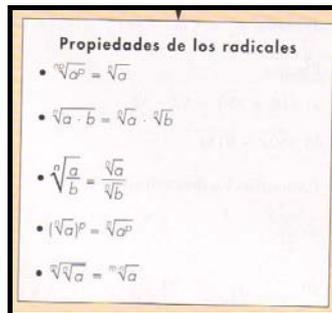


Figure 5. 4^o Secondary (10th grade), Anaya, 2006 b, p. 36

2. In the Rumanian textbooks reviewed, the sign $\sqrt{\quad}$ is associated with the radical notion. The radical with an index two of a positive number a is defined as the positive solution of the equation $x^2 = a$ and is denoted by \sqrt{a} . (Figure 6)

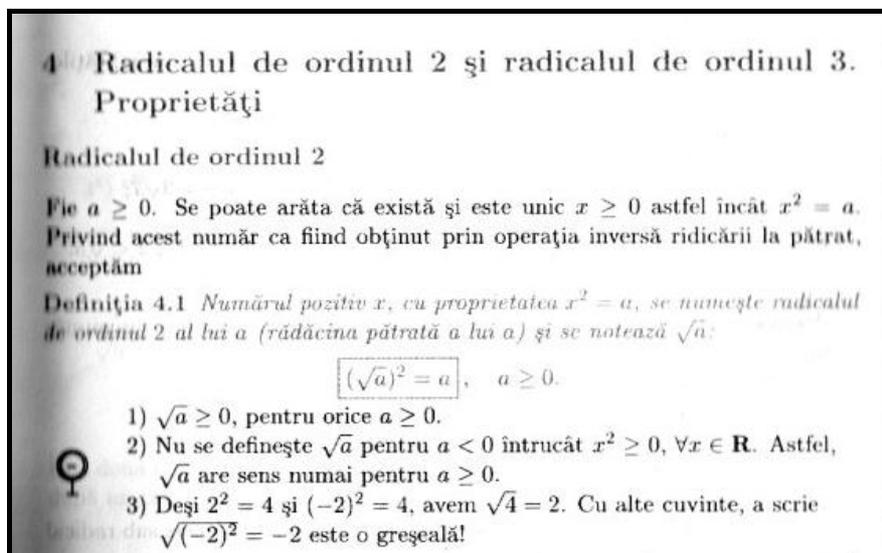


Figure 6. 10th grade, Fair Parteners, 2005, p. 13

It is taken into account that $\sqrt{a^2} = |a|, \forall a \in \mathbf{R}$, and the domain of validity of the radical's properties is specified. (Figure 7)

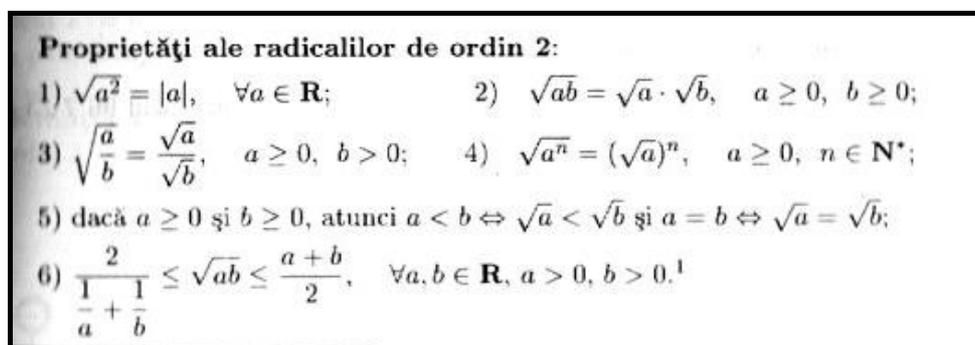


Figure 7. 10th grade, Fair Parteners, 2005, P. 13

FINDINGS AND CONCLUSIONS

As for the first objective, the review of textbooks shows that there are substantial differences in dealing with the $\sqrt{\quad}$ sign. Specifically, it can be said that in the Spanish textbooks studied, the conception associated with this sign is operational, whereas in Rumanian texts it is structural.

As regards the second objective, Patricia and Iulian's mathematical knowledge with respect to the radical sign shows significant differences.

In tasks 1 and 2, Patricia identifies $\sqrt{4}$ with the set of two solutions (2 and -2), and does not see the radical as the positive root when the index is even. In the interview, to emphasize this in task 2, she indicated that in reality there are not two solutions, but there are contexts in which it is replaced by +2 and others in which it is replaced by -2.

Iulian does not agree with $\sqrt{4} = \pm 2$, arguing that the radical of an even index of a positive number belongs to the interval $(0, \infty)$ and specifies that, in any context $\sqrt{4}$ represents a number, that is, the positive square root of 4.

In task 3 and 4, Patricia does not take into account that $\sqrt{a^2} = |a|, \forall a \in \mathbf{R}$. On the other hand Iulian correctly applies the restriction of the property of radicals and he realizes the error that the hypothetical student commits.

In conclusion, it can be said that Patricia has procedural knowledge of the $\sqrt{\quad}$ sign, whereas Iulian has structural knowledge, and that these conceptions are consistent with what is shown in the textbooks studied.

As for the third objective, this part of the work was restricted to Patricia's conflict, the answers to the questionnaire and the interviews that provide indications suggesting the validity of the hypotheses.

(H₁), Patricia does not distinguish between operational and structural use of radical sign.

(H₂), the review of Spanish texts evidences that the teaching proposal reflects the ambiguity of the radical sign, used in the expression $\sqrt{4} = \pm 2$, and does not use the formal definition of radicals, so that it is plausible to think that they encourage the appearance of Patricia's conflict.

(H₃), in the revised Rumanian texts, the formal definition of the radical sign is observed, so that it is possible to think that they support Iulian's way of acting, which does not encounter the conflict that Patricia expresses.

Finally, the important educational implication that should be pointed out is that in any educational proposal that aims to avoid conflicts such as the one expressed by

Patricia, the formal definition of radical must be considered, and it must be guaranteed that students understand the reasons for this definition.

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BEHIND STUDENTS' SPREADSHEET COMPETENCIES: THEIR ACHIEVEMENT IN ALGEBRA? A CASE STUDY IN A FRENCH VOCATIONAL SCHOOL

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Research on the use of spreadsheet in mathematics education usually points out its potentialities in the learning of algebra. The link between spreadsheet and algebra is thus often seen in the direction “spreadsheet for algebra”. This paper follows the opposite direction, i.e. “algebra for spreadsheet”, by questioning the role of algebra in students’ spreadsheet competencies. It reports a case study, based on computer tests, in the framework of a French research project studying students’ spreadsheet uses and competencies. The results of the test show algebra raising out again, playing a role behind students’ achievements and actions with spreadsheets. Algebra, Spreadsheet competences, Computer tests

INTRODUCTION

What role can technology play in mathematics education? Usually, didactic research approaches ICT questions through this direction, i.e. “technology for mathematics”. This is the case for many studies on spreadsheets which consider this latter as a good tool to help pupils understanding algebraic concepts.

Here, we take the opposite direction: “what about algebra for spreadsheet?” by questioning the role algebra plays in students’ mastery of spreadsheets. This issue comes from the analyses of experimentations in the context of *DidaTab*, a French research project studying students’ spreadsheet competencies. To identify the basic competencies students have acquired, the *DidaTab* project realised tests of competencies in several classes. In the analyses of the results, the relation with algebra stands out again, raising issues on the relations between students’ achievements and actions with this kind of software and their mastery of algebra.

In the first part, we give a quick description of the *DidaTab* project. The second part focuses on relationships between spreadsheets and mathematics learning. Then, to get a more concrete view of spreadsheet mastery problems, we detail the results of a computer test administrated to 17 y.o. students in a vocational marketing school. The results of this test put in perspective students achievements, actions, and software interaction understanding, with their knowledge (or their lack of) in algebra.

THE DIDATAB PROJECT

According to educational authorities of many countries, ICT has to be used in classrooms. In the case of secondary education, all countries have established detailed recommendations (Eurydice, 2004, p. 24). In general, using ICT to enhance subject knowledge or learning correct use of a word processor or a spreadsheet are part of the objectives at lower secondary level. But, if ICT seems to be included in prescribed curricula, we only have very few data about effective practices in classrooms and ICT

competencies of students. Some data from PISA 2003 (Eurydice, 2005) provides interesting results (for example, that less than half of students are familiar with using a spreadsheet to plot a graph) but rely on declarative statements. We don't know whether students under or over estimate their competencies. To get a more comprehensive picture, we considered that it was not fruitful to take into account ICT as a whole, and decided to focus on specific software. Spreadsheets, prescribed in French curricula for ten years now, were a good indicator of ICT mastery. What do students learn about spreadsheets? Which basic competencies do they have acquired at the end of their schooling?

DidaTab (didactics of spreadsheet^[1]) was a three year project (2005-2007) founded by the French ministry of research and dedicated to study personal and classroom uses of spreadsheets in French context. The methodology combined questionnaires, interviews (students and teachers), classroom observations, computer tests, content analysis of official curriculum texts, websites and resources, and some comparative studies with other countries (Belgium, Greece, Italy) have been made. As results (Blondel & Bruillard, 2006), we have an almost complete cartography of spreadsheets uses in the French secondary education, including an overview of personal uses, and we began to describe kinds of genealogy of uses, according to subject matters (e.g. mathematics, technology, social sciences, experimental sciences...). But we have not yet built a theoretical framework to explain spreadsheets uses and competencies of students. Some of these competencies relate to knowledge of mathematical nature, especially algebraic one. In a next part, we discuss this particular relation between spreadsheets and mathematics.

SPREADSHEET AND MATHEMATICS COMPLEX RELATIONSHIPS

In the title of this section, we play on the word “mathematics” to relate two points: mathematics as a school subject, this questions the place of spreadsheet within syllabus, or mathematics as knowledge that spreadsheets may bring into play, this questions the place of mathematics within the spreadsheet objects.

Spreadsheets within mathematics syllabus

Spreadsheets have been introduced at many different teaching levels and courses of the French Educational system. Part of the mathematics syllabus since 1997, first in middle school (grade 6 to 9) then in high school (grade 10 to 12), their place varies according to the school streams, as mathematics education appears under different aims. Two main tendencies can be distinguished, each of them promoting a different use of spreadsheets.

In the scientific streams, mathematics is a very theoretical discipline also used to select students. In this “abstract” approach of mathematics, spreadsheets appear as a

^[1] In French, spreadsheet is “tableur”. See <http://www.stef.ens-cachan.fr/didatab/en/index.html> for other information and results in English about *DidaTab* project

tool to serve the learning of mathematical concepts. Then spreadsheets' role is to support and enhance learning.

In some other streams, as vocational or literary, mathematics is considered as a more experimental subject oriented towards everyday life problems. This objective favours the use of different kinds of software such as spreadsheets, which allow a more concrete approach of mathematics opened on its everyday applications.

First vision leads to a very small place for working spreadsheet competencies. Moreover, as we will elaborate in the next section, using spreadsheet to enhance mathematical learning is "double-faced" as far as spreadsheet is not neutral on mathematical concepts. Second vision opens a larger place for building some spreadsheet competencies. In these streams, a hypothesis would be that students' difficulties in mathematics could be counterbalanced by some instrumental abilities and some mastery of this software^[2]. But the situation is not as simple because of the specific relationships existing between spreadsheet and mathematics: spreadsheet mastery requiring mathematical knowledge.

Mathematics within spreadsheet objects

ICT use in mathematics education is a question among the more general problematic of technology use in human activity, studied in the field of cognitive ergonomics. A theory of instrumentation (Vérillon & Rabardel, 1995); developed in this field, provides a frame to tackle the problematic of the learning in complex technological environments. In this frame, an instrument is not *given* but *built* by the subject (Vérillon and Rabardel, 1995) through a progressive individual *instrumental genesis*. This genesis, is not *neutral*, instruments have impact on the *conceptualisation*. This idea of non neutral «mediation» between subject and tools provides a way to report on the strong imbrications that exist, and have always existed, between mathematics and the instruments of the mathematical work. It led to an instrumental approach in didactics that has been used in several researches on symbolic calculators (CAS) in mathematics education (Artigue 2001, Lagrange 1999, Drijvers 2000, Guin, Ruthven & Trouche 2005). What about spreadsheets?

Some "computer" objects, characteristics of spreadsheets, do not strictly correspond to mathematical knowledge transposed in a computer environment, even not to a computer transposition of school knowledge, but are *however linked with* mathematics. The basic principle of spreadsheet, which consists in connecting cells between themselves by "formula", gives an example of these objects, linking spreadsheet to the domain of algebra. Such a particular relation with mathematics is precisely the reason why many research in didactics from different countries (Ainley (1999); Arzarello et al. (2001); Capponi (2000); Dettori et al. (1995) or Rojano and Sutherland (1997)) give spreadsheets a role in the learning of algebra. at elementary stages, iden-

^[2] For instance, in the literary stream, a place is given to concrete aspects of mathematics and this is precisely a stream where spreadsheets take an important part in the mathematics syllabus.

tifying them as tools of arithmetic-algebraic nature. Haspekian (2005a), having adopted an instrumental approach, showed that in spite of an apparent simplicity of use, it is not so evident for teachers to take benefit from these characteristics. The tool generates some complexity: spreadsheets transform the objects of learning and the strategies of resolution by creating new action modalities, new objects, and by modifying the usual ones (as variable, unknown, formula or equation...). Here are some examples.

In a paper and pencil environment, variables in formulae are written by means of symbols (a letter generally for the school levels concerned here). This variable ‘letter’ relates to a set of possible values (numerical here) and exists in reference to this set. In spreadsheet, let us take for example the formula for square numbers. The Fig.1 shows a cell argument A2 and a cell B2 where the formula was edited, referring to this cell argument.

	A	B
1		
2	5	=A2^2

Figure 1 A2 is the cell argument; B2 calculates the square of the value in A2

Here again the variable is written with symbols (those of the spreadsheet language) and exists, as with paper and pencil, in reference to a set of possible values. But this referent set (abstract or materialised by a particular value, e.g. 5 in Fig.1) appears here through an intermediary, the cell argument A2, which is both:

- an abstract, general reference: it represents the variable (indeed, the formula does refer to it, making it play the role of variable);
- a particular concrete reference: here, it is a number (in case nothing is edited, some spreadsheets attribute the value 0);
- a geographic reference (it is a spatial address on the sheet);
- a material reference (as a compartment of the grid, it can be seen as a box)

So, where in paper and pencil environment, we stick a set of values, a cell argument overlaps here, embarking with it, besides the abstract/ general representation, three other representations without any equivalent in paper and pencil (Fig.2).

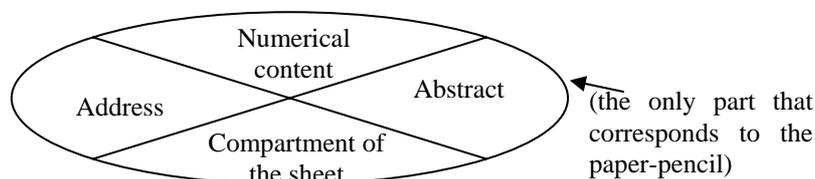


Figure 2: The “cell variable”

Other examples of the changes due to spreadsheets are given in Haspekian 2005a.

From an institutional point of view, these changes have different impact following the different way chosen to introduce algebra. As the recent ICMI study showed (Stacey et al., 2004), different aspects of algebra can be focused on: as a tool of generalisation, a tool of modelling, or a tool to solve arithmetical, geometrical or everyday life

problems through what is called since Descartes, the « analytical method ». Following the case, different mathematics is brought forward: variables, formulae and functions on one hand, unknowns, equations and inequations on the other hand. In the French school culture, it is traditionally the analytic way that is chosen, the resolution, though equation solving, of various problems appears as emblematic of pupils introduction to algebra. Table 3 gives a quick insight of the distance between the algebraic culture in the French secondary education and the algebraic world carried out by spreadsheets.

"Values" of algebra	In paper pencil environment	In spreadsheet environment
Objects	unknowns, equations	variable, formulae
Pragmatic potential	tool of resolution of problems (sometimes tool of proof)	tool of generalization
Process of resolution	"algorithmic" process, application of algebraic rules	arithmetical process of trial and refinement
Nature of solutions	exact solutions	exact or approached solutions

Table 3: distance between different "algebraic worlds"

More generally, the mathematical culture sustained by spreadsheets is an « experimental » one: approximations, conjectures, graphical and numerical resolutions, implementing everyday life/ concrete problems, statistics... Thus, this vision fits with the aim of mathematics in particular streams of the French Education, especially where students not very good at mathematics are supposed to use spreadsheet with stronger objectives. It is thus interesting to investigate with students of non scientific streams and test them at the last year of their schooling (grade 12).

As we will see next, the computer test confirms the complex relation between spreadsheet and mathematics. Algebraic aspects; especially the use of cell-variables in formulae, stand out again as one of students' main difficulties with the tool.

STUDENTS' SPREADSHEET COMPETENCES: A CASE STUDY

We report here the example of a one hour computer test administrated in 2005 in a class of 13 students of vocational school^[3] (17 year old) preparing a marketing diploma. After presenting the objectives and a brief description of the test, we first give an overview of the general results and then an analysis on the algebraic aspects that these results lead to focus on.

Objectives and description of the test

For this part of the *DidaTab* project, the objective was to assess students' spreadsheet competences in a computer test. In order to design such tests, a first step consisted in the identification of basic spreadsheet skills, that have been actually organised in five categories (see below), then the definition of some general and simple tasks corre-

^[3] This school is identified as rather difficult in the sense that students have behavioural difficulties and social problems.

sponding to each ability, and finally the construction of a database of skills, questions and tasks (for more details on this step of the project, and especially on the design of the tests, see Tort and Blondel, 2007).

From the database we selected 24 exercises relevant to the school year of the students and covering all categories of skills. Then, the students' mathematics teacher chose 11 exercises from this list according to the competences she assumed that her students have. With regard to the classification, the 11 exercises are divided in the following way:

1. "Cells and Sheets Editing" (3 were selected)
2. "Writing of formulae" (4 were selected)
3. "Translating data into graphs" (1 was selected)
4. "Managing data tables" (2 were selected)
5. "Modelling" (1 was selected)

The tasks were proposed in the computer test with increasing order of difficulty, in a spreadsheet file. Students had to answer directly within the tool and record their work at the end of the test. The collected data are constituted by these file records, observation and the complete recording of the actions for one of the students.

An overview of the results

Among the five categories of skills, clear differences between basic skills linked to superficial manipulations (not requiring knowledge of the contents) and abilities requiring deeper knowledge appear.

The best rates of success for the 13 students, concern cell formatting: italic (10), bold (11), date format (9). The results decrease then as the tasks require more understanding of spreadsheets objects. Some tasks requiring deeper knowledge of spreadsheet functionalities have been moderately achieved: recopying a format (6), sorting out data (6), or representing data with a graph by choosing the best type of representation (4). Finally, more specific knowledge as the conditional format (0) or specific displays either of numerical data (fractional format: 1) either of graphics (displaying labels on the X axis: 2) seem rather unknown from these students.

All exercises of the formulae category are part of the competences that have been failed in. Actually, the success rates for the four tasks of this category are the lowest of the test, varying from 0 to 2 good answers for each item: Writing a formula to calculate the AVERAGE of a line of data in adjacent cells (2), Writing a formula calculating a subtraction (0), a product (0), a division (0), Writing and copying down a formula using relative and absolute references (0, only 1 student answered: he gave a number...), Writing a conditional formula (using the IF function) (0).

Three main issues can be raised from these observations:

- 1) The inadequacies between the skills we thought students have and their actual level of competence. Students' abilities were clearly lower than expected.
- 2) The teacher tended as well to overestimate the skills of her students. The exercises she has chosen were globally too difficult.

3) The very bad results concerning the formulae category raise the question of spreadsheet's relation to algebra. Obviously, the formulae, the copying of formulae, the use of relative/absolute references as variables in formulae and the conditional formulae appear in students' results, as the less achieved competences.

In the next section we analyse this last point in more details.

Algebraic aspects in students' achievements

Competences just mentioned are all linked with algebraic knowledge of students, their understanding of the concepts of variable and formula. These results join other research in didactics of mathematics (Capponi, 2000, or Dettori & al. 2001). For Capponi, benefiting from spreadsheet potentialities requires from the user the understanding of some algebraic knowledge such as the notion of formula, and students' difficulties with spreadsheets show their needs in this domain: the work remains at the numeric level (data tables, numbers, operations) without reaching the level of an algebraic treatment (dynamic sheet of calculations, formulae).

About formulae

Looking further the tasks of the formulae category, we note that sometimes, not only the correct formula had not been found, but not even wrong formulae have been tried. Some students edit, instead of formulae, the corresponding arithmetic operations, some others edit directly the results they calculate by hand, but most of them do not answer anything. Another surprising point concerns the calculus of the average: we did not find any formula such as $(A5+B5 + \dots + N5)/ 14$ or equivalents and only 2 students achieved the calculus of this average.

Observation during the test brings out some more elements. One of the students who succeeded in the average used the AVERAGE functionality (and seemed yet surprised to have directly the response). This can seem paradoxical, but to calculate the average of the given numbers, he directly used the function "AVERAGE" provided by spreadsheet; the references to the adequate cells are then automatically made. The student has to calculate an average, he has an "average" function (as a key of calculator), and he uses it without controlling more what this feature produces. The use of "AVERAGE" can thus mask its lack of understanding of what is really a formula in spreadsheet and the way it can be used. We have the same observation for the other student who used the average function. Finally, in the whole test, we did not find any other formula at all except these automatic formulae as average or sum. And the very surprising result that is coming to light with these analyses is that no student used a single relative reference in the entire test! According to us, this is precisely linked to the problem of the cell variable. Very few students used formulae which send back automatically the cell references^[4] (such as SUM or AVERAGE) and not even a sin-

^[4] The spreadsheet used in this experiment is Microsoft Excel. The interface provides buttons that you can directly activate and obtain the writing of a formula including cell references

gle student was able to write a formula which requires finding and entering the cell variable.

About the cell variable

The use of cells as variable in a formula seems more difficult than the use of formulae itself. In tasks which require a formula which does not automatically send back the cells references, either students do not find any formula or they use again an “automatic” formulae (AVERAGE or SUM) even when these functions have nothing to do with the task! For example, a correct answer to a task was a formula with a multiplication and one student has written the following formula: “=SUM (C12*10)”. SUM is here totally useless and used in a non standard way. The student invoked the function and turned the usual argument automatically written by the software (“=SUM (C12)” in this case) into a multiplication. By using this automatic formula, he did not enter himself the cell reference in the formula.

We have exactly the same phenomenon in another task: 2 students used SUM in both columns although the answer has nothing to do with a sum. One of them, after using the function SUM transformed the separating sign ":" in the syntax of this function into a subtraction (and the result is then correct)! All the others had not answered or had put either an operation or directly the numerical result instead of a formula. Once again, the use of cells as variables in a formula seems to be problematic, the type of functions as SUM or AVERAGE being apparently the only type of formula those students manage.

The problem of the cell-variable is also revealed by the use of the recopy. Here again, a deeper analyse of the answers of the whole class shows that it is not so much the fill handle that raises problem than copying downwards formulae. The recopy becomes problematic when it puts at stake some cells references which have to be incremented. This principle of the spreadsheet functioning, which is one of its most basic interesting feature, but which has an algebraic nature (the recopied cell playing the role of variable in the formula and the spreadsheet keeping the structure of the formula during the recopy), seems not to be understood by students. Results concerning recopy are quite different whether the recopy does concern cell-variables (copying a formulae with references: 0) or not (as copying down a date: 6).

In conclusion, it seems clear that these students do not master the ability of self editing a cell-variable in a formula or the way the recopy affects the cell references.

DISCUSSION AND PERSPECTIVES

The computer test reveals difficulties of grade 12 students, not so much in surface manipulation skills, but in their lack of understanding of algebraic concepts. Using a formula in a spreadsheet requires having understood the concept of "variable" in the spreadsheet (the cell argument in the formula). Using a recopy of a formula requires seeing the increment of the references produced by the recopy as a means for the spreadsheet to preserve the algebraic structure of the formula along the copy. The

syntactic writing varies in every line but the algebraic structure is preserved. These types of knowledge were analyzed as algebraic competences which constitute a difficulty for students at the pre-algebraic level (Capponi, 2000, Haspekian 2005b).

In an exploratory study with younger students (grade 7), which consisted of a first approach to algebra through the use of spreadsheet, Haspekian (2005) found similar results. The students were asked to write, interpret or transform formulae. The observations have shown that the technique of using a formula and copying it down was the competence the longest to acquire and created most difficulties to the students. The difficulties were the following ones:

- comprehension of formulae (some remained in a use of arithmetical level of the spreadsheet);
- use of the fill handle, in particular at the beginning. But even afterward, when they experienced it several times, they had difficulty in appropriating it and its use was not systematic.

The experiment in vocational high school shows that students of grade 12 have the same difficulties as regard to these algebraic concepts embarked in the tool. It would be interesting to make paper-pencil test on their level in algebra to validate this hypothesis.

Another interesting point is the question of the modalities of spreadsheet learning. In the experiment of Haspekian (2005b), half of the students had followed a training course about spreadsheet (hands-on work) some months before the experiment. In particular they had seen formulae and recopy of formulae, and the teacher of this course had asserted that these students would have no difficulty with the tasks of the experiment. The results showed that they had the same difficulties and took the same time to answer the exercises that the other half of the students, those who had never used spreadsheet previously. Our computer test points out the same difficulties.

It is also interesting to compare with students of other professional fields or students of general fields. In *DidaTab*, another computer test has been administrated in a class of literary stream. Results show that students have less difficulty with recopy and formulae but have much more difficulties with manipulation skills. Yet in France, this stream is the general stream where spreadsheets use is the most strongly prescribed by curriculum... Certainly, as mentioned in part II, spreadsheets change too much the traditional mathematics that live in the general streams, teachers do not seriously enough take into account spreadsheet learning (not enough time devoted to spreadsheet learning, lack of structured training sessions, etc.) in these general streams, and many students are not able to manage important spreadsheet features. This result is confirmed by many interviews of students in the *DidaTab* project. Thus, our small experiment with 12th graders gives a rather different picture from general discourses about students great competencies. It seems that intrinsic difficulties of spreadsheet concepts are not sufficiently taken into account in mathematics education, even in the school streams where mathematics objectives and views are connected to every day

life. In conclusion, students of professional fields who are mostly supposed to use spreadsheet due to their school profile are unfortunately those who are precisely blocked by their difficulties in algebra, and students of general streams with a better level in mathematics are those who will not “meet” spreadsheets enough because of the specificity of their stream...

To go further, it would be interesting to deepen the research with more computer tests in different levels and settings, and try to define thus kinds of students trajectories of uses.

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DEVELOPING KATY'S ALGEBRAIC STRUCTURE SENSE

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In this paper we follow one student through a sequence of tasks and describe our observations of how her algebraic structure sense develops.

Key words: algebraic structure sense, high school algebra

INTRODUCTION

In this paper we take a close look at how one Israeli 11th grade high school student (age 16) performed during a series of teaching interviews designed to develop algebraic *structure sense*.

The term *structure sense* was coined by Linchevski and Livneh (1999). Subsequently the idea was developed and refined by Hoch and Dreyfus (2006) who arrived at the following definition.

Students are said to display structure sense for high school algebra if they can:

- Recognise a familiar structure in its simplest form.
- Deal with a compound term as a single entity, and through an appropriate substitution recognise a familiar structure in a more complex form.
- Choose appropriate manipulations to make best use of a structure.

See Hoch (2007) for a full definition and examples.

In an earlier paper (Hoch & Dreyfus, 2007) we showed how, through a simple intervention, students acquired the ability to recognise and exploit the properties of algebraic expressions possessing the structure $a^2 - b^2$. We described what is structural about $a^2 - b^2$, and showed how a student can learn to recognise structure. Hoch (2003) discussed and analysed structure in high school algebra, considering grammatical form (Esty, 1992), analogies to numerical structure (Linchevski & Livneh, 1999) and hierarchies (Sfard & Linchevski, 1994), culminating in a description of algebraic structure in terms of shape and order. In this research we took a similar approach, relating to any algebraic expression or equation as possessing structure, which has external and internal components. External components include shape and appearance. Internal components are determined by relationships and connections between quantities, operations, and other structures.

We designed a series of tasks with the aim of facilitating the improvement of structure sense. The tasks were deliberately devoid of any context other than the structural and technical, because the students had shown themselves unable to use certain algebraic techniques in different contexts, a phenomenon also noted by Wenger (1987). If a meaningful context had been chosen, then the issue of whether the students were familiar with the context and how well they understood it would have had to be considered.

The tasks were based on five structures that Israeli students meet in high school: $a^2 - b^2$; $a^2 + 2ab + b^2$; $ab + ac + ad$; $ax + b = 0$; and $ax^2 + bx + c = 0$. Hoch and Dreyfus (2006) identified students' difficulties with these structures. The creation of the tasks was based on the first author's analysis of structure sense and supported by her teaching experience. She placed emphasis on verbalising about mathematical concepts. In order to speak about a mathematical concept (or object), students must be able to deal with the result of some process without having to think about the process itself. The process is performed on a familiar object and then the result becomes another object (Sfard, 1991; Sfard & Linchevski, 1994). For example, in exercise 3 below the term $3xy$ is the result of the process of multiplying three elements. The student is required to relate to this result as an entity, in order to find its value.

In one task, the aim is to familiarise the student with equations that could be considered to have linear or quadratic structure when a product is related to as the variable. The student is presented with the following exercises in sequence:

1. Find xy : $8xy + 15 = 0$.
2. Find xy : $8x^2y^2 + 6xy - 9 = 0$.
3. Find $3xy$: $17xy - 25 = 13 + xy$.
4. Find $2xy$: $34xy - 4x^2y^2 = 10xy - 13$.
5. Find x : $17x^2 - 45 = 0$.

The student is asked to say which structure each equation possesses, to make up similar equations, and in some cases to devise efficient ways of solving them. The fifth equation is obviously quadratic, but the student is asked whether it could be considered to have a different structure if the instruction was "Find x^2 ".

In another task the student is required to describe each of the five structures listed above in words, and make up expressions or equations similar to those shown. The idea here is that the need to explain a structure in words causes the student to think more carefully about it. Gray, Pinto, Pitta, and Tall (1999) considered the use of language a powerful method of dealing with complexity. The student is asked to create expressions or equations that might be difficult for a friend to recognise. The rationale for this is that the act of creating more examples deepens the personal relationship with the structure. Rissland (1991) and others (e.g., Bills et al., 2006) said that generating examples is an important cognitive activity and that the ability to generate examples as needed is a cognitive tool of experts, often lacking in novices.

TEACHING INTERVIEWS

A series of three teaching interviews was designed, comprising tasks including the ones described above, with the purpose of improving students' structure sense. A pre-test measuring structure sense was administered to two 11th grade classes of intermediate to advanced students. Ten students who performed badly on the pre-test were chosen to participate in individual sessions of approximately 45 minutes each, over a period of up to two weeks. Throughout the sessions the researcher encouraged the students to verbalise about what they were thinking and doing, with emphasis placed on the correct naming of each algebraic entity and structure. A post-test was administered individually in a separate session a few days after the third session, and several months later a delayed post-test was administered.

All ten students displayed considerable improvements in structure sense, as measured by the immediate post-test. These improvements were maintained over time, to varying extents. We chose to report on Katy because she displayed the highest level of retention of learned abilities, and also because she was enthusiastic and highly verbal. On the pre-test Katy displayed technical skills such as opening parentheses, collecting like terms, and factoring trinomials. However her structure sense was poor—she was unable to factor an expression without first converting it into an equation and could not recognise a common factor. We will present here some excerpts from Katy's interviews. The excerpts are presented in chronological order: excerpt 1 is from the first session, excerpts 2 and 3 are from the second session, and excerpts 4 and 5 are from the third session.

EXCERPT 1: DIFFERENCE OF SQUARES

Katy displayed difficulties in factoring $49 - y^2$ as $(7 - y)(7 + y)$, and only reluctantly agreed that the expressions $x^2 - 16$ and $49 - y^2$ belong in the same structure group. When asked to give a general formula for the expressions in this group, she first suggested the formula $a^2 - b$. She observed that $49 - y^2$ confused her, "because for me the 'squared' is always plus". With a little help she arrived at the formula $a^2 - b^2$. However she was confused when asked to give a name to the structure represented by $a^2 - b^2$. The following extract is typical of students' difficulties when trying to explain mathematical concepts in words. (K = Katy; I = interviewer)

K The expression is made up of ...

I How did you decide that these belong together? [Points to $x^2 - 16$ and $49 - y^2$]. What characterises them?

K That squared minus that squared. Of the first degree.

This is an example of careless use of terminology. Earlier Katy had described linear equations as being of the first degree, yet here she assigns this name also to a quadratic expression, despite the fact that she first mentioned the squared terms.

I You called them $a^2 - b^2$.

K Ah. So ... eh ... how to give it a name?

- I Um , a description.
 K Can I call it $a^2 - b^2$?
 I Yes.
 K Is that a name?
 I No, that's a formula. You have a number squared minus a number squared. What do we call the result of a number minus a number?
 K A ratio?
 I No, that's a number divided by a number.
 K Difference?
 I That's right. So we can call this the difference of two squares.
 K Ah, I understand, the difference of two squares.

Many of the students were unable to name the result of subtraction without heavy prompting.

EXCERPT 2: COMMON FACTOR

In the pre-test Katy failed to answer any of the questions that required extracting a common factor. In the first session different types of factoring were mentioned, though not practised, including extracting a common factor. Subsequently, in the second session Katy had no problem factoring the expression $36axy - 16aby$. She was able to relate to the common factor $4ay$ as a single entity. However the expression $16x + 40xy + 50x^2$ presented her with more of a challenge. She rewrote it as $50x^2 + 40xy + 16x = 0$, and extracted a common factor to get $x(50x + 40y + 16) = 0$.

- I Why did you write “equals zero”? I don't see an equation.
 K [Scores out “equals zero”.] I can't do anything else.
 I You extracted a common factor. I don't think you extracted the greatest common factor.
 K Ah. Two. [Writes: $2x(25x + 20y + 8)$.]
 I Fine, but why did you change the order?
 K It's just simpler for me to have the x squared at the beginning.

The above extract illustrates Katy's diffidence about what she can “do” with an expression, although she knows what to do with an equation. It mirrors her performance on the pre-test. She does not, probably cannot, justify her preference for having “the x squared at the beginning” other than that she feels it is simpler. This preference was shared by other students, and perhaps reflects the manner in which textbooks and teachers present quadratic expressions. Although Katy succeeded in factoring the expression, she did not relate to $2x$ as an entity—she extracted first x , then 2.

EXCERPT 3: EQUATIONS

When it came to equations, Katy was overconfident, making some instant decisions that were not always correct. She was asked to copy each equation under its structure (quadratic or linear). Here is her response to $(2x^2 - x)^2 + 2(2x^2 - x) - 35 = 0$.

K Wow. This also doesn't belong here (pointing to $ax^2 + bx + c = 0$) but

I If it doesn't belong, don't write it there.

K No, it does belong, if we use t, where t is 2x squared

I Why?

K Because there will be x to the third.

I Yes, I agree you need to use a substitution, what will your t be?

K 2x squared.

Here followed a brief discussion about the viability of such a substitution.

K [Thinks] Then I'll get an equation with t equals x and tx squared and t squared. x to the third can be t squared.

I How would you solve such an equation?

K Eh ...

I I don't know either. Can you think of a different way?

K [Thinks]

I Continue with the idea of t.

K Oh I didn't look. 2x squared minus x is t.

Substituting t in place of a compound variable in an equation is taught in 10th grade and using it without regard for the appropriateness of the substitution is typical of many students. The fact that Katy said "I didn't look" rather than "I didn't see" suggests that she is self-reflecting and aware of what she should have done.

Katy very quickly classified $(x^2 + 3x)^2 = 2x^2 + 6x + 15$ as having structure $ax^2 + bx + c = 0$. The interviewer asked her to write down the appropriate quadratic equation.

K The quadratic equation? The equation ...

I Let's see. What will t be?

K Eh. [Writes $(x^2 + 3x)^2 = 2x^2 + 6x + 15$] To open and solve?

I How would you solve it?

K [Writes $x^4 + 6x^3$]

Eventually Katy was led to make the appropriate substitution. It seems that her original perception of the equation's structure was based on a guess, probably provoked by the fact that the term in parentheses is squared, or perhaps by looking only at the right hand side of the equation.

EXCERPT 4: NAMING A STRUCTURE

After Katy factored $(x + 3)^4 - (x - 3)^4$ correctly the interviewer pointed out that most students found that extremely difficult, and asked Katy why she thought that might be.

- K Because of the fourth power? They didn't identify ...
- I Uhm.
- K They didn't see the structure.
- I But there was this expression x to the fourth minus y to the fourth that nearly everyone succeeded in factoring. [Writes $x^4 - y^4$].
- K Because, in my eyes, it's different. Simply, that's clear [points to $x^4 - y^4$] and that's not [points to $(x + 3)^4 - (x - 3)^4$].
- I And now, with new eyes?
- K That's also clear [points to $(x + 3)^4 - (x - 3)^4$].
- I Are they different?
- K Yes, because of the words.
- I What?
- K Because in my head I see "difference of squares".

This extract clearly shows that being able to think about structure and give it a name helped Katy identify it.

EXCERPT 5: EXEMPLIFYING

Table 1 shows Katy's responses when asked to describe each structure in words and create more examples. Katy only managed to give the name of each structure (note that she said common denominator instead of common factor, a mistake made by many students) rather than a more wordy explanation. This, too, was typical of all the students. She displayed enthusiasm over the task of creating new examples, and made an effort to produce something out of the ordinary.

Table 1 Verbalising and exemplifying

Structure	Explanations	New examples
$a^2 + 2ab + b^2$	It's sum squared	1. $(3 + 2x)^2 + 6(3 + 2x) + 9$ 2. $(4x^2 + 12x + 9)^2 + 6(3 + 2x) + 9$
$a^2 - b^2$	Difference of squares	3. $z^2x^2 - 9$ 4. $x^2(3x + 2)^2 - 64$
$ab + ac + ad$	Common denominator	5. $(x + 2)y + (x^2 + 5x + 6) + (x + 2)(x + 5)$
$ax + b = 0$	Eh ... linear equation	6. $2(2x + 4)^2 - 9 = (4x^2 + 16 + 16x) + 5$ Find $(2x + 4)^2$
$ax^2 + bx + c = 0$	Quadratic equation	7. $9x^2y^2 + 6xy + 2 = 0$ 8. $9x^2y^2 + 6xy + 4 = 0$ 9. Solve for $(x^2 + 2x)^2$ $(x^2 + 2x)^4 + (3x^2 + 6x)^2 + 9 = 0$ 10. $(x^2 + 2x)^4 + 3(x^2 + 2x)^2 + 9 = 0$ 11. $(3x + 2)^6 + 9 = (3x + 2)^3$

Katy wrote example 1 and, when asked to write another one even more difficult, adapted it to get example 2, commenting, “I would never be able to solve that”. The interviewer asked her why she thought these examples might be difficult for other students.

K Because when you come to an exercise, you don’t look at the general structure, unless it is really obvious to the eye.

I Uhuh, okay.

K And because ... I wouldn’t get it. I would have to figure out how the 9 got there, in order to extract $3 + 2x$.

It seems that here Katy was talking about how she behaved before the teaching interviews.

In between writing examples 3 and 4 Katy said, “Just a minute, something more complicated? Now this was the one I really didn’t understand the most, now it seems the simplest, it’s impossible to make it more difficult.” We consider this a testimony to her structure sense development.

Katy changed example 7 into example 8 because she thought that the former had no solution while the latter had a solution. She seemed surprised to be informed that it was perfectly permissible to write a quadratic equation with no real solution. “Oh,” she laughed, “I didn’t know.” In fact she should have known, since in class she had learned to analyse quadratic equations, and in fact mentioned this kind of analysis at the end of the first session. This is an example of how Katy has compartmentalised her knowledge.

Katy corrected example 9 to example 10. She stated, “I meant this. Like $x^2 + 3x + 9$ ”.

At the end of the session the interviewer commented on how well Katy had done, and asked her if she had been practising.

K [Laughs] The penny dropped.

I How did the penny drop? Do you think you could tell me?

K I don’t know. But at least three times in class I found myself using this.

I Yes? I am very pleased.

K I said to myself, here are connections, suddenly I recognised a structure.

Katy’s self-reflection and enthusiasm were a foreshadowing of her performance in the post-tests.

POST-TESTS

In the immediate post-test, Katy answered all the items correctly. After the test she commented that she felt it had taken her too long because of, “The common factor. I don’t think about that. I will have to think about the common factor.” (Note that this time she said factor, not denominator.) When asked to account for her excellent performance:

K Do you know what helped me the most? It's the order; three different things. Everything I see I categorize. And in addition it helps – how it sounds, subtraction of squares, that's ... like ... Now that we're doing trigo, that appears a lot, a lot, a lot a lot, in identities.

I And you think of the ...?

K Today, there were three exercises, like, I work ahead with two boys, and I see that I'm three exercises ahead of them, and I stop to look what they've got stuck on, and I see that they're stuck on the subtraction of squares, and I said, but it's obvious what to do.

In the delayed post-test, several months later, Katy answered almost all the items correctly. Overall, Katy's structure sense improved considerably, and this improvement was sustained over time. Although the improvements in structure sense of the other participating students were less than that of Katy, their improvements also stood the test of time, providing evidence for the efficacy of the teaching interviews.

DISCUSSION

A close look at Katy's transcripts reveals that she displayed much typical behaviour: confusion between expression and equation, denominator and factor, ratio and difference; tendency to change the formulation of quadratic expressions; difficulty with verbalizing. She showed a clear improvement in structure sense from session to session, yet there is no instance that pinpoints the actual learning process. However, naming a structure helped her to use it, and she actually said that she succeeded "because of the words" that she sees in her head. Naming the structure is an important part of learning it – the name is part of the definition. One of the roles of a definition is to introduce a concept and convey its characterising properties. Another is to create a uniformity that allows easier communication of mathematical ideas (Borasi, 1992; Zaslavsky & Shir, 2005). A known concept or object can be given a definition by describing a few characteristic properties (De Villiers, 1998; Shir & Zaslavsky, 2001).

In conclusion, there is evidence that learning has taken place. Since there is no way of pointing to any one incident of knowledge acquirement, it can be surmised that the learning occurred as a process over time.

After the first post-test Katy said, "I think you should tell the teachers to do this with all the students. It would help them so much. Really." Of course, one-on-one intervention is not possible in a classroom situation, so the tasks would have to be adapted to make them suitable for group work, and yet enable the teacher to intervene when necessary. These tasks were designed as a form of remediation, to be used with 11th grade students who were assumed to be familiar with the algebraic structures. This raises the question whether it would be more effective if students' attention were drawn to structure at a much earlier stage, perhaps even before they practised using the formulae. Answering this question requires further research.

Further research is also required to answer other questions arising when attempting to develop students' structure sense. For example, can the teaching interviews be adapted for whole class activities? At what stage in the learning of algebra would this kind of intervention be most appropriate? Could the improved structure sense manifest itself in other subject areas, with other structures? The improvements in structure sense were maintained over a period of a few months. What would a longitudinal study show?

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CHILDREN'S UNDERSTANDINGS OF ALGEBRA 30 YEARS ON: WHAT HAS CHANGED?

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In this paper, we outline the design and method of the research project Increasing Student Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS). Phase 1 consists of a large-scale survey of attainment in algebra and multiplicative reasoning, using test items developed during the 1970s for the Concepts in Secondary Mathematics and Science (CSMS) study (Hart, 1981). This will enable a comparison of the current attainment of students aged 11-14 with that of 30 years ago. Phase 2 consists of a collaborative research study with 8 teachers extending the investigation to classroom / group settings and examining how formative assessment can be used to improve attainment. Although the focus of this paper is on reporting the research design, some early analysis of data from the initial survey data from 2008 ($n = 2400$) is reported.

INTRODUCTION

Over the past 30 years, there has been a great deal of work directed at, first, understanding children's difficulties in mathematics and, second, examining ways of tackling these difficulties. Yet, there is no clear evidence that that this work has had a significant effect in terms of improving either attainment or engagement in mathematics. Indeed, children continue to have considerable difficulties with algebra and multiplicative reasoning in particular (e.g., Brown, Brown & Bibby, 2008; Wiliam et al., 1999). In this paper, we describe the project Increasing Student Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS), a research study designed to address these problems.

ICCAMS is a 4-year research project involving a research team from King's College London and Durham University together with eight teacher-researchers from four schools. The project consists of a large-scale survey of 11-14 years olds' understandings of algebra and multiplicative reasoning in England followed by a collaborative research study with the teacher-researchers extending the investigation to classroom / group settings and examining how formative assessment can be used to improve attainment and attitudes. Although the project is in its early stages, we report some initial tentative results later in this paper. These initial results compare children's current understandings with a similar survey, the Concepts in Secondary Mathematics and Science (CSMS) study (Hart, 1981), which was conducted 30 years ago. When completed, the full results will enable us to examine what gains, if any, have been made over the intervening period. The Phase 2 findings will extend the results to children's understandings in group and classroom settings.

BACKGROUND

Mathematics education in the UK²¹ is facing a crisis; insufficient students are choosing to continue studying mathematics post-16, whilst university teachers and others point to falling standards in the subject (CBI, 2006; Smith, 2004). There is considerable research in the UK addressing reasons for non-participation in mathematics - students stop studying mathematics because they experience it as difficult, abstract, boring and irrelevant (e.g., Osborne et al., 1997). The most recent findings relating to 16 year-olds (Brown, Brown & Bibby, 2008) suggest that students' attainment and attitudes are strongly inter-related. A major factor is that even relatively successful students perceive that they have failed at the subject and lack confidence in their ability to cope with it at more advanced levels, especially in comparison to the perceived 'clever core' of fellow-students. When pressed about the reasons for their feelings of failure, students suggest that they do not understand parts of what they have been taught; this commonly relates to algebra and to aspects of multiplicative reasoning (e.g. percentages, and ratio) and its applications (e.g. in trigonometry). Students' negative attitudes commonly relate to the predominance of routine and formal work on algebra and multiplicative reasoning. Performance in these topics has been shown to be particularly weak in England relative to other countries (e.g. Mullis et al., 2004). Yet algebra and multiplicative reasoning are both essential for further study in mathematics, in science & engineering (as well as health and medicine, economics, etc.) and for mathematical literacy in the workplace and elsewhere (e.g., CBI, 2006).

The original CSMS study was conducted 30 years ago. The study made a very significant empirical and theoretical contribution to the documentation of children's understandings and misconceptions in school mathematics (e.g., Booth, 1984; Hart, 1981). In the intervening period, there have been various large-scale national initiatives directed at improving mathematics teaching and raising attainment: e.g., the National Curriculum, National Testing at age 7, 11 and 14, the National Numeracy Strategy and the Secondary Strategy²². Many of these initiatives have drawn directly on the CSMS study. During this period examination results have shown steady and substantial rises in attainment: e.g., the proportion of students achieving level 5 or above in Key Stage 3 (KS3)²³ tests has risen from 56% in 1996 to 76% in 2006 and the proportion of students achieving grade C or above at GCSE has risen from 45% in 1992 to 54% in 2006. However, independent measures of attainment suggest that that these rises may be due more to "teaching to the test" rather than to increases in genu-

²¹ This crisis in mathematics education is not confined to the UK. It is also a concern in the US and elsewhere in Europe.

²² These initiatives are particular to England. However, similar initiatives relating to testing (and accountability) and to national curricular are evident elsewhere in the world.

²³ In England, compulsory secondary school consists of two Key Stages: KS3 (11-14 years) and KS4 (14-16 years). In 2008, and for more than a decade previously, 14 year olds took a 'high stakes' test at the end of KS3, although this assessment has been abandoned for 2009 and future KS3 assessment arrangements are currently under review. GCSE (General Certificate in Secondary Education) is the examination taken at age 16, the end of compulsory schooling. Almost all 16 year olds in England take GCSE mathematics.

ine mathematical understanding. Replication results from the science strand of the CSMS study (using a test on volume and density) suggest that students' understanding of some mathematical ideas as well as the related science concepts has declined (Shayer et al., 2007). Studies at the primary level indicate that any increases in attainment due to the introduction of the National Numeracy Strategy have been at best modest (Brown, Askew, Hodgen et al., 2003; Tymms, 2004). Results from the Leverhulme Numeracy Research Programme suggest that any increase in attainment at Year 6 is followed by a reduction in attainment at Year 7 (Hodgen & Brown, 2007). Further, Williams et al. (2007) find that, following this dip at Year 7, there is a plateau in attainment across Key Stage 3.

AN ALTERNATIVE APPROACH: FORMATIVE ASSESSMENT?

National initiatives in mathematics education in England have largely focused on specifying what mathematics should be taught (e.g., the National Curriculum), how mathematics should be taught (e.g., the Secondary Strategy) and summatively assessing what mathematics has been learnt (e.g., National Tests). However, research suggests that a much more effective approach to increasing attainment and engagement would be formative and diagnostic assessment: the tailoring of teaching to students' learning needs (Black & Wiliam, 1998). In an extensive meta-analysis study Hattie (1999) found that interventions involving feedback are more effective than any other educational intervention, with an effect size of 1.13. Further, Wiliam (2007) calculates that, for the achieved effect size, the cost of formative assessment is lower than for other comparative educational interventions. Yet, whilst there has been a great deal of activity nationally and internationally in formative assessment, there is also considerable evidence that teachers have substantial difficulties implementing these ideas (Bell, 1993). These difficulties in implementation relate to three issues. First, formative assessment has largely been described generically rather than in subject-specific terms (Watson, 2006). Second, formative assessment has been poorly described theoretically and pedagogically (Black & Wiliam, 2006). Third, teachers' ability to use formative assessment in mathematics is limited by their knowledge about key ideas, and the likely patterns of progression in student learning. Thus if teachers focus on teaching mathematical procedures they may find it difficult to see what is causing problems for students in mastering and applying these, and though aware of the importance of questioning, they may not know what questions they should ask (Hodgen, 2007).

THE NEED FOR A COLLABORATIVE APPROACH TO DISSEMINATION

Much of both the research and the implementation of initiatives in these areas of mathematics have been "done to" teachers, which may in part explain the limited influence in schools. Leach et al. (2006) found that research evidence cannot simply be presented to teachers; research findings need to be "re-worked" as teaching materials. However this process of re-working, or recontextualisation, is not straightforward

(Ruthven, 2002). We hypothesise that in order for change to occur teachers must have greater insight into the problems of student understandings and attitudes, a profound understanding of fundamental mathematics (Ma, 1999), and understanding of how available resources relate to student understandings and underlying mathematical ideas. These approaches have been tried before in e.g. diagnostic teaching experiments – also based on the CSMS research - and have proven success (Bell, 1993; Swan 2006). Existing experience of collaborative research methods (e.g., Black & Wiliam, 2003) suggests that disseminating these approaches more widely and implementing them in ordinary classrooms is more likely to be successful if these approaches have been grounded in teachers' practices.

THE RESEARCH STUDY

ICCAMS is investigating engagement and achievement by focusing on the two topics at KS3 that are central to the current mathematics curriculum: algebra, and multiplicative reasoning. These topics are also fundamental to further study in mathematics and other numerate disciplines (e.g., science, engineering, economics²⁴, etc.) The study will focus on KS3, because this is where students first meet algebra and more abstract multiplicative reasoning, and where attitudes begin to deteriorate (Mullis et al., 2004). There is also evidence of a plateau in student achievement at KS3 (Williams et al., 2007).

Phase 1: The large-scale survey of algebra and multiplicative reasoning 11-14

In Phase 1, we are conducting a large-scale survey of attainment in algebra and multiplicative reasoning and attitude to mathematics, involving both cross-sectional and longitudinal elements. This will use test items first developed during the 1970s as part of the CSMS study (Hart, 1981). Based on a representative sample of schools and students in England, the survey will provide a comprehensive and detailed analysis of current student attainment in algebra and multiplicative reasoning. It will provide up-to-date information on student understandings of basic ideas in the areas of algebra and multiplicative reasoning enabling us to plot where changes have occurred since the original study. It will extend the CSMS study by linking understanding of concepts and student progression to student attitudes, to teaching, and to demographic factors. Analysis is being conducted using a variety of techniques, extending those used in the original CSMS study with Rasch and other techniques.

The full survey will consist of both cross-sectional (n=6000) and longitudinal (n=600) samples identified using the MidYIS database (Tymms & Coe, 2003). Three original CSMS tests (Ratio, Algebra, Decimals) will be administered with some additional items relating to fractions (drawn from the CSMS Fractions test) and spread-

²⁴ ICCAMS is funded by the Economic and Social Research Council in the UK as part of a wider initiative aimed at identifying ways to participation in Science, Technology, Engineering and Mathematics disciplines.

sheet items. Piloting indicated that only minor updating of language and contexts was required.

The test items range from very basic to sophisticated, allowing broad stages of attainment in each topic to be reported, but also each item, or linked group of items, is diagnostic in order to inform teachers about one aspect of student understanding.

Phase 2: The collaborative research study investigating formative assessment

In Phase 2, we are conducting a collaborative research study with teachers, which will indicate how they can best use a formative assessment focus within these curriculum areas to improve student confidence and competence, and thus participation, engagement and attainment. In this phase, we adopt a design research methodology (Cobb et al., 2003). Central to our approach will be the analysis of children's difficulties from both teaching and research perspectives.

Initially teachers will be supported in interpreting and acting upon the survey results of their students; later they will use classroom-based formative assessment based on the frameworks for learning provided by the tests, and assessment for learning approaches. They will also draw on research-informed approaches to the teaching of these curriculum areas. This study will, first, examine how teachers can make use of existing resources and initiatives to respond to students' learning needs, and, second, develop and evaluate an intervention designed to enable a wider group of teachers with much less support to do this. In the final year of the study, the approach will be implemented and evaluated with a further group of teachers and classes.

The Phase 1 findings will provide up-to-date information on student understandings of basic ideas in the areas of algebra and multiplicative reasoning to inform the teachers and teacher-researchers in Phase 2 both about their own students and about where they lie relative to the general population.

A central question for Phase 2 is how the generic approach of formative assessment can be adapted to the particular needs of mathematics teaching and learning. This will be done in several ways. First, the diagnostic results for individual students assessed against the learning and progression framework developed by CSMS will guide teachers in planning appropriate work for students and in further formative assessment. The CSMS tests were carefully designed over the 5-year project starting with diagnostic interviews in order to focus on student progression in understanding of key concepts such as variable and rational number. (See below for a fuller description of the Algebra test.) Second, we will identify and link existing teaching resources into the developmental and diagnostic learning structure provided by CSMS, building on and extending our existing work in this area which is underpinned by a combination of Piagetian and Vygotskian theories (Adhami, et al., 1995; Brown, 1992). There is extensive research evidence relating to the teaching and learning of both algebra and multiplicative reasoning that can inform this intervention (e.g. Bednarz et al., 1996; Sutherland et al., 2000; Ainley et al., 2006), but these research findings and resources

have only made a limited impact on teaching practices in classrooms. The solution lies not in designing yet another resource for the teaching of algebra and multiplicative reasoning, but in supporting the judicious use and interpretation of existing resources by teachers (Askew, 1996). Third, we will develop our existing work in this area (Hodgen & Wiliam, 2006).

THE WORK TO DATE AND EARLY ANALYSIS

In June 2008, tests were administered to a sample of around 3000 students in each of Years 7, 8 and 9²⁵. Approximately 2000 of these students took the Algebra test. The full cross-sectional sample will be completed in Summer 2009 when a further subsample of around 2000 students will be tested. We report here on the early analysis of this data. We note that these early results should be treated with caution. In particular, the current sample of students appears to be slightly higher attaining than the general population in England. This early analysis suggests that student attainment in algebra at age 14 is broadly similar to that of 30 years ago, although the patterns across the attainment range and in earlier years are more complex.

Students' understandings of letters

We now focus on just four linked items due to space constraints: 9a-d, illustrated in Figure 1. These items have been chosen to give a flavour of the test.

The CSMS algebra test was carefully designed over the 5-year project starting with diagnostic interviews. The original test consisted of 51 items. Of these 51 items, 30 were found to perform consistently across the sample and were reported in the form of a hierarchy (Booth, 1981; Küchemann, 1981). The test items range from the basic to the sophisticated allowing broad stages of attainment to be reported, but also each item, or linked group of items, is diagnostic in order to inform teachers about one aspect of student understanding. The focus of the test was on generalised arithmetic, and in particular it looked at different ways in which pronumerals can be interpreted (Collis, 1975). Items were devised to bring out these six categories (Küchemann, 1981):

Letter evaluated, Letter not used, Letter as object, Letter as specific unknown, Letter as generalised number, and Letter as variable.

The four items, 9a-d, were amongst the consistently performing items that formed part of the original hierarchy. Item 9a, at Level 1 in the hierarchy, and items, 9b and c, at Level 2, can be solved without having to operate on the letters as unknowns; the letters can be treated as objects (i.e., the name of the various sides of the figures). Items 9b and c additionally require the explicit use of some mathematical syntax. Item 9d, at Level 3, was designed to test whether students would readily 'accept the lack of closure' (Collis, 1972) of the expression $2n$, where the given letter, n , has to be treated as at least a specific unknown. The proportions of 14 year old students an-

²⁵ Key Stage 3 is made up of three academic years: Y7 (age 11-12), Y8 (age 12-13) and Y9 (age 13-14).

swering these items correctly in 1976 reflect this variation in difficulty: 94% for 9a; 68% for 9b; 64% for 9c; 38% for 9d.

The item facilities for 1976 and 2008 are presented graphically in Figure 1. This suggests that the *pattern* of progression is similar in 1976 and 2008: an initial relatively steep rise is followed by a much smaller rise subsequently. However, although the initial steep rise now appears to take place a year earlier, this initial advantage is not sustained and by age 14 students' attainment appears similar in 1976 and 2008. The results for item 9a are more of an anomaly: this relatively easy item appears to be more difficult now than in 1976.

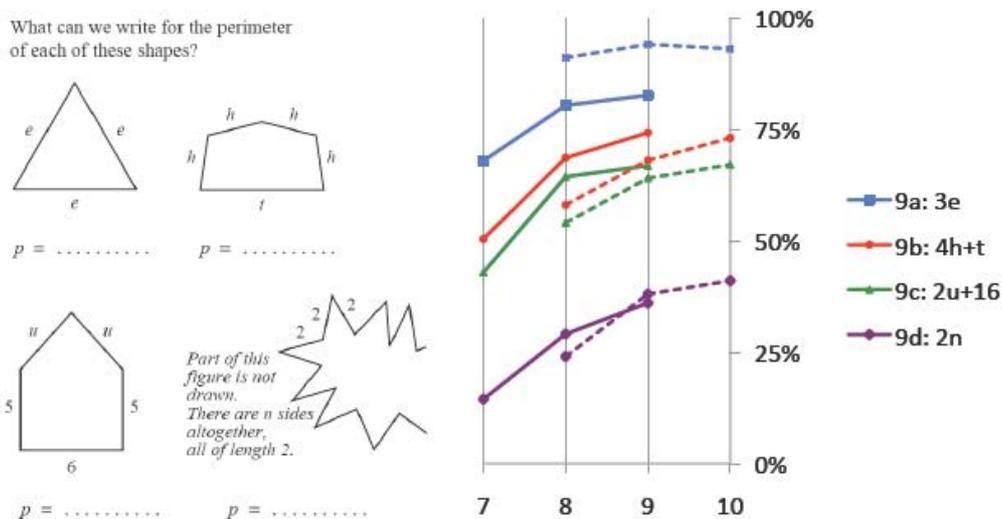


Figure 1: Items 9a-d. Facilities for items in both 2008 [continuous] and 1976 [dotted] for Year 7 to Year 10 (ages 11-14). In 2008 data were not collected for Year 10; in 1976 data were not collected for Year 7.

DISCUSSION

In comparison to 30 years ago, in England, formal algebra is taught to all students earlier. This is partly as a consequence of the introduction of a National Curriculum. The initial results of the study reported here suggest that, whilst this practice confers an initial advantage to students, this increased attainment may not be sustained. Our early analysis suggests that, by age 14, current performance in algebra is broadly similar to that of students in 1976. Moreover, it is worth noting again that the sample of students tested in 2008 is in general a relatively high attaining group. Hence, the data presented here suggest that increases in examination performance are not matched by increased conceptual understanding and, thus, add weight to the research reported earlier in this paper.

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PRESENTING EQUALITY STATEMENTS AS DIAGRAMS

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I describe a diagrammatic computer-based task designed to foster engagement with arithmetic equality statements of the forms $a+b=c$, $a+b=b+a$, and $c=a+b$. I report on six trials with pairs of 9 and 10 year old pupils, highlighting how they talked about distinctive statement forms and used these distinctions to discuss strategies when working towards the task goals. These findings stand in contrast to how pupils typically view and talk about equality statements as reported in the literature.

INTRODUCTION

The design of tasks that engage pupils with mathematical ideas in an open and exploratory manner presents a significant challenge. Constructionism offers a vision of mathematics learning in which learners explore, modify and create mathematical artefacts on a computer screen (Turkle, Papert and Harel 1991). The term “microworld” (Edwards, 1998) is often used to describe software that supports learners “discovering” mathematical rules through experimentation, mental reflection and discussion. The intention is to engage learners with mathematical ideas in a way that is meaningful to them. However, this can be difficult when the conventions of formal notation are the intended domain of learning because they are not so readily meaningful to learners. A way forward is offered by diagrammatic task designs in which learners explore, modify and create *notational* artefacts (Dörlfer 2006). This paper reports on trials with a diagrammatic computer-based task designed to engage primary children with arithmetic equality statements.

CHILDREN’S CONCEPTIONS OF EQUALITY STATEMENTS

In typical primary classrooms, arithmetical equality statements are presented and talked about as commands to work out a result. This leads most children to expect a term comprising numerals and operator signs on the left of the equals sign, and a single numerical result on the right (Behr, Erlwanger and Nichols 1976; Dickson 1989). This expectation can prove stubborn (McNeil and Alibali 2005), and lead to difficulties with equation solving (Knuth, Stephens, McNeil and Alibali 2006).

Presenting young children with a variety of statement forms leads to more flexible thinking about mathematical notation (Baroody and Ginsburg 1983; Li, Ding, Capraro and Capraro 2008). Interventionist studies have focussed on the careful selection of statements that appeal to structural readings, as in $50+50=99+1$, $7+7+9=14+9$, $246+14= _+246$ and so on (Carpenter and Levi 2000; Molina, Castro and Mason 2008; Sáenz-Ludlow and Walgamuth 1998). The intention is that pupils can notice and exploit arithmetic principles in order to assess or establish numerical balance, without the need to generate results. Such interventions produce encouraging

findings, but the long term impacts remain an open question (Dörfler 2008; Tall 2001).

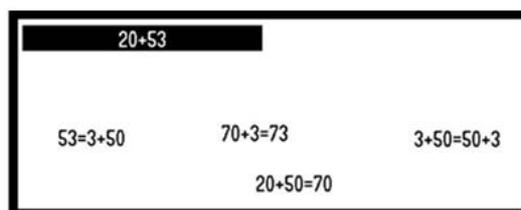


Figure 1: Screenshot from the computer-based task

A DIAGRAMMATIC APPROACH

An alternative to presenting statements as isolated questions of balance is offered by Dörfler's (2006) "diagrammatic" approach. The essence of diagrammatic notating tasks is learners manipulating conventional representations ("inscriptions") in an open, exploratory manner. This renders mathematical notating an empirical and creative activity, based in seeing potential actions (i.e. transformations). Generalisation can arise from noticing both visual patterns and patterns of repeated actions. As such, diagrammatic tasks offer learners an investigative, concrete notating activity that stimulates discussion, congruent with constructionist approaches. Note that "diagram" is being used here more loosely than everyday associations with "drawings" rather than "writings" would suggest. In another sense, however, it is more restrictive, referring only to those "inscriptions" that form precise mathematical structures with grounded rules for making transformations. From a diagrammatic perspective, arithmetic statements can be presented in parallel, forming relational systems akin to simultaneous equations (e.g. Figure 1). Numerals and their transformations, rather than numbers and arithmetic principles, are the intended "objects of the [learners'] activity" (p.100).

When pupils exploit shortcuts to establish the equivalence of presented statements they do engage in activities that are to some extent diagrammatic. Their attention is on the structural relationships of numerals, rather than computed results, and this can stimulate rich discussion (Carraher, Schliemann, Brizuela and Earnest 2006). However, such designs exclusively promote an "is the same as" meaning of the equals sign due to the task goal of establishing equivalence. There is no appeal to a "can be exchanged for" meaning, which is central to the nature of reversible equivalence relations (Collis 1975), and supports the transforming aspect of diagrammatic notating tasks.

The tasks used in the studies reported here presented pupils with sets of equality statements ("diagrams") on a computer screen. A screenshot from the task is shown in Figure 1 (an online example of the software is available at go.warwick.ac.uk/ep-edrfae/software). Each statement stands in isolation, but, as with an algebraic equation, can also combine with others in a collective, relational system. The task goal is to transform the term in the box at the top-left of the screen, $20+53$, into a single nu-

meral using the provided statements. For example, we might start by selecting $53=3+50$ and using it to transform the boxed term into $20+3+50$, then use $3 + 50 = 50 + 3$ to transform it into $20+50+3$, and so on until 73 appears in the box.

The tasks offer learners new ways to view and talk about statements. Working through notational diagrams (such as Figure 1) requires looking for matches of numerals across statements and the boxed term in order to determine where substitutions can be made, and this is quite distinct from viewing statements as isolated questions of numerical balance. Observing and predicting transformational effects ($20+53 \rightarrow 20+3+50$ and so on), when a statement is selected and visually matched notation is clicked, promotes making distinctions of statements by form. Notably, $a+b=b+a$ can be seen as commuting the inscriptions a and b ; and $c=a+b$ can be seen as partitioning the inscription c . If pupils articulate such distinctions when working towards the task goal this would stand in contrast to children's left-to-right computational readings of statements reported widely in the literature.

I report on six trials drawn from three studies. In each trial pupils were set a sequence of diagrams to solve, similar to that shown in Figure 1. These studies varied in the specific research questions addressed and the diagrams presented. The intention here is to present common and contrasting findings from across the trials (for a detailed discussion of the first two studies see Jones 2007, 2008).

METHOD

The method used was paired trialling and qualitative analysis for evidence of talking about mathematical ideas in novel ways (Noss and Hoyles 1996). Pairs of 9 and 10 year old pupils were presented with sequences of notational diagrams comprising statements of the forms $a+b=c$, $a+b=b+a$, and $c=a+b$. These began with simple diagrams comprising two or three statements of the forms $a+b=c$ and $a+b=b+a$, followed by more complicated diagrams comprising up to nine statements and including $c=a+b$ forms. Pupils were shown how to select statements and click on notation to see if a substitution occurs, and were given a few moments to get to grips with the software's functionality. I then set the task goal of transforming the boxed term into a numeral, and remained present to offer encouragement and ask for verbal elaborations ("what do you think?", "how did you know that would work?", and so on). Each trial lasted around 30 to 40 minutes.

Data were captured as audiovisual movies of the pupils' onscreen interactions and discussion. Data were transcribed and analysed using *Transana* (Woods and Fassnacht 2007). Occurrences of pupils computing results, looking for numeral matches and articulating the distinctive transformational effects of statement forms ("swap", "split" and so on) were coded. A trace of each trial was constructed to examine how such articulations arose, and how they were used by pupils in order to discuss strategies when working through the diagrams.

The six trials reported will be referred to as Trial A through to Trial F. The pupils in trials A to C were deemed mathematically able by their class teachers, and the pupils in trials D to F were deemed average. The trials can usefully be grouped as A, B, C and D, E, F in terms of the extent to which pupils (i) articulated distinct statement forms, and (ii) used these distinctions to work strategically with the diagrams.

FINDINGS

The data are presented here to illustrate the similarities across all trials, and the differences across trials A to C and D to F. I present a visual overview of the six trials, and offer illustrative transcript excerpts.

Visual overview

Figure 2 shows a time-sequenced map of codings across the six trials and was

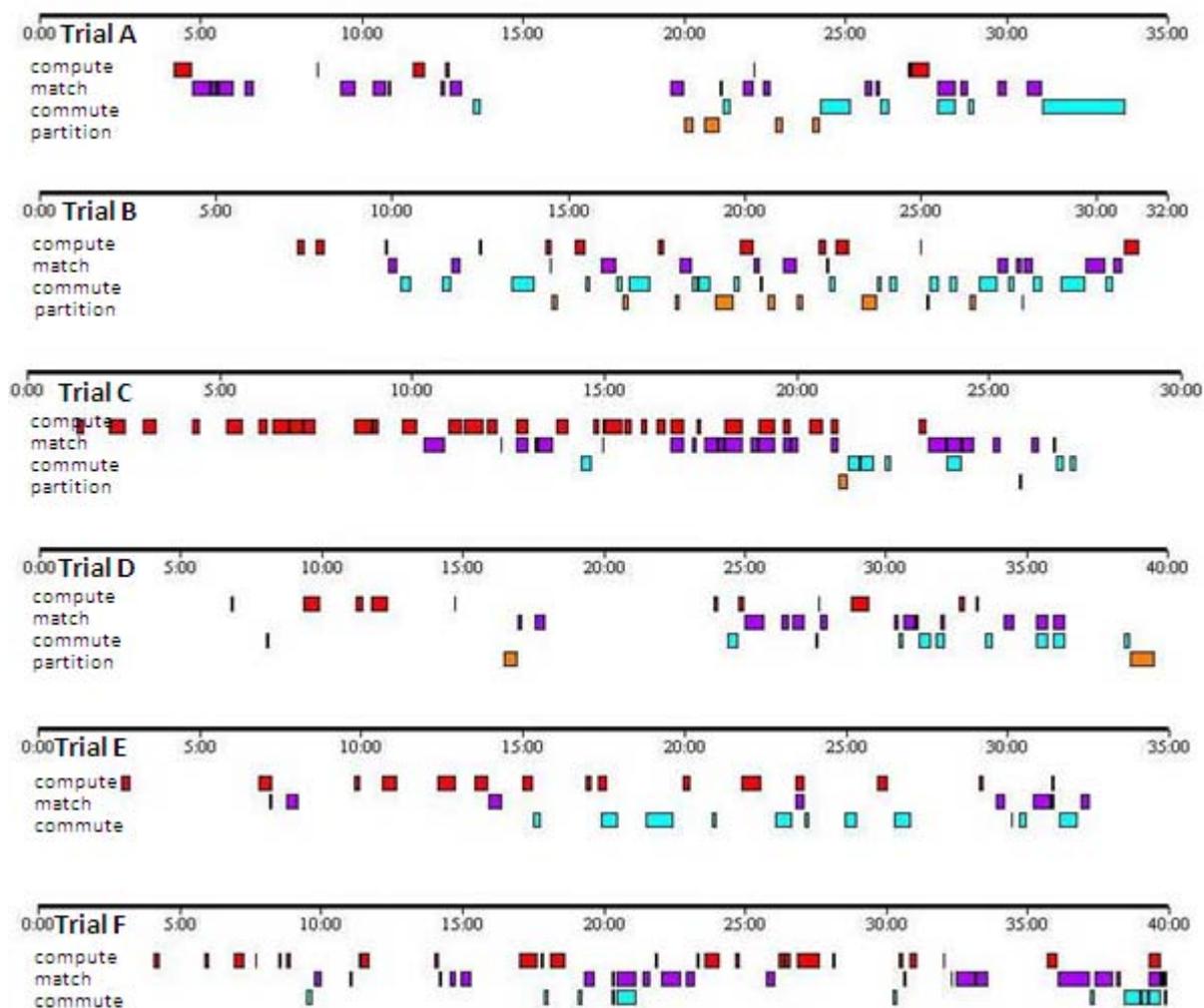


Figure 2: Time-sequenced coding of the data for computing results, looking for matches of numerals, and articulating commuting and partitioning transformations.

produced using *Transana*. Each block shows an occurrence of pupils computing results, looking for matches of numerals or terms across statements and the boxed term (Figure 1), or articulating the distinctive commuting (“swapping”) or partitioning

(“splitting”) transformational effects of presented statements. The length of each block is somewhat arbitrary. For example, one block of (say) “commute” might reflect pupils working in a trial-and-error manner with one of them suggesting they “swap” numerals, but offering no reason. Another block of similar length might reflect pupils discussing which numerals to commute, and how and why, as part of a shared strategy. As such, Figure 2 provides a useful visual aid for summarising the trials, but does not convey the quality, or the precise quantity, of the pupils’ articulations and strategising. Non-coded segments are those times when either I was speaking, or pupils’ discussion was ambiguous (“Click that one”, “Let’s try this one, no, that one” and so on).

The first thing to note is how little the pupils computed results across the trials (with the exception of Trial C, in which the notably enthusiastic pupils appeared keen to impress me with their computational prowess). Conversely, the pupils did engage in looking for matching numerals, and articulating the commuting properties of $a+b=b+a$ statements. Figure 2 shows that “compute” was prominent in the first ten minutes of each trial (bar Trial A), but was less present than the other codes in the final ten minutes. This reflects how most pupils began by computing results, as would be expected, but changed, sooner or later, to more diagrammatic views.

“Partition” is less prominent across the trials, and does not appear at all in trials E and F. The pupils in trials A to C came, sooner or later, to articulate partitioning transformations as part of their shared strategy for achieving the task goal. After a little practice, they would generally begin a new diagram by identifying partitioning statements, then using commuting statements to shunt the numerals in order to compose them. However, the pupils in Trials D to F rarely articulated partition if at all, and did not use it strategically, instead relying on a less efficient approach characterised by trial-and-error statement selection. It seems, then, that articulating partition is key to strategic discussions when working collaboratively with the diagrams.

Illustrative transcript excerpts

Early on in the trials, after the pupils had been introduced to the software’s functionalities, they articulated computational readings of statements. The following is from Trial E:

- John: 9 add 12 add 1 equals 22.
 Derek: 21.
 John: No it’s 22. 13 add 9.
 Derek: Hm, no 9 add 12. 9, 13 add 12. No, 13 ...
 John: 12 add 1 is ...
 Derek: Yeah 22 because it's 9 add 12 add 1 is 22

Searching for matches of numerals arose across all the trials as the pupils discussed why the software sometimes allowed a selected statement to make a substitution and other times did not. Often they looked for matches of single numerals, rather than terms. The following is from Trial C:

Barbara: 31 plus 19.

Nadine: 19. What's that?

Barbara: 31 ... look for a 31 somewhere.

Nadine: Well I found a 19 and another 19.

Barbara: But we need something that will equal 19. Aha, I found a 31.

At other times pupils attempted near matches, such as trying to use $5+18=23$ to transform $5+8+18$ (Trial C). However, often these near matches were attempted doubtfully when pupils were momentarily stuck, and, overall, they showed greater confidence when attempting exact matches. With prompting, the pupils were often able to explain why a given substitution did not work. From Trial A:

Researcher: Why do you think that wasn't working?

Terry: Maybe because ... 1 and 9 is ...

Arthur: Oh, because it hasn't got that sum in it.

Researcher: What do you mean?

Arthur: Well, because that's got 1 add 9 but then the end of that's got 9 add 1.

Pupils across all the trials readily came to articulate the observed or predicted transformational effects of $a+b=b+a$ statements as "swapping" or "switching" or "changing round". Some pupils did not initially see that this could be helpful for achieving the task goal. For example, when the pupils in Trial F used $31+35=35+31$ to transform $31+35+8 \rightarrow 35+31+8$ they commented:

Colin: That just swapped it.

Imogen: Swapped it around.

However, most pupils came to see a use for commuting numerals sooner or later, as articulated by John (Trial E) when prompted to explain why $16+32=48$ would not transform $13+32+16$:

Researcher: It's not working. Why not?

John: Because we haven't got a 13 yet.

Derek: Yeah we have look.

John: No, in these.

Derek: No.

John: It equals 48. But there is 48 in some things. Yeah, there is in this one.

Researcher: That's not actually the reason. It's not because of that 13.

John: Hm. [Doubtfully] Is it because we went wrong on one of these?

Researcher: No, no.

John: Is it because it's the wrong way round? The 16 and the 32?

Researcher: Is there anything you could do about that?

John: Oh yes, yeah, yeah, yeah, yeah. I thought this was useless but now it's useful. These bits. Okay. Right, now we've just changed it round. Now try. There we go. Now, 13 add 48. Now that one.

All pupils, to a greater or lesser degree, came to articulate potential commutations one or more steps ahead in order to use further statements to make transformations. From trial B:

Yuri: If we can swap them two around.

Linda: Yeah.

Yuri: And swap them with the 33 so we can get the 50 and 11. Go on, that one.

Linda: Huh?

Yuri: That one. Now swap them two around. Now you can get 50 add 11.

At times, some pupils commented on the physical appearance of the boxed term when transformed by $c=a+b$ forms. From Trial C:

Barbara: Now change the 53 into 41 plus 12.

Nadine: Okay now it's a big sum.

However, partition was explicitly articulated only in trials A to D. For example, in Trial A, when the pupils first encountered a diagram containing the form $c=a+b$, Terry inferred its transformational effect, and its use for achieving the task goal:

Terry: Oh! That's the one that you do first! It has to be.

Researcher: Why?

Terry: Because it's splitting up the 40 and the 1.

In trials A to C, the pupils adopted a strategy of starting with $c=a+b$ forms to partition the numerals in the box, then using $a+b=b+a$ and $a+b=c$ forms to commute and compose the term into a single numeral. From trial B:

Yuri: Try splitting the 37 first. Um, you have to click on that. No, hit [i.e. click] all the numbers ...

Linda: 29 add 8.

Yuri: So, 73. 29 add 73 that said so, split, no wait. How do you get that for... Unless you got to switch them two around. So it's...

Linda: Which two around?

Yuri: Them two. Go, go on that. Now switch them two. Now you got that one.

Linda: 29.

However, in trials D to F, this start-with-partitioning strategy was not discovered or adopted by the pupils. They relied on trial-and-error when selecting statements to a greater extent than the pupils in trials A to C. The following example is from Trial D:

Zoë: Try that on the other one.

Kitty: No, it's just swapped them.

Zoë: Shall we try swapping and then we can try ...

Kitty: What shall we try?

Zoë: That one.

Researcher: Why that one Zoë?

Zoë: I don't know.

The contrast across trials was most marked in the later stages when the diagrams are more complicated and so strategic approaches are significantly more efficient.

DISCUSSION AND FURTHER WORK

The data show that the presentation of equality statements as transformational rules enables pupils to explore and talk about arithmetic notation in non-computational ways. Left-to-right readings of individual statements, as widely reported in the literature, are replaced by looking for matches of numerals across statements and terms. The task offered pupils a *utility* (Ainley, Pratt and Hansen 2006) for equality statements, namely making substitutions of notation towards a specified task goal. This utility arose because statements were presented as reusable rules for diagrammatic activity rather than isolated questions of numerical balance.

All the pupils distinguished the commuting transformational effects of $a+b=b+a$ forms, and used this distinction to discuss possible transformations one or two steps ahead. Only half the pupils distinguished the partitioning transformational effects of $c=a+b$ forms, and these pupils were able to use this distinction as part of a strategy that proved advantageous for later, more complicated diagrams.

When the pupils articulated commuting and partitioning effects this does not mean they had a conception of the underlying arithmetic principles. Baroody and Gannon (1984) found that young children can appear to exploit commutation to reduce computational burden, but are often merely indifferent to consistency of outcome. Trial B came from a study in which the last few diagrams contained some false statements, such as $77=11+33$, and the value of the boxed term was not conserved across transformations. Interestingly, the pupils did not comment on this, and when asked afterwards if diagrams had contained false statements were unable to say (Jones, 2008).

This suggests pupils do not coordinate ‘sameness’ and ‘exchanging’ meanings for the equals sign when working with the task.

Current work is exploring how these two meanings for the equals sign might be coordinated using a constructionist approach to task design. Trials C, E and F are from a study in which the pupils subsequently went on to make their own diagrams using provided keypad tools. This requires ensuring numerical balance when inputting statements, and testing that these statements can be used to make substitutions when placing them into a diagram. A second aim of this current work is to find out whether pupils can translate verbalised calculations into notational diagrams. These calculations usually contain implicit partitioning and commuting (as in “ $34+23$. 3 plus 4 is 7, and 30 plus 20 is 50, and 50 add 7 is 57”), which learners must identify and make explicit as statements on the screen in order to achieve the task goals. Early analysis suggests that again articulating partition is key to success.

A future aim, then, is to explore how the selection and sequencing of arithmetic diagrams can help all pupils to notice and articulate partitioning effects.

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APPROACHING FUNCTIONS VIA MULTIPLE REPRESENTATIONS: A TEACHING EXPERIMENT WITH CASYOPEE

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Abstract: *Casyopée is an evolving project focusing on the development of both software and classroom situations to teach algebra and analysis at upper secondary level. This paper draws on our current research in the ReMath European project focusing on the approach to functions via multiple representations. In this paper, we present the design of an experimental teaching unit for the 11th grade and some preliminary results.*

INTRODUCTION

The notion of function plays a central role in mathematics and for many authors technology can help students to learn about this notion especially because of the representational capabilities of digital environments. Recently, authors extended the range of representations by considering functional dependencies in a non symbolic domain. Falcade and al. (2007) proposed for instance to use Dynamic Geometry as an environment providing a qualitative experience of covariation and of functional dependency in geometry.

An aim of our team in the ReMath project is to develop a teaching unit taking advantage of a wealth of representations of functions offered by technology. In this aim, our software environment - Casyopée - has been extended, adding to the existing symbolic window a geometrical window with strong connections between them. Casyopée's symbolic window is a computer environment for upper secondary students. The fundamental objects in this window are functions, defined by their expressions and domain of definition. Other objects are parameters and values of the variable. Casyopée allows students to work with the usual operations on functions like: algebraic manipulations (factoring and developing expressions, solving equations ...); analytic calculations (differentiating and integrating functions); graphical representations; supports for proof The new window offers the usual dynamic geometry capabilities, like defining fixed and free geometrical objects (points, lines, circles, curves) and constructing others. It also offers distinctive features: geometrical objects can depend on algebraic objects and it is possible to export geometrical dependencies into the symbolic window, in order to build algebraic models of geometrical situations (Lagrange & Chiappini, 2007).

SOLVING A PROBLEM OF FUNCTIONAL DEPENDENCY WITH

CASYOPEE

In order to explain this extension, we expose now the type of problem whose resolution can take advantage of Casyopée, and how. This is an example:

Consider a triangle ABC. Find a rectangle MNPQ with M on [oA], N on [AB], P on [BC], Q on [oC] and with the maximum area

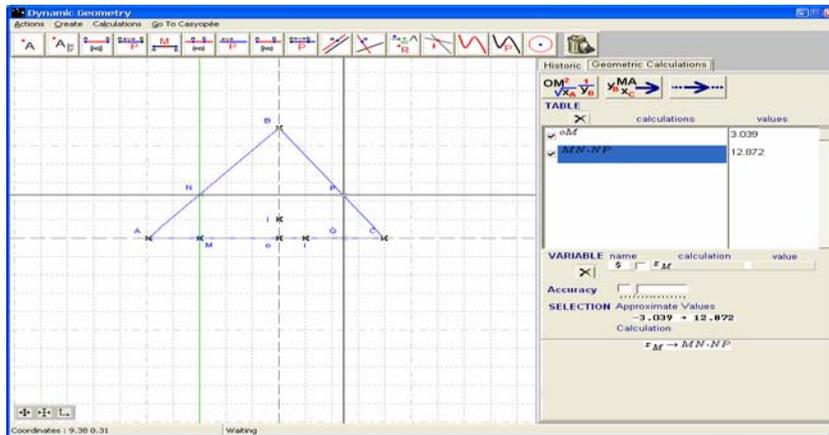


Fig. 1: The geometrical window of Casyopée

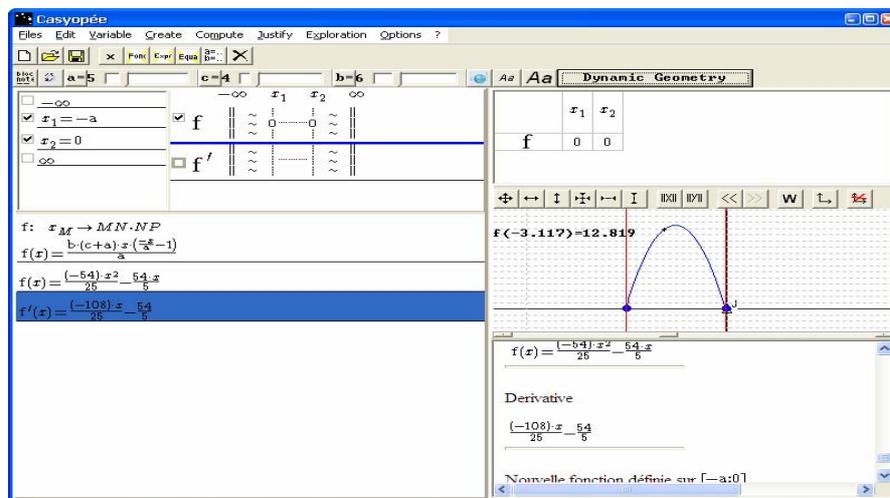


Fig. 2: The symbolic window of Casyopée

Constructing a generic triangle ABC in the geometrical window can be done after creating parameters in the symbolic window. For instance, the points can be $A(-a;0)$, $B(0;b)$ and $C(c;0)$, a , b and c being three parameters. Then one can create a free point M on the segment $[oA]$ (o being the origin) and the rectangle can be constructed using dynamic geometry capabilities.

In the Geometric Calculation tab (Fig.1) one can create a calculation for the area of the rectangle MNPQ and then define an independent variable. Numerical values of

calculations and of the variable are displayed dynamically when the user moves free points. The user can then explore the co-dependency between these values. If this co-dependency is functional (i.e., the calculation depends properly on the variable) it can be exported into the symbolic window and Casyopée automatically computes the domain and the algebraic expression of the resulting function. Otherwise, Casyopée gives adequate feedback.

After exporting into the symbolic window, one can work on various algebraic expressions of the function and on graphs. For instance, one can use properties of parabolas, or algebraic transformations or Casyopée's functionality of derivate to find the answer to the question. One can also use the graph of the function to conjecture about the area maximum.

QUESTIONS AND THEORETICAL FRAMEWORKS

As the above example shows, Casyopée offers very varied functionalities and representation of functions:

- means for creating generic dynamic figures,
- geometrical calculations to express a range of quantities that can be considered as dependant variables,
- possibilities of choosing an independent variable like a distance or an abscissa involving free points,... feedbacks about this choice of a variable,
- means to observe numerical covariation between points and calculations, or between an independent variable and a calculation,
- means to export a functional dependency between the chosen variable and a calculation to the symbolic window, resulting in an algebraic form of the function,
- means for treating this algebraic form in various registers.

The overarching question addressed by the Casyopée team is: how to exploit these varied functionalities of representation in order to develop students' understanding of a functional dependency, particularly by articulating a geometrical situation with its algebraic model?

To investigate this question, we built an experimental teaching unit at 11th grade. In this paper, we present first the frameworks that helped us to build this experiment and to interpret our observations. Then we present the experiment and we report on the observation of the last session where students used the wider range of representations.

The first framework is based upon the notion of "setting" introduced by Douady (1986). According to Douady, a *setting* is constituted of objects from a branch of

mathematics, of relationship between these objects, their various expressions and the mental images associated with. When students solve a problem, they can consider this problem in different settings. Switching from a *setting to another* is important in order that students progress and that their conceptions evolve. Students can operate these *changes of setting* spontaneously or they can be helped by the teacher. The setting distinguished here are geometry and algebra,

We also rely upon the notion of *registers of representations* from Duval (1993). Duval stresses that a mathematical object is generally perceived and treated in several registers of representation. He distinguishes two types of transformations of semiotic representations: *treatments* and *conversions*. A treatment is an internal transformation inside a register. A conversion is a transformation of representation that consists of changing of a register of representation, without changing the objects being denoted. It is important that students recognize the same mathematical objects in different registers and they get able to perform both treatments and conversions.

Here we distinguish the geometric and the algebraic settings corresponding to Casyopée's two main windows. In these two settings, the functions modeling a dependency are different objects: a relationship between geometric objects or measures in the geometric setting, and an algebraic form involving a domain and an expression in the algebraic window. In the above problem, students have to switch from the geometric to the algebraic settings and back, to be able to use symbolic means for solving questions that were formulated in the geometric setting. As explained by Lagrange & Chiappini (2007), we expect that, working in the geometric setting, students would understand the problem and the objects involved, and that after switching to algebra, this understanding would help them to make sense of the objects and treatments in the algebraic setting.

Inside each of these two settings the functions can be expressed in several registers. In geometry, especially with dynamic geometry, functions can be represented and explored in different registers: covariations between points and measures, or between measures, or functional dependency between measures. In algebra, functions can be expressed and treated symbolically, by their expressions, by way of graphs and of numerical tables. Mastering these expressions and treatments, and flexibly changing of register are important for students' ability to handle functions and acquire knowledge about this notion.

A third framework is the *instrumental approach*, based on the distinction between artefact and instrument. An artefact is a product of human activity, designed for specific activities. For a given individual, the artefact does not have an instrumental value in itself. It becomes an instrument through a process, called *instrumental genesis*, involving the construction of personal schemes or the appropriation of social pre-existing schemes. Thus, an instrument consists of a part of an artefact and of some

psychological components. The instrumental genesis is a complex process; it requires time and depends on characteristics of artefacts (potentialities and constraints) and on the activities of the subject (Vérillon & Rabardel, 1995).

In the case of an instrument to do or learn mathematics like Casyopée, the instrumental genesis involves interwoven knowledge in mathematics and about the artefact's functionalities. Artigue (2002) showed how this genesis can be complex, even in the case of simple task like framing a function in the graph window. More generally, the many powerful functionalities of CAS tools have a counterpart in the complexity of the associated instrumental genesis (Guin & Trouche, 1999). We are then aware that we must take care of students' genesis when bringing Casyopée into a classroom. Moreover, Casyopée offers a multiplicity of representations in two settings and in several registers. Understanding and handling these representations involves varied mathematical knowledge. Students have then to be progressively introduced to these representations, taking into account the development of their mathematical knowledge.

Constructing the sessions of the experiment, we also used the Theory of Didactical Situations as basis for designing tasks. According to this theory, learning happens by means of a continuous interaction between a subject and a milieu in an *a-didactical situation*. Each action of the subject in milieu is followed by a retro-action (feedback) of the milieu itself, and learning happens through an adaptation of the subject to the milieu. Thus, with regard to Casyopée use, learning does not depend only on the representational capabilities of this software, but also on tasks and on the way they are framed by the teacher. Within this perspective, we looked for situations in which students interact with Casyopée and receive relevant feedbacks. For example, to solve the above problem, students have to choose between different independent variables to explore functional dependencies in the geometrical window and to export a dependency into the algebraic window. In case the variable is inadequate, the feedback they receive is a message from Casyopée. In other cases, the algebraic expression automatically produced by Casyopée can be more or less complex, which is another feedback: too complex expressions have to be avoided in order to ease the subsequent algebraic work.

Concerning the methodology, we use *didactical engineering* (Artigue, 1989), a method in didactic of mathematics, to organize and evaluate the experimental teaching unit, and to answer the research questions. The treatments and interpretations of collected data based on an internal validation which consists in confronting *a priori* analysis of the situation with *a posteriori* analysis. This method produces an ensemble of structured teaching situations in which conditions for provoking students' learning have been planned.

THE EXPERIMENT

Our experimental teaching unit consisted of six sessions. It was experimented in two French 11th grade classes. It was organized in three parts. Consistent with our sensitivity to students' instrumental genesis, each part was designed in order that students learn about mathematical notions while getting acquainted with Casyopée's associated capabilities:

- ❖ The first part (3 sessions) focused on capabilities of Casyopée's symbolic window and on quadratic functions. The aim was that students became familiar with parameter manipulation to investigate algebraic representations of family of functions, while understanding that a quadratic function can have several expressions and the meaning of coefficients in these expressions. The central task was a "target function game": finding the expression of a given form for an unknown function by animating parameters.
- ❖ The second part (two sessions) aimed first to consolidate students' knowledge on geometrical situations and to introduce them to the geometrical window's capabilities. The central task was to build geometric calculations to express areas and to choose relevant independent variables to express dependencies between a free point and the areas. It aimed also to introduce student to coordinating representations in both algebraic and geometrical settings, by way of problems involving areas that could be solved by exporting a function and solving an equation in the symbolic window.
- ❖ Finally, in the third part (one session) of the experimental unit, students had to take advantage of all features of Casyopée and to activate all their algebraic knowledge for solving the optimization problem presented above.

Below, we give some insight on how we are currently exploiting this experiment with regard to our question about Casyopée' potential for multi-representation. We limit ourselves to the final session for which the problem and the students' instrumental genesis should allow to take full advantage of this potential. We draw some elements of a priori analysis of this session and we compare with the a posteriori analysis of the functioning of a two student team.

THE SITUATION IN THE FINAL SESSION: ELEMENTS OF A *PRIORI* ANALYSIS

Tasks

The problem is presented by the teacher by animating a figure in Casyopée's geometrical window:

Let a , b and c be three positive parameters. We consider the points $A(-a;0)$, $B(0;b)$ and $C(c;0)$. We construct the rectangle $MNPQ$ with M on $[oA]$, N on $[AB]$, P on $[BC]$ and Q on $[oC]$. Can we build a rectangle $MNPQ$ with the maximum area?

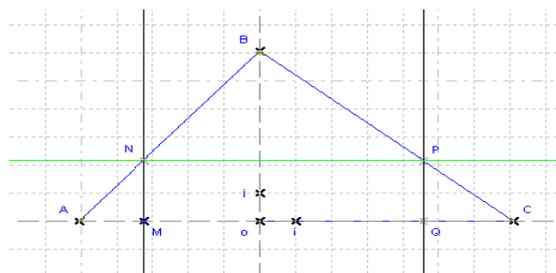


Fig. 3: The figure built in Casyopée

The tasks proposed to students are then:

- The construction of the rectangle MNPQ: students are required to load a Casyopée file with the parameters' definition and the triangle, then to complete the figure by building the segments $[oA]$, $[AB]$, $[BC]$ and $[oC]$ and to create the free point M and the rectangle's vertexes.
- To create a geometrical calculation for the area of the rectangle MNPQ: this can be obtained by the product of the lengths of two adjacent sides, e.g. $MN \times MQ$
- To explore the situation by moving the point M on the segment $[oA]$.
- To prove the conjecture by algebraic means.

The teacher also asks students to write the proof, indicating their choice of variable and using results displayed by Casyopée. Finally, students are expected to visualize the answer in the geometrical window.

Covariations and representation of functional dependencies

This situation involves two settings and different registers. Students can conjecture the answer to the question by exploring numerical values of the area in the geometrical setting. They can explore the variation of the area in different ways corresponding to different registers of representation. First, they can observe co variation between the point M and the area, looking at the values of the calculation they created for the area of the rectangle, noting that when M moves from A to B the value grows then decreases, with a maximum value when M is the middle of $[oA]$. They can also observe co variation between a measure involving the free point M and the area. For instance, they can observe together the values of the distance oM and of the area. Finally, they can choose an independent variable involving M and observe the functional dependency between this variable and the area.

In the algebraic setting students can apply different algebraic techniques to the algebraic form of the function in order to find a proof. Exporting a function with Casyopée, one obtains a more or less complex algebraic expression reflecting the calcula-

tion's structure. Students then need to expand this expression to recognize a quadratic function. They can then apply their knowledge about these functions to prove the maximum. It is possibly not easy for them, because of the three parameter involved.

They can also use the graphical representation in this algebraic setting to explore the curve, complementing the exploration they did in the geometrical setting: the parabola is familiar to the students and they can easily recognize a maximum.

The situation is partly a-didactical. In each setting, students interact freely with Casyopée and use the feedbacks to understand the situation. Nevertheless, some key points like passing from a co variation to a functional dependency are expected to be difficult for students, although the corresponding action (choosing an independent variable) has been presented in the preceding sessions. Passing from one setting to the other is expected to be far from obvious for students. The corresponding actions in Casyopée (exporting a function in the symbolic window, interpreting a symbolic value in terms of position of a point) have also been presented before, but it is the first time that students have to do it by themselves.

Students can choose their own independent variable between possible choices (oM , x_M , MN , $MQ\dots$) with consequences upon the algebraic expression of function. They can do it alone but it is expected that the teacher mediation will be necessary. It is also possible that they will want to change their choice of a variable in order to obtain a simpler algebraic expression of the function.

We expect a great variety of uses of representations reflecting students' free interactions with the situation. Some students can stay a long time exploring co variations and need teacher mediation to go to functional dependency while others pass more or less quickly to the algebraic setting to consider the function. In this setting, some can prefer to explore graphs, while others prefer working on algebraic expressions. It is possible that some students find too difficult to apply algebraic techniques to the general expression (i.e. with parameters) and prefer to work by replacing these parameters by numbers. In any case, we expect that students will consider several representations, make sense of them and make links between them.

ELEMENTS OF A *POSTERIORI* ANALYSIS: THE CASE OF A TEAM

During the experiment, we observed selected teams of students. In this paper, we focus on a team of two students, which according to the observation in the first five sessions had a favorable instrumental genesis. According to their teacher they were good students.

The explorations in different settings and registers

Creating a geometrical calculation for the area of the rectangle, they typed $MNxMP$ instead of $MNxMQ$ by mistake. They moved M and observed growing numerical

values of this calculation, while, for some positions of M the area was visibly decreasing. This first feedback allowed them to correct the geometrical calculation.

Like most students they had difficulties in choosing an appropriate independent variable, confusing the independent variable and the calculation. They needed help from the teacher to activate the correct button. They chose at first NP . They moved for a long time the point M and observed how numerical values of this variable and of the area $MN \times MQ$ changed. They found an optimal value and interpreted it: "(the optimum) is when N is the midpoint of $[AB]$ I believe, and P is the midpoint of $[BC]$ ". The teacher asked them for a proof. A student suggested an equation in an interrogative tone. Actually, the problems solved in sessions 4 and 5 were about equalities of areas and have been solved by way of an equation.

The teacher guided them to export the function, but they found the resulting expression too complicated. Then they choose another independent variable MQ , and got the same expression after exporting again the function. Finally, they chose x_M as an independent variable, obtained the algebraic expression $b(x-1/ax)(a+c-a(x-1/ax))-c(x-1/ax)$ and expanded it into a quadratic polynomial.

Proving the maximum

The team graphed the function, recognized a parabola, and said that they do not know how to determine the maximum's x -coordinate. Then they wanted to apply an algebraic formula to get this x -coordinate and used Casyopée to expand the expression. For some reason they got a non parametric expanded expression, the parameters being instantiated. Then it was easy for them to obtain by paper/pencil a numerical value of the maximum's x -coordinate. Then they returned to the geometrical window, checked this result and generalized, saying that the maximum is for $x_M=a/2$.

They did not attempt to prove this generalized property by working on the parametric expression and then they only partially solved the problem. Other teams did, but had much difficulty to apply the formula to the parametric quadratic expression.

SYNTHESIS

The observation reported above is globally consistent with the a priori analysis. The students used more or less all registers of representation. The independent variable was recognized as the central feature of the solution, allowing connections between registers. Casyopée offered means for exploration and various feedbacks that helped this recognition. The students' instrumental genesis helped them globally to interact with Casyopée, but important actions like choosing a variable and exporting a function were still unfamiliar. They were influenced by the problems they solved before and it was difficult for them to have a clear approach of an optimization problem. Although they used parameters before and they understood the generalized problem, using parametric expressions was still difficult.

With regard to our question on how to exploit Casyopée's varied functionalities of representation, we can say that, in spite of remaining difficulties, the teaching experiment helped this team to develop an understanding of a functional dependency. We have of course not now a more definite conclusion and we are currently analysing the other teams' observation as well as productions after the experiment. We are especially sensible to the teacher's help to students. In the above observation, we saw this help in crucial episodes, like changing settings and we want to know whether this help was efficient for students' learning, beyond the solution of the problem.

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EQUALITY RELATION AND STRUCTURAL PROPERTIES [1]

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We present the results of a questionnaire on equality we administrated to a large and vertical sample of Italian students. Some of the questions were devised to investigate the presence of relational thinking.

INTRODUCTION – THE SCENARIO OF THE RESEARCH

This paper emanated from an international study of arithmetical misconceptions in primary schools (Cockburn & Littler, 2008) part of which considered equality (Parslow-Williams & Cockburn, 2008). One way to detect whether a wrong answer can be attributed to a misconception or a slip (Schlöglmann, 2007), is to analyse the persistence of the same wrong answer through a range of school grades. Here we focus on a questionnaire on equality administered to 1,147 Italian seven to sixteen and a group of university students in their first year. (cf. table 1 below).

THEORETICAL FRAMEWORK AND THE AIM OF RESEARCH

It has been well documented that an understanding of equality is crucial to the development of algebraic thinking (Alexandrou-Leonidou & Philippou, 2007; Attorps & Tossavainen, 2007; Puig, Ainley, Arcavi & Bagni, 2007).). Here we focus on formal number sentences, building on the work of Molina, Castro & Mason (2007) and, in particular, *relational thinking* – a term that Molina *et al.* (2007) borrow from Carpenter, Franke & Levi (2003). The student employs *relational thinking* if s/he

“makes use of relations between the elements in the sentence and relations which constitute the structure of arithmetic. Students who solved number sentences by using relational thinking (RT) employ their number sense and what Slavit (1999) called “operation sense” to consider arithmetic expressions from a structural perspective rather than simply a procedural one. When using relational thinking, sentences are considered as wholes instead of as processes to carry out step by step.” (Molina *et al.*, 2007, p. 925)

The term *relational thinking* here is the opposite of procedural thinking. Although it sounds similar to Skemp’s (1976) *relational understanding*, i.e. “knowing what to do and why” (Skemp, 1976, p. 21), in this context it focuses on different aspects of learning. In our opinion *relational thinking* is very similar to *relational interpretation* of equality detected by Alexandrou-Leonidou & Philippou (2007) and very closely related to *conceptual knowledge*, as proposed by Attorps & Tossavainen (2007) as opposed to *procedural knowledge*. The latter adopted the framework of Sfard (1991) and focused on the mathematical properties of the equality relation, i.e. reflexivity, symmetry and transitivity and, using a sample of 10 qualified and 75 pre-service secondary mathematics teachers, concluded that a lack of understanding of these proper-

ties impairs the development of the concept of equation. In Italy the structural approaches to arithmetic and algebra, together with equations, are usually introduced in grade 9. Early structural approaches and equations are, however, in the curricula for grades 6, 7 and 8. In the light of the above, this study investigated

- whether there was evidence of relational thinking in grades 2 - 5;
- how the structural notions taught of pupils in grades 6 - 11 influenced the responses;
- misconceptions about aspects surrounding equality amongst the students.

METHODOLOGY

The questionnaire

All pupils were given a written questionnaire containing a series of equality problems. Our questionnaire comprised simple number sentences using similar questions and symbols to those found in the literature (cf. Radford (2000), Hejný & Slezáková (2007), and Behr, Erlwanger & Nichols (1980)).

Zan (2000) suggested that misconceptions may exist in a sort of ‘grey’ zone beneath the complete consciousness of the person. Our questions were intended therefore to be sensitive enough to reveal misconceptions and relational thinking without being too direct, since this can make the subjects aware of their errors, resulting in an immediate correction before they commit themselves to writing an answer.

We decided to avoid the issue of having both signs ‘+’ and ‘-’, in the same calculation, as an awareness of both algorithms was required to find the solution. The questionnaire was four pages in length [2]: 2a and 2s presented addition and subtraction problems respectively, using mainly single digit numbers; in 3a and 3s numbers were between 20 and 100. The first six questions on each page were designed to build confidence and involve two given numbers, one operational sign, ‘+’ or ‘-’. On all pages a firm knowledge of symmetry of equal relation can help solve the first six questions; in 2a form, two of them focus explicitly on the symmetry of the equality relation. The next four questions have three given numbers, two operational signs (cf. Behr *et al.* (1980), Sáenz-Ludlow & Walgamuth (1998) and Alexandrou-Leonidou & Philippou (2007)). These were followed by ‘open’ questions [3] with two operational sign, two boxes and two given numbers, as $a+\square = \square+b$; $\square+a = \square+b$; $a-\square = b-\square$ (we have yet to come across such examples in the literature). These were intended to reveal the possible use of reflexivity of equality, the commutative property of addition and awareness of 0 and its formal properties. We also tested the presence of the ‘commutative property’ of subtraction. Other less common open and closed questions were devised to detect the possible awareness of the transitive property of equality, with two-equality schema such as $a\pm b = c\pm\square = \square$ or with three-equality schemas such as $a\pm b = c\pm\square = d\pm\square = \square$ and $a\pm\square = b\pm\square = c\pm\square = \square$. These can be solved correctly by direct calculation showing the *non-RT behaviour* or by the use of structural properties, ap-

plying the different *RT behaviours* of Molina *et al.* (2007). For the reflexivity of equality in forms 2a and 3a we included a question of the schema $a = \square$.

The questionnaire instructions were intentionally open-ended: we asked “Can you complete these number sentences?”, without specifying which type of number could be used (naturals, relative integers, rationals or reals), thus leaving the possibility that older students could apply their knowledge about the various numbers systems.

The sample

As we had to rely on volunteers teachers, our sample was determined by their response. The number of returned questionnaires was as shown in Table 1.

Grade	2	3	4	5	6	7	8	9	10	11	Univ
No.	76	131	58	228	282	172	161	62	22	47	112

Table 1: The sample structure

The size of the sample (1,147 respondents giving 62,898 answers) and its breadth (11 different grades) allowed us to compare our data with the research literature; observe whether such findings might be extended to older students and detect any new phenomena. Due to the scope of the study, the conditions of the test administration were largely un-specified (time, day, duration of the test, surveillance during the proof, and so on) except in case of university students who were given 15 minutes to complete the questionnaire.

THE RESULTS

Interestingly, regardless of age, the majority of solvers only used natural numbers. Due to the lack of space we focus on sample questions (while retaining the original questionnaire ‘numbering’).

1. The first six questions on each page and symmetry of equal relation.

A-priori analysis. In the questions 2a. (b) $5+\square=8$ and 2a. (f) $8=5+\square$ the role of symmetry is evident, since the numbers involved are the same (‘strict’). In other examples we can speak of a symmetry ‘at large’ for the structure of the number statements, but not for the numbers involved. This gave us the opportunity to examine whether some pupils were ‘blind to the symmetric property of the equality’ (Attorps & Tossavainen, 2007), in the ‘strict’ sense and/or the ‘at large’ meaning. For each pair the correct answers to both questions can be obtained by computation; in case of 2a. (b) and 2a. (f), the result is 3, for both. For this pair, a difference in the result or the lack of one answer can be attributed to an incomplete mastery of the formal property of equality. For the remaining pairs we presume that a right answer to one question of the pair and the firm awareness of equality relation symmetry may suggest a good strategy for solving the other question of the couple, even if the numbers are different: a solver of $79-\square=25$ who has trouble with $53=78-\square$, can think of this second task in the form

78-□=53 to find the right answer. A right answer of only one question of these couples can suggest an ‘at large’ non-application of the symmetry in the pair.

A-posteriori analysis. The case simplicity of 2a. (b) and 2a.(f) resulted in high success rates: 98,14% and 95,40% respectively. People responding differently to the two tasks, certainly gave an incorrect answer. Individuals who responded incorrectly, are highly likely to a lack of their understanding of symmetry. However, in the case of the other pairs, the situation is more complex since we cannot exclude wrong computations even if symmetry was being used. In table 2 we distinguish between the ‘strict’ symmetry non-application and the ‘at large’ non-application. Data in the latter case are obtained cumulatively for the other eleven pairs (sample no. 12,993).

	number of at least one wrong or missing answer		rate of symmetry non-application		rate of contemporary success	
	<i>strict</i>	large	<i>strict</i>	large	<i>strict</i>	large
Grades 2-5 [4]	46	891	93.48%	79,91%	90.67%	78.33%
Grades 6-8	18	1090	83.33%	74,86%	97.07%	83.90%
Grades 9-11	4	231	75.00%	62,77%	96.95%	83.94%
University		47		89,36%		93.01%
χ -test	8.68E-6	3.38E-94	0.29	1.27E-7	3.29E-6	3.04E-24
Global sample	68	2259	89.71%	75,92%	94.51%	82.61%

Table 2: The non-application of symmetry of equality.

Values of the χ -test less than 0.05 (0.01) show that difference among grade classes are statistically significant; the result 0.29 is consequence of small numbers.

Reference the sum of the numbers of all the wrong and missing answers to at least one of two tasks suggests that a lack of awareness of the formal property is the greater source of error.

2. The task 2a. (k) $5 + \square = \square + 7$

A-priori analysis. The task is open with the choice of one of two missing numbers determining the other. The location of the boxes invites, possibly, the reflexive property of equality without the need for any sort of calculation e.g. $5 + \boxed{7} = \boxed{5} + 7$. The neutral role of 0 with addition could inspire the answer $5 + \boxed{2} = \boxed{0} + 7$. Other structural answers using the formal property of negative numbers (and 0) are $5 + \boxed{0} = \boxed{-2} + 7$ and $5 + \boxed{-5} = \boxed{-7} + 7$. Relational thinking offers a criterion for revealing a wrong answer: the given numbers are odd, therefore the two inserted numbers must have the same *even parity*. The repetition of a box could prompt (wrongly) younger pupils, in particular, into thinking that the numbers they are required to insert must be the same.

A-posteriori analysis. Of 1,143 students that were given this question, 1,057 responded, of which 842 gave the right answer (73.76%) suggesting that the task was relatively easy. Each answer given (right or wrong) used natural numbers. It is interesting to note the distribution of the structural answers by age of pupils. We suspect

that the infrequent use of zero to solve the problems e.g. $5 + \boxed{2} = \boxed{0} + 7$ could be due to a ‘fear’ of 0 - i.e. the complex acknowledgment of 0 as a number - or, simply reflect that individuals were unacquainted with this mathematical character.

2a. (k) $5 + \square = \square + 7$	correct response	presence of the answer $5 + \boxed{7} = \boxed{5} + 7$	presence of the answer $5 + \boxed{2} = \boxed{0} + 7$	commonest correct response (with frequency)
Grades 2-5	62.47%	7.26%	7.66%	$5 + \boxed{4} = \boxed{2} + 7$ (26.21%)
Grades 6-8	78.21%	6.86%	6.24%	$5 + \boxed{3} = \boxed{1} + 7$ (34.93%)
Grades 9-11	86.26%	2.65%	0.88%	$5 + \boxed{3} = \boxed{1} + 7$ (46.02%)
χ -test	4.79E-10	0.21	0.04	
Global sample	73.67%	6.41%	5.94%	$5 + \boxed{3} = \boxed{1} + 7$ (32.30%)

Table 3: The relational thinking presence and the commonest right answers to 2a.(k).

The commonest incorrect response was $5 + \boxed{2} = \boxed{7} + 7$ (with 18.14% of the 215 wrong answers). To interpret this we can consider the application of “Three First Numbers – TFN” and then “Answer After Equal Sign–AAES” modalities of Alexandrou-Leonidou & Philippou, (2007). The presence of two equal boxes, did not appear to be highly relevant as only the 8.37% of incorrect responses used the same number twice: $5 + \boxed{a} = \boxed{a} + 7$ ($a=1$ or $a=2$ having the greatest frequency). The *even parity* criterion was found in all of the 842 exact answerers and in 26.98% of the wrong answers, giving a total rate of 85.15% of the answers. We have also an *echo effect*: when the given numbers are odd, the percentage of correct answers using a pair of odd numbers is 62.59%.

3. The task 2s. (k) $6 - \square = 8 - \square$

A-priori analysis. This task is also open with the first number determining the second. Moreover, if restricted to natural numbers, the subtrahend must be less than minuend. The location of boxes may invite the following answer $6 - \boxed{6} = 8 - \boxed{8}$, a solution using 0 as result of both members of equality. Alternatively the neutral role of 0 when subtracting could be employed e.g. $6 - \boxed{0} = 8 - \boxed{2}$. For other aspects the a-priori analysis of this task is similar to the previous one. We expected a wrong relational thinking answer in the ‘commutativity’ of subtraction, i.e. the answer $6 - \boxed{8} = 8 - \boxed{6}$.

A-posteriori analysis. 1,056 students were given the question; 953 responded, 762

2s.k) $6 - \square = 8 - \square$	rate of success	rate of re- sponse $6 - \boxed{6} = 8 - \boxed{8}$	rate of re- sponse $6 - \boxed{0} = 8 - \boxed{2}$	rates of commonest right answer $6 - \boxed{2} = 8 - \boxed{4}$
Grades 2-5	61.09%	3.16%	2.76%	31.58%
Grades 6-8	76.06%	2.78%	1.05%	38.12%
Grades 9-11	80.15%	1.90%	3.43%	28.57%
χ -test	9.33E-7	0.24	0.24	0.09
Global sample	72.16%	2.76%	2.86%	35.17%

Table 4: The relational thinking presence and the commonest right answers to 2s. (k).

did so correctly (success rate 72.16%), suggesting that this task was relatively easy even if slightly harder than 2a. (k). Table 4 summarises the use of relational thinking. The *echo effect* appeared to be present as 56.17% of the right answers used pairs of even numbers. The *even parity* criterion is present in 87.20% cases. In this case the commonest correct answer is similar for all grades. Again we could argue that the commonest right answers were influenced by the fear of using 0 combined with the *echo effect*. The commonest wrong answer was $6-\boxed{2}=8-\boxed{2}$ (12.04% of the 191 wrong answers) and we could consider this kind of response motivated by application of TFN twice assuming that the second box is filled in first. Of the wrong answers, the structural, but incorrect, response $6-\boxed{8}=8-\boxed{6}$ was given in 4.19% cases. The value 0.09 of the χ -test show that the differences among grades classes are not statistically significant.

4. The task 3a. (k) $\square + 21 = \square + 11$

A-priori analysis. As above the task is open and has ‘freedom grade one’. The location of boxes may invite the use of commutative property of addition, i.e. $\boxed{11}+21 = \boxed{21}+11$. Moreover the neutral element of addition could reduce computation e.g. $\boxed{0}+21 = \boxed{10}+11$. Questions 2a.(k) and 3a.(k) have the same quantity of given numbers and addition symbols, but the boxes are differently placed: in 2a.(k) reflexivity of equality is at stake while in 3a.(k) the commutativity of addition is involved.

3a.k) $\square + 21 = \square + 11$	rate of success	rate of response $\boxed{11}+21 = \boxed{21}+11$	rate of response $\boxed{0}+21 = \boxed{10}+11$	rate of commonest right answer $\boxed{10}+21 = \boxed{20}+11$
Grades 3-5	60.52%	9.22%	2.84%	26.24%
Grades 6-8	72.17%	10.76%	4.48%	28.92%
Grades 9-11	80.92%	11.32%	4.72%	31.13%
University	94.64%	11.32%	6.60%	42.45%
χ -test	1.04E-10	0.94	0.57	0.03
Global sample	73.03%	9.14%	4.51%	30.54%

Table 5: The relational thinking presence and the commonest right answers to 3a. (k).

A-posteriori analysis. This task was administered to 1,094 students from grade 3 to first year of University: 979 responded with 799 of them giving the right answer (success rate 73.03%), comparable with the success rate for 2a. (k). Here *RT* appears to become more evident with increasing age. The use of 0 as the neutral element in addition is similar to that in task 2a. (k) but the commutativity of addition is more prevalent. Multiples of ten - excluding 0 - were found in 53.82% of the correct answers. The *even parity* criterion occurred in 83.86% responses. In 3a. (k) question the *echo effect* was not evident as 60.33% of the right answers had a pair of even numbers. The commonest wrong answer is $\boxed{32}+21 = \boxed{53}+11$ (6.11%). We hypothesize that the first box is filled in when the task is interpreted as $\square = 21+11$, in a sort of “Left Side Sum-LSS” modality. The completion of the second box is suggested by AAES modality (Alexandrou-Leonidou & Philippou, 2007).

In our opinion, the presence of two digit numbers had a double effect: the attempts decrease from 92.48% of 2a. (k) to 89.49% of 3a. (k) and this may be significant as the latter sample excluded 2nd graders but included first year university students. Secondly it may be that the presence of two digit number in this task activates a more attentive approach to the computation (the answers to other questions support this) and we could attribute to this attitude the greater presence of *RT*.

5. The tasks of type $a = \square$.

A-priori analysis. Behr *et al.* (1980) include examples of the type $a = a$, with given numbers and so we incorporated 2a. (l), $9 = \square$, and 3a. (n), $42 = \square$. To solve them one needs only apply the reflexive property of equality. These are closed tasks and do not require computation.

We anticipated that the absence of operational symbols would be destabilizing and

	9= \square success rate	9= \square with operational signs	42= \square success rate	42= \square with operational signs
Grades 3-5	76.67%	35.00%	70.82%	43.48%
Grades 6-8	73.17%	59.68%	72.98%	39.60%
Grades 9-11	67.94%	86.67%	67.94%	57.89%
University			92.86%	100%
χ -test	0.10	2.11E-6	1.53E-5	0.02
Global sample	74.01%	54.70%	73.70%	44.77%

Table 6: Comparison of results of the tasks 2a. (l) and 3a. (n).

The result in no answer or the use of operational symbols (cf. Behr *et al.*, 1980). location of the two tasks in their form allowed us to explore if there was a *tiredness effect*, influencing the rates of answer and success.

A-posteriori analysis. Task 2a. (l) was given to grades 2 - 11 (1,239) with 1,151 responding with 917 of correct (74.01%). The majority of incorrect answers (54.70%) express the result with operational symbols and the computation on the proposed numbers gives 9, showing a procedural interpretation of the sign =. The commonest answer of this kind is $9 = \boxed{3^2}$, in 44.53% of all ‘operational’ answers and was given by the majority of 6th graders and above.

Task 3a. (n), $42 = \square$, was given 1,190 grade 3-11 and 1st year university students, 1,049 responded with 877 of them giving the right answer (73.70%). The ‘operational’ answer rate is 44.77% and the commonest ‘operational’ responses were, globally, $40+2$ (19.48%) and $21+21$ (18.18%).

6. The task 2a. (m) $5 + 4 = \square + 6 = \square$

A-priori analysis. This task is the first which presents more than one equality sign. It is a closed task. The ‘chain’ of equality asks for the transitive property of equality.

Wrong answers suggest a lack of awareness of it. The most probable incorrect response is $5+4 = \boxed{9}+6 = \boxed{15}$ (cf. Alexandrou-Leonidou & Philippou, 2007).

A-posteriori analysis. 1,104 students responded with 718 giving the right answer (62.82%). As was expected the commonest wrong answer (70,47%) was $5+4 = \boxed{9}+6 = \boxed{15}$. This suggests either the pupils filled in the second box before completing the first or that they worked step by step from left to right. In either cases such results bring into question their intuition as Semadeni (2008) states:

“The transitivity of equality: “if $A = B$ and $B = C$ then $A = C$ ” was regarded by Fischbein (1987, pp. 24, 44, 59) as intuitively true. Piaget et al. (1987b, p.4) regards transitivity as an example of a systematic type of necessity...Transitivity is part of the deep intuition of equality (for numbers, for geometric points, for sets), involved in a multitude of deductive inferences.” (p.10)

7. The task 3s. (m) $48 - \square = 47 - \square = 46 - \square = \square$

A-priori analysis. This task is complex: it is open-ended, involves two-digit numbers, three subtraction signs and three equalities. Despite having four boxes to fill, it has ‘freedom grade one’.

3s.m) $48-\square=47-\square=46-\square=\square$	rate of success	$48-\boxed{48}=47-\boxed{47}=46-\boxed{46}=\boxed{0}$	$48-\boxed{2}=47-\boxed{1}=46-\boxed{0}=\boxed{46}$	commonest right answer rate $48-\boxed{3}=47-\boxed{2}=46-\boxed{1}=\boxed{45}$
Grades 3-5	56.99%	2.73%	9.09%	20.00%
Grades 6-8	56.91%	4.29%	12.29%	27.14%
Grades 9-11	60.77%	0%	7.59%	48.10%
University	83.04%	2.15%	6.45%	61.29%
χ -test	3.64E-6	0.22	0.29	6.5E-12
Global sample	60.19%	3.16%	10.28%	33.54%

Table 7: The presence of relational thinking regarding 0 and the commonest right answers to 3s.m).

To solve these questions correctly an explicit awareness of transitive property seems to be required. The task allows simple solutions involving *RT* and formal properties of 0 in many ways: $48-\boxed{48}=47-\boxed{47}=46-\boxed{46}=\boxed{0}$, or $48-\boxed{2}=47-\boxed{1}=46-\boxed{0}=46$. It is also possible to apply negative numbers, or fraction and so on, but no one did.

A-posteriori analysis. 896 – out of a possible 1,050 - responded with 632 giving the right answer (60.19%). The commonest correct answer reveals that the learners are at different levels of understanding, growing with age, taking care of the additive decomposition of numbers by fives: $48 = 45+3$, $47 = 45+2$ and so on. The structural properties of zero were most common in first eight grades of schooling. 41 pupils gave incorrect answers (22.65%) applying the transitive property of equality only once. 47.51% of those who were incorrect responded $48-\boxed{1}=47-\boxed{1}=46-\boxed{1}=\boxed{45}$.

CONCLUSIONS

The questionnaire enabled us to explore a phenomenology linked to relational thinking expressed by the reflexive, symmetric and transitive property of equality, the roles of zero respect to addition and subtraction and the commutativity of addition.

Our study is peculiar in the variety of schools and age range sampled. In this sense other similar experience known in literature took place in smaller school, segments. Another feature of our paper is that we are interested here in the right answers, even if sometimes we quote, also, wrong answers. Our research would have been more rigorous had we selected the sample statistically. Therefore our paper cannot be used for drawing general conclusions, statistically sound, about relational thinking, nevertheless in our feeling it might open a new trend of study about the equality, pointing out that this subject needs an attentive reflection regarding the way and the time in which the concept of equality is presented (in itself), let it grant that is introduced somewhere and somehow.

Overall primary school pupils were slightly better (even if in many cases differences are not statistically significant) than the older respondents in their application of relational thinking in specific tasks, but the presence of two-digit numbers appeared to hindered them. Nevertheless, a small but significant group demonstrated structural thinking provoking the question of how to extend such thinking to others. The transmissive teaching methods in Italy may explain why relational thinking does not appear to improve between grades 6 and 11 even if the structural properties of operations are taught explicitly, suggesting a parallel presence of relational and procedural thinking, independent from teaching. For symmetry our pupils confirmed the Attorps & Tossavainen (2007) results with prospective teachers.

There was a global score progression with increasing age. Addition questions were easier than subtraction; generally, pupils responded more appropriately to one digit answers than to two digit problems. Answering more complex questions under conditions of stress (e.g. tiredness) suggests that the students possessing a 'reified understanding' (described by Sfard (1991) as 'being able to see something familiar in a different light') of formal properties have an important tool which saves time and mental energies. Students who were aware of formal properties tended to cope better than others under conditions of complexity and stress. The prevalence of such knowledge was low however and in some cases appeared to decrease with age despite such topics being introduced in Italian Secondary School. Few participants (even from University) reificated the reflexive property of equality, and the function of zero in addition and subtraction. The commutative property of addition was more apparent. The more complex nature of the statements of symmetry and transitivity of equality do not necessarily indicate their presence, but only their absence. The sub-sample of university students appeared to have the awareness of these arithmetic tasks, but, surprisingly, more than 1/5 of the sub-sample responded to 3s. (m) incorrectly with

more than $\frac{1}{3}$ of them revealing a lack of a global view, answering $48 - \boxed{1} = 47 - \boxed{1} = 46 - \boxed{1} = \boxed{45}$, and of the transitive property of equality!

NOTES

[1] The authors gratefully acknowledge the support of the British Academy (Grant no. LRG-42447) which provided a platform for this study.

[2] The questionnaire presents 54 questions divided in four forms: 2a, 2s, 3a, 3s (the digit refers the grade of primary school and the letter 'a' is for addition and 's' is for subtraction). The integral version of questionnaire and the report of results are available at the web-site <http://www.unipr.it/arpa/urdidmat/M2ip>.

[3] When a solution is uniquely determined, e.g. $32 + 25 = \square + 16 = \square$ we use the adjective 'close'; whenever the solver is free to choose the suitable numbers, e.g. $48 - \square = 47 - \square = 46 - \square = \square$ we use 'open'.

[4] Italian children start school 6-years-old. Primary school comprises five grades; stage one of secondary school, grade 6, 7 and 8, and the final stage of secondary school 9 to 13.

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STRUCTURE OF ALGEBRAIC COMPETENCIES

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This paper reports a research study that aims at understanding interrelationships between algebraic abilities. Theoretical considerations drawn from the literature suggest various interconnections. To gain empirical evidence a test was developed and the findings analyzed by fitting different statistical models.

INTRODUCTION

Ideally, algebra lessons lead students to develop a profound understanding of algebraic concepts and the ability to see algebra as a central and connected branch of mathematics and the ability to apply algebra to a wide range of topics. If this happens, then students can be said to have a high algebraic competency. Even with this aim in mind, it is not clear how to design algebra courses. There are many approaches to the teaching of algebra (see e.g. Bednarz et al. 1996) and they obviously differ in the algebraic concepts that are given priority. The field of algebraic concepts is very broad, e.g. mastering the concept of an equation is a long process in which various aspects of the equation concepts are learned and they all interact with other algebraic concepts. To help in planning the algebraic learning process, it would thus be useful to gain more insight into the inner structure and dependencies of these algebraic concepts.

Such insight can be expected from empirical studies of various designs. Interpretative studies are valuable and some have been performed, especially as they allow to link theory and observations. However, they usually focus on a small number of students and it often remains unclear, how representative they are. Quantitative studies, on the other hand, often lack a deeper connection to theories.

The quantitative study reported here tries to apply advanced statistical models on a test that was developed to reflect certain theoretical assertions about the learning of algebra. In this paper, only results from a single use of the test are reported but this study is part of a larger research project that will collect longitudinal data as well.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

Algebra deals with a lot of objects, including numbers, variables, expressions, functions and relations, and each of these can play many different roles. School algebra thus is composed of many ingredients. Several theories have been developed that give some structure to this large field and we will mention some of them that were used implicitly in our study.

Variables play the central role in our investigation because they are a link between most of the other objects mentioned. Variables are used in many different ways in algebra. Küchemann (1979) gave six ways of using variables. From the perspective of

integrating these modes of variable usage into a scheme we found that these modes, although useful in explaining students results, are bit unhandy. When looking at algebra problems from textbooks we found that his “Letter ignored” is not of great importance and test items regarding it seem always a bit artificial. Moreover, it may be subsumed to the aspect of a variable as generalized number. The use of a variable as a reference to a non-arithmetical object “Letter as object” is (which restricts itself to standard school algebra) an important misconception that is viable only in a very limited subset of algebra. As a misconception it should not be included into the structure of abilities that are to be mastered by the students. Malle (1993) gave a short list of three aspects which proved a bit coarse when classifying textbook problems and test items. A synthesis of these approaches that works well for the classification of the role of variables in different problems turned out to be very similar to the one found by Drijvers (2003) in his empirical study, see below. It is worth to make explicit the operations that are linked with the different roles of variables. This shall emphasize the fact that the role of a variable is not only determined by the algebraic context but also by the subject working with it, e.g. the x in $2x+1=4$ may be viewed as an unknown which is to be determined or as a placeholder where one can insert numbers or expressions.

- Placeholder P (operation: substitute (not only numbers but general expressions))
- Unknown U (operation: determine)
- General number G (undetermined; operation: expressing relations)
 - Ga: General number used in analyzing expressions
 - Gm: General number used for modeling (describing)
- Variable as changing quantity V (operation: change the value)
 - Vi: independent variable (operation: change at will)
 - Vd: dependent variable (operation: observe change)
 - Vr: variable in a relation without predetermination what variables is changed independently as in Ohm’s law $U=RI$.
- Variable as a symbolic element of the symbolic algebraic calculus: C (i.e. operation: use as structure-less object in symbolic manipulations)

Different researchers have advocated the point of view that mathematical objects are constructed from operations (Sfard 1991, Dubinsky 1991, Gray & Tall 1994). While the theories of these authors differ in detail, the broad picture seems similar and naturally explains e.g. the creation of symbolic expressions as encapsulated calculation sequences. It is not as clear to which processes the concept of a variable is linked. Therefore, we associated the above mentioned processes to each aspect of variables. Obviously, different operations lead to different objects, but nevertheless, mathematicians look at variable as a single concept which can be used under different aspects. It is therefore interesting to note that the operation of substitution has tight relations to all the other operations except those operations associated with the last aspect of the above list. We therefore formulate the following hypothesis:

Substitution is a central operation in algebra and the competence to use it properly is at the heart of algebra in the sense that it makes other operations easy as well, with the exception of the symbolic calculus aspect. Put in more technical language, this states that the ability to use substitutions should be a good indicator variable for performance in other algebraic tasks.

Checking the validity of this hypothesis is one of our research questions. The next question is much more open: To what extent do these aspects of 'variable' depend on each other?

METHODOLOGY

There exist many tests for algebraic achievement but most items test syntactic term rewriting or formal equation solving capabilities. Far fewer test items exist that assess algebraic understanding and algebraic concepts developed by the students. A notable exception is Küchemann's work in the late 70s and early 80s. For this study we developed a new test that is somewhat in the spirit of Küchemann and uses many of his items, but most items were developed to reflect the various aspects of variables described above. In addition, there were test items on the relation between equations and functions.

The study was conducted at the beginning of grade 11 (age approximately 16 years) of a German high school (Gymnasium). There were 141 students from six classrooms in the study. Unlike most other German schools this particular high school starts at grade 11 and thus collects students who were recently at a large number of different schools. Although this sample is not representative of German students, it can be expected to span the breadth of the population better than samples from classes that had the homogenizing effect of a common school culture. However, the mean achievement level is supposed to be below that of an average grade 11 high school.

The test was compiled for this study but most of the items had been used in our research group before. The test consists of 43 items, two of which are multiple choice items, while the others ask for a free form response. The answers were rated on a point scale as the following example of a rating rule indicated:

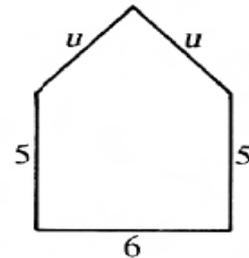
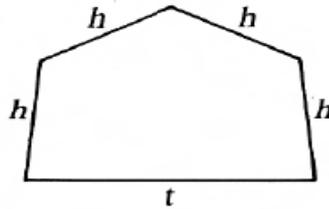
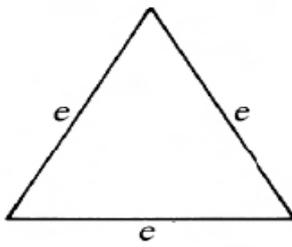
- Item 2a (from Küchemann 1979): Give a short answer and explanation: What is greater? $n+2$ or $2n$?
- 0 Points= no response; false response without argumentation
 - 1 Points= example; some explanation; wrong answer with detailed explanation
 - 2 Points= example with explanation; detailed explanation without case distinction
 - 3 Points= almost correct with case distinction
 - 4 Points= completely correct

Some examples of the test items are shown below; their association to aspects of 'variable' are shown in square brackets:

Item 4: (based on Küchemann 1979) Let r be the number of rolls and c the number of croissants bought at a bakery. A roll costs 30ct, a croissant is 70ct.

- a) What is the meaning of $30r+70c$? [G]
- b) How many parts have been bought all together? [G]

Item 6a,b,c (from Küchemann 1979): Work out the circumference of the following figures:



[G]

Item 9: a) Assume that the equation $a=b+3$ always holds. What happens to a if b is increased by 2? [V] (from Küchemann 1979)

c) Assume that the equation $a=2b+3$ always holds. What happens to b if a is increased by 2? [V]

Item 13: It is known that $x=6$ is a solution of $(x+1)^3 + x = 349$. How then can one get a solution of $(5x+1)^3 + 5x = 349$? [G] (from Küchemann 1979)

Item 14: Simplify the following expressions a) $(a-3)^2 - a^2$ b) $(x-x^3) \cdot (x+x^3)$ c) $\sqrt{36+4a^2}$ d) $\frac{1}{n} - \frac{1}{n+1}$ [C]

Item 16: Given the examples $7 \cdot 9 = 8^2 - 1$ and $11 \cdot 13 = 12^2 - 1$, formulate a general rule and justify it. [G]

Item 17: A function is defined by: $f(x) = x^3 - 2$. Determine

a) $f(2) =$ b) $f(y) =$ c) $f(x+1) =$ d) $x \cdot f(x) =$
[P]

Item 19: What must be substitute for x in the expression $2(x^2-1)$ to obtain the desired result? [P]

Desired Result	Substitute $x = \dots$
6	
-2	
$2((a+1)^2-1)$	
$2(b^2+2b)$	

The test items were classified by the aspects of variables they involve and by the relevance of the abilities to handle functions (Fun), relations (Rel), syntactical expression manipulation (Syn), working with unknowns (Unk), handling substitutions (Sub) and translating between algebra and geometry (Geo). Of course, this classification is build upon assumptions about typical solution strategies.

Besides more traditional statistical methods, this study uses structural equational modeling as a tool to model dependencies. While this technique is frequently used in many empirical sciences, it seems that its use in the mathematics education community not as widespread and I know of no application of this technique to gain insight

into concepts of algebra. However, I believe that this statistical tool is appropriate here, because it allows us to work with hidden variables that cannot be observed directly (e.g. the person's understanding of a variable as a general number) and to model relations among latent and observed variables.

RESULTS AND INTERPRETATION

The test contained several items developed and used by Küchemann 30 years ago. Despite the passage of time, our results were very similar, thereby underpinning the validity of his study. The order of empirical difficulty of the items turned out to be precisely the same as that found by Küchemann. Also the percentage of students that solved the items were remarkable close (despite the fact that we tested 16 year old students while Küchemann tested 14 year olds), with one interesting difference regarding the 'letter as object' aspect. We found Item 6a was solved by 74% while Küchemann found 94% (for 6b and 6c we found 74%, 58%, Küchemann found 68%, 64%). These numbers become interesting when combining with the result that item 4a was solved only by 14% and 4b only by 7%. Most students that failed on 4a showed a clear object interpretation reading $30r+70c$ to mean 30 rolls and 70 croissants. However, many more students were able to solve 6a and 6b, which are described by Küchemann as items that can be solved successfully using 'letter as object'. Using a variable as reference to an object should be differentiated into two aspects: The misconception that a variable can stand as shorthand for any object, and the conception that a variable stands for some measureable quantity, such as the length of a segment. This latter interpretation is at the heart of an approach to algebra by Davydov, Dougherty and others (see Gerhard 2008) that is suitable also for younger children. Interestingly, the sum of points of 6a and 6b show a correlation with the total test score of $r=0.62$ indicating that the ability to solve these items show much more than a misconception.

Next we gather some results from analyzing cumulative variables as described above. Together, these variables accounted for approximately 70% of all test items. According to the Kolmogorov-Smirnov-Test they can be considered to be normally distributed. Then a multivariate regression of the total score to these Variables was performed. The standardized beta-weights (with standard errors) were:

Variable	Standard. Beta(SE)
Syn (syntactic manipulation)	0.15(0.03)
Geo (geometry)	0.28(0.03)
Sub (substitution)	0.26(0.04)
Gen (working with general numbers)	0.22(0.04)
Fun (functions)	0.07(0.04)
Rel (relations)	0.38 (0.04)

The interpretation of these numbers must of course take into account that they reflect to some extent the composition of the test. There were eight items that were taken together to form the Rel variable, but only four that formed the Fun variable. Yet this can't explain the dramatic difference in beta weights. We conclude that understanding of algebraic relations is an important component of algebraic competency. It is also interesting that the Geo variable that consists of only five items is that important. One may draw the conclusion that expressing relations among quantities is at the heart of algebra. It is therefore justified to exercise this extensively in introductory algebra lessons.

Then an analysis of covariance gave first insight into interdependences. The interesting findings were: There is almost no correlation between the syntactic manipulation (Syn) and Geo ($r=0.09$), Sub ($r=0.09$), Gen ($r=0.10$), Fun ($r=0.13$), Rel ($r=0.02$). The scale Syn consists of item 14 (which has two more sub-items than shown) on the simplification of expressions and of two items on solving linear equations. The result means that syntactic manipulation and conceptual understanding are two different dimensions. The assumption implicit in some teachers position on teaching algebra that learning the symbolic algorithms will lead to insight seems thus to be false. To further support this point we give the following two-way table:

Number of students		Score on syntactical items	
		Above average	Below average
Score on other items	Above average	37	27
	Below average	36	41

The χ^2 -test gives $p=0.19$ on that, compatible with the assumption of independence (which is certainly not correct, but there is only a very weak relationship.)

This almost-independence result was stronger than expected and future studies should investigate this again. An interesting observation is that the connection is somewhat stronger for higher achieving students.

On the other hand the highest correlation ($r=0.63$) is between Rel and Subs. Subs also correlates with Geo ($r=0.44$), Gen ($r=0.54$) and Fun ($r=0.54$). All of these correlations are highly significant ($p<0.01$). This supports the hypothesis about the fundamental role of substitution given above.

Next, we report some results from the path model study. Although this interpretation was not intended by Drijvers (2003) we made up a structural equational model (more specific, a path diagram) from his diagram given below (Fig. 1). The model fit was acceptable according to Hair's (Hair et al. 1998) recommendations with $CMIN/df=1.96<2.0$ and Parsimony-Adjusted Measure $PCFI=0.56$. We found that the concept of placeholder loads most on the changing quantity (our role V of a variable; path weight and standard error: $1.14(0.48)$), then on Unkown (U, weight $0.37(0.12)$) and negligible on the generalizing aspect (G, weight $0.08(0.04)$). The other arrows

carry small weights as well. While the first two results are plausible, the question arises what influences the important aspect of a variable as a generalized number if not the placeholder aspect.

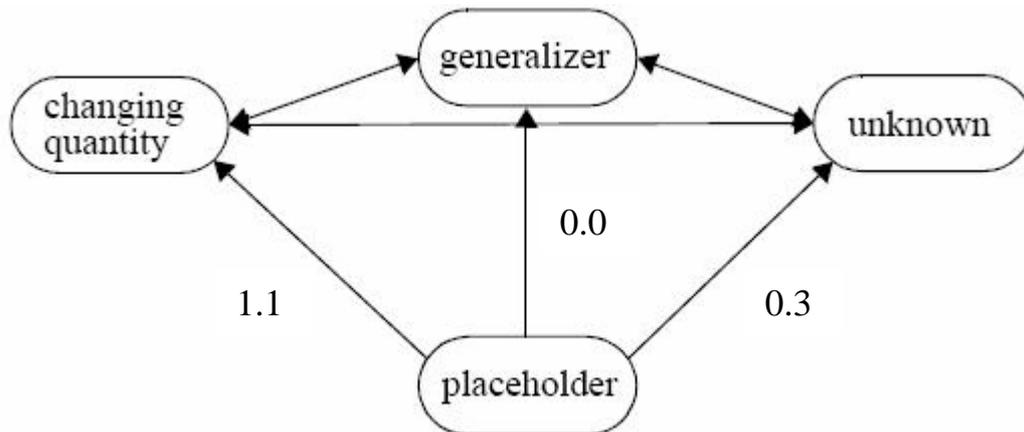


Fig. 1

The following model (Fig. 2) includes all of our five variable aspects. The latent variables are named by the short cuts of the variable aspects defined in the theory section. This model provides almost good model fit $CMIN/df=1.53$, $PCFI=0.67$. Nevertheless many of the estimates for regression weights are rather small and we will refine and modify the model shortly to get better results. Nevertheless this model shows some interesting results. First the arrows that relate the calculus aspect C with other aspects carry small weights. This feature is common to all models we tried and reflects the fact mentioned above, that syntactic manipulation is almost independent from the rest of the test. Another interesting fact is that there is a substantial (and significant) weight for the arrow from G to V. This is naturally interpreted as the implication that a general number can be viewed as standing for changing numbers. On the other hand, students learning algebra may first master the aspect of changing quantities and only later develop the general concept of a variable that stands for a general number without reference to a particular number. Therefore we omit this arrow in later models.

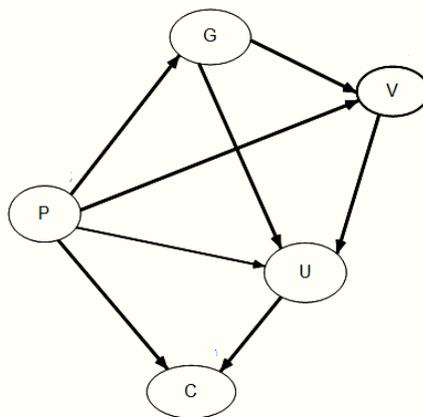


Fig. 2

The above path-model can be refined by splitting the aspect of general number as indicated in the theory section into the aspect of using the general number for analyzing

or for modeling. Furthermore, we will omit the syntactic aspect of a variable as an element of algebraic calculus, because it is essentially independent from the rest. With these decisions made we tried out many linear structural equational models but concluded that the following one is the best choice. Some other models provide a slightly better model fit, but this model (Fig. 3) has two important properties: It is plausible from the theoretical point of view and can therefore be easily interpreted. Its advantage from the statistical point of view is that most of its path coefficients are either significant or close to significant. The model fit is adequate with $CMIN/df=1.92$ and $PCFI=0.55$. The estimates for regression weights (with standard errors in parentheses) are:

Place holder P	→	Unknown UK	0.15 (0.07)
Place holder P	→	General Number Ga	0.031 (0.024)
Place holder P	→	General Number Gm	-0.003 (0.341)
≈0			
Place holder P	→	Variable V	-0.76(0.42)
Unknown UK	→	General Number Ga	0.30 (0.16)
Unknown UK	→	General Number Gm	-0.64(2.8)
≈0			
Unknown UK	→	Variable V	6.6 (2.0)
General Number Ga	→	General Number Gm	2.5 (1.9)
Variable V	→	General Number Gm	0.077 (0.44)
≈0			

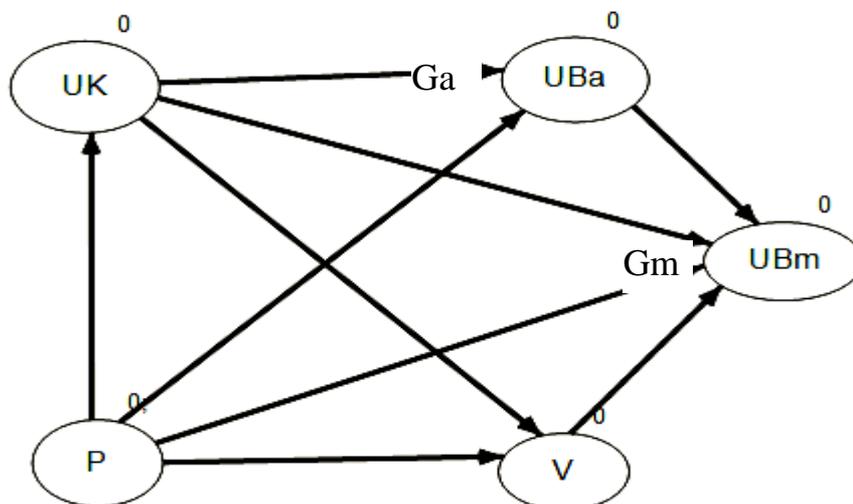


Fig. 3

Compared to the above model based on Drijvers diagram it may seem strange that the arrow $P \rightarrow V$ has a negative weight. This result does not claim that there is a negative correlation between these abilities but only that the direct influence is negative taking into account the large influence from the arrows $P \rightarrow UK$ and $UK \rightarrow V$ which both have positive weights. In fact, when omitting the $UK \rightarrow V$ arrow from the model, the arrow $P \rightarrow V$ gets massively positive (1.6). The negative weight in our model is therefore plausible: Learning to handle variables as placeholders may pave the way to seeing

variable as unknowns and this in turn helps develop the full concept; however students who can only deal with placeholders are unlikely to see variables as quantities that can change because a placeholder once filled with a number is constant.

The path weight for $G_a \rightarrow G_m$ was 2.5(1.9). When reversing the arrow it became negligible. This can be interpreted to mean that learning to analyze situations with variables is a prerequisite to modeling situations that are initially free of algebraic symbolization. On the other hand the aspect V is not helpful for algebraic modeling. This may give a hint that at the level of modeling situations by algebraic equations one is working at a rather high level where individual values of variables and their change is not considered. We hypothesize that the aspect of change is not important in forming the model but in its validation. But this conclusion can't be drawn from the data of this study.

Is it possible to assign students a single latent variable “algebraic competence”? To test this we fitted two simple models to the data. One model with only one latent variable “algebraic competence” and one model with latent variables “Univariate” and “Multivariate”. The model with two latent variables has a model fit of $CMIN/df=1.78$, while the model with a single latent variable has a model fit of $CMIN/df=2.99$. This substantial difference may be seen as support for the hypothesis that algebraic competency is a higher dimensional construct, because here we have a higher dimensional modeling that fits the data better. Nevertheless, the test as a whole fits the assumptions of the one-dimensional Rasch model. Hence we conclude, that structural equational models can reveal detailed results.

CONCLUSION AND OUTLOOK

The findings of this study lead to two different kinds of conclusions. The first kind concerns the results from analysis of covariance and fitting the structural models. They indicate that the activities of describing general geometric situations algebraically are good indicators for overall performance. Similarly, substitution is a fundamental operation in algebra that shapes the meaning of algebraic constructs.

The second kind of conclusion concerns the level of algebraic competency reached in grade 11 and this is more specific to the situation in Germany (although the study does not claim to be representative for all German schools). While some areas (in particular, solving linear equations and using binomial formulas) show acceptable results, other parts of algebraic thinking, especially those that serve as a backbone in introductory calculus courses, reveal a serious lack of competence. Either a solution has to be found to cure the algebra decrease or one should consider curricular changes in grades 11 and later that eliminate the need for those kinds of algebraic thinking; however, this would mean dropping calculus from the curriculum.

The future work of this research project is aimed at improving the situation. In collaboration with schools we aim to use this test as diagnostic instrument to help us assign tasks that will improve the construction of algebraic meaning. This includes the

use of new algebraic technology (Oldenburg 2007) and the use of experiments (Ludwig & Oldenburg 2006).

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GENERALIZATION AND CONTROL IN ALGEBRA

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This study addresses the importance of a pedagogical approach that contemplates generalizations students make spontaneously, due to the high value generalizations have in the learning of algebra and the construction of mathematical rationality. I consider the problem of the control of spontaneous generalizations, from the perspective of both didactic interventions and student's learning. I analyze the problem of the internal validation in the case of algebraic writings. I show various examples of pre-university students' (17-18 years) spontaneous generalizations and handling of control. The study suggests the necessity to face this problem from the beginning of the secondary school.

INTRODUCTION

Algebra constitutes a domain which favours the progress of mathematical rationality from the beginning of secondary school, through reasoning involving generalization. Moreover, generalization processes are of a great value in the production of knowledge (personal and scientific) (Garnham & Oakhill, 1993).

The ability to generalize is a common faculty of human reasoning, not specific of any content, which raises (not content-specific) learning questions. However, the ability to generalize in a particular domain involves specific learning problems within this domain. Various authors have considered the question of generalization in algebra, and favouring generalization activities is now seen as being an approach to algebra (see Bednarz, Kieran, Lee, 1996). Specially, justification related to generalization processes has been considered by Radford (1996) and, from a different perspective, by Balacheff (1987, 1991), amongst others.

However, students do not generalize only when faced to **generalization activities** (so as to find numerical or geometrical patterns, laws governing numbers, or the construction of formulas, etc). They also make generalizations in the context of tasks which do not require finding any regularity. This is what we call **spontaneous generalizations** (Panizza 2005a, 2005b).

From the point of view of the teacher's interventions, this sets the problem of anticipation. How can the teacher be attentive to the emergence of such spontaneous processes? Moreover, the student perceives differently the necessity to justify generalization, according to the more or less spontaneous character of the generalization, inasmuch as mathematical rationality is under construction.

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On the other hand, algebraic environment differs clearly from numerical and geometrical environments from the point of view of the *feedback* given to the student's activities.

It is important to consider this question in a systematic way through the various approaches to algebra (described in Bernardz, Kieran, Lee, 1996), which provide very different contexts for the emergence of such processes; in particular, from the point of view of the possibilities of control within algebraic environment or by means of conversion to other semiotic 'registers' (Duval, 1995, 2006).

I claim that such a pedagogical approach in the domain of algebra may favour the construction of mathematical rationality in secondary school.

RESEARCH METHODS

The data presented in this paper were obtained through qualitative methods: observation of regular classrooms and case studies, focusing on student's reasoning when analysing statements written in symbolic language. The research was conducted within four different pre-university (17-18 years) algebra courses.

The observations were conducted in a systematic way. A set of tasks was selected to be administered in class by the teacher, in order to observe the procedures of students when analysing statements written in symbolic language, especially when trying to determine conditions under which algebraic statements are true. Special attention was directed to: the verbal and symbolic descriptions students produced, based on their observations and descriptions of objects of reference of statements (instantiations); its influence on the processes of statements (re)formulation; the *treatments* (in the sense of Duval) they do within the algebraic writings register and the capacity for going over from the formulation of statements in symbolic language to a representation of the statement in other register (*conversions*, in the sense of Duval), very especially the use of this capacity for control. The data consisted of notes from classroom observations and the student's written works.

The study allowed identifying some phenomena among which the different kinds (according to its origin) of *spontaneous generalizations* presented in this paper.

For the case studies, four students that were considered representatives of the studied phenomenon were chosen from the algebra courses (their real names have been changed in this paper). The intention was to find specific features related to spontaneous generalizations, through mini-clinical interviews, all of them audio recorded. The reactions of students facing counterexamples provided by the interviewer in the context of their spontaneous generalizations, together with their perception (or lack of it) of the necessity of control and their processes of control inside or outside the register of algebraic writings, were observed.

The study showed that students often do (new) spontaneous generalizations based on the counterexamples provided by the interviewer and that their spontaneous generali-

zations are based on *local associations* of few examples which are not representatives of the objects of reference of the statements.

SPONTANEOUS GENERALIZATIONS: WHICH? WHY? WHAT? WHERE? HOW?

What are the spontaneous generalizations? Why it is important to take them into account in the class of mathematics? In what contexts do they emerge? How?

Spontaneous generalizations: which?

Let us see some examples, taken from the observations in the algebra courses:

Faced to the problem “Find the real values of x such that $x^2 \geq x$ ”, Belén and María answered that “ $x^2 \geq x$ is true for every real number” without solving the equation, but they arrived there by different ways. Inquired by the teacher, Belén argued “*it is evident, the square of any number is always greater than the number itself!*”. María, instead, argued “*I have tried with several examples, 1, 2, 3, -1, -2, -3, and so...*”

Belén seems to have generalized to real numbers the property valid for natural and integer numbers (*extension of schemes of knowledge*, see Vergnaud, 1996). María seems to have done an *induction process*.

I wish to point out that both have done a generalization **even if the activity was not a generalization one**. It is also important to notice that both arrived to the **same conclusion** by **different ways** of reasoning. I will come back to this point. Nevertheless, both examples are very familiar. But let us turn to another one.

The problem:

“Decide if the following implication is true or false:

$$\forall x \in \mathbf{R}: (2x^2 > x(x+1) \Rightarrow x > 1)”$$

was given in class in order to analyze the algebraic competence of students to decide the relation between the solution sets of two inequalities - in an implication context -. Brenda’s production is especially illustrative of the “problem” of **spontaneous generalizations** arising within the frame of a task.

When solving it, Brenda considers diverse examples, $x = 0$, $x = 1$, $x = 2$, $x = 3$, $x = -1$, $x = -2$, $x = -3$, $x = -4$ analyzing the value of truth of the antecedent and the consequent in each case. She concludes, correctly, that the statement is false, because “*it is possible to find values of x smaller than 1 that fulfil $2x^2 > x(x+1)$* ”

The professor asks her to explain how she arrived at the answer.

Brenda says that “*-2, -3, -4 are counterexamples, because for them the antecedent is true and the consequent is false*”.

According to the task, Brenda could have finished there, but she adds, immediately:

“*Ah, it was $|x|$ what we should have put!, what is true is:*

$\forall x \in \mathbb{R}: (2x^2 > x(x+1) \Rightarrow |x| > 1)$ ".

According to my interpretation, Brenda makes a spontaneous generalization of the set of counterexamples used by her to argue ($x = -4, x = -3, x = -2$), and proposes a statement that she considers true. It is to note that the task did not require to find any regularity. Brenda does it spontaneously, perhaps with the intention of finding a true statement (Balacheff, 1987).

I want to draw attention to the fact that from the point of view of the logical complexity, Brenda could have analyzed the value of truth of her statement, since the original task was correctly solved and both statements required the same logical competences. Even though we can think about a greater difficulty to find the counterexamples - in as much these are in the interval $[-1,0)$ -, I want to point out that Brenda does not consider it **necessary** to analyze her statement, she does not even consider it at all. She displays her affirmation beyond. So?

So, spontaneous generalizations: why?

Because a large part of the learning achievements resides in the capacity to generalize. By generalizing students construct knowledge. The emergence of these processes in the class is most important, as much for the learning of algebra as for the development of the mathematical rationality.

But conclusions require validation. This necessity –as it is well known -, is acquired, if it ever is, in the very long term.

On the other hand, when the generalization is a spontaneous one and therefore it is not directly related to the task to be solved- as in the cases of Brenda, María and Belén- it is difficult for the professor to anticipate it. In addition, a same result can come from different processes of generalization, as in the case of María and Belén. This is about something that usually occurs in the class of mathematics, and it is difficult for the teachers to have appropriate resources of intervention. So?

So, spontaneous generalizations: what?

This problem has led me to consider the generalization trying to deal with this phenomenon in its diverse manifestations. To do so, I tried to find the student's processes of generalization in their amplest sense, such as those of transference of a domain to another one (see Sierpinska, 1995). I also consider extension of knowledge schemes as generalization, as it has been studied by Vergnaud (1996) in the domain of mathematics, by Leonard and Sackur (1990) through the notion of local bits of knowledge; and by Harel and Tall, -quoted by Mason (1996)- through expansive, reconstructive, and disjunctive generalization. So?

So, spontaneous generalizations: where?

I consider that the different contexts of use, the nature of the task, the forms that are used for representation, the meaning granted to the letters, can originate different types of spontaneous generalizations.

The contexts provided by different approaches to algebra must be studied from this point of view: these contexts, give rise to specific spontaneous generalizations? Are there particularities of these contexts in relation to the control possibilities? (Balacheff, 2001). So?

So, spontaneous generalizations: how?

Up to now, I have found a lot of spontaneous generalizations, and I find it fruitful to consider them as of different kinds. According to its origin (for a particular student in a particular moment), a spontaneous generalization may be of nature:

2. conceptual (based on the content to which the statement refers to), as Belén did in extending the range of an existing scheme (“*it is evident, the square of any number is always greater than such a number!*”);
3. logic (based on an inadequate understanding of logical connectors or rules of reasoning), as María did when considering that with several examples she had arrived at a true conjecture (“*I have proved it with several examples, 1, 2, 3, -1, -2, -3, and so...*”)
4. semiotic (based on an analysis of the content of the semiotic representation (Duval, 1995, 2006).

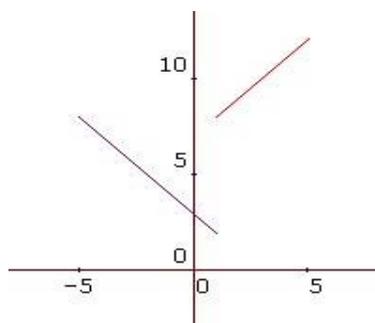
I think that this typology is interesting because it helps the teacher in the identification of leading elements of spontaneous generalizations on the part of the students, in the possibility of interpreting them and making them evolve.

Let us see an example of the later (semiotic) kind

Problem: Study the properties of the function

$$f(x) = \begin{cases} -x+3 & \text{if } x < 1 \\ x+7 & \text{if } x \geq 1 \end{cases}$$

Taking into account the habitual scales that students use to plot functions I posed the hypothesis that -looking at those graphs- :



students would decide the *injective character* of the function. And it is what 40% of the group of students actually did. They generalized the *content* of the graphic semi-

otic representation and decided that it was representative of the function in its complete domain.

As in the case of Brenda, the students who responded to the problem in agreement with our anticipation did not consider it even necessary to make a control.

In order to advance in this point, clinical interviews were made. Let us see the processing of control that Ana Paula makes, faced to a counterexample provided by the interviewer. Ana Paula had stated that the function is injective, having done an incomplete analytical study (she analyzed each branch ($x < 1$ and $x \geq 1$) of the function in isolation) and looking at the plot.

Let us see (minor episodes have been skipped):

The researcher suggests her to analyze the pair of values $x_1 = -6$, $x_2 = 2$

Ana Paula does some calculations

Ana Paula: Oh, yes, it's true...it is not injective... (*she thinks*)...What should I have put to see it was not injective? A negative number and a positive one?

Researcher: I don't know, you find out.

Ana Paula: I am searching so that they are the same... (*she thinks*)

Ana Paula: Of course, as $-x$ changes the sign it is as if I had two positives, one adds up 3 and the other 7, I must get the same result... (*she equals to 10, she thinks and finds -7 and 3*)

Ana Paula: -7 and 3... $(-7) + 3 = 3 + 7$, and thus I prove it is not injective

Researcher: Wasn't it proved with -6 and 2 ?...

Ana Paula: Yes, of course I had already verified it (*she still searches for counterexamples*)

Researcher: Why are you searching other counterexamples?

Ana Paula: Because if I had to do it again I would do it wrongly once again, because before I did it analytically, I verified it in the plot and I got the same result in both of them. Even more, I did a value table and I didn't put -6 and 2 . I don't understand where was my mistake (reviewing her previous works).

Researcher: aha...

Ana Paula: Has the difference between x_1 and x_2 to be constant?

Let's see, $x_1 - x_2$ equals to image

Researcher: Which image?

Ana Paula: Of both!... (*she gets at a loss in the calculations*).

Ana Paula: Oh no! There are going to be infinite providing the image is greater or equal to 8. What can I do to find them?

Researcher: The image of x_1 has to be the same as that of x_2 .

Ana Paula: I've already said it, it is the definition.

Researcher: You've said it but you didn't use it...

Ana Paula: Aha! (*she finally does some calculations and arrives to the equation*).

$$-x_1 + 3 = x_2 + 7$$

$$x_2 + x_1 = -4, x_1 < 1, x_2 \geq 1$$

To make control, Ana Paula analyzes the problem in various representations (graphical, algebraic, by tables) without integrating them. This example is representative of what happens with many students. Next I set out to analyze this problem, specially the problem of control related to the **algebraic writings**.

PROBLEMS OF CONTROL

Two aspects seem essential; on the one hand, the problem of the recognition of the **necessity** of control of the conclusions; on the other hand, supposing that the student has this ability, the problem of the **possibility** of making this control is posed (Panizza 2005b).

The problem of the necessity of control

In relation to the first point, perceiving the **necessity** of control is different according to whether generalization is a spontaneous one or it is obtained as asked for by the task. In the latter case, necessity of control is intrinsic to the task. Indeed, when someone **must** make a generalization, a suitable representation of the task should include the control necessity, that is to say the need to adjust the conjecture to the data. In addition, as Radford (1996) indicates "representations (in generalization) as mathematical symbols are not independent of the goal. They require a certain anticipation of the goal". That means, according to my interpretation, that in the **generalization activities** the control occurs like a process, during the resolution itself, through the re-representations that are made on the data, based on the analysis of the goal. On the contrary, for **spontaneous generalizations** the necessity of control is not intrinsic to the task, since generalization is not directly related to the goal. The examples of María, Belén, Brenda and Ana Paula are representative of this claim. However, many students may perceive this necessity. Ana Paula, faced to a counterexample provided by the interviewer, tries to control by shifting to other representations (graphical, algebraic, by tables). Anyway she does not succeed. This leads us to the problem of the **possibility** of control.

The possibility of control within the algebraic writings register

I claim that the **possibility** of control within the algebraic writings register is difficult as the retroaction does not work in the same way that in the arithmetical writings register or the material geometrical figures domain (Panizza & Drouhard, 2002).

In fact, in the arithmetical writings register, when students arrive by reasoning at an equality of the type $2 = 3$, this writing in itself gives them information that plays the role of an element of control.

In the same way, in the material geometrical figures domain, when, faced to the famous problem of extension of a puzzle of Nadine and Guy Brousseau (1987), the pupils make inadequate extensions, the fact that the resulting pieces do not fit, constitutes an element of control.

Algebra is quite different. As Drouhard (1995) shows, when students arrive at $(a + b)^2 = a^2 + b^2$ they believe that the teacher just “prefers another rule”, for instance $(a + b)^2 = a^2 + 2ab + b^2$ (“*You made a transformation and I made another one...*”).

This example illustrates a general problem: that the register of the algebraic writings does not offer the students good elements of feedback and control.

Rojano (1994) establishes a similar conclusion (quoting Freudenthal), when analyzing the differences of feedback of the errors in arithmetic and natural language - provided by numerical contexts and daily communication -, unlike the feedback in the register of algebra. However, these characteristics of algebra are not sufficient to determine the conduct of control of a particular student in a particular context. The possibility that certain information can act as a feedback also depends on:

3. the student’s abilities to “see” such information;
4. his possibilities to enter in contradiction (see Balacheff, 1987);
5. his capacity to deal with different types of statements (of existence, individuals, generals);
6. his linguistic skills on letters (syntax and semantics) (see Kirshner, 1989, Duval, 1995, Durand Guerrier, 1996, Panizza, Sadovsky & Sessa, 1998, Drouhard, Panizza, Puig & Radford, 2006);
7. his conceptual and operating skills on numbers, variables, unknowns and parameters (see Janvier, 1996).

I consider that an education that contemplates the fact that these skills are developed in parallel and in an interrelated way, must find didactic strategies for helping students to develop control means *inside* and *outside* the register of the algebraic writings. I adhere to the didactic frame of reference provided by Duval (ibidem) with the notion of *conversion* between different semiotic representation registers, especially for what control possibilities concerns.

CONCLUSIONS AND PERSPECTIVES

This study shows that pre-university students make different types of *spontaneous generalizations* in contexts of explanation, proof or discovery, without neither having acquired conscience of the necessity of justification of the conclusions, nor abilities for making control. From my point of view, this suggests the need of a pedagogical

approach at secondary school that considers educational interventions in front of the students' spontaneous generalizations, in order to help them to improve mathematical reasoning.

I think that much more research is still needed for that. Specially, concerning the spontaneous transferences -such as analogies and metaphors- of algebra domain to another one, and the different approaches to algebra as contexts of emergence of spontaneous generalizations, their particularities and problems of control.

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FROM AREA TO NUMBER THEORY: A CASE STUDY

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In this paper we examine the way two 10th graders cope with a tiling problem that involves elementary concepts of number theory (more specifically linear Diophantine equations) in the geometrical context of a rectangle's area. The students' problem solving process is considered from two perspectives: the interplay between different approaches relevant to the conceptual backdrop of the task and the range of executive control skills showed by the students. Finally the issue of the setting of modeling problem solving situations into number theory tasks is also commented.

INTRODUCTION

Modeling problem solving situations into generalization tasks related to number theory is useful for learning mathematics and includes two stages: modeling and solving the number theory tasks that emerge. On the one hand, solving generalization tasks dealing with number theory serves as a tool for developing patterns, as a vehicle towards appreciation of structure, as a gateway to algebra, as a rich domain for investigating and conjecturing at any level of experience (Zazkis, 2007). However despite of their significance number theory related concepts are not sufficiently featured in mathematics education. Consequently many issues related to the structure of natural numbers and the relationships among numbers are not well grasped by learners (Sinclair, Zazkis & Liljedahl, 2004). On the other hand according to Mamona-Downs and Papadopoulos (2006) when students have an accumulated experience on problem solving they can affect changes in approach and are able to take advantage of overt structural features appearing within the task environment. Moreover they can show a deeper understanding of the nature of mathematical generalizations. In their work which lasted 3 years they followed some students from the 5th grade up to their 7th with emphasis on problem solving techniques relevant to area. Three years later we follow two of these students who currently attend the 10th grade (15 years old) during their effort to cope with a non-standard task concerning problem solving activity relevant to elementary number theory concepts. The case is interesting since it displays executive control skills related to the way the students proceed when they have to work on a new domain and to the handling and establishment of a 'model' that could lead to the generalization. This is why we try to explore in this paper the interplay of the students among different approaches during their problem solving path towards generalization and at the same time to refer to the actions of the students concerning decision making and executive control. In the next section we present the task and describe the students' background. After that in the next two sections we present the problem solving approaches followed by our students (Katerina for the first, Nikos for the second). These are followed by a discussion section trying to shed

light on these two axes (i.e., the interplay and the control issues) and finally the conclusions.

DESCRIPTION OF THE STUDY AND STUDENTS' BACKGROUND

Katerina and Nikos were 10th graders and they had participated in an earlier study conducted by Mamona-Downs and Papadopoulos (2006) aiming to explore and enhance the students' comprehension of the concept of area with an emphasis on problem solving techniques for the estimation of the area of irregular shapes. Their participation in this resulted in the creation of a "tool-bag" of available techniques as well as in an accumulated experience on the usage of these techniques. The conceptual framework now mainly lies in number theory. However in the official curricula (for 10th graders in Greece) the only reference to number theory concepts is a tiny one commenting the divisibility rules for the numbers 2, 3, 5, 9, 10.

This is the problem we posed to the students:

Which of the rectangles below could be covered completely using an integral number of tiles each of dimensions 5cm by 7cm but without breaking any tile?

Rectangle A: dimensions 30cm by 42cm

Rectangle B: dimensions 30cm by 40cm

Rectangle C: dimensions 23cm by 35cm

Rectangle D: dimensions 26cm by 35cm.

For each rectangle that could be covered according to the above condition show how the tiles would be placed inside the rectangle.

Now, we want to cover a rectangle with an integer number of (rectangular) tiles. Each tile is of dimensions 5cm by 7cm. What could be the possible dimensions of the rectangle?

The mathematical problem is: define a set of necessary and sufficient conditions on a , b so that there exists a rectangle of dimensions a by b , that can be covered completely with tiles of dimensions 5 by 7. Look at the side of length a : if there are s tiles that touch it with the side of length 5 and k tiles that touch it with the side of length 7, then $a = 5s + 7k$. The same reasoning applied to b gives $b = 5s' + 7k'$, where s , k , s' , k' , are non negative integers. Now if c denotes the total number of tiles used then the area ab of the rectangle should be $35c$. Therefore 35 divides ab . Thus, there are three cases: i) 35 divides a , ii) 35 divides b , or iii) non of the previous, but since 35 divides ab , 7 must divides a and 5 divides b (or vice versa). Consequently, a and b should satisfy one of the following necessary conditions: i) $a = 35m$, $b = 5s' + 7k'$, ii) $b = 35n$, $a = 5s + 7k$ iii) $a = 7q$, $b = 5t$ (or vice versa). It is easy then to be shown, that these conditions are also sufficient. Thus, even though the context of the task seems to be geometrical with its relevance to area, however a crucial aspect in solving the task is the usage of a Diophantine linear equation $ax + by = c$ where the unknowns x and y are allowed to take only natural numbers as solutions. The task consists of two parts. In the first part

four rectangles have been carefully selected to help the solver when finishing the first part to be able to reach the generalization asked in the second part.

The problem solving session lasted one hour, without any intervention from the researchers, and the students were asked to vocalize their thoughts while performing the task (for thinking aloud protocol and protocol analysis, see Schoenfeld, 1985). Protocol analysis gathered in non-intervention problem-solving session is considered especially appropriate for documenting the presence or absence of executive control decisions in problem solving and demonstrating the consequences of those executive decisions (Schoenfeld, 1985). The students' effort was tape-recorded, transcribed, and translated from Greek into English for the purpose of the paper.

THE FIRST PROBLEM SOLVING APPROACH - KATERINA

Katerina's first criterion for deciding whether the four rectangles can be covered completely by the tile was based on whether the dimensions of the four rectangles were multiples of the dimensions of the tile. This is why her answer was positive only for the rectangle A (since $30=5*6$ and $42=7*6$) and negative for the remaining three ones. She used the quotient of their areas ($E1/E2$, $E1$ the area of rectangle A and $E2$ the area of the tile) as a way to determine the number of the tiles required for the covering and not as a criterion to decide whether the tiling is possible). She tried then (according to the task) to show how the tiles will be placed inside the rectangle. The visual aspect of this action made the student to realize her mistake and to re-examine the four rectangles:

K.1.23. The tiles could be placed in any orientation in the interior of the big rectangle.

K.1.24. It is not necessary to be placed all of them in a similar orientation.

After that she verified that the rectangle A could be covered according to the task's statement. For the rectangle B she worked with an interplay between an arithmetical and geometrical-visual approach and she realized that the case of tiles with different orientation could mean that she could work with an 'equation' since she was not able to proceed geometrically. Now, it is the first time a linear combination is involved:

K.1.37. It could be $5x+7y=30$

K.1.38. It must be a rectangle with length of 30cm and this has to be expressed with tiles of length 5cm and 7cm.

She was not able to express her thought using proper mathematical terms. Her intention was to say that this equation did not have integer solutions (the case for an unknown to be equal with zero is excluded). So she decided to use terms such as 'round numbers' to show that it is needed for x and y to be integer numbers:

K.1.42. However this case is not possible... (*the above mentioned equation*)

K.1.43. We could not expect to have 'round' numbers for x and y .

For the rectangle C she decided to rely on the question whether the length of the side of the rectangle could be written as a linear combination of the dimensions of the tile. The lack of relevant knowledge on this domain provoked a certain technique for

overcoming this difficulty. She worked with successive multiples of 7 plus the remainder (expressed in multiples of 5). She followed the same line of thought for the rectangle D. The criterion of the linear combination was already established and by the technique of the successive multiples she founded that:

K.1.67. For the side of 26cm it is necessary to have 3 tiles of length 7cm and 1 tile of 5cm.

Immediately she turned to the visualization in order to verify that indeed this can be done, working independently on each dimension of the rectangle D (Fig. 1, left).

For the second part of the task she started with two steps that according to her opinion could help her:

K.1.74. I will use drawings because it seems to me easier in that way

K.1.76. How could I use the findings of the first part of the task?

She rejects the condition of E1 being an integer multiple of E2 as the unique criterion since:

K.1.87. ...it might be necessary for a tile (or some tiles) to be split.

Her model for finding the possible dimensions of any rectangle that could be covered by tiling using an area unit (tile) with dimensions 5 by 7 includes two cases exploiting her previous findings of the first part of the task.

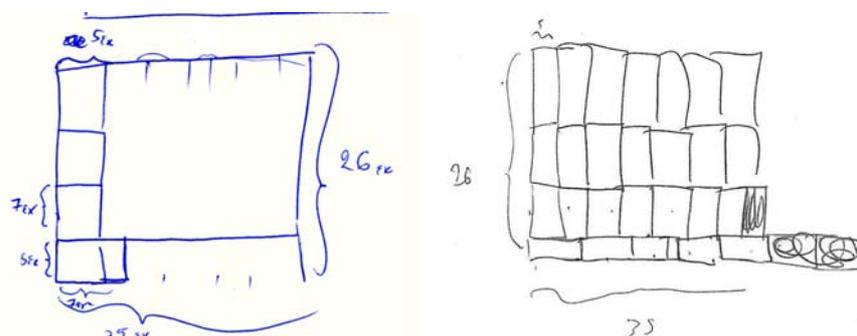


Fig.1 Katerina's (left) and Nikos's (right) visual approach on rectangle D

So, in the first case:

K.1.92. If all the tiles are oriented uniformly then the asked dimensions of the rectangle could be multiples of 5 or 7.

K.1.93. I will make a draw

K.1.94. It is a shape whose length is multiple of 7 and its width multiple of 5.

The second case resulted mainly as a consequence of the rectangle D and two conditions must be satisfied: one side must be multiple of the Least Common Multiple of the dimensions of the tile and the second dimension linear combination of them.

K.1.101. Length must be common multiple of 7 and 5 whereas width must be sum of tiles that are oriented some of them horizontally and some vertically.

She tried then to refine her model asking for a rule that governs the common multiples of 5 and 7 (i.e., of 35). For the number 5 she knew the divisibility rule (the last

digit must be 0 or 5). However she could not give any rule for the 7 or the 35. Finally she concluded with a recapitulation of her model trying to describe in a more formal way the second case of the model:

K.1.110. The rectangle in the second case should have one of its dimensions common multiple of both 5 and 7 and the other one sum of multiples of 5 and 7 at the same time.

THE SECOND PROBLEM SOLVING APPROACH - NIKOS

Nikos's first step was to interpret the statement of the problem in terms of conditions for the correct tiling: a) there is a rectangular region that has to be covered and b) the tile is a structural element of the task:

N.1.5. It means that each rectangle must be covered and for the measurement I must use an integer number of tiles

N.1.6. So we could consider this rectangle of 5 by 7 as a measurement unit

In his work and for each one of the four rectangles we can distinguish a concrete line of thought. For the rectangle A, his criterion was (as in Katerina's case) the proportionality of the sides, i.e. whether the dimensions of the rectangle were multiples of the dimensions of the tile. We have to mention here that his way of reading the task was non-linear in the sense that he did not follow the instructions of the task in the given order. Thus, he did not initially give answers for all the rectangles but after deciding for each rectangle, he proceeded to the specification of the way the tiles could be placed in the rectangle. In case there was not proportionality among the lengths of the sides of the rectangle and the tile -as it happened in the rectangle B- he used the criterion of $E1/E2$ as a way to ensure a negative answer. This quotient was not an integer number and this meant that there could not be coverage according to the task's statement. As he explained:

N.1.20. Because the ratio of their areas is not an integer

Now, in the rectangle C, the $E1/E2$ was an integer but the dimensions were not proportional. It is interesting the fact that his decision about $E1/E2$ is justified by the fact that $E2(=35\text{cm}^2)$ is a factor of $E1(=23*35)$, a relationship often overlooked even by pre-service elementary school teachers (Zazkis & Campbell, 1996). In their study and in an analogous quotient, teachers first calculated the product and then divided. At that point, Nikos asked for the linear combination that satisfies one of the dimensions since the second is multiple of 5:

N.1.24. When the area is 23 by 35, then obviously this product is divided by 35 which is the area of the unit (*tile*)

N.1.27. The point is **the way** the tiles must be placed

N.1.29. We could have $3*7+2$, $2*7+9$

N.1.34. $5+5+5+8$, $4*5+3$,....

N.1.35. For the 23 cm I can't make any **combination** of 5s and 7s.

In the rectangle D, he applied directly the rule of the linear combination that could satisfy the side of 26cm since the other one (35cm) was multiple of 5 (Fig.1, right). Trying to describe how the tiling will take place he worked initially independently on each side. However the way the tiles will be placed in one dimension affects the way the tiles will be placed in the second. This made him to turn towards a consideration of both dimensions at the same time. Despite this method could be considered adequate for him to give an answer for each rectangle, he preferred to re-check all the given rectangles, to verify his answers before making his final decision.

For the second part of the task he started with an impressive conjecture:

N.1.83. Obviously, if we want to cover a rectangle with this specific unit of dimensions 5 by 7, then the rectangle's sides must be the sum of multiples of 5 and 7 at the same time.

N.1.84. The case of $0*5$ and $0*7$ must be included in this.

However he still considers the two dimensions separately. Trying to figure out what would be the general case for the asked dimensions of the rectangle he created some arithmetical examples, fulfilling the need for linear combination for each dimension, without considering the fact that there is an interrelationship among the two dimensions since the area of the rectangle must be a multiple of 35:

N.1.102. We could say that $a=5x+7y$ (where 'a' is one of the rectangle's dimensions)

N.1.103. and similarly $b=5z+7w$

N.1.104. The product of these dimensions a and b will be the area

N.1.105. I can choose for a and b any sum of multiples. For example, $a=5+14=19$, $b=15+28=43$. So, the area is $19*43$

N.1.106. However in that case I have for the area a number that is not divided by 35.

N.1.107. So, 35 must divide the product $a*b$ which is the area of the rectangle.

N.1.112. Thus, $a=5x+7y$, $b=5z+7w$ and the quotient $ab/35$ must be an integer.

Trying to establish a model that would describe all the possible cases he was also influenced by the four rectangles of the first part of the task. He decided that his model would include two types of rectangles:

N.1.141. The first type concerns rectangles with one side multiple of 5 and the other multiple of 7. So, $a=5x$ and $b=7y$, which is $a=5x+0*7$ and similarly $b=0*5+7y$.

N.1.142. Consequently the area of such a rectangle divided by 35 gives an integer number as quotient.

N.1.154. And it is in accordance with the general form I conjectured earlier

For the second type he decided that:

N.1.159. One of the rectangle's side will be a sum of multiples of 5 and 7 at the same time

N.1.160. whereas the second side will be a multiple of 35

N.1.171. that is $a=5x+7y$ and $b=35z$

N.1.172. I think that these latter conditions form the most general form for the dimensions of any rectangle able to be covered with rectangular tiles 5 by 7.

After that, Nikos applied this most general form for each of the four rectangles examined in the first part to check the validity of this form. Furthermore he made clear that the first type of rectangles could be incorporated in the second:

N.1.188. ...to incorporate the first type which essentially is a special case in the second type which is more general..

Finally Nikos proceeded to a refinement of his model determining the circumstances that do not allow a rectangle to be covered according to the task giving a certain counterexample:

N.1.213. The second side must be always multiple of 35 and it can be constructed using either 5s or 7s.

N.1.218. This is the only solution because 35 is the Least Common Multiple of 5 and 7

N.1.219. This means that it is not possible to have a rectangle for which both its dimensions are linear combinations of 5s and 7s.

N.1.220. When I say that a is a linear combination of 5s and 7s, I mean that $a=5x+7y$ but not a multiple of 5 or 7.

DISCUSSION

In relevance to our research questions we could make some comments on our field-work.

1. Interplay among differing modes of thinking

During their attempts to solve the problem the students worked in tandem with two pairs of modes. The first pair included the arithmetical mode and visualization. Both students started arithmetically even though the context of the task was relevant to area that is geometrical. Katerina from the very beginning used the visual aspect as a tool. She started arithmetically but when she was unable to proceed with numbers she preferred to make drawings that would help her (K.1.74). In the same spirit some times she moved from the visual context to algebra. At some point she clarified that the tiles could be posed not necessarily with the same orientation. However she was not able to proceed geometrically and she preferred to turn to algebra asking for an equation (K.1.37). Nikos did not choose to work with this pair of modes. He mainly worked arithmetically and he turned to the visual aspect only to show the way the tiles could be placed in the interior of the four rectangles in the first part of the task. The second pair of modes has to do with the way students dealt with the dimensions of each rectangle. Working with the first mode dimensions were considered by the students separately as two unconnected objects (arithmetical mode). Thus, they made calculations (they summed, multiplied, divided) to determine the way the tiles should be placed in one dimension. In the second mode the dimensions were interrelated (geometrical mode, relevant to area). The fact is that the way the tiles will be placed in the first dimension influences the way the tiles will be placed in the second dimension. Working independently in two dimensions does not guarantee that the total area of the rectangle will be integer multiple of 35 which is the tile's area. Both students made successive movements between these two modes. Their initial approach was to

work separately for each dimension and only then they made the connection about the interrelation of the two dimensions. For example in Nikos's work (N.1.102-N.1.112) it is clear that his working on the two dimensions separately resulted in a rectangle that could not be covered with integer number of tiles since its area was not multiple of 35.

As a result of this interplay emerges -for Nikos in particular- the issue of putting forward a set of conditions (N.1.112) that are evidently realized as being *necessary* and later an equivalent set of conditions (N.1.172) that are seen as *sufficient* (because the covering of the relevant rectangles can be explicitly constructed).

2. Executive control and decision making issues

The students realized many actions that indicate interesting executive control and decision making skills. Katerina rejected her initial approach which was based only on the criterion of proportionality among the rectangle's and the tile's dimensions. This was because her turn to visualization made her to realize that it was not necessary for the tiles to be placed in a uniform orientation. This turn seemed to be in practice an important act of control. The task's statement did not give any direction concerning the way the tiles could be placed inside the rectangle. It was up to her to interpret correctly the statement. Later when she tried to solve the Diophantine equation she applied the technique of the successive multiples. According to this technique if one has to solve the equation $ax+by=c$ starts with positive multiples of a and then examines whether c minus ax is multiple of b or vice versa (i.e., one starts with multiples of b). This is an act of control since the solving of the equation was dealing with the task's limitation to use an integer number of tiles without breaking any of them. When she decided to deal with the second part of the task her first thought was to use her previous results (K.1.76). Moreover, an important act of control was the 'model' she proposed for estimating the possible dimensions of any rectangle that could be covered with an integer number of tiles according to the statement of the task (K.1.92, K.1.110). She exploited her previous findings (the four rectangles of the first part), and progressively she established this 'model' checking step by step its accordance with these rectangles as also with examples generated by herself. The choice of examples is especially important since not every example facilitates a successful generalization. Nikos also made an analogous proposition of a 'model'. He was also based on the four rectangles of the first part of the task. The steps followed by his line of thought reveal presence of control: First look if there is proportionality among the dimensions. See also whether $E1/E2$ is not an integer. This means that your answer has to be negative. It is not necessary always to make the long division $E1/E2$. Instead, see whether $E2$ is factor of the $E1$ (N.1.24). Now if sides are not proportional and $E1/E2$ is an integer, then construct the Diophantine equation and apply a strategy to find integer solutions. He also used to check always the consistency of his generalization model against particular examples and this is important. The continuous checking of their steps that both students showed is especially significant as an act of

control since students checking is not usually part of the algebraic thinking of the students when they make generalizations (Lee and Wheeler, 1987). A capable problem solver recognizes a correct approach and insists on it. This evaluation of a specific approach could also be considered as an act of control. Nikos recognized the applicability of the linear combination and he used it to check the plausibility of his answers always according to task conditions (N.1.154). This often turn to the tasks' statement was a common pattern for both students. However, perhaps the most important act of control of both students was their effort to refine their model regardless of whether they succeeded. Katerina tried without success to achieve a condition for the second side to be common multiple of 5 and 7. Nikos however did manage to refine his 'model' determining whether it is impossible for a rectangle to be covered according to the task's requirements (N.1. 219). Such an asking for a counterexample actually is an important act of control.

CONCLUSIONS

According to Douady and Parzysz (1998) when a problem allows the solver to move between different modes during the problem solving process then an interplay between these different modes is caused. They claim that the effort of the solver to reach the solution results to the relations of these modes as well as to the usage of some tools that belong to each of them. Additionally "...this interaction provides new questions, conjectures, solving strategies, by appealing to tools or techniques whose relevance was not predictable under the initial formulation..." (p. 176). Both of our students were able to apply this interplay among two pairs of modes. In the first pair (arithmetical-visual) this interplay was used as a way that allowed overcoming difficulties about how to proceed or for verifying or checking the validity of an argument. In the second pair of modes the one mode (arithmetical, working on one dimension) was indicative of a surface understanding of the structural elements of the task. However it seemed that finally the students did show a deeper understanding of these elements through the other mode considering both dimensions at the same time (geometric, interrelated dimensions).

'Executive control' and 'decision making' constitute in general the issue of control in problem solving. Executive control is concerned with the solver's evaluation of the status of his/her current working vis-à-vis the solver's aims (Schoenfeld, 1985). In general, this requires mature deliberation in projecting the potential of the present line of thought, married with an anticipation how this might fit in with the system suggested from the task. In our study and despite their age, these 15-years old students showed considerable control skills in relation to the task's requirements on the one hand and the specification of the 'model' they proposed for solving the task on the other. The existed experience enabled students becoming capable to make generalizations.

Concluding we could refer to some final remarks that emphasize the significance of our results. It is common thesis that the task design is a crucial parameter for teaching

and learning algebra at every level. So, in reference to our work, we could claim that the setting of modelling problem solving situations into number theory tasks allows students to:

5. transfer knowledge from one domain to another during their successful interplay among different modes of thinking (algebraic thinking and geometrical one).
6. construct and propose a ‘model’ that possibly describes the situation and facilitates the generalization
7. generate examples that check the consistency of their model, and
8. generate counterexamples that result to the refinement of the proposed ‘model’.

Obviously it would be an exaggeration for these conclusions to be generalized since we dealt with two students and this study could be better considered as a case study. However these findings were encouraging enough to call for a design of a future research on these aspects of problem solving.

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ALLEGORIES IN THE TEACHING AND LEARNING OF MATHEMATICS

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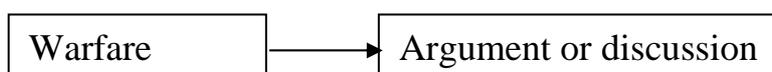
This paper explores how the concept allegory from literature theory can be used in the teaching and learning of mathematics. A cognitive allegory theory is developed in analogy with the metaphor theory of Lakoff and Johnson (1980). The theory differs from the traditional view. For instance an allegory is also a cognitive mapping and not only a narrative. The paper draws upon data from a study of how teacher training students learn the concept of linear congruence equations. The students are given word problems which were translated to congruence equations and later used to solve other word problems.

INTRODUCTION

Researchers like Lakoff, Núñez, Sfard and Presmeg have elaborated the role of metaphors in mathematics and mathematics education, see for instance Lakoff and Núñez (2000), Sfard (1994) and Presmeg (1997). Allegory is another concept from literature theory which so far has been sparsely used in mathematics education. In this paper we suggest that the concept of allegory can be applied to this field. Our contribution is to develop allegory as a part of mathematics education theory, in a way similar to how metaphors have been used in the tradition initiated by Lakoff and Johnson (1980).

METAPHORS AND ALLEGORIES

Traditionally a metaphor is a figure of speech in which a phrase denoting one kind of object or idea is used in place of another. An example is “You are straight on target with your reply.” In this view of metaphors “straight on target” is a figure which means something else. The phrase can be translated to literal speech, for instance “precise and relevant”. In cognitive metaphor theory metaphors are not as in the old traditional view, seen as isolated phrases, but as systems structuring concepts and thought. Such systems map one domain into another such that the target domain inherits structure from the source domain. An example is “argument is war”, Lakoff and Johnson (1980, p. 4).



Target is a concept from warfare. If an argument is compared to an arrow or a bullet, we can characterize the argument by describing how the arrow aims at the target. But,

the metaphorical mapping can also express lots of other things. An example is “The teacher went into a defensive position when faced with critique”. ‘Defensive’ is also part of military jargon, just like ‘targets’ is. In the tradition initiated by Lakoff and Johnson it is stressed that metaphors create or modify abstract concepts. The metaphor “argument is war”, is modifying or giving a special interpretation of what argument is. In other cases metaphors create a complete new concept.

Allegories are similar to metaphors, but have the structure of narratives and are usually more extensive. The New Encyclopædia Britannica has this definition:

...allegories are forms of imaginative literature or spoken utterance constructed in such a way that their readers or listeners are encouraged to look for meanings hidden beneath the literal surface of the fiction. A story is told or perhaps enacted whose details when interpreted – are found to correspond to the details of some other system of relation (its hidden, allegorical sense) (Fadiman, 1986, p.110)

Like metaphors, an allegory maps one domain onto another one, but the source domain is a narrative. Different parts of the source narrative are mapped into different parts of the target domain. An example from the Bible is Galatians 4:24, in which the word ‘allegory’ appears in the King James Version of the Bible. Two covenants are compared to the first two sons of Abraham by a freewoman and a bondwoman. An allegory maps objects and persons of a narrative to a more abstract domain. Each woman is mapped to a covenant, and the story told by the Apostle Paul gives flesh and meaning to the rather abstract concepts of a new and an old covenant. Both this allegory of Paul and the parables of Jesus have clear didactical purposes. They are designed by a teacher. These kinds of allegories are the focus of this paper, but of course mathematical ideas and conceptions are the goal, not spiritual ones. The word ‘conception’ is used to avoid non-cognitive interpretations of the alternative word ‘concept’, see Sfard (1991, p. 3) and Rinvold (2007, p. 4). A conception is a cognitive network in which several allegories and metaphors can be nodes. It isn’t uncommon to think that concepts are primarily given by formal definitions. Such definitions are just an aspect of conceptions and not at all a complete description.

We restrict our attention to allegories which include a timeline. This means that the narratives move in time. All the parables of Jesus are like that and so are most narratives. In mathematics education many text problems have the form of narratives. Such problems will be called narrative text problems. In this paper ‘text problem’ will always mean ‘narrative text problem’. On the other hand, narrative is a wider concept than text problem. A narrative is neither necessarily a problem nor given by a text.

Not all narrative text problems are allegories. This is only the case if a narrative text problem is going to represent or create something else, which usually is more abstract. Consider the following text problem: “John was hiking in the mountains. The first day he walked 20 km and the next day 25 km. What is the total distance he walked these two days?” This problem isn’t likely to represent something outside it-

self. Most students will solve the problem, forget it, and go on to the next one. The following narrative is different:

Peter has an urn containing balls. On each ball it's written a prime number. The urn may contain more than one ball with the same number. Peter asks Andrew to draw as many balls as he wants. Then Andrew is asked to find the product of the numbers on the drawn balls. When Andrew has told what the product is, Peter starts calculating. After a while he says: "I know which balls you have drawn". How is this possible? What would happen if composite numbers had been written on the balls?

With possible guidance from a teacher, this story can help the students to understand unique factorization in prime numbers and the role of such numbers. Drawing of a ball represents a factor. The information that Peter is able to tell which balls Andrew has drawn, corresponds to uniqueness of prime factorization. The fact that he isn't able to tell the order the balls were drawn, represents the commutative law. The narrative is used to create understanding of an abstract property of numbers. The problem isn't just a problem among many, but may have a lasting effect.

Even if the teacher tells a story intended to be an allegory, the learners don't always understand it in this way. Relating to a constructivist epistemology, it is the learners themselves who develop allegories. An allegory may be idiosyncratic and has elements of individual variation.

METHOD

This paper uses data from a study of how teacher training students learn the mathematical concept of linear congruence equations. The study was conducted in March 2008 by the authors on our own students. The data consist of participant observation of a teaching and learning session and three videotaped and partially transcribed interviews. One of the researchers interviews the teacher of the lesson, who at the same time is the other researcher. Both researchers together interview groups of either two or three students. As part of the sessions, students work together with a text problem. The researchers then ask questions helping the students to describe their reasoning process. The student interviews were conducted two days after the lesson, and the teacher interview the day after that. The transcriptions, descriptions and interpretations of the teacher interview were read by the interviewee, discussed with the researcher and then adjusted. Later, in the process of writing the paper, the teacher sometimes remembered thoughts and events from the lesson which can't directly be read from the data. Such thoughts aren't presented as data, but have without doubt influenced interpretations and directions of the paper.

CONGRUENCE CALCULUS

The lesson is based on several text problems given to the students. The first problems concern week days. Exercise 1 asked them to calculate the weekday of 31st March,

given that 1st January was a Tuesday. The teacher gave comments and discussed solutions in between. The students themselves made tables resembling calendars.

0	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30				

After their work the teacher showed them the table above on a blackboard. Then he introduced the signs ‘ \equiv ’ and ‘mod’ for congruent numbers. From 1st January to 31st January is 30 days. He pointed to the numerals 2 and 30 in the table and connected them with a red line. Then the teacher said that 2 and 30 are in the same column and wrote $30 \equiv 2 \pmod{7}$. Mathematically this means that 30 and 2 have the same remainder upon division by seven. In other words, the difference between 30 and 2 is an integer multiple of 7. Practically, the meaning is that 30 days from now and 2 days from now differs with a number of integer weeks. The identity was followed by $29 \equiv 1 \pmod{7}$ and $31 \equiv 3 \pmod{7}$ since 2008 is a leap year and March has 31 days. Finally the teacher wrote

$$30 + 29 + 31 \equiv 2 + 1 + 3 \equiv 6 \pmod{7}.$$

The move of six days forwards from a Tuesday gives a Monday, so that is the weekday of 31st March.

NARRATIVE TEXT PROBLEMS

After three other text problems having to do with calculation of week days, the students were given what we call the Duckburg problem:

A ship arrives at the harbour of Duckburg today, which is a Monday. Then the ship arrives at the harbour every third day. Some days later the ship arrives at Duckburg harbour on a Wednesday (two weekdays later). How many arrivals later can this be?

According to our observations, all students made a table with the weekdays from Monday to Sunday in the first row. There was some variation in the content of the tables, but in some way all students marked the days when the ship arrived. They all discovered the first solution of the problem, and some even found a formula for the number of arrivals when the ship arrives on a Wednesday. The student work was followed up by the teacher in a plenary session. As support for the introduction of congruence equations, he made a protocol for the arrivals of the ship by writing the identities

$$3 \cdot 1 \equiv 3 \pmod{7}, 3 \cdot 2 \equiv 6 \pmod{7}, 3 \cdot 3 \equiv 2 \pmod{7}, 3 \cdot 4 \equiv 5 \pmod{7}, \dots$$

He simultaneously said things like “three times four is in the same column as five”, referring to the table. Then the teacher related the text problem to the mathematical

formulation “which multiplies of three are in the same column as two when divided by seven”. Finally the congruence equation $3x \equiv 2 \pmod{7}$ was presented as a translation of the Duckburg text problem. The lesson continued with the demonstration of algebraic techniques for solving the equation. These techniques are part of the motivation for the translation, but we don’t discuss the solving process in the paper.

The Duckburg narrative is built upon the culturally shared concepts of days, weeks and calendars and the well-known phenomenon of ships regularly arriving at harbour cities. The name Duckburg, which is the domicile of the Disney figure Donald Duck, is used to make it clear that we are talking about a fantasy world in which details can be changed. Duckburg is a name which is easy to remember and with positive associations for most students. Also, this cartoon city is placed close to the sea, Grøsfjeld (2007), so arrivals of ships are relevant.

Later in the lesson the students were given the running track problem:

An athlete runs intervals of 300 m on a 400 meters running track. She starts at the starting line, runs 300 meters and stops. She continues this way. After a while she stops 100 meters after the starting line. How many 300 meter intervals has she run?

The students at first worked with the task themselves using a table. A drawing of the track was introduced afterwards by the teacher in the plenary. He used the drawing to simulate the intervals of the runner. This problem also corresponds to a linear congruence equation, but the situation is sufficiently different from the Duckburg problem to supplement it.

FROM NARRATIVE TO PROTOTYPE

Lakoff and Johnson (1980) claim that usually we place things and phenomena in categories by comparing with a typical or prototypical member. A prototypical bird has wings, is able to fly, lay eggs and has a beak. A picture of a blue jay is used by some dictionaries when defining birds. The blue jay is a candidate for a prototypical bird in countries where this bird is well known. We will use interview data to argue that the Duckburg problem has the potential to be a prototypical text problem for linear congruence equations. The argument is based on the way the Duckburg problem is used by the group of three students to solve the following text problem which also corresponds to a linear congruence equation:

Oda is sick and has to take a tablet every fifth hour, both day and night, in order to get well. She takes the first tablet at five in the morning. A friend calls her when her watch has just passed one o’clock. Her watch is analogue, that is, has rotating hands. How many tablets has Oda taken? There are several correct answers.

The students work for about twenty minutes with the problem and are then interviewed. In the interview one of the students was passive and seemed to participate only to a restricted degree. The active ones were Kari and Lise. A reason why they used so much time is that the problem is structurally more different from the Duck-

burg problem than we intended. In particular, Lise mentioned several times in the interview that she was confused because the problem was unclear. In fact, one of the researchers had forgotten to specify that Oda had just taken a tablet when the friend called. However, the students demonstrated understanding of the problem and were able to solve it with the extra constraint when asked to. Some statements by Kari support the claim that the Duckburg problem and some of its structure were used in the solution process. One example appears when Kari and Lise had written the congruence equation $5x \equiv 8 \pmod{12}$ on their sheets. When asked why the right hand side is 8, Kari said:

Kari: I remember when we worked on the problem with the ships which arrived at the harbour, we started with a Monday. Then we were going to find Wednesday, which was two days later, so we would have two there.

The student refers to the Duckburg problem which corresponds to the equation $3x \equiv 2 \pmod{7}$. The numeral 2 on the right hand side corresponds to 2 days later. In analogy, one o'clock is 8 hours later than five. We think this is the reason why the students wrote 8 on the right hand side of the equation $5x \equiv 8 \pmod{12}$. This is supported by another statement from the interview:

Kari: Then we draw a table with 12 columns. We started with the hour she took the first tablet, which was at five o'clock. (...) Then we counted every fifth hour...

The students made the same type of table as in the Duckburg problem. Five was the first column in the tablet case, as Monday was the first in the Duckburg problem. They counted how many hours after the first tablet she takes the next and would have got the same result if the first one was taken at for instance two o'clock. Another argument is this mentioning of the running track case:

Kari: Recognizing the running track task. Then 0 and 12 were the same. It was the starting line. Do we have to start with 0 then? But, now 0 is at 5 o'clock. If she starts at 5 there, then...

Without doubt, Kari now uses five o'clock as the zero point. The students however, didn't notice a minor difference between the questions in the problems. In the Duckburg problem the question is how many days *after the first* arrival the ship arrives at Wednesday. In the tablet case we asked how many tablets she has taken, *including the first* one. If Oda had taken the first tablet at hospital, and we had asked how many tablets Oda has taken at home, their equation had been correct. Then $5 \cdot 1 \equiv 5$ would have meant that she took the first tablet at home 5 hours after the one at hospital. The identity $5 \cdot 4 \equiv 20 \equiv 8$ would have meant that she took the fourth tablet at home 20 hours after the one at hospital. In some sense the wrong equation is more convincing than $5x \equiv 1 \pmod{12}$, which has $x \equiv 5$ as solution. In the latter case the students could just have put in the numbers 5, 1 and 12 given in the problem, without any understanding.

The students' use of the Duckburg problem and its structure is an argument that the Duckburg problem is on its way to becoming a prototype for a category of narratives. A more thorough study would have been necessary in order to claim with strength that some text problem has been established as a prototype. A possible weakness in our study is that only one student orally indicates this kind of reasoning. However, the students wrote the equation $5x \equiv 8 \pmod{12}$ collaboratively and Kari said that "we worked with the problem". This may indicate that at least Lise also shared her ideas.

ALLEGORIES AND GENERALIZATION

The transformation of a narrative text problem to a prototype for a class of such problems is an important step in giving a problem lasting value in mathematical thinking. This is one aspect of making the special case represent something general. In the Duckburg problem we can change the involved numbers without changing the structure of the narrative. Clearly, there is nothing special in "every third day" or "two weekdays later". The general is represented by the special case. The related "principle of generalization" is investigated in Rinvold (2007). To change the numbers of weekdays from seven to something else is also possible, but needs more imagination because weeks with seven days are so deeply established in our culture.

We think that the narrative of Duckburg has the potential of becoming an allegory for linear congruence equations with one unknown. When the narrative is turned into a prototype, each part of the narrative represents a part of a generalized narrative. For instance "arrives every third day" represents "a tablet every fifth hour" and "runs an interval of 300 meter" in the two other example problems. But, the parts of the Duckburg narrative also represent parts of a formal linear congruence equation. These representations can be made clearer with the help of mappings. In the latter case the source domain is the Duckburg problem and the target domain is the class of congruence equations.



"Every third day" is mapped onto $3x$, "two days later" onto 2 and the number of weekdays is mapped onto 7 in the equation $3x \equiv 2 \pmod{7}$.

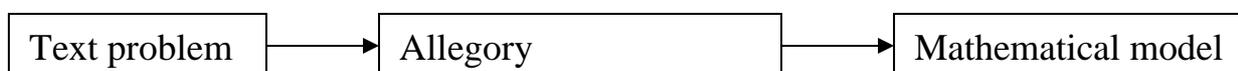
In the lesson the students were given some context free congruence equations and told how to translate these into Duckburg problems. When given the equation $2x \equiv 3 \pmod{8}$, one of the groups introduced a new weekday and drew a table. They quickly realized that the problem had no solution. With eight columns, steps of two weekdays can't lead to the same place as a change of three weekdays. The students said that it was a cheating exercise since there was no solution. In the beginning of the interview the students were asked about their experience of the lesson.

Kari: When we used the practical situations as starting points, we could in the end see a congruence equation, and then the numbers gave meaning. We could know what $4x$ really represents. When I recalled the ships, it gave meaning.

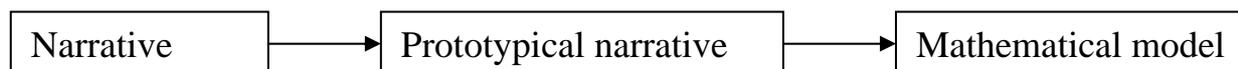
This could refer to the equation $4x \equiv 1 \pmod{7}$ which was one of the translation problems from the lesson. The formal congruence equation in the beginning seems to give little meaning to the students. The ships were part of the Duckburg problem and are used by the student to refer to that problem. We infer that translation of context free congruence equations into variants of the Duckburg problem was a main source of the meaning which emerged.

ALLEGORIES AND THE SOLVING OF TEXT PROBLEMS

When solving text problems allegories can be intermediate stages between the given narrative and a mathematical model.



The idea of prototypes means that new text problems given to the students won't be directly mapped to a mathematical model, but first to a prototype like the Duckburg problem.



A prototypical narrative in the learning of mathematics is a mathematized narrative. The given text problem or narrative also has to be mathematized to some degree in order to be mapped onto the prototype.

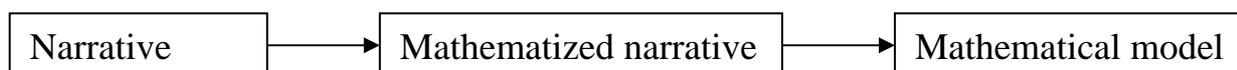
A crucial question is which qualities these mappings have for the students. Certainly, their versions of the mappings can differ from the intentions of the teacher. At best, the mappings reflect the mathematical structures effectively, but the mappings may also be based on superficial aspects of the text problems. Clement (1982) identified a syntactic and a semantic way of thinking when students tried to solve word problems for equations. The syntactic variant consists of a word by word translation of the text problem to algebraic language. Another kind of syntactic translation is based on possibly superficial similarities between the text problem and other text problems known to give a specific mathematical model. When working with the tablet problem, the student Kari made the following utterance:

Kari: We thought that it was $5x$ because it was every fifth hour she had to take the tablet and that was because the ship arrived every fifth day.

We see that the phrase “every fifth” appears in both problems. This may be interpreted as a sign that the student compared the appearance of words in the two problems. However, the ship in the Duckburg problem arrives every third day, not every fifth. In fact, some of the students, certainly including Kari, during the lesson also

solved a variation of the Duckburg problem corresponding to the congruence equation $5x \equiv 3 \pmod{7}$. This at least indicates that she compared with the appropriate version. Another argument that the translation has a semantic flavour is that in the lesson Kari, together with a group of students, generalized the Duckburg problem. They investigated what happens when the interval between arrivals or the number of weekdays ahead were changed.

Part of our theoretical thinking is that allegories are one of the sources for semantic meaning. When one text problem has been transformed to an allegory, the comparison with other text problems will no longer be just syntactical. Clement's semantic way of thinking means a mapping from a narrative or text problem to a mathematized version of the problem, and then a mapping to the congruence equation.



This is similar to the mappings of Parzysz (1999):



In the case of the Duckburg problem, the emphasis on the table, the columns and the introduction of mathematical signs means that the teacher intended to support the development of a mathematical structuring of the narrative. In the lesson the teacher explicitly sets up a mapping from the pseudo-concrete model to the congruence equation. For instance “the numbers which are multiples of three” was translated to ‘ $3x$ ’. The text problem is still a real situation for the student, but mathematical language has been introduced in order to change the students’ interpretation of the situation, making the translation to formal mathematics precise and smooth.

The term “real situation” is not as clear as commonsense language may suggest. We interpret ‘real’ as “real for the student”, as in RME, the Dutch approach to mathematics education (see van den Heuvel-Panhuizen, 2003). One point is that ‘real’ doesn’t have to mean practical or related to everyday life. The student Kari used the phrase ‘practical situation’ referring to the Duckburg problem, but imagined situations such as weeks with eight days can also be ‘real’. A situation isn’t something objective, but an experienced or imagined phenomenon. A narrative may create a situation in the mind of the student, but the process of mathematization also has a role in shaping the situation for the student. The degree of mathematization and semantic interpretation decides the quality of the mappings.

QUESTIONS FOR RESEARCH

This paper introduces the idea of cognitive allegories in mathematics education and supports this by discussions based on one limited empirical study. Obviously there is a need for more studies to establish that the concept of allegories is a fruitful one for

the use of narratives and text problems for conceptual learning in mathematics. It is necessary to have more thorough studies to establish that students transform introduced narratives into prototypes and allegories and how they do this. Other mathematical concepts and other potential allegories have to be studied. We also need to develop criteria for the design of such narratives. Another interesting task is to study how several allegories can be used for the same concept. We think that a single allegory usually isn't enough to develop a rich intuition. In our study the running track problem is a candidate for a complementing allegory, but only very limited evidence for this can be inferred from the data.

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ROLE OF AN ARTEFACT OF DYNAMIC ALGEBRA IN THE CONCEPTUALISATION OF THE ALGEBRAIC EQUALITY

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In this contribution, we explore the impact of Alnuset, an artefact of dynamic algebra, on the conceptualisation of algebraic equality. Many research works report about obstacles to conceptualise this notion due to interference of the previous arithmetic knowledge. New meanings need to be assigned to the equal sign and to letters used in algebraic expressions. Based on the hypothesis that Alnuset can be effectively used to mediate the conceptual development necessary to master the algebraic equality notion, two experiments have been designed and implemented in Italy and in France. They are reported in the second part of this paper.

Keywords: *Alnuset, semiotic mediation, conceptualisation of algebraic equality*

INTRODUCTION

The research reported in this paper is carried out in the framework of the ReMath project (<http://remath.cti.gr>) addressing the issue of using technologies in mathematics classes “*taking a ‘learning through representing’ approach and focusing on the didactical functionality of digital media*”. The work is “*based on evidence from experience involving a cyclical process of a) developing six state-of-the-art dynamic digital artefacts [DDA] for representing mathematics [...], b) developing scenarios for the use of these artefacts for educational added value, and c) carrying out empirical research involving cross-experimentation in realistic educational contexts*”. This paper presents the research concerning Alnuset, one of the 6 DDA developed within the project. First, some theoretical considerations related to the notion of algebraic equality, at stake in this paper, are presented. Next, our research hypotheses are discussed and Alnuset is briefly presented. Finally, two experiments involving this artefact are described and the main results are discussed.

THE NOTION OF ALGEBRAIC EQUALITY

Important conceptual developments are needed to pass from numerical expressions and arithmetic propositions to literal expressions and elementary algebra propositions. As a matter of fact, in arithmetic only numbers and symbols of operations are used and the control of what expressions and propositions denote can be realized through some simple computations. In elementary algebra, instead, letters are used to denote numbers in indeterminate way and new conceptualisations are necessary to maintain an operative, semantic and structural control on what expressions and

propositions denote (Drouhard 1995; Arzarello et al. 2002). The necessity of this conceptual development emerges clearly with the construction of the notion of algebraic equality. On the morphological plan, equality is a writing composed by two expressions or by an expression and a number connected by the “=” sign. On the semantic plan, equality denotes a truth value (true/false) related to the statement of a comparison. When the expression(s) composing the equality is (are) strictly numerical, it is easy verifying its truth value through some simple calculations (e.g., $2*3+2=8$ is true while $2*3+2=9$ is false). Experiences with numerical equality contribute to structure a sense of computational result for the “=” sign. This sense can be an obstacle in the conceptualisation of algebraic equality as relation between two terms, as highlighted by several researches (Kieran 1989, Filloy et al. 2000). When the expression(s) composing the equality is (are) literal the equality can present different senses because the value assumed by the letter can condition differently its truth value. In these cases the “=” sign should suggest to verify numerical conditions of the variable for which its two terms are equal. There are cases where the two terms could never be equal whatever the value of the letter is, as in $2(x+3)=4x-2(x-1)$. In other cases to interpret equality on the semantic plane, it is necessary to distinguish if it has to be considered as equation or as identity. The “=” sign assigns to the equality the sense of equation when its two members are equal only for specific values of the letter. For example, the equality $2x-5=x-1$ is true only for $x=4$ and it is false for all other values. Instead, the “=” sign gives to the equality the sense of identity when its two members are equal whatever the numerical value of the letter is, as in $2x+1=x+(x+1)$. In order to master algebraic equality, a conceptual development of notions of equation, identity, truth value, truth set and equivalent equation is necessary. Moreover, to express the way in which a letter can condition the truth value of an equality, it is necessary to develop a capability to use universal and existential quantifiers, even though in implicit way.

RESEARCH HYPOTHESIS

Traditionally, conceptual construction of algebraic equality is pursued through solving equations using techniques of symbolic manipulation. Empirical evidence and results of research have highlighted that in many cases this approach does not favour a construction of an appropriate sense either for the notion of algebraic equality or for that of solution of equation. In more recent years, a functional approach to algebra has been introduced within the didactical practice allowing to articulate algebraic and graphical registers of representations (Duval 1993). Even in this approach difficulties emerge. These regard the interpretation of a graph. For example, for the solution of equations of the type $ax+b=cx+d$, the intersection of the two lines in the graph has to be interpreted as indicator of the fact that the equation has a solution. Moreover this solution has to be read on the x-axis in correspondence of the intersection point of the lines. As Yerushalmy and Chazan (2002) observed, this approach is not devoid of obstacles: students can interpret the graph as comparing two functions ($y=ax+b$ and

$y=cx+d$) or as a solution set of a system of two equations in two unknowns, instead of an equation in a single variable. Our research hypothesis is that Alnuset, an artefact of dynamic algebra recently developed, can be effectively used to mediate conceptual development necessary to master the notion of algebraic equality. Further in the paper we discuss this hypothesis referring to some results of two experimentations.

SHORT DESCRIPTION OF ALNUSET

Alnuset is constituted of three components, Algebraic Line, Symbolic Manipulator and Functions, strictly integrated with each other. They enable quantitative, symbolic and functional techniques to operate with algebraic expressions and propositions.

The main characteristic of Algebraic Line component is the representation of an algebraic variable as a mobile point on the numerical line, which can be dragged with the mouse along the line. This feature has transformed the number line into an algebraic line where it is possible to operate with algebraic expressions and propositions through techniques of quantitative and dynamic nature. These techniques focus on numerical quantities indicated by an expression when its variable is dragged along the line or on numerical quantities that make true a proposition. These techniques make a dynamic algebra possible. The main characteristic of Symbolic Manipulator component is the possibility to transform algebraic expressions and propositions through a set of particular commands. These commands correspond to basic properties of operations, properties of equality and inequality, logic operations among propositions, operations among sets. Another characteristic is the possibility to create a new transformation rule once it has been proved. These characteristics support the development of skills regarding the algebraic transformation and they contribute to assign a meaning of proof to it. The main characteristic of Functions component is the possibility to operatively integrate Algebraic Line with Cartesian Plane, where graphs of expressions can be represented automatically. Moreover, dragging the point corresponding to the variable on the algebraic line makes the expression containing the variable move accordingly on the line. On Cartesian Plane, the point defined by the couple of values of the variable and of the expression moves on the graph. These characteristics support two integrated conceptions about the notion of function: a dynamic conception developed on Algebraic Line and a static one associated to the graph on Cartesian Plane. For a more detailed description of Alnuset, we refer to the work of Chiappini and Pedemonte presented in this edition of CERME within the working group 7.

EXPERIMENTATIONS

As we mentioned above, the development of DDAs was followed by a design of learning scenarios involving these tools and the implementation of these scenarios “*in realistic contexts*”. ReMath partners decided that each DDA would be experimented not only by the designer team, but also by an other team that did not participate to the DDA development. Such “cross-experimentation” of the DDA was intended to highlight the impact of theoretical frameworks and of contextual issues on the design of

both DDA and learning scenarios. Indeed, each team was free to set up educational goals taking account of institutional constraints and to choose theoretical approaches to frame the scenario design process. Thus, the experiments involving a given DDA were not meant to be compared, but rather to validate design choices related both to the DDA and the learning scenarios.

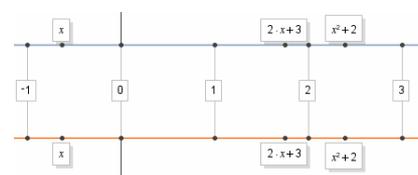
Italian experimentation

The experimentation activity reported below, lasting 1h40, has involved a class of 15-16 year-old students (Grade 10) attending a Classic Lyceum. The students worked in pairs using Alnuset. Previously, they had carried out 6 activities with Alnuset centred on notions concerning algebraic expressions. The whole teaching experiment lasted about 20 hours. The activity considered in this paper is centred on solving a 2nd degree equation. In the previous school year, students had learnt to solve 1st degree equations through symbolic manipulation. In this activity notions of conditioned equality, solution of an equation, equivalent equations, truth value of an equality and truth set of an equation are addressed. The didactical goal is the conceptual development of these notions while the research goal is the study of Alnuset mediation in this conceptual development. The activity comprises several tasks. The first task aims at allowing students to explicit their own conception of the algebraic equality notion.

Task: Consider the following two polynomials: x^2+2 ; $2x+3$. Explain what it means putting the equal sign between them, or, in other words, how you interpret the following writing $x^2+2=2x+3$.

Many students attribute to the “=” sign the meaning of computation result. Nevertheless they were already faced with 1st degree equations. A typical students’ answer is: “*To put the equal sign between two polynomial expressions means that these expressions have the same result*”. For many students inserting the equal sign between two expressions suggests the idea that the computation result of the two terms has to be equal when a value is assigned to the letter.

In the following task students were asked to represent the two expressions on the algebraic line of Alnuset to verify their answers. Dragging the mobile point x along the line (and observing that the points corresponding to the two expressions move accordingly), all students noted that there are only two values of x for which the points of the two expressions are close to each other, almost coincident. Through this exploration students experienced that equality of two expressions is conditioned by numerical values of the variable, which is crucial to develop the conditioned equality notion. In previous activities with Alnuset, students experienced that every point of the algebraic line is associated to a post-it that contains all expressions constructed by the user denoting that point. In order to verify equality of two expressions, the students tried to find values of x for which the two expressions belong to the same post-it. Since these irrational values had to be constructed on the line, the students could not verify this directly: “*we don’t*



understand what is the number...it will be 2 point something...even if we use zoom in we don't understand ...". The technique mediated by Alnuset to find these irrational numbers requires transforming the equation into its canonical form ($x^2-2x-1=0$), representing its associated polynomial on the line and using a specific command to find roots of this polynomial. Our hypothesis was that this technique could favour a conceptual development of notions of equivalent equations and of truth value of an equation. The transformation was realized in the Symbolic Manipulator and was guided by the following task:

Task: Select the equation and use the rule $A=B \Leftrightarrow A-B=0$ to transform it. Translate the result produced by this rule into natural language.

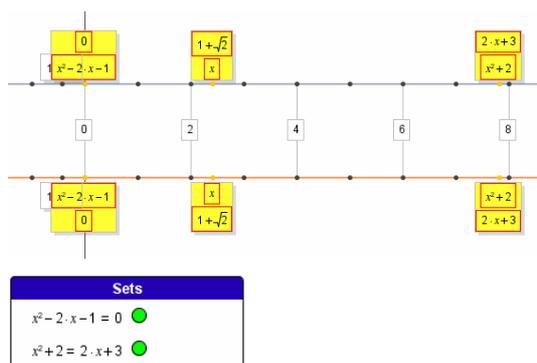
This task focuses on the rule $A=B \Leftrightarrow A-B=0$ of the manipulator through which it is possible to transform the equality preserving the equivalence. We report two students' answers: *"If two terms are equal, then their difference is zero"*; *"it means that if two expressions are equal, subtracting them the result will be zero"*. The conditional form of these sentences reflects a construction of an idea for the notion of conditioned equality used to justify the result produced by the rule. This does not mean that the students have understood the equivalence between the two equations in terms of preservation of the same truth set. Such understanding is the aim of the whole activity and its achievement requires several conceptual developments. First of all, students have to understand that the values of x for which x^2+2 is equal to $2x+3$ are the same for which x^2-2x-1 is equal to 0.

The following task was assigned to favour exploring such quantitative relations:

Task: Make a hypothesis about the relationship among the three polynomials x^2+2 ; $2x+3$; x^2-2x-1 imagining what you could observe if you represented them on the algebraic line and if you dragged x . Use algebraic line to verify your hypothesis.

A posteriori, we realized that the formulation of this task was misleading since it oriented the students to search for a relation among the three polynomials rather than between couples of terms of the two equations. Some students dragged the variable to explore if there were values of x for which the three polynomials could denote the same value on the line. They verified that such a value does not exist. Even if this exploration was not expected, it proved an important reference to overcome the following misconception, quite common in the students, concerning the equivalence of equations: two equations are equivalent if all their terms are equal for some values of the variable. A new formulation of the task by the experimenters allowed students to focus on couples of terms of the two equations. Exploiting the drag of the variable x they understood that, in order to find values of x for which x^2+2 is equal to $2x+3$, it is sufficient to find values of x for which x^2-2x-1 is equal to 0. Subsequently they used the command $E=0$ to find the irrational roots of the polynomial x^2-2x-1 and to automatically represent them on the line (the student drags x to approximate the polynomial to 0 and the system automatically produces the exact value of the root). Through

this experience an idea of equivalent equation begin to emerge. This idea will be consolidated through the exploitation of a new dynamic feedback offered by the system. We note that in the algebraic line environment expressions are represented on the line while equalities are represented in a specific window named “sets” and they are associated to a marker (a little dot) whose colour is managed automatically by the system. The marker is green if, for the current value of the variable on the line, the equality is true and, conversely, it is red if the equality is false. Dragging the variable allowed students to explore the truth of equalities and to construct a meaning for this notion, as shown in the following dialogue.



Student: If I drag x on $1 + \sqrt{2}$ and on $1 - \sqrt{2}$, the expressions of the first equation belong to the same post-it, namely $x^2 - 2x - 1$ and 0 are coincident for these values of x .

For the same values of x even $x^2 + 2$ and $2x + 3$ belong to a same post-it.

Student 1: When x is $1 - \sqrt{2}$ the two expressions are equal and these [dots] are green. So, since the solution of this equation is $1 - \sqrt{2}$ then also for the other equation is the same.

Student 2: and for the other value [$1 + \sqrt{2}$] it is true the same

Student 1: yes, for these values the two equations are true

To support the conceptual development necessary to master the notion of truth set of an equation, two other operative and representative possibilities of the algebraic line were exploited: a graphic editor to construct the truth set of an equality and a new feedback of the system to validate it. The graphic editor allows to operate on the line to define a numerical set that the system automatically translates into the formal set language associating it to a coloured marker. We note that the green/red colour of the marker means that the current variable value on the line is/is not an element of the set. As expected, students used this feedback to validate the defined numerical set as truth set of the equation, verifying the green colour accordance between equation marker and set marker during the drag of the variable on the line: “for the values $1 + \sqrt{2}$ and $1 - \sqrt{2}$ the two equations $x^2 + 2 = 2x + 3$ and $x^2 - 2x - 1 = 0$ have the same truth set. In our opinion, the two expressions from one side and the other side of the $=$ sign belong to the same post-it when x assumes the values of their solutions”.

French experimentation

Let us remind that the French team that experimented activities described in this section was not involved in the development of Alnuset. Therefore, a preliminary step before designing a learning scenario with Alnuset consisted in an analysis of the tool

from the usability and acceptability point of view (Tricot et al. 2003). This analysis brought to light main functionalities supposed to enhance learning of functions and equations, notions at the core of the Grade 10 math curriculum: dynamic representation of the relationship between a variable and an expression involving this variable and possibility to articulate different registers of representation of algebraic expressions (Krotoff 2008). In addition, praxeological analysis (Chevallard 1992) of the above mentioned mathematical objects allowed identifying types of tasks and comparing techniques available in Alnuset with institutional techniques identified in the Grade 10 textbook. This analysis shows that while institutional techniques are based on algebraic transformations on algebraic expressions, Alnuset techniques rely on visual observations of expressions (their position on the algebraic line, colour feedback...), and (almost) no algebraic treatment is needed when applying these techniques (Krotoff 2008). Thus, Alnuset seemed to be an appropriate tool to help students develop conceptual understanding of notions of function and equation, without adding difficulties linked to algebraic treatment that many students do not master well enough.

Although the French experiment was designed independently from the Italian one presented above, both experiments shared some didactic goals, in particular conceptual understanding of notions related to the notion of equation: meaning of a letter as variable or as unknown and of the “=” sign, understanding of what a solution of an equation means. Therefore, below we present only activities and results related to these common concerns. Our research goal was both to investigate to what extent the new representation of algebraic expressions provided by Alnuset contributes to the conceptual understanding of the notions at stake, and to study instrumental geneses (Rabardel, 1995) in students when interacting with Alnuset.

The experiment took place in a Grade 10 class with 34 students (15-16 years old), during two sessions lasting 3 hours altogether, held in a computer lab where students worked in pairs on a computer. Their work was framed by worksheets describing tasks and asking questions the students had to answer. Written productions are one kind of gathered data. Moreover, a few student pairs’ verbal exchanges were audio recorded and this data provided us with the possibility to carry out case studies, namely as regards studying instrumental genesis in students. Results reported below draw mostly on these case studies.

The first task involving equations was finding solutions of $f(x)=4$, with $f(x)=x^2$, after having studied the function f with Alnuset. The task was intentionally quite simple: the students could either solve the equation algebraically and verify the result with Alnuset, or solve the equation with the tool by dragging x along the algebraic line and looking for values for which x^2 coincides with 4. Both strategies appeared to almost the same extent. However, students who used the exploration strategy to find solutions with Alnuset succeeded better than those who used the tool just to verify the results found by solving the equation algebraically, since these often provided only one,

positive, solution. Alnuset turned out to be an efficient tool helping students to overcome their conception $x^2=k^2 \Leftrightarrow x=k$.

The next task, solving the equation $x^2=3x+4$, was proposed to prompt students to use Alnuset technique of dragging x on the line and searching for values for which the equality is true. Indeed, the students did not know yet algebraic techniques for solving such 2nd degree equation. Using the Alnuset technique requires to make sense of the “=” sign as meaning that the two expressions have the same value for some value of x , and thus also to distinguish between a letter standing for a variable and for an unknown. The students were first asked to determine whether 1, -1 and 2 are solutions of the equation. This question was intended to reveal students’ conceptions of the notion of solution of an equation. Almost all students succeeded the activity. However, the following dialogue between two students reveals the student’s S1 conception of a solution linked to the arithmetic sense of the “=” sign:

S1: You have to find 1. No, $3x+4$ must be equal to 1, the solution.

S2: No, you have to put x on 1 and the... what do you call it [*pointing at $3x+4$*]... Because x^2 should be equal to... the thing, equation and this isn’t the case (Fig. 2a).

S1: But it’s the result this [*pointing at 1*].

Indeed, it seems that S1 considers a solution of an equation to be the “result” or the value of the expressions: if 1 is a solution of $x^2=3x+4$, then $(x^2=) 3x+4=1$. This conception emerged also when the students checked for -1 . The student S2 grasped the targeted technique: “*On the other hand, -1 is the solution since $f(-1)$ equals this equals this equals this*” (Fig. 2b), and explains it to S1: “*To find the solutions, you drag x until x^2 and $3x+4$ overlap*”.



Figure 2. (a) 1 is not a solution since x^2 and $3x+4$ do not overlap when x is on 1; (b) -1 is the solution.

The students were then asked to find other solutions of the equation if there are any. This task was much more difficult for the students. Only half of the pairs succeeded it. The main obstacle was the fact that when $x=4$ (the other solution), the expressions x^2 and $3x+4$ went out of the screen. The students did not spontaneously resort to using “tracking” functionality allowing to keep visualising the expressions taking bigger values, which the students had used previously. Teacher’s intervention was necessary to remind the availability of this functionality, which helped the students to successfully finish the task. Such observations point to the issue of instrumental genesis in students, which can be a rather long-term process, especially in the case of

innovative functionalities such as “tracking” or “E=0” command as we will see in the following example.

Next, the students were asked to find solutions of the equation $x^2=x+3$. This equation has irrational roots, therefore the technique based on dragging x and making the expressions overlap is not efficient anymore. The aim was to introduce the E=0 command allowing to find irrational roots of the expression x^2-x-3 and thus bring the idea of equivalent equations $A=B$ and $A-B=0$. Most students used first the strategy relying on dragging x on the line and either provided approximate values of solutions (e.g., 2,3 and $-1,3$) or framed the solutions by integers (e.g., $-2 < x < 0$ and $2 < x < 4$). Teacher intervention was necessary to clarify that exact solutions were to be found and suggest using the E=0 command. Students encountered two main difficulties with using this command. The first difficulty was making a link between the expression $E(x)$ they needed to find to be able to solve the given equation of the type $A=B$ (the question intended to guide them was “What equation of the type $E(x)=0$ allows solving the given equation? Explain.”). The teacher had to state more precisely that Alnuset only provides a tool for solving equations with the right side equal to 0, and that it is then necessary to transform the given equation in a way to have 0 on the right side. Such intervention helped most students to find an adequate expression and use the E=0 command. The other difficulty was linked to the use of the E=0 command. In fact, to solve an equation with Alnuset, one has to use this command as many times as the equation has solutions. Although the students were aware that the equation has two solutions (most of them provided two approximate values at the beginning of the task), they did not think of using the command twice in order to find both solutions, and thus provided only a single solution. This difficulty is linked to the development of a scheme of using the E=0 command, which supposes to anticipate the number of solutions of a given equation and to be aware of the fact that applying the command gives a single solution at a time. This is quite unusual comparing to traditional algebraic techniques.

CONCLUSION

These two experimentations enable a first evaluation of the mediation offered by Alnuset. In both experiments Alnuset was exploited both as a tool to verify already developed conjectures and as a tool to explore algebraic phenomena in order to arise and validate new conjectures. It allows designing learning scenarios with characteristics that are deeply different, according to given contexts (institutional, cultural, social...) and educational goals to be pursued. The two experimentations lasted differently and this allowed to evidence that: (i) the instrumental genesis of the Alnuset instrumental techniques may be quite short for some of them (e.g., using drag mode for determining equivalence of two expressions) and longer for others (e.g., using E=0 command to solve polynomial equations and interpreting associated feedback); (ii) the instrumented techniques can be controlled by mathematical justifications and previous knowledge, correct or not. On the other hand, the French experiment showed that when the previous mathematical knowledge is rather fragile and the students are not very confident with it, resorting to the tool can help them carry out successfully the tasks they would not succeed without using the tool; (iii) the instrumented techniques produce representative dynamic events that can be easily related to algebraic notions and meaning involved in the activity.

Both experiments evidenced the importance of teacher’s role in supporting the development of students’ instrumental genesis at the beginning of the activity with Al-

nuset. Moreover, the role of the teacher remains very important during the whole activity to orient discussions and considerations about instrumental issues that have to be intertwined with algebraic knowledge involved in the activity.

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COMMUNICATING A SENSE OF ELEMENTARY ALGEBRA TO PRESERVICE PRIMARY TEACHERS

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This article reports on a university course for preservice primary teachers on ‘patterns and structures in primary school to prepare algebraic thinking’. We believe, if arithmetic is taught with an algebraic awareness, e.g. looking for patterns within arithmetic problems, algebraic thinking could be enhanced in primary school and the ‘cognitive gap’ between arithmetic and algebra would be reduced. In order to teach with an algebraic awareness the teachers must have developed such awareness themselves. We present the design of a course with which we contributed to this. The course serves us as a pilot experience for gaining hypotheses on the needs of teacher students and on good teaching interventions. We conclude the article with research questions in this field of teacher education.

THEORETICAL FRAMEWORK AND FOCUS OF THE PAPER

It is well known that there are many-faceted difficulties in learning algebra (see for example the contributions in Bednarz et al., 1996). Also the working group on algebraic thinking of CERME 5 has considered many features constituting elementary algebra and problems of learners. Some of the contributions are concerned with problems of constructing new mathematical objects (as formal or as abstract, cognitive objects) when dealing with algebraic expressions (e.g. Dörfler, 2007; Fischer, 2007a; Lagrange, 2007). Others point to students’ often limited or inappropriate ways of interpreting symbolic arithmetic or algebraic expressions (e.g. Alexandrou-Leonidou and Philippou, 2007; Molina et al., 2007; Papaieronymou, 2007). What do these many-faced difficulties have in common with the learning of algebra? The working group agreed on one central theme of algebra underlying all other aspects discussed: ‘expressing generality’ (Puig et al., 2007). However, students often do not experience this feature in their algebra classes.

One reason for these difficulties is the so-called ‘cognitive gap’ between arithmetic and algebra. Herscovics and Linchevski (1994) highlight some aspects of it. Features like the manipulation of variables occurring twice or more in a formal expression demand truly new cognitive abilities or constructions as compared to an arithmetic viewpoint. Similarly, they suggest a new viewpoint is required to comprehend formal arithmetic expressions as entities in their own right, or to look for patterns and structures in arithmetic problems. As a consequence of the observed gap, students have to cope with several changes to their habit of solving problems, their ways of interpreting signs, their ideas on what mathematics is about.

In this article we propose that some of the features of this gap between arithmetic and algebra are not so much due to the given characteristics of the two areas of mathemat-

ics, but to a tradition of teaching arithmetic common to many countries. This tradition focuses on ways of interpreting arithmetic expressions and treating them, which cannot be extended to the algebraic sign system. What is more, the tradition of teaching arithmetic narrows the focus of mathematics to calculations and results, giving little scope for the search for general patterns and the discussion of structures. Things can be done differently. The way formal expressions are interpreted in algebra can also be used for interpreting arithmetic expressions. For example the expression $3+4$ need not only be understood as a description of an activity but also as a sign for a number. Many other characteristics of algebra could effectively first be established within arithmetic contexts. A lot of research exists on including algebraic activities in mathematical learning environments for primary school children. For example several studies (e.g. Carraher et al., 2008; Fischer, 2007b; Söbbeke, 2005) report on the understanding of arithmetic or geometric patterns by young children who are not yet familiar with the conventions of the formal algebraic sign system. When they become familiar with activities of this kind in primary school children might be better prepared for the step to algebra.

But how can primary school teachers be persuaded to teach these issues? For a pilot experience we designed a university course aimed at preparing (future) primary teachers for integrating algebraic aspects in the math classes. In this article we will explain our grounds for the design of the course and report on our experiences. At the end we suggest ideas for further research to help evaluate the course and develop it further.

A central issue for our course was how to persuade primary school teachers to engage in algebraic ideas. Understandably, primary school teachers tend to focus on the goals set by curricula for the first school years. Often they are not aware of the consequences of their attitudes for the children's learning of further mathematical concepts. Moreover, many of them do not see a connection between learning mathematics in primary school and algebra in secondary schools. And those who do are not aware of different ways of dealing with arithmetic. Therefore, we consider it a necessary prerequisite to help (future) primary teachers look at the mathematics in primary school from an algebraic perspective and to show them how they can integrate pre-algebraic thinking without losing track of their primary goals.

Mason (2007) gives some ideas on how teachers can learn to deal with the subject of expressing generality. One central point is the highlighting of typical mathematical processes involved in the search for general patterns and in their representation and use. This is one important connection between the general goals of mathematics and our specific interest in advancing algebraic thinking in primary school. We recognised different though interwoven aspects of 'algebraic awareness':

- *Experience with problem solving activities*, e.g. analysing and describing patterns and structures, continuing patterns, using structures for calculations and problem solving,

- *Knowledge of different mode of representations and structures* of problems, solution methods and solutions,
- The *disposition* to look for patterns and structures in arithmetic problems and to argue with them and to perceive arithmetic expressions as processes and as objects.

All of these aspects can be provoked within arithmetic and geometric contexts in primary school (grade 1 to 4).

CONCEPTUAL DESIGN OF THE COURSE

In the course we had four main goals:

- The students experience algebraic thinking within arithmetic and geometric contexts. They are encouraged by personal success and gain a broadened view on mathematical tasks.
- The students understand challenges of (pre)algebraic thinking as part of mathematics fitting in the goals of primary school.
- The students design and analyse mathematical problems concerning arithmetic or geometric patterns in a context of primary school either within a case study or while analysing schoolbooks.
- The students reflect upon learning mathematics themselves and by children.

Organisational frame

The class met three hours each week for one semester (14 weeks) and was open for advanced students who had already taken some mathematics and mathematics education for primary school. Twenty three students attended the course. To obtain credits each student had either to undertake and write a report of a short empirical study with one or more children, or write a theoretical theses comparing two series of schoolbooks.

Progression

1. Introducing the course subject

During the first weeks of the course the students were presented with mathematical problems, which comprised different aspects of algebra and algebraic thinking. With this activate approach the students experienced algebraic thinking instead of dealing with a theoretical definition. We chose problems which highlighted characteristic aspects of algebraic thinking. Quite a number of these problems dealt with the discovery and expression of patterns. The students had to solve them with their preferred problem solving strategy and with at least one strategy that children in primary school might use. The class reflected upon the solutions, the solution methods and different ways of presenting both. Furthermore, problem solving strategies were elaborated and

differences were highlighted between problems which appeared to be very similar at first sight but turned out to have very different algebraic potentials.

1) What number belongs to the brick on top?

6	8	
1	5	3

2) Complete the number walls. Write down your notices.

2	3	5

2	5	3

3	2	5

3	5	2

5	2	3

5	3	2

3) Complete the number walls. How do you approach them? Do you detect alternative completions? How many?

20		
10		1

20		
5		

$a+2b+c$		
$a+b$	$b+c$	
a	b	c

figure 1

figure 2

Figure 1 shows problems from a worksheet on “number walls”. Three-layer number walls involving additive structures within integers are an often used format in German school books. They are constructed as indicated in figure 2 (where a , b , and c are integers).

The first task on the worksheet presents a typical arithmetic task: the sum of integers has to be calculated. Note, however, that if used to *introduce* number walls, this already demands some degree of structural analysis. The second task also starts with the calculation of sums. But the request to write down observations leads to a closer examination; the different walls have to be compared. Describing differences and commonalities of the six walls with the same integers in the bottom bricks demands a careful study of the walls. Verbalising the observation and explaining the findings helps the discovery of a mathematical pattern. Finally, the number walls of the third task cannot be worked out in the same straightforward way. They present disconnected problems (one of them is not solvable within integers) which can be tackled in different ways. Asking for the approach implies an explicit reflection on it; asking for other solutions and for the number of other solutions guides students towards a structural approach to the task.

Other problems given to the students offer different views of symbolical terms like the equal sign and expressions like the sums of two numbers. Given “ $3+4=$ ”, say, whereas one view sees the equal sign as an instruction to calculate ($3+4$ adds up to 7), another promotes the view of the equal sign as a balance and of the sum as being a number ($3+4$ is the same number as $2+5$). Cognitively the latter demands a view of an arithmetical expression as a number as well as a process (cf. Gray and Tall, 1994). Furthermore, the students were given problems on number sequences, geometric visualisations of such, arithmetic laws and (dis)connected arithmetic word problems.

Although the problems were basically taken from German schoolbooks for classes 1 to 4, the students had numerous difficulties solving them. Many of them made very formal use of variables, often with little or no understanding of the meaning. This caused mistakes on the one hand and impeded discussion of mathematical relations on the other hand. Moreover, the students frequently had difficulties to think of strategies without using variables. Often they thought of only one alternative strategy: systematic trial and improvement. Yet, they did not always acknowledge this as a valuable mathematical strategy.

Working on the given problems, the students were surprised by their experiences:

- 8.** There are mathematical tasks with different ways of solving them, some problems can even have different solutions.
- 9.** Strategies can be found which do not involve the formal algebraic sign system are possible. But to find such strategies requires insight into the structure.
- 10.** The inherent structure of similar looking problems can be very different.
- 11.** These problems offer challenges on different levels. Some of these challenges are revealed to the students only when working on them.

These experiences were facilitated by questions attached to the mathematical problems, which emphasised mathematical activities like visualising, comparing and arguing.

Besides solving the problems the students reflected upon the mathematical activities required by the children. Through this, we raised ideas of what algebraic thinking is about.

We concluded the introductory unit by taking a more theoretical standpoint. In class we discussed the paper of Lorenz (2006) on possibilities and challenges in using geometric representations of arithmetic patterns for illuminating the structure and solving problems about them. The claims of the text could well be investigated through some of the examples the class had worked on in the previous weeks.

The class then developed a notion of ‘good’ mathematical problems in general and in respect to algebraic thinking. The class agreed on the following features to constitute ‘good’ problems:

A ‘good’ mathematical problem must be

- 9.** open to different approaches or different solutions,
- 10.** given with a mathematical goal,
- 11.** easy enough for every child in class to start solving the problem and to obtain a (partial) result, but also
- 12.** challenging even for high achieving children.

The feature specifically relevant for the course is the encouragement of algebraic thinking. We listed the following characteristics of algebraic thought which can be found within arithmetic or geometric contexts:

- unknowns not only at the end of an expression,
- equal sign as balance sign,
- arithmetic expressions as representations of numbers,
- describing patterns,
- calculating big numbers effectively using structures instead of extensive calculations.

These criteria are neither original or exhaustive. But they reflect the views the students had developed at this point on the course and used as basis for their own work. Throughout the rest of the course these criteria served as an orientation for the students when developing and evaluating mathematical problems for primary school.

2. Preparing and realizing the individual projects

The students then started with their own projects. Seven carried out a case study with a child in primary school. Each of them prepared a short sequence of problems he or she was going to use in the interview. This sequence had to be analysed with respect to its algebraic potential. There was opportunity in class to have these sequences discussed in small groups and to work them through before they were used in the interviews.

After the interviews were accomplished the students had to transcribe interesting parts and analyse the children's performance. The students in Frankfurt have plenty of experience with carrying out interviews and analysing them with respect to interaction. Therefore we decided not to elaborate on these issues. Nevertheless we devoted one lesson to tools for analysing transcripts. We focused on gaining mathematical knowledge through working on representations. For this we read a paper on the epistemological triangle of Steinbring (2000). In this text two analyses are presented in which students explain and develop ideas on a mathematical problem. However this text turned out to be very difficult. It is too theory laden for our students to enable them to extract general principles and apply them for their own analyses.

Students who aimed for a theoretical thesis each had to analyse two series of schoolbooks for classes 1 to 4. Each student had to select two formats of problems like a sequence of problems with a common pattern or number walls recurring in his or her schoolbooks in different classes. He or she had to give an analysis of these formats pointing to their algebraic potential. On the ground of this analysis he or she had to evaluate the way the schoolbook makes use of these formats and compare the two series of schoolbooks. The students of this group, too, were given the opportunity to

have some examples from their schoolbooks discussed in class. In addition, throughout the whole course such formats served as examples for different aspects.

The individual projects were mainly worked on at home. Meanwhile, we were able to introduce several theoretical articles on mathematics education which discuss issues related to our subject. Our main focus was to interrelate educational theories with the students' own mathematical activities as well as with their design and analysis of problems. Through this, we also deepened the students' algebraic understanding.

We covered topics like learning, practising and problem categories. In particular, we compared learning mathematics via instruction to learning via discovery (cp. Wittmann, 1994) and related the findings to previous class sessions. Practising – not only algorithms of calculation but also mathematical processes like problem solving, representing mathematical ideas, argumentation – was connected to the different learning theories (cp. Winter, 1984) and discussed for one specific problem. The task of determining whether problems are open (for different solutions and solution methods) informative (regarding the learner's thinking) and process-oriented (which means, if they support mathematical activities like discovering, arguing and further elaborations; Sundermann and Selter, 2006), leads to reflecting on problems, varying and exploring them.

These articles addressed general principles of teaching mathematics in primary school. We found plenty of opportunities to interpret and understand them in respect to our subject of inducing algebraic thinking. Thus this subject appeared in the general context of teaching mathematics in primary school not as an exotic theme but as one way of complying with these general goals that are commonly shared.

3. Presenting the students' projects

In the last unit of the course the students presented some of their results. Those writing a theoretical thesis chose examples of their analytical work and some theoretical aspects related to it. Those doing an empirical analyses presented crucial aspects of their interview analyses. All of them were asked to look for ways of presentation that would actively involve the class.

The students who analysed schoolbooks had to think of criteria for their analysis first. It turned out that they used the criteria listed in the introduction only as a starting point. In order to build their criteria most of them chose one or more topics on learning mathematics we discussed during the second part of the course. It is pleasant to see that they altogether made careful analyses covering important aspects of algebraic thinking which proved a good insight into the formats.

For example one student gave an overview on which pages the formats occur in the schoolbooks before she went into quantitative and qualitative analyses. She did not only list the pages but stated the type of task linked to it, like discussing calculation rules, completing the format and comparing numbers of neighboured formats. This

affected her quantitative analysis: She put the frequency of a format into perspective with the aligned task. While she noted that in one book the format was used more often she also claimed that a lot of the tasks merely practise calculating.

At the beginning of the term another student commented on a schoolbook she had seen in use in primary school. She reported that the school children would love to work on the book and do their work autonomously. Her submitted analysis of this schoolbook shows that she gained a broadened view on mathematics teaching. She stated that this particular schoolbook is based on a theory of mathematics education of tiny steps but little structural understanding of mathematics problems.

CONCLUSION AND FUTURE PROSPECTIVES

Overall we are satisfied with this course since we met our goals for most part. The students gained (more) competencies solving mathematical problems with an algebraic notion. They intend to integrate (pre-)algebraic thinking in their mathematics classes through designing adequate mathematical tasks and an appropriate attitude. They gained competencies in judging maths problems in school books and their own, as well as reflecting on their interventions. Our evaluation corresponds well with the students' feedback.

It turned out that the aspects of algebraic thinking were best understood when they were directly linked to their own experiences – and more than once – and reflected upon afterwards. For example the students had to solve a variety of problems with patterns during the first sessions which were originally designed for primary school. We reflected upon them: The students had to present their results, find different solution methods, vary the tasks, compare it with other tasks, etc. The attitude to look for patterns became an important issue for the group and the focus on patterns can be traced to the students' projects. In contrast some algebraic characteristics were not understood quite as well, like the notion of the equal sign as a balance sign. This is perhaps because we did not mention those characteristics quite as often, or because we looked at them from a more theoretical perspective.

We believe that it was not only the students who learnt a lot about (pre-)algebraic thinking: we also benefited from this course. We learnt something about the thinking of university students, gained perspectives on teaching them and at the same time got deeper insight of the potential of mathematical tasks for teaching algebraic thinking.

This teaching experience serves as a pilot study for us. On the basis of this experience we see several research questions that would be worth following up.

- The course seems to indicate that student teachers do need help to get an algebraic awareness, even though they have used much algebra in their own time at school. A quantitative empirical study of teachers' performances in observing patterns and structures in geometric or arithmetic contexts should

give hard evidence on this issue. One could also investigate how, during a course like ours, students' ideas about arithmetic lessons change.

- We do not know very much about the inner representations student teachers have of principles of algebraic notation and algebraic argumentation. A qualitative empirical investigation on this issue might help us to better understand some of the underlying difficulties. In connection with this, the effects of some of the principles we applied during the course should be evaluated by empirical studies. The results of these studies might inform the development of curricula for teacher education.
- An underlying assumption of our course is that children who work on describing and using patterns in the context of arithmetic problems will be better prepared for algebra than students who only do calculations in their arithmetic classes in primary school. This conforms with theoretical positions on the nature of algebraic thinking in scientific literature. However, more empirical evidence is needed to investigate this claim.

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CONCEPTION OF VARIANCE AND INVARIANCE AS A POSSIBLE PASSAGE FROM EARLY SCHOOL MATHEMATICS TO ALGEBRA

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Change and invariance appear at the very early stages of learning mathematics. In this theoretical paper, examples of topics and tasks from primary school mathematics with various kinds of interplay between variation and invariants are presented. Application of this approach might be a tool that helps to improve non-formal algebraic thinking of students. We present some examples of pre-service teachers' reasoning in terms of variances and invariance.

INTRODUCTION

For over fifty years, mathematics educators have studied ways of teaching algebra. Beyond viewing algebra as generalized arithmetic, various classifications for meaning of algebra, algebraic symbolism, procedures and skills have been proposed (Usiskin, 1988). In algebra, students have to manipulate letters of different natures such as unknown numbers (Tahta, 1972), parameters, and variables. Required skills include specific rules for manipulating expressions and an ability to construct and analyze patterns. These components form the basis for the structure of school algebra, which appears to students to be abstract and rather artificial. Through dealing with transformation of algebraic expressions, students can hardly recognize the core ideas of algebra, such as application of standard arithmetic procedures to unknown or unspecified numbers.

From the point of view of primary school teachers, algebra is comprised of letters, rules of operations with expressions, and formulas to solve equations. Moreover, the term *pre-algebra* in the school math curricula stands for some “advanced arithmetic” topics that are linked with future algebra, mostly chronologically but not conceptually.

Since 2005, the awareness of pre- and in-service teachers about algebra has been one of the “hot” issues of annual conferences on training primary school math teachers in Israel. In order to match the course *Algebraic Thinking* to the needs of pre-service primary school mathematics teachers, a systematic study on their vision of algebra has been initiated. Preliminary results of this research show that only a few of these students are aware of non-formal components of algebra (Sinitsky, Ilany, & Guberman, 2009).

What mathematical concept could help pre- and in-service teachers to construct relevant algebraic comprehension? School algebra is a combination of generalized arithmetic, calculations with letters, and properties of operations (Merzlyakov &

Shirshov, 1977). In general, it requires reasoning on connections and relations between objects, for example, finding similarities and dissimilarities between objects. The question “what changes and what does not change?” seems to be fruitful in a meta-cognitive discourse that concerns problem-solving activity (Mason, 2007; Mevarech & Kramarski, 2003). We propose to apply this question at the very early stages of mathematical learning as a possible tool to connect primary school mathematics with algebra.

WHY VARIANCE AND INVARIANCE?

The two notions of *variance* and *invariance* are strongly linked, since “invariance only makes sense and is only detectable when there is variation” (Mason, 2007). Mason claims that “invariance in the midst of change” is one of three pervasive mathematical themes. Watson and Mason (2005) have elaborated the theory of possible variation and permissible change for the needs of mathematical pedagogy. The use of the concept of variance and invariance with pre-service teachers can develop their algebraic thinking and provide them with tools to construct examples.

The issue of learning processes is related to the human ability to associate and to distinguish between different characteristics of the same object. Research (Stavy & Tirosh, 2000; Stavy, Tsamir, & Tirosh, 2002) shows that reasoning patterns “same A then same B” and “more A then more B” are prevalent among students, and direct analogy causes deep misconceptions in the learning of mathematics. Refining comprehension of various types of interconnections between change and invariance may be fruitful for improving cognitive schemes of students.

Starting from secondary school, students systematically face algebraic notation and formalism. The most significant feature of algebra for students is manipulating with letters. It seems to them (and to their teachers) as a switch from four arithmetic operations with numeric operands into *terra incognita* of some quantities that are both unknown and tend to change.

Although the abilities to deal with varying objects, to explain, and to formulate are the very essence of secondary school algebra, students are expected to grapple with these based on their experience in primary school. In the framework of systematic construction of formal algebraic concepts, pre-algebra is responsible for the development of pre-abstract apprehensions of algebra (Linchevski, 1995).

In this paper, we bring up some issues from primary school mathematics and observe these problems in terms of change and invariance. We refer, at a non-formal level, to the main components of school algebra mentioned by Linchevski, i.e. using variables and algebraic transformations, generalization, structuring, and equations.

We proposed related mathematical activities for pre-service primary school mathematics teachers, and discuss some relevant classroom findings in the last paragraph and in the appendix.

VARIANCE AND INVARIANCE INTERPLAY IN PRIMARY SCHOOL

Word problems and algorithms of school algebra often have an origin, or an analogy, in primary school mathematics. Despite the concrete numerical form of arithmetic problems, they usually enable some algebraic generalizations into patterns for several number sets with suitable restrictions. For example, the property of being divisible by 9 is invariant in relation to any change in order of digits. Analysis of mathematical problems of primary school from the point of view of algebraic concepts may be fruitful for students as a step to constructing their algebraic thinking.

A consideration of variation, change, and invariance may help to provide a non-formal algebraic vision of arithmetic issues. Every mathematical situation provides a variety of variance–invariance links. Moreover, a suitable set of variations and related invariants that describe a task may provide a way to solve it. We illustrate the appearance and application of the “change and invariance” concept in a number of topics from primary school mathematics.

Quantities and numbers

The most fundamental example of invariant is human ability to count (Invariant, n. d.). It starts with the transition from objects to quantities and develops through numerous activities of counting objects of different nature. At this stage, quantity is invariant of physical properties of specific objects. Children also learn to count a given set of objects in different ways, and discover that the result is invariant of various (correct) counting procedures.

Thus, a basic conception of equality of quantities arises: the equality represents the fact that the same quantity is obtained or described in two different ways. There is also the possibility of inverting the problem: which changes are allowed within a given quantity? This question seems concerns a misconception of equality. Linchevski and Herscovics (1996) have connected cognitive difficulties in the transition from arithmetic to algebra to dual procedural-structural algebraic thinking. A well-known example of such difficulties is the comprehension of the expression $34+7=$ as a command to carry out an action (Gray & Tall, 1991). Accordingly, in the equation $8+4=\Delta+5$ the unknown is interpreted by students as the result of adding $8+4$. In contrast, the idea of equality as an idiom of invariance invites possible changes.

An appropriate didactical scheme for primary school students is to focus on problems of decomposition of given number into a sum of two addends. Typical questions require producing additional presentations based on a given one as demonstrated in this activity:

- $8=3+5$ How can you split the same number 8 into another sum of two addends?
- How does a change in the first addend influence the second one?

- How does the change of addends of two “adjacent” decompositions vary? (At a higher level this leads to a conclusion on invariance of parity for differences of addends for several decompositions of the same number)
- For a given odd (or even) number, what can you say about the parity of addends in each decomposition?

This activity invites students to discover the role of invariant quantities in a game of changing in.

In discussions with pre-service teachers, the same questions were followed by further generalizations. For instance, the last question on parity leads to a conclusion on the invariance of parity of algebraic sums of numbers, with arbitrary distribution of +/- signs, through an analogy to the arithmetic expression. A choice of signs +/- does not influence the parity of the expression $a_1 \pm a_2 \pm \dots \pm a_k$ (for integers a_1, a_2, \dots, a_k). At an advanced level, the same mathematical situation leads to combinatorial tasks, such as:

- In how many ways can we split a given natural number into the sum of equal addends?
- Can you arrange any presentation of an arbitrary multiple of three as a sum of consecutive addends by first splitting it into a sum of equal addends?
- In how many ways can we split a given natural number into sum of consecutive addends?

In the appendix, we present examples of pre-service primary school math teachers' response to some of these questions.

With this cluster of problems, we explored the concept of permissible changes within a given invariant in a variety of mathematical questions and levels.

Comparison of quantities in terms of change and invariance

In addition to invariance, the very basic process of counting deals with *variation* of quantity. Adding each new object to a given set of objects generates a new quantity that is greater than the given one. These examples are taken from the Curricula for Primary School in Israel (Curriculum, 2006): the sum $5+1$ is greater than 5, and the sum $67+2$ is less by 1 than the sum $67+3$.

From the point of view of invariance and change, students try “to find the same” in a pair of arithmetic expressions. The same operand plays a role of a parameter, i.e. arbitrary but the same number. The only cause for different values of given expressions is the difference in second operands. Therefore, to compare two quantities one looks at them in a structural manner: namely, noting the similarity and the difference between them. For example, comparing the results of other arithmetic operations when one of the operands is the same for both expressions:

- Which one of the differences is greater: $856 - 47$ or $856 - 44$?
- What is the difference between the two products: 84×123 and 83×123 ?

- Shirli arranged dolls in nine rows with the same number in every row. She added two dolls to each row. By how many dolls did the total number of dolls increase?

In school algebra, the presence of an unknown quantity typically turns the simple problem of comparing two similar expressions into a difficult one for students. For example, the comparing the pair $a-7$ and $a+7$ as opposed to the pair $7-a$ and $7+a$.

Further, in order to compare more “remote” arithmetic expressions, one can try to interpret them as a different change of the same connecting expression. When pre-service teachers discussed how to compare two differences, i.e. $1234-528$ and $1243-516$, they constructed intermediate expressions, $1234-516$ or $1243-528$. In a similar way, they proposed using the product 83×123 for comparison of products 84×123 and 83×124 . This method of comparison is also an algebraic one: two expressions $a*b$ and $c*d$ are interpreted as changes of the same basic structure $a*d$ or $c*b$.

Computational algorithms and techniques

In school algebra, most procedures cause changes in algebraic expressions yet preserve equality or inequality. This issue is not new for students. Almost every process of computation includes some transformation of a given arithmetic expression to another one. The transformation is valid provided it keeps invariant the value of the expression. In fact, both the rules of arithmetic operations and standard computational algorithms preserve the invariants:

- To calculate the sum $123+456$, one groups similar units of addends, $123+456=(100+400)+(20+50)+(3+6)$ – this is a direct analogy of gathering similar terms in algebraic expressions.
- The difference $123-49$ can be replaced by a new expression that retains the value of the given one: $123-49=124-50$.

Fraction reduction and expansion are additional examples in elementary school of variation that preserves value.

The ability to find a suitable variation of a given expression that preserves its value is a useful starting point for oral calculations. A necessary condition to apply is the invariance of the value under the change of form of the calculated expression.

We have studied the strategies pre-service primary school math teachers apply to calculate sums of arithmetic progressions (Sinitsky & Ilany, 2008). Only 5% of the students succeeded in recalling a suitable formula and applying it correctly. After taking part in series of assignments concerning interplay of change and invariance, the students were given similar tasks. They tried to calculate sums by reducing them to

known series in various ways

$(2+3+\dots+26 \rightarrow 1+2+\dots+25; 3+6+9+\dots+60=[2+4+\dots+40]+[1+2+\dots+20])$.

Number properties and range of generalization

When students manipulate algebraic expressions, the application of natural intuitive reasoning schemes “same A then same B” or “more A then more B” leads them to false reasoning: “ $x^2 = y^2$ implies $x = y$ ”, “ $-x > 2$, therefore $x > -2$ ”. In terms of change and invariance, this is a problem of connection between different invariants.

There are numerous examples of correct ways of reasoning when letters A and B stand for the property of numbers. Examples of correct propositions concerning squares of natural numbers: “If the unit digits of two numbers are the same their squares have the same unit digit”; “The squares of numbers with the same parity are also of the same parity”; “As natural numbers increase so do their squares”.

Such a convenient tie between invariants and changes invites a wide generalizing. Accordingly, questions that lead to counter examples and determination of range of possible changes or invariants are crucial: “Does changing the order of a sum change the result?”; “Does equal square/rectangle/parallelogram area imply the same perimeter?”; “Does multiplying a number by 2 increase the number of its divisors?”

Generalizing regularities and solving problems without algebraic formalism

An equation composed to solve a word problem algebraically expresses an invariance of some (typically unknown) value. For example, in problems that concerns motion, the same distance that two vehicles cover in different manners is the invariant of the two processes involved. Hence, the ability to identify invariance through some changes is useful for solving mathematical problems.

At primary school level, the search for invariance is an effective tool to discover regularities in numerical tables and in tables of arithmetic operations. For example, in the hundred table (see appendix, example 2) numbers increase constantly, but the change between adjacent cells in any row or column is invariant of the cell position. Similarly, the difference of products of diagonals of any 2×2 square is an invariant of the choice of square.

The next stage of proving those propositions typically involves some algebraic manipulation. Detecting a proper invariant for the problem can help avoiding formal algebra and provide a transparent proof with a generic example (Mason & Pimm, 1984). This type of reasoning is presented in the appendix.

Coming back to word problems and relevant equations, we illustrate another aspect of interaction between variation and invariance in pre-algebra mathematics. This interplay may provide non-algebraic solutions for some word problems. For example:

John bought two kinds of items: pencils that cost 30 cents each and pens that cost 50 cents each. He paid 6.20 euro for 16 items. How many pencils and how many pens did John buy?

We restate here a well-known arithmetic solution of the problem with an emphasis on variation and invariance. We start with the possibility that John bought 16 pencils at a cost of 4.80 euro. Now we need *to vary* the cost, keeping *invariant* the number of items. The answer to the question “How many pencils do we need to exchange for pens to increase the total price by 1.40?” provides the solution of the problem. In this approach, the total number of items is an invariant of the process. An alternative method of solution starts from any combination of items that provides the desirable cost (for example, 10 pens and 4 pencils). The next step is to vary the number of items keeping the total cost invariant.

A taxonomy of change and invariants

Due to many characteristics of each object or process, every variance results in several changes and introduces invariants as well. Alternatively, preserving some invariant permits variances of other properties. Thus, there are many possibilities of interrelation between change and invariance. The same sort of connection can occur in various mathematical problems and topics.

From the above and other examples, we have derived a suggested taxonomy for change, variance, and invariance:

- An invariant is given *a priori*, and the focus is on possible changes and related invariants.
- To understand the action of *prescribed change*, we look for imposed variations and for given invariants.
- To solve a problem, it is necessary to *find some key invariant* of all the procedures involved.
- To treat a mathematical situation, we *introduce a suitable variation* or a sequence of variations.

Within this classification, the two latter cases seem to be more complicated since they involve *construction* of relevant objects or procedures. On the other hand, a specific kind of relation between variation and invariance is connected more with the method of solving the problem than with the problem itself. Thus, various solutions of the same problem may bring into play different kinds of interaction of change and invariance or even a combination of those interactions.

PEDAGOGICAL ASPECTS OF THE APPROACH

We require that primary school mathematics teachers be competent to recognize relevant kinds of variations and invariants in various issues and problems of elementary mathematics. We need to start introducing this concept in teachers' education to en-

sure that they can construct an additional didactical tool for mathematical discourse in a classroom.

To test the influence of discourse in terms of interplay between variance and invariance on algebraic thinking of students, we designed an experimental study. The research involved future and current teachers of mathematics at elementary school. We tried to learn if, and to what extent, discourse on variance and invariance influenced beliefs and knowledge on the ability of further application of non-formal algebraic reasoning. In addition to checking the validity of our conjectures, we would like to improve the awareness of school educators about the use of variation and invariance at primary school level.

So far, pre-service teachers have participated in the study through problem solving activities in the framework of their courses in pedagogical colleges. Throughout these activities, they have discussed the ideas of variance and invariance with specific mathematical issues. We have found that future teachers have begun to construct examples for teaching in elementary school that invite algebraic thinking and argumentation in terms of change, comparison and invariants (Sinitsky & Ilany, 2008).

To promote this concept, we designed additional mathematical assignments. Each task includes a cluster of math problems on different issues at various levels of difficulty *united by the same relation of variance and invariance*. The starting point is part of the school curriculum, should be familiar to every pre-service teacher, and is a basis for further generalizations and analogies. The style of all the assignments is that of open problems in order to stimulate various approaches and strategies.

CONCLUSION

In this paper, we discussed applications of conception of changes and invariants in primary school mathematics. We looked at numerical problems from a point of view that is general and in many cases algebraic. The same types of connection can be detected in different mathematical issues. The ability to recognize variation and invariants may be an effective tool in constructing non-formal algebraic thinking of students. However, as a necessary stage, it requires the awareness of teachers on the subject. Some preliminary evidence on pre-service teachers' activities seems encouraging and invites further wide-scale research.

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APPENDIX: IT LOOKS LIKE ALGEBRA

Two samples of reasoning involving variance and invariance interplay are presented.

1. Representations of natural number as a sum of consequent addends – fragment of transcript of discussion with pre-service primary school mathematics teachers

Students wrote down all the pairs with the given product, 30, and constructed sample sums of equal addends.

Student A: *“I start with equal addends. Now, for $30=10+10+10$, I keep the total sum but vary the addends: (she moves a finger from the first term to the third one and has marked it with an arrow) $30=10+10+10$. We get $30=9+10+11$, and it is possible to do this for each of these sums of equal addends! For example, I can derive from this sum (she points $30=6+6+6+6+6$) another sum of consequent addends: $30=4+5+6+7+8$ and... No, it does not work with $30=15+15$: we need the sum to be invariant but also keep a middle term, and there is no middle addend here. Ah, I can try to split each one of 15s, but it changes the number of addends...”*

Students also obtained representation of 30 as a sum of four consequent addends: $30=6+7+8+9$, and tried to derive sum of consequent addends from the sum of fifteen equal ones.

Student B: *“But we need negative numbers. Aha, after the cancellation we get exactly the same sum! It means that for every presentation of natural number as a sum of consequent natural numbers we can make more sums if we use integer numbers that will be cancelled after that, for example, $12=3+4+5$ and also $12=(-2)+(-1)+0+1+2+3+4+5$, because $(-2)+(-1)+0+1+2=0$ ”*

2. Divisibility of differences of two-digit numbers with “inverted” digits – sample proof

Conjecture: The difference of two two-digit numbers, where the second number has the same digits as the first one but in inverted order, is a multiple of 9.

How can we introduce the justification of this proposition without algebraic formalism in the framework of discussion with the students?

Let us check, what is the same in each pair of these numbers? They have the same digits, therefore also the same sum of digits. Now, let us mark an arbitrary pair of these numbers in a hundred table, for instance, 62 and 26. Their difference is just a distance between cells. Can we construct the route from 26 to 62 that keeps **invariant** the sum of digits? The route passes through 35, 44 and 53 before reaching 62. Each step increases the number by 9 (see “decomposition” of one of the steps in the table), therefore the total difference is a multiple of 9. Moreover, the difference between inverted two-digit numbers equals the number of such steps multiplied by 9.

1	2	3	4	5	6	...
11	12	13	14	15	16	...
21	22	23	24	25	26	...
31	32	33	34	35	36	...
41	42	43	44	45	46	...
51	52	53	54	55	56	...
61	62	63	64	65	66	...
...

GROWING PATTERNS AS EXAMPLES FOR DEVELOPING A NEW VIEW ONTO ALGEBRA AND ARITHMETIC

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Sequences of growing patterns play an increasing role in the context of introducing terms. In this paper we reflect a new view onto the role of those particular visualisations for arithmetic and as well for algebra. By using a pupil's document we illustrate in this paper the theoretical framework of our concept.

Keywords: *representation/growing pattern, pre-algebra, children's interpretation, building structures and relations into diagrams*

1 Perspectives on the Mathematical Knowledge on the Way to Algebra

On their way from arithmetic to algebra, students have to develop a new awareness for the general, for the variation and the variable. At this period a new way of thinking, a new understanding of the previously acquired mathematical concepts, symbols and operations and thus a new interpretation of old knowledge becomes necessary. Students of elementary school become acquainted with equations in arithmetic lessons primarily in the context of calculating. In a special kind of lesson culture they learn more or less subconsciously that by dealing with equations they have to calculate the part on the left of the equal sign and after that to note the result on the right ("Task-Result-Interpretation"; Winter 1982). In many cases the equal sign is interpreted as a sign demanding to calculate. In many cases its function as a symbol of equality is not spoken about or used in every day arithmetic lessons. Such restriction in the interpretation, understanding and use of arithmetic terms and symbols is an obstacle not only for the later algebraic comprehension, but also for developing successful calculation strategies for the elementary arithmetical operations in the following school years.

Today algebra is seen as the lingua franca of higher mathematics (Hefendehl-Hebeker & Oldenburg 2008). However, algebra does not obtain the meaning and power of such a superior language if its status is restricted to the transformation and calculation of terms. Algebra has to be a "system characterised by indeterminacy of objects, an analytic nature of thinking and symbolic ways of designating objects" (Cooper & Warren 2008, 24). Therefore it is indispensable for the construction of algebraic comprehension not merely to calculate terms, but increasingly to see them in their structures, in order to understand formulae and principles. "The equation (or formula) must not be perceived as a sort of calculation shorthand note but rather as a type of scheme, which can in different ways be rearranged and be filled with concrete content" (Winter 1982, 210).

Various studies are concerned with the transition from arithmetic to algebra, which is accompanied by ruptures and discontinuities from the arithmetical to the algebraical

view (cf. Bednarz & Janvier 1996). In our paper we focus not only on ruptures in the transition from one view (e. g. arithmetic, geometric) to another but also on reinterpretations and developments within one view in the context of growing patterns.

2 Growing Patterns and Mathematical Visualizations as Mediators between old and new Mathematical Knowledge

If the substance of algebra is seen in the way it represents the principles and structures of mathematics and not in terms of the “behaviours“ of algebra (such as simplification and factorisation) (...) (cf. Cooper & Warren 2008, 24), then it is important for the introduction to algebra to make meaningful learning possible for the students, which at the same time constructs basic ideas that are sustainable in the long term. That means that such learning and exploring of algebraic ideas is always situated in the difficult balance between a rather empirical view on concrete objects and actions on the one hand and a certainly more challenging but in the long run necessary and profitable view on relations and structures on the other hand.

On their way to algebra it is necessary especially for young students to open a learning arrangement and an exploring field in which they can move between these poles of an empirical view on concrete objects and actions and a more abstract view on relations and structures. Structured mathematical visualization and growing patterns constitute such a learning environment, which merges those poles in a natural way.

Mathematical visualization and growing patterns - as a special type of mathematical visualization (for example to represent mathematical principles) - can mediate between the mathematical structure and the student’s thinking because of their special “double nature” (they are on the one hand concrete objects, which can be dealt with, which can be pointed at and counted, which can be manipulatively changed, and at the same time they are symbolic representatives of abstract mathematical ideas).

Mathematical visualizations and growing patterns are well-known to elementary and secondary school children from their daily mathematics classes. Geometrical patterns, which must be interpreted arithmetically, are used in class for various purposes. Steinweg (2002) notes that in text books dot patterns appear to practice calculating skills and thus function as visualizations, while sequences of dot patterns are to be explored as a separate and independent subject (cf. Steinweg 2002, 129-151). It is obvious that in everyday mathematics lessons dot patterns have predominantly the function of a methodological-didactical aid. Here is a parallel to the restricted view on equations and the equal sign mentioned above. Only in rare and isolated instances the structures incorporated in mathematical visualizations and growing patterns as well as equations are being purposefully explored and mentioned by the children. Against this backdrop Schwank and Novinska (2008) complain that didactic materials must be rescued from their shadow existence as mere aids and acquire a role as playing fields, in which genuine thinking processes can develop. Central questions such as “How many” and “if ... then” in dealing with this type of materials open a smooth transition to algebraic thinking - at first based on representations which become ac-

cessible through interaction, speech and graphics (cf. Schwank und Novinska 2007, 121).

3 Features in the exploration of growing patterns on the way to Algebra

If sequences of patterns support this new view – not only to figure out arithmetic terms, but to notice the underlying structure, transpose, re-organize and reinterpret them in a positive manner, then the following five aspects seem to be of particular importance. These categories were developed by connecting first results of a case study in progress (cf. Böttinger 2007) and the results of a completed case study (cf. Söbbeke 2005). In order to interpret representations more and more in the function as a representative of relations and structures and thus to focus on the abstract and generalizable “pre-algebraic aspects” it was necessary to connect in this paper two analysis instruments and to use them both to analyse the interpretations of student Ron. In order to describe the interplay between the geometrical, the arithmetical and the algebraic view it was necessary to develop an analysis instrument (cf. Böttinger 2007) by analysing the transcriptions of the interviews. While the analysis instrument “*Four levels of VISA*” (cf. 3.5) combines various aspects of structuring and interpreting a visual representation, in the analysis instrument “*Model of categories*” (cf. 3.1-3.4) these particular features were separated, adapted to sequences of growing patterns and the gradation was worked out by analysing the interviews.

The aim of the first case study (cf. Böttinger 2007) is to describe more precise on the basis of 20 interviews with 4th-grade children, in which way children translate geometrical relations in a sequence of growing patterns into arithmetic terms and in which way generalisations are carried out. The hypothesis is that there is no direct way from the geometrical representation to an arithmetical one and finally to an algebraic view. Instead there will be an interplay between these different views. In order to describe this interplay an analysis instrument (cf. Model of categories, Fig. 1; cf. Böttinger 2007) had been developed on the basis of the interview data.

3.1 Features concerning the structuring of single patterns

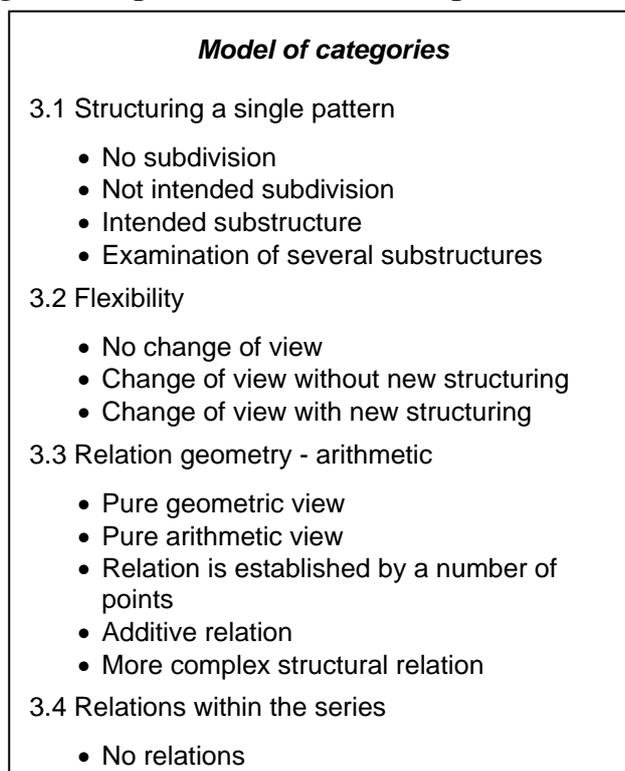


Fig. 1

In order to continue and examine the sequence a single pattern has to be structured. A subdivision can correspond to the *intended* structure of that person who composed the assignment on the one hand. On the other hand it can be an *individual* one, which does not correspond to a priori intended ideas.

3.2 *Features concerning the flexible re-organisation of single patterns*

In order to generate the idea of an equation one must be aware of different perceptions of a single pattern in the sequence. The aim is to identify the equality of arithmetic or algebraic expressions on the basis of the corresponding underlying geometric structure. Closely connected to this view is that transformations of equations correspond to changing the view on geometric structures. In analysing the children's interpretation one has to consider the flexibility during the process of work. It is essential to draw a comparison to the preceding interpretations of the child and to verify, to what extent a change of view occurs. This can be without new arrangement within the single pattern, e. g. when the number of dots is solely calculated in different ways. On the other hand a proper structural reinterpretation and re-organization exists, when the child builds fundamentally different structures into the diagram as in the step before.

3.3 *Features concerning the relation between geometric and arithmetic structures.*

Within her study Steinweg (2002) has worked out by what criteria children continue sequences of growing patterns. She distinguishes between a continuation by a figural aspect or by an arithmetical aspect. The figural aspect is concerned with the location of the dots and the external form built by the dots and the arithmetical aspect with the total number of dots in a single pattern. Steinweg accents that only the combination of figural and cardinal aspects lead to the intended continuation. Besides the distinction between a pure geometric view and a pure arithmetic view one has to regard the possible connections between both parameters. This can happen by a number of points, but also additive or more complex relations (e. g. multiplicative ones) can be identified.

3.4 *Features concerning relations within the patterns*

If sequences of patterns are used for algebraic investigation, one has to distinguish two totally different views. While the explicit formula uses the inner structure of a single figure, which must be suitable for all following figures, a recursive formula uses relations between consecutive patterns (cf. Carraher & Schliemann 2006). With the help of recursive formulas it is described, how the number of points changes from one pattern to the next. This view can be a great obstruction if the number of points in the 10th pattern is to be figured out. The student has to calculate step by step each particular pattern and simultaneously he has to control the number of steps. In addition, the indication of the recursion alone is incomplete to describe the building principle, because an initial condition is needed (Carraher, Schliemann, 2007, 697). From the

union of both perspectives interesting formulas can arise. Furthermore a dependence e. g. between the width and the height of a figure leads to dependent variables that describe exactly these features of the pattern.

3.5 Features concerning the interpretation visualizations (VISA)

In the second study (cf. Söbbeke) on the basis of detailed case studies with children of elementary school four levels of children's ability to build structures into mathematical representation (ViSA) had been distinguished. The underlying assumption of the study was that learning of mathematics has to be understood as a process of the children's more and more differentiated way of understanding and interpreting abstract patterns and structures (cf. Steinbring 2005). Visual representations are a tool to represent abstract mathematical concepts as well as to think about them or to talk about these with children. Growing patterns, as a special type of visualization, are often used to represent structures and relations in order to understand elementary mathematical principles (for example triangle numbers as an example to explore sums of odd numbers, etc.). The important information is not based in the concrete features of the material, but on the abstract, the relations and the structures within the material. Thus, what is decisive for a mathematical cognition in the figures is not the colours or the number of points; it is rather the function, which the concrete feature of the material takes for something. This means, the structure of the representation makes the understanding of a mathematical legality possible, but it cannot be read directly or immediately perceived with one's senses; it must be actively *interpreted* into the representation. In the empirical study (cf. Söbbeke 2005) it had been analyzed in how far the learning child succeeded in building such abstract structures and relations into the diagram. On this basis *Four Levels of Visual Structuring Ability* had been distinguished. These four levels characterize the children's interpretations in a spread of concrete and empirical interpretations on the one hand (cf. level one, left pole of the spread) and relational und structural interpretations on the other hand (cf. level IV, right pole of the spread) (cf. Söbbeke 2005).

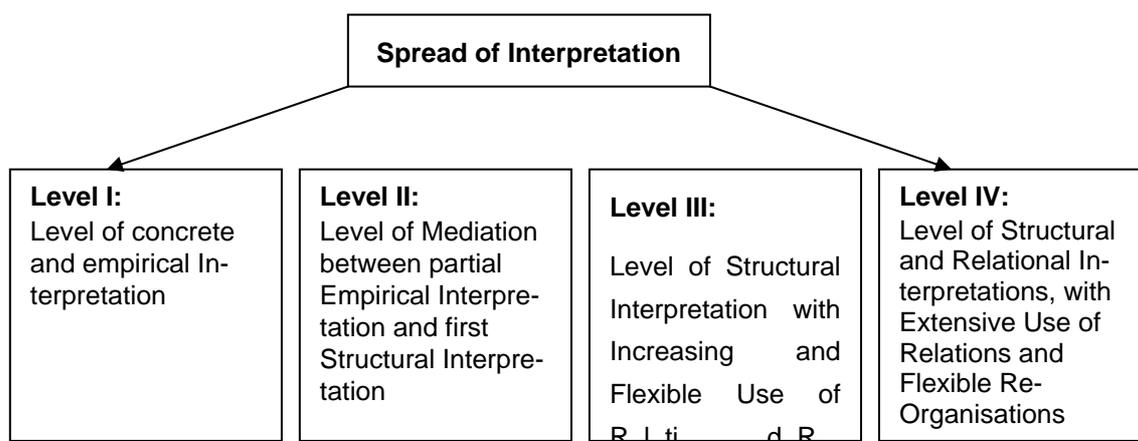


Fig. 2: Four Levels of Visual Structuring Ability (ViSA).

4 Using Growing patterns to Support Students' Way to Algebra

- Ron on his Way to an Abstract and Multi-relational View of the Pattern -

The following examples are to show how the student Ron (4th grade) deals with the challenge to use growing patterns and to interpret them more and more in the function as a representative of relations and structures and thus to focus on the abstract and generalizable “pre-algebraic aspects” in the representation. For this we connect in this paper for the first time two different analysis instruments and use them both to analyse the interpretations of student Ron. The scenes presented are not to deliver a thorough methodical analysis. Instead the analyse in this paper can be seen as a first approximation to grasp and to describe the fundamental elements of the children’s way to algebra by using growing patterns, which had been pointed out in 3.1 to 3.5. The analysis is not extracted from a finalized study, but it is an example of a new approach to the theme, to the underling structure and to a more detailed view onto sequences of growing patterns. In the *first* part of the different interview phases (*beginning, in course, end*) the elements of the aspects 3.1 to 3.4 had been described with the instrument “*Model of Categories of Changing Modes of Representation*“ (see fig. 1). In the second part of the interview phases Ron’s interpretations had been assigned to the “*Four Levels of Visual Structurizing Ability (ViSA)*” (cf. 3.5, fig. 2).

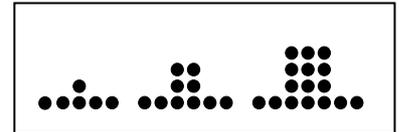


Fig.

At the *beginning of this interview* scene, Ron is presented the first three figures of the growing pattern and he is asked to describe what he can see (Fig. 3)

Ron	(16 seconds break) Mhm. (5 seconds break) Mhm (laughing). (10 seconds break) There at the bottom there is always one more (he points to lowest the row of dots in the first, the second, the third pattern). Five, six, seven (he touches the lower part of the first, the second, the third pattern) This next row. There are always some more.
Ron	Here there are, there are three more (he touches with his pencil the upper part of the second pattern). Here there are five more (he touches the third pattern with his pencil). (..) Since those I can remove (he puts his forefinger onto the third pattern), I can take away, because these are still there (he touches with the pencil the second pattern, afterwards he points to the not covered points of the third pattern). (...) Three, five. (6 sec. break, he moves the left forefinger to both left points of the bottom row in the third pattern, stops for a moment and takes the finger away from the paper) Mhm.

After 30 seconds reflecting about this task Ron starts to compare the three patterns. He structures the three figures into two parts: the horizontal row of dots at the bottom of the pattern and the field of dots placed at the top. In his first approach Ron does not pay attention to the part at the top of the pattern, but describes that the row of dots increases from one figure to the next and names the numbers “five”, “six”, “seven”. In the analysis, considering the aspects 3.1 - 3.4, Ron shows that at the beginning of the interview he had developed an idea of the structure of the lower part of the pattern. Ron determines the number of dots in this part of the pattern and finds a recursive relation between the figures: “five, six, seven. ... There are always some more”. He builds a relation between the geometrical figure and the arithmetic in finding out the number of dots in the lower part of the pattern. Ron does not make it explicit, but

his repetition of the number series can be seen as an indication that the number series and in association the structure of the lower part could always go on in this way. Against the background of his first interpretations, the number series can be understood as a preliminary stage of a recursive building principle: from one figure to the next you always have to add one point. Already at this early stage of the examination of the pattern you can see a first level of generalization.

After reflecting about 30 seconds about the upper part of the figures, Ron starts to describe the increasing of dots from the second to the third pattern. Ron structures the upper part into two groups: on

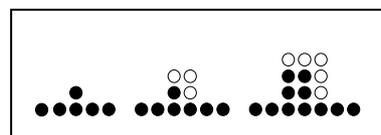


Fig.

the one hand, he sees the group of dots that had been seen in the previous figure, and on the other hand those, that had been added in the new following one: “Since those I can remove (he puts his forefinger onto the third pattern), I can take away, because these are still there”. In his approaches to understand the structure of the upper part, Ron shows a first re-organization of the pattern. He does not analyse the two parts of the figures separate, but tries to understand in what way the first pattern could be identified in the second one and the second one in the third one. In the meantime he points with his finger on special areas of the lower part of the pattern, which he had described before in his first analysis of the pattern (the vertical row of dots). The numbers “three” and “five”, he denominates, correspond presumably to the numbers of dots in the upper part of the pattern, marked for a better understanding here in white colour (see Fig. 4). Ron uses the numbers of dots and structures and builds first elemental relations between the different patterns into the diagram (he covers with his hands parts of the previous patterns etc.). As a kind of arithmetical information, Ron determines the number of dots in the particular figures. At the beginning of this interview the analyse shows a first recursive view on the pattern; however, Ron does not generalize this recursive view further, but applies it solely to the partly figures.

Altogether Ron’s interpretation of the pattern could be attributed to the 2nd level of *ViSA* (cf. 3.5). The child moves away from the concrete aspects of the representation (numbers of dots) and focuses increasingly on abstract relations and structures (two parts of the pattern; angle-structure of the added dots in the new figure). But the elements of interpretation often stand isolated as concrete objects, without building *rich* relations between them (for example relations between the structure of the part at the bottom and at the top of the pattern; relations between the different figures). Sometimes only sections of the diagram are taken into consideration. In interpretations on this level there is a typical mediation between partial empirical interpretations and first structural interpretations. But often the children’s interpretations are still inflexible and they do not look at the representation as a multi-faceted structural diagram.

In the *course of the interview*, Ron notices that he had always forgotten to pay attention to one point in the lower part of the pattern, while analysing the increasing of the patterns:

Ron	O no, I didn't count those (he taps the bottom row of points in the second pattern). That means, there would be four new ones (he touches the second pattern) and here there would be six new ones (he touches the third pattern).
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After that Ron constructs a recursive geometrical building principle into the growing pattern and tries to translate it into an arithmetical building principle. In the course of the interview Ron has been asked to find an arithmetical task, which corresponds to the given pattern. For this he finds calculation tasks, which correspond with the result ("16") to the number of given dots in the third pattern. Ron interpretes and explains the proposal of the potential task "3·3+7", given by the interviewer, solely against the background of the calculating result und does not indicate a relation between the structure of the arithmetic task and the structur of the pattern. For Ron it is crucial that the number of the dots corresponds with the result of the calucating task.

He finds the calculating task "10·3+4" in the 5th pattern, that can be seen als an analogon to the proposal of the interviewer in the 3th pattern ("3·3+7"). Presumably Ron takes the aspect "number of dots" on and tries to build an analog construction (second factor of multiplication is "3" or a task with a multiplative term) like in the task of the interviewer. Finally, at the end of the interview Ron is asked to determine the number of dots in the sixth pattern. He starts to draw the sixth pattern onto the interview sheet.

Ron	<p>Five, six, seven, eight, nine, ten (in the meantime he draws 10 points in a row beside the 5th pattern).</p> <p>The first new points, this would be here, one, two, three, four, five, six (while speaking he draws a row of 6 points directly over the row of 10 points; cf. Fig 5). One, two, three, four five, six, (he draws - always counting until six - four further rows consisting of 6 points). One, two, three, four, five (with his pencil he touches the dots of the first column, but omits the corresponding dot at the bottom). Now another one (he draws a further row consisting of 6 points over the 5th 6-row). Six. Ready.</p> <p>One, two, three, four, five, six, seven (he touches the dots of the first column including the corresponding dot at the bottom), seven. Six times seven is 42 plus four, 44.</p>
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At first Ron divides the 6th pattern into two parts: At the bottom he builds a long horizontal row consisting of 10 dots, in the upper part a rectangular field consisting of six rows of six dots. Subsequently he carries out an interesting new interpretation of the pattern. He structures it into a rectangle of seven rows of six dots, which reaches into the horizontal line at the bottom. Beside this 6x7-field of dots he regards two points at the left and two at the right-hand side – at whole 4

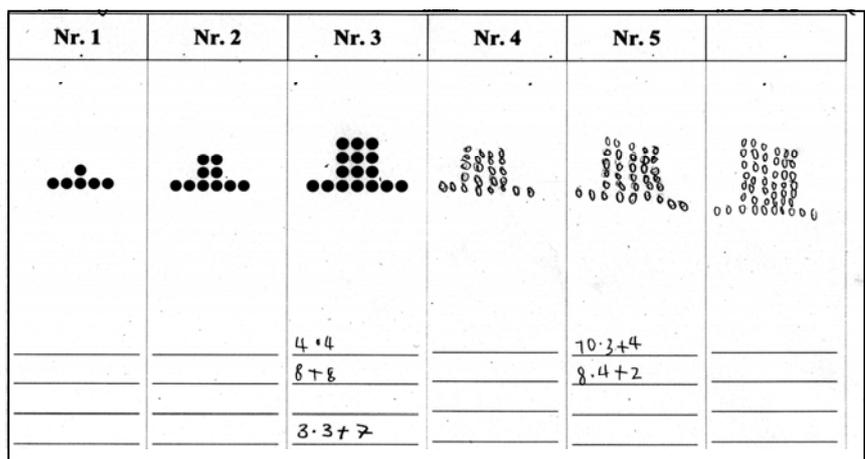


Fig. 5

points. To figure out the total amount of numbers in the 6th pattern Ron uses for the first time the inner structure of a single pattern. In comparison to his proceeding before this represents a change of view in connection with a new structuring. The relation between the geometric arrangement of the dots is no longer determined by the cardinality of a set of points but by a complex structural relation – namely a multiplicative one. By that Ron changes from his formerly recursive view onto the sequence and considers a single pattern in an explicit manner. The structure he uses is an intended one and in principle it is applicable to all patterns. But at this stage of the interview Ron does not express or indicate this generalisation.

Ron's interpretation of the pattern could be attributed to the *third level of ViSA*. In interpretations on this level *intended structures* and *relations* can be identified (for example relation between the part of the bottom and at the top of the figure; field of 6x7 dots; constancy of 4 dots in the part at the bottom). On this occasion different and multi-faceted aspects of the representation are recognised. In comparison to level II, the structures are *manifoldly coordinated* and more *flexibly re-organised*. The structures are no longer isolated, but seen as part of the whole and separated and put together in a structural way. You always find the use of *structural relations*, *coordination* and *re-organisation* of elements. In all, this level III of ViSA can be characterized by the combination of building structures with the increasing use of relations and re-organisations.

5 Conclusion

For a fundamental pre-algebraic comprehension it is indispensable to focus on structures, on the abstract and the general, right from the start of children's mathematics education. In this paper, growing patterns have been discussed and analysed as exploring fields on the way to focus on structures and relations. Structure sense seems to be a fundamental requirement to interpret sequences of growing patterns in an algebraical manner. Both analysing instruments examine in different ways how young children deal with the challenge to interpret this special visualization in a more structured, generalized and elementary "algebraic" way.

The examples of Ron indicate that this kind of structuring, translation and generalization does not take place in a direct and straight way. The child can partly understand the geometrical structures, translate them into arithmetic ones. It can change the view back to the geometric pattern and re-organise and re-structure the diagram. It seems that generalization is not always the "end" of this process; in fact ideas of generalization can be developed before comprehending the whole structure of the patterns.

An analysis of selected parts of the interview shows that in the *process of the examination* and the *interaction* between the student and the interviewer the child gradually develops a more differentiated, relational and generalized view onto the used diagrams, which can be described in detail by the system of categories and in a more summarising manner by means of ViSA (see e.g. the development of Ron's interpre-

tation from level II to level III). Altogether the excerpts of the interview with Ron serve to demonstrate the change in children's interpretations in a exemplary way and to accompany and better understand their way – to an increasingly open, general and flexible view onto relations and structures within diagrams.

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STEPS TOWARDS A STRUCTURAL CONCEPTION OF THE NOTION OF VARIABLE

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If students acquire a new mathematical notion, according to Sfard (1991), they pass through different phases: an operational and a structural phase. At a grammar school in Bremen, Germany, students of age 12 to 14 first came into contact with the notion of variable using a simple programming language without a computer. As a part of the learning environment the students wrote imaginary dialogues in which they let two protagonists talk about different tasks. The imaginary dialogues of the students are analysed against the background of Sfard's theory of the dual nature of mathematical conception. In particular, the different steps towards a structural conception of the notion of variable in the context of the programming learning environment are elaborated.

INTRODUCTION

If we look at a mathematical notion, we can think about what it is in the mathematical world, how it is defined, which properties it has, and how it relates to other parts of mathematics or we can consider how a human being thinks about it and what kind of inner picture has been built. Anna Sfard (1991) distinguishes here between the word *notion* or *concept* on the one hand and *conception* on the other hand.

The whole cluster of internal representations and associations evoked by the concept - the concept's counterpart in the internal, subjective "universe of human knowing" - will be referred to as a "conception". (Sfard, 1991, p. 3)

According to Sfard, a conception of a mathematical notion has two complementary sides, an operational and a structural one, in which a learner first passes through operational phases until a structural conception can be developed. She also points out that

without the abstract objects all our mental activity would be more difficult. (Sfard, 1991, p. 28)

In this article the development of the conception of variable is considered. The underlying question of the presented analysis is: what are steps towards a structural conception of the notion of variable? To approach an answer the findings of a qualitative analysis of imaginary dialogues written by students of age 12 to 14 from one class will be presented.

THEORETICAL FRAMEWORK

The theory of reification

Sfard (1991) presents a theoretical framework for the acquisition of a mathematical notion. She distinguishes between an *operational* and a *structural conception* of the same mathematical notion. If a learner has acquired an operational conception, she or he will know how to operate with the notion, i.e. with algorithms, processes and actions. For a structural conception it is necessary to recognise the notion as a mathematical object. Sfard expects that the operational conception precedes the structural. In this process from operational to structural three steps occur: *interiorization*, a process with familiar objects, *condensation*, where the former processes become separate entities and *reification*:

to see this new entity as an integrated, object-like whole. (Sfard, 1991, p. 18)

While a learner can come gradually from interiorization to condensation, Sfard speaks of a *leap* when it comes to reification:

“Reification (...) is defined as an ontological shift – a sudden ability to see something familiar in a totally new light. Thus, whereas interiorization and condensation are gradual, quantitative rather than qualitative changes, reification is an instantaneous quantum leap: a process solidifies into object, into a static structure.” (Sfard, 1991, p. 19-20)

Sfard & Linchevski (1994) used the framework of the theory of reification to study the case of algebra. In particular, they focused on the transition from operational to structural regarding a variable as a fixed unknown on the one hand and in a functional context on the other hand. Sfard (1991) asks the question how to diagnose the stages towards a conceptual development and proposes:

"It seems that we have no choice but to describe each phase in the formation of abstract objects in terms of such external characteristics as student's behaviour, attitudes and skills." (Sfard, 1991, p. 18)

Mathematical writing

Mathematical writing by students has been the issue of several studies, compare Borasi & Rose (1989), Clarke, Waywood & Stephens (1993), Gallin & Ruf (1998), and Shield & Galbraith (1998). Gallin & Ruf investigated the use of *journals* (in German: Reisetagebücher) in order to establish a written dialogue between the students and the teacher. While writing their journals the students can approach the regular mathematics in their singular way.

Imaginary dialogues are a different type of mathematical writing (Wille, 2008). In an imaginary dialogue the student lets two protagonists solve a mathematical task or talk about a mathematical question. Usually one protagonist understands the task better than the other. In this way the student can decide what particular themes she or he addresses. Unlike in journal writing, in an imaginary dialogue, one finds a lot of exploratory writing. On the other hand, in contrast to pure exploratory writing, like

writing a letter to someone and explaining something, in imaginary dialogues the protagonists can develop a solution of a task and the protagonists can point at possible learning difficulties.

LEARNING ENVIRONMENT

The learning environment is designed for first experiences with the notion of variable. The students do not start with a single variable as a fixed unknown. Instead, they get to know a simple programming language which is executed by the students without a computer but with a little wooden robot on a sheet of paper with a coordinate grid. The programming language has similarities to LOGO (Papert, 1980). Here, as a “memory” each robot needs matchboxes on which letters for the names of variables like “a” and “b” are written. These matchboxes serve as *preset reifications* of the notion of variable, which the students fill by hand instead of assigning a number to a symbolic variable. For example to move three steps forward, the program will look like this

a ← 3

forward(a)

While executing the first line it must be assured that exactly three matches are in the matchbox named “a”. In the second line, the robot will be moved into the direction it faces. The matchboxes must be used in order to move a robot, since the direct command “forward(3)” is not part of the programming language. Next to these commands there is also the command “turnaround()”, which lets the robot turn by 180°. Furthermore there are a right and a left turn, commands to place the robot on a certain intersection point on the coordinate grid and different command loops. That way students can write and execute programs in order to move their robot on the grid while assigning variable by filling matchboxes with matches.

In the learning environment the programming of the robot can be combined with writing imaginary dialogues. One of the first tasks can be the following: The students get a sheet of paper with “a ← “ and “b ← “ on top and “turnaround()” in the middle. On another sheet of paper eight paper commands “forward(a)” and eight paper commands “forward(b)” can be cut out. The students get the following exercise with the name “cut out and explore”:

On the next sheet of paper you see a program that is not finished yet. You can use commands out of a construction kit and put them above and below the command “turnaround()”. 1. Cut out as many commands as you need and write a program with them. 2. Execute your program with the matchboxes and the robot. 3. Try to write such a program that the robot comes back to his starting point. 4. For which values a and b does your program function? Are there different possible values? 5. Write your favourite program and name many values with which it works.

Right after this lesson the students get the following homework (*dialogue A*):

Two students talk about the last task “cut out and explore”. One of the students can do it easily, the other has more difficulties. Write a dialogue in which the two students talk about the task. Write at least one page.

In the next task a simple program is presented, where over the turnaround command there are two commands “forward(a)” and under it one command “forward(b)”. There is also a table given for a and b with values (1,2), (2,4), (3,6) and (4,7). A beginning of a dialogue is also part of the task where two students talk about whether the numbers in the table should be switched. One protagonist draws also the following picture:

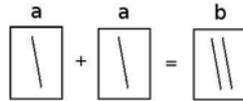
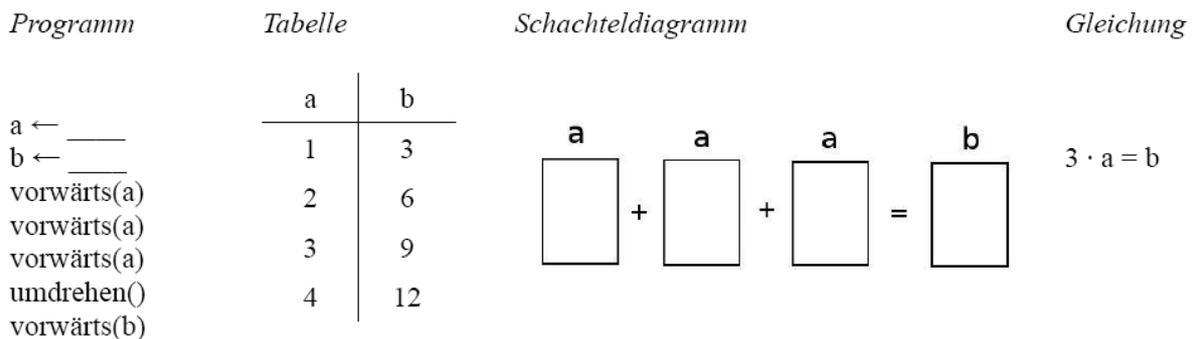


Figure 1

The students are asked to work with the program first, decide, if the table is correct and finish the dialogue (*dialogue B*). After further tasks with the robot a third imaginary dialogue task (*dialogue C*) is given. The students get the following picture:



Fig

ure 2

Now the students are asked to think of an interesting program of a similar form, find the proper presentations like in Figure 2 and write an imaginary dialogue about it.

METHOD

The study was carried out in a class of a grammar school (Gymnasium) in Bremen, Germany, in 2008 with the above mentioned learning environment. The students wrote three different dialogues A, B and C. Dialogue A was written after the second lesson, dialogue B after three more days and dialogue C after about three weeks. The imaginary dialogues A and B were given as homework, dialogue C was written in the classroom. Since not all students did their homework or some let the protagonists talk about only non-mathematical tasks, for the analysis 16 A-dialogues, 15 B-dialogues and 22 C-dialogues could be used. For the qualitative analysis of the imaginary dialogues the framework of Sfard's theory of reification was used. The analysis was carried out in three steps:

13.examination by *four criteria*: recognised structures, occurring aspects of the notion of variable, phase in which the student is (i.e. interiorization, condensation, mixed form/indistinct, or reification), mentioned preset reification

14.creation of a *mind map* of the seen structures for each dialogue A, B, and C

15.creation of *tables* that includes the information of the mind maps and the phases

In order to examine by the four criteria, most dialogues were first transcribed and then interpreted in detail. The students' development was classified according to the phases according to these criteria:

- *interiorization*: the student can handle the program: processing the program, filling matchboxes with matches, etc.
- *condensation*: the student deals with variables as with objects but does not see them as objects, the input and output is more important than the process itself
- *mixed form/indistinct*: it cannot be decided if the student already reificated the notion of variable, variables are used in a tight relation to preset reifications
- *reification*: variables are seen as independent objects

FINDINGS

All imaginary dialogues mentioned here were written in German and translated by the author.

Mini-statistics

We can observe a shift of the students of this class from interiorization to reification as Sfard predicted. It must be mentioned that the tasks for the dialogues A, B and C were similar, but different. Thus, there is the possibility that the observed shift also depends on the different tasks. In the following table, the number of students in a certain phase of a certain dialogue is denoted:

	i	c	m	r	Total
A	11	2	2	1	16
B	5	5	3	2	15
C	4	5	6	7	22

Table 1: number of students in a certain phase

Structures recognised by the students

The structures that were recognised by the students are shown in the tables of the Figures 3 and 4. The tables should be read like a tree from left to right where each

row is a branch. It is also listed which phase is assigned to the specific imaginary dialogue, in which the student recognised the structure. The letters i, c, m and r stand for the phases interiorization, condensation, mixed form/indistinct and reification. There are several crosses, if several students see the same structure. Some of the structures that can be seen as examples of *preliminary steps of reification* are discussed below. In the following, for example “Figure 3, structures in A, 7” refers to the seen structure in A written in row 7 which is here “segmentation of the distance – in segments a and b”.

structures in A

		i	c	m	r
comparison of the number of steps		x			
independence of the notation	name of variable is free to choose	x			
segmentation of the commands	search for a segmentation	x			
	same amount of commands with a and b with an example			x	x
segmentation of the distance	in steps without a and b	x			
	in segments a and b		x		
preform of abstraction	forward(a) and box content synonymical			x	
preform of substitution	preform of “a is 1”			x	
preform of reification	abstraction of matchboxes		x		
	paper commands “a’s” and “b’s”			x	
	relationship between the number of “a’s” and “b’s”			x	
	a and b as steps			x	
abstraction	a as a value in different contexts				x

structures in B

		i	c	m	r
correlation of the values in the table		x			
pre-understanding of equations	preform of symmetry in equations	x			
	correlation between program and substitution in equation	x			
	preform of the understanding of equations		x		
structure of explanation	abstract, concrete, example				x
correlation between term and distance			x		x
correlation between a and b	comparison of undefined step sizes with the notion “a” and “b”		x		x
	b is the double of a				xx
abstraction	variables as independent objects				x

Figure 3: structures in A and B

structures in C

		i	c	m	r
1	commutativity of the commands			x	
2	different substitutions possible				x
3	structure in table	multiple addition	x		
4		as the times table		x	
5		proportion between a- and b-values		x	
6		correlation between values in the table		x	
7	variable as common name for something	">a<" as name for a mathbox, column and field		x	
8		generic term for values, commands and distances		x	
9	correlation between a and b	as values			x
10		b is the double of a		x	xxx
11		it is "the same" with other values			x
12		how many of "a" yield "b"			x
13		concrete assignment unimportant		x	x
14	letters a and b	instead of a number of intersection points		x	
15		as a fixed unknown		x	
16		stand for a number of steps			x
17		stand for numbers in the language use	x		
18		stand for numbers	xx		x
19	step size as a scale				xx
20	preform of reification	"a)" and "b)" instead of concrete numbers in the language use	x		
21		notation "a=7" for values of the table		x	
22		a and b as generic term for values in the table		x	
23	understanding of equations	symmetry		x	
24		cancellation			x
25		equation as a summary of examples			x
26	expansion of the number of variables				x
27	abstraction	variables as independent objects			xxxx
28		terms as independent objects			x

Figure 4: structures in C

Independence of the notion

In the imaginary dialogue of a student (Figure 3, structures in A, 2) we can read that for him the name of the matchbox is free to choose. One of his protagonists explains:

"You put arbitrarily many matches of the 16 and label the matchbox with a letter, let me say an example: "N". You position the robot on the sea bottom and now you must give commands to the robot: for example: forward (for example N). Hence, he goes forward as much as you have put matches into the matchbox."

The student writes “forward(for example N)” which shows that he points out that he could have chosen another name for the matchbox. If we transfer this to variables, we can call it an aspect of the independence of the name of variable. This aspect has its relevance, if we think about students who might know for example the binomial formulas with a and b , but have difficulties, when different variable names are used.

Name of variable as a generic term for multiple objects

Variables can simultaneously represent multiple values and can be abstracted from multiple real objects, like distances or the quantity of something. Hence, a preliminary step for this abstraction is to use different objects synonymously or to use a variable as a generic term for multiple objects. We can see the use of different objects synonymously in a dialogue by a student (Figure 3, structures in A, 8) who first wrote:

“because (a) and (b) are most probable of different size.”

After this she inserted the words “forward” from above, such that the sentence looks like this:

“because forward(a) and forward(b) are most probable of different size.”

We do not know, if she means by “(a)” the box content or a value of an abstract a , but we might consider that she uses the command “forward(a)” and whatever she thinks of as “(a)” synonymously.

The next step is to use a variable as a generic term for multiple objects as in the following dialogue (Figure 4, structures in C, 7). Here, the protagonists are named “S” and “D”.

S: Well, the table has two columns. $A+b$. As the two matchboxes. In $\langle a \rangle$ are two matches, and in b 8. In column $\langle a \rangle$ 2 are added in each row. In column $\langle b \rangle$ it is the same.

D: Like a times table? Where in each row it increases by 2 or 8 respectively?

S: Yes! Precisely. Now to the matchbox diagram. The field $\langle a \rangle$ stands for the number $\langle 2 \rangle$. The field $\langle b \rangle$ stands for $\langle 8 \rangle$. That way the diagram is eventually: $2+2+2+2=8$.

When the student mentions her notation “ $\langle a \rangle$ ” the first time it means a matchbox. After this it is a column and the end a field which can be substituted. We can also observe that the student does not use the letter a without relating it to an object. It does not appear in a complete abstract manner.

A different student (Figure 4, structures in C, 8) uses variables as a generic term for commands,

“We have the commands A, B, & turnaround.”

values,

“But how do I know, what is the value of A & B?”

and distances:

“If you go the distance $a() + b()$, then it makes no difference, if you go back $a() + b()$ or $b() + a()$.”

Talking about a and b as talking about objects

A student talks in his dialogue (Figure 3, structures in A, 11) about a and b as if they were objects. Possibly he thinks about the paper commands while talking about them.

“If a is equal to 1 and b is equal to 2: First you must (you can) go with all a’s forward and with the half of the b’s backward and you are again on the same point.”

Since he says “with the half of the b’s”, the “b’s” are some kind of objects to him.

Correlation of different variables

Several students discuss the correlation between different variables (compare Figure 3, structures in B, 7-9 and Figure 4, structures in C, 9-13). One example is where the student recognises that b must be the double of a (Figure 3, structures in B, 9):

“If the robot moves two half steps (a) and he must go back steps which are bigger, then b must have the double, thus an entire step.”

A different student formulates the correlation by fitting a number of a into b (Figure 4, structures in C, 12):

S2: Well, if a and b stand for the number of steps and you can turnaround only once, then you must find out how many of a yield b.

S1: Thus, if a is 1 and b 4 then one must find out how often a fits in b.

S2: Exactly!

What are a and b?

Some students discussed the topic of what the letters a and b are. Most often they used the words “stands for” instead of “is”. We find passages, all in dialogue C, saying for example that a or b stand for a number of steps (compare the preceding example), or for numbers (Figure 4, structures in C, 18):

2: Exactly and for the equation you must do this in a multiplication exercise.

1: Without numbers?

2: The letters stand for numbers, for example out of the table.

1: But there are multiple numbers. Which ones do I take?

2: That is easy. You can take every number you like. Just make sure that a has the double value.

SUMMARY

The analysis of the imaginary dialogues written by the students indicates the process from the phase interiorization, passing condensation to reification, as predicted by Sfard (1991). In the tables we see all structures that were recognised by the students. Among those structures we can also identify several preliminary steps toward a structural conception of the notion of variable: the independence of the notion, using the name of variable as a generic term for multiple objects, talking about variables as about objects, recognising correlations between different variables, and actually discussing what a letter stands for. Whether these preliminary steps eventually lead to a complete reification or not, we cannot predict. But we can observe that several students in dialogue A are tight to the preset reification of the notion of variable in form of the matchboxes or paper commands, while reading the dialogues B and C, the preset reifications disappear in many writings and the language use becomes more and more regular.

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CERME 6 – WORKING GROUP 5

GEOMETRICAL THINKING

TABLE OF CONTENTS

Introduction.....	671
<i>Alain Kuzniak, Iliada Elia, Matthias Hattermann, Filip Roubicek</i>	
The necessity of two different types of applications in elementary geometry.....	676
<i>Boris Girnat</i>	
A French look on the Greek geometrical working space at secondary school level	686
<i>Alain Kuzniak, Laurent Vivier</i>	
A theoretical model of students’ geometrical figure understanding	696
<i>Eleni Deliyianni, Iliada Elia, Athanasios Gagatsis, Annita Monoyiou, Areti Panaoura</i>	
Gestalt configurations in geometry learning	706
<i>Claudia Acuña</i>	
Investigating comparison between surfaces.....	716
<i>Paola Vighi</i>	
The effects of the concept of symmetry on learning geometry at French secondary school.....	726
<i>Caroline Bulf</i>	
The role of teaching in the development of basic concepts in geometry: how the concept of similarity and intuitive knowledge affect student’s perception of similar shapes.....	736
<i>Mattheou Kallia, Spyrou Panagiotis</i>	
The geometrical reasoning of primary and secondary school students	746
<i>Georgia Panaoura, Athanasios Gagatsis</i>	
Strengthening students’ understanding of ‘proof’ in geometry in lower secondary school	756
<i>Susumu Kunimune, Taro Fujita, Keith Jones</i>	
Written report in learning geometry: explanation and argumentation.....	766
<i>Silvia Semana, Leonor Santos</i>	
Multiple solutions for a problem: a tool for evaluation of mathematical thinking in geometry.....	776
<i>Anat Levav-Waynberg, Roza Leikin</i>	
The drag-mode in three dimensional dynamic geometry environments – Two studies	786
<i>Mathias Hattermann</i>	

3D geometry and learning of mathematical reasoning	796
<i>Joris Mithalal</i>	
In search of elements for a competence model in solid geometry teaching	
Establishment of relationships	806
<i>Edna González, Gregoria Guillén</i>	
Students' 3D geometry thinking profiles	816
<i>Marios Pittalis, Nicholas Mousoulides, Constantinos Christou</i>	

INTRODUCTION

GEOMETRICAL THINKING

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The Working Group 5 on Geometrical Thinking had around 30 participants from 14 countries all over Europe and from America too (Mexico, USA and Canada). During its sessions, the participants discussed 16 papers prepared for the Working Group and selected among 23 initial proposals and 15 have been retained for publication. The participants, and it's a strength of the group, worked within the continuity of the former sessions of Cerme. Some points can be considered as a common background known by ancient participants to the Working Group and the discussions among people were facilitated by this common culture. The readers are invited to have a look on the former general reports made at Bellaria (Dorier et al., 2003) and Larnaca (Kuzniak and al, 2007) when they want to know more about the common background.

This report insists on the questions of theoretical supports in Geometry, which can be seen as local theory in comparison of more general theoretical frameworks used in Mathematics Education. It would be interesting to explore the relationships between both local and global viewpoints. This part results from a collective work of a small group managed by Iliada Elia.

Then, all the accepted papers are briefly introduced for giving an idea of problems the group was concerned by.

Theoretical and methodological aspects of research in geometry

During the working group, we distinguished two approaches of using theory in research: First, theory can serve as a starting point for initiating a research study. For instance, the need to empirically validate or extend specific theories may motivate an investigation. Second, theory can act as a lens to look into the data. For example, different phenomena and behaviours observed in mathematics classes may evoke ideas to the teacher or the researcher for starting research. To start from phenomena or data is a valid first approach to research. In this case, theory may enable the teacher or the researcher to better understand and interpret the collected data.

Certainly, if one has a dual approach to research (data or theory) s/he can start with theory or data. This has methodological implications, that is, the methodology has to be appropriate to a chosen theory or to the collected data. The collection of data is

very important, though, for both types of research. But to have substantial and long-standing effects to the research community's endeavour, the data, their use and interpretation should have a theoretical contribution (e.g. add or suggest modifications to an existing theory or develop new theory).

The most important theories in geometry education that were identified and discussed are the following: Van Hiele's levels, Geometrical Working Space and Geometrical paradigms and Duval's semiotic approach. Each line of theory approaches geometry learning from a different perspective and thus is helpful for different purposes. Van Hiele's theory is mainly helpful for evaluating students' reactions, productions and solutions to problems (phenomenological approach). Houdement and Kuzniak's (2003) theory about Geometrical Working Space and Geometrical Paradigms (e.g. Geometry I: Natural Geometry, Geometry II: Natural Axiomatic Geometry and Geometry III: Formal Axiomatic Geometry) is mainly helpful for classifying approaches, e.g. the types of argumentation used and to understand students' difficulties and errors (epistemological approach). Duval's (2005) theory is mainly helpful for examining the registers (e.g. geometrical figures, verbal representations-language) used in the field of geometry and their treatment in geometry tasks (semiotic approach).

Furthermore, there are psychological approaches to geometry that are often linked to spatial abilities, e.g. Gestalt and Piaget's theories, but are not very well taken into account in the mathematics education research community. Connecting these approaches with geometry theories and/or using them as a tool to look into the data in future studies could be a first step towards addressing this gap.

Future research on geometry theories and their articulation could use Geometrical Paradigms in a more operationalized manner to analyze existing curricula, to analyze students' behaviour and in investigating modelling and problem solving. Van Hiele's levels could be extended by proposing and empirically validating new (sub-)levels within their scale.

Educational goals and curriculum in geometry

The discussion on this general and fundamental topic was introduced by two papers. Using an epistemological approach, Boris Girnath criticized some present approaches in the learning of Geometry (especially in Germany) which leave aside the classical ontological aspect of Geometry. He claims that there are two different types of applications in geometry and that they both are necessary and not exchangeable by each other: The first one contains simple applications which are paradigmatic examples to learn basic geometrical concepts; the second one includes more complex ones and refers to transcendental aspects.

Then Laurent Vivier and Alain Kuzniak described a French viewpoint on the Greek Geometrical Work at Secondary level. Beyond some similarities between France and Greece, it appears that the Euclidean tradition stays stronger in Greece but only for cultural reasons. Due to the lack of evaluation at the entrance on the university, the teaching of geometry is not viewed as important by the students and we can notice

again the effects of evaluation on the real curriculum. In their study, the authors used a theoretical frame based on paradigms and geometrical working spaces and Greek people present in the group reacted and agreed with the conclusions. The presentation made at Cerme was thought as an important part of the research project.

Understanding and use of geometrical figures and diagrams

The study presented by Eleni Deliyianni investigated the role of various aspects of apprehension, i.e., perceptual, operative and discursive apprehension, in geometrical figure understanding. Based on a statistical exploration of data collected from 1086 primary and secondary school students, the existence of six main factors revealing the differential effect of perceptual and recognition abilities, the ways of figure modification and measurement concepts. However, findings revealed differences between primary and secondary school students' performance and in the way they behaved during the solution of the tasks.

In her presentation Claudia Acuna used the old but always pertinent viewpoint on the treatment of geometric diagrams as Gestalt configurations. In geometry, the figural aspects of diagrams as symbols are used to solve problems. When figural information are treated, Gestalt configurations emerge: auxiliary figural configurations, real or virtual, that give meaning and substance to an idea that facilitates the proof or solution to the problem. In the paper, some arguments are given to acknowledge the existence of these resources.

Understanding and use of concepts and “proof” in geometry.

The work presented by Paola Vighi is concerned by the comparison of surfaces which need some mereological transformations in the sense of Duval. The same problems were given to two groups of pupils 10-11 years old having followed different ways of learning geometry: one traditional and the second more “experimental”. She concludes with some observations about teaching geometry and suggestions for its improvement.

Caroline Bulf studied some symmetry's effects on conceptualization of new mathematical concept at two different levels at French secondary school, with students who are 12-13 years old and 14-15 y.o. From the study, the concept of symmetry makes students confused with the transformations of the plan introduced at the beginning of secondary school. Students seem to be more familiar with metrical properties relative to the symmetry and develop mathematical reasoning at the end of secondary school.

Mattheou Kallia investigated the basic geometrical knowledge of students of the Pedagogical Department of Education. She investigated mainly how they define similarity of shapes and how the intuitive knowledge affects their perception of similar shapes. The results showed that a large percentage of students are not in a position to correctly define the similarity of shapes and that initial intuition affects their responses and their mathematical achievement.

Two other papers were focused on the question of geometrical reasoning. Georgia Panoura and Athanasios Gagatsis underlined that the geometrical reasoning of primary and secondary school students can be compared mainly on the way students confronted and solved specific geometrical tasks: the strategies they used and the common errors appearing in their solutions. This comparison shed light to students' difficulties and phenomena related to the transition from Natural Geometry (the objects of this paradigm of geometry are material objects) to Natural Axiomatic Geometry (definitions and axioms are necessary to create the objects in this paradigm of geometry). They stressed the inconsistency of the didactical contract implied in primary and secondary school education and they conclude on the need for helping students progressively move from the geometry of observation to the geometry of deduction.

Based on a different framework, Taro Fujita seems to study the same problem in the case of geometry in Japan. This paper reports findings that indicate that as many as 80% of lower secondary age students can continue to consider that experimental verifications are enough to demonstrate that geometrical statements are true - even while, at the same time, understanding that proof is required to demonstrate that geometrical statements are true. Further data show that attending more closely to the matter of the 'Generality of proof' can disturb students' beliefs about experimental verification and make deductive proof meaningful for them. It could be interesting to interpret these results with the same tools as Panoura and Gagatsis: didactical contract and geometrical paradigms. It seems that the conclusions are very close but in different context.

Communication and assessment in geometry

In the two following papers, original tools were used to assess geometrical abilities and in the same time to help students in developing their skills in argumentation. Silvia Semana examined how the written report, within the context of assessment for learning, helps students in learning geometry and in developing their explanation and argumentation skills at the 8th grade in Portugal. This study suggests that using written reports improves those capabilities and, therefore, the comprehension of geometric concepts and processes. These benefits for learning are enhanced through the implementation of some assessment strategies, namely oral and written feedback.

Anat Levav developed an approach based on the presumption that solving mathematical problems in different ways may serve as a double role tool - didactical and diagnostic. She described a tool for the evaluation of the performance on multiple solution tasks (MST) in geometry. The tool is designed to enable the evaluation of subject's geometry knowledge and creativity as reflected from his solutions for a problem. The example provided for such evaluation is taken from an ongoing large-scale research aimed to examine the effectiveness of MSTs as a didactical tool. Anat Levav argued that this method could be extended to other domains in mathematics.

3D Geometry: Teaching, thinking and learning

The working group was concerned by some studies on 3D Geometry with new viewpoints due to the use of dynamical software in the learning of these specific parts of geometry which is often left aside in the real curriculum. Dynamic Geometry Environments (DGEs) in 2D are one of the well researched topics in mathematics education. DGEs for 3D-environments (Archimedes, Geo3D and Cabri 3D) were designed in Germany and France. Mathias Hattermann studied the specific drag-mode in 3D Geometry environments. He showed that pre-service teachers with previous knowledge in 2D-systems prefer to work with a real model of a cube instead of the 3D-system to solve certain problems. Previous knowledge in 2D-systems seems to be insufficient to handle the drag-mode in an appropriate way in 3D-environments. In a second study, he introduced the students to the special software before the investigation and distinguished different dragging modalities during the solution processes of two tasks.

The approach of Joris Mithalal is more on the transition to formal proof in 3D Geometry. Teaching mathematical proof is a great issue of mathematics education, and geometry is a traditional context for it. Nevertheless, especially in plane geometry, the students often focus on the drawings. As they can see results, they don't need to use neither axiomatic geometry nor formal proof. He tried to analyse how space geometry situations could incite students to use axiomatic geometry. Using Duval's distinctions between iconic and non-iconic visualization, he discussed the potentialities of situations based on a 3D dynamic geometry software.

In the two last papers, the authors focused on the traditional way of teaching and learning 3D Geometry. Edna Gonzalez presented part of the analysis of a Teaching Model for the geometry of solids of an initial Education Plan for elementary school teachers, and its implementation in the University School of Teaching of the Universitat de València in Spain.

In a statistical analysis of the results of 269 students (5th to 9th grade) in Cyprus, Marios Pittalis tried to show that 3D geometry thinking can be described across the following factors: (a) recognition and construction of nets, (b) representation of 3D objects, (c) structuring of 3D arrays of cubes, (d) recognition of 3D shapes' properties, (e) calculation of the volume and the area of solids, and (f) comparison of the properties of 3D shapes. With these factors, he identified four different profiles of students. In the future, it would be useful to make these kinds of studies in various contexts with other theoretical frameworks to validate the conclusions.

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THE NECESSITY OF TWO DIFFERENT TYPES OF APPLICATIONS IN ELEMENTARY GEOMETRY

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This article connects the results of an ontological investigation on elementary geometry to normative questions on educational goals of modelling. The main thesis consists in the assumption that there are two different types of applications in geometry and that they both are necessary and not exchangeable by each other: The first one contains simple applications which are paradigmatic examples to learn basic geometrical concepts; the second one includes more complex ones. It is claimed that a normative discussion on education goals of modelling is only possible as far as the second type is concerned. As a result, the debate on modelling differs in the scope of geometry significantly from similar considerations relative to other parts of mathematics, and that by an ontological and not normative reason.

A CASE STUDY TO RETHINK THE ROLE OF APPLICATIONS

This article is a result of a qualitative study concerning teachers' beliefs (Calderhead 1996) about teaching geometry at German higher level secondary schools (the so-called Gymnasien) including goals, contents, methods and connections to the teachers' broader understanding of mathematics as a whole system. The theoretical framework follows the psychological construct of subjective theories which are defined as systems of cognitions containing a rationale which is, at least, implicit (Groeben et al. 1988). The method depends on case studies. Data are collected by semi-structured interviews and interpreted according to the principles of classical hermeneutics. The construct of subjective theories and its adaption to the didactics of mathematics are briefly summed up by Eichler (2006).

In the following, a small part of this study will be presented. We will describe the difficulty of making sense of a teacher's utterances concerning geometrical applications. This difficulty was the initial point to *rethink the role of applications in elementary geometry in general*. Such a way of rethinking is one of the typical goals intended by the construct of subjective theories: This approach proposes, amongst others, to establish an exchange between individual opinions of "practising semi-specialists" and the theories of the scientific community.

A TEACHER'S OPINION ON APPLICATIONS IN GEOMETRY

The teacher of the case study presented here – let us call him Mr. B – has been taught mathematics, physical education, and computer science at a German secondary school for approximately 25 years. The age of his pupils ranges from 10 to 19 years. He seems to be well grounded in mathematics education and equipped with an elaborated concept of school-compatible mathematical applications. As a part of his position, he is involved in the education of trainee teachers in mathematics. This may be a

further indication for the assumption that he is familiar with recent theories and perspectives of didactics.

As far as applied mathematics is concerned, his criteria for “good” applications match a lot of the attributes which are discussed and accepted by professional didacts (cf. Jablonka 1999). He demands that “the result [of a model building process] has to be useful for practical acting and reasoning” and that the real-world problems have to be “authentic and realistic, and not artificial and constructed” fulfilling their educational functions by being “challenging, but solvable – possibly after and due to simplification” (all quotations are translated by the author). He mentions the concepts of modelling and model building processes explicitly and approves the new style of arguing which is introduced to mathematics education by mathematization. He concludes: “Modelling and mathematical applications – these are things for which I would never abandon just a minute to discuss an automorphism instead.”

AMIDST A STRUGGLE OF TENDENCIES?

At first sight, Mr. B seems to be a true advocate of model building processes and mathematization. But later, when asked how significant applications are for his everyday lessons taught in geometry, he admits that it is “not easy to find good geometrical applications.” He refers to some examples taken from computer-aided design, navigation and traffic routing, but – as the main surprise – he does not expect that these applications are the ones his students should keep in mind. They should rather gain “an understanding of spatial relations” and forms and symmetries and they ought to deal with “rather simple applications” like drawing and folding figures or “reading a city map”; and finally, he does not ask which abilities can be conveyed by modelling and mathematization, but, instead, in which cases modelling is “more necessary for the students” – and one can add: to understand geometry.

At this point, there appears to be a rupture, possibly an inconsistency in Mr. B’s perspectives concerning geometrical applications. On the one hand, he stresses the abilities and capacities in modelling and problem solving, which could be enforced by using authentic and challenging real-world problems; on the other hand, he regards “simple” geometrical applications as a tool to understand the concepts and theorems of elementary geometry – highlighting the knowledge of geometrical objects, of their attributes and dependencies as an educational goal on its own, and not as a device to manage practical challenges and to build up general skills beyond the scope of mathematics. The parts of goals and means seem to be suddenly switched over.

At first sight, there might be a simple and obvious explanation for Mr. B’s ambivalent statements: He could be influenced by two different schools which Kaiser claims to have located within the discussion on mathematical applications (Kaiser 1995). She distinguishes between a *pragmatic* and a *scientific-humanistic approach*: In the pragmatic view, mathematics is a tool to solve practical problems. Applications are deemed as practices to achieve problem solving capacities in managing real-world issues (Kaiser 1995, p. 72). Therefore, applied mathematics is seen from a *procedural*

point of view and modelling and model building *processes* are stressed as a content of the curriculum. The scientific-humanistic school, in contrast, emphasizes the principle of “conceptual mathematization”, that means that real-world situations are used to discover and develop mathematical concepts and insights and to receive mathematical ideas based on manifold associations (Kaiser 1995, p. 72).

GEOMETRICAL WORKING SPACES

To clarify the ideas of the scientific-humanistic school as far as geometry is concerned, it is suitable to use the theoretical framework of *geometrical working spaces* (summed up by Houdement 2007). By this approach, geometry is split into three different paradigms (Houdement & Kuzniak 2003):

1) Geometry I (Natural Geometry): Geometry is seen as an *empirical science* which refers to physical objects. To proof or to refute conjectures, both *deduction and experiments* are allowed, whereas measurement is the main experimental technique. This theory is *not axiomatic*, and its type of deduction is similar to inferential arguments between “local ordered” propositions in ordinary language discussions.

2) Geometry II (Natural Axiomatic Geometry): Geometry is treated as an *axiomatic theory*. The axioms are supposed to refer to the real world and, therefore, to describe physical figure and objects (with some idealization). Insofar, Geometry II is *empirical*, too. But to proof or to reject propositions, no empirical argument is permitted, but only a *deductive* one based on the axioms.

3) Geometry III (Formalist Axiomatic Geometry): Geometry is seen as an *axiomatic and deductive theory*, and no connection to the real world is intended.

With reference to this approach, the main goal of the scientific-humanistic school can be described as the project to prevent a sudden transition from Geometry I in primary school to Geometry III in the higher level secondary school in Germany. Such a sudden transition was enforced by the scientific tradition of this type of school and even increased by the New Maths movement until the early 1980s (Schupp 1994).

The alternative drift of the scientific-humanistic school was to fortify Geometry II, to establish a tender segue from Geometry I to II, and finally to achieve Geometry III or, at least, an idealistic interpretation of Geometry II which replaces the reference to physical objects by the platonic idea of *idealistic objects* not being present in the physical world. This project was mainly motivated by two reasons (cf. Kaiser 1995, p. 73): On the one hand, the ontological binding to real-world objects should be an *intermediate* stage on the way to an idealistic or formalist view of geometry to prevent a not understood formalism. On the other hand, it should establish an understanding of the role geometry plays as a tool in natural sciences. In both cases, the ontological foundation in real-world objects was primarily not intended to enforce model building processes and skills, but to build up a “field of associations” in order to understand geometry or natural science more proficiently.

NORMATIVE ISSUES OF APPLIED MATHEMATICS

Concerning applied mathematics, the pragmatic and scientific-humanistic approach differ in weighting *normative parameters*: One of them sets priorities in practical relevance and abilities to deal with model building processes; the other one stresses the theoretical aspects of mathematics (and natural sciences) and uses the associations to real-world situations as a tool to achieve a deep and connected understanding of mathematical concepts. The origin of this controversy appears to be nothing else but a disagreement about educational goals; and the different role of applications does not seem to arise from a specific character of geometry or geometrical applications, but only from disparate normative points of view – a situation which seems to have the same consequences in every part of mathematics and mathematics education, and not only in matters of geometry.

Exactly this opinion is called into question by our following considerations. We will propose an alternative assumption to explain the main statements of Mr. B. Our explanation is based on two arguments: Firstly, we will discuss an investigation on the *ontology* of geometry to clarify the question whether geometrical applications can be treated in the *same way* as other ones. Secondly, we will concern transcendental arguments to elaborate the issue to what extent the use and choice of geometrical applications are within the scope of *normative* deliberations.

THE STRUCTURAL THEORY OF EMPIRICAL SCIENCES

Our ontological consideration is influenced by a particular kind of *philosophy of science* which is called the “structuralist theory of empirical sciences”, primarily established by Sneed and later elaborated by Stegmüller and others (Sneed 1979 and Stegmüller 1973/1985). The core assumption of this approach is the idea that empirical theories can be described by two components, namely by a *set-theoretical predicate* and a *set of intended applications* (Stegmüller 1973/1985, pp. 27–42). The set-theoretical predicate contains all of the formal and axiomatic aspects and is defined by the same method used by mathematicians in succession of Bourbaki: In the same manner, how it is possible to define the concept of a group as a pair $(G, *)$ so that every element of G fulfils certain axioms relative to $*$, the axiomatic background of classical mechanics can be expressed by a quintuplet so that every element of the carrier set fulfils the well-known Newtonian axioms (Stegmüller 1973/1985, pp. 106–119).

At this stage, there is no difference between an empirical and a non-empirical theory (for example a mathematical theory from a formalistic point of view): They both can be defined by set-theoretical predicates. The difference arises from the set of intended applications: In case of non-empirical theories, this set is empty. In case of an empirical theory, it contains the applications which are claimed to be describable and explainable by the concerned theory. For instance, some of the intended applications of classical mechanics are pendulums, solar systems and especially apples falling from a tree. The set of intended applications cannot be defined extensionally, but only by enumerating paradigmatic examples and by declaring that every entity also belongs to

this set which is “sufficiently similar” to the paradigmatic examples – leaving vague what “sufficiently similar” means (Stegmüller 1973/1985, pp. 207–215).

The concept of geometrical working spaces is a useful framework to establish a connection between geometry and the structuralist theory of science: Geometry I and II are empirical theories insofar they are intended to refer to real-world objects, and they even share the same set of intended applications: physical objects of middle dimension, especially drawing figures and tinkered matters which are used at school. But despite sharing the same set of intended applications, these theories fundamentally differ in their set-theoretical predicates: Whereas Geometry II is assumed to fulfil an axiomatic system of Euclidean Geometry, the propositions of Geometry I may be so vague and psychologically motivated and so variable relative to different times and persons that they certainly cannot be transferred to a system of axioms and accordingly to a defining set-theoretical predicate. In contrast, Geometry III is not an empirical theory, since it is regarded in a formalist manner, presupposing not to have any applications; that means, in this case the set of intended application is empty. But on the other hand, Geometry III shares the same defining set-theoretical predicate with Geometry II: They both are intended to be a Euclidean Geometry.

The set of intended applications is not just an “illustration”, a nice, but useless thing which can be left out; it rather fulfils two indispensable functions: From a logical point of view, the set of intended applications is a conceptual attribute and a part of the *definition of an empirical theory*. It distinguishes an empirical theory from a non-empirical one and declares the “part of the world” to which the theory is connected. Exactly this is the difference between Geometry II and III.

The second function results from the fact that every non-trivial empirical theory is based on idealization. For example, classical mechanics presupposes the existence of point particles without any spatial dimension. However, such entities do not exist in a strict sense of the word, but only “approximately” – and this is the second task of the set of intended applications: Since there is no way to explain explicitly under which condition and to what extent an approximation is allowed to make an empirical theory applicable (Stegmüller 1973/1985, pp. 207–215), i. e. under which condition an application belongs to the set of intended application, the paradigmatic examples of this set provides a number of “case studies” by which the *limits of approximation* are implicitly defined and novices of the scientific community can become familiar with the scope and borders of their coming occupation.

In geometry, the problem of approximation will typically arise, if infinity or dimension zero occurs; straight lines, planes, and angles are paradigmatic examples of this case (Struve 1990, p. 43). For instance, if there is a line drawn on a paper, there will be two ways to deal with the question “Is this a straight line, a segment of a straight line or neither of them?”: From a formalist or idealistic view of geometry, this is a trivial question, since geometry does not refer to physical objects; a physical line is neither a segment nor straight line; at most, drawings could be symbolic tools to think about geometrical objects or propositions. But if it is taken serious that geometry can

be interpreted as an empirical theory (as supposed in Geometry I and II and as being common and necessary for geometrical applications as we will see later), the pupils will have to learn to treat a line sometimes as a segment and sometimes as a straight line. To deal with these decisions is a notorious problem in geometry. The intended applications like drawing figures are the paradigmatic examples by which pupils are supposed to learn to manage these questions.

Hence, the knowledge of the set of indented applications and the handling of its vagueness is not optional, but an integral part of a particular empirical theory and, therefore, one of the aspects of “possessing” and being able to apply a certain theory. The educational task of paradigmatic examples is primarily described by Kuhn as far as philosophy of science is concerned (Kuhn 1962/1976, pp. 59–62). It is also a common thesis in psychology that paradigmatic examples play a major role in learning a theory (e. g. Seiler 2001, pp. 144–225).

ONTOLOGICAL ASPECTS OF ELEMENTARY GEOMETRY AT SCHOOL

At this point, we will come back to didactics. Struve has investigated how elementary geometry is presented in secondary school following the philosophy of science structuralism sketched above (Struve 1990, p. 6). Expressed in terms of the theory of geometrical working spaces, he comes to the conclusion that the didactical changes which were established to avoid a sudden switch from Geometry I to Geometry III by stressing Geometry II (as mentioned above) factually took the effect that the new textbooks present rather Geometry I than Geometry II and (even if Geometry II is reached) geometry is continuously taught as an empirical theory, and never as a formalistic or idealistic one as intended: “students learn an empirical theory in the geometry lessons held at secondary school” and “concerning the empirical theory, as we want to call the theory the students learn in their geometry lessons according to our investigation, figures created by folding and drawing are the paradigmatic examples” (Struve 1990, pp. 38–39).

THE ISSUE OF MODELLING

Struve has mentioned some of the consequences of his result – foremost some consideration on the fact that proofs have different functions in empirical and non-empirical theories observing that students typically treat proofs in the same manner as they are used in empirical sciences (Struve 1990, pp. 38–49). In this article, we will add a consideration concerning modelling. If we can follow Struve’s results, Mr. B’s distinction between two types of geometrical applications is not confusing, but an obvious implication of the empirical character of geometry as it is taught in secondary school: The figures created by drawing and folding and the “simple” applications based on these figures can be regarded as the paradigmatic examples which define the set of intended applications and constitute geometry as the empirical science of the spatial environment surrounding us in everyday life.

In this view, the supremacy of simple applications is not based on a normative decision about the role of application in mathematics education, but on the *specific ontology of geometry*: The knowledge of and the familiarity to these examples of applications are defining attributes of geometry as an empirical science. Hence, with regard to these “basic” applications, geometry differs from the other parts of mathematics taught at school. In the other cases, the amount and choice of applications is a normative question guided by arguments which Kaiser has combed through. In geometry, however, the task of normative deliberations begins not before the set of intended applications is left. Therefore, it is not astonishing that the (rare) cases which Mr. B mentions as “real” examples of modelling in geometry are quite different from the paradigmatic examples of folding and drawing: computer-aided design, navigation and traffic routing. In these cases and after some basic courses based on “simple” applications, geometry may no longer differ in modelling and mathematization.

TRANSCENDENTAL ASPECTS OF GEOMETRY

Our last task concerns the question if the dominance of an empirical view of geometry at school (as Geometry I or II) is an aberration caused by psychological circumstances and enforced by “misguided” textbooks or if there are good reasons to teach geometry as an empirical theory (to some extent). We will argue for the latter, accentuating a special role of geometry in contrast to other parts of mathematics and aiming for the conclusion that therefore two different types of applications are needed.

Let us start with an example: In 2003, a new national curriculum framework called “Bildungsstandards” (educational standards) was established in Germany. In contrast to former resolutions, this declaration stresses general skills, abilities and competencies – and among others, abilities in mathematical modelling. The relevant paragraph closes with the following sentence: “This includes translating the situation which is to be modelled into mathematical concepts, structures and relations” (KMK 2004, p. 8). This is a formulation ranging over all parts of mathematics taught at secondary school. A specific statement focussing on geometry is not declared.

Let us deliberate what this sentence presupposes: There is a real-world situation which can be described by mathematical concepts, but need not to be treated in this way. For instance, you can cross the road without thinking about the probability to be knocked over and you can look at the carps in a lake without having a function in mind to describe their growth process. Normally, a mathematical description is *not necessary* and will only be introduced, if it promises deeper insights as a description in ordinary language. Besides the general skills, this is a typical educational goal of modelling: the awareness that mathematics is a useful tool to achieve knowledge of the external world and to formulate this knowledge in a very precise manner.

In geometry, the case is quite different. If geometry could be treated like other mathematical theories, it would be possible to describe a situation geometrically *only on demand*. But this assumption fails since it is inevitable to use, at least, rudimental

geometrical concepts to describe a situation at all. You cannot cross the road or look at the carps in the lake without possessing, at least, a broad understanding of basic geometrical concepts. For instance, a (vague) understanding of relative positions is necessary to individuate the different things, persons or objects which are part of a specific situation.

The idea that space is not a thing of human perception among others, but the conceptual framework which allows to describe real-world phenomena was primarily introduced by Kant as a part of his *transcendental* philosophy (Kant 1781/1998). In contemporary ontology the conceptual framework of space (and time) is broadly accepted as a condition to describe real-world situations (for everyday perceptions see Runggaldier and Kanzian 1998, pp. 17–52, as a condition of empirical sciences see Bartels 1996, pp. 23–71, or Stegmüller 1973/85, p. 60).

CONCLUSION: TWO TYPES OF GEOMETRICAL APPLICATIONS

Now, it is possible to connect both arguments: Following transcendental considerations, it is necessary to possess basic concepts to describe real-world situation and to establish the conditions under which model building processes are possible. That means, for mathematical reasons it may be passable to interpret geometry as a formalist or idealistic theory; but for model building processes or in contexts of natural sciences, it is necessary to understand geometry as an empirical theory. For some simple model building processes, an understanding on the level of Geometry I may be sufficient, but for more elaborated tasks or as a tool of natural sciences, Geometry II seems to be indispensable.

Against this background, we attain a “two step view” of geometrical applications: Since concepts of an empirical geometry are necessary to apply mathematics and, in a structuralist view of science, these concepts correspond to a set of intended applications taken from the world of folding and drawing, the first type of applications consists of very “simple” applications whose function is completely defined by learning and applying elementary geometry, especially by learning to manage the reference of concepts like “straight line” which can only be applied due to approximation. Hence, geometrical applications of a “simple” kind are *inevitable ingredients* of teaching geometry; and there is no reason to criticize the simplicity of these applications. At this stage, a normative debate about goals of teaching “applied geometry” is inadequate, since according to the empirical character of school geometry, there is no difference between teaching applied geometry and teaching geometry at all. This shall be our first conclusion: To some extent, it is necessary to deal with simple geometrical applications; and this necessity is not an inference from a normative decision about the goals of teaching applied mathematics, but a consequence of the specific ontological situation of geometry and its transcendental function as a condition of natural science and ordinary perception. No other part of secondary school mathematics possesses this ontological and transcendental function. For this reason, the status of geo-

metry is unique, and the debate on geometrical applications cannot be held in the same way as it is possible in the scope of other parts of mathematics.

The second conclusion is related to the other type of geometrical applications: If the “simple” and intended applications are the only ones which students get to know, there will be an obvious deficit in teaching general skills and model building capacities in the sense of the pragmatic view of applied mathematics. Exactly this is the function of the second type of geometrical applications. It is comprehensible that applications which are intended to fulfil this task are quite different from the first ones. Mr. B mentions examples taken from computer-aided design, navigation and traffic routing. A list of similar examples is collected by Graumann (1994). Applications of this kind are typically not “pure geometrical”, but includes concepts or hypotheses taken from natural or social sciences, basic economics or empirical tedium platitudes. This fact can be regarded as a further indication for our claim that there two different types of applications with distinct functions: Whereas the simple ones are used to built up geometrical concepts and to manage the vagueness of applying geometrical concepts to real-world situations, the more complex ones are intended to use *pre-existing* geometrical concepts and insights to reach some of the many educational goals which Kaiser sums up for model building processes in general (Kaiser 1995). For this purpose, a real-world problem only providing geometrical aspects often does not appear to be multifarious enough to allow a model building process whose challenges lie in this process (including mathematization, simplification, validation and hypothesis testing), and not in geometrical deliberations and calculations.

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A FRENCH LOOK ON THE GREEK GEOMETRICAL WORKING SPACE AT SECONDARY SCHOOL LEVEL

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Based on the geometrical paradigms approach, various studies have shown some tension in French Geometrical Working Space between institutional expectation and effective implementation. In this paper, we examine the Greek system from this point of view and we find the same kind of tension but in a certain sense stronger than in France even if both countries have an ancient Euclidean tradition.

FROM SPECIFIC FRENCH CASE TO THE PARTICULAR GREEK CASE

Since several years, it seems that curricula and syllabi converge to promote a close link between mathematics teaching and the “real world”. The idea of “mathematical literacy” is especially strong in the PISA evaluation which aims to organize this general trend among European countries. At the same time and close to this conception of mathematics, the constructivist approach is favoured by national educational institutions and teachers are asked to substitute “bottom up” teaching methods to the traditional “top down” entrance in mathematics.

In France, till today, and at lower secondary school level the prominent way suggested by the intended curriculum is based on “inquiry methods” and “activities” and relationships between mathematics and other scientific or technological domains are always pointed up. But the link to sensible world is only mentioned and the emphasis is put on the logical rigour of mathematics. The relationship to the “real world” seems really far off and into everyday classroom, inquiry based methods are left aside.

In the special case of geometry, we were concerned with the contradiction between official expectation and the crude reality of the classroom. To understand and explain the phenomenon, the notion of geometrical paradigms (Houdement and Kuzniak, 1999) and of geometrical working spaces (Kuzniak, 2007) have been used to explicit the different meanings of the term geometry. The field of geometry can be mapped out according to three paradigms, two of which – Geometry I and II – play an important role in today’s secondary education. Each paradigm is global and coherent enough to define and structure geometry as a discipline and to set up respective working spaces suitable to solve a wide class of problems.

This first idea is completed by the following hypothesis on the possible influence of these paradigms in geometry education and on the poor implementation of new teaching method. The spontaneous geometrical epistemology of teachers enters in contradiction with mathematical epistemology embedded in the new teaching methods. In other words: the geometrical work done and aimed by teachers could be

of another nature than the institutional expected one. The teacher's geometrical thinking is led by another geometrical paradigm as the paradigm promoted by the institution. Moreover this way of thinking leads to prefer pedagogical methods in contradiction with inquiry based methods.

Our investigation work has its roots in the French context but some comparative studies showed us that such a tension could exist in other countries. Houdement (2007) has presented in CERME 5 a comparison of magnitude measurement problems in Chile and in France. The social and economical contexts are quite different in both countries and so, we were interested to have a look on other European countries to verify if this kind of tension really exists and how it was managed. We have had the opportunity to work with Greek colleagues and to be aware of a great change in the curriculum based on the real world and turning back to the Euclidean tradition. We present the first part of our work which gives our analysis of the Greek situation through our viewpoint.

GENERAL FRAME OF THE STUDY

The theoretical frame we used has been soon described in detail in former CERME sessions (Houdement and Kuzniak 2003, Houdement 2007) and we refer to these papers for complements. We retain only here some particular elements used in our description of the Greek situation.

As we are interested in the awkward relationships between reality and mathematics education, we will focus on the role the reality plays in the different paradigms. In the first one, Natural Geometry or Geometry I (GI), the validation depends on reality and the sensible world. In this Geometry, an assertion is accepted as valid using arguments based upon experiment and deduction. The confusion between the model and reality is great and any argument is allowed to justify an assertion and convince. This Geometry could be seen as an empirical science and it is possible to build empirical concepts depending on the experience of the "real world". Natural Axiomatic Geometry, or Geometry II (GII), whose archetype is classic Euclidean Geometry is built on a model that approaches reality. Once the axioms are set up, proofs have to be developed within the system of axioms to be valid. In the formal Axiomatic Geometry, or Geometry III (GIII), the system of axioms, which is disconnected from reality, is central and leads how to argue. The system of axioms is complete and unconcerned with any possible applications in the world. In that case, the system creates its reality. Concepts are given *a priori* and come "from the Book" and so "top down" form of mathematics education seems well fitted to this conception. The study of Greek mathematical education will show that this dichotomy GII / GIII is not so simple.

To find a possible tension or contradiction between the institutional expectation and the teacher's approaches, we will describe what we call the personal teacher's Geometrical Working Space (GWS) faced to the GWS expected and promoted by the national institution in charge of mathematics education. More precisely (Kuzniak

2006), the Geometrical Working Space (GWS) is the place organized to ensure the geometrical work. It makes networking the three following components: the real and local space as material support, the artefacts as drawings tools and computers put in the service of the geometrician and a theoretical system of reference possibly organized in a theoretical model depending on the geometrical paradigm. To ensure that the components are well used, we need to focus on some cognitive processes involved into the geometrical activity and particularly the visualization process with regard to space representation and the material support, the construction process depending on the used tools (rulers, compass, etc.) and on the configuration, and finally reasoning in relation to a discursive process.

THE NEW CURRICULUM IN GREECE

Since 2007, a new curriculum for compulsory education is implemented in gymnasium (grades 7 to 9) in Greece and summarised in a list of ten highlights. It is presented as cross-thematic (1st and 5th highlights) and aims to connect the academic disciplines, everyday life, working world, history, technological improvement, etc. Within the flexible zone (4th highlight), some hours are planned for reaching this specific goal. Primary school learning explicitly rests on the Bruner's constructivist theory and assessment is now an essential part of the learning process (8th highlight). Sources and goals of connection with reality are in the 9th highlight, "A Broad Spectrum of Literacies":

Successful living in post-modern times presupposes that one is fully literate in many areas, such as reading, science, technology and mathematics in order to face international evaluation (PISA, TIMS, etc.) which demand more connections between school knowledge and the life reality.

The present mathematical syllabus expands the ancient one with no change in the content. It is written in a three columns table where some more detailed mathematical sections appear into the traditional blocks (arithmetic, algebra, geometry). Mathematical skills, which have to be learned by pupils, are described in the first column, the main mathematical notions are in the second and in the third one some activities are proposed, often to introduce some mathematical notions.

New textbooks are conformed to syllabus with no surprise since they are chosen by the curriculum designer Pedagogical Institute, one for each level. Textbooks structure is quite the same as the syllabus structure and activities coming from the syllabus third column can be found with few changes in textbooks. For these reasons, institutional GWS means the GWS presented by the curriculum including the official textbooks.

A SO FAR REALITY

We will highlight some internal slides into the institutional GWS itself. First, in spite of the curriculum demand, new technologies have to be used (7th highlight), syllabus

and textbooks do not mention software, computers or Internet. Beside this slide inside the curriculum, the reality is concerned by a second and less obvious one.

According to the cross-thematic curriculum, reality and everyday life have to be embedded in the learning process. But when everyday life is mentioned in syllabus it is without any details and only one syllabus activity could be described as real : *measure the width of the street and pavement in front of the school*. But the difficulty to follow this curriculum directive is more obvious in textbooks. This real activity in syllabus does not appear in the A' textbook (grade 7), and if there are numerous activities based on a “real picture”, they are not relevant for this purpose for several reasons:

- The 3D/2D problem: angles and distances on the textbook are not the good ones. For these kind of activities, geometry does not seem to be able to give the right answer!



Students have to find the lower distance between the point A at the house to the water. (A' page 184)

- A lot of activities refer to the macro-space but authors represent reality – probably under editorial constraint – with an image or photography. On these pictures, most of the time, some geometric element are placed and the reality is already mathematized. However, we often find activities and exercises with geographic maps, as it is stated in syllabus. But reality is once more already mathematized.



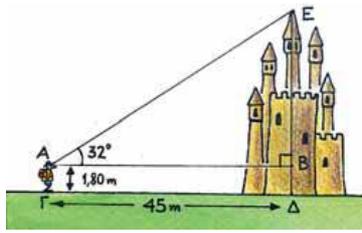
Why an airplane realize a lower distance than a boat to go from Athens to Samos? (A' page 164)



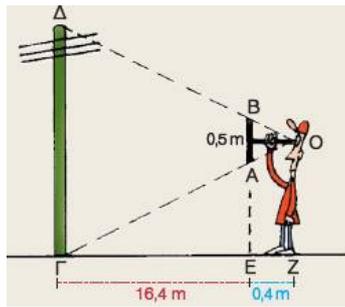
$\Gamma A=6371$ km, $\angle \Gamma = 89,05^\circ$, find ΓZ . (B' page 151)

- Activities and exercises are most of the time based on a picture of a real problem with a geometric diagram with all the measures needed to solve the problem, no more

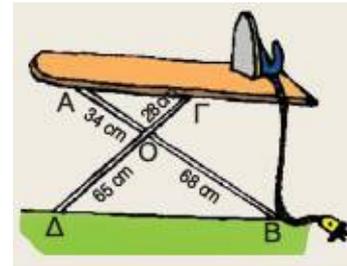
no less. Reality is not the point and is viewed through a picture already turned into a geometrical task support.



Find ΔE . (B' page 139)



Find $\Delta \Gamma$. (G' page 223)



Why it is not horizontal?
(G' page 209)

As we notice it, the geometrical local space is almost always the micro-space of a sheet of paper which is sometimes a representation of a macro-space problem (geographic maps, pictures, etc.). Actually, the reality in textbooks appears from a relevant point of view only in the GI paradigm [1], on a sheet of paper. And so we can characterize this internal slide: everyday life is not taking into account and reality is only treated within the GI paradigm, inside geometry.

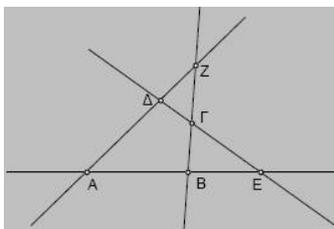
GYMNASIUM INSTITUTIONAL GWS

Since reality is not actually present in institutional GWS, except within the GI paradigm, we study the institutional GWS all along the gymnasium.

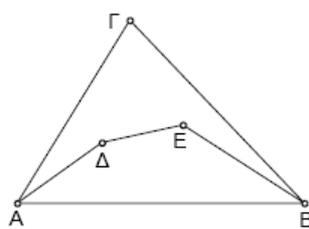
Artefacts, visualization and diagrams constructions: the GI paradigm

Geometric tools (ruler, compass, protractor, square, tracing paper) are only mentioned in syllabus at the A' class (grade 7). However, construction activities are present all along the gymnasium (much more at the first class). In the A' textbook, tools are pictured in many places, especially for showing how to construct. Tracing paper is used in many geometry sections, often to introduce a new concept. In the B' and G' textbooks (grades 8, 9) geometric tools are never drawn, sometimes mentioned.

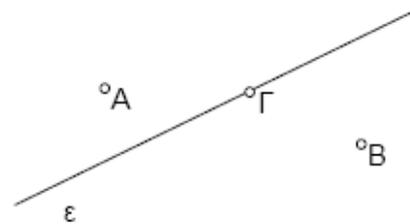
There is no freehand construction in syllabus, no freehand diagram in textbooks and we do not find any exercise where pupils have to draw such a kind of diagram. Some activities proposed in syllabus (third column of A' class) are in GI, excluding, or not, visualization:



How many angles?



Find in measuring the lower distance between A



Draw the perpendiculars to ϵ passing by these points.

An aim of syllabus, at B' class (grade 8), section trigonometry, is to construct an angle whose sinus, cosine or tangent are known. But we do not find any activity on

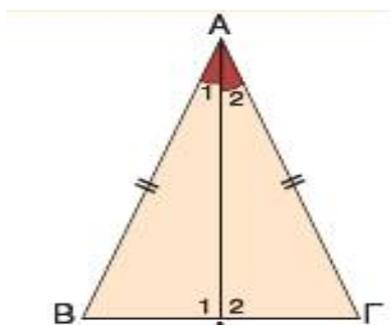
this topic in textbook. At the final class (grade 9) the section on dilation is directed by the GI paradigm with numerous drawing activities (7 exercises of the 9 at the end of the section ask for drawing).

Formal proofs: the GII paradigm

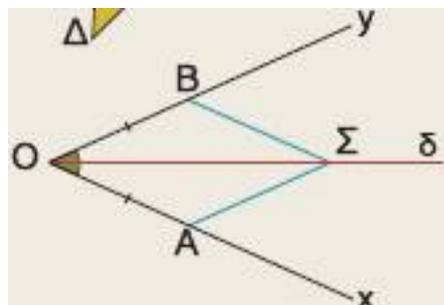
Proof process should start as it is written in syllabus preamble, but no formal proof is mentioned in the detailed table of mathematics syllabus. There are some theorems, definitions, properties.

Very few examples of formal proofs are given in the A' textbook (grade 7) and their solutions are always completely written. It is quite the same situation in the B' textbook (grade 8), except the proof that a dodecagon is regular (exercise 8, page 185). In the B' area section, a lot of exercises ask to "show that" but, in fact, the solution is always given by a calculation of an area or a length.

In G' textbook (grade 9) there is a great change with a lot of exercises where pupils have to prove. At the section on triangle congruence, the 21 exercises at the end of the section ask for a formal proof and the theoretic system of reference, with the three criteria of triangle congruence, is clearly directed by the GII paradigm. In this section, there are four solved exercises (pages 191, 192) which ask for a formal proof on triangle congruence (see below, for example, the figure on the left). At the end of the section (pages 194-196) some similarly exercises are given (see below, for example, the figure on the right). One could thought that the solutions of the four solved exercises could give a proof model to students to solve exercises at the section end.



Prove that $\Delta B = \Delta \Gamma$ (AA' is the bisector of \hat{A}). With solution. (G' page 191)



Prove that $A\Sigma = B\Sigma$ ($OA = OB$, $O\delta$ is bisector of \hat{O}). Exercise without solution. (G' page 194)

The diagrams similarity section is also in GII paradigm (half of the exercises ask for a formal proof, the others are on ratio and length calculation).

Gymnasium paradigm

At the first class A', both in curriculum and textbook, the main paradigm is GI and it is generally well assumed. However, the paradigm in which pupils have to work is not always clear. For example, the following syllabus activity starts in GI and finishes, with questions g) and h), necessarily in GII:

- a) Let O a point and a line ε and the point A so that OA is the distance from O to ε .

- b) Let B another point on ε , find the symmetric A' and B' of A and B through O and let ε' the line $A'B'$.
- c) Which is the symmetric of ε through O ?
- d) Which is the symmetric of the angle \widehat{OAB} ?
- e) How are the angles \widehat{OAB} and $\widehat{O'A'B'}$?
- f) How is the angle $\widehat{O'A'B'}$?
- g) How are ε and ε' with respect to AA' ?
- h) How are ε and ε' ?

Didactic contract is not very clear for the intermediate questions c), d) and e): GI, with tools or visualization, or GII paradigm? This activity is given in textbook with only one question and a complete solution below. The task paradigm is clearly GII: the answers corresponding to questions e) to h) are formal proofs. This example is a non explicit slide from GI to GII in a class where GI is the main paradigm [2].

Artefacts and diagrams constructions are used in many activities to discover geometrical properties, as it is written in the curriculum according to the bottom-up point of view: from the GI paradigm arises the GII paradigms. Some activities given in the third column of syllabus are in GI, to construct, to observe a property (sometimes in first class with the use of tracing paper and folding). This kind of activities can be find in all gymnasium textbooks (grades 7 to 9).

In gymnasium, from grade 7 to 9, geometrical tasks are very different. The GWS depends on the class and the section. In the first class GWS is clearly directed by GI but there are some slides in favour of the GII paradigm. In the last class, the GWS of the triangle congruence section is directed by GII while it is directed by GI in the section on dilation. In this last class, there are several very different GWS which seem not to be connected.

EUCLIDEAN PRESSURE ON TEACHER'S PERSONAL GWS

This section is supported by six secondary teachers' interviews where we focused on the new curriculum and more specifically on reality, geometrical tools, diagram constructions and formal proofs in textbooks and in classrooms. We turn out to teacher's personal GWS which is quite different from the institutional one as we will show it. Before studying the GWS teachers, we point out the particular importance of Euclidean Geometry in the Greek syllabus and for Greek teachers.

The paradoxical place of Euclidean geometry

According to the Lyceum syllabus, students have to learn a geometry based on axioms with formal reasoning (grade 10) and measurement of magnitudes becomes the main geometric topic at grade 11. The unique geometry textbook is entitled "Euclidean Geometry" and it is used in the two first classes (grade 10 and 11). Its content is close to the syllabus and to the classical Euclidean Geometry with a strong axiomatic point of view, except for measurement. In textbook, and for lyceum

teachers, geometry starts from zero with Euclidean axioms. Construction problems are of theoretical nature with letters and magnitude, such as $AB=a$, without any measure: geometrical tools are virtual and consist of compass and ruler according to the Euclidean tradition.

If Geometry is taught in compulsory education and during the two first lyceum classes (till grade 11), geometric knowledge is not assessed at the very important lyceum final test: the University where students will enter depends of this final test. Students know this fact and are less concerned with geometry than the others mathematics domains and do not work geometry especially in the numerous private institutes (*frontystiria*) where they could follow additional and expensive courses after the class time. It is a quite great contrast: a lot of geometry teaching times for nothing at the end? Teachers we interviewed told us that geometry is not important in the curriculum because of the hidden curriculum and, finally, “geometry is taught for culture, for Euclid”.

Teachers’ personal GWS

Gymnasium teachers think that pupils have to learn how to construct geometric diagrams, but they think that it is not the main point of mathematics learned in gymnasium. So as they have no time to teach all the syllabus, teachers often choose to teach very quickly diagrams constructions despite its importance and the fact that students have troubles with the use of drawing tools (especially the protractor) and with constructions. In the personal teacher's GWS, directed by GII, the aim of a diagram is to set a conjecture and the proof do not need an exact figure. That explains why teachers think that a freehand drawing is equivalent to a drawing with geometrical tools and the first one is done more quickly. Teachers’ local space could be anywhere they can draw a freehand diagram, for example a pack of cigarettes as two teachers told us. We see here a great difference between teachers’ beliefs and institutional content: in syllabus, nor in textbooks, there is none freehand drawing.

Another example of the prominence of GII in the personal teacher GWS is the importance they give to properties of quadrilaterals and triangles. They all think that these properties are fundamental even if they do not know the role of these geometric objects in mathematics class. As teachers rate highly Euclidean Geometry, a sufficient reason to teach triangles and quadrilaterals is given by their importance in the theoretic system of reference.

To conclude this part, we can say that the teachers’ GWS is clearly directed by a strong GII, almost GIII because of the axiomatic theoretic system of reference.

GWS TENSION

The new Greek curriculum demands to take into account reality. But the interviewed teachers told us how it is difficult for them: they do not know how to teach in a constructive way which is often opposite to their top-down learning conception. They concluded that Greek teachers do not like this new way of teaching and do not

understand it. Teachers' learning beliefs agree with the internal slide we pointed out about the everyday life in curriculum.

In the case of diagrams constructions, teachers' GWS is clearly against the institutional GWS, and not only in considering freehand drawing. Teachers do not only prefer teaching others geometric topics but they give all the diagrams in tests too to go over the lack of their pupils [3]. The same opposition to the institutional GWS can be seen with the use of tracing paper. According to syllabus, tracing paper has to be used as a geometric tool in A' class (grade 7). It is used in many places with a particular and original graphical representation in the A' textbook and it is explained how to use it. But creativity stops at the school border and tracing paper is never used in class!

In gymnasium, formal proof is usually taught during the last class year (grade 9), more specifically, in a Euclidean section about triangle congruence. In order to know how teachers could initiate their students to the formal proof in one year, we asked them about the possible use of the four solved activities we spoke about in the "Formal proofs: the GII paradigm" section. They are indeed proof models and, for assessment, students have to learn ten lesson proofs by heart which one of them is asked in test. This proof process initiation is again opposite to the curriculum expectation.

In gymnasium, there is a distance between institutional and teachers GWS. That creates a tension which is supported by the different beliefs on learning and geometry among teachers and curriculum writers. Moreover, teachers do not really deal with the existing and remaining students' difficulties with diagram constructions and the proof process initiation is based on a learning by heart. This tension between institutional and teachers' GWS is specific to gymnasium, it completely disappears at lyceum, but what about pupils?

CONCLUSION

Geometry positions in Greece and in France are closed even if we point out some main differences. In both countries, even if curriculum emphasizes its place, reality is not taken into account. Similarly, the transitions between paradigms GI and GII are most of the times ambiguous and implicit and give rise to fuzzy GWS.

The GI paradigm seems to be more assumed in Greece than in France and in France formal proofs are taught all along the junior high school. But the main curriculum difference takes place at the lyceum: in Greece, axiomatic Euclidean geometry is taught, not in France, and in France geometry is assessed in final test for some sections, not in Greece. Geometry is taught in Greece only for cultural reasons, for Euclid, whereas in France the geometrical work is oriented by the GII paradigm and university studies. However, according to the six teachers' interviews, the Greek teachers' GWS is quite different from the French teachers' GWS because of the axiomatic theoretic system of reference: GII paradigm is well structured and stronger in Greece than in France. In Greece, the cultural tradition of Euclid is more important

than in France and geometry knowledge seems to come from the Book [4]. This last point strengthens the GWS tension in junior high school which seems to be stronger in Greece than in France.

NOTE

1. The exercise on a map are in GI, but it could be solved by visualization or measurement, pupils have to choose.
2. This non explicit slide can also be seen, for example, at page 227 of A' textbook, examples 1 and 2.
3. In the A' final test we studied there is no construction; lyceum pupils have problems with geometric diagrams constructions, even for the equilateral triangle whereas it is a skill of the A' gymnasium class (grade 7).
4. According to Toumasis (1990) the Book is not Euclid's Elements but Legendre's geometry elements.

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A THEORETICAL MODEL OF STUDENTS' GEOMETRICAL FIGURE UNDERSTANDING

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This study investigated the role of various aspects of apprehension, i.e., perceptual, operative and discursive apprehension, in geometrical figure understanding. Data were collected from 1086 primary and secondary school students. Structural equation modelling affirmed the existence of six first-order factors revealing the differential effect of perceptual and recognition abilities, the ways of figure modification and measurement concepts, three second-order factors indicating the differential effects of the various aspects of geometrical figure apprehension and a third-order factor representing the geometrical figure understanding. It also provided support for the invariance of this structure across the two age groups. However, findings revealed differences between primary and secondary school students' performance and in the way they behaved during the solution of the tasks.

INTRODUCTION AND THEORETICAL FRAMEWORK

In geometry three registers are used: the register of natural language, the register of symbolic language and the figurative register. In fact, a figure constitutes the external and iconical representation of a concept or a situation in geometry. It belongs to a specific semiotic system, which is linked to the perceptual visual system, following internal organization laws. As a representation, it becomes more economically perceptible compared to the corresponding verbal one because in a figure various relations of an object with other objects are depicted. However, the simultaneous mobilization of multiple relationships makes the distinction between what is given and what is required difficult. At the same time, the visual reinforcement of intuition can be so strong that it may narrow the concept image (Mesquita, 1998). Geometrical figures are simultaneously concepts and spatial representations. Generality, abstractness, lack of material substance and ideality reflect conceptual characteristics. A geometrical figure is also possesses spatial properties like shape, location and magnitude. In this symbiosis, it is the figural facet that is the source of invention, while the conceptual side guarantees the logical consistency of the operations (Fischbein & Nachlieli, 1998). Therefore the double status of external representation in geometry often causes difficulties to students when dealing with geometrical

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problems due to the interactions between concepts and images in geometrical reasoning (e.g. Mesquita, 1998).

Duval (1995, 1999) distinguishes four apprehensions for a “geometrical figure”: perceptual, sequential, discursive and operative. To function as a geometrical figure, a drawing must evoke perceptual apprehension and at least one of the other three. Each has its specific laws of organization and processing of the visual stimulus array. Particularly, *perceptual apprehension* refers to the recognition of a shape in a plane or in depth. In fact, one’s perception about what the figure shows is determined by figural organization laws and pictorial cues. Perceptual apprehension indicates the ability to name figures and the ability to recognize in the perceived figure several sub-figures. *Sequential apprehension* is required whenever one must construct a figure or describe its construction. The organization of the elementary figural units does not depend on perceptual laws and cues, but on technical constraints and on mathematical properties. *Discursive apprehension* is related with the fact that mathematical properties represented in a drawing cannot be determined through perceptual apprehension. In any geometrical representation the perceptual recognition of geometrical properties must remain under the control of statements (e.g. denomination, definition, primitive commands in a menu). However, it is through *operative apprehension* that we can get an insight to a problem solution when looking at a figure. Operative apprehension depends on the various ways of modifying a given figure: the mereologic, the optic and the place way. The mereologic way refer to the division of the whole given figure into parts of various shapes and the combination of them in another figure or sub-figures (reconfiguration), the optic way is when one made the figure larger or narrower, or slant, while the place way refer to its position or orientation variation. Each of these different modifications can be performed mentally or physically, through various operations. These operations constitute a specific figural processing which provides figures with a heuristic function. In a problem of geometry, one or more of these operations can highlight a figural modification that gives an insight to the solution of a problem.

Even though previous research studies investigated extensively the role of external representations in geometry (e.g. Duval, 1998; Kurina, 2003), the cognitive processes underline the four apprehensions for a “geometrical figure” proposed by Duval (1995, 1999) have not empirically verified yet. Keeping in mind the transition problem from one educational level to another universally (Mullins & Irvin, 2000), our main aim was to confirm a three-order theoretical model concerning the primary and secondary school students’ geometrical figure understanding.

HYPOTHESES AND METHOD

In the present paper four hypotheses were examined: (a) Perceptual, discursive and operative apprehension influence primary and secondary students’ geometrical figure understanding, (b) There are similarities between primary and secondary school students in regard with the structure of their geometrical figure understanding, (c)

Differences exist in the geometrical figure understanding performance of primary and secondary school students and (d) Differences exist in the way primary and secondary school students behave during the solution of the perceptual, discursive and operative apprehension tasks. It should be mentioned that the influence of sequential apprehension in geometrical figure understanding is not investigated since the figure construction is not given much emphasis in the Cypriot curriculum.

The study was conducted among 1086 students, aged 10 to 14, of elementary (Grade 5 and 6) and secondary (Grade 7 and 8) schools in Cyprus (250 in Grade 5, 278 in Grade 6, 230 in Grade 7, 328 in Grade 8). The a priori analysis of the test that was constructed in order to examine the hypotheses of this study is the following:

1. The first group of tasks includes task 1 (Pe1a, Pe1b, Pe1c, Pe1d, Pe1e, Pe1f, Pe1g) and 2 (Pe2a, Pe2b, Pe2c, Pe2d, Pe2e, Pe2f) concerning students' geometrical figure perceptual ability and their recognition ability, respectively.
2. The second group of tasks includes area and perimeter measurement tasks, namely task 3 (Op3), 4 (Op4), 5 (Op5) and 6 (Op6a, Op6b, Op6c). These tasks examine students' operative apprehension of a geometrical figure. The tasks 3, 4 and 5 require a reconfiguration of a given figure, while task 6 demands the place way of modifying two given figures in a new one in order to be solved.
3. The third group of tasks includes the verbal problems 7 (Ve7), 8 (Ve8), 9 (Ve9), 10 (Ve10) and 11 (Ve11) that correspond to discursive figure apprehension. On the one hand, the verbal problems 7 and 8 demand increased perceptual ability of geometrical figure relations and basic geometrical reasoning. On the other hand, tasks 9, 10 and 11 are verbal area and perimeter measurement problems. In verbal problem 9 visualization (e.g. Presmeg, 2007) facilitates its solution process, while in verbal problems 10 and 11 the concept of epistemological obstacles (Brousseau, 1997) may interfere the way of solving them.

Representative samples of the tasks used in the test appear in the Appendix. Right and wrong or no answers to the tasks were scored as 1 and 0, respectively. The results concerning students' answers to the tasks were codified with Pe, Op and Ve corresponding to perceptual, operative and verbal problem tasks, respectively, followed by the number indicating the exercise number.

In order to explore the structure of the various geometrical figure understanding dimensions a third-order confirmatory factor analysis (CFA) model for the total sample was designed and verified. Bentler's (1995) EQS programme was used for the analysis. The tenability of a model can be determined by using the following measures of goodness-of-fit: χ^2 , CFI and RMSEA. The following values of the three indices are needed to hold true for supporting an adequate fit of the model: $\chi^2/df < 2$, $CFI > 0.9$, $RMSEA < 0.06$. The a priori model hypothesized that the variables of all the measurements would be explained by a specific number of factors and each item would have a nonzero loading on the factor it was supposed to measure. The model

was tested under the constraint that the error variances of some pair of scores associated with the same factor would have to be equal. A multivariate analysis of variance (MANOVA) was also performed to examine if there were statistically significant differences between primary and secondary school students concerning their understanding in the various geometrical figure dimensions. For the analysis of the collected data the similarity statistical method (Lerman, 1981) was conducted using the statistical software C.H.I.C. (Bodin, Coutourier, & Gras, 2000). A similarity diagram of primary and secondary school students' responses at each task or problem of the test was constructed. The similarity diagram allows for the arrangement of the tasks into groups according to the homogeneity by which they were handled by the students.

RESULTS

Confirmatory factor analysis model. Figure 1 presents the results of the elaborated model, which fitted the data reasonably well [$\chi^2(220) = 436.86$, CFI = 0.99, RMSEA = 0.03, 90%, confidence interval for RMSEA 0.026-0.034]. The first, second and third coefficients of each factor stand for the application of the model in the whole sample (Grade 5 to 8), primary (Grade 5 and 6) and secondary (Grade 7 and 8) school students, respectively. The errors of variables are omitted.

The third-order model which is considered appropriate for interpreting geometrical figure understanding, involves six first-order factors, three second-order factors and one third-order factor. The three second-order factors that correspond to the geometrical figure perceptual (PEA), operative (OPA) and discursive (DIA) apprehension, respectively, are regressed on a third-order factor that stands for the geometrical figure understanding (GFU). Therefore, it is suggested that the type of geometric figure apprehension does have an effect on geometrical figure understanding, verifying our first hypothesis. On the second-order factor that stands for perceptual apprehension the first-order factors F1 and F2 are regressed. The first-order factor F1 refers to the perceptual tasks, while the first-order factor F2 to the recognition tasks. Thus, the findings reveal that perceptual and recognition abilities have a differential effect on geometrical figure perceptual apprehension. On the second-order factor that corresponds to operative apprehension the first-order factors F3 and F4 are regressed. The first-order factor F3 consists of the tasks which require a reconfiguration of a given figure, while the tasks demanding the place way of modifying two given figures in a new one in order to be solved constitute the first-order factor F4. Therefore the results indicate that the ways of figure modification have an effect on operative figure understanding. The first-order factors F5 and F6 are regressed on the second-order factor that stands for discursive apprehension, indicating the effect measurement concept exerts on this type of geometric figure apprehension. To be specific, the first-order factor F5 refers to the verbal problems which demand increased perceptual ability of geometrical figure relations and basic

geometrical reasoning, while the first-order factor F6 consists of the verbal perimeter and area problems.

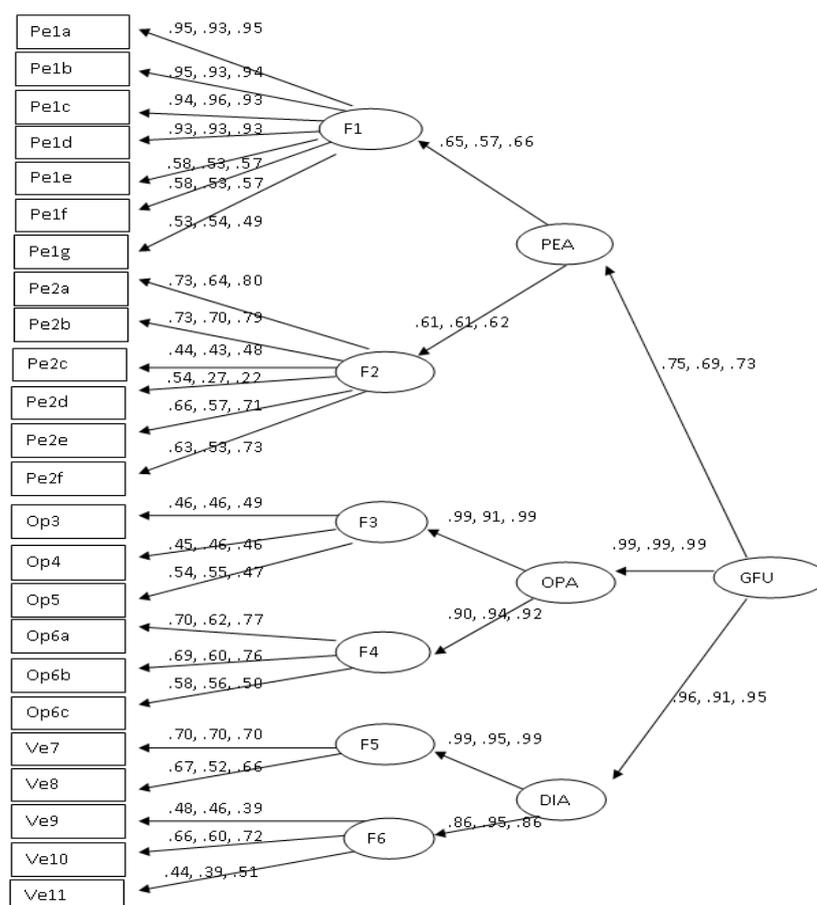


Figure 1. The CFA model of the geometrical figure understanding.

To test for possible similarities between the two educational level groups' geometrical figure understanding, multiple group analysis is applied, where the proposed three-order factor model is validated for elementary and secondary school students separately. The model is tested under the assumption that the relations of the observed variables to the first-order factors, of the six first-order factors to the three second-order factors and of the three second-order factors to the third-order factor will be equal across the two educational levels. The fit indices of the model tested are acceptable [$\chi^2(485) = 903.78$, CFI= 0.97, RMSEA= 0.04, 90% confidence interval for RMSEA= 0.036, 0.044]. Thus, the results are in line with our second hypothesis that the same geometrical figure understanding structure holds for both the elementary and the secondary school students. It is noteworthy that some factor loadings are higher in the group of the secondary school students suggesting that the specific structural organization potency increases across the ages.

The effect of students' educational level. In order to determine whether there are significant differences between primary and secondary school students concerning their performance in the different aspects of geometrical figure understanding, a

multivariate analysis of variance (MANOVA) is performed. Table 1 presents the means and the standard deviations for these dimensions in the two educational levels.

Overall, the effect of students' educational level (primary or secondary) is significant (Pillai's $F_{(6,1079)}=34.43$, $p<0.001$). In particular, the mean value of secondary school students' geometrical figure perceptual ability (F1) is statistically significant higher ($F_{(1,1079)}=79.51$, $p<0.001$) than the mean value of primary school students. Similarly, the mean value of secondary school students' recognition ability (F2) is statistically significant higher ($F_{(1,1079)}=38.81$, $p<0.001$) than the mean value of primary school students.

In tasks demanding reconfiguration (F3) secondary school students' performance is statistically significant higher ($F_{(1,1079)}=74.34$, $p<0.001$) than primary school students' performance. In the same way, the mean value of secondary school students' performance in place way modification tasks (F4) is statistically significant higher in comparison with primary school students' performance ($F_{(1,1079)}=36.03$, $p<0.001$).

Concerning primary and secondary school students' performance in verbal problems the results are quite different in the two dimensions. Particularly, in verbal problems 7 and 8 (F5) the performance of secondary school students is statistically significant higher ($F_{(1,1079)}=105.38$, $p<0.001$) than the performance of primary school students. In contrast, although the performance of secondary school students in verbal problems 9, 10 and 11 (F6) is also higher than the performance of primary school students this difference is not statistically significant ($F_{(1,1079)}=0.03$, $p=0.85$).

Therefore, the above findings verify the third hypothesis stating that differences exist in the geometrical figure understanding performance of primary and secondary school students. In particular, secondary school students' performance is higher in all the dimensions of the geometrical figure understanding relative to the primary school students' performance.

Level	F1		F2		F3		F4		F5		F6	
	\bar{X}	SD										
Primary	0.45	0.41	0.62	0.26	0.32	0.31	0.31	0.38	0.38	0.40	0.247	0.28
Secondary	0.66	0.38	0.72	0.27	0.49	0.35	0.45	0.42	0.63	0.40	0.251	0.31

Table 1: Means and standard deviations in geometrical figure apprehension dimensions in primary and secondary school students

Similarity diagrams. Figure 2 and 3 present the similarity diagrams of primary and secondary school students' responses to the tasks of the test. Particularly, in Figure 2 two clusters (i.e., groups of variables) can be distinctively identified. The first cluster consists of the variables corresponding to the perceptual tasks (Pe1a, Pe1b, Pe1c, Pe1d, Pe1e, Pe1f, Pe1g). In the second cluster the variables representing the recognition, operative and verbal problem solving tasks are included (Pe2a, Pe2c,

Pe2b, Pe2e, Ve11, Pe2d, Pe2f, Op6c, Ve7, Ve8, Ve9, Ve10, Op6a, Op6b, Op3, Op5, Op4).

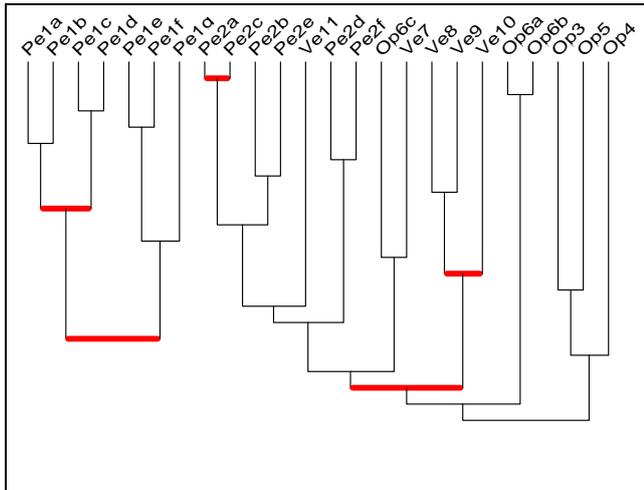


Figure 2. Similarity diagram of primary school students' responses to the test

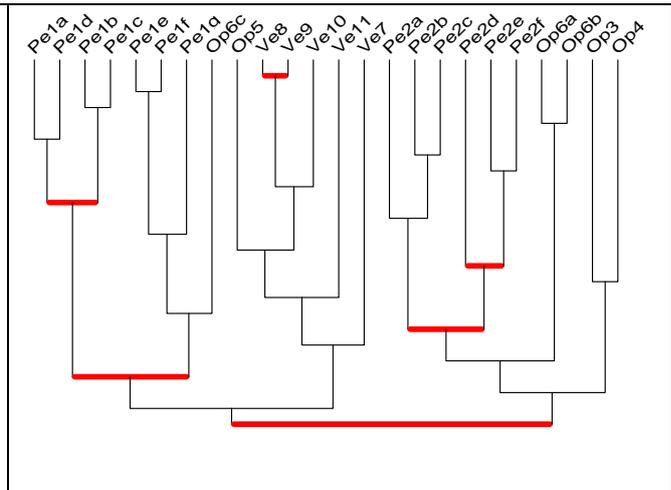


Figure 3. Similarity diagram of secondary school students' responses to the test

In Figure 3, three clusters can be identified. The first cluster includes the perceptual tasks and an operative task (Pe1a, Pe1b, Pe1c, Pe1d, Pe1e, Pe1f, Pe1g, Op6c). The second cluster consists of an operative task and the verbal problem solving tasks (Op5, Ve8, Ve9, Ve10, Ve11, Ve7). The third cluster involves the recognition tasks and some of the operative tasks (Pe2a, Pe2b, Pe2c, Pe2d, Pe2e, Pe2f, Op6a, Op6b, Op3, Op4). Comparing the two diagrams some relations between the variables remain invariant indicating a stability of the way the primary and secondary school students behave during their solution process (e.g. Pe1a, Pe1b, Pe1c, Pe1d, Pe1e, Pe1f, Pe1g and Ve8, Ve9, Ve10).

However, differences are observed in many relations of variables. For instance, primary school students behave in a similar way during the solution of the recognition and verbal problem solving tasks, while secondary school students behave in a similar way during the perceptual, some operative and verbal problem solving tasks. Furthermore, in Figure 3 the three clusters are strongly connected with each other indicating that secondary school students behave in a consistent way during the solution of the perceptual, operative and discursive tasks. In contrast, primary school students deal with perceptual tasks in isolation indicating a compartmentalized way of thinking (Duval, 2002). The similarity diagrams' results provide evidence for differences in the way primary and secondary school students behave during the solution of the perceptual, discursive and operative apprehension tasks, verifying the fourth hypothesis.

CONCLUSIONS

This study investigated the role of perceptual, operative and discursive apprehension in geometrical figure understanding. Structural equations modelling affirmed the existence of six first-order factors indicating the differential effect of perceptual and recognition abilities, the ways of figure modification and measurement concept, three second-order factors representing perceptual, operative and discursive apprehension and a third-order factor that corresponded to the geometrical figure understanding. It also suggested the invariance of this structure across elementary and secondary school students. Thus, emphasis should be given in all the aspects of geometrical figure apprehension in both educational levels concerning teaching and learning.

Furthermore, differences existed in the geometrical figure understanding performance of primary and secondary school students. Particularly, secondary school students' performance was higher in all the dimensions of the geometrical figure understanding relative to the primary school students' performance. The performance improvement can be attributed to the general cognitive development and learning take place during secondary school. In fact, secondary school curriculum in Cyprus involves many concepts already known and mastered during primary school. This repetition of concepts leads to higher performance even though primary and secondary school instructional practices differ.

Concerning the way students behaved during geometrical tasks solution process it was observed that the behaviour of primary and secondary school students was similar during the solution process of some of the tasks. This finding revealed that geometrical figure understanding stability existed to a certain extent in these students' behaviour. However, in some cases differences were observed in the way the two age groups of students dealt with geometrical figure understanding tasks. To be specific, secondary school students behaved in a consistent way during the solution of the perceptual, operative and discursive tasks. In contrast, primary school students dealt with perceptual tasks in isolation indicating a compartmentalized way of thinking. In fact, the results provided evidence for the existence of three forms of elementary geometry, proposed by Houdement and Kuzniak (2003). We may assume that in this research study, primary school teaching is mainly focused on Geometry I (Natural Geometry) that is closely linked to the perception, is enriched by the experiment and privileges self-evidence and construction. On the other hand, secondary school teaching gives emphasis to Geometry II (Natural Axiomatic Geometry) that it is closely linked to the figures and privileges the knowledge of properties and demonstration. As a result, in the case of primary school students geometrical figure is an object of study and of validation, while in the case of secondary school students geometrical figure supports reasoning and "figural concept" (Fischbein, 1993).

It seems that there is a need for further investigation into the subject with the inclusion of a more extended qualitative and quantitative analysis. In the future an investigation of the way students who master perceptual, operative and discursive

apprehension behave in complex activities that require a coordinated approach to these geometrical figure understanding dimensions should be conducted. It would be also interesting to compare the strategies primary and secondary school students use in order to solve perceptual, operative and discursive apprehension tasks. Besides, longitudinal performance investigation in geometrical figure understanding tasks for specific groups of students (e.g. low achievers) as they move from elementary to secondary education should be carried out.

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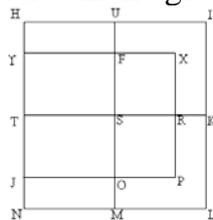
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APPENDIX

1. Name the squares in the given figure:



(Pe1a, Pe1b, Pe1c, Pe1d, Pe1e, Pe1f, Pe1g)

3. Underline the right sentence:

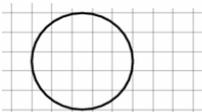


Figure 1

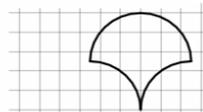
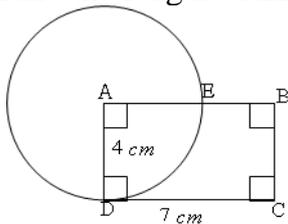


Figure 2

(Op4)

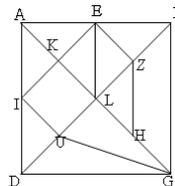
- a) Fig. 1 has equal perimeter with Fig. 2
- b) Fig. 1 has smaller perimeter than Fig. 2
- c) Fig. 1 has bigger perimeter than Fig. 2

5. In the following figure the rectangle ABCD and the circle with centre A are given. Find the length of EB.



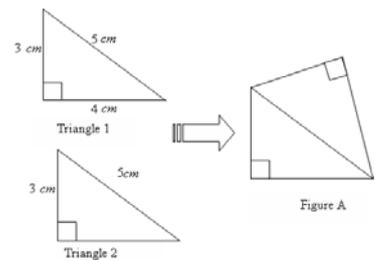
(Ve7)

2. Recognize the figures in the parenthesis (KEZL, IEZU, EZHL, IKGU, LGU, BIL)



(Pe2a, Pe2b, Pe2c, Pe2d, Pe2e, Pe2f)

4. Peter combines Triangle 1 and Triangle 2 making Figure A. Calculate the perimeter of Figure A. (Op6a)



6. Themistoklis has a square field with side 40m. He wants to construct a square swimming pool which is far from each side of the field 15m. Find the swimming pool perimeter. (Ve9)

GESTALT CONFIGURATIONS IN GEOMETRY LEARNING

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ABSTRACT

The treatment of geometric diagrams requires the handling of the figural aspects of the drawing as much as the conceptual aspects contained in the figure¹. In geometry we use the figural aspects of diagrams as symbols to prove or resolve problems. When we interpret figural information, what we call Gestalt configurations emerge: auxiliary figural configurations, real or virtual, that give meaning and substance to an idea that facilitates the proof or solution to the problem. In this work we give arguments to acknowledge the existence of these resources, identify their symbolic nature and consider the reasons behind their existence, sometimes ingrained, sometimes superficial.

INTRODUCTION

To conceive representation as “one thing in place of another, for someone” Pierce (1903) allows us to interpret it as a semiotic mediator between the abstract object of study and the cognizant individual.

In this sense the symbolic aspect in terms of the syntax of the representation must be considered as much as its semantics. The semantics are grasped by the individual through meaningful problematic practices.

In this work our aim is to identify the role played by the auxiliary constructions related to the use of diagrams, which we call Gestalt constructions and which are built by the users when they figural manipulate drawings in order to treat them as figures, Laborde and Caponni², (1994).

We hold that these configurations are profoundly ingrained in our students, that they are intentional but often unstable. They can be a particularly valuable resource in heuristic tasks of figural investigation.

THEORIC FRAMEWORK

From the point of view of Duval (1995):

¹ In the sense of Laborde and Caponni

² The treatment of the graph as a drawing or figure, is based, firstly, on observing its properties as an actual pictorial representation or, secondly, considering the mathematical properties associated with the graphical representation.

One figure³ is an organization in marked contrast to the shine. It emerges from the background through the presence of lines or points, governed by Gestalt law and perceptual indications p.142

In terms of the Gestalt relationship the figure has “form, contour, and organization,” while its preceding appears as an “amorphous and infinite continuity”, Guillaume (1979) p. 67.

Pictorial representations may be considered external and iconic, Mesquita (1998); they are also defined as inscriptions, Roth & McGinn (1998); or diagrams, Pyke (2003). The unifying idea is that the graph is an external representation that is materialized through the use of pencil and paper, the computer or other means and is, therefore, available through these means, in contrast to mental representations which are not accessible, op cit.

Below we consider the graphic representation as a diagrammatic representation or diagram that preserves the relationships of the objects involved. Diagrams from the viewpoint of sense will be observed in themselves and interpreted from the point of view of the reference between them.

On the other hand, diagrams are figural concepts that, in the words of Fischbein (1993) can be thought of as concepts and as objects: this duality emphasizes the different interpretations associated with graphic representations.

Thinking of a diagram as an object means associating specific figural properties with it, such as position or form. These considerations on what thinking about it as an object means, in Fischbein’s way, refer to a mathematical object, this is abstract. The dichotomy between object and concept is related more to a theory need to include non-formalized mathematical aspects, such as position or form, than to the mathematical objects in themselves.

For the purposes of this work we refer to the treatment of representations in geometry based on their iconic or figural properties centered on visual image and to their external nature as embodied materially on paper or other support.

The nature of diagrams in geometry learning is ruled by two types of properties as Laborde (2005), observes:

Diagrams in two-dimensional geometry play an ambiguous role: on one hand they refer to theoretical geometrical properties, while on the other, they offer spatial-graphical properties that can give rise to a student’s perceptual activity p. 159

The treatment given to the diagram as an object in geometry learning is closer to that given to a drawing as a current instance, and not as an

³ The word “figure” in this quote has a meaning close to diagram, distinct from how we use it in the rest of the work.

abstract mathematical object in the concept-object duality. It takes students some time, in fact, to incorporate the idea that drawn objects (representations) have properties which are distinct from those of real life objects.

In terms of learning, Laborde *op cit.* warns:

The distinction of the two domains, the spatial-graphical domain and the geometrical one, allowed us to show that the intertwining of the spatial aspects of diagrams with the theoretical aspects of geometry is especially important at the beginning of learning geometry *op. cit.* p. 177.

It is in the spatial-graphical domain where spatial and figural relations are developed that give shape to the thought structures that are developed around the Gestalt. First, as a relation between the background and the form and later, as resources in the explanation, construction or solution of problems, they give rise to Gestalt configurations.

Studies related to visualization and, most recently, visual perception, have addressed the role played by Gestalt relations between background and form in the pictorial representation that accompanies the mathematics, and the importance of considering it on a certain type of perceptive perception, Duval (1995)

In the work of Nemirovsky and Tierney (2001), regarding spaces of representation, we observe a special interest in establishing the existence of distinct ways of interpreting the same space of representation based on its use and meaning relative to the objects represented.

From the above we can say that the use of diagrams depends not only on what is represented in them, but also on the relations we can establish from them, including spatial information which includes Gestalt relations.

Gestalt configurations

In the work of Dvora and Dreyfus (2004) we have unjustified assumptions based on diagrams in geometry due to students confusing a mathematical motive and a purely visual motive. In addition, when problem solving they base themselves more on their beliefs about the topic in question than on the available propositions. The authors find that diagrams affect students' way of thinking because, among other things, they use diagrams as evidence.

The Gestalt configurations dealt with here have no evidential connotation, they are, instead, auxiliary constructions that complete or give shape to an idea and have their origin in the need to solve problems which involve a diagram.

Gestalt configurations are not related to all the possible pictorial tests that claim to find a solution helped by the drawing, whether the lead is promising or not.

A Gestalt-type configuration, as well as the intentionality of solution, should contain a reference to the relation between background and form, that is, Gestalt configuration “adjusts” to the general composition of the diagram. In other words, Gestalt configuration manifests as a cognitive resource to give substance to a thought and is distinguished by its figural relation between the background and the form of the diagram in question.

The symbolic relations of a Gestalt configuration are determinant: it is dependent on them whether this configuration can be built or not. By way of example, Acuña (2004), we have the case in which without the presence of a graphic reference the very existence of the geometric or graphic object is in doubt, as in the following cases:

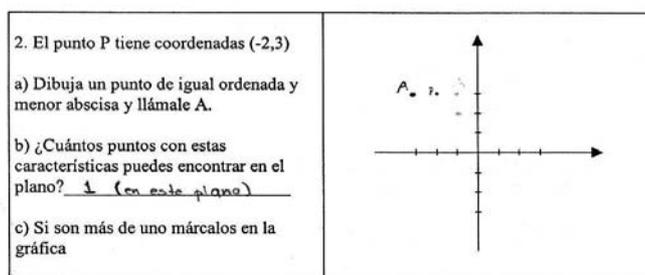


Fig. 1 Point A is the only one with equal ordinate and smaller abscissa than P, in this plane

In the student’s answer to the question about the number of points that have an equal ordinate and smaller abscissa than the point (-2,3) in which he (or she) affirms: 1 on this plane, we can see that he is trapped by the actual representation since the picture offers only one unit mark on the abscissa axis. The student does not consider alternative solutions other than that point located above the mark of the whole abscissa unit. The absence of the mark combines with the idea that a point should have a whole abscissa unit. This student was unable to build neither of a suitable configuration for the solution or a Gestalt configuration.

In the following case, Acuña (1997) we have (see Figure 2) a question about whether the suggested points are on the drawn straight lines or not. If we look at the point (-2, 3) we see that the straight line proposed does not reach the position where a perceptive solution could be given, that is, one perceived “by eye”. This fact makes the student doubtful and answers that if we lengthen the straight lines, the point is on it, otherwise it isn’t.

Our student is unsure of the existence of the point in spite of knowing its coordinates, thus the Gestalt configuration cannot be built because of the absence of the graphic reference that gives it substance. In this case, if the

straight line does not reach the indicated place, there is no security about its existence, which impedes the acceptance of the relation between the straight line and the point.

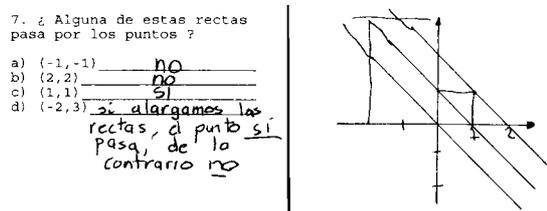


Fig. 2 Problem of points on the straight lines

Constructions with appropriate Gestalt configuration

In relation to the construction and use of geometric figures, Maracci (2001) has observed that students insist on making constructions that possess certain, from their point of view, appropriate aspect.

This insistence is accompanied by the preference for the horizontal-vertical position, or the choice of graphs that appear to be, for example, a straight line Mavarech and Kramarsky, (1997) or a segment of a straight line with an slope equal to 1, Acuña (2001), as well as students' penchant for using prototypes⁴ Hershkowitz (1989), or the use of the "best" examples from among one same category of possible cases, Mesquita (1998).

This phenomenon can be explained by the students' need to find a good orientation and familiar representation. In other words, they prefer to build "appropriate" configurations in general and Gestalt configurations in particular that give meaning to the actual figural relation.

In some tasks with qualitative instructions, as in figure 3, we have identified a tendency to recognize and build graphs in a certain position and with a certain peculiarity, forming prototypes, Acuña (2001). A large part of the students surveyed with the question for draw straight line with only points with positive abscissa, responded with a half-line that reaches the origin, with a slope of 1. This answer was more frequent than any other, correct or incorrect, in high school students.

⁴ We call prototypical figures those which correspond to a regular organization of contour, orientation and form; prototype figures tend to respect laws of enclosure (closed limits are preferably perceived), favoring some directions (such as horizontal and vertical) and forms (that tend to be regular, simple, and symmetrical); the components of the figure (sides, angles for example) have approximate dimensions.

5. Draw a straight line where all the points have a positive abscissa, that is, where $x > 0$ is true for all points on the line.

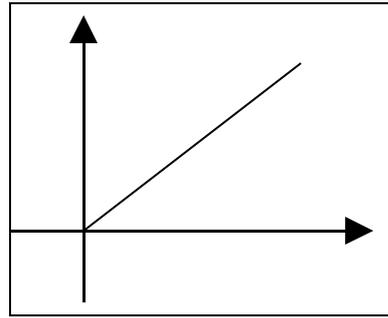


Fig. 3 Answer to a qualitative-type construction task

The students' answer presupposes that the straight line built does not cross to the other side of the vertical axis, as if it were a barrier, so that it will not take negative values for the abscissa.

The non-ostensive nature of the straight line related to the infinite extension of its extremes contributes to the incorrect interpretation of the answer that, in strictly figural terms, has a plausible logic, especially since it is not possible to have a representation of a straight line, only parts of it.

The non-ostensive aspect on the infinite extension of the line can be accepted theoretically by the students, but the impossibility of building theoretical straight lines leads them to accept the segments of a straight line as if they were straight lines themselves.

In figure 4, Acuña, (2002) students are asked to draw the graph of the straight line that would have an ordinate equal to the origin of the original straight line that appears on the left.

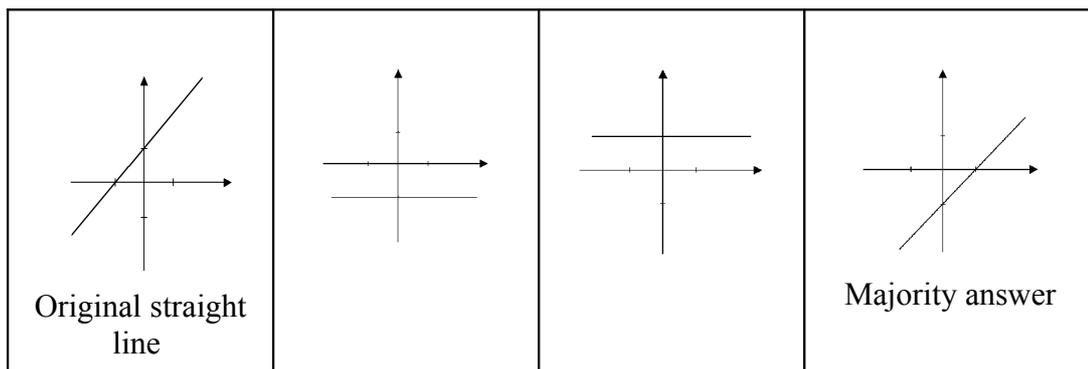


Fig. 4 Gestalt configuration combining figure and form

The majority of our students drew the graph on the far right. Many of them had correctly recognized the ordinate of the origin in straight lines given earlier; nevertheless, here they choose to conserve the “triangular” image formed in both graphs, preferring to relate the two graphs with a similar Gestalt.

This type of answer is strongly conditioned by the situation of the exercise, in particular given that this perception is unstable, as we can see in other exercises.

In the following exercise, Sosa (2008) two high school students have been asked to build the height corresponding to the side marked with X in each case.

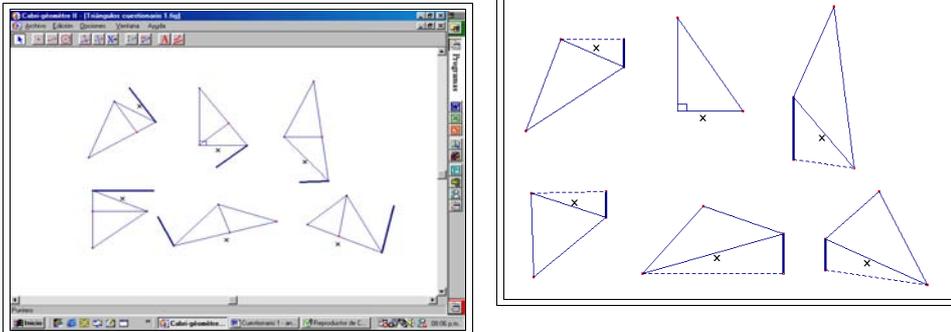


Fig. 5 Exercises on height construction

In these two cases, we have the application of a Gestalt configuration to solve the problem of the construction of the height of the marked side. In the answer on the left, the height is thought of as a conformation formed by the vertex of the obtuse angle, or what looks like it. The student also uses an auxiliary parallel line which we suppose was in the image the student recalled.

In the case of the constructions on the right (see figure 3) we have an auxiliary construction that includes the line marked with X but where this is a part of another auxiliary construction that presents a right-angle triangle where we observe some of the characteristics relevant to height, but its construction is unknown. The marked line is included in his construction, but its role in the construction is reinterpreted and he does everything he can to make it look good.

In the following case we ask students to mark the straight lines with a different slope to that of the one given.

The formation of this configuration not only appears when the definitions of the geometric objects are unknown or is recalled inexactly, but also when globalizing an idea of position, as in the following example. In the case of figure 6 and 7, we ask high school students to choose from the lower graphs that which have a different slope to the one proposed initially.

The results allow us to see their idea of a slope in this exercise. Despite having correctly compared, based on perception, the slope of the given lines, here they conceive it as the Gestalt configuration formed by the position of the straight line relative to the axes, that is, the line is positioned from left to right and from up to down.

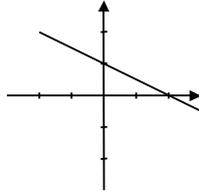


Fig. 6 Straight line with given slope

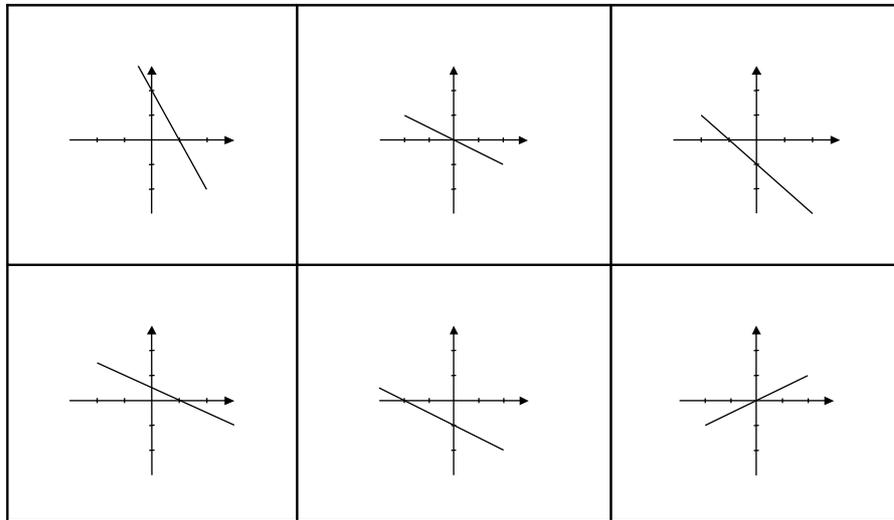


Fig. 7 Gestalt configuration on a slope

The 19.3 % of our sample only marked the straight line that is positioned from left to right, leaving aside the idea of slope that they used before.

The preference towards a “good” Gestalt appears to impose itself in tasks of identification of figural properties. This recourse may signify an advance or a backward step for solution strategies. What does appear to be constant is the use of this type of configuration to test solutions to problems with diagrams.

These configurations may disappear quickly with better instruction, but they also have aspects of profound rooted as in the case of Moschkovich’s (1999) investigation, regarding the use of the y-intercept. She finds that when observing the graph of a straight line students may expect the x-intercept to appear in the equation because on the graph it is a salient as y-intercept although this is not necessarily convenient in the case of the equation $y = m x + b$ however, they are important for the equation that considers two points on the straight line. The appeal of the x-intercept is so big than could think it as a preconception; in her investigation she affirms that:

The use of x-intercept is not merely the result of choosing or emphasizing the form $y = m x + b$ over other forms but is instead an instance of students making sense of the connection between the two representations and reflection on the conceptual complexity of this domain p. 182

We believe from the above that it is possible to suppose the existence of figural resources that take the form of Gestalt configurations that respond on one hand, to the necessity of giving substance to figural ideas, and on the other, that these configurations are ruled by the relations between background and form on which rests the figural representation of mathematical and, more concretely, geometric diagrams.

CONCLUSIONS

A Gestalt configuration is a mental or real construction utilized by the user to resolve, complete or give meaning to a given problem through a diagram that can be treated as a drawing or figure.

Gestalt configurations have a personal character, but on occasions reflect epistemological obstacles that are supported by the non-ostensive nature of the properties of the objects represented by the diagrams, as in the case of the infinite character of some of these representations.

The formation of some Gestalt configurations is characterized by having an ephemeral life, although there are some that persist; as they are personal productions of the user. In general, they are considered productive and reliable for confronting familiar graphic settings towards resolving problems that include diagrams.

In all cases, the construction of the Gestalt configurations is intentional in spite of the inability to ensure its pertinence. Gestalt configurations do not only appear as visual traps but as a diversity of resources to solve figural problems or proving.

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INVESTIGATING COMPARISON BETWEEN SURFACES

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This work is based on a geometrical problem concerning comparison between surfaces, presented to 58 pupils 10-11 years old. We present a worksheet aimed at revealing children's reasoning about visualisation in geometry. We compare the ways in which various problems are tackled by two different groups of students: Group E (experimental) and Group T (traditional). We conclude with some observations about teaching geometry and suggestions for its improvement.

INTRODUCTION

During a lecture to future teachers about fractions, I observed as they were analysing suitable geometric figures, drawn using computer graphics. I realised that these drawings could be useful for investigating geometrical learning. My attention was particularly attracted by different representations of the half of a rectangle. I mentioned my idea to a group of experienced Primary School teachers, and one of them, when she saw figures A, B and C (Figure 1), said: “If the pupils have already worked with fractions, they will certainly use and recognize the concept of half.” As in my experience this conclusion is rash and not entirely obvious, I decided to investigate it. Working with the teachers, we prepared a worksheet based on Figures A, B, and C and on a fourth Figure D, expressly created.

The aim of the research is twofold: to investigate the use of the concept of ‘half,’ and chiefly to study geometrical thinking observing pupils behaviours, with particular reference to registers of representation (Duval, 1998-2006), especially the figural register.

THEORETICAL FRAMEWORK

The concept of *half* and related notations are present in five and six-year-old children (Brizuela, 2006). At this age, children use different semiotic representations (Duval, 1995), but it is difficult for them recognise a half in different representations (Sbaragli, 2008). According to Duval, the passage from a semiotic representation to a different representation is fundamental for a conceptual learning of objects. In particular, he distinguishes two possible kinds of transformation of representation: *conversion* (from a semiotic representation to another, in a different register) and *treatment* (from one semiotic representation to another, in the same register). The half of a geometrical figure is usually presented to children when we introduce fractions, as one of the first examples. Subsequently, teachers move on to writing fractions and to calculating with them, moving from conversions to treatments.

Traditionally in Primary School we use geometrical figures as a suitable tool for teaching and learning geometry. Figures involve a fundamental action for the

pupil: looking. The didactical contract (Brousseau, 1986) based on showing requires that

“the pupil understands what the teacher expects that s/he will see, with the false illusion that both must see the same” (Chamorro, 2006).

If both parties do not see the same, the contract is broken and learning does not take place. So we need to ... “teach to see”. In geometry, a first problem is created by *perception*, which may hinder the ways of seeing figures. In other words, the perceptive indicators may be misleading for the qualitative evaluation of the extension of surface or of other magnitudes. Gestalt theory deals with laws of organisation of visual data that lead us to see certain figures rather than others in a picture.

More recent researches show that

“...it is the task that determines the relation with figures. The way of seeing a figure depends on the activity in which it is involved.” (Duval, 2006).

Duval (2006) analyses and classifies the different ways of seeing a figure depending on the geometrical activities presented to pupils. He distinguishes four ways of visualising a figure: by a botanist, a surveyor, a builder or an inventor. Botanists and surveyors have ‘iconic visualisation’, and perceive the resemblance between a drawing and the shape of an object. Builders and inventors on the other hand have ‘non-iconic visualisation’, and their perception is based on the *deconstruction of shapes*. Duval analyses the introduction of supplementary outlines, which he thinks fundamental in ‘non-iconic visualisation’, in particular he discusses *re-organising outlines* which allow to reorganise a figure and thus to reveal in it parts and shapes that are not immediately recognizable.

He also discusses the *méréological decomposition*¹ of shapes, a division of the whole into parts which can be juxtaposed or superimposed, with the aim of reconstructing another figure, often very different to the starting figure. This allows the detection of geometrical properties needed to solve a problem, using an exploration purely visual of the figure initial. He distinguishes three kinds of méréological decomposition: material (with cutting and rebuilding as in a jigsaw puzzle), graphic (using reorganising outlines) and by looking (with the eyes, not “mentally”). We tackled the problem of “which is ‘visual’ in geometry?” in a research paper (Marchini *et al.*, 2009) where we analysed in-dept the literature on this argument.

In Italian Primary School, *comparison between surfaces* is often reduced to evaluating areas (measurements of extension of surfaces) and to comparing numbers. Teachers tend to determine equivalence of the magnitude of two objects by means of measurement. But “transferring the comparison to the numerical field, we are in fact working with numerical order which doesn’t consider the criterion of quantity of

¹ In mathematical logic, mereology is a theory dealing with parts and their respective whole. The term was coined by Łeśniewski in 1927, from the Greek word μέρος (*méros*, "part").

magnitude” (Chamorro, 2001). An epistemological slide from geometry to arithmetic occurs. The comparison between surfaces and, in particular, the “equivalence of magnitude” is a fundamental but difficult concept, which requires specific teaching. In previous research we wrote:

“We did not predict that determining shapes of the same area would be difficult, But in fact there were cases where pupils failed to recognise that two congruent rectangles, set at a different way on the sheet of paper, had the same extension.” (Marchetti et al., 2005).

The comparison between surfaces is also influenced by the relationship between *shape and surface*: when we present a surface, we present something that has a specific shape. If the shape changes, a younger child might think that the surface changes too. Research shows clearly that pupils under 12 have difficulty in understanding that the shape and the surface of a figure are different (Bang Vinh & Lunzer E., 1965).

RESEARCH METHODOLOGY

We presented the worksheet at the end of the last year of Primary School, to three classes of students 10-11 years old, which had followed two different approaches to geometry. One class had already taken part in an experimental project and the other two classes had received only traditional teaching. We named the first group ‘Experimental’ (Group E) and the second group ‘Traditional’ (Group T). Group E consisted of 26 pupils; they had followed a Mathematics Laboratory Project (MLP)², during the last three years of Primary School. It focussed on activities that started from a practical problem, such as fencing in a field or tiling a room, and led to the introduction of specific instruments by the teacher as the children perceived the need for them. The early activities involved concrete materials and children using their hands, and geometric instruments and theoretical concepts were introduced in later activities. So Group E did not follow traditional curricular teaching; we presented new activities that were different in terms of both methodology and content. Group T consisted of 32 students from two classes which had followed the traditional mathematics curriculum. Both groups had previously studied and worked with fractions and areas. For Group E, however, the project had opted to present area before perimeter, which is unusual in Italian schools.

Pupils’ behaviours were observed as follows: when the teacher presented the worksheet, s/he explained that not was possible to use a rubber, but if necessary children could write their notes and opinions on another sheet of paper. I then analysed the protocols.

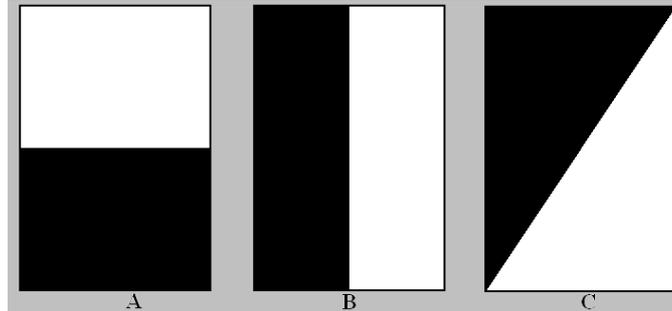
² The project was carried out by two researchers, D. Medici and P. Vighi, and two teacher-researchers, P. Marchetti and E. Zaccomer.

THE TASK AND ITS ANALYSIS

In the following pages we present and discuss the worksheet.

A pizza-maker with a lively imagination displays these slices of pizza.

All the slices have one part with only tomato (dark) and one part with only mozzarella (light).



One child wants a slice of pizza with a lot of tomato.

Which slice do you think he or she should choose? Why?

Does the slice of pizza below have more mozzarella or more tomato?

Why?

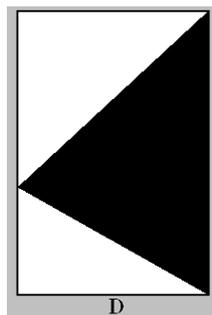


Figure 1: the worksheet

This activity on geometrical figures in the first part lies on the first level of van Hiele's theory, in the final part it lies on the second level, which involves the possibility of seeing inside geometrical figures and seeing and/or making a subdivision into parts (van Hiele, 1986). In the paradigmatic perspective introduced by Houdement and Kusniak (2003), the activity is situated in Geometry I.

Notice that the passage from A to B or C requires 'treatments' inside the register of visual representations. The first question is deliberately ambiguous; the form of the question could lead the child to opt for only one of the slices and, consequently, give a wrong answer. In other words, the question could lead the child to exclude the equivalence of surfaces. The second part of the task presents an unusual geometrical problem. The slice is divided into three parts and the comparison concerns only two quantities of food (two surfaces). There is a different subdivision in half of the same rectangle as before. The question is formulated differently from the first: the problem

is the comparison between tomato and mozzarella. Using a supplementary outline helps to find the answer. The main information is in the drawings: rectangles A, B, C and D are congruent ($8 \text{ cm} \times 5.3 \text{ cm}$) and, in particular, in A and B we used the middle point of a side, without specifying this; in other words, we gave implicit data. Figures play an essential role: they are shown against a grey background, with the aim of distinguishing between the whole slice and its parts.

The context of the problem is intended to focus attention on surfaces. The figures in the first part, rectangles and triangles, are familiar; the pupils know the formulas for the calculation of their areas. The last ‘slice’ is made up of a dark triangle, representing tomato, and two other white triangles, not contiguous, representing just mozzarella. It is an unusual figure which does not appear in textbooks (it may not in fact appear in pizza shops either), but if the sheet of paper is rotated, it probably becomes more familiar as a drawing related to the formula of area of a triangle. For Figure D too, children need to use the concept of half, or they need to “see” congruent parts, or draw supplementary outlines, or calculate areas and verify their equality.

The analysis of A and B by *méréological decomposition* is simpler than for C. In effect there is a difference in the geometry of transformations: in A and B it is sufficient to translate some pieces, while in C rotation is also required. As we saw, D implies cutting the figure and reconstructing congruent parts. We present slice D to investigate pupils’ strategies. We want to establish whether children use the same methods for answering both questions, or if D encourages them to try different methods. We also want to observe whether solving the second problem leads pupils to rethink their answers to the first.

RESEARCH RESULTS

The activity is presented in a geometrical context, which often seems to imply the use of specific geometrical tools. In many of the protocols the shift from the geometric register to the numerical register of fractions does not occur: ‘conversion’ between the registers does not take place.

Only a few answers to the first question (12% in Group E, 6% in Group T) use the concept of “half”: “Figures are divided in half”, or “Half the space is filled with tomato”. The question draws pupils’ attention only to the black shapes, or tomato. In other words, children focus on and compare particular parts, rather than looking at the slices globally. It is not by chance that the few answers which are based on “half” make recourse to the relation part-whole (Hart, 1985): “All slices are perfectly divided in the middle and the whole is equal for all figures”. Notice that the children use words that are usual in speaking about fractions, not the symbol $\frac{1}{2}$. In some cases the concept of half is questionable and ‘relative’: “I choose pizza C because tomato occupies the “biggest half.” The relation shape-surface also emerges: “Even if the pizzas are divided into different shapes, it is still half a slice and the slices are equal”.

The “equal extension” of tomato surfaces in A, B, C was recognised by only 6 pupils in Group E and 4 in Group T.

We now analyse different procedures observed for the first part of worksheet.

- *by perception*: children choose slice C because the tomato appears bigger (or “It looks like a piece of pizza”) (30% in E and 37% in T). In some cases, the choice is based on exclusion, which may be due to the question: some children verify that A and B have equal quantities of tomato, and they conclude that C must be bigger, without checking. Two pupils choose A because “it is larger,” taking account of one dimension only.
- *by subdivision*: here we notice very different behaviours according to the teaching methods adopted. In Group T, only 1 pupil uses *méréological decomposition*, while in Group E 6 do so. Pupils divide figures B and C by drawing (graphic decomposition) or imagining (decomposition by looking) a continuation of the horizontal line present in slice A which divides the white and black parts. They observe that it is possible to shift some black pieces of B or C in order to obtain A. It is significant that some of them write “If I cut in half ...”, although they did not see the half in Figures A, B and C.
- *by calculation of area*: only 4 pupils in Group E and 3 in Group T calculate 21.20 cm^2 as measure of three surfaces covered by tomato. There is also a problem of approximation: for figure B, in calculating $5.3 : 2$ they stop at the first digit after the decimal point obtaining 2.6 and 2.6×8 make 20.8. Slice B thus seems to have less tomato.
- *by calculation of perimeter*: 6 children in Group E use this method (maybe because perimeter was most recently studied) and 5 in Group T. Their procedures are based on measuring the sides of the black figures and their addition: in this way C appears biggest. This is a manifestation of perimeter-area conflict. (Chamorro, 2002), (Marchetti et al. 2005).
- *by flooring with squares*: based on reproduction of figures on squared paper, often without respect for shapes and measurements, or based on the superimposition of a squared grid, often not regular. Answers are based on counting the number of squares.

In the second part of the worksheet, we recorded 58% correct answers in Group E, and 34% in Group T. Obviously the use of half in the first part of the task is a successful strategy, as it is for the second part.

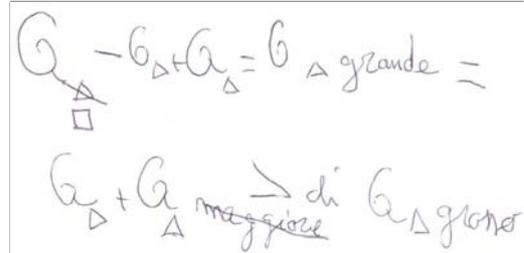
In Group E, previous methodological decisions and their experience of manipulation led children to tackle the problem in different ways. Some children took scissors, cut the pieces and superimposed two white pieces on the black. They still worked with real and not geometrical objects. Their conclusions may be “They are equal,” or not, because there is a problem of approximation: “They differ by a small amount”. Recourse to *méréological decomposition* promotes fast and correct answers, based

simply on the drawing of a horizontal segment, and the height of the dark triangle. An interesting observation is that a few pupils use the expressions “triangle” or “height of triangle” in their explanations; they write: “I connected the vertex of triangle with the opposite side ...” or “I drew a horizontal line ...”.

Some pupils make a rough estimate, and make recourse only to perception (26% in Group E, 40 % in Group T). They support their answers as follows: “I can see it,” “The part with tomato is slightly bigger.” In some answers the decision is based on the number of pieces, not on areas: “Mozzarella, because two pieces occupy more space than one.”

Both groups make little use of calculation. One girl wrote: $5.3 \times 8 = 42.4$ and $42.4 : 2 = 21.2$ tomato piece; $5.3 \times 5 = 26.5$ and $26.5 : 2 = 13.25$; $5.3 \times 3 = 15.9$ and $15.9 : 2 = 7.95$; so $13.25 + 7.95 = 21.20$ mozzarella piece. This is an example of rigorous application of rules, without geometrical reasoning.

Another boy uses ‘pre-algebraic’ notation and reaches an incorrect conclusion based only on intuition or perception. He tries to explain (Figure 2) that, starting from the area of the rectangle, we can subtract the areas of two white triangles and we obtain the area of the big triangle (black). In the second part, he observes that the sum of the areas of the white triangles is bigger than the area of the ‘big triangle’, but he doesn’t explain why.



Handwritten mathematical notation showing area calculations for a rectangle and triangles. The first line shows $A_{\square} - 6_{\Delta} + A_{\Delta} = 6_{\Delta} \text{ grande} =$ with a small square and triangle symbol. The second line shows $6_{\Delta} + 6_{\Delta} \text{ mozzarella} > \text{di } 6_{\Delta} \text{ grande}$ with a triangle symbol.

Figure 2: pre-algebraic notation

Some pupils measure two or all sides and multiply them: the idea of multiplication in area calculation is strong, which may be a result of the didactical contract, but there is no understanding of its meaning. We also find mixed procedures: $(8 \times 5.3) - (8 + 6 + 7) = 42.4 - 21 = 21.4$ area tomato, $42.4 - 21.4 = 21.0$ area mozzarella; the idea is to subtract from the rectangle area the dark triangle area, but the formula for finding the area of a triangle seems not to be known and the pupil calculates the perimeter. Nevertheless one child has a good idea: to obtain the white area as complementary to the black in the rectangle. Only this one boy used this strategy: in fact in school we usually present exercises involving only one shape, and the possibility of calculating an area by subtraction is not introduced.

The solution based on *méréological decomposition* appears the best, and is a successful strategy especially in Group E. We presume that the previous work with Tangram and a different methodological approach helps in the case of Figure D and its parts. Reasoning is based on the use of a supplementary outline (Figure 3).

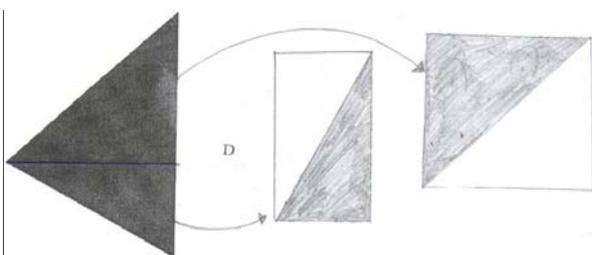


Figure 3: méréological decomposition

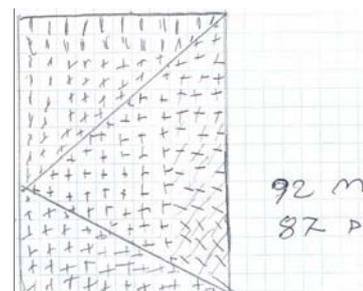


Figure 4: flooring with squares

The idea of measuring with squared paper also appears. In particular, in the protocol reproduced in Figure 4 there is evidence of a lack of understanding: the child counts both squares and pieces of squares and he concludes that the mozzarella area is bigger. In the case of *surface* measurement, schools usually make use of subdivision with squares; there is often no explanation of this method. Moreover it is not suitable for figures with sides that are neither ‘horizontal’ nor ‘vertical’.

Perimeter is used a lot by Group T (18%), but only two pupils use it in Group E (0,07%). It seems that Figure D, which is unusual in traditional teaching, causes the “perimeter-area conflict” and reveals this hidden misconception.

GENERAL CONCLUSIONS

In both groups there were pupils who made no use of geometrical reasoning, but only their eyes. The pizza problem is in fact unusual in that it requires observation of more than one shape and no explicit calculation of its perimeter or area. Often in real life we compare two quantities and we choose the bigger, using common sense rather than mathematics. So one child wrote: “From shapes A, B, and C, I choose C, since it looks like a slice. He was maybe thinking of the shape of a slice of cake. One significant answer came from a child imagining a real pizza, who observed that comparison is impossible, because there is no information about the thickness of the tomato and mozzarella. The analysis of answers confirmed the gap between ‘scholastic’ and ‘real’ problems (Zan, 1998). In other words, the same problem presented in the school or a snack bar may have different solutions. Canapés, in fact, are triangular, obtained by cutting a square along the diagonal, and it could well be that we think we are eating more than if the square of bread were cut in other way.

One week later, the teacher of Group E re-presented the worksheet to her class and encouraged a discussion of pupils’ own solutions. Many quickly recognized the concept of half as a key to the problem and modified their answers. But some children wrote an explanation clearly without conviction. As we wrote previously, in our experience the concept of half does not seem to have been acquired by pupils 10-11 years old. In our opinion, the concept of half needs to be constructed gradually and it is important to work on it with regularity so that it can successfully prepare the ground for introducing fractions.

We also notice that children often use whole numbers as measures of triangle sides: unfortunately in Italy the problem of approximation is neglected. In some cases pupils understand that different numerical results, can be given simply by approximated measurements, but in other cases the children are closely tied to numerical results, even where this conflicts with common sense.

The global analysis of protocols reveals the influence of different teaching methods. Comparison between the protocols of two groups shows clearly the existence of two different behaviours, closely connected to the “social norm” established in classroom (Yackel, Cobb, 1996) according to the “didactical contract”. In Group T, the necessity of following the rules leads to measurement by ruler and the calculation of

perimeters and areas. But in Group E, familiarity with manipulation, scissors and so on encourages the use of hands (and the head) (Chamorro, 2008). We observe the presence of an explicit, real geometrical aptitude in Group E, which was probably a result of the MLP. In Group T, traditional geometry and its formulas are prevalent. We surmise that the better results in Group E are closely connected with didactic choices. In other words, the fact that Group E children worked as ‘builders’ and ‘inventors’ supports the use of a ‘supplementary outline,’ which for Duval is fundamental in seeing figures; our experiment confirms his theory of different kinds of visualisation in geometry. Future research will feature an activity based on the same figures but focussing on ‘dimensional deconstruction,’ defined by Duval as a ‘cognitive revolution’ for visualisation.

Another important suggestion arises from pupil’s approach to the task. Protocol analysis shows that children who use the half or decomposition in shapes A, B and C, use the same concept to investigate D, with the same tools. Vice versa, those who ‘found’ the half in D, maybe by calculating the area, do not go back to modify their answer to the first part of the task. This points to another critical aspect of traditional teaching, not only in the field of mathematics: exercise books always have to be tidy, with no rough work or scribbling, and children are not encouraged to rethink or reflect on work or activities carried out previously. But often sketches and rough drafts can in fact help develop reasoning. We also feel that there should be more encouragement to write up reasoning in the classroom.

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THE EFFECTS OF THE CONCEPT OF SYMMETRY ON LEARNING GEOMETRY AT FRENCH SECONDARY SCHOOL

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This paper relates a part of a bigger research from my Phd (Bulf, 2008) about the symmetry's effects on conceptualization of new mathematical concept. We focus here on the results from students' productions at two different levels at French secondary school, with students who are 12-13 years old and 14-15 y.o. We find out different figural treatments according to the transformation at stake. The results work out the concept of symmetry makes students confused with the transformations of the plan at the beginning of secondary school whereas students seem more familiar with metrical properties relative to the symmetry and develop mathematical reasoning at the end of secondary school.

Key word: secondary school, geometry, transformations of the plan, symmetry, Geometrical Working Space, conceptualization.

INTRODUCTION

The constructivist wave suggests that a new knowledge is built from the old one. According to the French curricula (1), the symmetry (reflection through a line) is taught since primary school (through folding and paving), and more deeply during the first year of the secondary school (students are 11-12 years old). Next, the rotational symmetry (reflection through a point) is taught during the second year of the secondary school; the translation is taught during the third year and finally rotation is taught during the last year of the secondary school (students are 14-15 y.o.). One of the specificity of the French curricula is to teach the symmetry as a transformation of the plan even if the term "transformation" is mentioned only at the end of secondary school. Others countries (Italy as for instance) deal with transformations of the plan in the frame of the analytic geometry at high school (students are older than 15 y.o). Then, in this French context, we suppose the concept of symmetry takes part into the learning of the new transformations of the plan. The question is **what are the effects of the symmetry on this learning process?** This paper is the rest of our research, already introduced in CERME 5 (Bulf, 2007).

We do not need to argue that symmetry is part of our "real world" but it is a scientific concept too. Bachelard (1934) points out that "nothing is done, all is building", he adds the notion of obstacles "to set down the problem of scientific knowledge". He describes different kind of obstacles: the obstacle of "the excessive use of familiar images", or the obstacle of "common meaning" and "social representations". Nevertheless, we can not ignore the "real world" may be a help for empirical reasoning. As far as our work is concerned, **we wonder if the concept of symmetry**

may be an “obstacle” or a “help” into the learning process of the new transformations of the plan at secondary school. Several French authors have already pointed out some resistant misunderstandings linked with the concept of symmetry (Grenier & Laborde, 1988) (Grenier, 1990) (Lima, 2006) or linked with the others transformations of the plan, and in particular deal with the dialectic global/punctual (Bkouche, 1992) (Jahn, 1998).

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

Our research focuses on the process of conceptualization during the learning of the transformations of the plan. The Vergnaud’s theory (Vergnaud, 1991), “the conceptual field theory”, analyses the human component of a concept in action. We refer to this framework in order to analyse the students who solve mathematical problem. We focus on the adaptation of the “operational invariants” which are actually defined by the concept-in-action (“relevant or irrelevant notion naturally involved in the mathematics at stake”) and theorem-in-action (“proposition assumed right or wrong, used instinctively in the mathematics at stake”). The set of these invariants makes the schemes (notion inspired by Piaget) operate. A scheme is the “invariant organization of behaviour for a class of given situation. The scheme is acting as a whole: it is a functional and dynamical whole, a kind of module finalized by the subject’s intention and organized by the way used to reach his goal”. The “signifiers” s (according to Pressmeg’s translation of Saussure’s meaning (Pressmeg, 2006) is the set of representations of the concept, its properties, and its ways of treatment (language, signs, diagrams, etc.). According to Vergnaud, learning is defined as the adaptation of the schemes from students in a situation of reference.

In order to complete the analysis of students’ activities through geometrical problems, we refer to the Houdement and Kuzniak’s theoretical framework of Paradigm of Geometry I and Geometry II, and the notion of *Geometrical Working Space* (Houdement & Kuzniak, 2006). Geometry I (GI) is the naive and natural geometry and its validity is the real and sensible world. The deduction operates mainly on material objects through perception and experimentation. Geometry II (GII) is the natural and axiomatic geometry, and its validity operates on an axiomatic system (Euclid). This geometry is modelling reality. The notion of *Geometrical Working Space* (*GWS*) is the study of the environment, organized on a suitable way to articulate these three components: the real and local space, the artefacts (as for instance geometrical tools), and the theoretical references (organized on a model). This *GWS* is used by people who organise it into different aims: the *reference GWS* is seen as the institutional *GWS* from the community of mathematicians, the *idoine GWS* is the efficient one in order to reach a definite goal and the *personal GWS* is the one built with its own knowledge and personal experiments.

Then the main research question is: **How does the concept of symmetry set up the organization and the inferences between the operational invariants relatives to**

the others transformations of the plan into the student's *personal GWS*? And how does this *personal GWS* evolve during secondary school?

METHODOLOGY

We propose a common test to students at two different levels: at the second year, after the teaching of the reflection through a point and, at the fourth year, after the teaching of the rotation. The students are 12-13 y.o. and 14-15 y.o. and have the same mathematics' teacher. We chose the situation of recognition of transformations because it is a usual task all along French secondary school. We define two different tasks from a same configuration with triangles but with different kind of graphical support. These tasks are given to students at two different times. The first task (Fig. 1) suggests a "Global Perception" (we will note GP) because triangles are indicated as a whole with numbers and the transformations are indicated with arrows. This does not mean the students are only involved on a global perception; they may use a punctual perception too. The terms of the problem are: *In each fallow case, indicate which reflection(s), translation(s), rotation(s) transform: a) $1 \rightarrow 2$ b) $2 \rightarrow 3$ and c) $1 \rightarrow 4$. Justify yours answers. If you add marks on the figure, please do not rub out.* The last question c) is only given to the students from the last year but we do not analysis the results because we are devoted to the case with reflection(s) and rotation(s). Furthermore, it is only indicated *which reflection(s)* (and not the other transformations) with the students from second year.

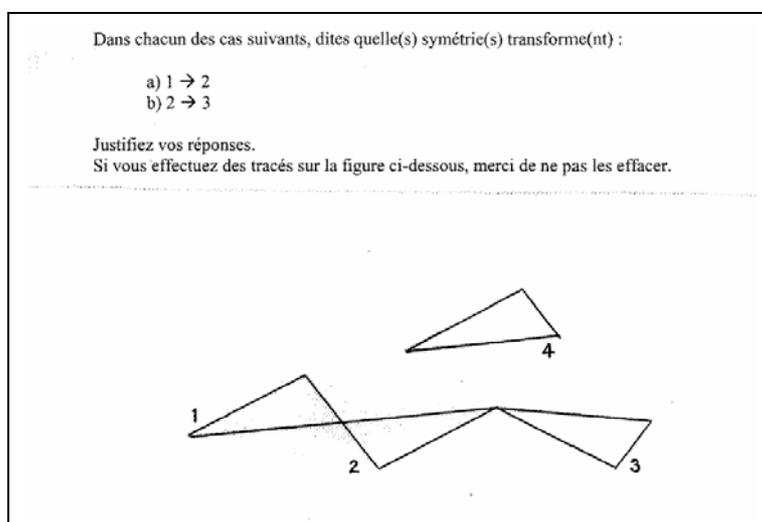


Fig 1: "The triangle situation" in the case called "Global Perception" (GP).

The second task, given one week later, is the same as previously but the terms of the problem suggest a "Punctual Perception" (we will note PP) to the students (Fig. 2). The configuration is given with a squaring and the triangles' tops are called by letters on the pattern and in the terms of the problem (*ABC in EDC*).

Dans chacun des cas suivants, dites quelle(s) symétrie(s), translation(s), rotation(s) transforme(nt) :

a) ABC en EDC
b) CDE en GFE
c) ABC en MNP

Justifiez vos réponses.
Si vous effectuez des tracés sur la figure ci-dessous, merci de ne pas les effacer.

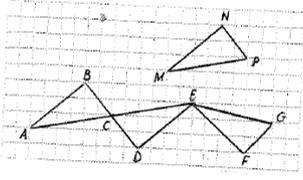


Fig 2: “The triangle situation” in the case called “punctual perception”.

These tasks are quite easy for these students (they have to recognize a reflection through a point or a rotation of 180° at the question *a*) and a reflection through an axis at the question *b*). Different didactical variables are convened and then different students' strategies are implied in both tasks. In particular, the graphical support is different in both case, in the GP one, students' adaptations are wider: they may involve arguments based on superimposition (folding or half-turn) or build strategies based on metrics' arguments (Euclidian Affine Geometry) with measurement or perception. We suppose these latter strategies (with metrical arguments) are more effective in the task PP since there is a squaring and figures are nominated. Mathematical properties are not given as hypothesis in the term of the problems, so different types of metrical properties are acceptable (as for instance “ $AC=CE$ ” or “ AC and CE are almost equals” or even “ AC is not equal to CE ”) but it is assumed a transformation has to be recognized. Moreover, the figural position is actually a didactical variable to consider and we should consider intermediate task (as for instance, without common point, etc.) in order to consolidate the results already got here. However, considering that, we show that students' behaviour changes according to the perception suggested by the task (as expected) but the adaptations imply a different way of figural treatment according to the transformation at stake and according to the students' grade. The aim of this paper is describe the differences between transformations and the influence from the concept of symmetry on these adaptations at these both levels at secondary school.

RESULTS AND DISCUSSION

Student's category according to stability of student's achievement

We collected $29 \times 2 = 58$ productions from students who are 14-15 y.o. and $26 \times 2 = 52$ productions from students who are 12-13 y.o. We classified students' productions according to the stability of their performance on both tasks, i.e. if student proposes a correct answer in the task GP and next if he changes or not his answer in the task

called PP. We will write RIGHT (R) or WRONG (W) the student's finale issue on these both tasks. Then, different profiles are exhibited according to the student's achievement at the question a) (the correct transformation is the reflection through a point - or a rotation of 180°) and at the question b) (the correct transformation is a reflection through an axis). Finally, the main student's profiles are presented on the table 3 and table 4, and count at least two students.

Recognition of the reflection through a point (question a)		Recognition of the reflection through an axis (question b)		Number of students	Indicative percentage of pupils %
GP	PP	GP	PP		
R	R	R	R	16	≈ 55
W	R	W	R	2	6,9
R	R	W	W	4	13,8
R	W	R	R	4	13,8
At least one WRONG				10	≈ 34,5

Tab. 3: Student's profile from the last year of secondary school (14-15 y.o) depending on whether student is successful.

Recognition of the reflection through a point (question a.)		Recognition of the reflection through an axis (question b.)		Number of students	Indicative percentage %
GP	PP	GP	PP		
R	R	R	R	9	≈ 34,7
R	R	W	W	3	11,6
W	W	W	W	3	11,6
R	W	W	W	4	15,4
W	R				
R	R	R	W	3	11,6
		W	R		
At least one WRONG				13	≈ 50

Tab. 4: Student's profile from the second year of secondary school (12-13 y.o.) depending on whether student is successful.

According to these results, only 34,7 % students from the second year recognize both transformations with successful, whatever the perception suggested by the task; and

only 55 % students among students from the last year of secondary school recognize both transformations with successful, whatever the perception suggested by the task.

The students' profiles from the second year are more fragmented than the students' ones from the last year. Therefore, we suppose the student's *Geometrical Working Space (GWS)* from the last year is more stabilized. What we need now is to determine what did each profile (especially what mistakes) and what kind of adaptations they made according to the perception and the transformation at stake.

Local analysis of the *Geometrical Working Space* through the figural treatment according to Duval's meaning

We analyse the *GWS* through its organization between the real space (marks on sheet of paper), the objects of reference from a mathematical model (Euclidian one), and the artefacts (tools, schemes). Inspired by Duval (2005), we focus on the way of treatment of the figure in order to describe these links into the *GWS*. Duval defines different kinds of "figural deconstruction". He opposes "instrumental deconstruction" which implies the use of tools to build the figure and "dimensional deconstruction" which implies links between figural units (for example the points A and B - dimension 0D - indicate the measure AB - dimension 1D) in order to exhibit mathematical properties. The latter deconstruction may imply a mathematical reasoning and suggests a geometrical paradigm closer to GII. Finally, we assume the fact the *GWS* is a favourable environment to analyse the process of conceptualization at stake because, according to Vergnaud's meaning, the notion of representation of the real world is at the heart of the process of conceptualization. Therefore, an analysis of students' productions in term of figural treatment (according to Duval's meaning) is a relevant way to describe the connection between the component of the *GWS* (Object of real world / tools / models of reference) and therefore allows us to approach the process of conceptualization at stake.

Results about students' productions at the end of secondary school (14-15 y.o.)

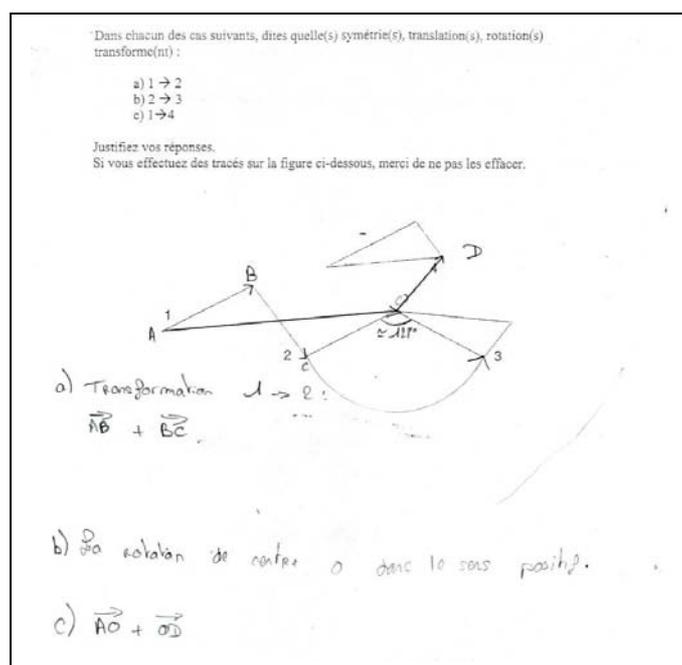
The student's *personal GWS* is adapted to the perception suggested by the task, as expected *a priori*. The operational invariants relative to the recognition of the reflection through an axis are different according to the task. The strategies of superimposition, folding or the use of common references are more present in the case GP than in the case PP.

Students may develop arguments from the Euclidian affine geometry with different kinds of "signifier" (Presmeg, 2006):

- signifier from an "instrumental deconstruction" (Duval, 2005), as for instance the *theorem-in-action of cocyclicity* : pupils use their compasses to test if a couple {point; image} of the figure belong to the same circle and therefore they infer it is a rotation. The language allows the denomination or describes the action.

- signifier from a “dimensional deconstruction” (Duval, 2005) through mathematical symbolism on the drawing (equality of measure, orthogonally, etc.). The language is used to announce the mathematical properties and make deduction.

These adaptations are used not only by students who propose correct answers but with students who propose wrong answers too. At the end of secondary school, we identify only one main kind of mistake made by students in these tasks. Students apply the *theorem-in-action of cocyclicity* at the question b) to recognize a rotation whereas it is actually a reflection through an axis (document 5).



Doc. 5: student's production with a wrong use of the theorem-in-action of cocyclicity.

We suppose this mistake is from a “cognitive conflict” about the dimension of the mathematical objects at stake with different transformations (between rotation and symmetry). With this theorem-in-action, students do not control the conservation of the measure of the angle with other couples {point; image}. They only refer to an instrumental deconstruction and not to relevant mathematical properties to recognize a rotation. This mistake could be expected if we consider the relative position between triangles (with a common top) but in the case PP, the transformation is given point by point (“CDE in GFE”) and several cases show stronger relation with the figure (because they still use this theorem-in-action) whereas these same students may adapt their strategies according to the task if the recognition of reflection occurs (namely they use a dimensional deconstruction in order to refer to mathematical properties in the case PP). We have already noticed this mistake, called ‘theorem-in-action of cocyclicity’ in a pre-test with others students with the same age (Bulf, 2007).

Results about students' productions at the second year of secondary school

If we compare the tab. 3 and tab. 4, students' profiles of 12-13 y.o. are more diversified. The personal *GWS* is still adapted to the perception suggested by the task but not as distinctly as for the students older, i.e. students use references to the real world mainly in the case GP but in the case PP too. On the other hand, they do refer to the Euclidian geometry in the case PP but sometimes in GP too. The mistakes are also more diversified because the adaptations to the perception suggested by the task are different than previously. We distinguish two main sorts of mistake:

- mistakes caused by “contract's effect” in the case PP. The notion of didactical “contract” is designed by Brousseau (1997) as a “relationship [...] [which] determines - explicitly to some extent, but mainly implicitly - what each partner, the teacher and the student, will have the responsibility for managing and, in some way or other, be responsible to the other person for managing and, in some way or other, be responsible to the other person for. This system of reciprocal obligation resembles a contract”. In our research, students propose mainly exhaustive explanations to solve the task in the case PP. They give too much mathematical properties to justify the transformation. Or, students change their mind and propose “institutional” properties on a wrong way to justify their choice in the case PP whereas their choice in the case GP was correct with naïve arguments from the real word. As for instance, one student justifies correctly the reflection through an axis (question b) in the case GP because he writes “it is possible to fold” but this same student writes, for the same transformation in the case PP, it is a reflection through a point because “in the reflection through a point, the image of a segment is a segment with the same length”. This student proposes this same “argument” at the question a) too, but in this case it is coherent. This “institutional” sentence is exactly the same which is given during the classroom. This kind of mistake lets think that the “dimensional deconstruction” (he mentions segments) suggested by students' activity is artificial, and confirm Duval's point of view who pretend this cognitive operation is not self-evident.

- mistakes caused by “amalgam between notion on the same support” according to Artigue's meaning (Artigue, 1990). Students are confused with the reflection through a point and the reflection through an axis, because these both transformations imply the same schemes as for example the global superimposition, cutting in two both sides, the properties of equal distances, etc. In particular, some students recognize a reflection through an axis instead of a reflection through a point in the case GP (question a). Some other students recognize a reflection through a point instead of a reflection through an axis in the task called PP (question b). This kind of amalgam suggests the reflection through an axis is crystallized in a “global perception”, at least at the beginning of secondary school.

CONCLUSION AND DISCUSSION

This research is devoted to the analysis of students' productions from two different levels at French secondary school. The students solved the same task given under two

different forms (one is called “Global Perception” (GP) and the other one is called “Punctual Perception” (PP)). This research points out that the *personal Geometrical Working Space* is more stabilized for a student at the end of secondary school than for a student at the beginning of secondary school. The schemes of the concept of symmetry are more flexible and can be adapted to the task (arguments can be empirical or from deduction in the frame of Euclidian Affine Geometry according to the perception suggested by the task). These adaptations show a relevant expertise of the dialectic of paradigms GI-GII when the reflection through an axis is involved, for the older students. However, the analyses of the mistakes of these students show a difference of conceptualization between the rotation and symmetry. Rotation involves an “instrumental deconstruction” only, whereas the symmetry may involve “dimensional deconstruction”.

The mistakes made by younger students imply a sort of amalgam between the different symmetries or imply the use of an artificial “dimensional deconstruction”. These mistakes make unstable the *GWS* of these students.

This variation of the use and the effects of the concept of symmetry in the *personal Geometrical Working Space* leave questions about how is managed the concept of symmetry by the teacher during secondary school and how is managed the figural deconstruction. Duval has already mentioned the problem of transmission of the different crossing of figural deconstruction (2D, 1D, 0D) in classroom (Duval, 2005). He points out these different crossings are not so obvious for students, and the difficulty of these crossings are underestimated by teachers and curricula. This point concerns the rest of our research.

NOTES

1. Official instructions: <http://eduscol.education.fr/>. BO n°10 Hors-Série, 15/10/1998, pp. 106-114 (3^e's instructions). BO n°5 Hors-Série, 09/09/04, pp. 4-16 (6^e's instructions). BO n°5 Hors-Série, 25/08/2005, pp. 9-16 (5^e's instructions).

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THE ROLE OF TEACHING IN THE DEVELOPMENT OF BASIC CONCEPTS IN GEOMETRY: HOW THE CONCEPT OF SIMILARITY AND INTUITIVE KNOWLEDGE AFFECT STUDENT'S PERCEPTION OF SIMILAR SHAPES

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ABSTRACT

In this research we investigate whether students of the Pedagogical Department of Education have the basic geometrical knowledge which is related mainly with the similarity of shapes. We also investigate how they define similarity of shapes and if the intuitive knowledge affects their perception of similar shapes. The results showed that students have developed certain structures in regard to some concepts in geometry based on the teaching that they have received in school. The results showed, as well, that a large percentage of students are not in a position to correctly define the similarity of shapes. Finally, research shown, that intuition affects their responses and their mathematical achievement.

INTRODUCTION

The role of geometry in the development of mathematical idea is very important. The geometrical skills and visual icons are basic instruments and source of inspiration for many mathematicians (Chazan & Yeryshalmy, 1998 in Protopapas, 2003). The content of geometry is appropriate both for the development of lower level of mathematical thinking, (i.e. the recognition of shape), as well as for higher order thinking, (i.e. the discovery of the properties of shapes), the construction of geometrical models and the solution of mathematical problems (NCTM, 1999). The representation of geometrical objects and the relationships between geometrical objects and their representations constitute important problems in geometry (Mesquita, 1998).

Geometry constitutes a basic part of the National Curriculum for Primary as well as Secondary Education. The concept of similarity between two shapes is taught in the 3rd grade in Secondary School and in the 1st grade in higher Secondary School, with special emphasis on the similarity of triangles. The teaching mainly concerns, understanding of the concept of similar shapes, i.e. that similar shapes are those which their sides are proportional and their angles that are created by the respective angles are equal.

Literature review has shown the concept of similarity is presented and taught through the environment of dynamic geometry and mainly through the use of applets. The concept is taught in coordination to the teaching of symmetry and transformations that can occur in a shape (<http://standards.nctm.org/document/eexamples/chap6/6.4>).

In addition, the properties of similar shapes are presented and in the proof of Thalís theorem. This theorem has some applications and proofs with the use of the Geometer Sketchpad. Although there are no relationships presented in regard to the results and consequences (proportion of relationships of line segments) of Thalís Theorem and the concept of similarity of shapes (beyond quadrilaterals).

The common teaching environment of geometry is very limited in formal education. For example, the constructions that the children are asked to deal with, the shapes are placed in a horizontal position, i.e. the sides are parallel to the sides of the object on which the construction is done. As a result most students develop an holistic and stereotype view of the geometrical shapes which is very affected by the intuitive rules.

At the university level, the students of the Department of Education are taught geometry through its historic evolution. In order to be able to follow and understand these lectures basic knowledge of geometry is required. This knowledge is mainly provided at the 3rd year of secondary school. Unfortunately, students appear to be lacking knowledge. This may be due to the long interval that has transpired since they dealt with geometry or due to the teaching in higher secondary school where it is mainly expected by the student to memorize relationships instead of understanding and applying them.

It is possible that the level of mathematical thinking may be influenced by some factors which are mathematics specific, such as the specific mathematical terminology which may be in conflict with the meaning we give to these terms in every day life or the conclusions that we reach based on the intuitive view of mathematical knowledge.

The aim of the present study is to investigate whether the students participating in EPA 171 (Basic concepts in mathematics) have the basic geometrical knowledge that is required for this specific course. It aims to investigate students' knowledge in regard to the similarity of shapes and how their intuitive knowledge may affect their perceptions about similar shapes.

THEORETICAL BACKGROUND

Geometry is comprised by three kinds of cognitive procedures which carry out specific epistemological functions (Duval, 1998):

- a) Visualization: Is the procedure which is related to the representation of space in order to explain a verbal comment, for the investigation of more complex situations and for a more holistic view of space and subjective confirmation.
- b) Construction with the use of apparatus. The construction of shapes can act as a model.
- c) Reasoning: Is investigated in relation to verbal procedures and the extension of knowledge for proof and explanation.

These different procedures can be carried out separately. Thus the visualization is not based on the construction. There is however access on the shapes and the way that they have been constructed. Even if the construction leads to visualization, the construction is based only on the connections between mathematical properties and technical restriction of the apparatus which are used. Furthermore although the visualization is an intuitive aid, necessary in some instances for the development of proof, still the justification is solely depended on a group of sentences (definitions, axioms, theorems) which are available. In addition to this visualization is sometimes more deceptive or impossible. Still these three kinds of cognitive procedures are closely linked and their cooperation is necessary for any progress in geometry (Protopapas, 2003).

The concept of similarity:

Similarity constitutes a basic link between algebra and geometry and it also has a close relationship to trigonometry. The theorem which expresses that two similar triangles have their sides proportional and Pythagoras theorem constitute two basic links between geometry and algebra. The connection of geometry and algebra is particularly construction as it allows using the visualization of geometry in algebraic problems and the flexibility of algebraic operations in geometrical problems. Similar triangles and the Pythagoras theorem constitute the cornerstone of Trigonometry. By using similar triangles we can calculate the sides and angles of an object by measuring the lengths of a smaller model.

According to Vollrath (1977) in geometry similarity constitutes a relationship between shapes/figures. A shape F_1 is similar to a shape F_2 if there is a transformation s such as $s(F_1) = F_2$. i.e. the square is similar to another one only when the ration of their sides is the same. In a didactical situation this constitutes a conclusion. Similar conclusions may be reached in regard to triangles and polygons. The proof is given based on the definition, using the properties of similar transformation. For a spiral approach of geometry it is important to know when it is possible to extract conclusions in regard to the understanding of similarity as it is defined through geometry or based on everyday language before teaching definition. Nevertheless, students do not seem to use the idea of sides' proportion to secure an exact answer about the similarity of shapes in enlargement or deduction in size of a shape (Kospentaris and Spyrou, 2005).

This can form the basis for a general definition of the concept of similarity. For the teaching of similarity at University level it is necessary, the lecturers to know in what extent the link between representation and expression of the concept of similarity can support or inhibit the cognitive procedure for this relationship. Furthermore it is important to know the explanation that the students give to similarity as it is used in everyday life or in a geometrical model (Vollrath, 1977). Kospentaris and Spyrou (2005) confirms in their study that the term similarity in everyday language does not in any way coincide with geometrical similarity, being more close to the meaning of having the same size.

The understanding of the concepts of similarity can be tested with exercises of classifying geometrical objects due to the fact that similarity constitutes a relationship of similarity between shapes/figures. In the teaching of mathematics the exercises of classification direct students in the study of properties and the properties that characterize concept and lead them to the extraction of definitions and they coordinate the understanding of definitions. Due to their importance we use exercises on classification to investigate students' understanding related to similarity irrespective of the mathematical definition. (Vollrath, 1977).

Intuition – and how it affects the teaching in mathematics:

As suggested by Fischbein (1999) intuition constitutes a theme that mostly philosophers are interested in. According to Descartes (1677) and Spinoza (1677) intuition appears to be a genuine source of pure knowledge. Kant (1781) describes intuition as the ability which leads directly to your goals and indirectly to the basic knowledge. Bergson (1900) made a distinction between intelligence and intuition. Intelligence is the way in which one may know the physical world, the world of stability, the extent of the properties of statistical phenomena. Through intuition we have a direct perception of the essence of spiritual life and control of the phenomena, time and motion (Fischbein, 1999).

Some philosophers, such as Hans Hahn (1956) and Burge (1968), have criticized intuition and its effect, in its scientific explanation. They believe that intuition leads to deceptive results and this has to be avoided in the scientific procedure.

The investigation of intuitive knowledge appears mainly in the work of people that are interested in scientific and mathematical understanding of students (for example Clement et al., 1989; DiSessa, 1988; Gelman and Gallistel, 1978; McCloskey et al., 1983; Resnick, 1987; Stavy and Tirosh, 1996; Tirosh, 1991 in Sierpinska, 2000). There is not an accepted definition of intuitive knowledge. The term: “intuition” is used mainly as a mathematical basic term such as the point or line (Sierpinska, 2000).

The importance of definition is probably respected just like the elements that are based on intuition. The basic common properties of these are based on individual proofs which are in conflict to logical and analytic attempts.

The problem of intuitive knowledge has earned an important place in scientific attempts. On one hand scientists need intuition in their attempt to discover new strategies, new theoretical and empirical models and on the other hand they need to be acquainted with what does not constitute intuition – according to Descartes and Spinoza – basic guarantee, fundamental basis for objective truth.

The interest in regard to intuition also stems from the teaching of science and mathematics. When you need to teach a chapter in science or mathematics you often discover that what was already a fact for you – after university level studies – comes in conflict with basic cognitive obstacles that the students exhibit in their understanding. As a teacher you often believe that students are ready to memorize what they have been taught, actually they understand and memories relative

knowledge. Intuitive perception of phenomena is often different that to their scientific explanation.

In mathematics, the belief that a square is a parallelogram is intuitively very strange for many children. The belief that by multiplying two numbers we may get a result that is smaller than one or both the numbers which we have multiplied is also difficult to be accepted. Intuition affects many of our perceptions. The educator discovers that the knowledge which s/he is supposed to transfer to the students is in conflict, very often, with the beliefs and explanations which are direct and solid and at the same time in conflict with the scientifically accepted perceptions.

THE STUDY

Aim:

The aim of the study is to investigate whether the students participating in EPA171 (Basic concepts in mathematics) have the basic geometrical knowledge that is related mainly with the similarity of shapes. How do they perceive the concept of similarity of shapes and how their intuitive knowledge may affect their understanding of similarity of shapes?

The three hypothesis of the study were:

1. The students have specific difficulties in basic concepts in geometry.
2. The students define similarity of shapes based on similar triangles or intuitive knowledge.
3. Intuitive knowledge affects their perception of similar shapes.

Subjects:

The participants in this study were 85 students of the Pedagogical Department of Education. 42 had mathematics as a major subject in higher secondary school, 39 had mathematics as a core subject and 4 did not specify.

Design of the study:

In order to examine the hypothesis of this study a test was administered to all the students that took part in the study. The students had 40 minutes available to respond to the test. The tasks of the tests were related with basic geometrical concepts (definition and construction of obtuse angle, application of properties of parallel lines and of the Pythagoras theorem in the solution of relevant exercises), definition of similarity of shapes, recognition of similar shapes as well as tasks which were used to examine whether the students had the necessary knowledge which is required to teach the lesson.

For the analysis of the results descriptive statistic as well as the implicative analysis have been used. More specifically for the data analysis the following elements of implicative analysis have been utilized: (a) The similarity tree-diagram which shows

the variables according to the similarity they show (b) the hierarchical tree-diagram which presents the implicative relationships according to the order of significance.

Results:

The first hypothesis is confirmed in that basic knowledge of geometry where no special attention is given in school, such as the ability to give the definition of concepts. For the examination of this hypothesis which concerns basic geometrical concepts four questions were posed.

The first two questions were related mainly to the mathematical terminology which the students use. Students were asked to give a definition and construct an acute angle and its supplementary. The analysis of the results shows that 83% can draw an obtuse angle but they only refer to the fact that it has to be bigger than 90° but they do not specify that it has to be smaller than 180° . 14% of the students who are mostly the ones that had mathematics as a major subject in higher secondary give a complete answer, whereas 3% of the students can not answer this basic question at all. In regard to the question related to the supplementary angles 95% give a complete answer since only one condition is requested (sum 180°) and only 5% does not answer or gives a wrong answer.

The third question of the test concerns the use of basic relationship between angles and is based on parallel lines and the solution of a problem. These relationships are used quite extensively in secondary education something that leads students to a direct recognition and use of the relationships. This is illustrated by the results in the test since the majority (90%) that dealt with the task in question 3 managed to give correct answers.

The fourth question of the test requires a direct application of Pythagoras theorem twice. The application of Pythagoras's theorem without its proof constitutes a basic chapter in the teaching of geometry in secondary school. Thus 82,5% of the students were able to solve the exercise, 4,5% were able to solve only half of the task and 13% either gave a wrong answer or did not provide a response.

The second hypothesis was not fully confirmed. More than a third of the students could give a complete answer and a significant percentage of students referred to the similarity of the appearance of the shapes or the similarity of triangles. In order to examine this hypothesis the questions 5a and 5b were given.

In the question 5a, which asked students to answer "what are similar shapes?" only 36,5% of the students were able to give a complete answer (5iv). 21% referred to the similarity in the appearance of the shapes (5iii) and 14% referred to the similarity of triangles (5ii) which plays a significant role in the teaching of similarity in secondary education. A significant percentage of the students 12% referred to equality (5i), whereas 16% of the students either did not provide any answer or gave a wrong response (5i).

In order to examine whether the students have the ability to use the definition of similarity of shapes in an exercise regarding similar triangles, the second part (5b) of exercise 5 was asking students to find the relationship of similarity between given triangles. Differently to their responses in the 1st part of the exercise where 53% could give a complete answer, only 30% were able to reach a mid way to the solution. 17% could not solve the problem or did not give any response.

For the application of the theory regarding the relationships of similarity and also for the examination of the third hypothesis, exercise 8 was presented where students were asked to find which polygons are similar. In contrast to exercise 5b where they had to write some relationships algebraically in order to prove the similarity of the shapes, in this exercise, they needed mental representations of the relationships so that the right choices could be made. Just like in question 5, some students confuse similarity with the relationships regarding the appearance of the shape. That is probably why 87% responded that the parallelograms that have equal angles one side proportional and one side equal are similar (8i). It is very likely that they have reached this answer because both of them are parallelograms. 13% of the students believe that the rectangles are similar to the square (8iv) in the shape. This may be due to the fact that all three of them are parallelograms (appearance of the shape). Similarly 6% believe that the right angle triangle is similar to the scalene triangle (8v), most probably because both of the triangles have the same appearance. 80% recognize the similarity of the rectangles that are presented (8iii) and of the right

<p>Question 5: What are similar shapes? 5i: referred to equality or no answer or wrong response. 5ii: referred to the similarity of triangles 5iii: referred to the similarity in the appearance of the shapes 5iv: complete answer</p>	<p>Exercise 8: students were asked to find which polygons are similar. 8i: responded that the parallelograms that have equal angles one side proportional and one side equal are similar. 8ii: recognize the similarity of the right angle triangles 8iii: recognize the similarity of the rectangles 8iv: believe that the rectangles are similar to the square in the shape. 8v: believe that the right angle triangle is similar to the scalene triangle</p>
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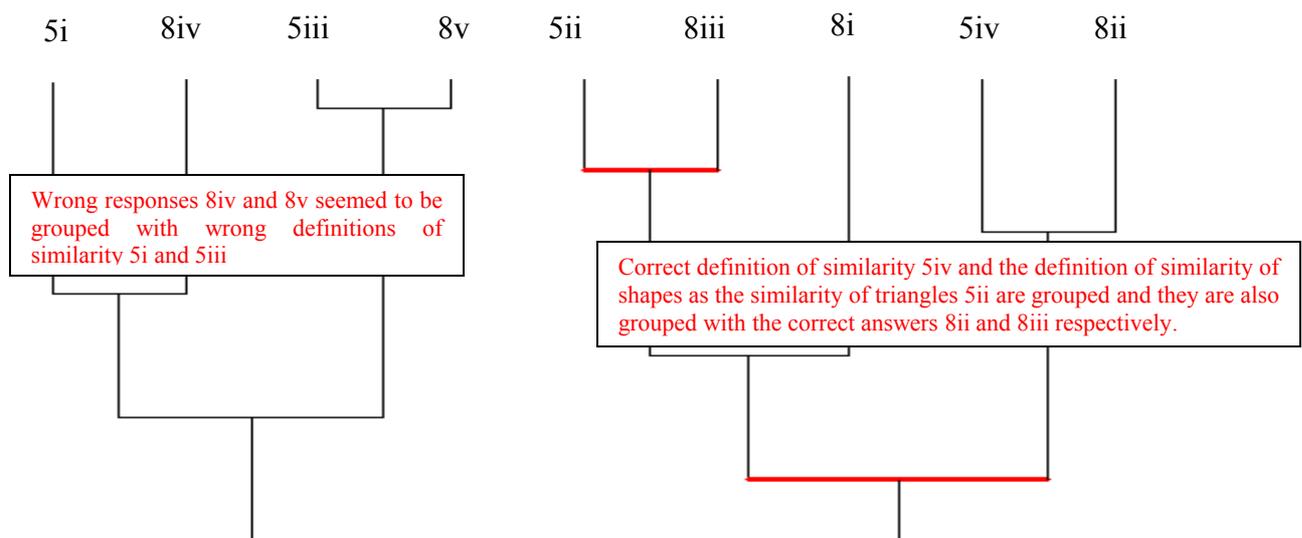


Figure 1: similarity tree diagram

In order to examine whether the definition that students give for the similarity of shapes affects their answer in exercise 8 where they are asked to recognize similar shapes we have used the similarity tree diagram (Figure 1). In the tree diagram the wrong responses in exercise 8 seemed to be grouped with the variables 8iv and 8v (similar shapes: square-rectangle, variable 8iv and right angle triangle and scalene triangle 8v) with the variables 5i and 5iii respectively of exercise 5 which refer to wrong definitions of similarity (5i: equality of shapes or wrong answer and 5iii: similarity in the appearance of the shape). In addition to this, the correct definition of similarity (variable 5iv) and the definition of similarity of shapes as the similarity of triangles (variable 5ii) are grouped and they are also grouped with the correct answers in exercise 8, and the variables 8ii and 8iii respectively. The variable 8i which is the wrong answer in 8 since it presents the similarity of two parallelograms that their sides are not proportional appear to be grouped with the correct definitions (mainly with the definition of similar triangles and the correct answer in regard to rectangles) and the correct answers. This may be due to the fact that most students perceive as the correct answer, something that indicates that students are depending on the perception of shapes and not the definitions and the properties of the shapes.

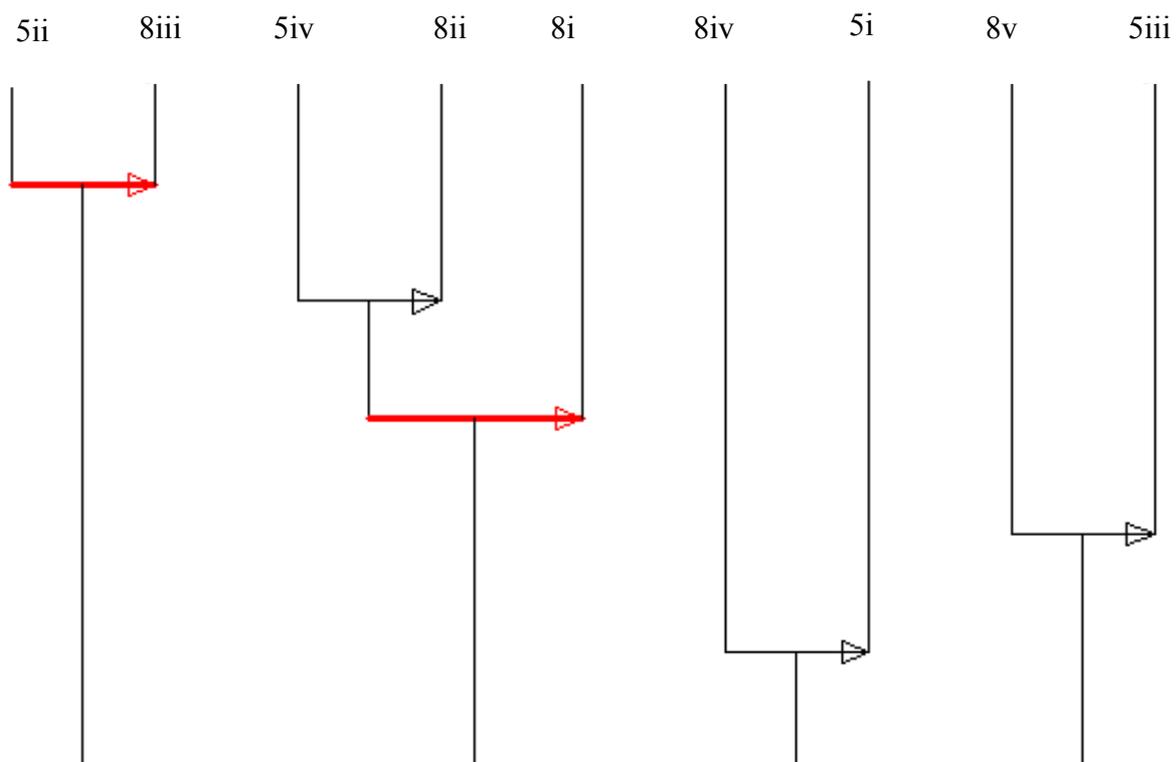


Figure 2: hierarchical diagram

The hierarchical diagram (Figure 2) shows that success in the definition constitutes success in the tasks in exercise 8, whereas in the wrong responses higher in line are

the tasks in exercise 8, something that results to difficulty in giving a correct definition for the similarity concept.

CONCLUSIONS

The data of the study suggest that students have developed certain structures in regard to some concepts in geometry based on the teaching that they have received in school. The fact that in secondary education more emphasis is placed on the practical application of theory and less on the understanding of concept, leads to students' difficulty in giving complete definitions that require conditions, which in the practical application are implied without being presented (for example, the representation of an obtuse angle is never presented opposite to angles bigger than 180° and that is why students never refer to the condition that an obtuse angle needs to be smaller than 180°).

Based on this it appears that students are in a position to carry out operations by using certain formulas (Pythagoras's theorem) or recognize relationships in shapes which they were taught in school and they are expected to apply these in exercises similar to exercises 3 and 4 of this test.

For a spiral approach and development of geometry, it is important to know when it is possible to extract conclusions in regard to the concept of similarity as it is defined in geometry. As it appears from the data, a large percentage of students are not in a position to correctly define the similarity of shapes. However they are able to apply the relationships of similarity in triangles since teaching in secondary education is related to the similarity of triangles

In the search for similarity relationships in exercise 8 students influenced by their intuition found relationships that were based on the similarity of the appearance of the shape but they were not mathematically similar. This indicates that intuition affects their responses and their mathematical achievement since a number of these students have not received adequate mathematical training in order to base their answers on definitions, properties of the shapes and not on the perceptual appearance of the shape.

The data suggest that the wrong similarity definition leads to wrong responses in the practical applications, whereas the wrong representations of concepts create students' erroneous structures and definitions of the specific concepts.

In conclusion, in regard to the teaching of geometry at University level it is important to give more attention in the teaching of basic geometrical concepts and skills. As it was shown by the results in this study the teaching that many students receive in secondary school is inadequate, something that affects their perception and achievement in geometry. The lack or limited knowledge that students have, lead, to the use and translation of mathematical definitions based on wrong mental representations which are affected by intuitive knowledge and not by the correct mathematical definitions and correct representations.

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THE GEOMETRICAL REASONING OF PRIMARY AND SECONDARY SCHOOL STUDENTS

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In the present paper comparing the geometrical reasoning of primary and secondary school students was mainly based on the way students confronted and solved specific geometrical tasks: the strategies they used and the common errors appearing in their solutions. This comparison shed light to students' difficulties and phenomena related to the transition from Natural Geometry (the objects of this paradigm of geometry are material objects) to Natural Axiomatic Geometry (definitions and axioms are necessary to create the objects in this paradigm of geometry) and to the inconsistency of the didactical contract implied in primary and secondary school education. These findings stress the need for helping students progressively move from the geometry of observation to the geometry of deduction.

INTRODUCTION

Teaching geometry so that students learn it meaningfully requires an understanding of how students construct their knowledge of various geometric topics (Battista, 1999). This means it is necessary that mathematics educators investigate and mathematics teachers understand how students construct geometrical knowledge as a result of their learning experiences in school. An important aspect of this research direction is the study of the strategies that students use in different geometrical tasks as well as the identification of their mistakes. In the work of Piaget and in the Geneva School we see that errors were for the first time viewed positively, in the sense that they allow the tracing of the reasoning mechanisms adopted by students.

The literature review reveals that the investigation of various issues related to students' geometrical reasoning (knowledge, abilities, strategies, difficulties) is in most cases restricted to the study of groups that come from one educational level. We believe that it is necessary to gather empirical data which would allow the comparison between groups of students in primary and secondary education and would be valuable sources of information regarding aspects of teaching in the two educational levels as well as the difficulties met by students of different age groups.

The transition from elementary to secondary school is recognized as a critical life event, since, progressing from one level of education to the next, students may experience major changes in school climate, educational practices, and social structures (Rice, 2001). Research results reveal substantial agreement that there is often a decline in students' achievement following this transition, but achievement scores tend to recover in the year following the transition (Alspaugh, 1998). In the case of Cyprus, students experience difficulty during the transition from elementary

to secondary school which is evident in their performance in most topics, especially in mathematics.

This paper is based upon a research project which investigated the transition from elementary to secondary school geometry in Cyprus, gathering data concerning students' performance in tasks involving two-dimensional geometrical figures, three-dimensional geometrical figures and net-representations of geometrical solids, as well as the students' spatial abilities. In the present paper we focus on the strategies the students used to solve specific geometrical tasks involving two-dimensional figures and we study the kinds of errors that we identified in the students' solutions.

THEORETICAL BACKGROUND

In the present paper we use as explanatory framework Duval's cognitive approach to geometry (Duval, 1995, 1998) and the framework of Geometrical Paradigms proposed by Houdement and Kuzniak (Houdement & Kuzniak, 2003; Houdement, 2007). We also use the concept of the didactical contract, introduced by Brousseau (1984) to interpret some of the students' wrong answers. According to him, the didactical contract is defined as a system of reciprocal expectancies between teacher and pupils, concerning mathematical knowledge. The didactical contract is in large part implicit and is established by the teacher in her teaching practice. The students may interpret the situation put before them and the questions asked to them on the basis of the didactical contract and act accordingly.

A cognitive approach to geometry

Duval (1998) argues that geometry involves three kinds of different cognitive processes – visualization processes, construction processes and reasoning in relation to discursive processes – the synergy of which is necessary for proficiency in geometry. Approaching geometry from a cognitive point of view, he has distinguished four cognitive apprehensions connected to the way a person looks at the drawing of a geometrical figure: perceptual, sequential, discursive and operative (Duval, 1995). Briefly, perceptual apprehension refers to what a person recognizes at first glance when looking at a geometrical figure, while sequential apprehension is required whenever the construction or description of construction of a figure is involved. Discursive apprehension refers to the mathematical properties that cannot be determined through perceptual apprehension of a figure, but must be given through speech or can be derived from the given properties. Operative apprehension depends on the various ways of modifying a given figure. Solving geometrical problems often requires the interactions of these different apprehensions, and “what is called a ‘geometrical figure’ always associates both discursive and visual representations, even if only one of them can be explicitly highlighted according to the mathematical activity that is required” (Duval, 2006, p.108).

The framework of Geometrical Paradigms

Keeping the idea of ‘paradigm’ from Kuhn, who used it to explain the development of science, Houdement and Kuzniak (2003) proposed that elementary geometry appears to be split into three various paradigms, characterizing different forms of geometry: Geometry 1 (natural geometry), Geometry 2 (natural axiomatic geometry) and Geometry 3 (formalist axiomatic geometry). The theoretical framework they have developed specifies the nature of the geometrical objects, the use of different techniques and the validation mode accepted in each of the three paradigms. Here we briefly describe the first two geometrical paradigms distinguished by Houdement and Kuzniak (Houdement & Kuzniak, 2003; Houdement, 2007), which mainly concern primary and secondary school students that participated in the present study.

Geometry 1 is intimately related to reality and reasoning is close to experience and intuition. The objects of Geometry 1 are material objects, graphic lines on a paper sheet or virtual lines on a computer screen. Drawing and measurement techniques with ordinary geometrical tools (ruler, set square, compass) as well as experimentation in the sensible world (using techniques such as folding, superposing) are used in this paradigm. New knowledge may be produced based on evidence, experience or reasoning, while a permanent motion between the model and the reality enables the student to ‘prove’ the assertions.

In Geometry 2 the objects are ideal, so reasoning relies on the mathematical properties of the abstract geometrical objects. A system of definitions and axioms is necessary for the creation of the objects. In this system the axioms are as close as possible to intuition, but making progress and reaching certainty demands demonstrations inside the system. Hypothetical deductive laws are the source of validation.

THE PRESENT STUDY

As noted in the introduction, this paper is based upon a research project which examined primary and secondary school students’ geometrical knowledge and abilities related to tasks involving different geometrical figures, as well as their spatial abilities in micro-space. Participants in our study were 1000 primary and secondary school students (488 males and 512 females) from 29 classes of 9 elementary schools and 12 classes of 8 secondary schools in four different districts of Cyprus. Specifically, the sample involved students from three grades (fourth grade – primary school: 332, sixth grade – primary school: 333 and, eighth grade – second grade of secondary school: 335). The mean age of the three grades was as follows: fourth grade, 9.8 years; sixth grade, 11.7 years; eighth grade, 13.9 years. Information concerning the instrument we constructed for the purpose of our research project and the procedure we followed can be found in Panaoura and Gagatsis (2008).

In the present paper we attempt to compare the geometrical reasoning of primary and secondary school students (the three age groups in our study) based on their solutions

to three specific geometrical tasks which involved two-dimensional figures (the three tasks are shown in the Appendix). At this point we have to stress that the comparison attempted here does not refer to the levels of success of the three groups of students, since we study students of different age, from different educational levels, with different learning experiences and different cognitive abilities. Using as explanatory framework the theoretical notions presented above, we focus on the strategies and the common errors we identified in students' solutions. In this direction first we present part of the results from our study concerning students' solutions of three geometrical items included in the test and then we discuss these results and students' difficulties under the light of didactic phenomena rising from our research.

RESULTS ON SPECIFIC GEOMETRICAL ITEMS

Item [A]

On the geometrical figure presented in item [A] a square and a right triangle can be identified. In order to give the correct answer, the students had to (a) identify, within the figure presented, the subfigures of the square and the right triangle, (b) pass from 2D to 1D and 'see' that the unknown segment [AC] is one of the square's sides and (c) recall and apply the cognitive unit referring to the property of equal sides in a square. At this point we must note that in the geometry test we included a multiple choice item to examine whether students possess the cognitive unit referring to the property of equal sides in a square. The results presented in Table 1 showed that while a high percentage of the students answered correctly to the specific multiple choice item (61.7% of 4th graders, 85.9% of 6th graders and 86.9% of 8th graders) – indicating they *know* that the four sides of a square are equal – a smaller number of students (especially from primary school) eventually gave a correct answer to the geometrical item [A].

Item	Answer	4 th graders	6 th graders	8 th graders
Multiple choice	Correct	61.7	85.9	86.9
Item [A]	Correct – using properties	36.4	71.8	66.9
	Correct – applying theorem	---	---	18.5
	Wrong – using ruler	8.4	2.1	---
	Wrong – arithmetical operations	6.0	4.8	2.4

Table 1: Students' answers to multiple choice item and item [A] by age group

Crosstabs tables of performance to the multiple choice item by performance to item [A] were obtained for each age group in order to examine what percentage of the students who answered correctly to the specific multiple choice item, did actually solve the geometrical item [A]. The crosstabs results indicated that half of the 4th

grade students and a percentage of 22% of the 6th grade students who gave the correct answer to the multiple choice item (know that the sides of a square are equal) were not able to produce a correct answer to item [A]. The corresponding percentage was 10% in the case of 8th grade students. So it seems that the secondary school students, working in the Natural Axiomatic Geometry paradigm, generally felt the need to use the properties and recalled the right one to solve item [A].

On the other hand, examining at the common errors identified in the students' solutions (Table 1), we notice some primary school students who gave (wrong) answers after using their ruler to measure the unknown segment on the geometrical figure presented on their paper. Additionally, a small number of students of the three age groups tried to combine the arithmetical data of the problem in a random way in arithmetical operations in order to come to an answer.

At this point it is interesting to state that, while the students could give the correct answer to item [A] by simply applying the property of equal sides in a square, we identified 18.5% of the secondary school students who solved the specific geometrical problem by applying Pythagoras' theorem in the subfigure of the right triangle. This performance is probably influenced by a part of the didactical contract according to which they are expected to apply Pythagoras' theorem any time a right triangle is involved in a geometrical figure. On the other hand, the specific performance indicates a difficulty concerning the transition from primary to secondary school. Specifically, the emphasis put on the use of algorithms during mathematics teaching in the secondary school seems to gradually result to the phenomenon that the students feel the safe of using an algorithm to be greater than that of a simple application of a geometrical property.

Items [B] and [C]

In Table 2 we present the results of students' attempts to solve two other geometrical tasks included in our test (item B and item C). Item [B] is a problem given to French students entering middle school (Duval, 2006). Item [C] was constructed for the present study, as an analogous problem to item [B], with two basic differences. First, on the geometrical figure presented in item [B], the subfigures of a circle and a rectangle appear, while on the geometrical figure presented in item [C] the two subfigures identified are a square and a rectangle. Second, the 'visibility' of the geometrical figure (and its subfigures) is less in the case of item [B] due to the specific configuration.

Facing the geometrical problem presented in item [B] a number of students in the present study relied only on a visual perception of the figure (perceptual apprehension) and either considered point E as the middle of [AB] (16.5% of 6th grade students and 9.3% of 8th grade students), or answered that the length of segment [EB] is equal to the circle's ray, "because *it seems to be equal to the ray*" (11.1% of 6th grade students and 9.0% of 8th grade students).

Answer	Item [B]			Item [C]		
	4 th graders	6 th graders	8 th graders	4 th graders	6 th graders	8 th graders
Correct – using properties	15.1	33.3	51.9	46.1	62.2	81.5
Wrong – visual perception (i)	6.6	16.5	9.3	3.3	4.5	0.6
Wrong - visual perception (ii)	8.7	11.1	9.0			
Wrong – using algorithms	10.2	5.4	0.9	11.4	9.9	2.1

Table 2: Students' answers to item [B] and item [C] by age group

In order to solve the item [C], the solver had to identify the two subfigures, to possess and to use the cognitive unit referring to the property of equal sides of a square. As in the case of item [B], a number of students relied only on the visual perception of the given figure and considering point E as the middle of [AB] answered that the length of segment [EB] is equal to 3.5 cm. In both cases perceived features of the geometric figures (relying on a perceptual apprehension of the given figure in each problem) have misled the students as to the mathematical properties involved in the problem solution and have obstructed appreciation of the need for discursive apprehension of the presented geometrical figure.

Finally, it is interesting to note that, as in the case of item [A], there are (mainly primary school) students who tried to give an answer to the items [B] and [C] combining in arithmetical operations the data presented in the geometrical problems. A possible explanation to the specific students' performance is that, according to the implicit didactical contract (Brousseau, 1984) established during the teaching and learning processes in the mathematics classroom – especially the aspect concerning the solution of routine arithmetical word problems – when those students are given a geometrical problem which involves arithmetical data, they suppose that they are expected to combine them in order to give an answer. They probably consider that in this way not only they can give an answer, but they also demonstrate that they have tried to solve the problem by identifying and using the data given in the problem. So, they assume that their teacher will be pleased with their performance!

DISCUSSION

Research about the learning of mathematics and its difficulties “must be based on what students do really by themselves, on their productions, on their voices” (Duval, 2006, p. 104). In this paper we presented some results from our research referring to the solutions of primary and secondary school students in three geometrical items,

focusing on the strategies they used and their common errors. Once again we stress that we did not seek to compare students' levels of success, since it is obvious that the students participating in our study have different learning experiences (as far as the amount of experiences and the teaching methods are concerned) and differ in their cognitive development. The comparison of the solutions of the different age groups students shed light to phenomena related to the transition from Natural Geometry to Natural Axiomatic Geometry and to the inconsistency of the didactical contract implied in primary and secondary school education.

The transition from Natural Geometry to Natural Axiomatic Geometry

The passage from Geometry 1 to Geometry 2 is a complex, sensitive and crucial matter (Houdement & Kuzniak, 2003), since these two paradigms are different as far as objects, techniques and validation mode are concerned (Houdement, 2007). Moving from Natural Geometry to Natural Axiomatic Geometry students have to change their theory concerning the nature of the objects and of the space. They are forced to adopt the notion of conceptual objects, the existence of which is based on a definition in an axiomatic system. Consequently, they have to foster new techniques to work relying on the mathematical properties of each abstract geometrical figure.

The findings of the present study indicate that students working in the paradigm of Natural Geometry (mainly primary school students in our study) tend to consider geometrical objects as material objects and specific pictures rather than as theoretical, ideal objects which bear specific properties. This difficulty results to the phenomenon of students trying to solve geometrical problems often relying on the visual perception of the given geometrical figure rather on a mathematical deduction based on the properties of the geometrical objects involved. This phenomenon is related to the students' difficulty to work with geometrical figures as 'figural concepts' (Fischbein, 1993). We call it 'geometrical figure to figural concept' difficulty. As Mariotti (1995) has noted, correct and effective geometrical reasoning is characterized by the interaction and the harmony between figural and conceptual aspects of geometrical entities. In the present study, students working in the Natural Geometry paradigm (mainly primary school students) base their geometrical reasoning on the perceptual apprehension of the geometrical figure presented in a given task and this results to erroneous solutions, since the geometrical properties cannot be determined only through the specific type of apprehension. The perceptual apprehension of a geometrical figure must be under the control of the verbal propositions (discursive apprehension) which are presented in a geometrical problem (Duval, 1998), in such a way that correct geometrical reasoning results through the combination and interaction of the verbal propositions and the geometrical figure. In contrast to the students working under the Natural Geometry paradigm, students working in the Natural Axiomatic Geometry paradigm (mainly amongst secondary school students) focus their efforts on geometrical relations and they confront geometrical tasks based on the properties of geometrical figures (Houdement & Kuzniak, 2003).

Inconsistency of the didactical contract in primary and secondary education

The strategies used by the students in the solution of the presented tasks indicate that the didactical contract which is established among teachers and students concerning geometry learning in primary school education does not discourage all the students from (a) extracting conclusions based on the visual perception of a geometrical figure and (b) giving an answer extracted from random combination of the arithmetical data given in a geometrical problem. These aspects of the didactical contract were not identified to be present in the secondary school education, in the Natural Axiomatic Geometry paradigm, where the emphasis is on the properties of geometrical objects. We call this phenomenon “inconsistency of the didactical contract” among the two education levels concerning the teaching of geometry and further investigation is needed in order to gather information regarding the actual teaching of geometry in primary and secondary schools.

The power of the didactical contract of Natural Axiomatic Geometry

In the case of geometry teaching in the secondary school, the emphasis on learning theorems and continuous practice with close tasks demanding the application of theorems may result in the application of these theorems even in cases that this is not necessary. For example, as a consequence of the continuous practice of the Pythagoras’ theorem and the didactical contract formed during teaching, students consider that they are expected to apply Pythagoras’ theorem any time a right triangle is involved in a geometrical figure. As we have noted in the results section, attempting to solve a task which could be solved with the mere application of the property of equal sides in a square, almost one fifth of the 8th graders in the present study applied Pythagoras’ theorem in the rectangular triangle they identified in the given geometrical figure. The power of the didactical contract in secondary school geometry concerning the application of theorems, leads students to mechanically apply the theorems, especially those that involve an algorithm, feeling safer to use an algorithm than a geometrical property.

Teaching implications and further research

Most of the difficulties that have been identified and discussed in the present study concerning primary and secondary school students’ attempts to solve geometrical problems are centred around the issue of the difficulties raised during the transition from Natural Geometry paradigm (where the objects are real, material) to Natural Axiomatic Geometry paradigm (where the objects are conceptual). Subsequently, one of the main goals during the teaching of geometry should be to help students progressively pass from a geometry where objects and their properties are controlled by perception to a geometry where they are controlled by explicitation of properties. But, as Houdement and Kuzniak (2003) note, students and their teachers are not necessarily situated in the same geometrical paradigm, so this is a source of educational misunderstanding. Therefore, we consider essentially important that (prospective) primary and secondary school mathematics teachers are aware of the

existence of the different geometrical paradigms (Houdement, 2007) and of the difficulties arising from the fact that plane geometrical figures on paper may be considered by the students in the teaching process during elementary school as if they were real objects (Berthelot & Salin, 1998). Further research is needed in order to prescribe and compare the way mathematics teachers in primary and secondary school approach geometry in their classrooms.

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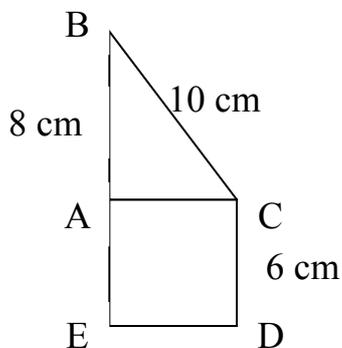
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APPENDIX

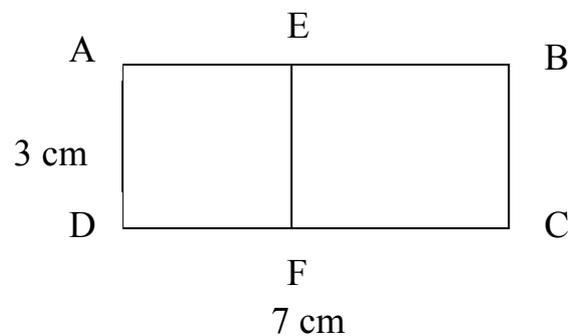
Item A

On the right triangle ABC, $BC=10\text{cm}$ and $AB=8\text{cm}$. ACDE is a square ($CD=6\text{cm}$). Find the length of segment AC.



Item C

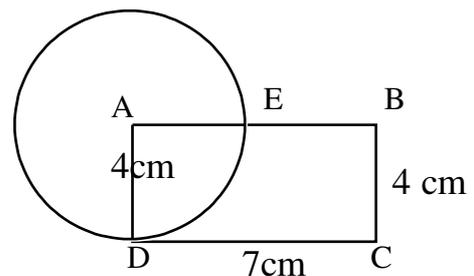
On the rectangle ABCD, $DC=7\text{cm}$ and $AD=3\text{cm}$. AEFD is a square. Find the length of segment EB.



Item B

On the figure sketched freehand here (the real lengths are written in cm), are represented a rectangle ABCD and a circle with center A, passing through D.

Find the length of segment EB.



STRENGTHENING STUDENTS' UNDERSTANDING OF 'PROOF' IN GEOMETRY IN LOWER SECONDARY SCHOOL

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This paper reports findings that indicate that as many as 80% of lower secondary age students can continue to consider that experimental verifications are enough to demonstrate that geometrical statements are true - even while, at the same time, understanding that proof is required to demonstrate that geometrical statements are true. Further data show that attending more closely to the matter of the 'Generality of proof' can disturb students' beliefs about experimental verification and make deductive proof meaningful for them.

Key words: Geometrical reasoning, generality of proof, cognitive development, lower secondary school, curriculum design

INTRODUCTION

School geometry is commonly regarded as a key topic within which to teach mathematical argumentation and proof and to develop students' deductive reasoning and creative thinking. Yet while deductive reasoning and proof is central to making progress in mathematics, it remains the case that students at the lower secondary school level have great difficulty in constructing and understanding proof in geometry (Battista, 2007; Mariotti, 2007). Our work focuses on researching, and comparing, the teaching of geometry at the lower secondary school level in countries in the East and in the West, specifically China, Japan and the UK (see, for example, Ding, Fujita, & Jones, 2005; Ding & Jones, 2007; Jones, Fujita & Ding, 2004, 2005). In our research we are interested in students' cognitive needs in the learning of geometrical concepts and thinking, and in principles for classroom practice which would satisfy such needs of students.

In this paper we report selected findings from a series of research projects on the learning and teaching of geometrical proof carried out in Japan where formal proof is intensively taught in the lower secondary school grades (Grades 7-9). We address the issue of students' cognitive needs for conviction and verification and how these needs might be changed and developed through instructional activity. We first present how students in lower secondary schools perceive 'proof' in geometry in terms of the levels of understanding of geometrical proof. We do this by using data collected in 2005 from 418 Japanese students (206 from Grade 8, and 212 from Grade 9). We then offer some suggestions that we have developed from classroom-based research (undertaken since the 1980s) about how we might encourage students' geometrical thinking and understanding of deductive proof in geometry.

Given our data is from studies conducted in Japan, we begin with a short

account of the teaching of proof in geometry in Japan.

THE TEACHING OF PROOF IN GEOMETRY IN JAPAN

The specification of the mathematics curriculum for Japan, the ‘Course of Study’, can be found in the *Mathematics Programme in Japan* (English edition published by the Japanese Society of Mathematics Education, 2000). It should be noted that no differentiation is required in the ‘Course of Study’, and mixed-attainment classes are common in Japan. ‘Geometry’ is one of the important topics (the other topics are ‘Number and Algebra’ and ‘Quantitative Relations’). The curriculum states that, in geometry, students must be taught to “understand the significance and methodology of proof” (JSME, 2000, p. 24). In terms of the *Paradigm of Geometry* proposed by Houdement and Kuzniak (Houdement & Kuzniak, 2003), Japanese geometry teaching may be characterized as within the *Geometry II* paradigm (in that axioms are not necessarily explicit and are as close as possible to natural intuition of space as experienced by students in their normal lives).

In terms of Japanese curriculum materials (such as textbooks for Grade 8 and Grade 9 students) our analysis indicates a varying amount of emphasis on ‘justifying and proving’ (see, for example, Fujita and Jones, 2003; Fujita, Jones and Kunimune, 2008). While the curriculum requires that the principles of how to proceed with mathematical proof are explained in detail, including explanations of ‘definitions’ and ‘mathematical proof’, our research repeatedly shows that many students difficulties to understand proof in geometry (for example, Kunimune, 1987; 2000¹).

In what follows we provide an analytical framework for students’ understanding of proof in geometry and then report on our data from three surveys carried out in 1987, 2000 and 2005.

ASPECTS OF STUDENTS’ UNDERSTANDING OF PROOF IN GEOMETRY

In our research, as summarized in this paper, we capture students’ understanding of proof in terms of two components: ‘Generality of proof’ and ‘Construction of proof’. The first one these, ‘Generality of proof in geometry’, recognizes that, on the one hand, students have to understand the generality of proof in geometry, including the universality and generality of geometrical theorems (proved statements), the roles of figures, the difference between formal proof and experimental verification, and so on. The second of these two components, ‘Construction of proof in geometry’, recognizes that, on the other hand, students also have to learn how to ‘construct’ deductive arguments in geometry by knowing sufficient about definitions, assumptions, proofs, theorems, logical circularity, and so on.

Considering these two aspects, we work with the following levels of student understanding (we do not have space in this paper to relate these levels to the van Hiele model):

Level I: at this level, students consider experimental verifications are enough to demonstrate that geometrical statements are true. This level is sub-divided into two sub-levels: Level Ia: Do not achieve both ‘Generality of proof’ and ‘Construction of proof’, and Level Ib: Achieved ‘Construction of proof’ but not ‘Generality of proof’

Level II: at this level, students understand that proof is required to demonstrate geometrical statements are true. This level is sub-divided into two sub-levels: Level IIa: Achieved ‘Generality of proof’, but not understand logical circularity, and Level IIb: Understood logical circularity

Level III: at this level, students can understand simple logical chains between theorems

We used the following questions to measure students’ levels of understanding:

Q1 Read the following explanations by three students who demonstrate why the sum of inner angles of triangle is 180 degree.

Student A ‘I measured each angle, and they are 50, 53 and 77. $50+53+77=180$. Therefore, the sum is 180 degree.’ Accept/Not accept

Student B ‘I drew a triangle and cut each angle and put them together. They formed a straight line. Therefore, the sum is 180 degree.’ Accept/Not accept

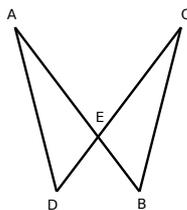
Student C Demonstration by using properties of parallel line (an acceptable proof) Accept/Not accept

Q2 In Figure Q2, prove $AD = CB$ when $\angle A = \angle C$, and $AE = CE$.

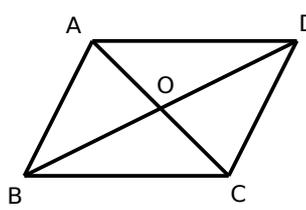
Q3 The following argument carefully demonstrates that *the diagonals of a parallelogram intersect at their middle points* (see Figure Q3). ‘In a parallelogram ABCD, let O be the intersection of its diagonals. In $\triangle ABO$ and $\triangle CDO$, $AB \parallel DC$. Therefore, $\angle BAO = \angle DCO$ and $\angle ABO = \angle CDO$. Also, $AB = CD$. Therefore $\triangle ABO \cong \triangle CDO$. Therefore, $AO = CO$ and $BO = DO$, i.e. the diagonals of a parallelogram intersect at their middle points’

Now, why can we say a) $AB \parallel DC$, b) $AB = CD$, and c) $\triangle ABO \cong \triangle CDO$?

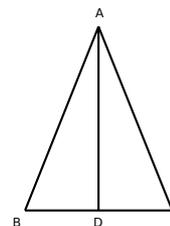
Q4 Do you accept the following argument which demonstrates that *in an isosceles triangle ABC, the base angles are equal?* (see Figure Q4). ‘Draw an angle bisector AD from $\angle A$. In $\triangle ABD$ and $\triangle ACD$, $AB = AC$, $\angle BAD = \angle CAD$ and $\angle B = \angle C$. Therefore, $\triangle ABD \cong \triangle ACD$ and hence $\angle B = \angle C$ ’. If you do not accept, then write down your reason.



Q2



Q3



Q4

In the above items, Question 1 (Q1) checks whether learners can understand difference between experimental verification and formal proof in geometry. Question 2 (Q2) checks whether learners can understand a simple proof. Q3 checks whether learners can identify assumptions, conclusions and so on in formal proof. Finally, Q4 checks whether learners can identify logical circularity within a formal proof (proof is invalid as ' $\angle B = \angle C$ ' is used to prove ' $\angle B = \angle C$ '). To achieve Level II, students have to answer Q1 correctly. Students who perform well in Q2 and Q3 can be considered at least at Level Ib as they achieve good understanding in 'Construction of proof'. Figure 1 summarizes the criteria and levels.

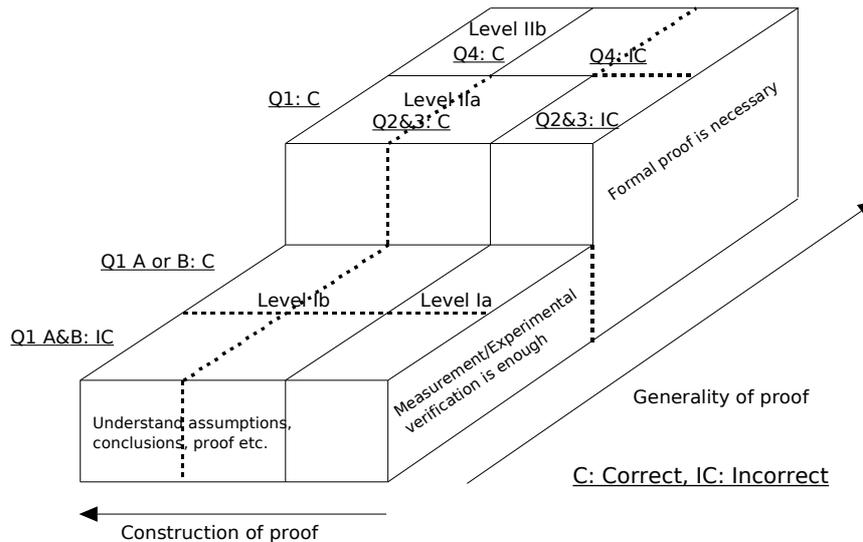


Figure 1: Criteria and levels of generality and proof construction

STUDENTS' UNDERSTANDING OF PROOF IN GEOMETRY

Student surveys were carried out in 1987, 2000 and 2005. One consistent result from these surveys is that over 60% students consider that experimental verification is enough to say it is true that the sum of the inner angles of triangle is 180 degree. Tables 1 and 2 show data collected in 2005 (with 206 students from Grade 8, and 212 students from Grade 9, collected from five different schools).

	Empirical argument using measures (Student A explanation)		Empirical argument using tearing corners (Student B explanation)		Proof (Student C explanation)	
	Accept	Not accept	Accept	Not accept	Accept	Not accept
Grade 8	62%	32%	70%	21%	74%	15%
Grade 9	36%	58%	52%	38%	80%	6%

Table 1: Results of Q1

The results in Table 1 indicate that, whereas students can accept (or understand) that a formal proof ('Student C' explanation) is a valid way of verification, many also consider experimental verification ('Student A' or 'Student B' explanation) as acceptable. There are, however, changes from Grade 8 to Grade 9, as, by the later grade, more students reject empirical arguments or demonstrations. The likely reason for this is that Grade 9 students have more experience with formal proof, whereas in Grade 8 the students are only just started studying proof (for more on this, see Fujita and Jones, 2003).

Turning now to students' understanding of 'Generality of proof' and 'Construction of proof', the results in Table 2 indicate the following:

- More than half of students can construct a simple proof (Q2).
- Students (in Q3) show relatively good performance for Q3a and Q3b, and these indicate that students have good understanding about deductive arguments of simple properties. Q3c is more difficult as students are required to have knowledge about the conditions of congruent triangles.
- The results of Q4 suggest that more than half of students cannot 'see' why the proof in Q4 is invalid; that is they cannot understand the logical circularity in this proof.

	Q2	Q3a	Q3b	Q3c	Q4
Grade 8	57%	82%	80%	53%	34%
Grade 9	63%	85%	81%	59%	49%

Table 2: Result of Q2-4

In summary, as shown in Table 3, some 90% of Grade 8 and 77% of Grade 9 students were found to be at level I. Data from surveys carried out in 1987 and 2000 show similar results (see Kunimune, 1987, 2000).

Level	Ia	Ib	IIa or above
Grade 8	33%	57%	9%
Grade 9	28%	49%	22%

Table 3: levels of understanding

The result from Grade 9 shows a slight improvement from Grade 8. Using a 2x2 cross-table in which the numbers of level Ia+Ib and IIa or above are considered, the chi-square value is 13.185 (df=1, $p < 0.01$), and this indicates that the significant improvement can be observed between Grade 8 and Grade 9.

	Level Ia+Ib	Level IIa or above
Grade 8	185	19
Grade 9	163	47

Table 4: comparing Grade 8 and Grade 9

MOVING STUDENTS TO DEDUCTIVE THINKING

As evident in a recent review of research on proof and proving by Mariotti (2007, p181), the ‘discrepancy’ between experimental verifications and deductive reasoning is now a recognized problem. Japan is not an exception to this. Our findings given above indicate that Japanese Grade 8 and 9 students are achieving reasonably well in terms of ‘Construction of proof’, but not necessarily as well in terms of ‘Generality of proof’ in geometry. There is a gap between the two aspects. This means that students might be able to ‘construct’ a formal proof, yet they may not appreciate the significance of such formal proof in geometry. They may believe that a formal proof is a valid argument, while, at the same time, they also believe experimental verification is equally acceptable to ‘ensure’ universality and generality of geometrical theorems.

Our data for Grade 9 students can be considered as quite concerning, given 80% of students remain at level I in terms of their understanding of proof even after they have studied formal proof at Grade 8 using textbooks where 90% of relevant intended lessons can be devoted for ‘justifying and proving’ geometrical facts’ (Fujita and Jones, 2003). However, we would like to stress that we are still encouraged by the result that 20% of Japanese students achieve relatively sound understanding of proof through everyday mathematics lessons.

Hence, in our research, we turn to the question of working with students on why formal proof is needed. Based on over 10 years of classroom-based research, Kunimune *et al* (2007) propose the following principles for lower secondary school geometry (Grades 7-9) designed to help students appreciate the *need* for formal proofs (in addition to the students being able to construct such proofs):

- Grade 7 lessons to start from problem solving situations such as ‘consider how to draw diagonals of a cuboid’, and so on; *this develops students’ geometrical thinking and provides experiences of mathematical processes that are useful in studying deductive proofs in Grades 8 and 9;*
- Geometrical constructions to be taught in Grade 8; *this replaces the practice of teaching constructions in Grade 7, and then proving these same constructions in Grade 8, as such a gap between the teaching of constructions and their proofs has been found by classroom research to be unhelpful;*
- Grade 8 lessons to provide students with explicit opportunities to examine differences between experimental verifications and deductive proof; *this helps students to appreciate such differences;*
- Grade 8 lessons to start the teaching of the teaching of deductive geometry with a set of already learnt properties which are shared and discussed within the classroom, and used as a form of axioms (a similar idea to that of the ‘germ theorems’ of Bartolini Bussi, 1996); *this provides students with known starting points for their proofs.*

While we do not have space in this paper to provide data to support all these

principles, in what follows we substantiate those related to differences between experimental verifications and deductive proof in geometry.

Constructions and proofs

In our experience (Shinba, Sonoda and Kunimune, 2004), while geometrical constructions (with ruler and compasses) can be taught in Grade 7, these constructions are often not proved until Grade 8 (after students have learnt how to prove simple geometrical statements). In a series of teaching experiments, we investigated the use of more complex geometrical constructions (and their proofs) in Grade 8. As an example, one of our lessons in Grade 8 started from the more challenging construction problem ‘Let us consider how we can trisect a given straight line AB’.

In our classroom studies, we observed that such lessons are more active for the students. The students could also experience some important processes which bridge between conjecturing and proving. Students could first investigate theorems/properties of geometrical figures through construction activities, and this led them to consider *why* the construction worked. By appropriate instructions by the teachers, the students then started proving the constructions. For example:

Student C: I thought that I could trisect AB when I constructed this (No. 11 in Figure 2), but I think I found this is not true. So I prove that we cannot trisect the line AB. We just saw the construction No. 8 is true, so I use this approach in my proof. Now, I draw an equilateral triangle on AB (No. 11'), and by doing this, we can trisect the AB, and proof is similar to No. 8. Now, compare to this (No. 11') to my construction, and C and D are not in the same place, as the height of the triangle ACB is shorter than the height of the square. We know we can trisect the AB by using this approach, and therefore, my method (No. 11) does not work.

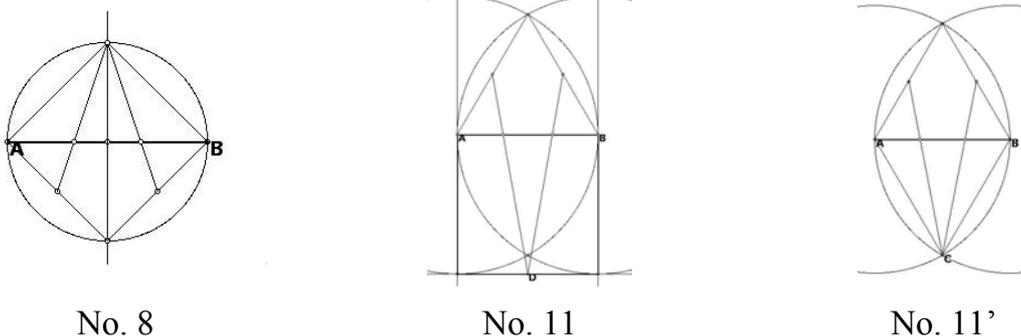


Figure 2: Constructions proposed by students²

The data extract above shows that some students in this class start using an already proved statement (i.e. a theorem) to justify why the construction (No. 11 in Figure 2) does not work to trisect the line AB.

Making explicit the differences amongst various argumentations

In a series of lessons for 41 Grade 8 students, tasks were designed and

implemented to disturb students' beliefs about experimental verification. In the lessons, students were asked, for example, to compare and discuss various ways of verifying the geometrical statement that the sum of the inner angles of triangles is 180 degrees (this relates to Q1 in the research questionnaire). The angle sum statement was chosen as way of trying to bridge the gap between empirical and deductive approaches, given that students often encounter the angle sum statement in primary schools and they study this again with deductive proof in lower secondary school. While we do not have space in this paper to provide the data from the study, we can provide a summary of ways which can be useful in encouraging students to develop an appreciation of why formal proof is necessary in geometry (for more details, see Kunimune, 1987; 2000).

- Students first exchange their ideas on various ways of verification; they comment on accuracy or generality of experimental verification; they discuss the advantages/disadvantages of experimental verifications.
- Students' comments such as 'A protractor is not always accurate ...', 'It takes time to measure angles, and we cannot see the reason why', 'The triangle is not general', and so on, often cause a state of disequilibrium in students (*viz* Piaget), and make students doubt the universality and generality of experimental verification.
- Students made various comments on the argument based on 'cutting each angles and fitting them together' (Q1-b). For example, 'I think this is an excellent method as I cannot see any problems in this method', 'This is an easy method to check (whether the sum of inner angles of triangles is 180 degree)', 'I think this is a good way, but because we use a piece of paper, I think it can be sometimes inaccurate', and so on.
- Advice from teachers is necessary to encourage students to reflect critically on different ways of verifications (*viz* establishment of 'social norm' in classrooms, Yackel and Cobb, 1996).

Kunimune (1987; 2000) found that, after such lessons, around 40% of students previously at Level Ib have moved to Level II (post-test I). They no longer accept experimental verification and start considering that deductive proof as the only acceptable argument in geometry. A later post-test (post-test II) carried out one month after the lessons found that about 60% of students are at Level IIa. Table 4 (below) summarises the result of the pre and post-tests with five types of cognitive changes observed among students in terms of the levels of understanding of proof in geometry.

An interesting observation is the type *d* in which three students show unexpected behaviour in terms of their cognitive development in that there was a regression from level IIa to Ib. A detailed reason for this is unknown, but, unlike the majority of students, it might be that their states of disequilibrium created rather a 'negative' effect for these students.

In summary, we conclude that the matter of the 'Generality of proof' could

usefully be explicitly addressed in geometry lessons in lower secondary schools.

Type	Pre-test	Post-test I	Post-test II	N
a	Level II	Level II	Level II	2
b	Level I	Level II	Level II	13
c	Level I	Level I	Level II	9
d	Level I	Level II	Level I	3
e	Level I	Level I	Level I	14
Level II	2	18	24	

Table 4: Results from Pre- and Post tests

CONCLUDING COMMENTS

This paper outlines research findings from Japan suggesting that, in terms of ‘Generality of proof’ and ‘Construction of proof’, many students in lower secondary school remain at Level I where they hold the view that experimental verifications are enough to demonstrate that geometrical statements are true, even after intensive instruction in how to proceed with proofs in geometry. Classroom studies have tested ways of challenging such views about empirical ways of verification which indicate that it is necessary to establish classroom discussions to disturb students’ beliefs about experimental verification and to make deductive proof meaningful for them.

NOTES

1. Some papers by Kunimune (1987; 2000) are written in Japanese; this paper, one of outcomes of our collaborative work over five years, contains his main ideas.
2. In No 8 AB is trisected by constructing a square whose diagonal is AB, and joining a vertex and midpoints; In No 11, an equilateral triangle and a square are constructed on AB; In No. 11’, AB is trisected by constructing equilateral triangles on AB, and joining a vertex and midpoints.

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WRITTEN REPORT IN LEARNING GEOMETRY: EXPLANATION AND ARGUMENTATION

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In this article, we examine how the written report, within the context of assessment for learning, helps students in learning geometry and in developing their explanation and argumentation skills. We present the results of a qualitative case study involving Portuguese students of the 8th grade. This study suggests that using written reports improves those capabilities and, therefore, the comprehension of geometric concepts and processes. These benefits for learning are enhanced through the implementation of some assessment strategies, namely oral and written feedback.

Key-words: Geometric thinking, explanation, argumentation, assessment for learning, written reports.

INTRODUCTION

Explanation, argumentation and proof are mathematics activities that assume a main role in the teaching and learning of geometry, but present a lot of difficulties to students (Battista, 2007). The need to implement an assessment that contributes to students' learning is also widely recognized: an assessment that guides the students and helps them to improve their learning (William, 2007). As such, in this study, we attempted to understand how the written report, as a tool of assessment for learning, contributes to learning geometry and, in particular, reinforces the development of students' explanation and argumentation processes.

The present study follows a wider one that aimed at understanding the key role of the written report as an assessment tool supporting the learning of 8th grade students (aged thirteen) in mathematics. The larger study was developed during the academic year 2007/2008 under the scope of project AREA [1].

EXPLANATION, ARGUMENTATION AND PROOF IN TEACHING AND LEARNING GEOMETRY

All over the world and in Portugal, in particular, the mathematics curriculum recognizes geometry as a privileged field for the development of explanation, argumentation and proof (NCTM, 2000; DGIDC, 2007). Battista and Clements (1995) notice the need to shape the curriculum in order to develop students' explanation and argumentation skills and so that students use proof to justify powerful ideas. According to Polya (1957) mathematical proof should be taught because it helps in: (i) acquiring the notion of intuitive proof and logical reasoning; (ii) understanding a logical system; and (iii) keeping what is learnt in one's memory.

Many authors have addressed geometrical thought based on Van Hiele's model. This model proposes a sequential progression in learning geometry through five discrete and qualitatively different levels of geometrical thinking: visual, descriptive/analytic, abstract/relational, formal deduction and rigor. However, according to Freudenthal (1991), these are relative levels, not absolute ones. Nevertheless, "the levels can help to find and further develop appropriate tasks (...) and they are obviously helpful for explorative activities to come across new, maybe even innovative ideas" (Dorier *et al.*, 2003, p. 2). This progression is determined by the teaching process, thus the teacher has a key role in setting appropriate tasks so that students may progress to higher levels of thought and walk towards proof. The learning of deductive proof in mathematics is complex and its progress is neither linear nor free of difficulties (Küchemann & Hoyle, 2002, 2003). As regards explanation, we may consider several modes, including non-explanations (where, for example, students refer to the teacher's authority), explaining how, explaining to someone else (spontaneously) and explaining to oneself (in response to a question) (Reid, 1999). Argumentation is viewed as an intentional explication of the reasonings used during the development of a mathematical task (Forman *et al.*, 1998).

ASSESSMENT FOR LEARNING

Current mathematics curriculum documents advocate an assessment whose main purpose is to support students' learning, and whose forms constitute, at the same time, learning situations (DGIDC, 2007; NCTM, 1995, 2000). "Assessment in education must, first and foremost, serve the purpose of supporting learning" (Black & William, 2006, p. 9). In this study, assessment for learning is seen as "all the intent that, acting on the mechanisms of learning, directly contributes to the progression and/or redirection of learning" (Santos, 2002, p. 77). Several studies show that the focus on assessment for learning, as opposed to an assessment of learning, may produce substantial improvement in the performance of students (Black & William, 1998).

In order to develop their own knowledge about thinking mathematically, students need to develop a conscious, reflective practice, which encompasses the processes of self-assessment. According to Hadji (1997), self-assessment is an activity of reflected self-control over actions and behaviours on behalf of the individual who is learning. Santos (2002) stresses that self-assessment implies that one becomes aware of the different moments and aspects of his/her cognitive activity, therefore it is a meta-cognitive process. A non-conscious self-control action is a tacit, spontaneous activity that is natural in the activity of any individual (Nunziati, 1990), and in this sense all human beings self-assess themselves. Meta-cognition goes beyond non-conscious self-control, for it is a conscious and reflective action (Nunziati, 1990).

Some assessment strategies can be adopted to promote learning, including: a positive approach of the error; oral questioning of students; feedback; negotiation of assessment criteria; and the use of alternative and diversified assessment instruments (Black *et al.*, 2003; Santos, 2002). In particular, the written report is a privileged

instrument to monitor students' learning. Students' work on written reports has advantages in terms of developing their explanation and argumentation skills, which are two intrinsic requests of this instrument; furthermore, written reports may help students to reflect upon their work, because time and space are given (Mason, Burton & Stacey, 1982). "Intensive approach to argumentative skills, relevant for mathematical argumentation, seems to be possible through an interactive management of students' approach to writing" (Douek & Pichat, 2003). The description of thinking processes, with the identification of the strategies used to solve a given task, including the difficulties that were encountered and the mistakes that were made, allows students to rethink their learning process. However, it is desirable that a report be done in "two stages" to allow for an effective opportunity for learning. This means that a first version of the report is subject to the teacher's feedback and then the student develops a new version, a second one, taking into account the feedback received (Pinto & Santos, 2006).

METHODOLOGY

This study was based on an interpretative paradigm and on a qualitative approach. We chose the case study for the design research, given the nature of the problem to study and the desired final product (Yin, 2002).

The research involved an 8th grade class, with 24 students. We selected four of these students based on different mathematical performances, and taking into account their mathematics communication skills. These students were Maria, Rute, Duarte, and Telmo, and they constituted a working group in the classroom.

Data were collected through lesson observation, namely, the lesson dedicated to the discussion of the guidelines for preparing the report and of the assessment criteria, and the lessons dedicated to carrying out tasks as well as the first and second versions of the reports. Three individual interviews to each of the four students were made, the first one at the beginning of the school year and the others after the establishment of the second version of each report. Two tasks led to the development of two written reports, each one with two versions.

The data were subjected to several levels of analysis that took place periodically (Miles & Huberman, 1994), based on categories defined a posteriori that arose from the data gathered, keeping in mind the focus of the study and the theoretical framework.

PEDAGOGICAL CONTEXT

Since the writing of a report was a novelty for the students, they were given a set of guidelines for writing the report and the assessment criteria. These two documents were discussed with the students. According to the guidelines, the organization of the report should include three parts: introduction, development, and conclusion. Both first two parts, and the tasks that originated the report, should be produced within the group. The last part should be held individually and it included students' self-

assessment. The reports were produced in two "stages", the students benefiting from the teacher's comments to the first stage in order to improve the second one. Students were not required to do any proof, but were asked to provide explanations for their thinking (Küchemann & Hoyle, 2003).

The first task proposed an investigation of possible generalizations of the Pythagorean theorem. Students were asked to remember and to reflect upon the relationship between the areas of the squares constructed on the sides of a right triangle, and to investigate what happens if they construct other geometric figures on the sides of a right triangle. The second task was a problem that involves the application of the Pythagorean theorem in space. Students were asked to construct a cone based on one of the three equal sectors of a circle, with a radius of six centimetres, and to determine the height of the constructed cone. They were also encouraged to explain how they could determine the height of a cone obtained from a circle with a radius r . These tasks were chosen based on the assumption that presenting students with unfamiliar questions can provide a rich context for classroom discussion which helps students in developing mathematical arguments (Küchemann & Hoyle, 2003).

The first report

In the first task, students reflect on the meaning and implications of the Pythagorean theorem and review some geometric concepts and procedures (such as what an equilateral triangle is and how it can be constructed with ruler and compass). Due to the nature of the task, the group is still required to formulate and test conjectures, and to argue in favour of their ideas, thus appealing to students' mathematical reasoning skills. In particular, when writing the report, the students, in group, explain how they exploited the first situation proposed in the task, concerning equilateral triangles built on the sides of a right triangle.

In the first version of their report, students described how they had built the equilateral triangles and stated how they had determined the areas of those triangles:

We started by making a right triangle, with the help of a compass we drew around it (at the endpoints of the right triangle) three equilateral triangles, because we couldn't obtain equilateral triangles nor a good graphic design by using rules. We determined the area of the triangles.

The justification for the use of compass comes in the wake of some oral feedback provided during the preparation of the report. This feedback may have helped the students to explain their options:

Rute: We did it like this: with the help of the compass, we made around it three equilateral triangles. Then we can put... ah...

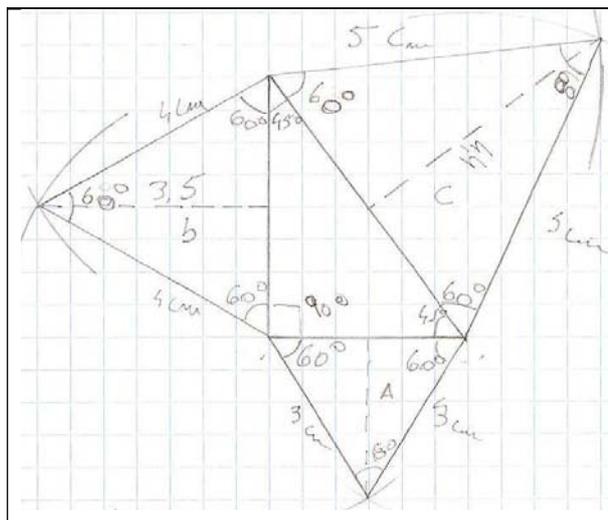
Teacher: Why did you use the compass?

Rute: Because we couldn't complete the task with the ruler only.

Teacher: So, couldn't you draw a triangle with the ruler only?

Rute: Yes, but in order to be an equilateral triangle, it had to have all equal sides.

In an attached document to their report, the group presented the construction of equilateral triangles, as well as the values of the basis and the height considered in each one. It also presented the calculations that were made to determine the corresponding areas.



However, in any part of the report, did the students explain how they had found the values of the bases and heights, nor what conclusions they obtained from the areas determined. Two different comments were provided to the first version of the report. On the one hand, the teacher praised students for their use of a compass and the reasons for their choice: "You did an excellent option. It's a good way to answer a problem that you had to overcome." In this way, the teacher identified positive aspects of the report, so that knowledge could be consciously recognized by students and their self confidence could be promoted (Santos, 2003). On the other hand, the teacher questioned students about the conclusions they had drawn from the areas obtained: "And what did you find?". Furthermore, the teacher still posed some questions written near the construction of the triangles, which sought to guide the work of students in order to include the missing information in the report: "How did you come to these figures? Which relationship may you establish?"

While working on the second version of their report, the students kept the description that had been praised and tried to answer the questions. They explained in more detail how they had proceeded, namely in finding the values of the basis and height of the triangles, in determining the corresponding areas in each equilateral triangle, and in making explicit the conclusions they had obtained for the first situation:

We determined the area of the triangles. We know that in order to determine the area of a triangle: $\frac{\text{basis} \times \text{height}}{2}$, we measured the height and the basis, we multiplied and then we divided by 2 (and likewise for the three triangles). We concluded that the sum of area A and area B is equal to area C.

In the final version, the students determined and identified the value of the area of each one of the considered triangles and explained the relationship found among the areas of the equilateral triangles constructed on the sides of the right triangle. This work was based on the figure of the first version:

$A_{\triangle} = 4 \times 3,5 = 7 \text{ cm}^2$
 $A_{\triangle} = 3 \times 2,5 = 3,75 \text{ cm}^2$
 $A_{\triangle} = 5 \times 4,3 = 10,75 \text{ cm}^2$
 $A_{\triangle} + A_{\triangle} = A_{\triangle}$
 $3,75 + 7 = 10,75 \text{ cm}^2$

A soma da área
 A e área B é
 equivalente a área
 C.

The sum of area A and area B is equivalent to area C.

Students still added a comment. They identified the negative aspects of the first version and they improved them in the second stage: “[In the first stage] we didn’t present the value for the areas, we messed up the computations, and we did not present the conclusions.” The students identified and corrected their own mistakes.

The second report

In the second task, the students review and apply the Pythagorean theorem as well as some mathematical concepts and procedures (such as, the height of a cone or the perimeter of a circle given its radius). Due to the nature of the task, it calls, mostly, for problem-solving and mathematical reasoning skills.

In the report, the students explained how they had built the cones and sought reasons for their actions. In particular, they explain how to determine the angle of each of the three circular sectors:

We started by reading the task and answering to what had been requested. We drew a circle of radius 6 cm. To divide the angle into three equal parts, we know that the angle measures 360° : (so $\frac{360^\circ}{3} = 120^\circ$). With the help of a protractor, we measured, on the radius, 120° three times and joined the points and we got 3 equal parts. Then, we cut the three parts, and with the help of some tape, we constructed three cones.

Then, the students described the strategy implemented to determine the height of the cones. Before moving to the resolution itself, they made a brief description of how the group had addressed the issue, referring various ideas discussed and some difficulties encountered, which they sought to overcome with the help of the teacher. Then they determined the radius of the basis of the cone, giving the necessary calculations (determining the perimeter of the original circle, the perimeter of the basis of the cone and, finally, the radius of the basis of the cone).

$2\pi r = \text{perimetro do círculo}$
 $2\pi 6 = 37,68$
 $\frac{\text{Perimetro do cone}}{3} = \frac{\text{perimetro do cone}}{3}$
 $\frac{37,68}{3} = 12,56$

$2\pi r = \text{perimetro do círculo}$
 $\frac{\text{Perimetro do cone}}{2\pi} = R$
 $\frac{12,56}{2\pi} = 1,9$

However they did not explain the calculations nor did they give reasons for those calculations; they did not distinguish the two circles involved (the original one and the basis of the cone), nor did they present units of measurement. Written feedback was provided with the intention of alerting students to these aspects: "Why did you do these calculations? You refer the perimeter of the circle several times. Maybe it would be better to distinguish which circle you are talking about in each situation. Attention to the lack of measurement units". The importance of students' explanation and justification of their calculations was further strengthened through oral feedback:

Teacher: "(...) you must try to explain the calculations you presented better and why you have done them". You presented these calculations, didn't you? For what? When? How?

Rute: The teacher wants to know everything!

Teacher: I want to know everything, no... Imagine that I'm teaching a lesson and I write something on the blackboard, and then you ask me "teacher, what is that?" and I say "You want to know everything!", right?

Rute: Teacher, but, here, we already know that this is the perimeter...

Teacher: You know, but you must write what you mean. I am not going to take Rute home to explain it to me, right?

It was also necessary to complement the written feedback with new clues, so that the students could distinguish the different circles considered in the resolution of the problem:

Rute: Teacher, how do we distinguish the circles?

Teacher: Which circles did you work with?

Rute: With the one with radius six.

Teacher: Yes. And didn't you work with any other circle?

Rute: With the basis.

Teacher: The basis?

Rute: Yes, of the cone.

Teacher: So, in the report, you only have to say which one you are referring to when you explain what you did.

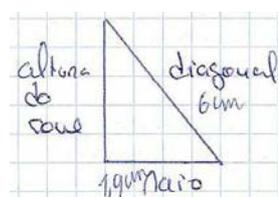
The students took into account the feedback received, both oral and written. In the final version of the report, besides adding the measurement units, they described how they had proceeded to determine the radius of the basis of the cone. They clarified the context, they explained the purpose of the calculations they had presented, and they also identified the circle referred in each case:

First we found the perimeter of the circle of the problem. Then we divided the perimeter of the circle of the problem into three equal parts, and we got the perimeter of

the basis of a cone. Knowing that to find the perimeter of the circle is $2\pi r$, to find the radius is the other way around: $P \div 2\pi = r$. And then, we obtained 1,9 cm.

In the first version of the report, students had already tried to describe in detail the right triangle used to determine the height of the cone and they explained how they had determined the length of the hypotenuse (which they refer to as diagonal) of that triangle:

If we draw the height of the cone, it will coincide with the radius forming an angle of 90° . If, at the endpoints of the lines, we draw a line segment, it will form a right triangle and, for our own luck, it was the diagonal, which we knew about.



We know that the diagonal measures 6 cm because the diagonal is the radius of the circle when we open the cone, and, as the radius of the circle is 6 cm, we got to know the diagonal.

Finally, the students presented the necessary calculations to determine the height of the cone, but they did not mention how they had concluded that “height of the cone² = diagonal² - radius²”. They were reminded of this fact through written feedback: "How do you achieve this equality?" In the final version of the report, the students considered the feedback received and stated that they had used the Pythagorean theorem to obtain the height of the cone.

DISCUSSION OF RESULTS

In this study, students were asked to describe and explain the strategies used in the implementation of two tasks and to submit the results, duly substantiated, under the form of written reports. Students, working in a group, were given constructive comments on the first version of their reports so that they could improve their work and develop a second version. In many cases, in the first version of the reports, students gave procedural explanations instead of providing a mathematical justification (Hoyle & Küchemann, 2003). In other words, they presented how they had done their work, but not why. For example, in the first version of the report regarding the first task, students described how they had built the equilateral triangle, but they did not mention the characteristics of this figure. In the second version of the report, students presented mathematical arguments for the choices made and for the results found in performing the tasks. They also used symbolic language of mathematics when necessary (it happened, for example, when they obtained the area of equilateral triangles in the first task or when they obtained the height of the cone in the second task). However, in both cases, they seemed to be, mainly, at the descriptive/analytic level of Van Hiele’s geometrical thinking model.

Feedback, both oral and written, allowed students to identify aspects to improve in the reports and provided clues about what students could do to develop their first productions. Indeed, feedback seems to have enabled students to produce a better report in the second version, especially regarding explanation and justification of the strategies adopted (it should be noted, for example, the explanation given, in the final

version, to the operation performed in the first phase to obtain the radius of the basis of the cone, starting from its perimeter). In addition, the feedback did not contain any information about errors; it only included guiding questions and comments (Black *et al.*, 2003; Santos, 2003). This led students to identify mistakes and to correct them (as is evident in the first task, in which the students relate what they had done wrong in the first version). Thus, feedback also promoted the development of students' reflection and self-assessment skills (Nunziati, 1990).

The need for students to explain and justify, in written form, the mathematical procedures and results involved in performing mathematically rich tasks caused a high level of demand and consequently of learning. These situations, which involve knowledge that students possibly know, but which they need to explain and justify, have a strong didactic purpose (Küchemann & Hoyle, 2003). The identified benefits associated with the written reports seem to be enhanced by investing on a type of report in "two stages", in which oral and written feedback gain prominence.

NOTES

1. The project AREA (Monitoring Assessment in Teaching and Learning) is a research project funded by the Foundation for Science and Technology (PTDC/CED/64970/2006). The main objectives of the project are to develop, implement and study practices of assessment that contribute for learning. Further information can be found in <http://area.fc.ul>.

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MULTIPLE SOLUTIONS FOR A PROBLEM: A TOOL FOR EVALUATION OF MATHEMATICAL THINKING IN GEOMETRY

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Based on the presumption that solving mathematical problems in different ways may serve as a double role tool - didactical and diagnostic, this paper describes a tool for the evaluation of the performance on multiple solution tasks (MST) in geometry. The tool is designed to enable the evaluation of subject's geometry knowledge and creativity as reflected from his solutions for a problem. The example provided for such evaluation is taken from an ongoing large-scale research aimed to examine the effectiveness of MSTs as a didactical tool. Geometry is a gold mine for MSTs and therefore an ideal focus for the present research, but the suggested tool could be used for different mathematical fields and different diagnostic purposes as well.

Introduction

The study described in this paper is a part of ongoing large-scale research (Anat Levav-Waynberg; in progress). The study is based on the position that solving mathematical problems in different ways is a tool for constructing mathematical connections, on the one hand (Polya, 1973, 1981; Schoenfeld, 1988; NCTM, 2000) and on the other hand it may serve as a diagnostic tool for evaluation of such knowledge (Krutetskii, 1976). In the larger study we attempt to examine how employment of Multiple-solution tasks (MSTs) in school practice develops students' knowledge of geometry and their creativity in the field. In this paper we present the way in which students' knowledge and creativity are evaluated.

Definition: MSTs are tasks that contain an explicit requirement for solving a problem in multiple ways. Based on Leikin & Levav-Waynberg (2007), the difference between the solutions may be reflected in using: (a) Different representations of a mathematical concept; (b) Different properties (definitions or theorems) of mathematical concepts from a particular mathematical topic; or (c) Different mathematics tools and theorems from different branches of mathematics.

Note that in the case of MSTs in geometry we consider different auxiliary constructions as a difference of type (b).

Solution spaces

Leikin (2007) suggested the notion of "solution spaces" in order to examine mathematical creativity when solving problems with multiple solution approaches as follows: *Expert solution space* is the collection of solutions for a problem known to the researcher or an expert mathematician at a certain time. This space may expand as new solutions to a problem may be produced. There are two types of sub-sets of expert solution spaces: The first is *individual solution spaces* which are of two

kinds. The distinction is related to an individual's ability to find solutions independently. *Available solution space* includes solutions that the individual may present on the spot or after some attempt without help from others. These solutions are triggered by a problem and may be performed by a solver independently. *Potential solution space* include solutions that solver produce with the help of others. The solutions correspond to the personal zone of proximal development (ZPD) (Vygotsky, 1978). The second subset of an expert space is *a collective solution space* characterizes solutions produced by a group of individuals.

In the present study solution spaces are used as a tool for exploring the students' mathematical knowledge and creativity. By comparing the individual solution spaces with the collective and expert solution spaces we evaluate the students' mathematical knowledge and creativity.

MST and mathematics understanding

The present study stems from the theoretical assumption that mathematical connections, including connections between different mathematical concepts, their properties, and representations form an essential part of mathematical understanding (e.g., Skemp, 1987; Hiebert & Carpenter, 1992; Sierpiska, 1994). Skemp (1987) described understanding as the connection and assimilation of new knowledge into a known suitable schema. Hiebert & Carpenter (1992) expanded this idea by describing mathematical understanding as “networks” of mathematical concepts, their properties, and their representations. Without connections, one must rely on his memory and remember many isolated concepts and procedures. Connecting mathematical ideas means linking new ideas to related ones and solving challenging mathematical tasks by seeking familiar concepts and procedures that may help in new situations. Showing that mathematical understanding is related to connectedness plays a double role: it strengthens the importance of MSTs as a tool for mathematics education and it justifies measuring mathematics understanding by means of observing the subjects' mathematical connections reflected from one performance on MSTs..

Why geometry

The fact that *proving* is a major component of geometry activity makes work in this field similar to that of mathematicians. The essence of mathematics is to make abstract arguments about general objects and to verify these arguments by proofs (Herbst & Brach, 2006; Schoenfeld, 1994).

If proving is the main activity in geometry, *deductive reasoning* is its main source. Mathematics educators claim that the deductive approach to mathematics deserves a prominent place in the curriculum as a dominant method for verification and validation of mathematical arguments, and because of its contribution to the development of logical reasoning and mathematics understanding (Hanna, 1996; Herbst & Brach, 2006). In addition to these attributes of geometry, which make it a

meaningful subject for research in mathematics education, geometry is a gold mine for MSTs and therefore an ideal focus for the present research.

Assessment of creativity by using MST

Mathematical creativity is the ability to solve problems and/or to develop thinking in structures taking account of the peculiar logico-deductive nature of the discipline, and of the fitness of the generated concepts to integrate into the core of what is important in mathematics (Ervynck, 1991, p.47)

Ervynck (1991) describes creativity in mathematics as a meta-process, external to the theory of mathematics, leading to the creation of new mathematics. He maintains that the appearance of creativity in mathematics depends on the presence of some preliminary conditions. Learners need to have basic knowledge of mathematical tools and rules and should be able to relate previously unrelated concepts to generate a new product. The integration of existing knowledge with mathematical intuition, imagination, and inspiration, resulting in a mathematically accepted solution, is a creative act.

Krutetskii (1976), Ervynck (1991), and Silver (1997) connected the concept of creativity in mathematics with MSTs. Krutetskii (1976) used MSTs as a diagnostic tool for the assessment of creativity as part of the evaluation of mathematical ability. Dreyfus & Eisenberg (1986) linked the aesthetic aspects of mathematics (e.g., elegance of a proof/ a solution) to creativity. They claim that being familiar with the possibility of solving problems in different ways and with their assessment could serve as a drive for creativity. In sum, MSTs can serve as a medium for encouraging creativity on one hand and as a diagnostic tool for evaluating creativity on the other.

According to the Torrance Tests of Creative Thinking (TTCT) (Torrance, 1974), there are three assessable key components of creativity: fluency, flexibility, and originality. Leikin & Lev (2007) employed these components for detecting differences in mathematical creativity between gifted, proficient and regular students in order to explain how MSTs allow analysing students' mathematical creativity, and thus serve as an effective tool for identification of mathematical creativity.

Fluency refers to the number of ideas generated in response to a prompt, *flexibility* refers to the ability to shift from one approach to another, and *originality* is the rareness of the responses.

In order to assess mathematical thinking in the Hiebert & Carpenter (1992) and Skemp (1987) sense, while evaluating problem solving performance of the participants on MSTs, we added the criterion of connectedness of mathematical knowledge which is reflected in the overall number of concepts/theorems used in multiple solutions of a MST.

In this paper we outline the use of MSTs as a research tool for evaluation of mathematical knowledge and creativity in geometry.

Method

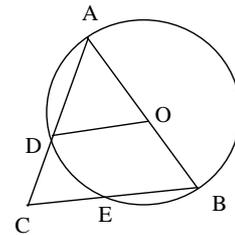
Following MST instructional approach, three 60 minutes tests were given to 3 groups of 10th grade, high-level students during geometry course (total number of 52 students). The first test was admitted in the beginning, the second in the middle and the third in the end of the course. Each test included 2 problems on which students were asked to give as many solutions as they can.

Example of the task

The following is one of the MSTs used for the tests

TASK:

AB is a diameter on circle with center O. D and E are points on circle O so that $DO \parallel EB$. C is the intersection point of AD and BE (see figure).



Prove *in as many ways as you can* that $CB=AB$

Examples of the solutions

Solution 1:

$DO = \frac{1}{2} AB$ (Equal radiuses in a circle) \Rightarrow DO is a midline in triangle ABC (parallel to BC and bisecting AB) $\Rightarrow DO = \frac{1}{2} AB = \frac{1}{2} BC \Rightarrow AB=BC$

Solution 2:

$DO=AO$ (Equal radiuses in a circle) $\Rightarrow \angle AOD = \angle ABC$ (Equal corresponding angles within parallel lines) $\Rightarrow \angle A = \angle A$ (Shared angle) $\Rightarrow \triangle AOD \sim \triangle ABC$ (2 equal angles) $\Rightarrow AB=BC$ (a triangle similar to an isosceles triangle is also isosceles)

Solution 3:

$DO=AO$ (Equal radiuses in a circle) $\Rightarrow \angle ADO = \angle A$ (Base angles in an isosceles triangle)
 $\angle ADO = \angle ACB$ (Equal corresponding angles within parallel lines), $\angle ACB = \angle A \Rightarrow$
 $AB=BC$ (a triangle with 2 equal angles in isosceles)

Solution 4:

Auxiliary construction: continue DO till point F so that DF is a diameter. Draw the line FB (as shown in the figure)

$DO=AO$ (Equal radiuses in a circle) $\Rightarrow \angle ADO = \angle A$ (Base angles in an isosceles triangle)
 $\angle F = \angle A$ (Inscribed angles that subtend the same arc) \Rightarrow
 $\angle F = \angle ADO \Rightarrow CD \parallel BF$ (equal alternate angles)
 DFBC is a parallelogram (2 pairs of parallel sides) \Rightarrow
 $DF=CB$ (opposite sides of a parallelogram), $DF=AB$ (diameters) $\Rightarrow AB=BC$

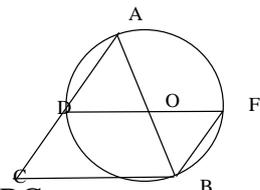


Figure 1: Example of MST

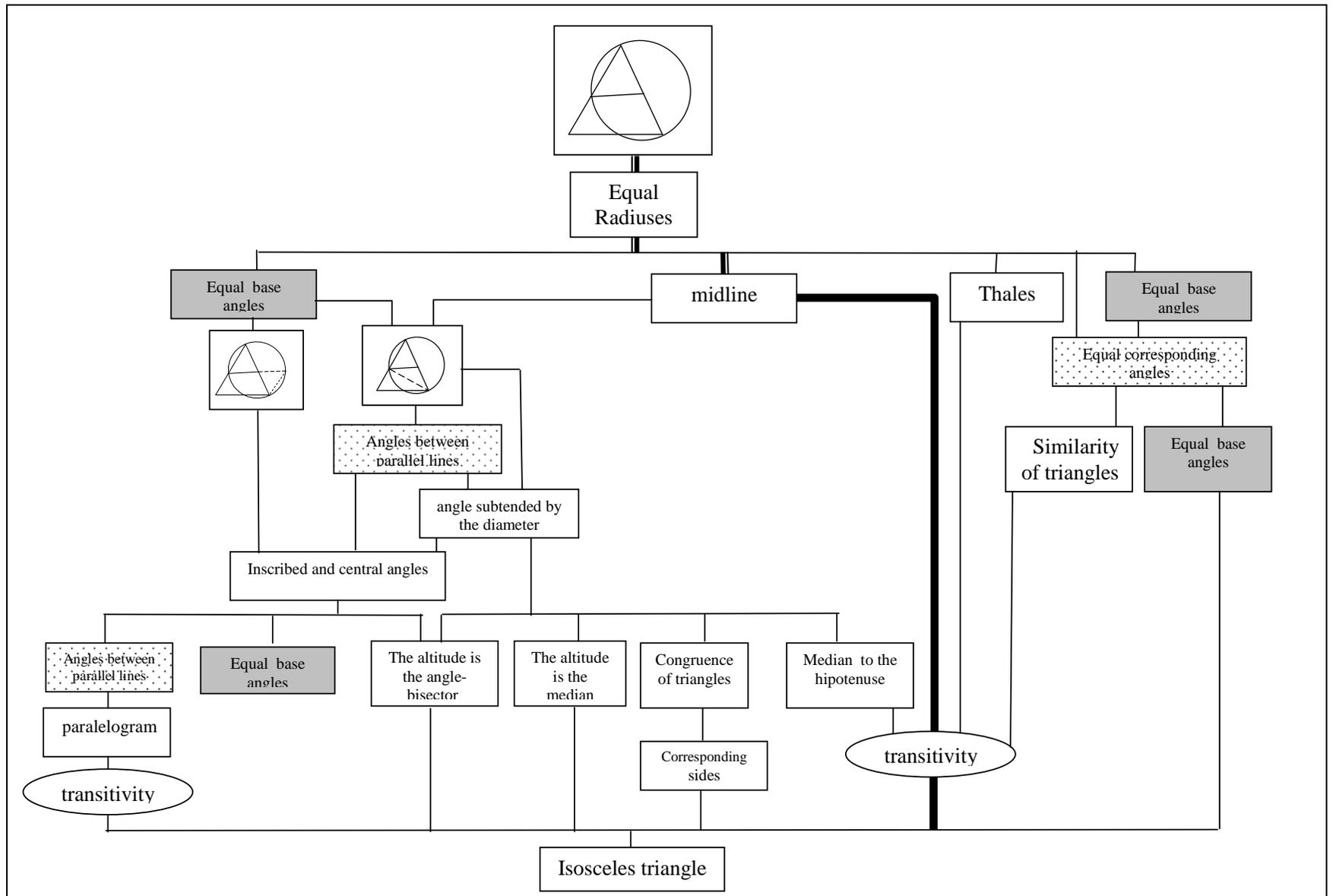


Figure 2: The map of an expert solution space for the task (see Figure 1)

Figure 1 presents an example of a task used in this study. Figure 2 depicts a map of the expert solution space for this task. The map outlines concepts and properties used in all the solutions as well as the order of their use in each particular solution (for additional maps of MSTs see Leikin, Levav-Waynberg, Gurevich and Mednikov, 2006).

The bold path in the map (Figure 2) represents Solution 1 of the task (see Figure 1).

Data analysis

	Correctness	Connectedness	Creativity		
			Fluency	Flexibility groups of solutions should be defined	Originality According to (P) frequency (conventionality) of a strategy used
Scores per solution	1-100	--	1	Flx=10 for the first solution Flxi=10 solutions from different groups of strategies Flxi=1 solutions from the same group – meaningfully different subgroups Flxi=0.1 solutions from the same group-similar subgroups	Ori=10 P<15% Ori=1 15%<P<40% Ori=0.1 P>40%
Final score for a solution group (strategy used by a student including its repetitions)	--	$\frac{t}{T} * 100$		$Flx = \sum Flx_i$	$Or = \sum Or_i$
				$Flx_i \times Or_i$	
Score per individual solution space for a problem (SPI)	Correctness= Max score among the solution for a problem	$Con = \frac{t}{T} * 100$		$n \times \sum (Fl_i \times Or_i)$	
<i>n</i> : number of solutions in the individual solution space <i>N</i> : number of the students in a group <i>T</i> : number of concepts and their properties used in the expert solution space <i>t</i> : number of concepts and their properties used in the individual solution space			<i>m_i</i> : the number of students who used the strategy <i>i</i> $P = \frac{m_i}{N} 100\%$		

Figure 3: Scoring scheme for the evaluation of problem-solving performance on a particular MST based on Leikin (forthcoming)

The analysis of data focuses on the student's individual solution spaces for each particular problem. The spaces are analyzed with respect to (a) Correctness; (b) Connectedness; (c) Creativity including fluency, flexibility, and originality.

The maximal correctness score for a solution is 100. It is scored according to the preciseness of the solution. When solution is imprecise but lead to a correct conclusion we consider it as *appropriate* (cf. Zazkis & Leikin, 2008). The highest correctness score in an individual solution space serves as the individual's total correctness score on the task. This way a student who presented only 1 correct solution (scored 100) does not get a higher correctness score than a student with more solutions but not all correct. Connectedness of knowledge associated with the task is determined by the total number of concepts and theorems in the individual solution space. Figure 3 depicts scoring scheme for the evaluation of problem-solving performance from the point of view of correctness, connectedness and creativity. The scoring of creativity of a solutions space is borrowed from Leikin (forthcoming). In order to use this scheme the expert solution space for the specific MST has to be divided into groups of solutions according to the amount of variation between them so that similar solutions are classified to the same group. The number of *all the appropriate solutions* in one's individual solution space indicates one's fluency while flexibility is measured by the *differences among acceptable solutions* in one's individual solution space. Originality of students' solution is measured by the rareness of the solution group in the mathematics class to which the student belongs. In this way a minor variation in a solution does not make it original since two solutions with minor differences belong to the same solution group.

Note that evaluation of creativity is independent of the evaluation of correctness and connectedness. In order to systematize the analysis and scoring of creativity and connectedness of one's mathematical knowledge we use the map of an expert solution space constructed for each problem (see Figure 2).

Results – example

In the space constrains of this paper we shortly exemplify evaluation of the problem-solving performance of three 10th graders – Ben, Beth and Jo -- from a particular mathematics class. The analysis provided is for their performance on Task in Figure 1. Their solutions are also presented in this figure. We present these students' results because they demonstrate differences in fluency, flexibility and originality. Solutions 1, 2 and 3 are classified as part of the same solutions group whereas solution 4 which uses a special auxiliary construction is classified as part of a different group.

Ben performed solutions 1, 3 and 4, Beth produced solutions 1, 2 and 3, and Jo succeeded to solve the problem in two ways: solutions 1 and 3 (Figure 1). Figure 4 demonstrates connectedness and creativity scores these students got on the Task when the scoring scheme was applied (Figure 3). Their correctness score for all the solutions they presented was 100.

We observed the following properties of the individual solution spaces for Ben and Beth: they were of the same sizes; they included the same number of concepts and theorems and contained two common solutions (solutions 1 and 3). However Ben's creativity score was much higher than Beth's one as a result of the originality of Solution 4 that was performed only by Ben, and his higher flexibility scores.

Beth and Jo differed mainly in their fluency: Beth gave 3 solutions and Jo only 2. Since their solutions had similar flexibility and originality scores their creativity scores are proportional to their fluency scores.

		Solution Type (in order of presentation in the test)	group	Connectedness	Creativity			
					Fluency	Flexibility	Originality	creativity
Ben	Scores per solution	1	1			10	0.1	1
		3	1			1	0.1	0.1
		4	3			10	10	100
	Final			50	3			303.3
Beth	Scores per solution	2	1			10	0.1	1
		3	1			1	0.1	0.1
		1	1			1	0.1	0.1
	Final			50	3			3.6
Jo	Scores per solution	3	1			10	0.1	1
		1	1			1	0.1	0.1
	Final			30	2			2.2

Figure 4: Evaluation of the solutions on the task for three students

Concluding remarks

MSTs are presented in this paper as a research tool for the analysis of students' mathematical knowledge and creativity. The tasks are further used in the ongoing study in order to examine their effectiveness as a didactical tool. The larger study will perform a comparative analysis of students' knowledge and creativity along employment of MST in geometry classroom on the regular basis. The scoring scheme presented herein can be considered as an upgrading of the scoring scheme suggested by Leikin and Lev (2007). Correspondingly we suggest that the scoring

scheme presented herein can be used for examination of individual differences in students' mathematical creativity and students' mathematical knowledge in different fields. We are also interested in employment of this tool for the analysis of the effectiveness of different types of mathematical classes in the development of students' mathematical knowledge and creativity.

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THE DRAG-MODE IN THREE DIMENSIONAL DYNAMIC GEOMETRY ENVIRONMENTS – TWO STUDIES

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Dynamic Geometry Environments (DGEs) in 2D are one of the well researched topics in mathematics education. DGEs for 3D-environments (Archimedes Geo3D and Cabri 3D) were designed in Germany and France. In a first study we could show that pre-service teachers with previous knowledge in 2D-systems prefer to work with a real model of a cube instead of the 3D-system to solve certain problems. Furthermore we could find out that previous knowledge in 2D-systems seems to be insufficient to handle the drag-mode in an appropriate way in 3D-environments. In a second study we introduced the students to the special software before the investigation and distinguished different dragging modalities during the solution processes of two tasks.

THEORETICAL FRAMEWORK

During the last three decades, several 2D-Dynamic Geometry Environments (DGEs) have been created to enrich and further the learning process in the mathematics classroom. The most popular DGEs are Cabri-géomètre, GEOLOG, Geometer's Sketchpad, Geometry Inventor, Geometric Supposer and Thales. In Germany, Euklid-DynaGeo, Cinderella, GeoGebra, Geonext and Zirkel-und-Lineal are popular, with Euklid-DynaGeo being the most widespread software in German schools. DGEs are powerful tools, in which the user is able to exactly construct geometrically, discover dependencies, develop or refute conjectures or to get ideas for proofs.

DGEs are characterised by three central properties: the "drag-mode", the functionality "locus of points" and the ability to construct "macros". The drag-mode is the most important feature available in these environments, because it allows to introduce movement into static Euclidean Geometry (Sträßer 2002). It is possible to drag basic points (points which are neither intersection points nor points with given coordinates). During this dragging process, the construction is updated, according to the construction commands which were used in the drawing. To the user, it looks as if the drawing is respecting the laws of geometry while the dragging process is in progress.

2D-DGEs are one of the best researched topics in mathematics education and especially within the PME-group (Laborde et al. 2006). For example, we find research on "DGE and the move from the spatial to the theoretical" (Arzarello et al. 1998, 2002) or "construction tasks" (Soury-Lavergne 1998). Noss (1994) has shown that beginners have problems to construct drawings, which are resistant to the drag-mode and it is reported that for pupils there exist two separate worlds, the theoretical one and the world of the computer. "The notion of dependency and functional relationship" (Hoyles 1998 and Jones 1996) is another interesting theme and it has been shown that pupils have heavy problems in understanding the notion of dependency. They have to

be encouraged to use the drag-mode to support the understanding of the spatial-graphical and the theoretical level, serving as a tool for externalising the notion of dependency. Several researchers showed that students do not use the drag-mode spontaneously and they have to be encouraged to do it. Most of the students are afraid to destroy the construction by using the drag-mode and they do not like to use the drag-mode on a wide zone (Rolet 1996 and Sinclair 2003). Arzarello and his group elaborated a hierarchy of several dragging modalities, which were linked to "ascending" and "descending" processes and reveal students' cognitive shifts from the perceptual level to the theoretical one (Arzarello 1998, 2002 and Olivero 2002). There is a great variety and number of research reports concerning the use of the drag-mode in proving and justifying processes (for example Jones 2000 and Mariotti 2000). Other fields of study were "the design of tasks" (Laborde 2001), "the role of feedback" (Hadas 2000) and "the use of geometry technology by teachers" (Noss, Hoyles 1996).

THE FIRST STUDY IN 2007

In the following we will give a brief summary of the research design and the results of our first study. For details see Hattermann, 2008. In July 2007, 15 pre-service teachers with previous knowledge in Euklid DynaGeo (2D-DGE) took part in our investigation. Some groups worked with Archimedes Geo 3D and others with Cabri 3D, their actions on the screen and their discussions and interactions were recorded by a screen-recording software called "Camtasia" and a webcam. We used a qualitative approach to get ideas about students' behaviour in 3D-DGEs. Some important research questions were the following:

- Do the students use spatial constructions like spheres or do they prefer elements from plane geometry? (Task 1)
- What are the preferred tools to work with (paper and pencil, real model, imagination, DGE) to work with? (Task 2)
- Do students use the drag-mode to validate a construction and to find solutions to problems? (Task 1 and 2)
- How do participants behave in 3D-environments and how do they use the drag-mode? (Task 1 and 2)

Task 1 and Results

The first task was: "Construct a cube without using the existing macro!" Five of seven groups constructed the cube. The Cabri groups needed between 20 and 25 minutes to construct the cube, whereas the Archimedes groups needed about 40 minutes. Different groups tried to utilise transformations as reflections or rotations. While the realisation of a reflection is quite easy in Cabri, rotations seem not to be easy to handle without any instructions. In the Archimedes environment students had problems with every transformation. The majority of the students used the drag-mode to validate their construction only on demand. This result is comparable to the results ob-

tained by Rolet and Sinclair who worked with school children in 2D-environments. Our probands preferred to measure several segments of the cube instead of dragging a basic point. During the construction, elements from plane geometry (circles, segments, straight lines) were preferred. Some groups used spheres to construct intersection points or to construct equidistant segments, but the majority of the groups worked with circles.

Task 2 and Results

The second task was: “A student affirms: The slice plane between a cube and a plane can be:

- an equilateral triangle
- an isosceles triangle
- a right-angled isosceles triangle
- a regular hexagon.

Construct (with the help of the function “cube”) a cube, check the student’s affirmations and justify your results!”

Every group tried to find validations for their conjectures with the help of the real model, the utilisation of the real model prevailed the use of the computer environment. Students preferred “the old strategy” to examine the cube and to try to imagine the intersection figure. The software was used to validate the conjectures, which were mostly generated outside the software environment. The students defined a plane with the help of three fixed points, so that no dragging was possible. Furthermore, the drag-mode was not understood and it is not sure, if these students did not understand it in the 2D-case or if they could not negotiate it to the 3D-environments. The possibilities of the drag-mode were not understandable to most students. They did not use the drag-mode in an expected manner (to use draggable points on an edge of the cube to define the intersection plane and to drag it to scrutinise different intersection figures). The approach of one group could illustrate this result: The students defined many fixed points on every edge of the cube and defined a plane with the help of three points. After verification, they deleted the plane and constructed another one with the help of other points. Only in exceptional cases the drag-mode was used and more often than not in a manner that a controlled dragging of the plane was impossible, which is the case when students used three arbitrary points in space to define the intersection plane. Students’ statements support the assertion that the “drag-mode” was not understood and previous knowledge in 2D seems to be insufficient to handle 3D-systems!

THE SECOND STUDY IN 2008

Methodology

Our second study took place in February 2008 at the University of Giessen and 15 pre-service teacher students participated in it. The participants had previous knowledge in Euklid DynaGeo (the most widespread 2D-DGE in Germany), but their experiences with DGEs were less than those from students who participated in our first study, because of changes concerning the content of different lectures following new study regulations. There were seven groups (six groups of two students and one group of three students). Three groups worked with Archimedes Geo3D while four groups utilised Cabri 3D to solve the given tasks. Each group worked in a separate room, the actions on the screen were recorded by utilising the screen-recording software “Camtasia”. Furthermore, a webcam and a microphone were used to record students’ voices and interactions.

In our second study we tried to create an environment in which we could observe different dragging modalities. Due to the results of our first study we opted for an approach with a preparation session in which students were introduced to the special software environment and were encouraged to use the drag-mode. Both groups were taught in:

- dragging basic points in 3D-space in the special software environment with the help of the keyboard
- the distinction between basic points, semi-draggable points and fixed points
- the construction of a midpoint of two points
- the construction of a “perpendicular plane” to a straight line through a given point beyond the straight line
- the construction of a “perpendicular line” in the x-y-plane to a given straight line in the x-y-plane through a given point , beyond the straight line
- in the construction of a circle in an arbitrary plane, devoid of the x-y-plane, with a given centre and through a new point on the plane
- in reflecting the circle on an arbitrary point devoid of the circle’s centre
- in constructing a plane which contains a given straight line
- in constructing a plane with the help of three points in such a way that one of these points can be dragged on a straight line

Archimedes-groups were especially introduced to the utilisation of transformations which is quite complicated in this environment. After the first introduction students were urged to solve five task which forced students to use the drag-mode. Here, we followed suggestions from the Centre informatique pédagogique (CIP 1996) for 2D-environments and adapted the ideas to our 3D-environment. There were five files and

every file contained a special task. Every task consisted of a body and one or several yellow points which had been constructed by the researchers before. The task was to find hypotheses concerning the construction of the yellow point(s) by dragging a special point which was marked in blue colour. With the help of these preparation tasks, we intended to weaken students' constraints to use the drag-mode and to encourage them. Because of the domination of the real model compared to the software environment in our first study, we decided to forbid paper and pencil and not to provide a real model of the cube.

In our preparation session, we tried to provide students with competencies to solve the tasks which were given in our study without giving them exact hints. So we broached the issue of constructing a perpendicular line to a straight line through a given point on a special plane without mentioning that this construction could be useful to construct a cube. For another example, students had to construct a plane in such a way that one point of this plane could be dragged on a straight line. The idea behind was to show students how to construct a "draggable plane" without telling them that it could be an appropriate way to scrutinise different intersection figures of a plane and another body by using three defining points of the plane on appropriate segments of the body, which seems to be a reasonable way to solve our second task in the study.

Research questions

First of all we are interested in the general behaviour of our students in a 3D-environment; especially we looked for differences in students' behaviour during the solution process of different tasks compared to the first group in July 2007 which had no preparation session. Are there important differences among the two DGEs? Because of the importance of the drag-mode in DGEs, we want to know more about the utilisation of it, especially we are interested in different dragging modalities in 3D-environments. Do students use the drag-mode to validate their construction in task one (construction of a cube)? A validation of the construction with the help of the drag-mode assumed, how do they use it? Are they more "courageous" than their predecessors in July 2007 and do they use the drag-mode on a "wider zone"? What are the preferred tools to construct a cube? Is one preparation session enough to get students familiar with a 3D-DGE in such a way that elements like spheres or 3D-reflections will be used to construct a cube or do constructions like circles (elements from planar geometry) prevail the construction?

Do students use the drag-mode to discover different intersection figures of a cube and a plane or do they try to avoid the utilisation of the drag-mode in task two? Is it possible to identify different "ways of dragging"? What solving strategies are preferred by students who do not possess neither a real model of a cube nor a paper and pencil environment?

Task one and Results

We used the same task as in our first study in July 2007: "Construct a cube without using the existing macro!"

Every group constructed the cube. The Cabri-groups needed 17, 19, 26 and 41 minutes for the construction, whereas the Archimedes-groups needed 34, 37 and 45 minutes. Furthermore every group utilised the drag-mode to validate their construction and two Cabri-groups did it in a "courageous way" so to say, they used it on a wider zone. One Archimedes-Group was very careful by dragging basic points. Every group was very happy by observing the invariance of the constructed cube under dragging and jubilation and pleasure were recognisable in nearly every group. This fact shows that dragging can motivate and emotionally affect students which underlines the importance of this feature.

By comparing the periods of construction it seems as if Cabri-Groups work faster. In our first study we came to the same statement and argued that one reason for this could be the "base plane (x-y-plane)" which exists in Cabri. In Archimedes this plane has to be constructed first. We can't support this hypothesis with our actual data, because during the preparation session the construction of the x-y-plane in Archimedes was mentioned and every Archimedes-group had no problems to construct it in a short time not exceeding 3 minutes.

No group tried to construct the cube with the help of spheres, only circles, planes and perpendicular lines were used to construct cube vertexes. An explanation for this result lies in the preparation session, in which circles, but no spheres were explicitly mentioned.

One Archimedes-group utilised reflections on a plane and reflections on a straight to construct cube vertexes. One Cabri-group utilised the function of a parallel plane to a given plane but furthermore no reflections were used by students. In our first study no Archimedes-group used reflections to construct the cube. Due to the fact that "transformations" are not easy to handle without instructions, this fact was not surprising to us. After an introduction in defining and utilising transformations in Archimedes, one of three groups used "reflections", but the size of the sample seems to be too small to interpret this fact in more detail.

Besides we observed students who had problems with "parent-child-relations" (see also Talmon 2004). Several situations occurred, which prove that dependencies of construction objects are not understood completely. Some groups did not understand that objects disappear by deleting an object on which they depend on.

Furthermore we could identify several dragging modalities in 3D-environments. Students used the drag-mode in our first task to

- validate the construction at the end of the construction process.

- see that there are only two draggable points (the points that define the first edge of the cube) and to see that the other points are fixed.
- find out the function of a semi-draggable point on the edge of the cube that had been constructed before. (Students forgot for what reason they had it constructed)
- adapt the length of a segment to the measure of the first edge. (students did not really construct a cube in this attempt, they created a cube which was not invariant under dragging)
- find out more about the degrees of freedom of draggable points, for instance to scrutinise if points are draggable on a plane or only on a straight line.
- find an error in the construction. (Actually the construction was correct, only one point was wrong and this fact was discovered by dragging)

Task two and Results

The second task was changed compared to the version used in July 2007. Task two was the following: “Construct with the help of the function “cube” a cube and try to find by experiment all Polygons ($n = 3, 4... n = \text{number of vertexes}$) which exist as intersection figures between the cube and a plane.” The second task was changed slightly in comparison to the first study, because we intended to further the need for the utilisation of the drag-mode. In the first study we gave four intersection figures and asked students to confirm or refute our statements, whereas the assignment is more open in our second study. We hoped that trying to discover new intersection figures would motivate students and moreover we tried to create an environment in which dragging could help students to find solutions. Finally we intended to observe and distinguish different “ways of dragging” during the solution process.

Except of one group, everybody found the equilateral triangle and the isosceles triangle as an intersection figure. Approximately the half of the participants mentioned an arbitrary triangle as intersection figure, whereas only one group could find a parallelogram. The rectangle and the square were the easiest figures which were found by every group. Half of the groups found the trapezoid as intersection figure, whereas the other participants found it was well, but did not identify this quadrilateral as a trapezoid. Nobody looked for an isosceles trapezoid. Three groups found a pentagon, four groups found a hexagon and four groups found the regular hexagon. There were groups that found the hexagon and not the regular hexagon and vice versa.

During the solution process we observed different dragging modalities. Students used the drag-mode by

- defining the intersection plane by one point on an edge of the cube and two vertexes.

- choosing two points in a Cabri-environment to define the plane (now a plane appears) and to observe the behaviour of this plane by moving the cursor on the screen. (a special type of dragging only available in Cabri-environments)
- defining three points on different edges of the cube to define the plane.
- using three arbitrary points in space to define the intersection plane.
- defining one draggable point on a straight line that is defined by two vertexes of the cube and to use two other points in space to define the plane.

Students used the drag-mode to:

- find out the function of a special point which had been constructed before. (a point was used to define a plane for example)
- vary the volume of the cube so that the intersection points between the cube and the plane become visible (which is not always the case).
- identify new intersection figures.
- get an idea how to construct the intersection figure afterwards with the help of fixed points to define the plane.
- identify more special figures/more general intersection figures from an existent figure. (find an equilateral triangle from an arbitrary triangle or vice versa)
- scrutinise if there are intersection figures with more than 4 vertexes. (with the special type of dragging in Cabri)
- move the cube, instead of varying the plane, to scrutinise different intersection figures.
- identify draggable and non draggable points.

It is really worth mentioning that we could observe happiness in every group by realising different intersection figures with the help of the drag-mode. “Wow” or “that’s really great” are only two short examples that underline our affirmation.

Conclusion

We succeeded in our second study to get the probands more familiar with the special DGE and to observe different dragging modalities in task one and two. There are still situations in which students utilised the drag-mode very careful and not on a wider zone, but the majority of our participants utilised the drag-mode to validate and to discover in a “courageous” manner without hesitation. So we claim that it is possible to prepare students in an appropriate time to use the drag-mode in 3D-systems and to encourage them.

For a classification of different dragging modalities it will be interesting to categorise them theoretically and to analyse the “instrumental genesis” of the drag-mode according to Rabardel’s theory (Rabardel 1995). It will be an exciting task for further re-

search to observe the progress of the utilisation of the drag-mode. It should be possible to define different theoretical stages in the utilisation of the drag-mode from a “beginner’s stage” which will be characterised by nearly no dragging or careful dragging up to an “expert’s stage”.

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3D GEOMETRY AND LEARNING OF MATHEMATICAL REASONING

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Teaching mathematical proof is a great issue of mathematics education, and geometry is a traditional context for it. Nevertheless, especially in plane geometry, the students often focus on the drawings. As they can see results, they don't need to use neither axiomatic geometry nor formal proof.

In this thesis work, we tried to analyse how space geometry situations could incite students to use axiomatic geometry. Using Duval's distinctions between iconic and non-iconic visualization, we will discuss here of the potentialities of situations based on a 3D dynamic geometry software, and show a few experimental results.

In mathematics education, resolving geometry problems is a usual way of teaching *mathematical proof*, and plane geometry is mainly used.

Nevertheless the students often focus on the properties of *drawings* — which are physical objects — instead of *figures* — the theoretical ones. In this case they may solve geometry problems by using empirical solutions, based on their own action on the drawing: *One can read the property on the drawing*. That is why using drawings as regards plane geometry is very confusing for many of them: since they are able to *see* results on the drawings, since they can work easily on it, mathematical proof seems to be useless, and may appear as a didactical contract effect (Parzysz, 2006).

On the contrary, in space geometry, it seems to be much harder for them to be certain of a visual noticing, and they may need new tools to study representations and to solve problems.

Our hypothesis is that it is possible, with specific situations, to make the students use tools concerning theoretical objects: working on figures, using geometrical properties... In order to control these new tools, mathematical proof is a very useful process the students can use to solve problems. This is why we assume that 3D geometry could be very helpful for proof teaching.

Nevertheless, formal proof is a complex process that not only involves hypothetico – deductive reasoning, but also (for instance) specific formal rules (Balacheff, 1999)

we will not study here. Therefore, we will only focus in this paper on the first hypothesis we mentioned.

We will present here a preliminary study in order to illustrate and test our theoretical hypothesis.

THEORETICAL FRAMEWORK

Resolving problems of geometry

As it is said in Parzysz (2006):

The resolution of a problem of elementary geometry consists of the successive working with G1 and G2, focusing on the “figure”. The figure has a central part in the process: even if it is very helpful in order to make conjectures, it may be an obstacle to the demonstrating process, as the pupils don’t know how to use data because of the “obviousness of the visual phenomenon”.

Parzysz refers to Houdement&Kuzniak’s geometrical paradigms, in so far as G1 is a “natural geometry” — where geometry and reality are merged — and G2 is a “natural axiomatic geometry”, an axiomatic model of the reality, based on hypothetico-deductive rules (Houdement, Kuzniak, 2006).

As we can see, demonstrating is really meaningful when working with both G1 and G2, but the sensitive experience may encourage the pupils to work only with G1. In order to describe more precisely what can be this *sensitive experience*, and the ways it is related to using — or not — G2, we chose to use the distinctions that Duval (2005) makes between the different functions of the drawing, and the different ways of seeing it.

A first way of using representations is the *iconic visualization*: in this case the drawing is a true physical object, and its shape is a graphic icon that cannot be modified. All its properties are related to this shape, and so it seems to be very difficult to work on the constitutive parts of it — such as points, lines, etc. Then, the drawing does not represent the object that is studied, it *is* this object, and the results of geometrical activities inform on physical properties.

The other way is the *non-iconic visualization*, where the figure is analysed as a theoretical object represented by the drawing, using three main processes:

Instrumental deconstruction: in order to find how to build the representation with given instruments.

Heuristic breaking down of the shapes: the shape is split up into subparts, as if it was a puzzle.

Dimensional deconstruction: the figure is broken down into *figural units* — lower dimension units that figures are composed of —, and the links between these units are

the geometrical properties. It is an axiomatic reconstruction of the figures, based on hypothetico-deductive reasoning.

These different possible ways of using the drawings lead us to two important consequences.

On the one hand, using G2 makes no sense with only iconic visualization, as geometry problems concern nothing but the drawings to the student's eyes.

On the other hand, carrying out the dimensional deconstruction means isolating subparts of the drawing and, at the same time, describing how these subparts are linked: this last part has no sense when using only G1. Therefore this operation implies a more axiomatic point of view, and the figure — described by the dimensional deconstruction — is likely to be used.

Finally, we assume that dimensional deconstruction would become an efficient tool if the iconic visualization weren't reliable any longer, as the pupil would have to make up for the lack of information in order to solve geometry problems. Using graphic representations is much more complex in space geometry, and then it seems to be an appropriate environment for the teaching of axiomatic geometry.

3D geometry

Using physical representations is very different in space geometry: there are various ways of representing figures, such as models or plane projections, and each kind of representation has specific properties and constraints. As the physical models are too restricting — for instance, adding new lines is generally impossible, and constructing models takes much time —, cavalier perspective representations are generally used. Then, visual information is no longer reliable: for instance, it is impossible to know whether two lines intersect or not, or whether a point is on a plane, without further information.

So in space geometry iconic visualization fails, and it is necessary to analyse the drawings in other ways. The problem is that using drawings is generally too difficult for the pupils. Chaachoua (1997) mentions that this involves the students' interpretation, based on their mathematical and cultural knowledge. They have to break down the drawing into various components, so that they can imagine the shape of the object. In fact, they would have to carry out dimensional deconstruction *before* any visual exploration. Therefore they are unable to understand that iconic visualization is not sufficient to solve geometry problems, as they only think that they *see* nothing.

Using 3D geometry computer environments may balance these difficulties, since the students could get more visual information, for instance by using various viewpoints as if the representations were models. It has to be noticed that, even in this kind of environment, visual information is usually not reliable, so that iconic visualization remains inadequate to solve geometry problems.

Hypothesis about Cabri 3D

With Cabri 3D, the user can watch the representation as if they were models. It is possible to adjust viewing angles by turning around the scene, to look at the drawing from various viewpoints, and then to be more easily conscious of the visual issues. For instance, it becomes possible to see that a point belongs to a plan, when the point *visually* belongs to it. Actually the user can get visual information to determine the shape and some properties of the figures, but generally this information is not sufficient to carry out geometrical works. For instance, as the representations are not infinite in Cabri 3D, two secant lines could have no intersection point on the screen, then it would be impossible to determine visually whether these lines are secant or not. Some operations are almost impossible too, like moving a point to reach a given line with no other tools than visual perception.

Then, the feedback from a Cabri 3D - based *milieu* — as described in Brousseau (1997) — may emphasize that, even if visual information is available, this information is partial. A Cabri3D drawing does not permit to *see* all the specificities of the object the student has to study – which is clearly not the drawing itself.

It seems that a problem any student would have to deal with, when using Cabri3D, is “How can I get information from the drawing, and how may I use it in order to deduce information I cannot see, and solve geometry problems?”. We showed that there are two main kinds of answers: the iconic visualization based ones, and the non-iconic visualization based ones.

Our first hypothesis is that with Cabri 3D it is much easier for the students to get information about the drawings, and then to start a research process, even if they only use iconic visualization. This research process may evolve because of the dynamic geometry software properties of Cabri3D.

Cabri 3D not only produces representations, it is a dynamic geometry software. In this way it is possible to use *hard* geometric constructions: these drawings are based on geometric properties, and keep it when the user drags a part of it. As an example, a hard square remains to be a square — with different size — when one of its vertexes is dragged. Therefore, the students may assume that the reason of simultaneous movements of figural units is the relation between them: if a point moves when another one is dragged, it may seem that they are *linked*, in a way that has to be elucidated by the students.

We can guess that this point is stressed in 3D dynamic geometry situations, since other visual information is generally not reliable: one can be sure of the simultaneous movement of two figural units, even if it can be quite difficult to determine *how* these units are linked. These links are in fact invariant properties when points are dragged, and then direct results in Cabri3D of geometrical properties (Jahn, 1998).

Our second hypothesis is that with dynamic geometry it is possible to stress the inefficiency of iconic visualization, and to support experimental studies of the properties of the figure. Therefore dimensional deconstruction and axiomatic geometry would become very efficient tools for the students to design research processes, to study a given representation and to solve geometry problems.

Nevertheless, these theoretical tools are not sufficient: any experimental work in Cabri 3D has to involve Cabri 3D's tools. Therefore we have to study their role and the way they could interact with the theoretical ones.

First, many tools are very linked to visual perception: changing viewpoint tools, drawing and measuring tools. If they are not used with other tools, there is no need for the student to control her/his work with G2. S/he can measure drawings, watch their shape and construct objects as *soft*, and not hard constructions. When a part of such a drawing is dragged, the shape changes and so do the geometric properties the user can see. Then the feedback from Cabri 3D invalidate this kind of construction to the user's eyes (Laborde, Capponi, 1994).

Secondly, other tools are more strongly linked to a theoretical control of the constructions: construction primitives — intersection, parallel, perpendicular, tetrahedron, etc. — and transformations. Even if using axiomatic geometry is not necessary to control the use of these tools, an empirical control may be very difficult in many situations (for instance, in order to use a transformation, the user generally has to choose the values of several arguments before any visual control). So using G2 would become an economical way of controlling it. Furthermore, these tools would be very helpful for the process of instrumental deconstruction, as they are designed with axiomatic definitions. Actually, for this reason, instrumental and dimensional deconstructions would be very linked in this case.

Eventually, we have to point out that the designer of a situation (teacher, researcher...) can choose the toolset available in Cabri 3D. This is a way for him to delete specific tools in order to design feedbacks. For instance, if the students have to construct hard squares, there is no feedback about the hardness of constructions when using the “square” tool. Therefore choosing the available toolset is often a very important choice for this *didactical variable*, to make strategies inefficient or impossible.

Then, our third hypothesis is that in some specific situations, with a specific Cabri 3D toolset, it is possible to provoke a particular instrumental deconstruction, strongly linked to dimensional deconstruction.

Research problem

As a consequence of our theoretical framework, it is now possible to make the problem mentioned in the introduction clearer and more accurate: is it possible to

design didactical situations with Cabri 3D that make iconic visualization inefficient and in which dimensional deconstruction can be a tool to analyse figures and solve problems? Then we have to wonder whether using dimensional deconstruction could be liable to make the students using G2.

The following example is a situation we designed in order to test our hypothesis, in which a student has to analyse a Cabri3D-drawing in order to explain to another student how to construct the same object with Cabri 3D.

AN EXAMPLE OF A RECONSTRUCTION SITUATION

Methodology

We used a qualitative approach to analyse the students dealing with this task. We referred to our theoretical study in order to distinguish different strategies they were likely to use. It was possible to foresee how they would analyse the drawings, as shapes or as geometrical constructions... Moreover we had to analyse how they design their construction strategies. For instance, anticipating the properties of the object constructed would reveal G2-based strategies. We will only detail below the three main kind of strategies we distinguished.

In order to analyse the students' work, we used a screen-recorder software (Camtasia), microphones, and a video camera. Then we could observe at the same time their dialog, their gestures (for instance to describe physical objects), and the way they used Cabri 3D.

The situation.

This situation involves 10th French graders (15 to 16 year-old students), working in pairs. Each student works on a computer. The first one (S1) has to analyse a model, a Cabri3D-drawing, and describe orally to the second student (S2) a way of reconstructing it. Using S2's computer is forbidden to S1, and S2 cannot see S1's screen.

There are four distinct phases, from the simple to the complex one (see Fig.1): first a prism with a rhombus as a base, and then are successively added its symmetrical with respect to a vertex, an edge and a lateral face. All these prisms are constructed from three directly movable points: a and b are in the base plane, and c is on the line perpendicular to the base plane at point O (the centre of the bottom face of the prism). All the other points are constructed using symmetries, so that the constructions are robust ones.

S2 is given a file with the three points, a, b and c, and the two students have to validate their constructions by themselves. The only condition is that the behaviour of the new object has to be the same as the model's one when point a, b or c are moved.

S2 doesn't see the prism and the polyhedron tool is not available, so it is much harder to solve empirically the three last problems by constructing symmetricals of the first prism.

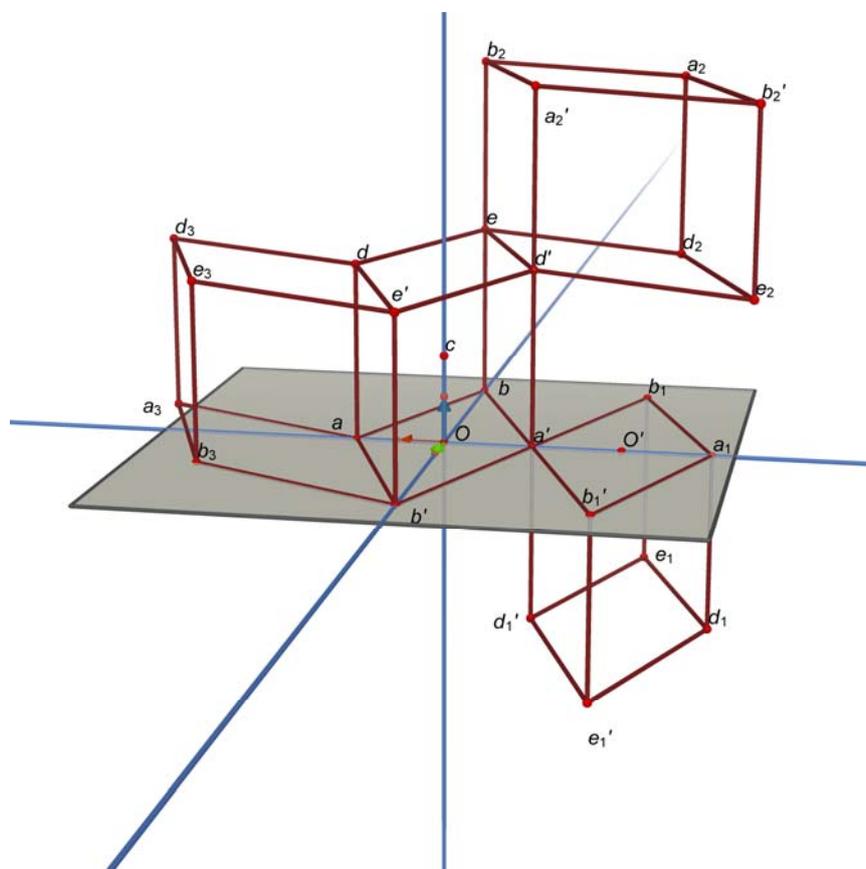


Fig. 1: Figure to analyse and reproduce in phase 4 (in previous phase, parts of the figure have been reconstructed)

Three strategies

First, if they worked using only G1, they would analyse the shapes and sizes of the models, and try to reproduce it by creating points and dragging it to the right positions. This is very difficult in a 3D space represented in 2D, and we can guess that construction primitives may be used as stands on which a visual control of the positions is possible. This is a basic strategy, and it fails in Cabri 3D whereas it wouldn't in a paper/pencil environment. We call it R1.

The second strategy (R2) is based on the use of construction primitives controlled by knowledge about "basis configurations" (Robert, 1998) learnt before. For instance, point O may be recognized as the centre of symmetry of the bottom rhombus not because a and a' seem to be symmetrical with respect to it, but because the student already know that the "centre" of a rhombus is its centre of symmetry. Therefore the

students may use locally plane transformations (on some planes). But in space, as they have no previous knowledge about symmetry in a prism, their strategy may be similar to R1. We expect that in this case, in the model analysis phase and in the interaction with S2 phase, S1 may focus at the same time on geometric properties and on size information. This strategy does not necessarily require dimensional deconstruction. The result of it is a partial failure, as the dynamic properties exist in planes, but not in space.

The third strategy (R3) may be based on transformations. In this case, we assume that the student use axiomatic geometry and dimensional deconstruction, then we can guess that their analysis would focus on invariant properties when they drag points, and their reconstruction strategy would be designed in order to reproduce these properties.

Experimental results

We experimented this task with three pairs of 10th French graders, who had been just introduced to Cabri 3D before. Our following analyse will mainly focus on the “reconstruction phases”, and not on S1’s analysis of the drawings.

First of all, it seems that the students could get information about the drawings by manipulating it. They were able to determine, visually, shapes and basic physical properties, and to try to find a solution to the problem. For instance, the Group 3 students only used iconic visualization, and they could construct the prism shape – but a soft construction, based on the length of the edges. They tried something, and their failure was not the consequence of the too high complexity but was linked to the expected properties: some points “*don’t move*”.

Secondly, all the students realized that iconic visualization was not sufficient to carry out the expected construction. We have to distinguish to main cases.

Groups 1 and 2 first used only R1, but they realised that this strategy was no longer efficient in 3D geometry. As they were able to use – more or less easily – non-iconic visualisation, they tried other strategies and could reproduce the dynamic properties. It has to be noticed that they used R2 and R3 because it was easier than R1, and not in order to make *hard* constructions (even if this was a consequence).

On the contrary, at the beginning, Group 3 students were not able to use anything but iconic visualisation. They constructed the first prism with R1, which led them to a failure: the points “didn’t move”. Iconic visualisation couldn’t help them to analyse this:

S1: Try to make the point move

S2: I can’t, there is no line [on which the point could move]

Then they started to use iconic and non-iconic visualisation at the same time, depending on their aim. For instance, they first tried to make b' , b_1 and b_1' while dragging b , but didn't care about a , a' ... They kept constructing a , a' , a_1 , etc., by measuring lengths, but constructed b_1 and b_1' by using geometrical properties, such as “parallel”, instead of adjusting positions. This second case underlines that using non-iconic visualisation can be strongly linked to the dynamic properties of the drawing.

Eventually, we have to point out that the students didn't use easily dimensional deconstruction, and then they first tried to use it as little as possible. For instance, it seemed to Group 2 students that $ed'e'$ and $ed_2'e_2'd'$ (see Fig. 1) were linked, and that (ed') had something to do with this link: “*a rotation*”. They tried to use the tool without any further analysis (basic instrumental deconstruction), and couldn't succeed. Then, they analysed more precisely the link, and discovered that they had to use “*symmetry*”. Actually, as instrumental deconstruction was not precise enough, they used dimensional deconstruction in order to control more precisely the way they used the tools.

CONCLUSION

Finally, our experimental results have a global consistency with the three hypothesis we mentioned.

The students used the representations as if they were models, and could get information from it. Even if they wanted to draw shapes, without any dynamical properties, they were able to get enough information by *looking* and *measuring* the models. Moreover, we could observe that, even to draw shapes, non-iconic visualization led them to more efficient strategies (Groups 1 and 2).

Nevertheless, because of the dynamic geometry, this process was inefficient, and they had to find a way of reproducing dynamical effects. With this new research process, they had not only to use iconic visualization but to find something else. Depending on the students' knowledge, most of them tried to use dimensional deconstruction and an axiomatic point of view, as the most efficient strategy – efficient for analysing, giving oral information, reconstructing, arguing... In every group, the strategies used by the students evolved and dimensional deconstruction was more and more involved, so that they were able to give an interpretation to dynamical effects.

It seems that Cabri3D's tools were very important in the evolution of strategies. Using of transformations appeared to be a way of solving the problems, but an empirical control was very difficult in most cases. Then, the students changed their strategies, and tried to find new ways of controlling it, by using dimensional deconstruction.

Therefore, these results give us informations about our research question: iconic visualisation failed, and dimensional deconstruction was necessary to solve the problem. Moreover, even the weakest students started using dimensional

deconstruction, whereas they were unable to do so at the beginning of the exercise. Then we could ask two new questions, more accurate. On the one hand, how did dimensional deconstruction appear, and how is it related both to the task and to instrumental deconstruction? On the other hand, we will have to study whether using dimensional deconstruction is liable to make the students use G2 in geometry, and not only in 3D geometry.

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IN SEARCH OF ELEMENTS FOR A COMPETENCE MODEL IN SOLID GEOMETRY TEACHING. ESTABLISHMENT OF RELATIONSHIPS

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ABSTRACT

In this paper we present part of the analysis of a Teaching Model for the geometry of solids of an initial Education Plan for elementary school teachers, and its implementation in the University School of Teaching of the Universitat de València, Spain. We have focused our attention on how the establishment of relationships among geometric concepts have been worked on. For this analysis we considered theoretical contents related to geometric contents (concepts, mathematical processes and different types of relationships). This study is part of a more extensive work that tried to elaborate the competent conduct features for a teacher teaching solid geometry in elementary school.

PRESENTATION

This work is part of a more extensive research project which uses as a methodological framework the theory of the “Modelos Teóricos Locales” (MTL) (Local Theoretical Models) (Fillooy, 1999). According to Filloy and col. (1999), to be able to take into account the complexity of the phenomena that take place in the educative systems, the MTL incorporate several interrelated theoretical components: 1) Competence Model; 2) Teaching Model; 3) Cognitive Processes Model, and 4) Communication Processes Model. Our work is focused on the first component in relation with the training process of elementary school teachers in the subject of solid geometry.

De Ponte and Chapman (2006) point out that this research line has given priority to the analysis of teachers knowledge or practice paying less attention to the analysis of the programs for their training. In our work we analyze a solid geometry training Program for elementary school teachers and its putting into practice; we want to establish some elements for the Initial Competence Model (ICM) in relation with the training of elementary education teachers in the geometry of solids. In previous papers we have presented elements of this competence model that show a competent conduct for teaching mathematical processes related with describing, classifying, generalizing and particularizing. In the present paper we focus on the elements related to the establishment of relationships among geometrical contents.

BACKGROUND AND FRAMEWORK

The analysis we present in this paper is part of a more extensive work - González (2006)¹-, which had the purpose of elaborating the elements for an ICM that can be used as a reference to interpret the teaching models proposed for teaching solid geometry in training programs for elementary school teachers. This work belongs to a project that aimed for the creation of a "Virtual Library"² that could help to teachers' permanent education.

In previous works (González and col. 2006, 2008; González, E. and Guillén, G. 2008) we have presented some results of the analysis. To group these results we have followed the distinction made by Climent and Carrillo (2003), who take into consideration teacher's knowledge and distinguish as different components the mathematical content knowledge (in our case contents *of* and *about* geometry) and the knowledge of the subject for its teaching.

In previous papers above mentioned we refer to results that have to do with the contents of "solid geometry" related to mathematical processes of classifying, describing, generalizing, and particularizing. We show how the attempt of organizing the surrounding objects and their construction, by means of different procedures, provides very rich contexts to develop these mathematical processes. We also present some of the reflections encouraged by the teacher concerning the learning process of both children and teachers, questions having to do with preparing the lesson, are related to the use of language, or the way to respond to the appearance of misconceptions.

The observations we present in this paper belong to the first group of contents *of* and *about* geometry, and complete the study; these observations refer to relationships among geometric objects of the same and different dimension; that is, relationships among solids, among their elements or among plane and space elements.

As we advanced in the presentation, we follow the Theory of the MTL as experimental methodological framework. We have commented that in this Theory four interrelated theoretical components can be distinguished. What differences each component from the others is, among others, the phenomena taken into account

¹ Work carried out to obtain the "Diploma de Estudios Avanzados" (Certificate of Advanced Studies) of the PhD program of Mathematics Education. Universitat de València, Spain.

² Project "Procesos de transferencia de resultados de investigación al aula: el caso del bajo rendimiento escolar en matemáticas". Research project, co-financed by the Consejo Nacional de Ciencia y Tecnología (CONACYT-G37301-S) (National Council for Science and Technology). México.

<http://www.pernodis.com/ptria/index.htm>. In the site dedicated to geometry, section "*Descubrir y matematizar a partir del mundo de las formas*", chapter *¿Cómo enseñan otros?* we present extracts of the class sessions with the corresponding analysis (<http://hipatia.matedu.cinvestav.mx/~descubrirymat/>).

regarding the concept subject of analysis. In this work in particular, the ICM includes elements of the knowledge of an ideal person, capable of carrying out tasks related to the teaching of solid geometry at elementary school level. This is, it includes the elements which should be part of the competent conduct of elementary school teachers when teaching the geometric topics regarding solids in their classes. We have already pointed out that the elements commented in this work refer to the establishment of relationships among geometric contents.

When we focus on solids, our theoretical framework is based on the studies made in Didactics of solid geometry (Guillén, 1991, 1997; Guillén and Figueras, 2005), we continue reorganizing these contents as referred to: a) geometric concepts, b) mathematical processes (to analyze, to describe, to classify, to generalize, etc.), c) relations among geometric contents. When we studied how these geometric contents were taught, we also paid attention on how the skills are used (to construct, to modify, to transform) to work the mathematical processes indicated or to develop skills (to communicate and/or to represent forms). The reorganization of the school contents has led to organize the observations as related to the teaching/learning of concepts, of mathematical processes, or of the establishment of relationships among different geometric contents. The observations made are detailed in Guillén (1991, 1997). These works take into account, on the one hand, relationships among solids and/or families of solids. These refer to inscription and duality relationships among families of solids, to composition or decomposition relationships, or to inclusion, exclusion or overlapping relationships among different classes established with different classification types (hierarchical partitions or classifications) taking into account several universes and criteria for classifying. On the other hand, we stand out the relationships among the solids elements that can be either of parallelism and perpendicularity or numerical relationships among them. Also were taken into account the relationships among geometric contents of several dimensions that emerge when solids truncate or during the construction of models parting from a plane surface. Moreover, attention has been paid to the establishment of relationships by analogy. In the work of González and Guillén (2006) the inclusion, exclusion or overlapping relationships among families of solids were studied. The rest of types of relationships are the ones that have been taken as reference to organize the observations that this report presents.

The studies above mentioned have been developed taking as a reference the works of Freudenthal (1973, 1983) and others, that have been carried out at the Freudenthal Institute (for example Treffers 1987). These works are the theoretical basis for our concepts over geometry and its teaching, over the relationships among the different geometric contents, and also provides us with information to organize the solids geometry teaching. In this framework one of the aims of geometry teaching is the development of mathematization through mathematical practice.

To carry out the analysis we have also taken as a reference other studies about the appropriate contents for the teachers training plans, emphasizing on the different

contents that should be discussed on a reflective level (Shulman, 1986; Climent y Carrillo, 2003; De Ponte y Chapman, 2006; González et. al. 2006).

DATA COLLECTION AND ANALYSIS

To create the MCI, we analyzed the available literature related to the mathematical content analysis and observation of the learning process for mathematical processes and the literature related to teachers' education, this enabled us to elaborate the Theoretical Framework of the work and define the criteria used to analyze the design and implementation of a Teaching Model of the teacher of Teaching with an extensive experience in introducing to the study of geometry having as a support solids geometry.

The work has been developed in several stages. In the first one, we examine theoretical works of the research lines we mentioned in the previous section and the teachers' training plan of the teacher who constitutes the study scope of our work (Guillen, 2000). In a second stage we analyzed the implementation of this training plan.

The data for this experimental study was obtained during the 2005-2006 school year. We attended and took notes of 22 class sessions the training teacher dedicated to solid geometry during the course she gave to a group of students belonging to the foreign language specialty at the University School of Teaching of the Universitat de València (Spain). Each session lasted 50 minutes approximately.

To control all the information that emerged during the teaching, the sessions were recorded in video and audio. These recordings were transcribed and from them, together with the notes taken during the classes, were obtained the extracts to carry out the analysis. These were considered the essential element and were defined taking as a reference the theoretical analyses performed during the first stage. They could be a sentence or a set of sentences that not necessarily had to match the answers or individual interventions of the teacher or of the students.

These extracts were organized in groups as it follows: i) On geometry and its teaching. Student and teacher; ii) On geometric contents; iii) How do some of those students learn? What for?; iv) The class planning; v) Interacting in the class and ... vi) What about language? In Gonzalez et al. (2006) we briefly detail observations related to each of them.

The school contents organization we carried out, mentioned in the previous section, show the distinction we made in the observations we included in group ii). We separated them as follows: ii.1) relative to concepts learning; ii.2) relative to mathematical processes; ii.3) relative to the establishment of relationships. We have already mentioned that in the following section we will refer to group ii.3).

To analyze the corresponding extracts for the establishment of relationships we used, on one hand, the diagram presented by Olvera (2007) and showed in figure 1. This diagram was constructed starting from the characteristics of Van Hiele levels for

solid geometry determined in the study by Guillén (1997). On the other hand, in its organization the families of solids and polygons implicated and the relations among flat geometric objects and space geometric objects were taken into account. Also different representations of the solids used as a context were considered and numerical relations were also underlined.

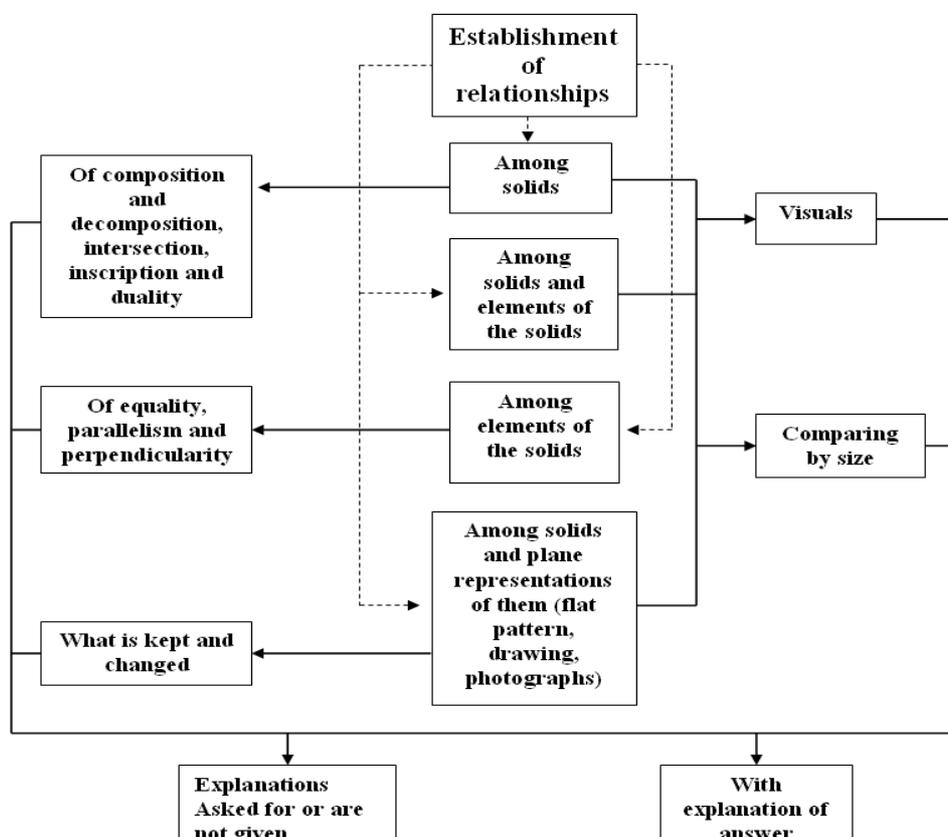


Figure 1

IMPLEMENTATION OF A TEACHING MODEL FOR SOLID GEOMETRY. OBSERVATIONS RELATED TO THE RELATIONSHIPS AMONG THE GEOMETRIC CONTENTS

In Figure 1 we show how the observations of relationships among geometric contents during the implementation of the analyzed training plan are grouped. Following, we present some examples.

Establishment of relationships

The observations that we present in this section have been organized taking into account, on the one hand, the solid families used as a support to develop the activity. On the other one, that the context can also consist of the different representations of solids. It is also necessary to take into account that the relations established could also be numerical.

1. Relations of inscription and duality among regular polyhedrons. When numerical relations are exposed in a table as shown in Figure 2, in which the number of faces,

vertexes, edges, order of the vertexes and number of sides of the polygons of the faces have been registered, it leads to the establishment of a wide variety of relationships.

For example, it comes to express that the number of faces of the dodecahedron is equal to the number of vertexes of the icosahedron; or that the number of vertexes of the octahedron is equal to the number of faces in the cube. From this type of relationships, it can be concluded that some polyhedrons can be inscribed in others. For example, the cube can be inscribed in an octahedron in such a way that the vertexes of the cube are in the center of the faces of the octahedron, or vice versa.

	C	V	A	O de V	Forma de C
Tetraedro	4	4	6	3	3
Cubo	6	8	12	3	4
Octaedro	8	6	12	4	3
Dodecaedro	12	20	30	3	5
Icosaedro	20	12	30	5	3

Figure 2



Octahedron inscribed in the cube

There are also relations established among elements of the dual regular polyhedrons when instead of considering models of pairs of dual regular polyhedrons inscribed, compound models are considered, which are intersections of pairs of dual polyhedrons. For example, the cube and the octahedron.



After encouraging students to imagine in a dynamic way how to pass from the inscribed model to the compound model when the size of the inscribed polyhedron is increased, the attention is focused on the fact that the edges of both polyhedrons cut perpendicularly at their midpoint.

2. Relations among regular polyhedrons and other solid families.

When trying to analyze regular polyhedrons, they have repeatedly been studied in relation to other families. For example, in the analysis of the icosahedron it is emphasized that it can be seen, on the one hand, as the composition of two pentagonal pyramids of regular faces and a pentagonal antiprism of regular faces or as the fitting of two caps that correspond, each of them, to a pentagonal bipyramid of regular faces, in which one of the pyramids has been opened.



3. Cylinders and Prisms. Cones and pyramids. Immersed in the situation of generating models with different procedures, in first place, the family of straight prisms was introduced through the truncation of a straight cylinder.



For example, questions raise such as: What form do we obtain if we cut perpendicularly the base? How many cuts, perpendicular to the base, should be done for the circle of the base to turn into a 5-sided polygon? What does the cylindrical

surface turn into? How are the cylinders obtained with parallel to the base cuts? Can we also obtain oblique prisms? And this problem extends to the establishment of relations between cones and pyramids.

Likewise, comparisons among naive ideas and properties of both families are established. For example, it is pointed out that with parallel cuts to the bases in both families (cylinders and prisms) the shape of the sections is maintained (same form of the bases), and these cuts divide the corresponding solid into other solids with the same form, with the same bases as the original one; and, when adding the corresponding heights, the original solid height is obtained. Immersed in this matter, it is concluded that some prisms can be inscribed in cylinders raising the question of which polygons can be inscribed in a circumference?

4. Comparing cylinders and cones. Prisms and pyramids. When considering a dynamic transformation of one family into another, this transformation is profited to establish relations among the elements of the families of implied solids. For example, when the attention is focused on the transition from a prism into a pyramid, one of the bases of the prism is reduced to a point in the pyramid and it results in the transformation of the lateral faces of the prism into triangles, or that the number of faces in prisms is reduced by one in the number of faces of pyramids, etc.

5. Families of solids and flat shapes. When we focus on counting the elements of regular polyhedrons paying attention to their layout in space, relations are established among this layout and the form of the cuts sections equidistant from opposite faces, vertices or edges. The study is completed with the determination of the different types of planes of symmetry and axes of rotation of each regular polyhedron and the number of planes and axes of each type.

In a context of truncation in cylinders, cones, spheres, prisms and pyramids the relations among the direction of the cut and the form of the sections are established. The process is also considered in a dynamic way; that is, it starts with the observation of a section shape and this is compared with the other sections obtained by parallel cuttings done to the original.

6. Different representations of the solids as a starting point. This situation enables setting relations among different representations or among the corresponding models and their representation. For example, when disassembling the straight cylinder model, the cylinder edges are related to the sides of the rectangle in the flat pattern, and to the length of the circumferences of the bases.

When comparing a model with its flat pattern, problems arise such as the following: To which vertex of the model corresponds a given vertex of its flat pattern? Observing the flat pattern of a cube, can we know the number of faces? Observing at the flat pattern of a solid, can we know the number of faces? How many cuts do we need to make to a model to obtain the flat pattern? Which sides of the flat pattern form an edge in the model?

In order to work on the establishment of relations among the different representations the teacher compares the model properties maintained and the properties that “are broken” in each of them. For example, in a perspective representation of a cylinder, the property of bases being circles is “broken”, or in a perspective representation of the cube, the property of all edges being equal and all angles being equal is “broken”, property that does show on the corresponding flat patterns.

7. Numerical relations. These types of relationships are studied in several contexts. For example, when finding the numerical characteristics of the prisms, we obtain certain relations such as: the number of edges of a n-agonal prism is equal to 4 times the number of lateral faces plus 2 times the number of sides of the polygon of the base; for regular polyhedrons: the number of edges (sides of polygons of the faces) is equal to number of polygons of the sides of faces multiplied by the number of faces and divided into two.

CONCLUSIONS

In Gonzalez et. al. (2006; 2008) we already pointed out that solids constitute a very important context for the development of mathematical activity and we have presented some features that characterize a competent conduct to teach solid geometry in primary school. These results complement those deduced from observations that we will refer to in the following paragraphs. To introduce the study of geometry in primary school, the competent conduct implies putting into practice the different contents recommended in a training plan for teachers related to the establishment of relationships among geometric contents:

- The use of different contexts with all the possibilities they offer for the establishment of relations among geometric contents of the same and different dimension.
- The establishment of relations among geometric contents of one, two and three dimensions.
- To emphasize about the multitude of relations among geometric contents. For example, those that arise when considering different solids families and/or their elements: i) cylinders and prisms, cones and pyramids; ii) some polyhedra families (prisms, pyramids); iii) solids families and flat figures, etc; iv) regular polyhedrons and other solids families; v) relations of inscription and duality among regular polyhedrons.
- To work on the transformation of some solid families into others with different objectives, such as: i) focusing attention on seeing them in a more dynamic way; ii) discovering the properties maintained and lost along the transformation; iii) discovering new knowledge; iv) using knowledge that we already have in order to discover new; v) working on the same geometric content in different contexts and times.

- To present the contents regarding the subject knowledge for its teaching without overlooking the contents of the subject itself. For example, to propose different questions with the intention of generating mathematical activity, emphasizing on the relations expressed and paying attention to the type of language used for this purpose; the use of different materials, diagrams and tables with the aim of facilitating the discovery and verbalization with a each time more specific geometric language of the relationships that arise.

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STUDENTS' 3D GEOMETRY THINKING PROFILES

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This article focuses on the construction, description and testing of a theoretical model for the structure of 3D geometry thinking. We tested the validity and applicability of the model with 269 students (5th to 9th grade) in Cyprus. The results of the study showed that 3D geometry thinking can be described across the following factors: (a) recognition and construction of nets, (b) representation of 3D objects, (c) structuring of 3D arrays of cubes, (d) recognition of 3D shapes' properties, (e) calculation of the volume and the area of solids, and (f) comparison of the properties of 3D shapes. The analysis showed that four different profiles of students can be identified.

INTRODUCTION

Geometry and three-dimensional (3D) thinking is connected to every strand in the mathematics curriculum and to a multitude of situations in real life (Jones & Mooney, 2004, Presmeg 2006). The reasons for including 3D geometry in the school mathematics curriculum are myriad and encompass providing opportunities for learners not only to develop spatial awareness, geometrical intuition and the ability to visualise, but also to develop knowledge and understanding of, and the ability to use, geometrical properties and theorems (Jones, 2002). However, it is widely accepted that the 3D geometry research domain has been neglected and efforts to establish an empirical link between spatial ability and 3D geometry ability have been few in number and generally inconclusive (Presmeg, 2006). Moreover, 3d geometry teaching gets little attention in most mathematics curriculum and students are only engaged in plane representations of solids (Battista 1999; Ben-Haim, Lappan & Houang, 1989). Thus, there is neither a well-accepted theory on 3D geometry learning and teaching, nor a well-substantial knowledge on student's 3D thinking.

The purpose of the present study is twofold. First, it examines the structure of 3D geometry abilities by proposing a model that encompasses most of the previous research in 3D geometry abilities and describes 3D geometry thinking across several dimensions. Second, the study may provide a worthwhile starting point for tracing students' 3D geometry thinking profiles based on empirical data with the purpose of improving instructional practices.

THEORETICAL CONSIDERATIONS

3D Geometry Abilities

For a long time studies on 3D geometry have concentrated mainly on the abilities of students to processes and tasks directly related to school curriculum (NCTM, 2000; Lawrie, Pegg, & Gutierrez, 2000). Following, we describe the main research findings on these 3D geometry abilities.

(a) The ability to represent 3D objects: Plane representations are the most frequent type of representation modes used to represent 3D geometrical objects in school textbooks. However, students have great difficulties in conceptualizing them (Gutierrez, 1992; Ben-Chaim, Lappan, & Houang, 1989). Specifically, students and adults have great difficulties in drawing 3D objects and representing parallel and perpendicular lines in space. Parzysz (1988) pointed out that the representation of a 3D object by means of a 2D figure demands considerable conventionalizing which is not trivial and not learned in school. He concluded that there is a need to explicitly interpret and utilize drawing 3D objects conventions, otherwise, students may misread a drawing and do not understand whether it represents a 2D or a 3D object.

(b) The ability to recognise and construct nets: Net construction requires students' ability to make translations between 3D objects and 2D nets by focusing and studying the component parts of the objects in both representation modes. Cohen (2003) supported that the visualization of nets involves mental processes that students do not have, but they can develop through appropriate instruction. The transition from the perception of a 3D object to the perception of its net, requires the activation of an appropriate mental act that coordinates the different perspectives of the object.

(c) The ability to structure 3D arrays of cubes: Tasks related to enumeration of cubes in 3D arrays appear in many school textbooks. For example, images of cuboids composed by unit-sized cubes are used to introduce students to the concept of volume (Ben-Chaim et al., 1989). The development of this ability is not a simple procedure and as a result primary and middle school students fail in these tasks (Battista 1999; Ben-Chaim et al., 1989). Battista (1999) support that students' difficulties to enumerate the cubes that fit in a box can be explained by the lack of the spatial structuring ability and the inability of students to coordinate and integrate to a unified mental model the different views of the structure.

(d) The ability to recognise 3D shapes' properties and compare 3D shapes: Understanding the properties of a solid equals to understanding how the elements of the solid are interrelated. This understanding may refer to the same object or between objects. The properties of the composing parts, the comparative relations between the same composing parts and the relations between different composing parts compose altogether the properties of a 3D object that students should conceptualize. Although the composing parts of polyhedrons are almost the same, the special characteristics of these parts vary between the different types of polyhedrons (Gutierrez, 1992).

(e) The ability to calculate the volume and the area of solids: 3D geometry ability is closely connected to students' ability to calculate the volume and surface area of a solid (Owens & Outhred, 2006). Research findings showed that students focus only on the formulas and the numerical operations required to calculate the volume or surface area of a solid and completely ignore the structure of the unit measures (Owens & Outhred, 2006). Based on these findings, researchers affirmed that students should develop two necessary skills to calculate the volume and surface area of a solid: (i) the conceptualization of the numerical operations and the link of the formulas with

the structure of the solid, and (ii) the understanding and visualization of the internal structure of the solid.

3D Geometry Levels of Thinking

In plane geometry systematic research efforts have described extensively progressive levels of thinking and define profiles of geometric thinking in various geometric situations. Most of these studies are grounded on Van Hiele's model (Lawrie, Pegg, & Gutierrez, 2000). The van Hiele model of geometric thought outlines the hierarchy of levels through which students progress as they develop of geometric ideas. The model clarifies many of the shortcomings in traditional instruction and offers ways to improve it by focusing on getting students to the appropriate level to be successful in high school Geometry. Gutierrez (1992) extended Van Hiele's model in 3D geometry by analyzing students' behaviour when solving activities of comparing or moving solids is the ground. Students of the first level compare solids on a global perception of the shapes of the solids or some particular elements (faces, edges, vertices) without paying attention to properties such as angle sizes, edge lengths, parallelism, etc. When some one of these mathematical characteristics appears in their answers, it has just a visual role. Students of the second level compare solids based on a global perception of the solids or their elements leading to the examination of differences in isolated mathematical properties (such as angles sizes, parallelism, etc.), apparent from the observation of the solids or known from the solid's name. Their explanations are based on observation. Students of the third level analyze mathematically solids and their elements. Their answers include informal justifications based on isolated mathematical properties of the solids. These properties may be observed in the solids' representations or known from their prior knowledge. Students of the fourth level analyse the solids prior to any manipulation and their reasoning is based on the mathematical structure of the solids or their elements, including properties not seen but formally deduced from definitions or other properties.

THE PURPOSE OF THE STUDY AND THE PROPOSED MODEL

The purpose of the present study is twofold: First, to examine the structure of 3D geometry thinking by validating a theoretical model assuming that 3D geometry thinking consists of the 3D geometry abilities described above. Second, to describe students' 3D geometry thinking profiles by tracing a developmental trend between categories of students. To this end, latent profile analysis, a person-centered analytic strategy, was used to explore students' 3D geometry abilities, allowing for the subsequent description of those patterns in the context of dealing with different forms of 3D geometry situations. In this paper, as it is highlighted in Figure 1, we hypothesized that students' thinking in 3D geometry can be described by six factors that correspond to six distinct 3D geometry abilities. Specifically, the hypothesized model consists of six first order factors which represent the following 3D geometry abilities: (a) Students' ability to recognise and construct nets, i.e., to decide whether a net can be used to construct a solid when folded and to construct nets, (b) students'

ability to represent 3D objects, i.e., to draw a 3D object, and to translate from one representational mode to another, (c) students' ability to structure 3D arrays of cubes, i.e., to manipulate 3D arrays of objects, and to enumerate the cubes that fit in a shape, (d) students' ability to recognise 3D shapes' properties, i.e., to identify solids in the environment or in 2D sketches and to realize their structural elements and properties, (e) students' ability to calculate the volume and the area of solids, i.e., to calculate the surface and perceptually estimate the volume of 3D objects without using formulas, and (f) students' ability to compare the properties of 3D shapes.

METHOD

Sample

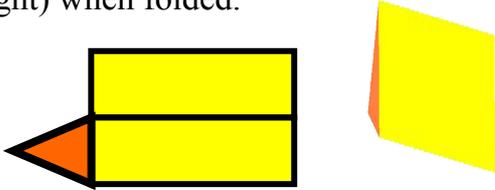
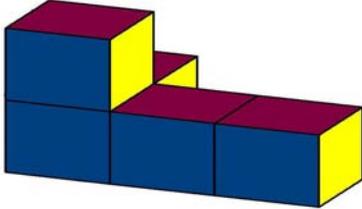
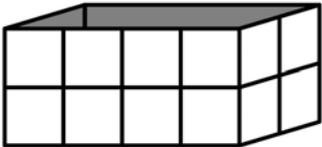
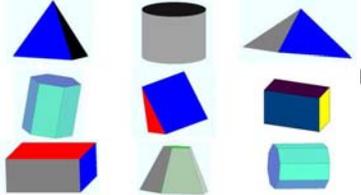
The sample of this study consisted of 269 students from two primary schools and two middle schools in urban districts in Cyprus. More specifically, the sample consisted of 55 fifth grade students (11 years old), 61 sixth grade students (12 years old), 58 seventh grade students (13 years old), 63 eighth grade students (14 years old) and 42 ninth grade students (15 years old).

Instrument

The 3D geometry thinking test consisted of 27 tasks measuring the six 3D geometry abilities: (a) Four tasks were developed to measure students' ability to recognise and construct nets. Two tasks asked students to recognise the nets of specific solids while the other two asked them to construct or complete the net of specific solids. For example (see Table 1), students had to complete a net in such a manner to construct a triangular prism when folded. (b) Six tasks were developed to capture the nature of the factor "students' ability to represent 3D objects", based on the research conducted by Parzysz (1988) and Ben-Chaim, Lappan, and Houang (1989). Two tasks required students to translate the sketch of a solid from one representational mode to another. For example (see Table 1), students were asked to draw the front, top and side view of an object based on its side projection. (c) Four tasks were used to measure the factor "students' ability to structure 3D arrays of cubes". For example (see Table 1), students were asked to enumerate the cubes that could fit in open and close boxes. (d) Five tasks were developed to measure the factor "students' ability to recognise 3D shapes' properties". For example (see Table 1), students were asked to identify the solids that had minimum eight vertices. The second task asked students to identify the solids that were not cuboids out of twelve objects drawn in a solid form. The other three tasks asked students to enumerate the vertices, edges and faces of three pyramids drawn in transparent view. (e) Four tasks were used as measures of the factor "students' ability to calculate the volume and the area of solids". For example, students were asked to calculate how much wrapping paper is needed to wrap up a cuboid built up by unit-sized cubes. Students should have visualized the object and split its surface area into parts. Two other tasks asked students to calculate the surface area and the volume of cuboids that were presented in a net form (proposed by Battista, 1999). (f) Three tasks were developed to measure the factor "students'

ability to compare the properties of 3D shapes". For example, students were asked to decide whether statements referring to properties of solids were right or wrong (see Table 1). The other two tasks asked students to explore the Euler's rule and extend it to the case of prisms.

Table 1: Examples of the 3D geometry thinking tasks.

The ability to recognise and construct nets	The ability to represent 3D objects
Complete the following net in a proper manner to construct the triangular prism (at the right) when folded.	Draw the front, side and top view of the object.
	
The ability to structure 3D arrays of cubes	The ability to recognise 3D shapes' properties
How many unit-sized cubes can fit in the box?	Circle the solids that have at least 8 vertices.
	
The ability to calculate the volume and the area of solids	The ability to compare properties of 3D shapes
Find the area of the box.	Which of the following statements are wrong? <i>-The cuboid is not a square prism.</i> <i>-The prisms' and cuboids' faces are rectangles.</i> <i>-The base of the a prism, a cuboid and a pyramid could be a rectangle</i>

Data Analysis

The structural equation modelling software, MPLUS, was used (Muthen & Muthen, 2007) and three fit indices were computed: The chi-square to its degrees of freedom ratio (χ^2/df), the comparative fit index (CFI), and the root mean-square error of approximation (RMSEA). The observed values for χ^2/df should be less than 2, the values for CFI should be higher than .9, and the RMSEA values should be lower than .08 to support model fit (Marcoulides & Schumacker, 1996).

RESULTS

In this section, we refer to the main issues of the study. First, we present the results of the analysis, establishing the validity of the latent factors and the viability of the structure of the hypothesized latent factors. Second, we present the exploration of the data for meaningful categories with respect to 3D geometry abilities, and then working up from those categories, we present the characteristics of each 3D geometry thinking profile.

The structure of 3D geometry thinking

In this study, we posited an a-priori (proposed) structure of 3D geometry thinking and tested the ability of a solution based on this structure to fit the data. The proposed model for 3D geometrical thinking consists of six first-order factors. The six first-order factors represent the dimensions of 3D geometry thinking described above: students' ability to recognise and construct nets (F1), students' ability to represent 3D objects (F2), students' ability to structure 3D arrays of cubes (F3), students' ability to recognise 3D shapes' properties (F4), students' ability to calculate the volume and the area of solids (F5), and students' ability to compare the properties of 3D shapes (F6). The six factors were hypothesized to correlate between them (see Figure 1). Figure 1 makes easy the conceptualisation of how the various components of 3D geometry thinking relate to each other.

The descriptive-fit measures indicated support for the hypothesized first order latent factors ($CFI=.95$, $\chi^2=375.88$, $df=301$, $\chi^2/df=1.25$, $p<0.05$, $RMSEA=.03$). The parameter estimates were reasonable in that all factor loadings were statistically significant and most of them were rather large (see Figure 1). Specifically, the analysis showed that each of the tasks employed in the present study loaded adequately only on one of the six 3D geometry abilities (see the first order factors in Figure 1), indicating that the six factors can represent six distinct functions of students' thinking in 3D geometry. The results of the study showed that the correlations between the six factors are statistically significant and high (see Table 3). The correlation coefficients between F1 with F2 ($r=.94$, $p<.05$), F1 with F3 ($r=.96$, $p<.05$), F2 with F4 ($r=.92$, $p<.05$), F3 with F5 ($r=.97$, $p<.05$) and F4 with F6 ($r=.92$, $p<.05$) were greater than .90.

Students' 3D Geometry Thinking Profiles

To trace students' different profiles of 3D geometry thinking we examined whether there are different types of students in our sample who could reflect the six 3D geometry abilities. Mixture growth modeling was used to answer this question (Muthen & Muthen, 2007), because it enables specification of models in which one model applies to one subset of the data, and another model applies to another set. The modeling here used a stepwise method-that is, the model was tested under the assumption that there are two, three, and four categories of subjects. The best fitting model with the smallest AIC and BIC indices (see Muthen & Muthen, 2007) was the one involving four categories. Taking into consideration the average class

probabilities (not presented due to space limitations), we may conclude that each category has its own characteristics. The means and standard deviations of each of the six 3D geometry abilities across the four categories of students are shown in Table 2, indicating that students in Category 4 outperformed students in Category 3, 2 and Category 1 in all 3D geometry ability factors, students in Category 3 outperformed their counterparts in Categories 2 and 1, while students in Category 2 outperformed their counterparts in Category 1.

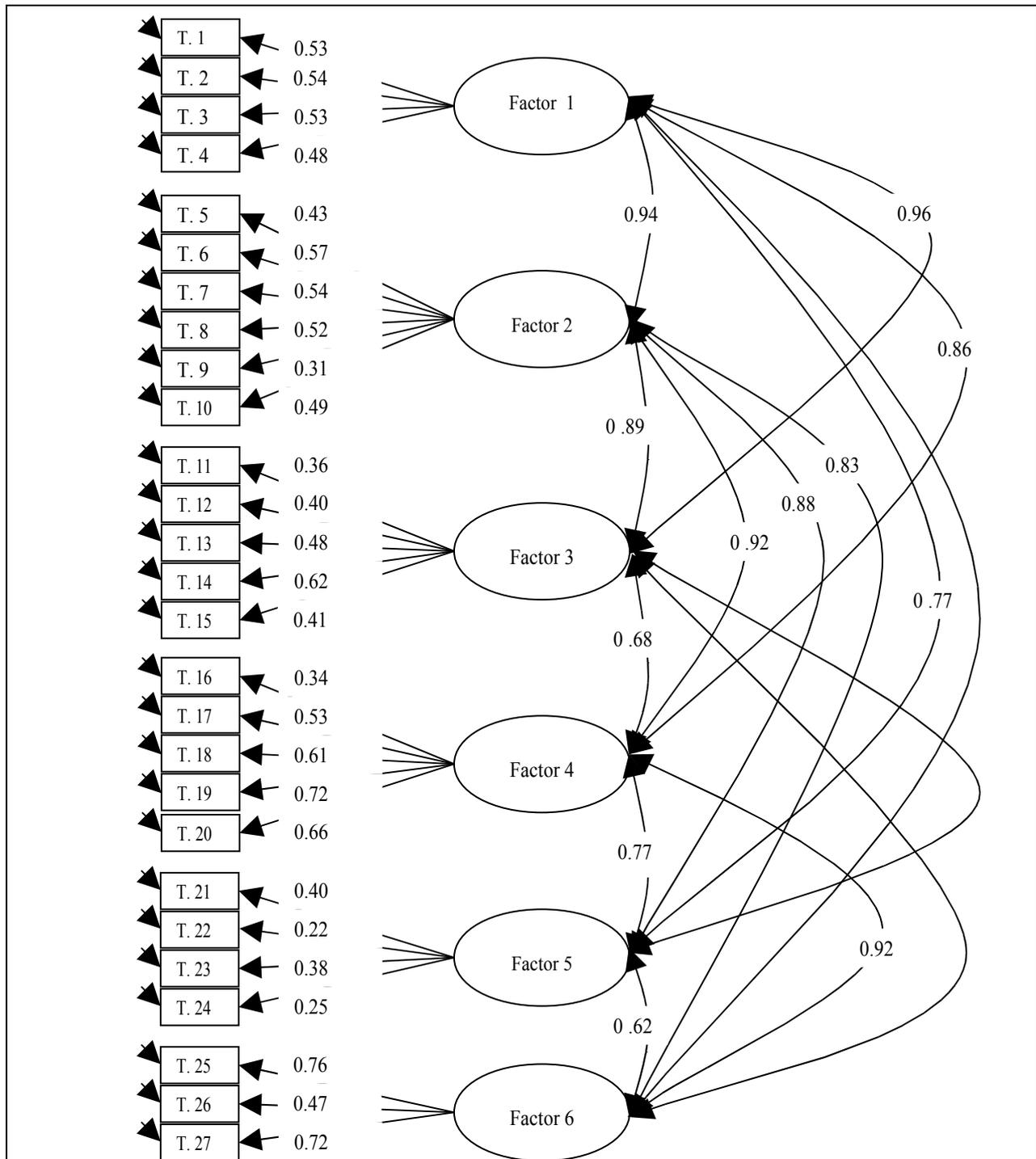


Figure 1: The structure of 3D geometry thinking.

From Table 3, which shows the problems solved by more than 50% or 67% of the students in each category, it can be deduced that there is a developmental trend in students' abilities to complete the assigned tasks of the six factors because success on any problem by more than 67% of the students in a category was associated with such success by more than 67% of the students in all subsequent categories.

Table 2: Means and Standard Deviations of the Four Categories of Students

Category	Factor 1		Factor 2		Factor 3		Factor 4		Factor 5		Factor 6	
Categ. 1				S.D.		S.D.						
Categ. 2	Mea	S.D.	Mea	0.1	Mea	0.1	Mea	S.D.	Mea	S.D.	Mea	S.D.
Categ. 3	n	0.16	n	8	n	3	n	0.17	n	0.18	n	0.17
Categ. 4	0.28	0.21	0.30	0.1	0.15	0.2	0.51	0.15	0.24	0.21	0.10	0.17
	0.54	0.20	0.51	7	0.29	0	0.71	0.14	0.40	0.23	0.28	0.24
	0.76	0.14	0.71	0.1	0.55	0.2	0.83	0.07	0.49	0.22	0.50	0.22
	0.88		0.86	7	0.83	2	0.92		0.77		0.78	
				0.1		0.1						
				3		9						

The data imply that there are four profiles of students' 3D geometry thinking according to the characteristics of the four categories of students. The first profile of 3D geometry thinking represents the students that recognize in a sufficient way 3D shapes but fail in the other 3D geometry tasks. The second profile of 3D geometry thinking represents the students that do not have any problems in recognizing 3D shapes and have some difficulties in recognizing and constructing nets and representing 3D shapes. Students that belong to the third profile of 3D geometry thinking grasp easily recognizing and representing 3D shapes tasks and recognizing and constructing nets tasks. However, students of the third profile have difficulties in structuring 3D arrays of cubes and comparing 3D shapes' properties. The fourth profile represents the category of students that successfully solves tasks related to the recognition of 3D shapes' properties, the comparison of 3D shapes' properties, the recognition and construction of nets tasks, the structuring of 3D arrays of cubes, the representation of 3D shapes and the calculation of volume and area of solids.

Table 3: Problems Solved by More than 50% or 67% of Students in Each Category

	F1 tasks	F2 tasks	F3 tasks	F4 tasks	F5 tasks	F6 tasks
Category 1				■		
Category 2	■	■		√		
Category 3	√	√	■	√		■
Category 4	√	√	√	√	√	√

■: Problems solved by more than 50%, √: Problems solved by more than 67%

DISCUSSION

The results of the study suggested that 3D geometry thinking can be described across six dimensions based on the following factors which represent six distinct 3D geometry abilities. The first factor is students' ability to recognise and construct nets, by deciding whether a net can be used to construct a solid when folded and by constructing nets. The second factor is students' ability to represent 3D objects, such as drawing a 3D object, constructing a 3D object based on its orthogonal view, and translating from one representational mode to another. The third factor is students' ability to structure 3D arrays of cubes by manipulating 3D arrays of objects, and enumerating the cubes that fit in a shape by spatially structuring the shape. The fourth factor is students' ability to recognise 3D shapes' properties, by identifying solids in the environment or in 2D sketches and realizing their structural elements and properties. The fifth factor is students' ability to calculate the volume and the area of solids. The sixth factor is students' ability to compare the properties of 3D shapes, by comparing the number of vertices, faces and edges, and comparing 3D shapes' properties. The structure of 3D geometry thinking suggests that students need to develop their own 3D geometry skills that integrate the six 3D geometry parameters described above. Based on this assumption, we could also speculate that the most common definition of 3D geometry by other researchers (Gutierrez, 1992) as the knowledge and classification of the various types of solids, in particular polyhedrons, is not sufficient. 3D geometry thinking implies a large variety of 3D geometry situations which do not correspond necessarily to certain school geometry tasks. The results of the study revealed that the six factors are strongly interrelated. The correlation coefficients between the first factor and the second factor, the first factor and the third factor and the third factor and the fifth factor were the stronger ones. This result could be explained by the fact that these factors are strongly related with spatial ability skills.

The second aim concerned the extent to which students in the sample vary according to the tasks provided in the test. The analysis illustrated that four different categories of students can be identified representing four distinct profiles of students. Students of the first profile were able to respond only to the recognition of solids tasks. Students of the second profile were able to recognize and construct nets and represent 3D shapes in a sufficient way. Students of the third profile did not have any difficulties in the recognition and construction of nets and the representation of 3D shapes and furthermore they were able in structuring 3D arrays of cubes and calculating the volume and area of solids in a sufficient way. Students of the fourth profile were able in all the examined tasks.

The identification of students' 3D geometry thinking profiles extended the literature in a way that these four categories of students may represent four developmental levels of thinking in 3D geometry, leading to the conclusion that there are some crucial factors that determine the profile of each student such as the ability to represent 3D objects and the ability to structure 3D arrays of cubes. These two

abilities are closely related to spatial visualization skills (Battista, 1991; Parzysz, 1988). This assumption promulgates the call to study in depth the relation of 3D geometry thinking with spatial ability by using a structured quantitative setting.

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CERME 6 – WORKING GROUP 6

LANGUAGE AND MATHEMATICS

TABLE OF CONTENTS

Introduction.....	828
<i>Candia Morgan</i>	
Imparting the language of critical thinking while teaching probability.....	833
<i>Einav Aizikovitsh, Miriam Amit</i>	
Toward an inferential approach analyzing concept formation and language processes	842
<i>Stephan Hußmann, Florian Schacht</i>	
Iconicity, objectification, and the math behind the measuring tape: An example from pipe-trades training	852
<i>Lionel LaCroix</i>	
Mathematical reflection in primary school education: theoretical foundation and empirical analysis of a case study	862
<i>Cordula Schülke, Heinz Steinbring</i>	
Surface signs of reasoning	873
<i>Nathalie Sinclair, David Pimm</i>	
A teacher’s use of gesture and discourse as communicative strategies in concluding a mathematical task.....	884
<i>Raymond Bjuland, Maria Luiza Cestari, Hans Erik Borgersen</i>	
A teacher’s role in whole class mathematical discussion: facilitator of performance etiquette?	894
<i>Thérèse Dooley</i>	
Use of words – Language-games in mathematics education	904
<i>Michael Meyer</i>	
Speaking of mathematics – Mathematics, every-day life and educational mathematics discourse	914
<i>Eva Riesbeck</i>	
Communicative positionings as identifications in mathematics teacher education.....	924
<i>Hans Jørgen Braathe</i>	

Teachers' collegial reflections of their own mathematics teaching processes Part 1: An analytical tool for interpreting teachers' reflections.....	934
<i>Kerstin Bräuning, Marcus Nührenbörger</i>	
Teachers' reflections of their own mathematics teaching processes Part 2: Examples of an active moderated collegial reflection.....	944
<i>Kerstin Bräuning, Marcus Nührenbörger</i>	
Internet-based dialogue: a basis for reflection in an in-service mathematics teacher education program	954
<i>Mario Sánchez</i>	
The use of algebraic language in mathematical modelling and proving in the perspective of Habermas' theory of rationality.....	964
<i>Paolo Boero, Francesca Morselli</i>	
Objects as participants in classroom interaction	974
<i>Marei Fetzer</i>	
The existence of mathematical objects in the classroom discourse	984
<i>Vicenç Font, Juan D. Godino, Núria Planas, Jorge I. Acevedo</i>	
Mathematical activity in a multi-semiotic environment	993
<i>Candia Morgan, Jehad Alshwaikh</i>	
Engaging everyday language to enhance comprehension of fraction multiplication	1003
<i>Andreas O. Kyriakides</i>	
Tensions between an everyday solution and a school solution to a measuring problem.....	1013
<i>Frode Rønning</i>	
Linguistic accomplishment of the learning-teaching processes in primary mathematics instruction.....	1023
<i>Marcus Schütte</i>	
Mathematical cognitive processes between the poles of mathematical technical terminology and the verbal expressions of pupils	1033
<i>Rose Vogel, Melanie Huth</i>	

INTRODUCTION

LANGUAGE AND MATHEMATICS

Candia Morgan, Institute of Education, University of London

The 21 papers presented to the Working Group were marked by a wide diversity of research focuses and theoretical perspectives. We therefore organised the discussion around five themes:

- Language and thought
- Classroom interaction
- Teacher development
- Theoretical perspectives to describe, analyse and interpret the semiotic aspects of students' mathematical activities
- 'Everyday' and mathematical language and learning

As will be seen from summaries of each of the sections below, there is some overlap between the issues considered in each theme. For example, the use of gesture has become of increasing interest and importance in the field and is found as a focus in papers in several of the themes. Similarly, while the relationship between everyday and mathematical language is a significant theme in its own right, it also emerges as an issue of relevance across other themes.

SECTION 1: 'LANGUAGE' AND THOUGHT

'Language' has a material, and therefore public, surface: either visible (writing and gesture - including sign language) or audible. On the other hand, thinking is invisible and inaudible. Therefore there is a challenge to render it observable, which must of necessity be by indirect observation. This sets up two fundamental tensions:

- Between the individual and the social
- Between implicit and explicit expression

The papers in this section propose different perspectives on how to make sense of the relation between language and thought.

- Focus on gestures, broad view on language (LaCroix)
- Reflection (Schülke/Steinbring)
- Inferential approach (Hußmann/Schacht)
- Argumentation: Toulmin model (Pimm/Sinclair)

- Critical thinking (Aizikovitsch/Amit)

SECTION 2: CLASSROOM INTERACTION

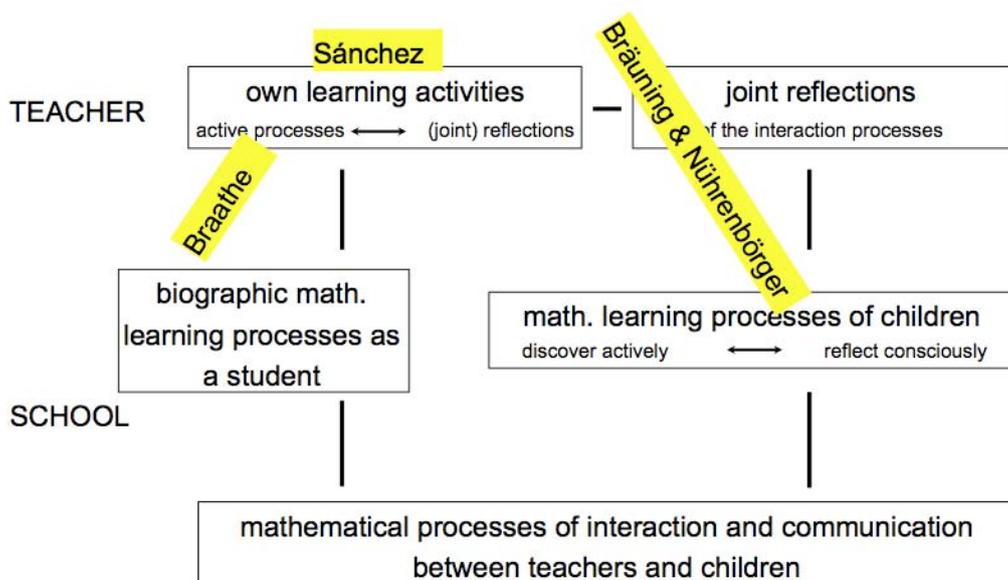
The theme “Classroom interaction” indicates that the papers in this section focus on the whole classroom, the relationships between teacher and students and among students and the role that language plays in establishing these relationships and in building mathematical discourse. The papers use a range of perspectives including the Wittgenstein’s language games, the notion of teacher as improviser, a focus on the use of gesture, shared thinking in group talk, and the interplay between everyday and mathematical discourse, aiming:

- to get deeper insight into processes of giving meaning to words in class (Meyer)
- to show how teacher and pupils co-construct new mathematical ideas using the improvisation metaphor (Dooley)
- to describe the communicative strategies of an experienced teacher when summing up pupil solutions (Bjuland et al.)
- to consider how discourse, as a theoretical and didactical concept, can contribute towards developing mathematics teaching (Riesbeck)

SECTION 3: TEACHERS’ PROFESSIONAL DEVELOPMENT

“Teachers’ professional development” is a major theme of the papers presented by HansJørgen Braathe, Kerstin Bräuning, Marcus Nührenbörger and Mario Sánchez. The understanding of different interaction forms of teachers` distanced view on communication and interaction processes is a necessary condition for their development, as Dewey (1916, 4) pointed out, “society not only continues to exist by transmission, by communication, but it may fairly be said to exist in communication.”

Each paper analysed ideas and thoughts expressed by teachers in written and oral form. But each paper deals with different aspects and schemas of professional development. The following diagram is separated in two levels: “teacher with distance to communication processes in school” and “the mathematical learning and teaching in school”. The level “Teacher” means that teachers are integrated in two different activities: On the one hand their own mathematical learning activities, and on the other hand their joint reflections. Each teacher has biographical mathematical learning processes. This aspect is located in-between the levels “Teacher” and “School”. The 2nd level “School” includes the mathematical learning processes of children and the interaction between teachers and children.



Each paper highlights not only different aspects and methodological approaches to teachers' professional development, but also refers to different theoretical frameworks – like positioning theory, inquiry cooperation model, epistemological and interactional theory. The variety of the theories deepens and broadens the insights in the special conditions of teachers' interactions and learning processes connected to language and mathematics.

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SECTION 4: THEORETICAL PERSPECTIVES TO DESCRIBE, ANALYZE AND INTERPRET THE SEMIOTIC ASPECTS OF STUDENTS' MATHEMATICAL ACTIVITIES

A common aspect of the four papers of this theme is the fact that their structure consists in the presentation of a *new* or *adapted* theoretical tool (or perspective), followed by some examples that are chosen to illustrate (and, possibly, discuss) the use and the potential of the proposed tool (or perspective). A *common*, problematic situation in mathematics education is particularly relevant in the *specific case* of these papers: the plurality of theoretical references (from different disciplines: linguistics, epistemology, psychology, sociology...) brings a proliferation of theoretical tools. Two legitimate questions are related to the previous remark: what educational need/problem should the theoretical tools (or perspectives) satisfy? And what effective educational implications do they have?

Boero and Morselli present a **comprehensive** tool derived from Habermas' construct of "rational behaviour" to describe and analyse student use of algebraic language. By integrating Blumer's "Symbolic interactionism" and Latour's "Actor-network-theory", Fetzer offers a perspective to analyse classroom interaction and discuss related interpretations. Font et al. present "Objectual metaphors", a particular kind of

(Lakoff & Nunez) “Grounding metaphor”, as a tool to analyze and discuss how the classroom discourse helps to develop students’ comprehension of the non ostensive mathematical objects. Morgan and Alshwaikh argue that a multi-semiotic environment not only affords rich potential for developing mathematical concepts, but may also affect more fundamentally the goals of student activity.

The discussion of the group of papers demonstrated openness to alternative theoretical perspectives. Not only may we consider what we can learn from others but attending to different perspectives serves to sharpen our understanding of our own theories. However, there are problems with the proliferation of theories that need to be managed, showing how various perspectives may be useful while being alert to the possibilities and constraints of combining or ‘merging’ theories. There is also felt to be a need to maintain links with the original sources of theoretical perspectives.

Theoretical ideas also have implications with respect to practice. They can provide language to help researchers see new aspects of practice. Moreover, through being introduced to theoretical ideas, teachers could develop awareness of complexities of the classroom

SECTION 5: ‘EVERYDAY’ AND MATHEMATICAL LANGUAGE AND LEARNING

All four papers of this theme group are in various ways occupied with links between everyday and mathematical concepts. Analysing classroom data the authors identify attempts to create such links. The discussion of the development of scientific concepts in children can be traced back to Vygotsky who describes this as a cooperative process between an adult and the child. Kyriakides discusses diagrams as a mediating tool in learning about fraction multiplication and points to an episode where the introduction of everyday language, instead of trying to remember an algorithm, proved to be an effective link to the scientific concept. On the other hand, Schütte describes an episode having to do with adding fractions, where the scientific concept *least common multiple* is lying behind. The teacher mainly uses everyday language, and the link to the scientific concept and her assisting function in the pupils’ development of mathematical language seem to be lost. In the paper by Vogel and Huth, the focus is on a combinatorial problem where two first graders, assisted by an adult, gradually start to use technical terms and the practical context become less and less important. Rønning studies a situation where the pupils are measuring milk, and where both teacher and pupils are moving back and forth between an everyday situation and a school situation. The two situations involve different semiotic representations and also different goals and actions, which can be seen to create a certain tension.

The following topics for discussion were identified.

- The function of everyday language in learning mathematics
- The function of diagrams in learning mathematics

- The teacher as a model for learning technical (scientific) language.

IMPARTING THE LANGUAGE OF CRITICAL THINKING WHILE TEACHING PROBABILITY

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*This paper reports a preliminary study of imparting to students a new kind of language, incorporating elements of **critical thinking (CT)**, in the course of a mathematics (probability) lesson. In the paper, we describe and analyse one probability lesson, which is part of an in-depth study that comprises fifteen math lessons of similar constitution. The purpose of this research is to determine whether the teaching methods we developed can improve students' critical thinking. Our approach favors immersion-teaching of CT, i.e. incorporating CT terminology and practice within the framework of a probability lesson, and is based on the specific taxonomy of CT skills proposed by Ennis. We focus specifically on critical thinking while distinguishing it from stochastic thinking, creative thinking and statistical thinking. This study involved 55 subjects. Analysis of interviews conducted with the students and an analysis of their submitted work indicated that students' critical and analytical capabilities greatly improved. These results show that if teachers consistently and methodically encourage CT in their classes, by applying mathematics to real-life problems, encouraging debates, and planning investigative lessons, the students are likely to develop the language of critical thinking as a result. This paper is a description of an initial study, a snapshot that focuses on one lesson and illustrates the orientation of the entire study.*

INTRODUCTION AND THEORETICAL FRAMEWORK

It has already been suggested that teachers should use a language of critical thinking as part of the attempt to change the method of teaching to enable meaningful learning of information (Perkins, 1992). This is an area in which a substantial research literature already exists.

Our focus in this paper is describing our approach and its initial results. In this paper, we are focusing on the language of critical thinking. When defining the term critical thinking (CT), it is important to realize that it is not a new concept; we can find it as early as ancient Greek times: Socrates, as reported by Plato, used to roam the streets of Athens asking people all kinds of philosophical questions about the purpose of life, morality, justice, etc., apparently for the purpose of stimulating a form of critical thinking. These questions and answers were collected and recorded in the Socratic dialogues. In the field of education, it is generally agreed that CT capabilities are crucial to one's success in the modern world, where making rational decisions is becoming an increasingly important part of everyday life. Students must learn to test reliability, raise doubts, and investigate situations and alternatives, both in school and in everyday life. Abundant definitions of critical thinking have been proposed, since

this is a multidisciplinary subject that engaged teachers, educators, sociologists, psychologists and philosophers in all eras, but we would like to focus on Ennis' taxonomy, because for our purposes we needed to employ a hierarchical set of critical thinking skills isolated from other definitions. Ennis (1962) defines CT as “a correct evaluation of statements”. Twenty-three years later, Ennis broadened his definition to include a mental element, defining CT as “reasonable reflective thinking focused on deciding what to believe or do” (Ennis, 1985). Our research is based on three key elements: a CT taxonomy that includes CT skills (Ennis, 1987); the learning unit "Probability in Daily Life" (Lieberman & Tversky, 2002); and the infusion approach of integrating subject matter with thinking skills (Swartz, 1992).

Ennis' Taxonomy (Ennis, 1987)

In light of his definition, Ennis developed a CT taxonomy of skills that include intellectual as well as behavioural aspects, e.g. judging the credibility of sources, searching for clarifying questions, defining the variables, searching for alternatives etc. In addition to skills, Ennis's taxonomy (1987) also includes dispositions and abilities. Ennis claims that CT is a reflective and practical activity aiming for a moderate action or belief. There are five key concepts and characteristics defining CT: practical, reflective, moderate, involving? belief and oriented towards? action.

Learning unit "Probability in Daily Life" (Lieberman & Tversky 2002)

In this learning unit, which is a part of the formal syllabus of the Ministry of Education, the students are required to analyse problems, raise questions and think critically about data and information. The purpose of the learning unit is to teach the students not to be satisfied with a numerical answer but to examine the data and its validity in order to arrive at a more valid answer and develop their critical thinking. In cases where there is no single numerical answer, the students are required to know what questions to ask and how to analyse the problem qualitatively, not only quantitatively. Along with being provided with statistical instruments, students are redirected to their intuitive mechanisms to help them estimate probabilities in daily life. Simultaneously, students examine the logical premises behind their intuitions, along with possible misjudgments of their application.

The infusion approach (Swartz, 1992)

There are two main approaches to fostering CT: *the general skills approach* which is characterized by designing special courses for instructing CT skills, and the *infusion approach*, according to Swartz (1992), is characterized by providing these skills through teaching the set learning material. According to this approach, there is a need to reprocess the set material in order to combine it with thinking skills. In this report, we will show, on the example of one lesson, how we combined the mathematical content of "probability in daily life" with CT skills from Ennis' taxonomy, and evaluated the subjects' CT skills.

METHODOLOGY

The main paradigmatic aspects of methodology in mathematics education research have been broadly established (Scherer & Steinbring, 2006). Our methodological challenge was to investigate the development of the "language of critical thinking" through critical thinking skills incorporated into a structured mathematics lesson, such as a probability lesson. In this regard, the methodological approach is closest to the "Design Experiment" (as discussed by Cobb, Confrey, diSessa, Lehrer and Schauble, 2003). Through careful instructional design, a lesson sequence was constructed with the goal of consistently and methodologically encouraging and promoting critical thinking by applying mathematics to real-life problems, encouraging debates and using investigative lessons, in order to develop the "language of critical thinking". The research process examined student classroom products (primarily student submitted work) and post-lesson interviews with students to document changes in students' analytical capabilities. These changing capabilities could then be related to classroom activities, which were documented by video.

Setting, Population, and Data

Fifty-five children between the ages of fifteen and sixteen participated in an extra curriculum program aimed at enhancing the critical thinking skills of students from different cultural backgrounds and socio-economical levels. An instructional experiment was conducted in which probability lessons were combined with CT skills. The study consisted of fifteen 90 minute lessons, spread out over the course of an academic year, in which the teacher was also one of the researchers.

Data sources were: Students' products, Pre and post questionnaires, Personal interviews and Class transcriptions.

The students' products (papers, homework, exams etc.) were collected. Five randomly selected students were interviewed at the end of each lesson and one week after. The personal interviews were conducted in order to identify any change in the students' attitudes throughout the academic year. Not only was the general attitude examined, attention was paid to the development of critical thinking language (e.g., by asking the student to define critical thinking and to explain how they viewed critical thinking in the scope of the lesson; furthermore, they were also asked to assess whether they considered themselves to be critical thinkers, and it was the answer to this question that was used to establish the nature and frequency of critical thinking among them). All lessons were video-recorded and transcribed. In addition, the teacher kept a journal (log) on every lesson. Data was processed by means of qualitative methods intended to follow the students' patterns of thinking and interpretation with regards to the material taught in different contexts. Following Ennis' taxonomy (Ennis, 1987), data was analysed by employing three principles: (1) As the student is asked to articulate the question dealt with in a particular lesson, the level of critical thinking was deciphered (as will be discussed later on); (2) students' reactions to the teacher's attempt to induce critical thinking were examined through their responses as well as

from the interviews; (3) proposition of alternatives was employed as an interview technique, in an attempt to identify critical thinking abilities.

The Intervention- Unit Description

As already mentioned, the probability unit combines CT skills with the mathematical content of "probability in daily life". This new probability unit included questions taken from daily life situations, newspapers and surveys, and combined CT skills. Each of the fifteen lessons that comprised the probability unit had a fixed structure: a generic (general) question written on the blackboard; the student's reference to the question and a discussion of the question using probability and statistical instruments; and, an open discussion of the question that included practicing the CT skills. The mathematical topics taught during the fifteen lessons were: Introduction to set theory, probability rules, building a 3D table, conditional probability and Bayes theorem, statistical connection and causal connection, Simpson's paradox, and judgment by representativeness. The following CT skills were incorporated in all fifteen lessons: A clear search for an hypothesis or question, the evaluation of reliable sources, identifying variables, "thinking out of the box," and a search for alternatives (Aizikovitsh & Amit, 2008). Each lesson followed the same four part structure.

1. Given Text

At the beginning of the lesson the teacher presented a short article or text.

2. Open Class Discussion in Small Groups

Discussion in small groups about the article and the question.

- Initial suggestions for the resolution of the question
- No intervention by the teacher

3. Further Discussion Directed by the Teacher

Open class discussion. During the discussion the teacher asked the students different questions to foster the students' thinking skills and curiosity and to encourage them to ask their own questions.

- Various suggestions from students in class.
- Interaction between groups of students.
- Reaching a consensus across the whole class (or just across the group).

4. Critical Thinking Skills and Mathematical Knowledge (Teaching)

The teacher referred to the questions raised by the students and encouraged CT, while instilling new mathematical knowledge: the identification of and finding a causal connection by a third factor and finding a statistical connection between C, and A and B, Simpson's paradox and Bayes Theorem.

Case study- The Aspirin Case

Below, I have provided a detailed description of one lesson called the Aspirin Case. Following the description, I outline the analysis of the lesson using the following techniques: referring to information sources, raising questions, identifying variables, and suggesting alternatives and inferences. The lesson topic was conditional probability. The CT skills practiced in the lesson were evaluating source reliability, identifying variables, and suggesting alternatives and inference.

1. A Given Text

Your brother woke up in the middle of the night, crying and complaining he has a stomachache. Your parents are not at home and you don't know what to do. You gave your brother aspirin, but an hour later he woke up again, suffering from bad nausea and vomiting. The doctor that takes care of your brother regularly is out of town and you consider whether to take your brother to the hospital, which is far from your home. You read from a book about children's diseases and find out that there are children that suffer from a deficiency in a certain type of enzyme and as a result, 25% of them develop a bad reaction to aspirin, which could lead to paralysis or even death. Thus, giving aspirin to these children is forbidden. On the other hand, the general percentage of cases in which bad reactions such as these occur after taking aspirin is 75%. 3% of children lack this enzyme.

(Taken from "probability thinking" p. 30+slight changes made by researcher)

2. Open Class Discussion in Small Groups

Discussion in small groups about the generic question:

Should you take your brother to the emergency room? What should you do?

Can aspirin consumption be lethal?

3. Further Discussion Directed by the Teacher

The generic question on the blackboard was:

Should you take your brother to the emergency room? What should you do?

21 Teacher: What do you think?

22 Student 1: Where is the information taken from? Can we see the article for ourselves?

23 S2: Is the source reliable? How can we check it?

24 S3: Where is the article taken from? What is its source?

25 S1: Should I answer the identification of the sources question?

26 T: Not yet. We are focusing on searching for questions. Please think of other questions.

27 S3: What connection does the article discuss?

28 S2: first we need to identify the variables!!!

29 T: Right. First, we ask what the variables are.

30 S4: You can infer it from the title that suggests that a connection exists between aspirin and death.

31 T: According to the data from the article, Can we find a statistical connection? (the student already know this subject)

32 S2: I know! We can ask: suggest at least 2 other factors that might

cause the described effect.

- 33 S5: The question is what causes what?
 34 S6: Can aspirin consumption be lethal?
 35 T: What do you think?
 36 T: How can you be sure?
 37 S6: Umm...
 38 S3: Are there other factors, such as genetics!?
 39 T: Very good. What did student 3 just do?
 40 S1: He suggested an alternative!!
 41 T: How can we check it? Do you have any suggestions? Can you make a connection between this problem and the material we have learned in the past few lessons? Can you offer an experiment that would solve the problem?
 42 S3: Of course. An observational experiment.

In paragraph 21 we encounter skills such as "searching for the question"- a fundamental skill. First there is a need to clarify the starting point for the interaction with the student. We also need to clarify to ourselves what is the thesis and what is the main question before we approach decision making. The paragraph also demonstrates relevance to daily life. In paragraph 26 the students are taking a step back, we refer to "identifying information source and evaluating the source's reliability" skill. This step is crucial, as it helps us to assess the quality and the validity of the article discussed. This skill was practiced in past lessons. See paragraph that summarizes the article. In paragraph 26 we encounter "searching for the question" skill again. We will continue searching for the main question through practicing the "variables identification" skill. Raising the search for alternatives. Posing questions enables the practice of this skill. Paragraph 30 deals with identifying the variables and understanding them by a 2D table and a conditional probability formula. In paragraph 36 the teacher builds the students' self esteem by encouraging them to express their ideas and opinions (even if they are not always correct or relevant). She prevents any intolerance of other students. The method of instruction that aims at fostering the confidence and the trust of the students in their CT abilities and skills is, according to Ennis "referring to other peoples points of view" and "being sensitive towards other peoples' feelings". In paragraph 23 the student is referring to other sets and finding the connection between them. Paragraph 31 depicts the skill of "Searching for alternatives". Paragraph 42 refers to a controlled experiment or an observational experiment. An additional grouping and finding the connection between the variables by Bayes theorem or a 2 dimensional table.

4. Critical Thinking Skills and Mathematical Knowledge (Teaching)

This phase of the lesson focused on encouraging critical thinking and instilling new mathematical knowledge (Bayes formula) statistical connections by referring to students' questions and further discussion.

A teacher-led discussion focused on methods of analysis using such Critical Thinking skills as: Source identification: Medicine book; Source reliability: High; Variable identification: A – enzyme deficiency, D – adverse reaction to aspirin; Mathematical Knowledge: Data: $P(D/A)=0.25$ $P(D)=0.75$ $P(A)=0.03$, To prove: $P(A/D)=?$

Using Bayes formula (or a two dimensional matrix) the result is:

Lesson Conclusion is that only 1% of the children without the enzyme develop an adverse reaction to aspirin, thus there is no need to go to the hospital.

Even so, is it worth taking the risk? What do you think? (question to the class).

DISCUSSION

Research analysis according to critical thinking skills in this case study

Through the infusion approach, students practice their CT while acquiring technical probability skills. In this lesson, the following five skills are exercised: raising questions – asking question about the article and probing on the main question about the connection between aspirin and death; referring to information sources and evaluating the source's reliability - the text took from Medicine book; the students skepticism and identification of variables – students identified the enzyme deficiency and adverse reaction to aspirin. Following these skills, another skill, searching for alternatives (paragraph 38), was presented. In class the teacher and the students spoke about suggesting alternatives, not taking things for granted, but examining what had been said and suggesting other explanations. Hence, the skills that were practiced in the described lesson were: raising questions, evaluating the source's reliability, identifying variables, and suggesting alternatives and inference. In order to understand and monitor the students' attitudes toward CT as manifested by the skills specified above, interviews were conducted with five students after the aforementioned lesson. In these interviews, the students acknowledged the importance of CT. Moreover, students were aware of the infusion of instructional strategies that advance CT skills. Examples from two of the interviews follow.

Student 4 was interviewed and was asked to define CT. His answer was:

"I think CT is important when you study Mathematics, when you study other topics and when you read the paper, but it is most important when you deal with real life situations, and you need the right instruments in order to do so (deal with these situations)."

When Student 2 was asked about important components during the last few classes and the present class, she answered: *"first we should check the information source's reliability and despite all the numerical data, I don't accept the researcher's conclusion."*

Additional data, consistent with these two examples suggest that infusion of CT into the formal curriculum in mathematics can equip students with CT skills that are applicable to wider disciplines.

RESEARCH LIMITATIONS

This case study presents one lesson which was designed in a fixed pattern – a generic question, a discussion of the question, the practice of statistical connection, introduction to causal connection and experiencing the use of CT skills such as: raising questions, evaluating the source's reliability, identifying variables, and suggesting alternatives and inferences. On the basis of the interviews conducted and questionnaires that were qualitatively analyzed, it is not established, at this stage, the extent to which these skills have been acquired. Skill acquisition will be evaluated in much greater detail at a later phase in this study, using quantitative measures – the Cornell Critical Thinking Scale and the CCTDI (Facion, 1992) scale. At this stage we have provided only an introductory picture of our approach and an indication of the form of our analysis and results. However, this case study provides encouraging evidence of the effectiveness of this approach and further investigation in this direction is needed.

CLOSING REMARKS

The small scale research described here constitutes a small step in the direction of developing additional learning units within the traditional curriculum. Current research is exploring additional means of CT evaluation, including: the Cornell CT scale (Ennis, 1987), questionnaires employing various approaches, and a comprehensive test composed for future research.

The general educational implications of this research suggest that we can and should lever the intellectual development of the student beyond the technical content of the course, by creating learning environments that foster CT, and which will, in turn, encourage the student to investigate the issue at hand, evaluate the information and react to it as a critical thinker. It is important to note that, in addition to the skills mentioned above, in the course of this lesson it appears that the students also gained intellectual skills such as conceptual thinking and developed a class culture (climate) that fostered CT. Students practiced critical thinking by studying probability. In this lesson, the following skills were demonstrably practiced: referring to information sources (paragraph 22), encouraging open-mindedness and mental flexibility (all questions), a change in attitude and searching for alternatives. A very important intellectual skill is the fostering of cognitive determination – to be able to express one's attitude and present an opinion that is supported by facts. In this lesson, students could be seen to be searching for the truth, they were open-minded and self-confident. In other words, they practiced critical thinking skills. A new language was being created: the language of critical thinking.

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TOWARD AN INFERENTIAL APPROACH ANALYZING CONCEPT FORMATION AND LANGUAGE PROCESSES

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This paper introduces a theoretical approach to study individual conceptual development in mathematics classroom. It uses the theory of a normative pragmatics as an epistemological framework, which Robert BRANDOM made explicit in 1994. There are different levels of research in mathematics education on which BRANDOM's framework offers a consistent theoretical approach for describing such developments: a linguistic perspective, the theory of conceptual change and the theory of conceptual fields. Using that framework, we will outline an empirical example to describe technical language developments as well as developments of conceptual fields and of the students' conceptualizations.

INTRODUCTION

Many results of large-scale studies monitoring the education system (PRENZEL et al. 2007, ARTELT et al. 2000, BAUMERT et al. 1997, BAUMERT et al 1998) show for mathematics education that German students have difficulties with tasks that challenge their conceptual understanding. These difficulties seem to be caused by the German classroom practices, which do not challenge enough the students' individual cognitive skills, which lack teachers' diagnosis abilities, and which do not offer enough room for creative and individual work (e.g. PRENZEL et al. 2004).

Research is required in both mathematical learning environments and in formation of concepts and conceptualizations in order to find out in how far (i) the use of the specific potential which certain tasks offer and (ii) the dealing with students' conceptualizations have an effect on the formation of conceptual thinking. In Germany, there are only some studies which focus on the analysis of individual concept-formation (HUßMANN 2006, BARZEL 2006, HAHN / PREDIGER 2008, Prediger 2008a/b). There is also a demand for research with regard to dealing with certain individual students' conceptualizations.

Because mathematical thinking is genuinely conceptual thinking, the formation of mathematical concepts has gained big interest in the mathematics education research community. The multiple approaches and theories for describing and explaining conceptual processes and developments differ a lot in terms of their theoretical framework, e.g. developmental psychology or cognitive psychology. In this study, we choose a social-constructivist approach (COBB, YACKEL 1996).

With his theory of inferentialism, the philosopher Robert BRANDOM (1994) has introduced a convincing, comprehensive and coherent theoretical framework to analyze such language processes.

EPISTEMOLOGICAL FRAMEWORK: INFERENTIALISM

In his influential book on reasoning, representing and discursive commitment “Making it explicit” (1994) BRANDOM chooses an inferential approach to describe semantic content of concepts in terms of their use in practice: it is the idea that propositional semantic content can be understood in terms of the inferential relations they play in discourse, which means for example to know what follows from a proposition or what is incompatible with it. BRANDOM gives an analysis of discursive linguistic practice, describing a model of social practice - and especially a model of linguistic discursive practice - as a game of giving and asking for reasons, which means a normative pragmatics in terms of deontic scorekeeping. Using his theory to describe linguistic practice and based on the theory of a normative pragmatics introduced by BRANDOM (1994), we will develop an analytic tool to describe the formation of concepts. For BRANDOM, understanding

can be understood, not as the turning on of a Cartesian light, but as practical mastery of a certain kind of inferentially articulated *doing*: responding differentially according to the circumstances of proper application of a concept, and distinguishing the proper inferential consequences of such application. (BRANDOM 1994, p. 120)

In this sense, discourse can be described as a game of giving and asking for reasons, a term that can be traced back to WITTGENSTEIN’S ‘Sprachspiel’ (language game). Therefore, every ‘player’ in the game of giving and asking for reasons keeps score on the other players. This deontic score keeps track on the claims that every player (including oneself) is committed to and it keeps track on the commitments each one is entitled to. With every assertion – so with every move in the game of giving and asking for reasons - which one player is making, the score may change.

The inferential relations are commitment - and entitlement- preservations and incompatibilities. BRANDOM’S normative pragmatics gives an understanding of conceptual content on the basis of using the concepts in practice. “The aim is to be able to explain in deontic scorekeeping terms what is expressed by the use of representational vocabulary - what we are doing and saying when we talk about what we are talking *about*.” (BRANDOM 1994, p. 496)

BRANDOM claims that the fact that propositions have a certain (propositional) content should be understood in terms of inferential relations. Accordingly, propositions are propositions because they have the characteristic feature to function as premises and conclusions in inferences (that means they function as reasons).

Thus grasping the semantic content expressed by the assertional utterance of a sentence requires being able to determine both what follows from the claim, given the further commitments the scorekeeper attributes to the assertor, and what follows from the claim, given the further commitments the scorekeeper undertakes. (...) In such a context, particular linguistic phenomena can no longer reliably be distinguished as ‘pragmatic’ or ‘semantic’. (BRANDOM 1994, pp. 591/592)

It is important to note that it is not necessary for an individual to know all the inferential roles of a certain concept to be regarded as someone that has conceptualized a certain concept. "To be in the game at all, one must make *enough* of the right moves - but how much is enough is quite flexible" (BRANDOM 1994, p. 636).

DERRY (2008) outlines the characteristics of an inferential view for education. Referring to BRANDOM and VYGOTSKY she notes that the

priorisation of inference over reference entails, in terms of pedagogy, that the grasping of a concept (knowing) requires committing to the inferences implicit in its use in a social practice (...). Effective teaching involves providing the opportunity for learners to operate with a concept in the space of reasons within which it falls and by which its meaning is constituted. (DERRY 2008, p. 58)

CONCEPTUAL DEVELOPMENT RESEARCH IN MATHEMATICS EDUCATION

Using Robert BRANDOM's ideas of a normative pragmatics, it is the aim of the project to develop a coherent theoretical framework within which the formation of concepts in mathematics education can be described. This theoretical framework uses inferential (instead of representational) vocabulary. There are different levels of research in mathematics education on which BRANDOM's framework offers a consistent theoretical approach for describing such developments.

Theory of conceptual fields

Using Robert BRANDOM's theory of a normative pragmatics as an epistemological background to describe formations of concepts, VERGNAUD's theory of conceptual fields offers a consistent framework within which long- and short-term conceptual developments can be analyzed. Within his framework, he gives respect to both mathematical concepts and individual conceptualizations.

WITTENBERG says that mathematics is "thinking in concepts" (1963). What distinguishes us as human beings is the fact that we are concept users (Brandom 1994). Accordingly, not only mathematics is thinking in concepts: everything obtains a conceptual meaning for us and concepts are the smallest unit of thinking and acting. This decisive linguistic perspective of conceptual understanding was pointed out by SELLARS: "grasping a concept is mastering the use of a word" (see BRANDOM 2002, p. 87). Accordingly, it is necessary to research concept formation, which means it is necessary to study the classroom discourse. For that, VERGNAUD (1996, 1997) offers a solid theoretical framework. With his theory of conceptual fields, VERGNAUD developed a theoretical framework which picks up BROUSSEAU's theory of didactical situations (1997) and which offers a tool to describe, to analyze and to understand both short- and long-term formations of concepts. For him, a conceptual field refers to a set of (problem) situations, conventional and individual concepts.

[A] *conceptual field* is a set of situations, the mastering of which requires several interconnected concepts. It is at the same time a set of concepts, with different properties, the meaning of which is drawn from this variety of situations. (VERGNAUD 1996, p. 225)

A concept is a three-tuple of three sets: $C = (S, I, S)$ where S is the set of situations that make it meaningful, I is the set of operational invariants contained in the schemes developed to deal with these situations, and S is the set of symbolic representations (natural language, diagrams (...)) that can be used to represent the relationships involved, communicate about them, and help us master the situations. (VERGNAUD 1996, p. 238)

In the latter definition, VERGNAUD points out that language is essential for focusing on conceptual fields. Language is the surface on which we analyze formations of concepts. Conceptual fields are equally related to situations, to mathematical concepts, to individual conceptualizations and to operational invariants such as *theorems-in-action* or *concepts-in-action*. On the one side, those operational invariants are *theorems-in-action* which are “held to be true by the individual subject for a certain range of the situation variables” (VERGNAUD 1996, p. 225). On the other side, they are categories- or *concepts-in-action*,

that enable the subject to cut the real world into distinct elements and aspects, and pick up the most adequate selection of information according to the situation and scheme involved. Concepts-in-action are, of course, indispensable for theorems-in-action to exist, but they are not theorems by themselves. They cannot be true or false (VERGNAUD 1996, p. 225).

In every new situation, the individual schemes develop. Because of the strong connection between situation and scheme, the short-term perspective on concept formation is important to study. At the same time, because of the individual development within the learning process and the different situations the individual deals with, the long term perspective is equally important to study.

Linguistic approach

Besides the theory of conceptual fields, there is a specific linguistic approach that can be drawn from BRANDOM’S epistemological framework. Therefore, SIEBEL (2005) refers to developments from colloquial to technical language by making implicit concepts explicit.

Thought and language is not the same, otherwise we would not be able to form sentences like “I don’t know how to say it” or “that is not what I meant”. Still, we can only get a precise picture of conceptual developments by observing the use of language, the discourse, that what’s made explicit. To get an idea of what is implicit in use, we have to ask for reasons and commitments.

In her linguistic approach categorizing and analyzing technical language used in elementary algebra books, Siebel (2005) picks up that distinction. She distinguishes between explicit and implicit technical terms. Explicit ones are explicitly defined, e.g. by “ x is called variable”. Explicit technical terms are characteristic for explicit knowledge (‘know-that’) which can be made explicit in either words or formulas. In

contrast, the meaning of implicit terms is characterized by their use (SIEBEL 2005, p. 120). Implicit technical terms are characteristic for implicit knowledge ('know-how') which can only be learnt by practical exercising. SIEBEL points out that most of our concepts are implicit and that we can only make some of them explicit (see SIEBEL 2005, p. 122). Referring to BREGER (1990), SIEBEL describes how knowledge and concepts develop from "know-how" to "know that" knowledge, from implicit to explicit knowledge – by making them explicit (2005, p. 122). That linguistic approach offers a description of developments from colloquial to technical language, lining out how implicit concepts and knowledge ("know-how") become explicit ("know-that").

Judgments as basic units

Following BRANDOM, the linguistic perspective cannot be separated from the propositional content. With every commitment and every judgment, we have taken on a certain kind of responsibility and committed ourselves to some explanation of the given phenomenon. Those explanations and judgments correspond to the theoretical schemes (see VERGNAUD 1996) which are intimately interwoven with the specific situation.

Theory of conceptual change

Following BRANDOM and VERGNAUD, learning and formation of concepts is closely linked to a specific situation. The developments that proceed in these situations are closely connected to the conceptualizations we have. These conceptualizations maybe have to be revised, expanded or modified in every new situation which we have to commit ourselves to, for example to a certain scheme or an explanation. The theory of conceptual change (e.g. DUIT 1996) picks up that distinction between individual conceptualizations and scientific conceptions.

The conceptual change theory is a constructivist approach to describe learning processes in terms of reorganization of knowledge (Duit 1996, p. 158, Prediger 2008b for an example in mathematics education). That means for the students to learn that their preinstructional concepts do not give sufficient orientation in certain scientific situations and for them to activate scientific conceptions in those situations (see DUIT 1996, p. 146). Learning scientific concepts often leads to conflicts with prior knowledge and familiar everyday concepts because certain features of both – familiar and new scientific concepts - seem to be incompatible. FISCHER and AUFSCHNAITER (1993) for example studied developments of meaning during physics instruction, focusing on the terms charge, voltage and field. Against the background of different levels of perception, they describe how the use of certain words changes during the learning process: "For this reason, at the beginning of the development of a subjective domain of experience it might be possible that words, as properties of objects, are not yet generated." (p. 165)

Summary

In all the perspectives above, there is a similar line of thought concerning the analysis and description of conceptual developments: intuitive concepts-in-action to consolidated mathematical concepts, implicit meaning of use to explicit technical language, pre-instructional conceptualizations to scientific concepts. The aim of our project is to follow those lines among linguistic descriptions of expressions in mathematics classrooms and to develop learning environments which considering the formation of concepts in mathematics classrooms.

For this purpose, we study the development of individual long- and short-term conceptualizations and of formations of mathematical concepts within learning processes: what is the connection between (problem) situations and operational invariants (such as *theorems-in-action* or *concepts-in-action*)? What is the connection between the formation of concepts and symbolic expressions? In how far is it possible to classify the (problem) situations against the background of individual operational invariants?

Three aspects can be inferred from those questions: How does technical language develop? How do individual conceptualizations develop? How do conceptual fields develop? To examine these questions, we develop an empirical study to describe the individual learning processes.

ONE EXAMPLE ON (TECHNICAL) LANGUAGE DEVELOPMENT

To give an example of how the research questions outlined above can be approached, we offer some results of a small-scale study on technical language development (SCHACHT 2007). This example shows how technical language in chance-situations can develop, how individual conceptualizations develop and how the conceptual field of chance-situations has developed.

Short introduction to the study

For this purpose a fifth grade mathematics classroom of 30 students was observed and videotaped over a period of about six weeks. The central goal of the unit for the students was to develop a concept of 'chance'. That means that in chance situations, the individual case will not be predictable, but focusing the long term, chance has a certain kind of mathematical structure (HEFENDEHL-HEBEKER 2003). Accordingly, one special focus of this unit was for the students to discover and experience the law of large numbers.

Main features of the unit concerning the research interests of the study were the focus on *discursive elements* in mathematics classroom, the focus on *reflection tasks* during the mathematical learning process and the focus on *student-* rather than *teacher-activity* (cf HUBMANN/PREDIGER 2009).

Based on a functional pragmatic approach, language was analyzed in terms of its use (e.g. EHLICH/REHBEIN 1986, KÜGELGEN 1994). The features of the unit mentioned above formed a solid base to analyze language developments of some students

especially because they were often challenged to make their concept of chance explicit (either in written form or verbally).

Some results of the study

The results of this small scale study show some interesting phenomena which could be observed. We will outline one prototypical example of the study and describe its main features concerning technical language development as well as individual conceptualizations and conceptual fields development.

In this example, the task for the student Ralf was to describe and compare results of dice throws in different situations (10, 100, 500 and 1000 throws). Because of the qualitative differences of the situations which he is working in (description of absolute values \square description of relative values), the technical language he uses leads to a paradox situation (distance (“Abstand”) is ‘small’ and ‘large’ at the same time). A couple of days after this situation, he uses a different and new term which seems more sufficient and viable.

More precisely, the student Ralf first uses the term ‘distance’ to compare some results of dice-throws. In the first scene, he uses the term ‘distance’ to compare absolute results.

Number	1	2	3	4	5	6
10 throws	3	1	0	4	0	2
1000 throws	165	174	169	161	171	160

Table 1:

Similar example of dice results in absolute values (10 and 1000 throws)

Comparing results similar to Table 1, Ralf observes:

- 102 And in the situation with small numbers of throws
103 the distances (“Abstände”) get smaller.

There are two aspects to point out concerning the use of the term ‘distance’. First, he compares the dice results by noticing that the “distances get smaller” (line 103) the smaller the number of throws is. In the example above, that means that there is a little distance between the one time ‘2’ and the two times ‘6’ but there is a greater distance between the 160 times ‘6’ and the 171 times ‘5’. Second, he uses the term distance to distinguish between situations with a high number of throws (e.g. 1000 throws) and a small number of throws (100 throws).

Later in the same lesson, he uses the same term (‘distance’) again to compare dice results, except now they are given in relative values (in percent). The teacher asks the students to compare a couple of histograms which show the results of 10-100-500-1000 throws. The histograms which show results of 10 throws of course look quite different to those with 1000 throws. The latter ones show the stabilization of the relative distribution (law of large numbers) while the others show that the results with for example 10 throws differ quite a lot.



Table 2: Similar example of histograms showing dice results in relative values (10 throws)

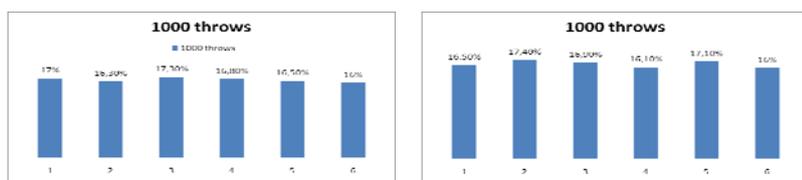


Table 3: Similar example of histograms showing dice results in relative values (1000 throws)

The teacher asks Ralf, what he noticed. Ralf answers:

- 8 Ralf: I observed that,
 9 given a small number of throws,
 10 the distances (“Abstände”) become larger
 11 and given a large number of throws,
 12 the distances (“Abstände”) become smaller.

In this situation Ralf describes that the distances become larger given a small number of throws. It seems plausible that he has a horizontal perspective and compares all histograms showing the results of 10 throws whereas the “distances become smaller” comparing the others with for example 1000 throws.

At the same time, like in the situation above, Ralf is using the term ‘distance’ again to distinguish the small and the large ‘number-of-throws situations’. Except that he uses the term conversely: in the first situation he described the distances to become smaller when the number of throws becomes *smaller* (lines 102/103), in the latter situation he observes the distances to become larger when the number of throws becomes *smaller* (lines 8/9).

Comparing both examples, the difficulty is that the quality of the situation changes: in the first situation, Ralf compares the absolute values of the dice results of 100 and of 1000 throws. He recognizes that the distances of the results with 10 throws are smaller than the ones with 1000 throws (lines 102/103).

Accordingly, although the term ‘distance’ is a quite helpful and viable term *in each situation* to distinguish between small and high number of throws, it is overall not sufficient because it seems to lead to paradox and incompatible results.

Some days later the students are asked to give a written comment on the following sentence: “You cannot predict the result of throwing a single dice, but in the long run you don’t have a random result.” Ralf writes:

- 130 Given a small number of throws
 131 you cannot predict
 132 chance, but
 133 given a higher number of throws, that works better
 134 because it is more distributed (“verteilter”) there.

The next day, he adds on a working sheet in a similar situation:

- 5* in the situation of thousand throws, the distribution (“Verteilung”) is: (...)

In both quotes, Ralf uses the term ‘distribution’ / ‘distributed’ to distinguish between small and large numbers of throws. For him, this term works without inconsistencies

to distinguish both situations. He is also able to predict a distribution in the large number-of-throws situation (line 5*).

Summary

Focusing on technical language development from a linguistic perspective, this example describes a development of the intuitive and implicit use of the term ‘distance’ to an explicit use of the technical term ‘distribution’ that is viable to distinguish between small and large number of throws.

There are two different concepts-in-action Ralf uses: in the first situation, he has a binary concept for comparing the results. In the other situation, Ralf observes a certain structure given a high number of throws. Here, his concept-in-action is that given a high number of throws and a certain mathematical structure, chance is predictable. That effects his theorem-in-action: given a high number of throws, the (relative) distribution can be predicted quite precisely.

This development shows his conceptual change regarding chance situations: whereas his intuitive conceptualization focuses on the term ‘distance’, he then is able to activate a mathematical conception on chance situation using the technical term ‘distribution’ which focuses on the long-term perspective on chance situations. The conceptual change is in line with the dynamic development of Ralf’s theorem-in-action: the new problem situation leads him to come up with a new theorem-in-action.

This example shows in how far all three levels are connected in terms of the inferential epistemological approach that BRANDOM introduces: both conceptual change and conceptual fields help to observe the formation of concepts. But these processes can only be studied because we are concept users (BRANDOM): language is the surface on which the linguistic analysis of the formation of concepts operates.

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ICONICITY, OBJECTIFICATION, AND THE MATH BEHIND THE MEASURING TAPE: AN EXAMPLE FROM PIPE-TRADES TRAINING ¹

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This paper examines an adult student's efforts as he works intensely, with the help of the researcher, to make sense of the fraction patterns on a measuring tape marked in inches. The multi-semiotic analysis of this encounter is framed using Radford's Theory of Knowledge Objectification. From this socio-cultural perspective, mathematics learning involves the social and semiotically mediated process of objectification, i.e. a process in which one becomes progressively aware and conversant, through one's own actions and interpretations, of the cultural logic of mathematical objects. This paper contributes to Radford's notion of iconicity by showing, through fine-grained analysis, relevant aspects of its dynamics as well as by calling attention to a form of iconicity that, to my knowledge, has not been reported elsewhere.

INTRODUCTION AND THEORETICAL FRAMEWORK

This paper is based upon a small part of an impromptu tutorial session involving a pre-apprentice in the pipe-trades with the researcher serving as mathematics tutor. It is part of a larger case study that focuses on the manner in which the pre-apprentice attempts to make sense of, and become fluent with, the mathematics embedded in a measuring tape marked in feet and inches—an essential skill for the pre-apprentice's chosen vocation. While Canada has officially adopted the metric system and most students study measurement exclusively using metric units in their mathematics courses in elementary and secondary school, the use of imperial units of linear measure (e.g. feet and inches) remains common practice in the construction trades. Consequently, is it not unusual to find students at the start of workplace training in the construction trades who struggle with the cultural practice of measuring lengths in fractions of an inch using a measuring tape.

The study draws upon Radford's (2002, 2008a, 2008b) socio-cultural theory of knowledge objectification (TO) to examine the manner in which the pre-apprentice begins to notice the mathematics embedded within the inscriptions on a measuring tape. In this theory, learning is conceptualized as the active and creative acquisition of historically constituted forms of thinking. Such an acquisition is thematized as a problem of *objectification*, that is, as a problem of becoming conscious of, and

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critically conversant with, the cultural-historical logic with which mathematical and other objects have been endowed. One of the aspects that makes the idea of objectification distinctive is the close relationship that it bears with the Vygotskian concept of consciousness and the mediated nature of it (Vygotsky, 1979, also Leont'ev, 1978). Consciousness is formed through encounters with other voices and the historical intelligence embodied in artifacts and signs with which we mediate our own actions and reflections. Within this context the efforts that the pipe-trade pre-apprentice undertakes to make sense of the mathematics of a measuring tape are seen as a process of objectification. One of the questions is to investigate how the cultural meaning of the mathematics behind the measuring tape becomes “recognized” by the pre-apprentice. The question is not only the manner in which personal and cultural meanings become tuned, for personal meanings can only arise and evolve against the backdrop of forms of activity. Here the TO departs from other approaches. The problem is precisely the very social formation and evolution of personal meanings as they evolve within goal directed activity and are framed by the cultural meanings conveyed by socio-cultural contexts.

Several contemporary approaches emphasize, for various theoretical reasons, the embodied dimension of thinking (see, e.g. Arzarello, 2006; Lakoff & Núñez, 2000; Nemirovsky & Ferrara, 2008) and the role of artifacts (Bartolini Bussi & Mariotti, 2008). In the TO, the sensuous and artifact mediated nature of thinking leads, methodologically, to paying attention to the semiotic means through which objectification is accomplished. These means are called *semiotic means of objectification*. Much more than being simple aids to thinking, semiotic means of objectification are constitutive and consubstantial parts of thinking and include kinesthetic actions, gestures, artifacts (e.g. rulers, tools), and/or signs, e.g. mathematical symbols, inscriptions, written and spoken language (see Radford, 2008c); they allow one to draw one's own attention and/or the attention of another to particular aspects of cultural objects (Radford, 2003; Radford, L., Miranda, I. & Guzmán, 2008).

In his recent work, Radford has identified two main (and interrelated) processes of objectification, namely *iconicity* and *contraction* (2008a). While contraction refers to the process of making semiotic actions compact, simplified and routine as a result of acquaintance with conceptual traits of the objects under objectification and their stabilization in consciousness, iconicity is a link between past and present action: it refers to the process of noticing and re-enacting significant parts of previous semiotic activity for the purpose of orienting one's actions and deepening one's own objectification (Radford, personal communication, September 29, 2008). One of the goals of this paper is to contribute to this idea of iconicity by showing, through fine-grained analysis, some relevant aspects of its dynamics as well as to call attention to a new form of iconicity that, to my knowledge, has not been reported elsewhere.

METHODOLOGY

Data collection

The data for this study was collected in a pipe-trades pre-apprenticeship training class being conducted at a trade-union run school in British Columbia, Canada. This program involved pencil and paper work in the classroom as well as practical work in the workshop. It was designed to give the pre-apprentices a head start with important skills that would be addressed subsequently in the early years of their formal apprenticeship training in a number of different pipe-trades.

Throughout this pre-apprenticeship course the researcher served as a math tutor for any pre-apprentices who sought out his help. At other times, the researcher observed pre-apprentices and engaged them in discussion about their mathematics related coursework as they were working on it. The activity of individual and groups of pre-apprentices, working either with the researcher or working on their own, was documented using a video camera. Copies of the course print materials and copies of pre-apprentices' written work were also retained for analysis. The data for this paper was selected from this collection of data.

The individual who is the focus of this analysis, was a secondary school graduate. He had been in the workforce and completed a small number of courses in an electronics-technician training program at a community college during the three and a half years between the time that he finished secondary school and the time he began the pre-apprenticeship program in the pipe-trades. Throughout the pre-apprenticeship course he actively sought out the researcher for help with his mathematics related work.

Data analysis

A *multi-semiotic analysis* was conducted of the pre-apprentice's and the researcher-as-tutor's joint activity during their one-on-one tutoring session to investigate process of knowledge objectification. This involved the construction of a transcript of the dialogue from the video-recording of the session, along with a detailed account of significant actions, semiotic systems, and artifacts used. This process required, at times, a slow-motion and frame-by-frame analysis of video tape to assess the role and coordination of spoken language with the use of artifacts and gestures during the encounter.

The analysis to be discussed here focuses on an excerpt from the beginning of the tutoring session with the pre-apprentice, who will henceforth be referred to as "C". The researcher will henceforth be referred to as "L". This session took place at a table in the classroom immediately after L discovered that C was having difficulty reading fractions of-an-inch from his measuring tape while working on a pipe-fitting project with his colleagues in the workshop. The focus here is on C's objectification of the difference in the fraction marking patterns on the measuring tape below and above 12 inches, or one foot, where they are marked to thirty-seconds of an inch and sixteenths of an inch respectively. These two marking patterns can be seen in figure one. This is

one of a number of mathematical patterns inscribed on the measuring tape that C comes to notice and coordinate as he becomes proficient with reading the measuring tape over the course of the entire thirty-two minute tutorial.



Figure 1. The marking lines to the left of one foot indicate fractions to thirty-seconds of an inch. On the right side of one foot the markings indicate fractions to sixteenths of an inch. (C has inscribed a line across the measuring tape with his pencil at $11 \frac{1}{8}$ "', partly obscuring the measuring tape inscriptions, and another short line over the marking at $11 \frac{5}{32}$ "'.)

RESULTS AND DISCUSSION

The shared goal of C and L's work together in the tutoring session is for C to learn how to read fractions on the measuring tape to sixteenths of an inch or, using the language of the TO, to objectify the system of fractions-of-an-inch crystallized within this cultural artifact (the measuring tape). C needs to learn this to be able to complete a pipe-fitting project that he is working on, as well as for his ongoing training, and for his future work as a trades person. L's immediate goal in this particular episode is for C to begin noticing differences and similarities in the marking patterns on the measuring tape.

Semiotic means of objectification using gestures and signs

The measuring tape from C's tool box is extended on the table top in front of both C and L and the session begins with L asking C what difference he notices between the pattern of spaces on his measuring tape below 12 inches and above 12 inches.

75. L: ... What do you notice here between the spaces here, up to twelve [Gesture-uses the index finger of his left hand to sweep up from the zero end of the measuring tape and pauses at 12" just before saying "up to twelve"]
76. C: Yeah its,
77. L: and the spaces after twelve? [G-now pointing with the fourth finger of his left hand to sweep through the exposed interval of the tape measure above 12"]

Here L asks C to explain what he notices while using two distinct sweeping gestures separated by a static pointing gesture at the twelve inch point. This in an attempt to draw C's attention to, and initiate his objectification of, these two intervals as distinct regions of the measuring tape. L emphasizes this distinction by using different pointing fingers to sweep through each of the intervals and a contrasting static pointing gesture at the end of his sweep up to 12 inches to highlight the boundary point between them. As every educator knows, posing a question like this one is an

effective means of drawing a student's attention to, and having him or her engage in a more critical way with, an object at hand. In this short excerpt L's question is framed through the coordinated use of spoken language to describe the two regions of the measuring tape, and the use of a static pointing gesture and two different forms of sweeping gestures. Together, spoken language and gesture serve as semiotic means of objectification for C.

Gestures dominate C's response to L's question. This is clear by considering his spoken words alone, which provide only a vague and partial response. It is only through C's use of spoken language, interspersed with an elaborate and coordinated sequence of ten gestures, each positioned in a precise way relative to the measuring tape that it becomes clear that he is, indeed, becoming consciously aware of the way in which the marking patterns on the measuring tape are different from one another.

(Transcript note: The spoken words in the transcript below are printed in bold to assist the reader to differentiate these from the descriptions of the accompanying actions.)

- 78 C: **There's**, [G(Video frame 1, 26:52)–sweeps up through the first few inches of the tape measure with the fourth finger of his left hand in a manner similar to the gesture just enacted by L]
there's more. [G(Video frame 2, 26:53)–makes two chopping motions aligned with the markings on the tape measure with his left hand, the first significantly larger than the second just before he says “there's more” in reference to the markings inscribed on the measuring tape.
 G(Video frame 3, 26:54)–points to the 12” mark with the fourth finger of this left hand before withdrawing it from the measuring tape].

		
<p>Video frame 1 (26:52). C sweeps up through the first few inches of the measuring tape.</p>	<p>Video frame 2 (26:53). C makes two chopping motions aligned with the markings on the measuring tape.</p>	<p>Video frame 3 (26:54). C points to the 12” mark.</p>

In line 78, C begins his description of the difference between the two marking patterns on the measuring tape. He starts by sweeping the fourth finger of his left hand upwards through the first few inches of the measuring tape (Video frame 1). This is the same type of one finger indexical sweeping gesture that L had just used

(albeit using a different finger) to draw attention to this region of the measuring tape. C embellishes L's original gesture sequence by including a chopping gesture midway up this interval. This chopping gesture is aligned with the series of parallel markings inscribed on the measuring tape and reflects the familiar action of physically dividing or chopping up the interval on the measuring tape in the same way as is indicated by the inscribed measuring tape markings (Video frame 2, 26:53). Immediately following this gesture C says "there's more" (line 78), a confirmation that he is, indeed, referring to the closely packed markings inscribed on this region of the tape measure. C resumes and finishes his sweep through this region of the tape measure by pointing with the same finger of his left hand to the 12 inch point, the endpoint of this interval (Video frame 3, 26:54), before taking this hand away from the measuring tape. This use of a static single-finger pointing gesture at the 12 inch point separating the two regions of the measuring tape is the same type of gesture that L used a few seconds earlier to separate his sweeping gestures at the 12 inch point as well.

(line 78 continues) **It's like it's more spread out** (in reference to the markings on the tape measure after the 12 inch point.) [G(Video frame 4, 26:55a)–points briefly to the 12" mark on the tape measure now with the first finger of his right hand, replacing the previous pointing gesture expressed by the fourth finger of his left hand.

G(Video frame 5, 26:55b and Video frame 6, 26:56a)–starting with his thumb positioned at the 12 inch point, sweeps his right hand up the measuring tape a short distance while holding an approximately 2.5" wide interval between the thumb and first finger.]



Video frame 4 (26:55a). C points again to the 12" mark on the measuring tape.



Video frame 5 (26:55b). C begins to sweep an approximately 2.5" wide interval up the measuring tape starting with his right thumb at 12".



Video frame 6 (26:56a). C continues his wide-interval sweep up the measuring tape.

(line 78 continues) **when** [G(Video frame 7, 25:56b)–grasps the tape measure with his right thumb and first finger on opposite edges at the 12" point and G(Video frame 8, 26:57)–sweeps his hand in this configuration upwards a short distance from 12"] **you pass one,**

79 L: **Yeah,**

80 C: **one foot** [G(not shown)–while maintaining the same grasping position, repeats this sweep upwards for a second time]



Video frame 7 (26:56b) C grasps the measuring tape at the 12" point.



Video frame 8 (26:57) C sweeps his hand in this configuration upwards a short distance from 12" and then repeats this motion a second time.

When line 78 continues, C replaces, briefly, his left hand pointing gesture at the 12 inch point with the first finger of his right hand (Video frame 4). This reflects, in part, L's earlier set of indexical gestures, i.e. using different pointing fingers to distinguish between the two different regions of the measuring tape. C then forms a wide-interval gesture using his right thumb and first finger and without hesitation sweeps this up the measuring tape with his right thumb starting from the 12 inch point (Video frame 5 to Video frame 6). As he does this he says "it's more spread out" (line 78). This reflects the wider interval spacings between adjacent fraction markings inscribed here. C then grasps the measuring tape at 12 inches with his right thumb and first finger in a position that looks like he is grasping or pinching it (Video frame 7), and then sweeps his hand up the measuring tape from 12 inches and Video frame 8) and then repeats this a second time (not shown). This series of three sweeps up the measuring tape from the 12 inch point (one wide-interval sweep and two grasping sweeps) serves to sustain both his own and L's attention on this region of the measuring tape.

(line 80 continues) **and when you're before one foot its more um**, [G(Video frame 9, 27:01)–makes a very brief and narrow-interval gesture with the thumb and first finger of his right hand with this hand now positioned above the region of the tape measure between 0" and 12".]



Video frame 9 (27:01) C makes a very brief narrow-interval gesture with the thumb and first finger of his right hand with this hand now positioned above the region on the tape measure between 0" and 12".

81 L: **Okay.**

82 C: [silence]

C's explanation comes to an end as he says "below one foot its more um" (line 80) while making a very brief but distinct narrow-interval gesture with the thumb and first finger of his right hand (Video frame 9). This gesture is positioned above the region of the measuring tape between 0 and 12 inches and reflects the narrower

intervals between adjacent markings on this region of the measuring tape in comparison to the intervals above 12 inches that C had described using a wide-interval gesture seconds earlier.

By responding to L's question in lines 78 and 80, C enacts a coordinated series of semiotic actions that serve to draw his own awareness to the marking patterns on the tape measure and thus mediate his thinking and deepen his consciousness of these patterns. This was, after all, the outcome L was aiming for by posing his initial question in lines 75 and 77. C's use of gestures and spoken language in this excerpt are examples of semiotic means of objectification for oneself.

Forms of iconicity and mathematics as reflexive praxis

Radford describes iconicity as the process of noticing and re-enacting significant parts of previous semiotic activity for the purpose of orienting one's actions and deepening one's own objectification. We can find three forms of iconicity within this brief and intense exchange between L and C.

The first form of iconicity involves C noticing and re-enacting all of the hand gestures and corresponding hand positions that L had used while posing the question to him at the start of their exchange. These included his use of different fingers for pointing at the different regions of the measuring tape in line 79–Video frames 1 and 4, the sweeping gesture for identifying the region of the measuring tape below 12 inches in line 78–Video frame 1, and the static one-finger pointing gesture directed at the 12 inch point in line 78–Video frame 3.

The second form of iconicity involves C noticing the different inscription patterns on his measuring tape below and above 12 inches and re-enacting these using different forms of semiotic actions, in this case using hand gestures. The examples here include C's chopping gesture to describe the closely packed pattern of marking lines below 12 inches in line 78–Video frame 2, his wide-interval gesture to describe the relatively wide intervals between markings above 12 inches also in line 78–Video frame 6, and his narrow-interval gesture to describe the relatively narrow intervals between the markings below 12 inches in line 80–Video frame 9.

The third form of iconicity to be found coincides with the second form of iconicity just described in this set of data. It involves C noticing a form of gesture that he has enacted himself and then re-enacting this within a different context. I refer here to C's use of a narrow-interval gesture using this thumb and first finger to describe the marking pattern below 12 inches on the measuring tape in line 80–Video clip 9. This occurs after he has enacted a similar wide-interval gesture using his thumb and first finger in reference to the marking pattern above 12 inches in line 78–Video frame 6.

We can infer that C became consciously aware of the possibility and/or usefulness of utilizing this form of interval gesture as a result of using it to describe the intervals above 12 inches because he then backtracked to elaborate on his previous description of the region of the measuring tape below 12 inches using this same form of gesture. The finding of this third form of iconicity—noticing and re-enacting parts of one's

own semiotic activity in a new context—is a new contribution to the theory of knowledge objectification.

CONCLUDING REMARKS

The brief excerpt that is the focus of this paper is taken from the beginning of a tutoring session involving a pre-apprentice in the pipe-trades learning to read the mathematical meaning embedded within a measuring tape marked in inches with the researcher serving in the role of tutor. This analysis illustrates the sensuous and artifact mediated nature of mathematical thinking and knowledge objectification. Particular features of the theory of knowledge objectification were evident including: examples of semiotic means of objectification—for another as well as for oneself—and three forms of iconicity: re-enactment using matching semiotic actions, re-enactment using different forms of semiotic action, and a newly reported form of iconicity, re-enactment of one's own previous form of semiotic actions in a different context.

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MATHEMATICAL REFLECTION IN PRIMARY SCHOOL EDUCATION

Theoretical Foundation and Empirical Analysis of a Case Study

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Abstract. *The paper presents the theoretical construct “mathematical reflection“ and elaborates its specificity with regard to the epistemological conditions of mathematical knowledge. This construct of “mathematical reflection” is the key concept in a wider research project. A conceptual grid with fundamental categories is developed that serves to carefully characterize the important components of “mathematical reflection” and that is used as an instrument for qualitatively analyzing students' mathematical collaboration in clinical interviews and for identifying different types of “mathematical reflection” in interaction.*

Key words: reflection, mathematical interaction, qualitative analysis, epistemology

1. INTRODUCTION: THE CENTRAL CONCEPT OF THE RESEARCH PROJECT – MATHEMATICAL REFLECTION

In several primary schools in Germany – also in North Rhine-Westphalia – teaching in grades 1 & 2 is organised comprehensively in the frame of experimental trials. It is assumed that “learning in grade-comprehensive groups [...] [offers] a lot of opportunities of using the different learning potentials for the mutual stimulation and support for the students as a whole” (North Rhine-Westphalia State Ministry for School, Youth and Children 2004)

The research project presented here refers to age-mixed mathematics learning and is oriented on the paradigm of interpretative instruction research. On the basis of the interaction-theoretic perspective (developed by Bauersfeld 1994) and the specific research approach of social epistemology of mathematical knowledge (developed by Steinbring 2005), this project deals in particular with the socio-interactive learning of mathematics in grade-heterogeneous learning groups in the flexible entrance phase of elementary schooling. The analyses of mathematical interactions, elaborated in this project, refer in a complementary way to individual-psychological and social processes and at the same time to the particularity of mathematical knowledge as the object of the interaction.

The fundamental concept of the analyses attempts to theoretically capture the *reflective mathematical thinking* of the children. We proceed on the assumption that, by means of the collaboration of younger and older children on mathematical problems, particularly the older children receive manifold opportunities of reflecting *mathematically*. With his concept of *observed mathematics*, Freudenthal characterized the (reflective) moment of thinking, where mathematics carried out and used on a lower level becomes observed mathematics on a higher level (cf. Freudenthal 1978, 64). In addition, Nührenböcker and Pust (2006) pointed out that, in

the interaction with the younger children, the older children, already used to school, are challenged to “verbalize their own thoughts and insights. In this process, existing knowledge is reflected and newly organized before it is handed on to others, and becomes further differentiated during the explanation process. For the children who are already used to school, a possible retrospection onto a previous learning process opens up opportunities for reflection on the meta-level” (Nührenbörger/Pust 2006, 24).

But how can *reflective thinking in mathematical* interaction processes be identified and what can be understood by *reflective mathematical thinking* as a conceptual element of an epistemologically oriented interaction-theoretical point of view onto learning mathematics and the nature of mathematical knowledge?

An initial foundation of the concept of “reflection” took place on the basis of already existing descriptions of “reflection” within the existing research literature, particularly in (actual) mathematics education literature. The examination of the status of research clearly showed the necessity of a precision of the theoretical construct “*mathematical reflection*”.

The elaboration of a broadened conceptual understanding of *mathematical reflection* is based on the (particular) epistemological nature and the conditions of the development of mathematical knowledge (cf. Steinbring 2005) as well as on the concept of reflection as a “change of standpoint”, which Freudenthal has developed in his article “How does reflective thinking develop?”: “The unfolding reflection shows different traits. One of them, I would like to call standpoint change – a mental standpoint change, where the standpoint itself can be local or mental, while the change can take place in space, time, or another, for instance mental, dimension” (Freudenthal 1983, 492).

Thus, by *mathematical reflection*, we understand a *cognitive activity, a process of thinking*, in the sense of a *change of standpoint or perspective*, on the basis of which *processes of re-interpretation* take place. Old, common mathematical knowledge and familiar ways of proceeding are thought through again intentionally, they are scrutinized and *newly or re-interpreted*. The construct “reflective mathematical thinking” corresponds with the epistemological character of mathematical knowledge as pattern-like, relational structures. With the assumption that stimulating reflective thinking aims at the development of mathematical knowledge, *mathematical reflective thinking* is not merely a repeated consideration, a remembrance, or a reference to familiar contents.

This specific characterization of *mathematical reflection* requires to take into consideration the following essential issues when trying to analyze whether one can observe within a mathematical interaction this kind of *mathematical reflection*. First, when analyzing a *change of standpoint or perspective* (in the sense of Freudenthal) within an observed mathematical interaction, we use the epistemological analysis and apply the epistemological triangle (see Steinbring 2005) to figure out whether one

can speak in a *proper epistemological* sense of a change of standpoint that introduces new mathematical relations or that generalizes mathematical relations. The second analysis instrument is the “analysis grid“ that tries to characterize the specific *type* of change of standpoint; this basic instrument is developed in the following section.

2. A GRID FOR THE ANALYSIS OF *MATHEMATICAL REFLECTIONS* WITHIN INTERACTION PROCESSES

The analysis grid (see Fig. 1) is divided into four fields, labelled “trigger”, “response”, “reaction” and “reflective level” together with sub-categories. The two fields “trigger” and “reaction” are *descriptive* elements in the analysis grid, and the fields “reaction” and the central category of the “reflective level” are characterized as *interpretative* elements.

In an interaction sequence, the question to which extent a new or re-interpretation of a mathematical content on the basis of a standpoint change becomes apparent, can only be examined in an exclusively interpretative way. In the frame of a sequential analysis of the scope of possible interpretation hypotheses, the convincing possibilities of interpretation, which can be justified by the direct reference to the transcript, are elaborated.

descriptive element	interpretative element	descriptive element	interpretative element
Trigger	Response	Reaction	Reflective Level
Question Exercise Remark (Interviewer /Teacher)	No Irritation	No Remark	Standpoint Change “foreign Perspective”
Question Discovery Way of proceeding Remark (Partner child)		Imitation	
Own way of proceeding/remark	Moment of Irritation	Recapitulation	Standpoint Change “Context”
		Construction	Standpoint Change “Retrospection”

Fig. 1: Analysis grid

The allocation to the descriptive elements of the analysis grid is exclusively oriented on the linguistic format of a remark and has a purely *descriptive* character.

The grid serves for the purpose of being able to focus on the central research questions and it allows on the basis of an epistemological analysis to examine the interactive processes taking place during a partner interview in a purposive and careful way. Even if, at first sight, the analysis grid might seem to present a chronological sequence of the fields, it is expressly not the aim of the grid to simply be used for the description of a temporal sequence.

During the real interaction proceedings, different sub-categories can overlap. For instance, a mathematical remark, which on the basis of its linguistic format is allocated to the sub-category of recapitulation, can at the same time contain a hint

towards a moment of irritation. The following more detailed explanations of the categories and sub-categories will further clarify the analysis grid.

The different elements of the analysis grid

- *The element “trigger”*: On a descriptive level, several possible triggers for reflection or thinking activities can be identified in the interviews. Examples: a question, a discovery or a way of proceeding can represent a triggering moment.

For the research it is important *which person* stimulates reflections. Is this rather true for the remarks by the interviewer, for one's own discoveries and ways of proceeding, or the remarks of a cooperating partner child? This relevant aspect is allowed for by the distinction of the three sub-categories.

- *The element “response”*: A first central research question concerns the identification of possible clues in the analysis of interactive processes, which suggest reflective thinking. When does a question or a mathematical problem not only initiate recapitulation or imitation, but a *reflective* process?

The research results up to now show that irritation or a moment of surprise is an important indicator in this context. If, for example, an exercise cannot be done spontaneously, if one does not agree with the previous proceeding of the answer or with the ways of proceeding, ideas or remarks by another participant, and if one shows irritation or surprise, that means that it is not possible to simply resort to common knowledge or familiar ways of proceeding. An irritating exercise can challenge to engage in a foreign perspective.

- *The element “reaction” (descriptive element)*: Children can react differently to the different triggers. In this regard, we distinguish between the sub-categories “no remark”, “imitation”, “recapitulation” and “construction”.

Besides “not remarking”, a possible reaction is “imitation”, which means the literal repetition of one's own or someone else's remarks or the direct imitation of familiar ways of proceeding or the partner child's strategies.

By “recapitulation”, we understand resorting to knowledge or ways of proceeding already familiar from the previous context, or the reference to remarks and strategies of a partner child in one's own words.

If the children also refer to mathematical knowledge, which had not been introduced by any of the interaction participants in the previous contexts, the category of “construction” is fulfilled.

The allocation of the children's reaction to one of the given categories takes place depending on the format of the remark and is oriented on the linguistic elements used, on a purely descriptive level.

If the children only refer to common knowledge or familiar ways of proceeding in phases of cooperation, the interaction remains on the level of reaction. But if new or

re-interpretations of old knowledge or new constructions take place, the level of “mathematical reflection” is addressed as well.

- “*Reflective level*” (*interpretative element*): The question whether new or re-interpretations are carried out within interactions or if new mathematical knowledge is constructed, can only be examined interpretatively. In order to do so, the epistemological triangle (Steinbring 2005) is used in the analysis.

The identification of the standpoint changes, which might follow, takes place with the help of the developed characteristics and features of differentiation.

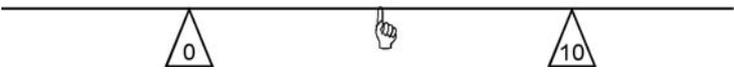
Three levels of changes of standpoint or perspective: The point of view developed by Freudenthal about reflective thinking as a standpoint or perspective change made it possible to characterize and distinguish three different forms of possible standpoint changes from the data material. Besides the theoretical clarification of the concept *mathematical reflection*, these represent an essential result of this research.

An important feature of the three levels of standpoint changes consists in the new or re-interpretation of a mathematical exercise, a mathematical content or a mathematical sign / symbol. A distinction is made with regard to the different possibilities or ways of changing one's own standpoint.

- Standpoint change “foreign perspective”: The children take a foreign perspective, someone else's standpoint, for instance they relate the ways of proceedings, discoveries and views of their partner child to their own points of view and ways of proceeding, test and evaluate these and are stimulated to newly or re-interpret their own mathematical knowledge.
- Standpoint change “context”: A mathematical challenge is put into and observed within another context and thus is subject to a new or re-interpretation. In contrast to the standpoint change “foreign perspective”, no concrete possibility of interpretation is given, which then might be followed, but rather the change of context *allows for* a new point of view. If, by means of such a context change, one of the participants develops a new interpretation perspective, a mathematical reflection according to the standpoint change “context” has taken place.
- Standpoint change “retrospection”: If there is an *intentional* resort to common knowledge and familiar ways of proceeding from a previous context in order to thus new or re-interpret a mathematical content, a standpoint change “retrospection” has taken place. Such a standpoint change can only be spoken of if a remark by a participant presents a way of proceeding or a mathematical context as familiar and relates this with the current exercise.

3. Analysis of an Exemplary episode: Gina & Sharon discuss a “Number line”– Problem

1	Int	(places the number cards 0 and 10 at the number line) I am placing the number cards at the number line, ...
2	S	[incomprehensible]
3	Int	Put this card, (places the number card “5” between Gina and Sharon onto the table)

4	G	(Gina takes the number card "5" with her left hand)
5	Int	at the number line.
6	G	# (leans forward / holds the number card "5" with both hands / looks at the number line)
7	S	# In fact the zero belongs in front # (places her left hand onto the left end of the number line)
8	G	# (holds the number card in her right hand / looks at Sharon's left hand)
9	S	or shall we now, well shall that now be like that the number line begins with this? (puts the edge of her left hand on the left of the number card zero on the table)
10	Int	Think about it together, how you can do that now.
11	S	You now certainly have (looks at L.) well. (...) (turns to Gina) She probably has chosen such a place (points over the section of the number line which is marked by the number cards 0 and 10 / Gina looks at the number line) where one could add that, so that we well that this, that this is supposed to be the beginning (places the edge of her right hand to the left of the number card "0" on the table) that this piece is then practically gone, (moves her right hand in the direction of the left edge of the table over the number line) in your mind, right? (looks at Gina / Gina continues to look at the number line) Well such a place, then the five would go here, right? (puts a finger between the numbers 0 and 10 onto the number line, see below. / looks at Gina / Gina continues to look at the number line) (...)
		
		because one two three four five. (while counting the numbers, she points at the spots marked in the diagram, see below)
		

This short episode originates from an interview about the topic "number line", which was conducted with Sharon and her classmate Gina in the second project year. For Sharon, this was the fifth interview during the research project, for Gina it was the first.

Before the children were introduced to the number line, which they had never used as means of visualisation. This scene of positioning of "5" takes 5 minutes.



Fig. 2: Section of the number line

On the children's desk, a string was attached as a number line. The interviewer had positioned the "0" and "10" (cf. Fig. 2) when asking the exercise question.

Analysis of the interview sequence

The exercise is opened by the interviewer. She positions the "0" and "10" thus providing the initial situation. This *task of the interviewer* is emphasised by the remark ("I am placing the number cards at the number line" (1)).

Sharon directly reacts to this action or remark (2). Maybe she already shows a first reaction to the positioning of the number cards. As Sharon's remark is incomprehensible, therefore this guess cannot be clarified definitively.

Gina immediately takes up the number card "5" and at the same time watches the number line (4, 6). While doing this, she shows that she is engaging with the exercise question and is considering where to put the number card "5".

Sharon exclusively refers to the current position of the number card “0” and wonders about the position of the “0” and “10” at the number line in her following remarks (7, 9, 11).

Sharon's remarks are of essential importance for the central research question and the identification of reflective mathematical thinking and thus represent the main focus and the starting point of the following interpretations. The *clarification of the position of the number card “0”* as an element of the number line (by Sharon) is at the centre of analysis.

In her first remark after the exercise question, Sharon points at the left end of the number line and explains that the “0” should be placed directly at the beginning of the number line (7 “In fact the zero belongs in front”). The positioning of the “0” by the interviewer does not correspond with her idea of the “correct position”. Her remark suggests that, according to her previous point of view, the position of the “0” on the number line is fixed and cannot be chosen freely.

The possible previous consideration of changing the position of the number card in the frame of the work on the exercise can be seen in particular in remark (9) “or shall we now, well shall that now be like that the number line begins with this?”. This is supported by the use of the words “in fact”, which underlines the discrepancy between the current and Sharon's “correct positioning” of the “0”.

The interviewer gives the question raised by Sharon back to the two students (10 “Think about it together, how you can do that now.”).

Sharon's remark (11) suggests that she now assumes an intentional positioning of the “0” by the interviewer and is challenged to find an explanation for the “unusual position” of the number card at the number line (“She probably has chosen such a place where one could add that, so that we well that this, that this is supposed to be the beginning”).

Applying the analysis grid “mathematical reflection” to the episode

The element trigger: The exercise question given by the interviewer as well as the given positioning of the number cards 0 and 10 at the number line (1, 3, 5) represents the trigger for the following cognitive activities by the two students.

The element response: Sharon makes a remark about the current position of the number card “0” at the number line directly after the explanation of the exercise question by the interviewer. The position of the number card does not correspond with her idea and she is probably surprised or irritated by the interviewer's way of proceeding. A clue for a possible moment of irritation becomes apparent in her remark (7): “In fact the zero belongs in front”. Sharon points out an alternative possibility of positioning the number card. Her remark “In fact” can be seen as an indicator for her not agreeing with the current position of the number card.

The element reaction: In her reaction to the triggering moment, which is the exercise question and the localization of the section of the number line to be observed, Sharon

refers to the positioning of the number card “0” and discusses this (not verbally expressed) action of the interviewer with her own words. Thus Sharon's reaction can be allocated to the sub-category recapitulation.

The levels of mathematical reflection

The question to which extent Sharon performs a change of view and carries out a new or re-orientation of her mathematical knowledge regarding the positioning of the “0” at the number line is examined with the epistemological triangle (Steinbring 2005) as an analysis instrument of relations between signs, reference contexts and concept.

If a change of standpoint or perspective can be identified, this will be allocated to one of the three levels of mathematical reflection on the basis of the characteristics described in the presentation of the analysis grid.

The analysis instrument “epistemological triangle”

Conventional interpretation: The *sign* to be clarified in the present interview sequence is the position of the number card “0” at the number line. In this first representation the original, conventional interpretation by Sharon regarding the position of the number card is made clear by referring to a familiar *reference context*.

In her remark (7) “In fact the zero belongs in front”, Sharon probably refers to the known “familiar” position of the number card “0” at the beginning of the number line. Maybe she remembers the positioning carried out previously to the interview and points at the left end of the number line as the only possible position for the number card up until this point. Two different aspects become manifest in her remarks. On the one hand, there seems to be a fixed position for the number card at the number line for Sharon, on the other hand the number card “0” belongs to the beginning of the number line, i. e. left of this number, neither does the number line continue nor can there be further number cards.

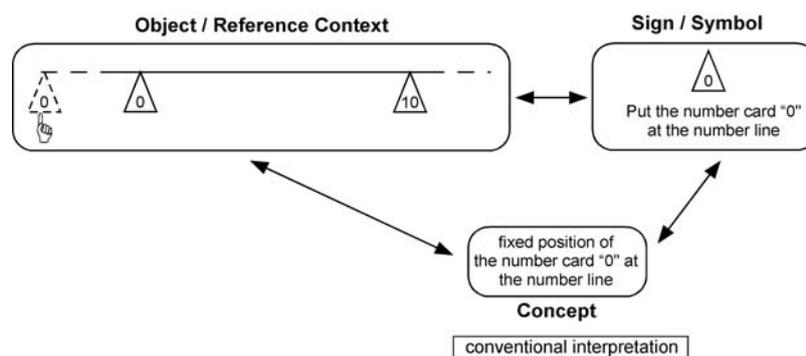


Fig. 3: Epistemological triangle: The original interpretation of the position of the number card “0”

Beginning of a relational interpretation: Besides the originally conventional view concerning the position of the number card “0”, a beginning mentally more flexible interpretation becomes apparent in this scene. Sharon tries to conciliate her previous point of view with the current position of the number card. In doing so, she refers to the reference context presented in Fig. 4. She explains the – for her point of view – still unfamiliar position of the number card “0” by placing her hand to the left of the

number card and remarking in the one hand (9): “or shall we now, well shall that now be like that the number line begins with this?”, on the other hand (11): (“She probably has chosen such a place where one could add that, so that we well that this, that this is supposed to be the beginning”).

The mentally changed number line thus forms the reference context, i. e. the current position of the number card is interpreted by referring to the theoretical picture of the number line, which Sharon has developed and in which the sequence in front of the number line is mentally ignored.

In this interaction of sign and reference context the beginning of a detachment from a purely empirical point of view concentrated on the concrete, towards a stronger mental use and change of the number line becomes apparent. The following remarks by Sharon can serve as concrete indicators of this more flexible point of view “in your mind” (11) and “would” (11: “then the five would go here, right?”). The positioning of the “5” which she suggests takes place depending on the current position of the “0” and “10”.

While at the beginning of the interview sequence Sharon still allocates a fixed position at the beginning of the number line to the number card “0”, she ultimately takes a more flexible point of view about this: By means of the possibility of putting the number card “0” at a random position of the number line, sections of the number line can be realized variably.

Still, the number card “0” remains the first card for Sharon, however, thus left of this number card there can be no other number cards. Furthermore, her way of proceeding when positioning the number card “5” (11) indicates that she continues to pay attention to the sequence and distance of the number cards.

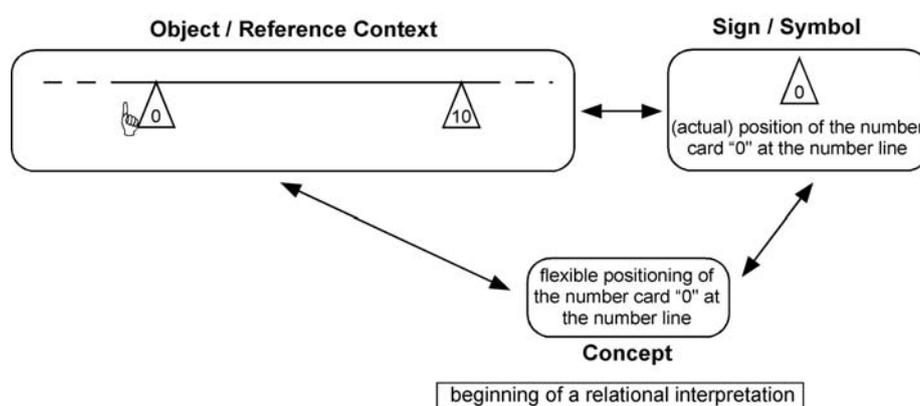


Fig. 4: Epistemological triangle: Beginning of a relational interpretation

Characterization of the standpoint change

As has already become clear in the first step of the analysis, Sharon performs a new or re-interpretation of the number line regarding the positioning of the number card “0” during the course of the interaction.

As previously to the present interview sequence, the number card was always placed at the beginning of the number line, its current position represents a *changed context* in this regard.

Concerning the position of the number card “0”, Sharon develops a new interpretation perspective and thus carries out a *standpoint change “context”* on the basis of this changed context given by the interviewer.

descriptive element	interpretative element	descriptive element	interpretative element
Trigger	Response	Reaction	Reflective Level
Question Exercise Remark (Interviewer / Teacher)	No Irritation	No Remark	Standpoint Change “foreign Perspective”
Question Discovery Way of proceeding Remark (Partner child)		Imitation	Standpoint Change “Context”
Own way of proceeding/ remark	Moment of Irritation	Recapitulation	Standpoint Change “Retrospection”
		Construction	

Fig. 5: Application of the analysis grid

4 SHORT RÉSUMÉ

The analysis grid developed in the course of the research project offers the possibility of presenting the results of the analyses and interpretations cohesively. The central element of the grid is the “reflective level”. The distinction of the three categories of standpoint changes is a fundamental result of the research up until now and allows for the analysis to pursue the question which specific form of a standpoint change provokes and stimulates the process of new interpretation of mathematical knowledge, which is essential for the learning of mathematics.

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SURFACE SIGNS OF REASONING

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Abstract

In this paper, we explore forms of verbal expression undergraduate mathematics students employ while working in pairs on geometric tasks in a computer environment, focusing in particular on the connectives (notably ‘because’) they use as well as the modal expressions in their talk as they discuss ideas with their partner. We use this data to bring together C. S. Peirce’s idea of abduction, the linguistic notion of hedging and Toulmin’s argumentation scheme, and argue that in trying to identify abductions, the presence of hedges (of which Toulmin’s ‘modal qualifiers’ are an instance) or a particular use of ‘because’ may provide some evidence.

It is a commonplace of philosophical logic that there are, or appear to be, divergences in meaning between, on the one hand, at least some of what I shall call the formal devices— \sim , \wedge , \vee , \supset , $(\forall x)$, $(\exists x)$ (ιx) (when these are given a standard two-value interpretation)—and, on the other, what are taken to be their analogues or counterparts in natural language—such expressions such as *not*, *and*, *or*, *if*, *all*, *some* (or *at least one*), *the*. (Grice, 1975/1989, p. 22)

In this paper, we wish to explore some of the natural language markers (in English) that are employed in students’ *spoken* mathematical reasoning. One motivation for doing so is a realisation of how different, on occasion, even experienced mathematical undergraduates *speak* when working on problems in pairs, from the conventional way formal mathematics is supposed to be *written* (e.g. Morgan, 1998). A second was the difficulty we had at times in identifying the nature of the reasoning from the speech of the participants. A third arose from our growing interest in the notion of *abduction*, which has been receiving attention in the past few years within mathematics education (e.g. Mason, 1995; Pedemonte, 2007; Reid, 2003; Rivera, 2008; Sinclair, Lee and Strickland, under review), as well as possible connections to the linguistic notion of hedging (see, e.g., Rowland, 1995) and Toulmin’s argumentation scheme (see, e.g., Inglis et al., 2007).

In mathematical discourse, there are significant differences between speech and writing. We are not claiming that there are disjoint vocabularies, but there are some words that are usually only spoken (including a few that require invented spellings for transcription e.g. ‘cuz’, ‘gonna’, ‘gotta’) and some that are much more commonly written (hence, therefore, consequently). The formal written mathematical register is quite tightly specified in terms of particular conjunctions to be used in proofs, particularly at the beginnings of sentences to mark the relation between the preceding and subsequent comments (e.g., ‘let’, ‘hence’, ‘therefore’, ‘if’, ‘since’, ‘conversely’). This is another level of difference beyond that to which Grice is drawing attention.

However, one linguistic challenge arises from the fact that mathematical purposes are not the only functions that these words encode. The language of ‘if ..., then ...’, for instance, so common in written mathematics, is also the language of threats. Many of the conventional connectives in other circumstances carry a space, time or sequencing connotation (e.g. then, since, when, hence) – for more on mathematics and time, see Pimm (2006). In conversation, the *then* of ‘if ..., then ...’ is often elided, and there are occasions when even the *if* marker can be absent.

In this paper we wish to go further than Paul Grice in differentiating logical operators from what he terms ‘natural language’, by distinguishing spoken from written natural language. Unlike Grice, however, we will offer attested speech data for consideration rather than invented data. In the opening chapter to his book *Text and Corpus Analysis*, linguist Michael Stubbs (1996) criticises the dominant tradition since Chomsky (and including Grice) for basing extensive theoretical arguments on no real language data. Nevertheless, Stubbs (see below) supports Grice’s specific claim about the non-congruence between logical connectives and English words and goes further, paying close attention to the role of modality in verbal communication.

This paper draws on data collected within a larger study of mathematical reasoning in undergraduate students. The data consist of twenty videotaped episodes (ranging from ten to twenty minutes in length) in which pairs of students are working at computers, using *The Geometer's Sketchpad* (Jackiw, 1991) to solve geometric tasks. These tasks include, among many others, using *Sketchpad* to construct a parabola, to identify the particular transformation that relates two given shapes, to solve the Apollonius problem and to figure out the fractal dimension of given curves.

SPOKEN MARKERS OF REASONING

A third case of the interaction of pragmatic and syntactic matters is provided by the so-called logical connectors (e.g. *and*, *but*, *or*, *if*, *because*). Their uses in everyday English are not reducible to their logical functions in the propositional calculus, but have to do with speakers justifying their confidence in the truth of assertions, or justifying other speech acts. (Stubbs, 1996, p. 224)

Any modal utterance contains both propositional information and the speaker’s attitude towards the information. Echoing Grice, Stubbs uses modality to distinguish between different functions of connectives. He claims *because* is representative in having two distinguishable uses, which he terms *logical* and *pragmatic*: the first has the structure of ‘effect plus cause’, the second ‘assertion plus justification’. Stubbs notes that the pragmatic use of *because* is often signalled by the addition of epistemic *must* (‘he must have been drunk because he fell down the steps’). In addition, He provides a number of syntactic criteria to help distinguish the two uses. He claims these points are also true for the pragmatic use of *if*, *or*, *but*, and *and*.

An example of the logical use comes from Birkhoff and Mac Lane (1941/1956) “Because of the correspondence between matrices and linear transformation, we need

supply the proof only for one case” (p. 227). Similarly, in Spivak (1967), we find: “Because this sequence varies so erratically near zero, our primitive mathematical instincts might suggest that $\lim_{n \rightarrow \infty} f_n(x)$ does not always exist” (p. 414).

There is no scope within this paper for a detailed corpus analysis of connectives in our data, though we wish to remark on the prevalence of ‘so’ and ‘which means’ as markers of deductive utterances. From our data, we find very few logical uses of because.

A: Well, **because** those two don't, for sure, lie in the circle, so if we rotate it around that point, it's not gonna be exact.

In A's statement above, the cause is signalled by ‘so if.’ Far more often, the uses of because are pragmatic, as in the following two examples.

D: No, **because** the rotation point is gonna be over here.

E: Yeah, the original one **because** then O_1 will convert to a line and through ... never mind. That didn't work. We did it wrong.

In both these and other similar instances, what we find is students hypothesising or positing justifications for claims they are making. This connects in an interesting manner to the theme we turn to in the next section, namely abduction as a form of inferring, which is proving challenging to us to identify confidently. This brief look at ‘because’ suggests that one place to look for abductions is in pragmatic uses of the connective ‘because’.

TWO SHORT EPISODES OF STUDENT REASONING

Here are two episodes of student mathematical problem solving where we found the form of reasoning less clearly identifiable, less likely to be deductive, and replete with modal utterances. We provide a brief contextualisation of each episode in this section, and then offer two tentative analyses—one using Peircean abduction and the other Toulmin's model of argumentation—of each episode in the following section.

Example 1

Two students (Lucie and Brad) are trying to solve the problem of geometrically constructing a parabola in *Sketchpad* given a focus point P and a directrix line j . The students have already constructed the envelope of the parabola by tracing the perpendicular bisector of PB where B is a point on j that can be dragged back and forth along the line. The students begin looking for ways to construct a point that depends on B so as they move B along j it will trace out the parabola.

At first, they place a point on the segment PB right where the segment first touches the envelope edge. When Lucie drags B , they both realise that this point does not always lie on the curve, so they delete this point. In turn 1 below, Brad notices that if the solution point is placed on PB , then it could never reach the upper parts of the parabola (given that PB is a segment). This seems to give rise to an anomaly for Brad

– that the point will have to be able to travel high up the sides of the parabola. Indeed, his expression is emphatic and strong-voiced and the modal verb ‘can’t’ is also strong: “We can’t have ...”. Indeed, he tries to convince Lucie of what he’s noticing: “see that point”. In turn 3, Brad makes a deductive inference, first using the word ‘so’ and then “which means” to indicate the implication that the point *cannot* be on *PB*.

- 1 Brad: We can’t have [...] [1] Well, like, [...] like, see that point has to be able to get up here, right? (He points to *j* with his pen and then points to the top left of the curve with his pen and then his finger.)
- 2 Lucie: Uhuh.
- 3 Brad: So, which means it can’t touch the line.

Lucie then proposes that this point lies on a line passing through *P* perpendicular to *j*.

- 4 Lucie: Yep [...] So then [...] Let’s say [...] (Constructs the line through *P* perpendicular to *j*, as in Figure 1.) Maybe that’s the line [...] ‘cause um [...] the distance from like [...] here to here would be the same as that one? (Points to distance between the envelope of the curve on the left and her new line.) But I don’t know if that’s right. (Points to her new line and the curve on the right.)
- 5 Brad: So what line did you just create?
- 6 Lucie: The perpendicular line to the bottom through *P*. But I don’t think it’s right.

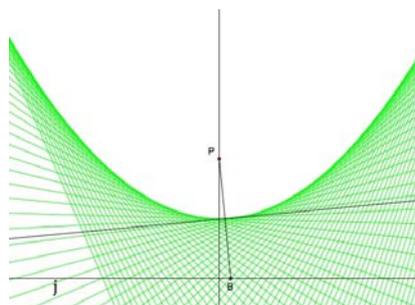


Figure 1: The envelope of a parabola with focus *P* and directrix *j*

Brad seems to think that Lucie’s line “couldn’t be” the right one, but acknowledges her statement about equidistance. At this point, the instructor intervenes and redirects the students’ attention to the more pertinent equidistance relationship (to point *P* and line *j*). The students eventually figure out how to construct the point on the parabola as the intersection between the perpendicular bisector of *PB* and the line perpendicular to *j*, passing through *B*.

Example 2

Two students (Gloria and Peter) are trying to figure out which isometry maps a given shape on the computer screen onto another and then to construct the specific transformation. The students have studied the composition of reflections (and found that the composition of two reflections gives a rotation, unless the two lines of reflection are parallel). In turn 1, Gloria has already identified two corresponding segments of the shape and asks “can we continue these two lines?”

- 1 Gloria: Rotation right? [...] Which is two reflections but I don't know how to do that. (Points to the right edge of top figure and top edge of bottom one – see Figure 2 below.) OK, can we continue these two lines?
- 2 Peter: Probably two reflections.
- 3 Gloria: Can we, yeah, or a rotation, same difference.
- 4 Peter: [inaudible]. (Gloria draws a straight line extending the right-hand vertical edge of the top figure.)
- 5 Gloria: Can we make this a straight line and find out what this angle is, and then rotate it that much? [.....] Um [...] That'd work, wouldn't it?

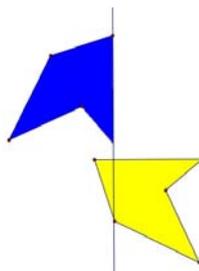


Figure 2: Line extending one side of the top shape

In turn 4, Gloria extends the line and then, in turn 5, infers that the intersection of the line and the horizontal side of the lower shape will form an angle that corresponds to the angle of rotation necessary between the two shapes.

INTERPRETING THE EPISODES

In each episode, we see mathematical reasoning that plays an important role in the problem-solving process of the pairs, but that does not fall easily into the two most commonly-discussed categories of inductive and deductive reasoning. We thus begin by interpreting the two episodes described above in terms of Peircean abduction. We then interpret the same episodes using Toulmin's (1958) structure of argumentation.

Focus on Peirce's different types of inferences

Deduction proves that something *must* be; Induction shows that something *actually is* operative; Abduction merely suggests that something *may be*. (Peirce, 1931/1960, 5.171)

Peirce's description of the three forms of inference, as quoted above, marks a shift in interpretation away from the logical *form* of a given inference (how it might be characterised through syllogistic propositions) toward its *use*, by the inquirer, in the process of inquiry. While researchers such as Reid (2003) and Cifarelli (2000) claim to have identified student abductions based on these logical forms, Mason (2005) cautions, "The tricky part about abduction is locating at the same time the appropriate rule and the conjectured case" (p. 5). In many cases, neither of these propositions will be uttered out loud in spoken conversation – they must be inferred from context.

While logical forms are sometimes easy to identify in written language (especially in mathematics texts), they can be much harder to identify in speech, which is frequently less planned and more emergent in real time, especially in the context of

pairs jointly co-constructing the talk. While some students will state that something “must be” (or ‘has to be’ or ‘gotta be’) true, others may choose to express their certainty through other means, both verbal and non-verbal. Peirce's emphasis on the *uses* of deduction, induction and abduction invites attention to the intentions of the inquirer, but these intentions, about what must be, what actually is, and what may be, can't always be clearly identified either. Thus, one challenge facing researchers is how to work with the surface elements of language in order to make interpretations about the type of inference demonstrated in particular conversational exchanges. The short list given by Grice in our opening quotation, which includes clear, propositional terms of inference, is completely insufficient when looking at real people reasoning in conversational pairs about mathematics.

Considering episode 1, we can see Brad's inference that the point cannot lie on PB as a deduction, since he states what *must* be the case. Here, the logical form is quite easy to identify, as are the linguistic features. By contrast, Lucie's proposal that the point lies on the perpendicular to j through P can be seen as an abduction, since it indicates what *may be* true, as exemplified by her own words “Maybe that's the line” and her later hedged statement of hesitation “But I don't know if that's right.” Lucie's inference satisfies two additional characteristics of abduction: (1) it involves the generation of a new idea (the line she constructs did not exist before, and stands as a genuinely new and plausible solution); and (2) it is not logically derivable from true statements (and, indeed, the line she proposes is not the right one). Further, the use of “cause” is a pragmatic one, in Stubbs's sense as described above.

We might also attempt to interpret Lucie's abduction in the following logical form, where the case is the only thing Lucie knows to be true, and the result has been hypothesised as a plausible situation in light of the novel rule.

case: The (solution) point has to go up

rule: If it's on that line, it would go up

result: The point is on that line

In contrast with the linguistic interpretation offered above, the logical form fails to capture the interlocutor's degree of conviction when she hedges her proposal both with ‘maybe’ and “I don't think that's right”. Additionally, there is a close link between this formulation of abduction and Stubbs's pragmatic category of connective use, as noted above in relation to “cause”. Curiously, Stubbs's term ‘pragmatic’ seems to evoke Peirce's work on pragmatism.

We turn now to episode 2, where Gloria and Peter are trying to identify the isometry relating two shapes. In turn 1, Gloria asks, after pointing to the two lines in question, “can we continue these two lines?” She has not explicitly stated that she is trying to identify the angle of rotation (or the angle between the two lines of reflection), but this becomes explicit in turn 5, where she asks (again): “Can we make this a straight line and find out what this angle is?” We see this as an abductive inference, since it

follows the use of what *may be* true, as evidenced by her questioning tone of voice, her use of the hedge tag phrase “can we” and the final, doubtful, tagged utterance “That would work, wouldn’t it?”

We find further evidence of this as an abductive inference by the fact that it introduces a new idea (the technique of extending lines had not been previously used in class), which, in this case, turns out to be fruitful. Once again, we could offer an interpretation based on the ‘underlying’ logical form of the inference, but the preceding analysis seems to offer an identification consistent with Peirce’s conceptualisation of abduction in its pragmatic function.

Focus on Toulmin’s forms of argumentation

In work on forms of argumentation and informal logic, Toulmin’s (1958) scheme has had its place. But, as Inglis *et al.* (2007) clearly point out, it is a reduced form of Toulmin’s scheme that has been commonly used in mathematics education, one which leaves out two of the six components: the rebuttal and, of greater relevance for us here, modal qualifiers. Inglis *et al.* worked with the production of individual oral arguments of graduate students in mathematics, exploring a range of mathematical conjectures. We were struck in their paper by the fact that modal qualifiers are precisely hedges, those statements of propositional attitude concerning the degree of conviction the speaker is willing to express. This made us wonder about the connection between overt hedging and abduction, which suggest that the student was to some extent aware of the making of an abduction that consequently required a more tentative assertion.

Inglis *et al.* (2007) give a visual summary to illustrate Toulmin’s model of argumentation (Figure 3). The argument would read: based on the data (D) given, the warrant (W) – which is supported by the backing (B) – justifies the connection between D and the conclusion (C), unless the rebuttal (R) refutes it. The modal qualifier (Q) qualifies the certainty of the conclusion by expressing degrees of confidence.

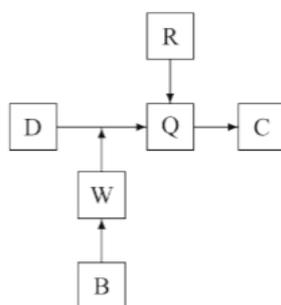


Figure 3: Toulmin’s model of argumentation

We now run the first episode above through Toulmin’s model to obtain Figure 4. The data include the point P, the directrix j , the point B on j , as well as the segment PB . Lucie’s conclusion, that the point lies on the line perpendicular to j and passing

through P is qualified by her hedged utterances “Maybe” and “But I don’t know if that’s right”. We see her statement regarding the equidistance of the line to each side of the parabola functioning as the warrant, even though it is offered *after* the argument – following some hesitation and speculate that it is the presence of her partner that makes her verbalise this at all. The backing includes the fact that the point must be on some line (instead of a line segment like PB), but one that should somehow involve both P and *j* (the givens in the situation). The rebuttal is not evident in her argument and may not exist at all.

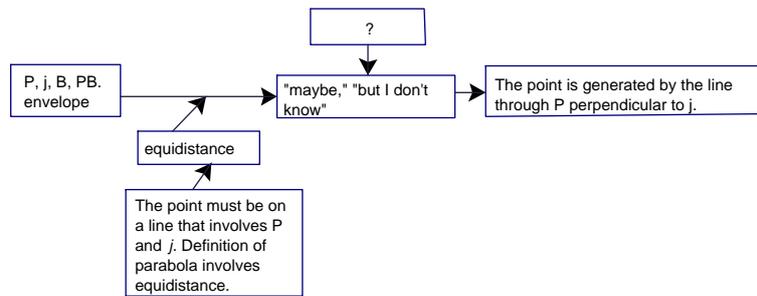


Figure 4: Lucie’s argument expressed using Toulmin’s scheme

Turning now to the second episode, we can also run Toulmin’s scheme on Gloria’s argument (in Figure 5).

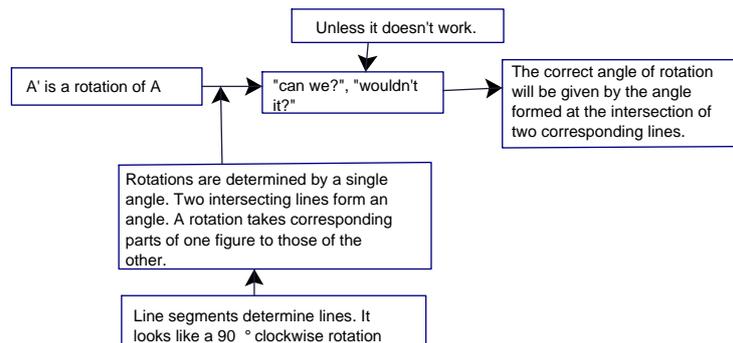


Figure 5: Gloria’s argument expressed using Toulmin’s scheme

This time the modal qualification is not expressed through specific words, such as ‘maybe’ or ‘probably’, but instead in the intonation of Gloria’s statement, which is made in question form: “Can we [...]?”. In this episode, we also find no evidence of a rebuttal, though presumably Gloria had an immediate and pragmatic rebuttal in mind, which was to actually see whether the angle of rotation created by intersecting the line and side segment would work to rotate the pre-image to its image. Filling in the scheme, Gloria’s conclusion is that the angle of rotation between the two shapes is the angle created by the intersection of two corresponding sides (one extended).

CONCLUSION

The above analyses show that it is possible to interpret the two excerpts of paired student reasoning in conversation using either Peirce’s idea of abduction or

Toulmin's model of argumentation. Both are challenging to use as interpretational frameworks, and this is so for several reasons. First, both Peirce and Toulmin tended to work with made-up examples to illustrate their inferences or arguments; and, as we have seen, real speech is much messier – some phrases are omitted, others are communicated non-linguistically, and so on. Second, and especially for abduction, we have already noted that the most important component of the abductive inference – the stating of the general rule – must often be inferred from context. However, even in Toulmin's case, what counts as data, warrant, and backing is not always obvious, and certainly not objectively knowable. Third, neither Peirce nor Toulmin has conversational reasoning in mind when articulating their theories. In some senses, Toulmin's emphasis on argument is *post hoc*, given that the interaction between two students (in our own data) frequently involved negotiation of meanings, and subsequent attempts to explain and/or convince.

The analyses we conducted reveal interesting similarities and differences. Most remarkable of the former related to the importance attached to the degree of confidence held by the reasoner. Toulmin includes modal qualifiers in his model in order to account for the variety of certainty that one might have about a claim. Peirce's abductions are seen as hypothetical *may be*'s. Their attention to uncertainty might seem strange in the context of mathematics, where one frequently seeks precisely the opposite. Yet both Peirce and Toulmin seem to care about how the reasoner can make advances in inquiry, and take it as given that many advances will be tentative. A particular resonance such a perspective has in mathematics education can be found in the work of Rowland, who has studied the notion of hedging in the mathematics classroom. We suggest that this notion could be used productively to help identify and analyse and interpret student reasoning in terms of Toulmin or Peirce. Lastly, the pragmatic use of 'because' also appeared as a surface marker in one of the two episodes that may help identify abductions in some cases.

Toulmin is concerned with trying to identify the structure and form of an existing argument, whereas Peirce is more concerned with examining the process of scientific discovery. Peirce draws attention to the way in which problem solving may require abductive 'leaps of faith', where one is reasoning ahead of more explicit or acknowledged deductive or inductive means. This seems to us an important awareness in educators involved in supporting and eliciting mathematical problem solving. Toulmin's analysis of an argument acknowledges the qualification involved in any emergent complex argument, and serves to draw attention to argument structures and resources that may not have been apparent in the more 'logical' literature analyzing the form and nature of mathematical arguments.

By juxtaposing the results of each analysis of the same two mathematical episodes, as well as identifying hedging as one surface linguistic phenomenon common to both, we have attempted to highlight how one might ground each theoretical account in the

specifics of moment-to-moment conversation, as well as thereby drawing attention to commonalities across the two accounts that have not been made before.

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A TEACHER'S USE OF GESTURE AND DISCOURSE AS COMMUNICATIVE STRATEGIES IN CONCLUDING A MATHEMATICAL TASK

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An experienced teacher has been observed in dialogue with her sixth-grade pupils when summing up their solutions to a mathematical task. The pupils have worked in small groups on this task, which is related to a transposition of data (age and height) from a figure to a Cartesian diagram and to a written text. The teacher's discourse has been analysed, using the dialogical approach to communication and cognition. Analyses of gestures are based on McNeill's classification expanded by Edwards, using the concept of embodied cognition and complemented by the work of Goodwin, taking into account the contribution of the environment to the organisation of the gesture. Some communicative strategies used by the teacher have been identified, for example, questioning (who, how, why, asking for other suggestions). Pointing gestures are used, but they are not prominent. Our findings suggest that gestures are more used and connected to the teacher's explanations than to other procedures.

INTRODUCTION

Gesture and discourse have, for a long time, been seen as two distinct ways of conveying meaning. The tendency today is to conceive these two modalities of expression of meaning as complementary. In teaching-learning situations, gestures can be considered as carriers of meaning having the function to *locate* ideas in space, to make them visually perceived. Meanwhile, discourse has the function of *transforming/making* ideas in words. These are privileged tools used by teachers when communicating, explaining, and discussing mathematical concepts in the classroom. The aim of this paper is to focus on a teacher's communicative strategies while summing up, in dialogue with her pupils, the solutions from the pupils' small-group discussion on a mathematical task (called the diagram task), emphasising the transition between three semiotic representations: figure, diagram and written text.

This study is related to the research and developmental project, Learning Communities in Mathematics¹ (LCM) which was designed at the University of Agder (UiA) in Norway. The project was implemented in the period from 2004-2007, and the theoretical framework for it was presented at Cerme 4 (Cestari, Daland, Eriksen, & Jaworski, 2006). The project aimed to “create *inquiry communities* of teachers and didacticians to both develop and explore the development of mathematics teaching and learning” (Jaworski, Fuglestad, Bjuland, Breiteig, Goodchild, & Grevholm, 2007, p. 7).

Inspired by ideas and discussions at workshops in the LCM project, the experienced teacher in focus (about 35 years in service, spring 2005) organised workshops in the

classroom with her pupils during one lesson a week. It is in such a workshop context that the diagram task was used in the classroom with the following structure in three parts: 1. Introduction of activities (00:00-04:28), 2. Working in groups of two and three (04:28-13:47), and 3. Summing up with the whole class (13:47-18:47). In Bjuland, Cestari and Borgersen (2008c) we identified the teacher's communicative strategies while presenting the task in a dialogue with her pupils (part 1). The teacher used both speech and gestures when focusing on the transition from the two different semiotic representations, figure and diagram. More specifically, she posed open *questions* while simultaneously "*pointing* to the diagram followed by a gradually decreasing *circular sliding* between the diagram and the picture" (op. cit., p. 190).

We were also concerned with the difficulties the pupils met in the solution process. One group (two girls) made incorrect suggestions without being attuned to each other, and they had difficulties in focusing on two dimensions in the diagram. The teacher visited the girls twice during the solution process (part 2). She posed different questions (yes-no, open, specific) in order to help them to express their difficulties. The teacher gave verbal explanations simultaneously with using gestures like *pointing* and *circular slidings* to make connections between figure and diagram (Bjuland et al., 2008c).

After having reported from the first two parts of the work on the diagram task, we are now concerned with the way the teacher sums up and concludes the mathematical activity (part 3). This paper addresses the following research question: What kinds of communicative strategies does an experienced teacher use in her dialogues with sixth-grade pupils, while summing up the pupils' solutions to a task that involves moving between different semiotic representations? In Bjuland et al. (2008c), we have illustrated that gesture and speech are natural mediating devices when this teacher introduced the diagram task and when she visited the girls' group. It is therefore important to ask how gestures are used in connection with speech in part 3.

THEORETICAL FRAMEWORK

Gestures and discourses are fundamental modalities in the interpretation of communicative strategies used by teachers in the classroom. According to Roth (2001), teachers employ many gestural resources crucial for understanding a concept. So, pupils need to attend to both their teachers' speech and their gestures in order to access information presented in a lesson. In Bjuland et al. (2008b), we have revealed how the multimodal components of expression, speech, gesture, and written inscriptions develop synchronically. These major components of the objectification process (Radford, 2003) have stimulated the pupils to come up with a solution. We have in our work mostly observed deictic gestures. These are defined by Mc Neill as "pointing movements, which are prototypically performed with the pointing finger" (1992, p. 80). This kind of gestures has an important function of locating in space the referent of the discussion. Likewise, Edwards (2005) reported that almost all gestures produced in the solution of a problem, related to fractions, by prospective teachers

were deictic. According to Edwards (2009), they constitute a particular modality of embodied cognition.

In this paper we take a complementary approach, inspired by the work of Goodwin (2003), and include the analysis of the structure of the task. He has introduced the concept of *symbiotic gesture* when investigating how gesture is related to the physical, semiotic, social and cultural components of the context where it is embedded. An example provided by Goodwin (op. cit.) refers to archaeological analysis related to patterns of earth. He explains that the finger of the archaeologist pointing to the ground shows the graphic structure in the dirt, and, at the same time, that structure provides the context, the place, for the precise movement of the gesture. Another example of a football player is a classic one: if taken in isolation, it is not evident what he is doing. However, if the player is placed in the context of the game, the meaning emerges naturally. According to Goodwin (op. cit.), the nature of embodied practices which promote the competence to act as a member of a community is basically interactive. So, instead of taking as an analytical focus the gesture and discourse by themselves, we include the object which gestures are referring to as part of the analysis. We include as well the activity where this object is inserted in a sequential organisation, taking into account contributions from participants assuming different roles at different moments in the lesson. We illustrate how the teacher makes use of these components in the dialogues with her pupils.

METHOD

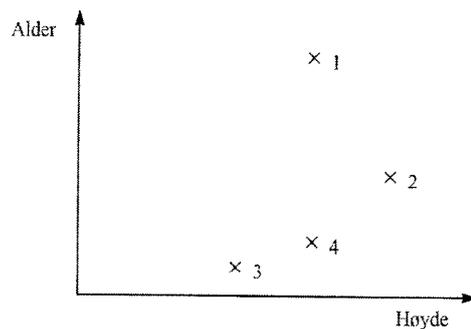
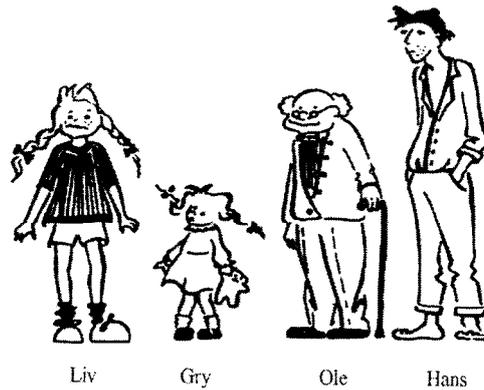
For analysing the discourses we have used a dialogical approach to communication and cognition (Bjuland, 2002; Cestari, 1997; Linell, 1998; Marková & Foppa, 1990) in order to identify an experienced teacher's communicative strategies used in the dialogue with her pupils. In this approach, there are some important principles: the *sequentiality*, *joint construction*, and *act-activity interdependency* (Linell, 1998). As far as the *sequential organisation* of discourse is concerned, "each constituent action, contribution or sequence, gets significant parts of its meaning from the position in a sequence. That means that one can never fully understand an utterance or an extract, if taken out of the sequence which provides its context" (op. cit., p. 85). In this case we have to take into account how a particular utterance is related to the *previous* utterance as well as to the *subsequent* one. The teacher's gestures are identified within a theoretical framework that considers cognition as an embodied phenomenon (Edwards, 2009) and as an interactional process (Goodwin, 2003). Further details about this multimodal approach can be found in Borgersen, Cestari, and Bjuland (in press) and in Bjuland et al. (2008b).

The dialogues presented in this paper are situated in a particular instructional context where the teacher, in dialogue with her pupils, sums up the mathematical solutions (part 3). In our analysis, we focus on the teacher's speech and gestures embodied and situated in the lesson. Part 3 of the selected 19-minutes video clip has been transcribed line by line, and we have divided the transcribed material into numbered

utterances/turns. “An utterance lasts as long as a speaker holds the floor” (op. cit., p. 281). The gestures are described in italics inside brackets [] within the utterances/turns where they occurred.

The task

The following task was given to the pupils: Write down which person corresponds to each of the points in the diagram (the Norwegian words *alder* and *høyde* mean age and height respectively).



Liv corresponds to point

Gry corresponds to point

Ole corresponds to point

Hans corresponds to point

In earlier papers (Bjuland et al., 2008a; Bjuland et al., 2008b) we presented a detailed analysis of the proposed task, emphasising the characteristics of the three mathematical representations *figure*, *diagram* and *written text* respectively. Here, we only present the task as a background for understanding the dialogue between the teacher and her pupils while summing up the mathematical solutions. The teacher-pupil dialogues therefore focus particularly on the third representation (written text), including questions asking for the number in the diagram corresponding to every person in the figure.

SUMMING UP IN THE CLASSROOM

The plenary discussion (part 3) could be summarised in one ongoing episode, consisting of five thematic sequences:

Sequence	Communicative Strategies	Time	Turns
1. The location of Ole – explanation	Open question: Who is number one, two, three and four respectively? Two how-questions, trigger pupil explanation. The answer is visualised on the overhead projector. One further how-question, and the pupil repeats his explanation. Question asking for other suggestions. The teacher uses gestures by pointing to point 1, 2, 3 and 4 on the transparency.	1.13 min	162–172a
2. The location of Gry – explanation and justification	Open question: What about the other points? How-question – triggers an explanation. The answer is visualised on the overhead. Why-question related to the two variables, height and age. Gestures are not identified.	0.43 min	172b–179
3. The location of Hans – explanation and justification	Open question: Other answers? The answer is visualised on the overhead. How-question – triggers an explanation. Question asking for other suggestions in combination with gestures, pointing to point 1. Why could Hans not be point 1?	1.21 min	180–200a
4. The location of Liv – explanation and justification	Question directed to a pupil, Do you have the last solution? The answer is visualised on the overhead. How-question – triggers an explanation. One further question, Was it just a guess or should it be like this? Gestures are not identified.	0.40 min	200b–206a
5. Teacher summing up	Do all of you agree with these answers? Other solutions? Give praise to the pupils. Focus on the unusual – height at the horizontal axis. Recapitulation of the two dimensions, height and age. Gestures are not identified.	0.54 min	206b

Table 1: Plenary discussion after the small-group work

In our analysis we have focused on the first sequence of the dialogue since it illustrates how the teacher initiates the discussion. We have also chosen an extract from the third sequence since this dialogue shows how the teacher focuses on the pupils' argumentation, emphasising the connection between the two dimensions, height and age in the diagram. This third sequence also shows how one of the pupils

(from the group with the two girls) that seemed to have most difficulties in understanding the task (Bjuland et al., 2008a; Bjuland et al., 2008b) responds to one of the teacher's questions, giving us some impressions of her understanding of the problem at this moment.

These sequences show the direction of the mathematical discussion between the teacher and her pupils, from a discussion of the location of Ole to the location of Gry and so on. This is based on the pupils' responses to the questions posed by their teacher.

The location of Ole

The dialogue below illustrates the first utterances in the teacher-pupil discussion of the mathematical solutions which have resulted from the collaborative small-group work. The teacher (Tea) initiates the dialogue, inviting her pupils of both sexes to be attentive to the task:

- 162 Tea: Girls and boys [*Turns on the overhead projector*]. What I wonder about, what I actually wonder about, where are the different persons? Who is number one? [*Points at point 1, diagram*], who is two? [*Points at point 2, diagram*], who is three? [*Points at point 3, diagram*], and who is four? [*Points at point 4, diagram*] Per?
- 163 Per: We think Ole is one.
- 164 Tea: Ole is number one. How can you be sure of that? How did you think that out?
- 165 Per: Since he's oldest, and then he is tallest [Hans] (...).
- 166 Tea: Yes.
- 167 Per: [Ole is] as tall as Liv.
- 168 Tea: Okay. But Ole he's then number one. Can you write it on [the transparency], so we know it? [Per goes to the overhead projector and writes "1" on the transparency] ... Ole is number one. [Per gives the pen/Indian ink to his teacher and goes down to his seat] But what did you think when you found out that Ole was number one?
- 169 Per: Since, when he is [oldest]
- 170 Tea: [Ssss]
- 171 Per: and then he is on the picture, then he is as tall as Liv. No one else is as old as him [Ole].
- 172 Tea: Okay. Mm. Did anyone think differently? Since he is oldest, okay.

The teacher initiates the discussion by using the same open questions as she did when she presented the task before the collaborative small-group work (Bjuland et al., 2008c). However, her gestures are a bit different. In Bjuland (op. cit.) we observed that she focused on the transition from the figure to the Cartesian coordinate diagram by making four consecutive *pointings* to the diagram with a gradually decreasing *circular sliding* between the diagram and the figure. The interplay between the teacher's gesture and her questions seemed to be a mediating device in her

presentation, showing the relationship between figure and diagram. She is here using the four *pointing gestures* to the diagram in connection with her questions without moving between the two representations (162). We observe from the dialogue that the teacher's use of gestures in part 3 is far less prominent than in the presentation of the task (part 1) and in her small-group dialogue (part 2) with the two girls (Bjuland et al., 2008b). This indicates that the teacher uses more gestures in connection with her explanations to the pupils than in relation to pupils' explanations. In the dialogue between the teacher and the pupil Per (162-172), he comes up with the group solution for Ole as a candidate for point 1 (163). This response guides the direction of the discussion, showing that the teacher-pupil dialogue begins to focus on one of the extreme locations. The two questions from the teacher (164) stimulate Per to give an explanation (165) by making a comparison between Hans and Ole related to both age and height and a comparison of Liv and Ole related to their same height (167).

After having been concerned with the third representation (*written text*), showing the written solution on the transparency, the teacher poses a third how-question (168), provoking Per to repeat his explanation (169), (171). The teacher invites the pupils to make other suggestions (172), but she does not wait for a response. It seems that the teacher has observed that her pupils are satisfied with the solution putting Ole at point 1.

The location of Hans

The dialogue below contains a particular extract from the third sequence.

- 194 Tea: But you [singular you], what did you [plural you] think when you found out that Hans should be number two?
- 195 Odd: We thought that he was tall, and he [Hans] was much younger than Ole.
- 196 Tea: Mm. Yes, so therefore he should be there. Is there anyone else that thought about it? [Silence, 6. sec.] Leo, what did you think?
- 197 Leo: Eeh, no I (...)
- 198 Tea: Eeh, yes, Is there anyone else that thought about it? Let's see, Hans is number two. He had to be there. Why couldn't Hans be there [Points at point 1, diagram] Why couldn't Hans be there, Eli?[The teacher chose Eli among several pupils who raised their hands]
- 199 Eli: Since he, or if Ole, he is the oldest and then couldn't he [Hans], since he [Hans] is the youngest [of these two].
- 200a Tea: Mm. Yes.

In the second sequence of the episode, one of the girls chooses Gry at another extreme location in point 3 and gives an explanation for the location of Gry (see Table 1). One of the boys has responded to the teacher's open question and told the class that Hans corresponds to point 2, the third extreme location. This answer has also been visualised on the transparency.

In the continuation of the dialogue, the teacher poses a question that stimulates the pupils to explain how they come up with this particular location for Hans (194). The pupils were not only to produce an answer, but they are also challenged to explain their thinking. Odd's response, starting with *we*, (195) shows that he explains the group's thinking. In his explanation Odd is concerned with the two variables, age and height, making a comparison between Ole and Hans. Since they have already discussed the location of Ole (first sequence), it is natural for Odd to explain how his group has discovered the relationship between the placement of Hans and Ole respectively.

After having evaluated this response, the teacher goes on to pose another question that provokes other suggestions (196). The pause indicates that the teacher allows a waiting time of six seconds, giving the pupils opportunities for individual considerations. Since the pupils do not respond to this initiative, the teacher repeats her question and directs it to the individual pupil, Leo (197). His response and the teacher's next question (198) show that the pupils do not have other suggestions. They seem to be convinced that Hans corresponds to point 2. We might wonder why the teacher is so focused on bringing other suggestions into the dialogue. One possible explanation could be that she wants to focus on possible misconceptions. The teacher seems to be aware of how complex it could be for pupils to realise how the two variables, height and age, are connected in the Cartesian coordinate system. By focusing on point 1 as a possible location for Hans, the teacher also triggers the visual misconception: the tallest person corresponds to the point, located highest in the diagram. In connection with this question she also uses gestures to make the pupils aware of the possible location of Hans at point 1. In the analysis of the dialogue of the two girls (Bjuland et al., 2008b), we identified this misconception.

When the teacher poses the challenging why-question twice, provoking the pupils to consider the wrong location of Hans, the pupil Eli (pupil 4 from our girl group) responds to the teacher's initiative (199). Eli makes a comparison of Hans and Ole due to their ages. In one respect, it is possible to argue that Eli is still just focusing on one dimension, the variable of age. However, if we situate the response in this particular context based on the teacher's way of posing the question and also the teacher's evaluation of the response (200), it seems as if Eli has given a proper explanation and developed her understanding from the group work.

CONCLUDING REMARKS

Through the analysis of dialogues from the teacher-pupil discussion of group solutions on the diagram task, we have identified the teacher's communicative strategies. Her use of *questioning* (who, how, why, other suggestions) is the most prominent strategy. The analysis has also revealed that her use of gestures is more restricted in part 3 compared to gestures used in connection with her explanations while presenting the task and in a small-group dialogue with the two girls (Bjuland et al., 2008c). We could wonder why this restriction happens in part 3. When the teacher

plays the role as a *presenter* (part 1) and as a *supervisor* (part 2), she uses gestures as a mediating device in combination with verbal explanations. In part 3 she uses mainly gestures, pointing to the diagram without circular slidings between representations, to initiate the discussion. Here (in part 3) the teacher plays the role as a *coordinator*, opening the floor for the pupils to write their answers. The teacher-pupil discussion focuses on the mathematical representation, *written text*, in which the pupils show their group solutions on the transparency, making explanations and justifications.

Concerning the contribution of the environment, supported by the concept of *symbiotic gestures* (Goodwin, 2003) we have observed that the nature of the task is influencing the different pointing gestures. It is indeed the pupils' responses that guide the direction of the mathematical discussion. Gestures and discourses are conceived as *meaning translators* between different mathematical and pedagogical ideas used by the teacher as communicative strategies.

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A TEACHER'S ROLE IN WHOLE CLASS MATHEMATICAL DISCUSSION: FACILITATOR OF PERFORMANCE ETIQUETTE?

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In the improvisation that occurs in a jazz ensemble, a soloist rarely develops a completely new idea but, instead, elaborates and builds on the previous player's input. From an emergent perspective, classroom mathematical practice is akin to such improvisation. How this might happen in a whole-class situation is unclear. In this paper, a description is given of a whole-class discussion that took an unplanned trajectory. The teacher did not impose a particular structure on the lesson but focused pupils' attention on productive mathematical ideas that emerged from the group. In the concluding discussion, it will be shown that the improvisation metaphor, while useful for describing mathematics as a socio-cultural activity, may have a different application in a whole-class situation than in small group settings.

INTRODUCTION

Although plenary sessions are common to mathematics lessons, they are often characterized by traditional approaches that endorse the position of mathematics as a kind of received knowledge and the teacher as sole validator of students' contributions (See, for example, Boaler, 2002; Cobb, Wood, Yackel, & McNeal, 1992) While research shows that whole-class discussion can be fertile ground for higher-order mathematical thinking (Cobb et al., 1992; O'Connor, 2001), the fast pace with which it is usually associated means that there is little scope for students to make comments and build on each others' mathematical ideas (Hodgen, 2007). One consequence of this is that students become disengaged from the subject, perceiving it to be one in which they have little opportunity for participation (Boaler, 2002). However, the orchestration of inquiry-based discussion in mathematics is challenging for teachers. Sherin (2002) alludes to two key tensions whereby teachers, on the one hand, are expected to encourage students to share ideas and, on the other, have to ensure that the lesson is mathematically productive.

In this paper the improvisation metaphor is used to show how a teacher and her pupils co-constructed new mathematical ideas in the context of a whole-class discussion in a primary school. In particular, attention is paid to the way provision can be made for different levels of understanding within the class. In the concluding discussion, reference will be made to limitations of some tools that are used to analyse such research.

THE IMPROVISATION METAPHOR

According to Lakoff and Johnson (1980), metaphors not only help us to understand one kind of thing in terms of another but they can also create a reality and thus act as

guides for future action. In relation to the teaching of mathematics the improvisation metaphor is one that serves both of these purposes. Consistent with a view of mathematics as a socially and culturally situated activity, the point of reference in mathematics education is the classroom mathematical practice, a perspective that has been described by Cobb (2000) as *emergent*. Sawyer (2004) maintains that this perspective implies that teaching must be improvisational and ‘that the most effective learning results when the classroom proceeds in an open, improvisational fashion, as children are allowed to experiment, interact, and participate in the collaborative construction of their own knowledge’ (p.14).

In theatrical improvisation, a group of actors creates a performance without using a script. Because it is characterized by a high level of unpredictability, the performance has associated with it what Sawyer describes as a ‘moment-to-moment contingency’ (Sawyer, 2006: p.153). As the actors play their parts, several potential possibilities are brought into the frame. What emerges is not decided by any one person but rather is a phenomenon that is produced by the group. In jazz improvisation, each soloist is assigned a number of measures to play before the next soloist takes over. Due to the rapidity of the transition, a player rarely develops a completely new idea but rather responds to and builds on the previous player’s input (Berliner, 1994).

Sawyer (2004) maintains that like the improvisation that occurs in theatre or in a jazz ensemble, creative teaching is both emergent and collaborative. It is emergent because the outcome cannot be predicted in advance and it is collaborative because the outcome is determined not by any one individual but by the participants of the group. Martin, Towers and Pirie (2006) used the improvisational lens to analyse collective mathematical understanding. They describe collective mathematical understanding as the kind of learning and understandings that occur when a group of any size work together on a mathematical activity. Central to their analysis is the idea of *co-acting* which they define as

...a process through which mathematical ideas and actions, initially stemming from an individual learner, become taken up, built on, developed, reworked, and elaborated by others, and thus emerge as shared understandings for and across the group, rather than remaining located within any one individual. (p.156)

They make a distinction between co-actions and interactions. While in interactions there is an emphasis on reciprocity and mutuality, co-actions concern actions that are dependant and contingent upon the actions of other members of the group (Towers & Martin, 2006). Through this co-acting, an understanding emerges that is the property of the group rather than any individual. It is not that all individuals bring the same understandings to the scene but rather that individual contributions will result in something greater than the sum of the parts. Neither does it preclude an individual making his or her own personal advancements.

In a more fine-grained analysis of the improvisational metaphor, Martin and Towers (2007) have introduced the notion of performance etiquette. In jazz terms this refers

to a situation where players drop their own ideas in deference to a better (in the view of the collective) idea if that works. It means that due attention and equal status have to be given to all players' ideas and intuitions. According to Martin and Towers, '(in) mathematics, 'better' is likely to be defined as a mathematical idea, meriting the attention of the group, which appears to advance them towards the solution to the problem' (p.202). Although much of the work done by Martin et al. concerns small groups there is evidence that the metaphor is also applicable to whole class discussion (See, for example, Dooley, 2007). King (2001) contends that in lessons where students and teachers co-create classroom discourse, 'one can view students as other participants in [the] improvisation, following the direction of the lead improviser, the teacher'(p.11). She proposes that the teacher is rather like the soloist who must modulate her performance to her instrumentalists and audience. There is some danger that this analogy leads to the teacher's role being perceived as centre of (as opposed to central to) the learning process. Sherin (2002) suggests that, in order to achieve a satisfactory balance between process and content, the teacher engages in *filtering* by which is meant a narrowing of ideas generated by students so that so that there is a focus on mathematical content. An implication for whole class discussion is that the teacher is more facilitator of group etiquette than lead improviser. This idea is pursued further in the account below.

BACKGROUND

The aim of my research is to investigate the factors that contribute to the development of mathematical insight by primary school pupils. The methodology is that of 'teaching experiment' which was developed by Cobb (2000) in the context of the emergent perspective and in which students' mathematical development is analysed in the social context of the classroom. For a period of six months, I taught mathematics to a class of thirty-one pupils (seven girls and twenty-four boys) aged 9 - 10 years. The school is situated in Ireland in an area of middle socio-economic status. Although I taught the lessons, the class teacher played an active role as co-researcher, advising on the suitability of lesson content, clarifying any confusion that arose in whole class discussions, working with pupils during group work and making observations in post lesson discussions. Many lessons took place over two or three consecutive days, each period lasting forty to fifty minutes. I visited the class on a total of twenty-seven occasions. All phases of the lesson were audiotaped. When children were working in pairs, audio tape recorders were distributed around the room. Each pupil maintained a reflective diary. Follow-up interviews were held with students who had shown some evidence of reaching new understandings over the course of a lesson.

Forman and Ansell (2001) contend that analysis based on isolation and coding of individual turns is too limited to bridge the individual and social. Therefore, I conducted ethnographic microanalysis, which according to Erickson (1992) is especially appropriate when the character of events unfolds moment by moment. The

approach adopted was top-down starting with the molar units (lessons) and moving to progressively smaller fragments. I transcribed all lessons and isolated those in which pupils showed evidence of constructing new mathematical insight. Thereafter I identified constituent parts of the lesson, starting with major events and moving progressively to the actions of individuals. A comparative analysis of lessons was also undertaken.

The lesson described here took place on a third consecutive visit to the class during a week of the Spring term. On the previous two days, the pupils had been working on a lesson entitled 'Chess', the object of which had to find the minimum number of games that could be played by participants in a competition where each competitor had to play all other players. At the conclusion of this lesson some pupils had found the answer for one hundred players (i.e., the sum of 1 - 99) by using a calculator while others had latched onto the discovery made by one pupil, David¹ that the solution could be found 'by multiplying by the number less than it and halving it' ($(100 \times 99) \div 2$). It was my intention on the third day to begin a new lesson but first told the story of Gauss (the mathematician who, as a boy, had amazed his teacher by his rapid calculation of the sum of integers from 1 to 100) in order to see if the pupils would make any connections between it and the chess problem. I expected that talk on this problem would last no longer than five or ten minutes. However, a rich discussion followed in which I truly had to improvise. Although this lesson is not being promoted as exemplary, I learnt from it something about the power of 'letting go' and ways in which group etiquette might be facilitated.

The focus of this paper is on the discussion that took place after I first related the story of Gauss. Although space does not allow the full transcript to be presented, an effort is made to give as full as possible a sense of the lesson trajectory (a problem described by O'Connor (2001: p.144) as 'the competing requirements of data reduction and interpretive explicitness'). The following transcript conventions are used: T.D.: the researcher/teacher (myself); Ch: a child whose name I was unable to identify in recordings;...: a hesitation or short pause; [...]: a pause longer than three seconds; (): inaudible speech; []: lines omitted from transcript because they are extraneous to the substantive content of the lesson.

THE IMPROVISATIONAL CREATION

On telling the story, some pupils suggested that Gauss may have found his solution by adding fifty and fifty or five twenties, considering addends of rather than the sum to a hundred. When I focused their attention on the problem conditions, Barry had this idea:

18 Barry: Eh, you add up all the numbers that are in ten like one, two, three, four, five, six, seven, eight, nine, ten...

¹ Pseudonyms are used throughout the paper.

- 19 T.D.: Hmm.
- 20 Barry: and then multiply by ten.
- 21 T.D.: Ok, so you would add up as far as ten and then multiply the answer by ten?
- 22 Barry: Or nine, I'm not really sure.
- 23 T.D.: Ok, why do you think it might be nine?
- 24 Barry: Eh, because you have already counted up to ten and it's ten tens in a hundred.

Here he was making an assumption that the sum of numbers between 1 and 10 would be the same for all decades. Brenda then asked if she could check the answer on the calculator which was interesting given that she had thus correctly established the solution for forty players in the Chess activity.

Anne and Fiona then built on the idea proposed by Barry:

- 48 Anne: I think it's thirty multiplied by ten.
- 49 T.D.: Sorry?
- 50 Anne: Thirty multiplied by ten.
- 51 T.D.: Thirty multiplied by ten, why would you say it's thirty? []
- 54 Anne: Because if you add from one up to ten it's thirty.
- 55 T.D.: How do you know if you add one up to ten it's thirty?
- 56 Anne: If you add one to five, that's fifteen...
- 57 T.D.: Hm, hm
- 58 Anne: and then fifteen and fifteen is thirty so then if you multiply that by ten.
- 59 T.D.: Ok, possibly that would get it for you. Fiona?
- 60 Fiona: Well, could you em, oh, em, do, eh, you could do one plus two and up to fifty and then double it...

I chose not to correct misconceptions at this point but wrote the suggestions on the blackboard. This proved a good judgement in this instance because a short while later two pupils commented on Anne's input:

- 66 Alan: Em, well, I don't think Anne's one is right.
- 67 T.D.: Why?
- 68 Alan: Cos ninety-nine plus ninety-eight plus ninety-seven plus ninety-six to ninety would be around over five hundred and when...
- 69 Ch: Oh!
- 70 T.D.: Ok, [] you are thinking ninety plus ninety one plus ninety two plus ninety three would give you approximately how much?
- 71 Alan: Em, I don't know.
- 72 T.D.: But it's...
- 73 Alan: But it would probably be over five hundred.
- 74 T.D.: It would be over five hundred, so in that section, if you are thinking about all those numbers there that would give you about, even just adding ninety to a hundred so you are thinking that would give you about five hundred. []. Barry?

75 Barry: Eh, well, I disagree with Anne as well because, eh, I counted, I counted up all the numbers up to ten and I got fifty-five.

Enda then said that multiplying five by twenty or adding fifty plus fifty (both ideas were written on the blackboard) didn't 'actually have much to do with this'. Anne now corrected her earlier idea:

91 Anne: I don't think...my answer wouldn't work.

92 T.D.: What were you thinking your answer was?

93 Anne: I thought it would be thirty multiplied by a hundred.

94 T.D.: Why would it not work?

95 Anne: Em, because you would have to, cos I did eh one plus two plus three plus four plus five and then em I got fifteen and then I added fifteen and fifteen equals thirty but then it would be more because you would have to add six, seven and that.

Anne seemed to have reached a new understanding about the addition of a series of numbers. It is possible that she began to reflect on her thinking because Barry and Alan disagreed with it. Colin then arrived at a new approach to the problem:

97 Colin: It could like eh add the, say you could have ninety-nine, add the closest and the furthest and then the second closest and the second furthest.

98 T.D.: So give me an idea what you are talking about now. Tell me, elaborate a bit on that. []

101 Colin: Eh if it was ninety-nine, you add one, if it was ninety-eight you add two, if it was...

102 T.D.: Ok, so you are thinking - very interesting because that's - you could have ninety-nine plus one, go on!

103 Colin: Ninety-eight plus two, ninety-seven plus three, ninety-six plus four, eh, ninety-five plus five, ninety-six or ninety-four plus six (*teacher records on blackboard*)...

104 T.D.: Ok, so what's that giving you, why are you putting those numbers together?

105 Colin: They all go up to a hundred.

106 T.D.: So what's that telling you then, what do you think it might be, have you any idea what the answer might be?

107 Colin: Eh, no.

108 T.D.: Do you see what Colin is doing there? He is matching up numbers, he is taking the numbers at the very beginning and he is matching them up with the numbers at the end.

I was quite excited when I heard this input as this was the method used by Gauss as a young boy, hence my remark, on line 102, 'very interesting because..'. I wrote his suggestion on the blackboard but also 'revoiced' his input (line 108), a teacher strategy that serves to repeat or expand a student's explanation for the rest of the class (Forman & Ansell, 2001; O'Connor, 2001). Enda then proposed a different way of grouping the numbers. However, I did not grasp his idea:

- 113 Enda: Eh, well, I think one possible way it would probably would be just as hard, it would be harder than one plus two plus three, it's probably not going to help us, what I was going to say is eh adding...when adding ninety plus ninety-one plus ninety-two and all that sort of stuff...
- 114 T.D.: Hm, hm.
- 115 Enda: It's the same every time, you would just, all you would probably, eh, you would probably need to go backwards and just take way ten from the answer above every time. That would () if you took away ten from the answer every time.
- 116 T.D.: Hm, hm
- 117 Enda: So add up the numbers going from a hundred backwards. []
- 120 T.D.: If you went a hundred plus ninety-nine plus ninety-eight plus ninety seven...
- 121 Enda: Yeah
- 122 T.D.: all the way back as far as one, would you still get the same answer?
- 123 Enda: The same answer, even though it would just be easier to do it backwards with that way em you just need to take ten away from it every time. If you were on ninety, if you got a hundred back to ninety and you were on eighty, just take ten away from the answer above.

Enda had found an interesting solution method, that is, adding from 100 to 91 and then finding the solution for the sum from 90 to 81 by subtracting ten. In fact this is a very viable method (if one hundred is subtracted each time). I had assumed he was talking about commencing the addition from a hundred rather than one. It is very possible that I did not comprehend his approach because it was one I had never considered. I did, however, ask him to pursue his idea in his diary.

Liam then made another observation about Colin's list:

- 135 Liam: I don't think like if you go back to Colin's way...if you go back, you wouldn't be able to do it, if you go back to one then you might double it, the whole thing.
- 136 T.D.: Sorry?
- 137 Liam: If you go all the way to one, then you double the whole thing.

Neal then suggested that the list should terminate at $50 + 50$ and I urged pupils to think about the number of 'hundreds' there might be. Anne then proposed that the answer would be a thousand and this led to an interesting contribution by Brenda:

- 166 Anne: I think the answer would be a thousand.
- 167 T.D.: You think it's going to be a thousand. Do you agree with Anne that it's about a thousand? Brenda?
- 168 Brenda: Eh, no cos when I em added up forty for it and, em, I got more than a thousand.

This is the first time in the lesson that a direct reference has been made to the chess activity. Fiona confirmed that the answer for 40 children (i.e., the sum from 1 to 39 although this was not as yet clear) was 780. Anne picked up on this idea:

183 Anne: Well, in the one we did yesterday, when the number of children was a hundred, then the number of games was four thousand, nine hundred and fifty so that there would be the answer.

I wrote 4950 on the blackboard as one other possibility. Hugh however noticed the error:

197 Hugh: I think it would be, em, five thousand, nine hundred and fifty.

198 T.D.: Where are you getting that from?

199 Hugh: Em, because eh yesterday we didn't add on the hundred.

200 T.D.: Ok [...] so

201 Hugh: So then it would be ...five thousand...and fifty.

Liam now saw that $50 + 50$ should not be included in the list:

209 Liam: Well on the last one in Colin's one you have to do a triple sum kind of () because it would be forty nine plus fifty one and then add fifty on to it.

David confirmed that the solution was 5050 and explained his reasoning as follows:

213 David: Em, well if you do Colin's way and then, em, you get, em fifty () and then when you get to forty nine plus fifty one and you have to add the fifty on and that gives you about five thousand and fifty.

At this point in the discussion the class teacher indicated that a small group of pupils had taken out their diaries and were working on solution methods in them. In particular, Declan seemed to be very keen to complete the listing suggested by Colin. The pupils embarked on paired/individual work during which the class teacher sat with Declan. In the plenary session that was held at the conclusion of the lesson, Fiona and Clare discussed possible answers for the sum of numbers up to 200 (they proposed 5050×2). Some pupils spoke about the solution they found on the calculator. Declan described how he solved the problem using Colin's method. Miles began to consider that the answer might be obtained by multiplying a hundred by a hundred and then halving it 'to take way the pluses that you add on to get one hundred'. David, however, did not use the formula he had found for the chess problem to add the numbers from 1 to 100.

DISCUSSION

There is evidence that co-acting took place in this lesson. For example, in the early part of the lesson, Fiona and Anne picked up on Barry's idea of adding a section of numbers and applying proportional reasoning (albeit incorrectly). Later Anne reconsidered her reasoning on the basis of input by Alan and Barry. Colin's idea may well have emerged because of the discussion around addition of numbers between 1 - 10 and 90 - 100 (see lines 68 and 75). Enda's method could be an elaboration of that proposed by Colin. Brenda made the explicit connection with the previous day's lesson which prompted solutions by Anne and Hugh. However, the co-acting is not as linear as might be the case in small group discussion. Rather there is a weaving in and out of ideas. Lines 135 and 209, where Liam broke the flow of conversation to

transform Colin's listing, are instances of this. It also seemed that some students who made no contribution to the dialogue reported above were nonetheless actively engaged. For example, Declan, a student who is not confident about his mathematical ability, pursued Colin's idea with great zeal. An implication of this is that tools used to analyse whole class discussion must extend to include those who are silent but participating in the enquiry.

O'Connor (2001) ponders the difficulties of looking objectively at transcriptions and attempting to discern the motives of the teacher in taking certain actions. As the researcher/teacher on this lesson, I am in a position to say, at least to some extent, why I took certain courses of action. A primary concern was keeping things, to continue with the jazz metaphor 'in the groove', for the group while at the same time respecting the input of individuals. Enda's idea (lines 115 and 123) did not become part of the collective because I did not understand it. Recourse to a diary allowed him to pursue his own investigation, however. My position in this lesson was not that of lead improviser because the lesson took an unexpected trajectory, but I feel that I facilitated group etiquette by drawing attention to ideas that would lead to solution to the problem.

With regard to the future direction of this research, the ways in which whole class discussion can impede or facilitate pupils' mathematical insight will be further analysed. In particular attention will be paid to the ways in which the making public of ideas by writing them on the blackboard and the revoicing of pupils' input stimulates the filtering process.

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USE OF WORDS – LANGUAGE-GAMES IN MATHEMATICS EDUCATION

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This article focuses on the introduction of new concepts in mathematics classrooms. A theoretical framework is presented which helps to analyse and to reflect on the processes of teaching and learning mathematical concepts. The framework is based on the theory of Ludwig Wittgenstein. His language-game model and especially its core, the primacy of the use of words, provide insight into the processes of giving meaning to words. The theoretical considerations are exemplified by the interpretation of a scene, in which students are introduced to the concepts of “perpendicular”, “parallel” and “right angle”.

INTRODUCTION

“Mathematics education begins and proceeds in language, it advances and stumbles because of language, and its outcomes are often assessed in language.” (Durkin and Shire, 1991, p. 3)

A lot of research has been done on communication in the mathematics classroom. Mathematical interactions have been analysed from many different perspectives (cf. Cazden, 1986). This text will focus on the teaching and learning of mathematical concepts in classroom communication. The importance of introducing mathematical concepts is underlined by the multitude of theories used for analysing concepts. In this paper only a few of them can be taken into account: de Saussure (1931), Peirce (CP 2.92) and Steinbring (2005).

By his concept of “language-game” Wittgenstein offers us an alternative view on the introduction of concepts in mathematics classrooms. His perspective has often been used to discuss problems concerning communication in the mathematics classroom (e.g., Bauersfeld, 1995; Schmidt, 1998). Sfard (2008) is using Wittgenstein’s theory within her “commognitive model”.

Wittgenstein presents considerations we can use to analyse language and especially the meaning of words. His theory of language-games and the construction of meaning will be considered in this paper, which presents first results of scientific research in progress. According to Wittgenstein, the expression of words does not constitute their meaning. Words have another function in the process of constructing knowledge. The main aim of the research is to analyse whether Wittgenstein’s theory is useful for reconstructing and thus understanding communication. In spite of the multiple Wittgenstein references, I only know a few examples of using Wittgenstein’s theory for analysing communication in the mathematics classroom (cf. the examples of Sfard 2008). More specific aims will be described in the course of this article. The core of the theory, the primacy of the use of words, will be exemplified.

USING WORDS IN LANGUAGE-GAMES

In his later philosophy (cf. the “philosophical investigations” and the “remarks on the foundation of mathematics”) Wittgenstein describes a pragmatic theory of language and meaning. He denies every fixed relation between language and objects. Also Wittgenstein is no longer searching for anything, which could be taken as something basically shared by all linguistic acts. Language is not an objective mediator between human beings and objects given. Nevertheless, he considers knowledge – and thus mathematical knowledge – not to be transmitted objectively:

“Language is a universal medium – thus it is impossible to describe one’s own language from outside: We are always and inevitable within our own language [...]. Knowledge appears as knowing, and knowing is always performed in language games. Language as languaging or playing a language game is equal to constituting meaning and, thus, constituting objects. There are no objects without meaning, and meaning is constituted by a special use of language within a respective language game” (Schmidt, 1998, p. 390).

For Wittgenstein the construction of knowledge takes place by playing language-games. The term “game” does not imply an option for those who are involved. We cannot choose in the first place whether we want to play the game or not. The problem is that Wittgenstein does not explain in detail what he means when speaking of “language-games”. As we will see, this is not because he does not care. Rather it is due to his theory of giving meaning to words.

Words have neither a consistent nor an objective meaning. In different language-games various meanings of a word can occur. Following Wittgenstein there is no direct transformation from a word to its meaning: “[...] experiencing a word, we also speak of ‘the meaning’ and of ‘meaning it.’ [...] Call it a dream” (Wittgenstein, 1958, p. 216). Moreover, it is the use of a word which determines its meaning:

“For a large class of cases – though not for all – in which we employ the word ‘meaning’ it can be defined thus: the meaning of a word is its use in the language” (Wittgenstein, PI, §43).

The term “use” is not limited to the application of words (e.g., in order to solve problems). If we exemplify a word, we also make use of it. One research-guiding problem will be to identify different forms of uses of mathematical words.

A word does not mirror objects and the meaning of a word cannot be observed while looking at its association with a specific object. The meaning of a word is nothing but the role it is playing in the specific language-game and accordingly can be observed only by looking at the use of words. This central thesis might be the reason why Wittgenstein does not define what he means using the term “language-game”. He stays consistent: He exemplifies the words he makes use of [1]. Language-games can be different in character. So Wittgenstein (PI, §23) presents the following examples among others:

- “Giving orders and obeying them”,

- “Forming and testing a hypothesis” and
- “Solving a problem in practical arithmetic”.

These examples may indicate that language-games are little “passages” or specific situations in our daily communication, but Wittgenstein also presents a larger field:

“I shall also call the whole, consisting of language and the actions into which it is woven, the ‘language-game’.” (Wittgenstein, PI, §7)

Language is constituted by a “multiplicity” (Wittgenstein, PI, §23) of language-games. And all these language-games bear a temporal dynamic:

“And this multiplicity is not something fixed, given once for all; but new types of language, new language-games, as we may say, come into existence, and others become obsolete and get forgotten. (We can get a *rough picture* of this from the changes in mathematics.)” (Wittgenstein, PI, §23)

The temporal dynamic indicates once more that there is no specific meaning for words fixed forever. Changing the meaning of a word is accompanied by a change of the language-game. Learning also means to realize changing meanings of words. Learning includes learning how to play different language-games. Thus, learning implies partaking in changing and new language-games.

USING WITTGENSTEIN

In mathematics education there has been a lot of research to consider and to analyse concepts and how students get used to them. Some work (e.g., Duval, 2006) is based on de Saussure’s (1931) relation between signifier and signified (fig. 1). The theory of de Saussure provides a subject-object dualism and thus implies some problems:

“If there would be a correspondence between language and reality, then, surely, one could arrive at true verbal statements about the world. Descriptions (and teaching), then, would become a case only of an adequate selecting and of providing for sufficient precision of the verbal means (denotations), as well as an adequate fit of these means with the object” (Bauersfeld, 1995, p. 277).

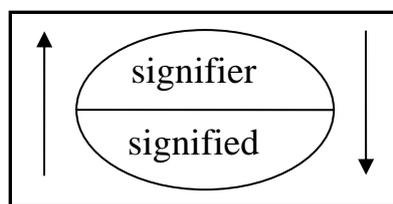


Figure 1: De Saussure’s (1931) relation between signifier and signified

Peirce (CP 2.92) offered a more detailed framework. His triadic relation between the sign, its object and its interpretant (fig. 2) has been used to analyse and to describe verbal or non-verbal interaction (e.g., Hoffmann & Roth, 2004; Presmeg, 2001; Sáenz-Ludow, 2006; Schreiber, 2004). The reconstruction of classroom interaction based on this framework has to deal with the difficulty that it is problematic to

determine the object to which the sign is related. Contrarily, Wittgenstein's theory is a more pragmatic one. He does not regard any ontology of a sign. According to his theory words only get their meaning by their use and do not transport any given meaning. There is no fixed relation between words and objects.

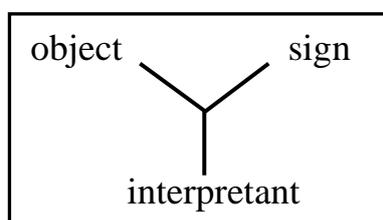


Figure 2: Peirce's triad

By his epistemological triangle (fig. 3) Steinbring (2006) provides a way to analyse static moments in the process of giving meaning to words. He presents a triadic relation between “sign/symbol”, “object/reference context” and “concept”:

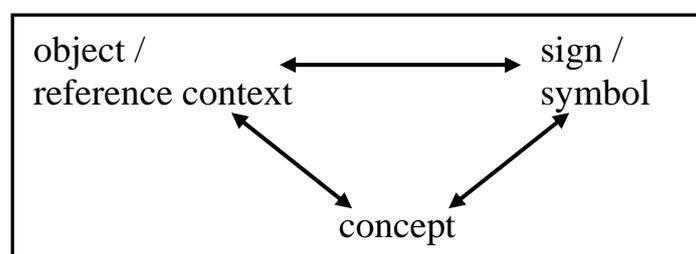


Figure 3: Steinbring's epistemological triangle (2006, p. 135)

The importance of the context can also be observed in Wittgenstein's writings, as he is considering the use of a word in the specific language-game. And language-games are depending on the situation:

“Here the term ‘language-game’ is meant to bring out into prominence the fact that the speaking of language is part of an activity, or of a form of life.” (PI, §23)

Wittgenstein points out that there is no direct transport of meaning from the teacher to the student, nor a direct understanding. We only can analyse the meaning of a word by looking at the use of that word in a specific language-game, which is at the same time influenced by other language-games. If we take a look at the language-game “mathematics education”, we are also confronted with influences of every-day language-games of the students (and the teacher) and, all the more, of the rather mathematical language-games the teacher is able to participate in with mathematics experts outside of the classroom.

Words can be used in more than one language-game and thus each word can exhibit different meanings. If the teacher is going to introduce a concept in mathematics education, the children might immediately associate some meaning to it – due to the use of that word in another language-game the student took part in. This might be an every-day language-game or a language-game of mathematics education of a

previous era (e.g., subtraction means to remove things, which does not work for negative numbers).

Words could be used in more than just one way. Accordingly, they can convey different meanings or meanings, which cannot be grasped only by knowing one form of their use. Thus, the use of a word in a specific situation must not lead to the whole range of possible meanings. Also, some concepts are restricted or expanded in the course of mathematics education (e.g., the concept of numbers). Therefore, this study is going to focus on the introduction of new concepts in the mathematics classroom and their development during following lessons. Some research-guiding questions are: How do students make use of words? What might be the meaning of a word for them? How do teachers influence the play of another language-game?

METHODOLOGY

The empirical data emerged from classroom observations in different grades (1-10) in Germany. Classroom communication has been videotaped by teacher students acting as researchers. The videographed units comprised 4-8 lessons of 45 minutes each. The teacher students were observers; they were told to exert no influence on the classroom communication and on the teachers' way to introduce the concepts. Altogether eight classes were visited.

The qualitative interpretation of the classroom communication is founded on an ethnomethodological and interactionist point of view (cf. Voigt, 1984; Meyer, 2007). Symbolic interactionism and ethnomethodology build the theoretical framework which is going to be combined with the concepts of "language-game" and "use".

According to Wittgenstein we should not ask: What is the meaning of a word? Rather we should analyse what kind of meaning a word gets in the classroom. Therefore, we have to analyse social processes. Thus, we have to follow the ethnomethodological premise: The explication of meaning is the constitution of meaning.

Analysing students' languaging for mathematical concepts, the development and the alteration of meaning by the use of the according words, we are able to reconstruct the social learning in the mathematics classroom [2].

The main aim of this study is to get a deeper insight into the processes of giving meaning to words in the mathematics classroom. Therefore, alternative ways of introducing concepts are going to be considered. Comparing possible and real language-games can help to understand the special characteristics of the actual played language-game.

THE USE OF WORDS IN CLASSROOM COMMUNICATION

The following scene emerged from a 4th grade classroom in Germany (students aged from 9 to 10 years). It is the first time that these students get in contact with the concepts of "parallel", "perpendicular" and "right angle" in this mathematics class.

The teacher starts the lessons by writing the words on the blackboard and asking the students to associate anything coming to their mind about these words. Afterwards a painting by Mondrian (cf. fig. 4) is presented on the blackboard [3].

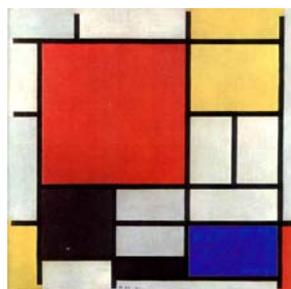


Figure 4: Painting by Mondrian on the blackboard

Teacher: Why do I fix such a picture on the blackboard? And why are these concepts written down on the blackboard? I have a reason to do so. Jonathan, it is your turn.

Jonathan: Because the painter has done everything in parallel, perpendicular and in right angles.

Teacher: You are right. You seem to know what parallel, perpendicular and right angle means. Maybe you can show it to us on the picture.

Jonathan: Perpendicular is this here (points first at a vertical, afterwards at a horizontal line). Parallel is this here (points at two vertical lines). A right angle is this (pursues two lines he former would have called perpendicular).

By pointing to different things on the blackboard, Jonathan makes use of the words “perpendicular”, “parallel” and “right angle”. He must have been in contact with practices of using them and thus with meanings of these words in a language-game outside this classroom. In this situation the words get a meaning by him pointing at something. This use can be described as an *exemplaric use*: An example is used to show the meaning of a word.

The use Jonathan makes of the words need not imply that those words could also be used in different ways, but *this* use and respectively *this* meaning get established in this classroom communication.

The teacher does not have any further questions. The teacher accepts the use of the words Jonathan must have known from another language-game. Thus, it seems that the exemplaric use is an acceptable one and that the meaning of the words is “taken-to-be-shared” in the classroom (cf. Voigt, 1998, pp. 203).

Certainly, in another language-game the meaning of the words “perpendicular”, “parallel” and “right angle” can be different. They can be defined by using other concepts. A right angle can be defined as an angle of 90 degrees. Also the word “right angle” can be used in coherence with Pythagoras’ theorem or in relation to the shortest distance of parallel lines. Perpendicular can be described by using the

concept of “right angle”. All of these uses describe other language-games and not all of them can be played in a 4th grade classroom. Altogether, the words can have different uses and, thus, different meanings. In this classroom the words are used in order to represent things (cf. de Saussure’s model).

In the next few minutes the students had to create a mindmap, which should contain “something which can fit to the picture”. Then, afterwards “perpendicular” gets exemplified on the picture again. Now the classroom communication goes on with “parallel” and “right angle”:

Teacher: Now we just have two problems: parallel and right angle.

Sebastian: Right angle is easy (holds the set square at the blackboard).

Teacher: Can you show it here (points at two lines on the painting by Mondrian which have been used to show “perpendicular”). (After five seconds) Doris just say it. Wait! Before you go ahead, let –

Doris: You can make out four right angles out of it.

Teacher: This is the sign for the right angle (draws  on the blackboard). Maybe you can just draw it into the picture? (After three seconds) You can also choose another one.

Doris: John

Teacher: John and Tim come here. Doris said you would be able to find four right angles.

John: You two, me two (speaks to Tim while pointing at two lines).

Teacher: That is not right. No. Doris, show him were they are.

John: There is a right angle.

Teacher: Ah, yes!

The class is going to consider the last two “problems” (parallel and right angle), which have not been exemplified a second time. Doris identifies four right angles on those lines, which had been used before in order to show the meaning of the word “perpendicular”. John shows an example for a right angle. Again we can speak of an *exemplaric use*. The meaning of the word “right angle” is connected to the examples on the blackboard. Now and again, it seems that the meaning of “right angle” is “taken-to-be-shared”, but the students do not yet express characteristics of right angles, they only have examples.

Now the scene is going on:

Tim: Ah, this corner which is coming from the right side (marks the angle with the teachers’ sign)

Teacher: Correct! Just make it a little bit thicker, so that the other ones can see it.

Tim: This is a left angle. (points at the opposite side of the vertical line)

Teacher: No!

Lisa: That is always a right angle.

Tim recognizes the examples as examples for the use of the word “right angle”. He explains why John’s example can be called “a right angle”. Thus, he abstracts from the concrete example and presents a use of the word “right angle” by a kind of definition: The word “right angle” can be used, if a line for the angle comes from the right side. Tim tries to give an *explicit-definitional use* (cf. Winter, 1983) of the word: The student describes a general characteristic when and how the word “right angle” has to be used. He relates the word “right angle” to other words. Contrarily to the former use of the word “right angle”, Tim uses another ethnomethod to constitute meaning.

The concept of the word “left angle” is used by an implicit reference. It is implicit, because the pair of concepts “left-right” indicates that an orientation in space is considered – a relation between observer and object. Thus, the word “left angle” gets an *implicit-definitional use*. The exemplaric use Tim makes of the word “left angle” can be seen as a test of his proposal. It is a probable consequence of his first definition. In other words: It is a hypothetic-deductive approach of verification (cf. Meyer, 2008).

Tim’s use of the word “right angle” can be explained only because there is use of the word “right” in common practice. Here the word “right” can be used to show a certain relation between observer and object. So Tim was able to combine the two uses of the words “right” and “angle” to establish a constructive meaning of the conglomerated word “right angle”. The comment of the teacher harshly shows that the new language-game is not acceptable.

Tim’s use shows that the former meaning of the word “right-angle” only seemed(!) to be “taken-as-shared”. It has not been shared. Tim has been trying to give a theoretical fixation of the concept. The language-game he initiated is not an acceptable one. Lisa does not take part in the new language-game. She seems to play the former game and to explicate a routine: We need to have more examples to grasp the meaning of the word “right angle”.

FINAL REMARKS

The episode shows that de Saussure’s model is not sufficient to analyse classroom communication. Mathematical concepts are in need of a fixation by other concepts (a theoretical fixation). An empirical way can be used to introduce words, but the language-game has to change afterwards. In this scene a student initiates another language-game, which is condensing in (not acceptable) theorems.

The use of Wittgenstein’s theory shows that concepts can be observed by looking at the way teacher and students make use of the words at hand in the specific language-

game. In this scene we have seen an *exemplaric*, an *explicit-definitional* and an *implicit-definitional* use. The exemplaric use consists of pointing at examples to illustrate the words. The explicit-definitional use consists in giving an explanation for the word in relation to other concepts. Thus, it provides a deeper insight in mathematical coherences: Characteristics of the underlying concept get expressed. The concept gets a general character, not being linked to special examples any more. An explicit-definitional use is also in need of a deeper mathematical insight, as it has to be known what counts as a definition. The implicit-definitional use in this scene requires a common pair of concepts (“left-right”) and an explicit-definitional use of the other word.

Wittgenstein’s theory itself is not a theory of interpretation. Rather he presents a theoretical framework, which can be used on top of a theory of interpretation. Symbolic interactionism and ethnomethodology fit to Wittgenstein’s considerations of social processes in languaging. Future analyses have to show the fruitfulness of this framework.

NOTES

1. “ ‘The meaning of a word is what is explained by the explanation of the meaning.’ I.e.: if you want to understand the use of the word ‘meaning’, look for what are called ‘explanations of meaning’.” (Wittgenstein, PI, §560).
2. As proposed by Bauersfeld (1995) I will speak of “languaging” to accentuate the connotation of language use.
3. Many thanks to Johannes Doroschewski and Philipp Heidgen for the video. The translation has been done and simplified by the author of this article. The original transcript will be sent on demand.

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SPEAKING OF MATHEMATICS – MATHEMATICS, EVERY-DAY LIFE AND EDUCATIONAL MATHEMATICS DISCOURSE

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The aim of this paper is to describe and analyze how discourse as a theoretical and didactical concept can help in advancing knowledge about the teaching of mathematics in school. The collection of empirical data was made up of video and audio tape recordings of the interaction of teachers and pupils in mathematics classrooms when they deal with problem-solving tasks. Discourse analysis was used as a tool to shed light upon how pupils learn and develop understanding of mathematics. The results underline that a specific and precise dialogue can contribute towards teachers' and pupils' conscious participation in the learning process. Teachers and pupils can construct a meta-language leading to new knowledge and new learning in mathematics.

INTRODUCTION AND AIM OF THE STUDY

This research deals with teachers and pupils discussing with each other in different situations within and about mathematics in school. The theoretical point of departure is first and foremost an in-depth study of the meaning of and relationships between concepts, words and signs in order to demonstrate how mathematical discussions can be understood. The concepts of context, mediation and artefacts are central to the socio-cultural perspective chosen and thus play an important role in this research, (Vygotsky, 1978, 1934/1986, 2004). The concept of context can be described as being the environment where our actions take place and thus create and re-create the context as such. Mediation implies that human beings interact with external tools in their perception of the world around them. Linguistic as well as physical artefacts are created by mankind to perform actions and solve problems. They are cultural resources which contribute towards maintaining and developing knowledge and abilities in society (Vygotsky, 1978, 1986). Using semiotic tools one can demonstrate how a linguistic element is connected to its meaning, (Ogden and Richards, 1923; Melin-Olsen, 1984; Johnsen-Hoines, 2002). We can picture a semiotic triangle made up of concept, expression and reference. If we look upon language as a medium for communication based on conventional signs it is by applying language that the reference to the world at large is created.

The relationship between thought and symbol is, like the one between thought and reference causal and direct in a semiotic triangle. The relationship between symbol and reference, on the other hand, is indirect and attributed. Concepts within a socio-cultural perspective which may be applied to the semiotic triangle are expression, content and reference. These three functions of a sign can only be understood when they are applied simultaneously. Thus we can see signs such as words, numbers, symbols, diagrams, equations and letters. The sign expresses something separate from the sign itself. Signs, objects are related to the meaning or conception of them. Mathematical knowledge must be actively constructed in relationship to signs, words and symbols.

I have chosen to describe mathematical discussions out of a discourse perspective. The concept of discourse can be understood in different ways. It can be interpreted as a set of conventional rules for discussing, understanding and conceiving the world and its different phenomena (Winther-Jørgensen & Phillips, 2000; Sfard, 2002). A discourse can be understood as a linguistic system which delineates issues of exclusion and inclusion, borders on what is excluded and inner standardization (Gee, 2005; Börjesson & Palmblad, 2007).

Foucault (1972/2002) wants to clarify how we are caught up in and blinded by lines of reasoning without really being conscious of what we say. We can refer to this as an invisible discourse. In the discourse on teaching mathematics there is an invisible element which is difficult to affect unless we make ourselves aware of its existence.

From a socio-cultural perspective discourse is defined as the language which gives and is attributed meaning in various contexts and which excludes and includes things to be understood (Säljö, 1999, 2000). In this work I have chosen to metaphorically regard discourse as a network where signs, concepts and references make up the nodes. Nets can be chosen or created in such a way that meaning is constructed in situated action as well as socio-cultural practices which transgress defined situations. Thus, a discourse can also be a set of rules for talking, writing and thinking about a specific content. Many discourses are mixed in school which both teachers and pupils must learn to become involved in, understand and master. This includes knowing when borders between different discourses are crossed. Mathematical instruction means that teachers and pupils are placed in different discourses, ranging from those applied to every-day life to purely mathematical ones. This means that they move over borders and between registers all the time. An example of this occurs when pupils work with concrete materials and are to express themselves using numbers and symbols. In doing so, they will move over different borders. When working in school we must learn to understand when we are situated in a specific discourse.

A mathematics lesson contains a number of words and expressions from every-day life. The language applied is rich and we talk departing from many different perspectives and towards many different aims. To be able to conduct conversations in a context as specific as school mathematics we have to develop a meta-language

which makes it possible to put what we want to express into perspective. In every-day life we build models in order to understand reality and we use every-day methods for solving problems in order to describe connections to mathematics. We seek the history of mathematics to be able to see how every-day application developed into pure mathematics. This paper mirrors how teachers and pupils apply different types of discussions to deal with problem-solving tasks in and about mathematics. In these discussions we develop our thinking and our methods for learning and it is in the same discussions that we shed light on the transitions required in order to move from concrete to abstract activities. A knowledge rendered in linguistic terms is required. This is something that I aim to disclose in my empirical studies. In the discussions in and about school mathematics an oscillating movement between reality and mathematical concepts and expressions is to be seen.

Communication in a mathematics classroom can be described in terms of learning a mathematical register, (Duval, 2006). It can also be looked upon as a situation where there are two parties involved – two individuals who speak, think, write, read and listen. It is therefore highly interesting to study what learners and teachers have to say in and about mathematical practices.

The over-riding aim here is to raise this issue: “How can discourse as a theoretical and didactical concept contribute towards further developing mathematical teaching?”

Method

I have for many years been interested in communication and interaction within and about mathematical teaching. In my studies I have chosen to monitor how teachers and pupils have generated knowledge in discussions on mathematical concepts, problem-solving and formal mathematics. I did so in order to be able to establish what happens in interaction between teachers and pupils and between pupils.

In these studies I have made use of video and audio recordings. Video recordings were applied in order to make sure that it became clearly visible what went on in the interaction within a classroom. It also proved to be fruitful in that the activities on both teachers’ and pupils’ part became evident. The audio recordings were used as a means of analyzing the discussions as interactive situations. Group interviews are a well-chosen strategy for trying to capture discourse as regards what they include and exclude. The table below describes the environment used to acquire data in the respective studies.

Design of the Empirical Studies

Study I	Study II	Study III	Study IV
7 teachers	14 groups	26 groups	68 groups
Teacher-pupil interaction Classroom	Pupil interaction Classroom	Group talk Three pupils	Group talk Three pupils
Video	Video	Audio tapes	Audio tapes

Mathematical content

The Area of a Triangle	The Area of a Triangle	Problem-solving	Rational Numbers
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Table 1. Data acquisition in the empirical studies I-IV

Seven teachers took part in my first study. They were assigned to plan and carry out an introductory lesson on the area of the triangle in year 5 in compulsory school. Choosing mathematical content was a regular concept to the teachers who took part. Focus for these video recordings lay on documenting the public and the teacher-led interaction in the classrooms involved. Each recording lasted between forty and sixty minutes. Twenty-five occasions were recorded and focused on interaction between teachers and pupils. The study further describes how teachers cross discourse borders in teaching on the area of the triangle and in what ways they carried out their lessons as regards interaction between teachers and pupils, as well as what types of questions they used in their talks with pupils.

The introductory lesson on the area of the triangle is carried on into this second study but here focus is on pupils' interaction in a laboratory situation, where the teacher gives explicit directives to the groups of pupils. Varying directives from the teacher in the classroom lead to different trains of action and linguistic concepts on the pupils' part. In total the interaction of fourteen groups has been recorded and analyzed in the classes involved. The groups were made up of five to six pupils. The laboratory situations are described as regards activity and linguistic interaction. The pupils are active in that they draw, cut and fold pieces of paper. Every-day language is used to a great extent and retains its every-day character.

The point of departure for the third study was to monitor 26 groups of pupils when they set about a written mathematical task. The task is of an open variety and contains different pieces of information that the pupils are to decide on. One of the concepts which stay in focus for the pupils is the word fairness. Pupils seek, talk, make guesses, test and calculate an answer. There is, however, no evident way to go about solving the task. On the one hand the pupils end up in an every-day discourse and on

the other hand in a mathematical discourse. They have difficulties making judgments as they reason with each other. Each group has been recorded on audio tape which has then been transcribed and analyzed. The pupils were put into groups on the basis of their mathematical skills as deemed by their teachers. The recordings took place in a small room next to the classroom.

For the fourth study one of the assignments from the National Test of mathematics for year five was used. The assignment deals with rational numbers. Five different partial studies were carried out. Sixty-eight groups of three pupils each and 120 individual pupils took part in the different studies. The first partial study was carried out with 30 pupils in year five who solved the assignment on their own and were asked to provide a written explanation. The second study took place in three classes of 30 pupils each. For the third partial study I used five schools from different parts of a large municipality. Thirty-one group interviews with pupils in year five were carried out, each group consisting of three pupils. When the pupils solve their assignment they rely on an every-day discourse. The next study involved 31 new groups of pupils. They were allowed to use a pocket calculator and they engaged in a solely mathematical discourse. The last part of this study was carried out with six groups of three pupils each and it deals with the issue of reasoning with the help of a numerical line. The results show that, depending on what tools are applied and what situation the pupils are in, the outcome turns out differently in different discourses.

I have used a discourse analysis to analyse the group discussions and the discussions in the classroom, (Wertsch, 1985, 1998; Kozulin, 1998; Fairclough, 1992, 1995, Gee, 2005). A discourse analysis is based on details in what is written and said in a particular situation. In the restricted discourse language can be seen as “language-in-action” which is always an active process in constructing knowledge. My study focuses on the interaction between individuals and in what ways knowledge, language and mathematical skills develop.

Results

Discourse analysis can be used as a tool with help of which descriptions of how pupils learn and develop their understanding of mathematics can be made clear. Looking at my empirical material I have come to discern the discourse in school mathematics which can provide the bridge upon which teachers and pupils can meet and become mutually involved.

In school mathematics teachers and pupils talk using every-day concepts and mathematical concepts, signs and words. This intercourse demands that a mutual understanding takes place. The analysis of what is said in the different groups shows that the discussions are situated somewhere on a scale between two extremes – on the one hand every-day concepts, on the other hand purely mathematical concepts. Words such as “put on” and “put together” are based in every-day practice whereas

words such as “add/addition” and complex numbers are situated in a purely mathematical discourse. Any individual is to be found somewhere in this continuum depending on how far this individual has come in the process of developing an understanding of abstract reasoning. If we consider signs and expressions the same thing can be said for them.

In my empirical data where teachers talk to pupils in whole-class discussions and in group talks, teachers utilize different signs and change registers in their teaching. They go from geometrical into arithmetical/algebraic discourse and back. Analysis of these talks clearly reveals how pupils talk about and understand the concepts. Most pupils use every-day language and it demonstrates that teachers are situated in one discourse and pupils in another. The same thing can be seen when pupils work with concrete materials, performing acts but not acquiring the mathematical concepts which the teacher had planned. Pupils find themselves in a distanced discourse rather than an inclusive one as the teacher had intended. In one of my excerpts the pupils are engage in a group discussion of how to move from a rectangle made of red paper to a triangle. The teacher has told the pupils to prove that the triangle’s square is half of the rectangle. Here we can follow their discussion:

Måns: Mine is so smeary. Nobody can think about that it is so smeary.

Kalle: We can fix this so it will be the half.

Beatrice: It’ll be a square.

Stina: Do you know how to fold all pieces of papers. I can’t fold anything.

Måns: You can learn how to fold if you know how to fold.

Kalle: The fundamental form to fold frogs, but I can’t, they don’t jump like this.

Stina: I can fold aeroplanes.

Here you can see pupils being in an every-day and distanced discourse. They try to follow the teacher’s goal to prove but they got into another discourse.

In another assignment of a problem-solving character about decimals the pupils first had to work with an every-day picture as a point of departure and their talks are thus carried out in an every-day discourse. Some pupils do not arrive at the mathematical terms and an understanding of them. Other groups are given a formal assignment to be solved using a pocket calculator and they remain there, locked up in the system of signs and decimals. Yet another group of pupils draw lines together in order to understand the decimals and can accommodate the mathematical signs and words, which makes them involved in the discussion and solving of the assignment. They start to speak, think and write “Mathematish”.

I: Now I want you to explain why you think that this is right.

H: Nine is a whole number, it’s one smaller, only a whole number. 9,12 is nine whole and one tenth and two hundredths, I think, 9,2 then there is nine whole number and two tenths.

E: Nine is such a whole one. 9,12 there is a tenth smaller than two tenths so then 9,2 will be bigger than 9,12.

N: Nine is a whole number the second number in 9,12 is a hundredth and 9,2 the second is a tenth.

The connections are created between every-day references and mathematical concepts and expressions and it becomes easier for pupils to leave the idea of “doing”. Meaning has been attributed to mathematical concepts and signs and these have been created for defined ends. But the meaning can only be understood by those who are able to take part in a mathematical discourse.

By analyzing how teachers and pupils talk about mathematical phenomena in different situations I can use the concept of discourse to establish that connections are often not created between every-day concepts and their mathematical counterparts. If pupils cannot interact and thus form networks of concepts which assist them on their path to conscious mathematical thinking this becomes a major problem for them. Consequently teachers and pupils must develop their mathematical language in concord with every-day language.

Discourse analysis can thus be used as a tool where descriptions of pupils’ learning processes and understanding of mathematics can be made clear. I have displayed the results of my documented discussions and will place discourse in focus and further develop it as a means of establishing a direction.

Discussion

If the discourse is viewed as a distinct means of establishing the direction for teaching mathematics, it becomes the teacher’s task to bring to a conscious level the different ways pupils use for passing borders between different discourses, so that pupils become aware of the nature of mathematical concepts. A discourse is made up of artefacts and products created by mankind for specific ends and the language used can be understood only if the discourse itself is understood (Säljö, 2005). Teaching should invite pupils to become participants in a mathematical discourse.

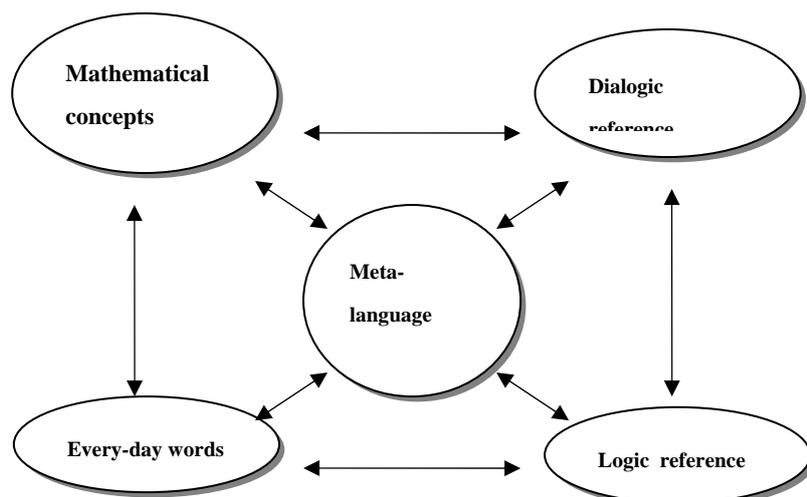
The words *speak*, *think* and *write* can be viewed as parts of a discourse and when teachers and pupils apply them in the teaching and learning process, it can reinforce consciousness and participation in mathematical thinking. This could constitute the formative discourse. Furthermore, teachers and pupils must learn to realize what is changed when going from one discourse to another in mathematics. To be able to discern whether the discussion is carried out in an every-day or a mathematical discourse, to be able to recognize whether one is situated in a geometrical or an algebraic discourse and how the movement between registers manifests itself in mathematics is important knowledge for teachers, student teachers and pupils. When an individual speaks the way language is applied can develop qualitatively by the process of learning to value, scrutinize and put forth arguments in both every-day and mathematical discourses. In these, thinking is developed and by using linguistic and concrete artefacts in interplay thinking is further prompted. We can thus create a

connection between every-day life and mathematics. Since mathematics started in a culture which used conventional signs and written language it has also developed texts and thus reading is a part of mathematics. The concepts of listening and reading should also be entered into the discourse, leading onto the concept of interpreting. In this perspective pupils will actively form and interpret their knowledge.

Discourse can be defined as a “way of speech” but I would prefer to widen the definition in so much that I view discourse as a network where teachers and pupils acquire knowledge by moving between and utilizing mathematical and every-day concepts, expressions and situations by talking, thinking, writing, listening and reading.

It has been my ambition to put the concept of discourse into perspective in the following manner. By adopting a discourse perspective we can direct attention to linguistic dimensions of mathematics teaching. It would also assist us in letting individual, silent calculation interact with a communicative aspect. By formulating and interpreting their mathematical knowledge pupils can acquire new knowledge. We will create a recognizing nearness through experience and distancing, fostered in a development and a familiarity with the system of mathematical signs. Through quality in the discussions which arise in a learning process we can develop the language concerned and thus improve understanding. In this context quality means that teachers and pupils use words, signs, concepts and situations in awareness of the specific discourse. We should also keep in mind that a mathematical discourse is something that develops over time.

Current research presents many images of the existent situation – “this is what it is like”. My discourse perspective, however, focuses possible changes. I want to present a discourse theory which recognizes qualities in language and knowledge from both the every-day world and the mathematical sphere and in doing so clarifies both every-day and mathematical concepts. In this context quality means that we communicate around a concept, a sign, a reference and a situation by looking critically at it, putting forth arguments for and against, and eventually arriving at understanding what I take with me from this learning process. It is absolutely clear that the further our acquisition of new knowledge develops into an issue of learning to apply abstract and complex intellectual and practical tools, the more essential it becomes to engage in communicative practices. Thus we can learn how to apply and co-ordinate these tools, both linguistic and physical, with an outside world to reach new mathematical knowledge. Models and symbolic representations can be tested critically as regards their connections to the every-day world and other concepts as well as their logical consequence and explanatory value. The table below reinforces discourse as a theoretical and didactical concept.



Model describing the passing of borders between discourses.

By placing focus in learning processes on the concept of discourse our teachers and pupils can grow to master a meta-language for school mathematics. This will then constitute a specific and precise language in and about mathematics. Language is constructed in our actions and how we express ourselves using the appropriate signs. By putting forth arguments and making interpretations in a dialogical environment we can acquire knowledge as regards knowing when borders between discourses are passed, as well as regarding the interplay between thought and experience in mathematics.

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COMMUNICATIVE POSITIONINGS AS IDENTIFICATIONS IN MATHEMATICS TEACHER EDUCATION

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Student teachers positioning related to own emotions and experiences, the mathematics and the teaching and learning of mathematics, and the classroom, teachers and others are theorised, and exemplified, as aspects of identifications as becoming mathematics teachers.

INTRODUCTION

As a teacher educator I have searched for signs of how the student teachers in the pre-service mathematics courses change from seeing themselves as students of mathematics to seeing themselves as teachers of mathematics. That is negotiating identities as mathematics teachers.

Teaching is not a knowledge base, it is an action, and teacher knowledge is only useful to the extent that it interacts productively and dynamically with all of the different variables in teaching. Therefore connecting the act of teaching and teacher identities focuses on identities as something people do which is embedded in social activities, and not something they are.

Identifications as teachers of mathematics, through acting, or performing, as teachers in mathematics, are closely associated with meaning making in mathematical contexts. In this paper I will outline descriptive devices in order to analyse the properties in texts and the technical skills of mathematical communication that are employed in the service of mobilizing teacher identities by student teachers.

Dewey (1916) examined the purpose of education in a democratic society. He writes: “society not only continues to exist *by* transmission, *by* communication, but it may fairly be said to exist *in* communication” (p. 4, emphasis in original). He further holds that “This transmission occurs by means of communication of habits of *doing*, *thinking* and *feeling* from the older to the younger” (p. 3, emphasis added by Ongstad 2006).

Conceiving teachers’ knowledge as part of a complex set of interactions involving *action*, *cognition* and *affect*, places teaching as a complex practice. A main perspective then is a view of teaching and learning as communication (Braathe, 2007; 2009; Ongstad, 2006; Sfard, 2008).

POSITIONING THEORY

“Positioning Theory” has been discussed and developed among others by Harré and van Langenhove (1999). Their concept of positioning is offered as a dynamic replacement of the more static concept of role. Role identity theory views society as made up of roles, and explains how roles are internalised, as cognitive schemes, as

identities that people enact and try to live up to (Stryker and Burke, 2000). “‘Position’ will be offered as the immanentist replacement [...] of transcendentalist concepts like ‘role’” (Harré and van Langenhove, 1999, p. 33).

Harré argues that during communicative interactions, people use narratives, or “storylines”, to make their words and actions meaningful to themselves and others. They can be thought of as presenting themselves as actors in a drama, with different parts or “positions” assigned by the various participants. Positions made available in this way are not fixed, but fluid, and may change from one moment to the next, depending on the storylines through which the various participants make meaning of the interaction.

In positioning theory, the concept of positioning is introduced as a metaphor to enable an investigator to grasp how persons are ‘located’ [...] as [...] participants in jointly produced storylines.

One mode of positioning of particular interest to us [...] is the intentional self-positioning in which a person express his/her personal identity (Harré and van Langenhove, 1999, pp. 61-62).

IDENTITIES

Identities have been used as a strategic concept in research addressing the relationship between individuals and society, and, related to this, in formulating how selves are socially constituted, and in explaining how social structures or processes affect individuals’ lives.

The kind of questions asked in traditional social science are *what* identities people have, what *criteria* distinguish identities from each other, and what *part* identities play in the maintenance of society and in enabling the functioning of social structures and institutions. In this respect social identities are assumed to have an overarching relevance (Stryker and Burke, 2000).

Underlying most of these approaches, whether sociological or social psychological, are concepts of identities that can be characterised as essentialist and realist. The concepts are essentialist in the sense that identities are taken to be properties of individuals or society; and realist in the sense that it is assumed that there is some kind of correspondence between identities and some aspects of social reality (Sfard and Prusak, 2005).

Across the social sciences, the main criticism of, and alternatives to, traditional models of identities are found in a variety of social constructionist approaches. The concept of identity produced is designed in part to deal with variability and flexibility and how even the most obvious identities are negotiable. Although they are various, these approaches share in common an emphasis on the multiple ways that social identities are constructed, negotiated and performed. Contrary to the use of identity for the purpose of classification, or as a causal variable related to other phenomena,

this view of identities, it is argued, enables a social constructionist to provide a more dynamic view of individual-social relations.

A social constructionist approach also draws on the idea that symbolic or cultural resources influence identities, and how identities are constructed through historical, political, cultural and discursive practices. It is argued that the symbolic or linguistic resources available in the discourses provide possibilities and constraints on identities individuals can take. Methodologically this is used empirically to identify the linguistic resources or repertoire available in a culture for individuals to construct their self-understanding. In other words, they aim to show how cultural narratives become a set of personalised voices and positions. This offers alternative ‘texts of identities’.

IDENTIFICATIONS

The positioning theory developed by Harré and van Langenhove (1999) is based on social constructionism. They see positioning in terms of a triad of interrelated concepts: storyline, positions and actions/acts. The storyline is the narrative that is being acted out in the metaphorical drama. Within it, the positions are the parts being performed by the participants. The actions of the participants are given meaning by the storyline and the positions available, and once given meaning become social acts. This positioning can be seen as interactors identifying themselves as actors, and being identified by others, in a metaphorical drama.

The focus on identifications as a participant’s resources generates different questions and a different focus. Thus, instead of asking what identities people have, the focus is on *whether*, *when* and *how* identities are used in social acts, for example performing as teachers of mathematics.

In their pre-service teacher education student teachers have to produce texts answering different tasks and reporting from group works and from practicing teaching in practice schools. Text in this connection will also include mathematical text. These texts can be seen as utterances in a dialogic relation to their teachers in the teacher education, or as social acts within the storylines of mathematics teacher education. These social acts are seen as positionings, or identifications as becoming teachers of mathematics.

I investigate student teachers’ identifications relative to the three aspects of action, cognition and affect. Instead of methodologically trying to identify available positions in these storylines as categories following a social constructionist methodology, I will use another related dynamic concept of communicative positioning derived from Bakhtinian thinking searching for these three aspects. This concept of positioning is used as an analytic tool to analyse the student teachers texts as they are seen as struggling for making meaning of teaching and learning of mathematics.

POSITIONING AS A TRIADIC DISCURSIVE CONCEPT

The communicative positioning developed and used by Ongstad (2006) is partly generated from Bakhtin's essay "The problem of speech genres" (Bakhtin, 1986, pp. 60-102). Ongstad identifies Bakhtin's communicative elements necessary for an utterance to communicate in dialogic relations. One of these is how the utterance is positioning, and positioned, as such by addressing someone, referring a semantic content, and expressing feelings and intentions.

Methodologically the utterance is seen as the unit of analysis. We communicate through utterances. Utterances are any sufficiently closed use of sign that makes sense. All utterances are uttered and interpreted related to expectations of genres, i.e. contexts that helps us to understand the utterance. Genres are ideological, i.e. they give tacit premises for the utterances' positioning in the communication (Bakhtin, 1986). Ideology is broadly defined as unspoken premises for communication (Braathe and Ongstad, 2001). It is something we think *from*, not on. Genres can be described as kinds of communication.

The genres are to be seen as triadic in the same sense as the positioning of the utterance, that they simultaneously give potential for the addressing, referring and the expressing. The three aspects are seen as parallel, inseparable, reciprocal, simultaneous processes (Ongstad, 2006).

In the mathematics teacher education context the three aspects are seen as positioning related to addressing the classroom, teachers and others, referring the mathematics and the teaching and learning of mathematics, and expressing own emotions and experiences. Students' different texts relate to different components of teacher education. Consequently they are positioned differently with dominance either on the expressive, referential or the addressive aspect. However, as utterances, all three aspects are simultaneously present, and consequently identifying the student as becoming teacher of mathematics related to all three aspects. This identifying process focuses identities as something the student teachers do, as communicative positioning, which is embedded in the social activity of teacher education.

MATHEMATICS AS GENRES

Seeing mathematics and mathematics education as a kind of communication will be to see mathematics and mathematics education as genres. I will hold the view that in their pre-service training student teachers are parts of different genres, kinds of communication, including mathematical, and potentially experiencing different ways to act as a teacher. It is helpful to call this process 'learning'. This will theoretically be connected to seeing learning as semiosis in the field of teaching mathematics. This connects to seeing learning as communication. This shifts seeing development from a psychological to a semiotic perspective so as to locate developmental principles in the making of meanings. As I see learning, or developing of identities, as being positioned in communicational genres, I locate identities as dialogically situated in, negotiated and formed by genres, and so can have many expressions dependent on

the context. Identity can then be seen dynamically combining the personal, the cultural and the social (Braathe, 2007).

Sfard (2002; 2008) takes a similar “communicational approach to cognition” (2002, p.26), where she holds that “[t]hinking may be conceptualised as a case of communication” (2002, p. 26), and even constructs the concept of “commognition” (2008, p. 296) to emphasise the necessary connection between the two. She further holds that “[l]earning mathematics may [...] be defined as an initiation to mathematical discourse, that is, initiating to a special form of communication known as mathematical” (2002, p. 28).

Furthermore Sfard holds that “[c]ommunication may be defined as a person’s attempt to make an interlocutor *act, think or feel* according to her intentions” (Sfard, 2002, p. 27, emphasis by me). Discussing factors that give discourses their distinct identities Sfard identifies meta-discursive rules as

usually not something the interlocutors would be fully aware of, or would follow consciously, [...] there are special sets of meta-rules involved in regulating interlocutors’ mutual positioning and shaping their identities (ibid. p. 30-31).

TELLING IDENTITIES

In Braathe (2007) I discuss the theoretical framework presented in Holland et al (1998), especially their use of the Bakhtinian diverted concept of “the authoring self”. I relate this Bakhtinian concept to Sfard and Prusak (2005) and their conception of identity (Braathe, 2009). They define identities as stories about persons. In a communicative and dialogic sense they adhere to that “[i]dentity [...] is thought of as man-made and as constantly created and re-created in interactions between people” (Sfard and Prusak, 2005, p.15). Stories about persons, the term identifying, is in their context to be understood as “the activity in which one uses common resources to create a unique, individually tailored combination” (p. 14). From seeing the processes of identifying as discursive activities, the activities of communication, they suggest that “identities may be defined as collections of stories about persons or, more specifically, as those stories about individuals that are *reified, endorsable and significant*” (2005, p. 16, emphasis in original). This definition is an attempt to avoid the problem of essentialism, the extra-discursive existence that often is either implicit or explicit in the use of the concept of identity in educational research.

Discursive acts of positioning, identifying, are seen in my context as communicative acts for establishing meaning. In the teacher education students’ produced texts can be seen as utterances that communicatively position the student teacher dynamically combining the personal, the cultural and the social.

These texts/stories are not about persons, but about the explorative mathematics activities in their pre-service training, where the students have to explain mathematical patterns, connections and reasoning. These texts are seen as utterances in the genres of teacher education, told by the students of “themselves” to their

teacher. Sfard and Prusak (2005) call these stories the student teacher's first-person identity. On the other hand my analysis of positioning of these texts will be called stories about stories. These stories about stories can also be seen as the student teacher's third-person identity told by me as the researcher. In teacher education the resources, voices, used by the student teacher when writing in the different genres of mathematics educational texts, are found in dialog both with practice, theory and experience, and as such seen as influencing the negotiation of their semiotic identifications as teachers of mathematics.

The analysis of positioning, applying the triadic discursive concept to these texts,

A

a) 2 - 7 - 12 - 17 → 22 - 27
 Man finner det neste tallet ved å legge til 5 til det foregående tallet.

B

Tallfølge a er en aritmetisk tallfølge og det vil si at differansen, d, mellom leddene er konstant. Rekursiv respektiv eksplisitt formel er som følger:

REKURSIV FORMEL: $A_{n+1} = A_n + \text{differanse} / 5$
 EKSPLOSITT FORMEL: $A_n = A_1 + (n-1) \cdot d /$
 $A_n = 2 + (n-1) \cdot 5$

C

De rekursive formlene er logiske og er allerede forklart med ord og viser altså hva vi må gjøre for å finne neste ledd i tallfølgen.

De eksplisitte formlene fungerer annerledes for de skal hjelpe oss til å finne et hvilket som helst ledd i tallfølgen.

Tallfølge a viser at vi må ha med det aller første tallet i tallfølgen (A_1), dette adderes med $(n-1) \cdot d$ (multiplisering først...) og $n-1$ er viktig, for vi skal finne f.eks det 10. leddet blir $n=10$. Vi må her trekke fra en hvis ikke regner vi ut det 11. leddet.

Eks fra tallfølge a der de 6. leddet er 27:

$$A_6 = 2 + 5 + 5 + 5 + 5 + 5 = 2 + 5^5 = 27$$

explores how the students position themselves in relation to 1) own emotions and experiences, 2) the mathematics and the teaching and learning of mathematics and 3) the classroom, teachers and others.

Analysis of positioning

To illustrate the analytical tool, I give a short extract of a text produced by a student teacher. The text is translated into English by me.

The student teacher, Ina, is solving a task on finding and describing the pattern of a given number sequence. This text is produced in her second semester in her teacher training.

The number sequence is given: 2, 7, 12, 17,....

The student teacher is asked to:

A: Find the next two numbers in the sequences.

B: Find the recursive and the explicit formulae for the sequences.

C: Explain why the formulae are correct.

The written text in **A** is:

a) One finds the next number by adding 5 to the previous number.

In **B**: The number sequence a is an arithmetic sequence and that means that the difference, d, between the terms is constant. Recursive respectively explicit formulae are as follows:

In **C**: The recursive formulae are logical and are already explained in words and shows what we must do to find the next term in the number sequence.

The explicit formulae functions differently because they shall help us to find any term in the number sequence.

The number sequence a shows that we must include the first number in the number sequence (A_1), this is added to $(n-1) \cdot d$ (multiplication first..) and $n-1$ is important, because if we shall find f . ex. the 10. term then $n=10$. Here we must subtract one if not we are calculating the 11. term.

Ex from the number sequence a where the 6. term is 27:

The expressive aspects of utterances are related to form and what this form symptomatically can express. One can read how Ina uses the arrow connecting the next two numbers in a) either as a (rough) draft she does to help her own thinking, and/or it can be read as a communicative utterance where she explains how the next number in the sequence is constructed. In both cases Ina uses an informal, illustrative, nearly oral, genre. The written text in a) is referring to an impersonal “one”, which is quite familiar in mathematical texts in textbooks. We can read it as a “rule giving” genre; written in an impersonal voice, in present tense and in general terms (it is about “the next number”).

In B Ina lists the two formulae. In her writing of the recursive formula she writes $/5$ to indicate that the difference is 5 in this case. The $/$ is kept in the explicit formula, but “difference” is replaced with the variable d . This form may be a symptom of insecurity in the mathematical terminology. It could be read as if the difference in meaning, expressed with written symbols, is not quite clear to her yet. In both cases, writing formulae, she is writing in what can be identified as from a technical genre, as in her mathematics textbooks. Ina seems to have grasped the ideas, but I read this as she has not yet acquired the genre as a cultural tool, and have difficulties in expressing these ideas in writing. This mix of genres could be seen as voices from her earlier school experiences and also from the lectures at the teacher college.

The referential aspects of the utterance are related to the mathematics in her text. She has got the answers correct. The notions of pattern and generalisation, in particular generalisation expressed in formulae, plays an important role both in the immediate context of situation through the instructions given in the statement of the task to “Find the [...] formulae” and to “Explain why the formulae are correct” as well as through the assessment criteria and more generally through the genre of investigation in which ‘spotting’ and generalising patterns is highly valued.

Her explanation of the recursive formula refers to what she has written in a), and she uses ‘logical’ as a self-explaining argument. Both formulae are given an authority as mathematical objects that can perform activity. The recursive formula “*shows what we must do*”, and the explicit formula “*help us to find any term*”. However when Ina presents the process she is also including actors in addition to the mathematical objects, as inclusive “*we*” and “*us*” respectively. This is also expressed in: “*One finds the next number*”, “*The number sequence a shows that we must include*”, “*because if we shall find*”, “*Here we must subtract one if not we are calculating the 11. term*”. These actors can also be read as a general “one” or “we”, rather than specific persons. Thus, the process of varying values in the problem is not shown as

something done by the author herself. It shifts from being a process that may be carried out by any mathematician, to a process performed by mathematical objects themselves or by some unspecified agent, and finally, using the grammatical metaphor of nominalization, to an object which may itself have properties and variations. This expression of agency in the utterance serves as construction of a picture of her mathematical world.

The addressive, or relational, aspects of the utterance are related to normativity, here in the sense of usefulness related to role of mathematics teacher in the primary school. Usefulness here includes ethical values concerning teaching and learning. Her explaining text in a) can be identified as “rule giving” genre within mathematics, and as such as part of the repertoire of the becoming teacher. In C she has included in brackets “(*multiplication first..*)”. This can be read as addressing the reader as a reminder of the rules for the priority of the numerical operations.

The normative claim can be understood as part of an instrumental view on teaching and learning mathematics. This can be seen as an element of Ina’s experience and praxis as part of her stories of mathematics as a subject where she has to learn the rules, and where you have true or false answers. That is an ideology within the genres of teaching mathematics.

In the utterance Ina uses a mix of genres. However, one genre seems dominant, the “Explaining” or “Introduction” genre. This is demonstrated by the explicit formula in C as she is both explaining the general by an example and by the nearly tactile metaphor she uses in explaining the explicit formula. This is a genre which is frequently used in the mathematics texts in her study. Explaining by examples is used frequently both in educational texts and also in teaching sessions, both at the college and in the practice schools. One could see this as a sign on her appropriating the voices of mathematics educational genres. This appropriation, making meaning of mathematical communication, is seen as the negotiation of identity as becoming teacher of mathematics. This shifts seeing development from a psychological to a semiotic perspective so as to locate developmental principles in the making of meanings.

THEORIES FOR RESEARCHING TEACHERS IDENTITIES

In this paper I have presented Positioning Theory as Rom Harré and associates have developed it. Their concept of positioning has been interpreted as persons’ identifications in a social psychological sense. From seeing teaching and learning as communication I have inserted a semiotic related concept of positioning based on Bakhtinian dialogism. This triadic discursive concept of positioning is then used as an analytic tool in analyzing identities according to the definition of identity proposed by Sfard and Prusak (2005). Here the utterance, as student’s text, in the genre of mathematics teacher education is used as the unit of analysis.

I see development of identities as learning, and theoretically investigating negotiation of identities from a semiotic perspective, not a psychological one. Therefore I explain

identifications exposed in student teachers' utterances as meanings within the genres, and the underlying ideologies, of teacher education. In the Norwegian mathematics classroom there are different ideologies simultaneously represented by different actors (Braathe and Ongstad, 2001). Essentially these are ideological conflicts within which the student teachers are struggling to create and negotiate their teacher identities. Going back to Dewey and seeing education as communication of doing, feeling and thinking from the older to the younger, has given me support for searching within theories of communication for a triadic understanding of learning to become mathematics teacher. Becoming a mathematics teacher includes building professional identities. This again includes knowledge of and identification with both mathematics and teaching and learning of mathematics.

The concern then is to focus on identities and the settings in which those can change, as a way of conceptualising mathematics teacher development as learning processes including the personal, the social and the cultural. Seeing development from a semiotic perspective, and learning as semiosis, all these aspects will have to be taken into consideration simultaneously.

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TEACHERS' COLLEGIAL REFLECTIONS OF THEIR OWN MATHEMATICS TEACHING PROCESSES

Part 1: An analytical tool for interpreting teachers' reflections

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Abstract. *The research presented in this paper offers a theoretical approach to the analysis of teachers' professional development by collegial reflection. The analysis of the reflections is applied to teaching episodes observed by videos and transcripts. The communication processes of constructing interactive mathematical knowledge with regard to develop together a more and more professional reflection of the student/ teacher mathematical interactions are seen here from a complementary perspective: (1) The construction process of an analytical tool for describing the reflection process of teachers; (2) The reflection process of mathematics teachers on the videos and transcripts of a diagnostic episode showing their own interviewing. This paper as the first of two papers concentrates on the first perspective.*

1. INTRODUCTION: THE RESEARCH PROJECT AND ESSENTIAL RESEARCH PERSPECTIVES

The presented research frame deals with discussion and results of the epistemological analyses of mathematical interactions in different social contexts (cf. Nührenbörger and Steinbring, 2009). In this article, we will concentrate on the development of teachers' professional learning by reflecting together their own teaching episodes. We will discuss an analytic tool for describing the reflection process with regard to a professional development of a more and more sensible interpretation and analysis of the students' mathematical interactions in the course of the teaching episodes observed. This research focus is one important element besides other research questions of two broader projects dealing with questions of the mathematical teaching and diagnosis of students' mathematical abilities in grades 1 and 2.

- a. *›Mathematics teaching in multi-age learning groups – interaction and intervention‹ (Malin). The question of this larger research report is: In which way do the teachers professional perspectives on their own role of teaching develop during the interactive lesson process with regard to the collegial reflections? For two years, eleven teachers from four elementary schools participate in the research project with their multi-age classes (grades 1 & 2). All teachers have been introduced to mathematics instruction in multi-age groups (cf. Nührenbörger and Pust, 2006). Each school year the partner work of two children (of different age) is video graphed in five lessons. The children work in pairs on open or structure-analogue tasks, which are supposed to permit an interaction and reflection from different points of view for both of them. After each term (four times over two years), the teachers of each school meet for a collegial reflection, in which video graphed episodes are watched out of their own instruction and analysed with the*

help of corresponding transcripts. The objects of their critical analyses are video episodes from their mathematical classroom that contain two types of mathematical communication in two different social contexts: “A short episode of two students interaction without the teacher's presence” and “A following short episode of the two students interaction with the teacher's participation”.

These interaction settings are taken as a productive opportunity for making sense of the students' processes of mathematical understanding within these two sub-settings and of constructing mathematical knowledge in view of their own interventions (cf. Nührenbörger and Steinbring, 2009).

- b. “*Mathematics talks with children – individual diagnosis and supporting*” (MathKiD). The question of this research report is: In which way do the teachers' professional perspectives on their own role of talking with one child develop during a diagnostic interview by means of structured talks of reflections? For one year, five teachers from two elementary schools participate in the research project with their children (grade 1 or 2). All teachers have been introduced to diagnostic situations in mathematics instruction. In one year, the interaction between the teacher and one child of his class is video graphed about six times. The teacher and the child talk about “pure” math situations or playing situations with implemented math situations. They are supposed to permit diagnostic findings about the mathematics abilities of the child. In one year, the teachers of each school meet three times for a structured talk in which video graphed episodes out of their own diagnostic talks will be watched and analysed with the help of belonging transcripts and the intervention of a moderator (project leader). The objects of their critical analysis are video episodes from their diagnostic talks that contain interesting situations under three different analytic perspectives: “Analysing the understanding of the child”, “Analysing the intentions and actions of the teacher” and “Analysing the interactions between the teacher and the child.”

The cooperative reflection of mathematics teachers constitutes a practice-orientated discourse for constructing professional teacher knowledge. This research approach aiming at the analysis and reflection of the teachers' *own* teaching activities in the course of their professional development differs from those approaches that offer exclusively theoretically elaborated patterns of teachers' activities for reflection and imitation. The main focus of this paper is on the problem of developing an adequate tool for describing the process of collegial reflection with regard to the construction of a more professional knowledge for the learning and teaching process of mathematics. This leads directly to the research question of this contribution:

In which way teachers become aware of and understand carefully the students' interactive mathematical interpretation processes in relation to their own intervention possibilities for stimulating students' mathematical understanding processes?

In the last decades, research studies on mathematics teachers' professional development have more and more emphasized the importance of video graphed

episodes of mathematics teaching and interactions for sensitizing the teachers for their own teaching and talking activity in and about math (i.e. Maher, 2008; Benke et al., 2008). In this frame it is important to recognize that teaching itself is not a mere routine task of transferring more or less finished mathematical knowledge, which the teacher has prepared, to the students. Steinbring (2008, 372) points out that “school mathematics, as finished given knowledge, is not the *actual subject of teaching* in an unchanged way. Mathematical knowledge emerges and develops only in an effectively new and independent way within the instructional interaction with the students. Thus, finished, elaborated mathematics is not an independent input of the teacher into the teaching process which could then become an acquired output by means of students’ elaboration processes.”

During the process of teaching, the teachers are involved directly in the interaction with the student(s) and cannot play the role of a distanced observer of the events. The teacher has to draw directly a conclusion of the situation. “Normally, whenever we hear anything said we spring spontaneously to an immediate conclusion, namely, that the speaker is referring to what we should be referring to were we speaking the words ourselves. In some cases this interpretation may be correct; this will prove to be what he has referred to. But in most discussions which attempt greater subtleties than could be handled in a gesture language this will not be so” (Ogden & Richards, 1972, p. 15). But the development and change of the activity of teaching requires a critical consideration and thus a distance of one’s own activity (cf. Krainer, 2003). Collegial reflections offer the teachers an “unusual” view of interaction processes. Possibly they will be irritated, they observe greater subtleties and thereby view the situation in another way (cf. Gellert 2003).

Otherwise one cannot see a typical dilemma of mathematical teaching routines: Mathematical teachers know, on the one side, of the importance of interactive learning processes during a learning environment, supporting the active-exploring work of students. But on the other side, the talk of the teachers during the teaching is affected by an attitude that mathematical knowledge is a *complete and clear product*, which can be developed directly by the students (cf. Steinbring, 2005). Hence, it might be the danger that teachers act on the assumption to support the students’ learning processes with open learning environments. But due to the direct involvement in the mathematical teaching process, teachers tend to their personal views on knowledge. Their spontaneous work bases on own experiences and routines: Their talk to students is characterized by leading, funnelling and product-orientating, so the students have no choice to develop active own mathematical interpretations (cf. Bauersfeld, 1995). The teachers involved in the teaching process cannot see this dilemma. It is only noticeable in the distance and in a critical-reflected talk with colleagues observing by a video of their teaching. The distanced observation of a communication process in the classroom can highlight causal relations between the learning and teaching process. “During the common systematic reflection in a group of teachers about their own teaching processes with students thus emerges a

further communication system, which again has to deal with the necessary interrelation between one's own consciousness and common communication. This communication now has communication processes as its subject and it is supposed to animate a professional consciousness" (Steinbring 2008, 379). However, the reflection of one's own activities that temporally separates from the teaching situation looks to future teaching activities. These future teaching processes can relate to the results of the distanced reflection (cf. Krainer 2003; Sherin and Han, 2004).

As a basis of professional teacher development we see an active, self-responsible and reflective elaboration of one's own practice with colleagues (cf. Altrichter, 2003, Krammer et al., 2006). „Systematic reflection on mathematical interactions that focus on the students' learning and understanding processes, as well as on one's own interaction behavior, represents an essential professional competence of teachers" (Scherer & Steinbring, 2006, p. 166, cf. Mason, 2002).

The growth of new insights refers to the active process of reflecting one's own teaching and learning. „If mathematics education is to be influenced in a positive way and ameliorated, the teachers have to be the ones who initiate these changes, and their reflection on their own activity is crucial" (Scherer and Steinbring, 2006, 165). Professional development needs to talk with the professional group about the own practice. In this sense, we mean with "collegial reflection" the common discussion and negotiation of teachers watching a video of a teaching episode and reading the transcript.

In this article, we will discuss the question, how the collegial reflections support teachers with the help of videos and transcripts to be sensitive to the power of the mathematical negotiating process of students: In which way teachers develop in the course of collegial reflections differentiated mathematical interpretations and interrelations? In which way teachers look to the possibilities to attend the students with open, mathematical focused and interactive orientated interventions?

2. THE DESIGN OF THE COLLEGIAL REFLECTIONS

In the context of the two research projects, the teachers take part on distanced collegial reflections of their own or of known (this means known lessons held by colleagues) teaching lessons. In this sense, the projects do not focus on the imitation successful teaching and learning strategies. Both projects aim at the commonly constructed reflection of interaction processes with the focus on the understanding of the students' mathematical thinking, on the role of interaction for constructing mathematical knowledge, and on the patterns of the interactive teaching and learning process. The collegial reflection focuses on classroom cases (Malin-Project) or diagnostic talks (MathKiD-Project).

Teachers can be encouraged to reflect their own talking activities and to make conscious decisions by learning how to "read" and interpret an episode of talks in a

complex classroom situation or in a diagnostic situation. In addition, the collegial reflection follows some guidelines for initiating joint analyses:

Continuity: The teachers meet more than one time a year. The long-term meetings are necessary to grow into and to stabilise the reflection process of exemplary cases. Furthermore, each teacher of the group of 3 to 5 teachers should be one or two times a year in the focus of the reflection.

Collegiality: The teachers work together and reflect their view of the real teaching episodes in a new way.

Familiarity: It is necessary to integrate the collegial reflection process in a trustful atmosphere to experience a positive learning community. A concentrate altercation of the teachers with the episode relates to the familiarity of the video episodes.

Concentration on teaching and learning: The analyses focus is on the teaching and talking activity, not on the teachers (cf. Stigler and Hiebert, 1999) - the teachers do not want to evaluate the teacher, they want to understand the teaching process and the practice of instructing - they give only alternative teaching offers (cf. Seago, 2004).

Concentration onto the teachers: The teachers will and should not analyse the transcripts like researchers. They have their own interests in working with the transcripts, just like the socio-cooperative possibilities of learning or the everyday constitutions of their practice.

The teachers can take different roles in the course of the analyses. The results discussed in this article bases on the research project “Malin”. The researcher takes the role of a cautious moderator to initiate the collegial reflections.

Cautious moderator

After an empirical analysis the researcher chooses one video episode of the classroom teaching lessons of one participant. The video episode contains a potential for discussing the interactive knowledge construction of the children in relation to the intervention of a teacher. At the beginning the teachers get an orientation of the teaching episode by the teacher involved. The researcher offers the video episode and the corresponding transcript. Furthermore, the teachers discuss different perspectives for the interpretation process – such as special features of the mathematical understanding of a student, of the interactive construction of mathematical knowledge, or of the teachers` attitudes and verbal interventions and their consequences of the students` behaviour and knowledge construction (cf. Scherer et al, 2004). The video episode is structured in three sequences and each sequence is an “object” of the teachers’ cooperative and joint reflection:

- a. Mathematical interpretation processes of two cooperating students
- b. Mathematical interpretation processes of the intervening teacher
- c. Mathematical interpretation processes of the two cooperating students after the leaving of the teacher

Firstly, the teachers see and discuss only the first sequence with the help of the transcript without knowing the teacher intervention. The researcher as a moderator

has mainly the task to choose and structure a comprehensive teaching episode and to moderate cautiously the collegial reflection. At the end, he animates the teachers to a short review – in form of a “flashlight” – on the collegial reflection and on their learning process. The cautious moderation guarantees a negotiation of deep structures that seems to be important for the professional development process of the teachers` group. Furthermore, the teachers have the opportunity to adopt the collegial reflection as a school-internal way of professional learning. In this sense, we hope that this may guide the teachers to understand their school as a place where also teachers can learn.

3. THEORETICAL COMPONENTS OF ANALYSING TEACHERS` COLLEGIAL REFLECTION

In this report we concentrate exclusively on exemplary cases in order to elaborate the particularities of collegial reflections that were analysed in the Malin-Project. The qualitative data is carefully evaluated in an interpretative way and analysed with regard to the classification of specific interpretation dimensions (for the research approach of qualitative and interpretative analyses of mathematical interaction processes see e.g. ZDM (2000)).

The collegial discourse creates a new context, in which the teachers talk in a different way of teaching mathematics as during the lessons. The teachers` interpretations during the different collegial reflections of their own teaching episodes can be compared with the reconstruction of a “case”. Their discussions are effected by the search for evidences to clarify the case. The results of the analyses lead to the assumption that the teachers construct an understanding of the interpretation to an agreed case – likewise teacher and students negotiate common mathematical interpretation during the lessons. For a collegial reflection, we will differ three main analysing aspects, which relate to the professional development of the teachers:

- The constructing of a case (What teachers are talking about the empirical event?)
- The reading (How teachers are speaking about the case?)
- The generation of case knowledge (Which knowledge teachers are expressing to make sense to their case?)

The constructing of a case: The teachers watch a video episode of a teaching sequence and read the corresponding transcript. Their discussions differ from spontaneously reflections in or after a teaching episode. The teacher involved in the case gives a lecture of his thinking of the named case. In the collegial reflection, the teachers frame firstly the empirical event in different ways. Here, we can mainly distinguish between three frames, which seem to be important for a professional development of mathematical teaching:

- An interactional frame containing utterances to the social learning of students, to their cooperative activities, to the dialogues between students or between students and teacher depending on their social roles (cf. Nührenbörger and Steinbring, 2009, e.g.: “*The starting situation, that [the student] Klaus decides and Sönke is*

in the role of working and writing, is changed, when a teacher comes to the students. Klaus is very orientated to the teacher telling him what they have already done”)

- An epistemological frame containing utterances to interactive construction of mathematical interpretations of the students and to the mathematical understanding of the teachers themselves in the distanced situation of the collegial reflection (e.g.: “*Ah, these four plus four idea.*” “*I think also this crux of the matter. Well, I mean, with six plus two and two plus six it is obvious, that they are exchange exercises which have the same result, but which are the other way round. And with four plus four. (...) It is in fact also an exchange exercise...*” “*But Ben, with your theory, well I am considering right now. If one puts them into a line and then you would have one plus seven, but also two exercises.*”)
- An organisational frame containing utterances to the conditions of teaching (i.e. presentation of a task, time management etc.) and to the development of their own teaching (i.e. the effects of diagnostic questions etc.)

The relation between the empirical event and the frame of the teacher describes the case which the teachers construct in their collegial reflection and which is the focus of their understanding. The teachers pick different cases as a central theme during the active reflection of the different sequences. Five main cases can be differed: learning of mathematics with focus on results and algorithmic or on arithmetical and geometrical processes, social learning of the students, teaching of the teachers, mathematical context, diagnose of competences.

However, the teachers construct a case in the collegial reflection, they do not discuss a staged case. The constructed case must be proved (on) by the empirical event.

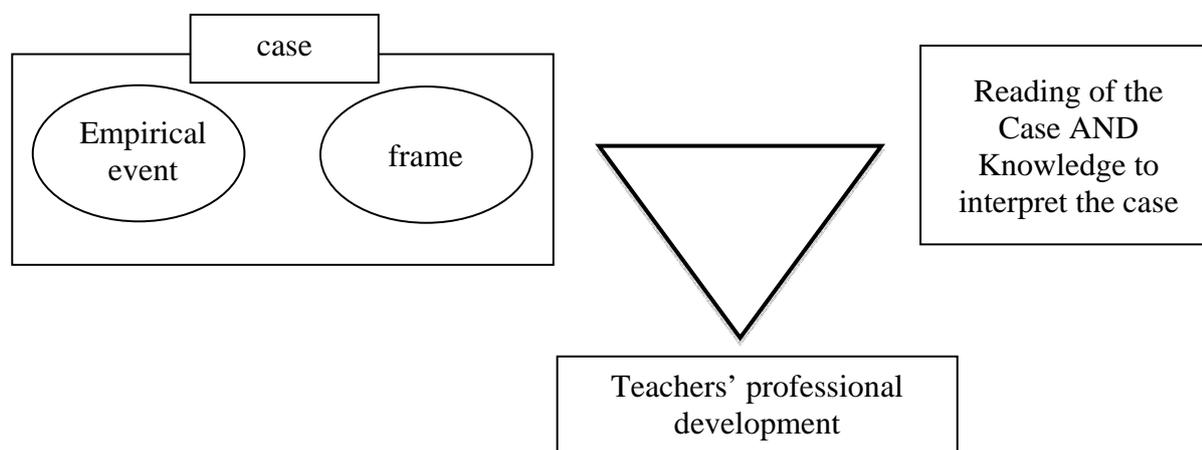
The reading of the case: The teachers can articulate the constructions of the cases in different ways. If teachers – after reading the transcript or watching the video - think to know and understand the interaction process, they *narrate* and *evaluate* the text in a biased-spontaneous way. A more open-reflected approach contains different *paraphrase* and *interpretations*. What will we mean with these notations indicating the access of the teachers to the case?

Description: The teachers concentrate on aspects of the episode and give a detailed or a short description. If the teachers illustrate the attitude or the talks as a clear and understandable learning episode, they tend to *narrate* the scene in a short way. But if the teachers illustrate different phenomena of the teaching and learning process in a neutral and accurate way, they tend to *paraphrase* the scene.

Evaluation: The teachers link their descriptions with personal views on the situation to evaluate the attitudes and talks in the teaching and learning process.

Interpretation: The attempt to clarify the teaching and learning episode must not go along with an evaluation. When the teachers describe the scene in a detailed way and try to analyse the different acts and utterances, they begin to interpret the scene. The interpretation leads to different explanations without regard to own experiences.

The readings of the case interrelate to a different case knowledge of the teachers. The analysis of the collegial reflection in the Malin-Project shows three different types of practice case knowledge (knowledge by observation, by experience, by transfer, by interrelation) that the teachers activate to clarify the case. However, in this sense the case relates to the common professional knowledge. The following diagram shows the coherences between the case and the construction of professional knowledge.



The generation of knowledge: During the reflection process the teachers bring in their knowledge to construct and understand a case. On the one hand, they use their common experiences and observations to clarify an utterance or an act of the students or of the teachers. This case knowledge relates to old knowledge (e.g.: “*I think it is typical. The older guy tells the younger one what to do. Klaus says to Sönke, how it will go.*”). In this sense, the interpretation of the case is used to confirm one owns pedagogical and mathematical beliefs. A teacher will use his case knowledge by *observation* to describe and reconstruct the empirical event. When teachers use experiences of their own teaching practice that relates to the empirical event observed by the video, they activate case knowledge by *experience*. This means that they construct retrospectively an adequate perspective to give a plausible explanation for the colleagues.

On the other hand, teachers can pick the case as a central theme for constructing new relations dynamically. If the case provides a basis for a productive irritation, it can inspire the previous knowledge of mathematical topics (e.g. see the discussion of the teachers above, if there exist an exchange task to $4 + 4$: The way of the students` interpretation of a mathematical task can lead to a new discussion about mathematical patterns), mathematical interpretations of children and mathematical interactions (e.g.: “*The schizophrenic thing is, I as a teacher have given them a partner work, but I do not lead the student-teacher-conversation as a partner-work-conversation*”). If a teacher reproduces the ideas of the other teachers in relation to his old knowledge, he constructs new case knowledge by *transfer and interrelation*.

4. CLOSING REMARKS: THE PROFESSIONAL DEVELOPMENT OF TEACHERS' IN RELATION TO THE COLLEGIAL REFLECTIONS

The teachers construct and negotiate different cases in different ways if they have the opportunity to reflect together their own teaching process. The analyses of the reflections in the Malin-Project (cf. Nührenbörger and Steinbring, 2009) showed that teachers activate different types of case knowledge to interpret the empirical events. We described a professional development of the teachers as a growth of the reading of a case in an open and reflected way (paraphrase and interpret). Likewise, one can see a growth of professional practice by the construction of relations between the case and the knowledge by transfer and interrelation based on a productive irritation by the teachers. Besides the organisational frame, the conditions and the trustful willingness of the teachers to open up for the exchange with their colleagues, it seems to be essential that the collegial reflections were founded on scenes from one's own teaching. But which role has the moderator?

The analysis of the collegial reflections showed that many times, the teachers discussed a scene without a mathematical orientated frame. They used the empirical event to talk about common pedagogical and organisational topics. What will happen if the moderator leaves the cautious role and takes a more active role? We have the hypothesis that the role of the moderator can focus on the discussions of the teachers on one case and can provoke a more open and reflected reading of a case with the use of knowledge by transfer and interrelation. An *active moderator* looked for special features which he wants to discuss with the teachers and which they shall notice. We will discuss a collegial reflection structured by an *active moderator* in the second part of this paper with regard to the MathKiD-Project.

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TEACHERS' REFLECTIONS OF THEIR OWN MATHEMATICS TEACHING PROCESSES

Part 2: Examples of an active moderated collegial reflection

Kerstin Bräuning & Marcus Nührenbörger

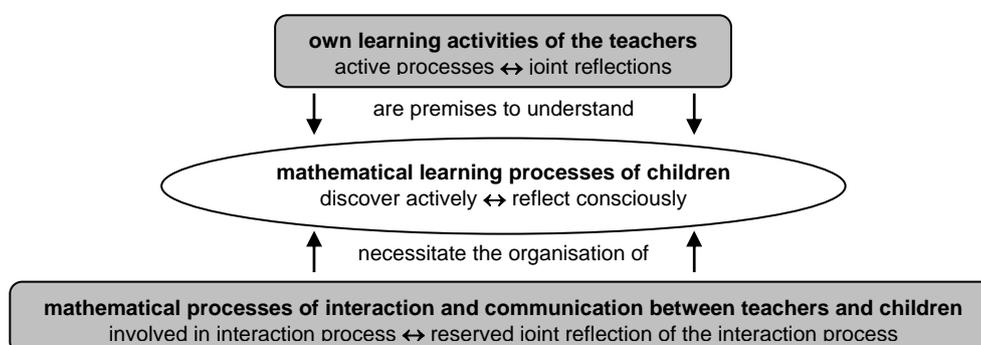
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Abstract. *The research presented in this paper offers a methodological approach to the analysis of teachers' professional development by collegial reflection. Collegial reflections are professional development meetings in which teachers watch and discuss excerpts from talking with their pupils. We'll present an example of collegial reflection based on a diagnostic talk between a teacher and a 2nd grade child. The instruments presented in the first part of this paper will be used for the analysis of the collegial reflection. Investigating the case knowledge participants' construct in professional development can further our understanding of how teachers interact to influence one another's learning. We'll see how participants make inferences about the events they noticed and how they use videos as evidence for their interpretations.*

1. INTRODUCTION: THE RESEARCH PROJECT AND COLLEGIAL REFLECTIONS

The presented research deals with the development of teachers' professional learning by analyzing video episodes. In this article we will concentrate on one example of a collegial reflection process and we will use the analytic tool presented in the first part of this paper for describing the reflection process.

Teacher professional development seems to be short-term, individualized and disconnected from practice (Ball & Cohen, 1999; McLaughlin & Mitra, 2002). An important aspect of teacher learning groups is that they engage in long-term collaboration with their colleagues, focusing on issues that relate to their daily teaching activities (Little, 2002). To promote and support teachers in attending to and interpreting students' mathematical thinking there should be interplay between activity and reflection (figure in: Steinbring, 2003, p. 217/218).



Lesson study provides such a possibility for teachers where they examine systematically their instructional methods, teaching content and also their students' processes of learning and understanding (Yoshida, 2008, p. 85). A small group of teachers plan together a research lesson, implement it and the other teachers observe this lesson. Afterwards they discuss about this research lesson. With the collegial reflection we try to offer the teachers of our projects a possibility to deepen and broaden their understanding of the teaching episode by an unusual view of the situation.

Our interest is to find out what kind of readings the participants use in the collegial reflections and what kind of case knowledge they develop when talking about the video episodes. In the first part of this paper we explained the different kind of readings: *biased – spontaneous* (narrate, evaluate) than *open – reflected* (paraphrase, interpret). The teachers construct knowledge by observation, experience, transfer and interrelation. If the teacher just refers to his own thinking, he will develop knowledge by observation or experience. If he takes account of the other participants' utterances, he will construct knowledge by transfer and interrelation. We also want to find out what impact the moderator has on the readings and the case knowledge the teachers develop in the structured talk. A structured talk is a collegial reflection with a moderator attending the meeting.

Sherin and van Es use a related approach for analysing their video clubs (Sherin & van Es, 2005) which are similar to our collegial reflections. They examine the teachers' role in the video club setting. In contrast to our research they do not identify the case knowledge the teacher construct when talking about the video episode. They analyse speaking turns along the dimension *specificity* (general or specific) and *focus on video* this means that they explore if the comments grounded in the events that occurred in the video or based on events outside of the video episodes.

This article is based on two research projects (“Malin” and “MathKiD”), which both deal with collegial reflections, but which differ in the way of support and moderation (see also first part of this paper).

- *Cautious* moderator („Malin-Project“) (Nührenbörger & Steinbring, 2008): The researcher chooses one video episode and provides the teachers with the video episode and the belonging transcript. Furthermore he introduces the methods of collegial reflection and presents a paper with analytic perspectives, which the teachers can use during the reflection process. The researcher moderates the reflection process in a *cautious* way. The teachers can discover and discuss independently the basic structures of their teaching. In the long-term they can adopt the collegial reflection as a school-internally way of professional learning. We hope that this may guide the teachers to understand their school as a place where also teachers can learn.

- *Active* moderator and *no* moderator („MathKiD“): The researcher chooses one video episode of a diagnostic talk, which one participant conducted. In every meeting the

chosen episode will be discussed from a different analytic perspective. The teachers are provided with the video episode and the transcript to the chosen episode. In the structured talk, where the project leader is an *active* moderator, the teachers first get a short introduction about the following meeting. They receive a paper with several stimuli to the specific analytic perspective, which they can use in the interpretation process for their orientation (Scherer, Söbbeke, & Steinbring, 2004). The project leader is an *active* moderator in the structured talk because she analysed the whole transcript sensitively before the meeting and looked for special features to be discussed with the teachers and which they shall notice. The structured talk is like a supervision where the external moderator is the supervisor (Lippmann, 2005, p. 10 ff.). In the informal talks the teachers meet each other without the project leader. You can compare the informal talk with intervision. If people meet each other without a moderator it is called intervision (Lippmann, 2005, p. 12). The structured talks and the informal talks are both audio taped. The informal and structured talks take place in an alternating fashion. In every meeting new transcript will be discussed.

In the following we will look at one structured talk of the project MathKiD. The influence of the informal talk prior to the structured talk will not be discussed in this article.

2. THE COMPOSITION OF THE STRUCTURED TALK

The composition of the structured talk is the following:

1. The teachers' feedback on the informal talk.
2. Analysis of the video episode with the belonging transcript from a specific analytic perspective:
 - a. Understanding of the child (first structured talk)
 - b. Intentions and actions of the teacher (second structured talk)
 - c. Interaction between the teacher and the child (third structured talk)
3. Flashlight to the new insights, which resulted from the analysis of the video episode.

Different stages of the structured talk are:

1. The teachers' feedback on the informal talk.

The moderator listens to the teachers and they report on the contents they discussed in the informal talk.

2. Analysing the video episode with the belonging transcript from a specific analytic perspective (understanding of the child, intentions and actions of the teacher, interaction between the teacher and the child).

First, the moderator asks the teacher who talked to the child in the video, what she expected from the child of her class before the diagnostic talk and what kind of feelings she had at the beginning of the diagnostic talk. Then all the participants watch the video episode and after that the teacher from the video has the possibility to express her first impressions of it. Then the other teachers can also express their

impressions. In the analysing process the moderator structures the discussion, 1) she encourages the others to express what they think about a statement of one teacher, 2) she tries to find out what every participant wants to express, 3) she points to different possibilities to interpret a situation and look deeper on special issues in the transcript, 4) she refers to the given stimuli on the paper the teachers got, 5) she focuses the conversation on mathematical interactions, 6) she reminds the teachers to talk about the transcript and 7) she remarks the teachers to provide an evidence from the transcript for their interpretation. The moderator is not assessing the interpretations of the teachers, is not changing her role into the didactical expert and is not insisting on her stimuli, which she offered to the teachers.

3. Flashlight to the new insights, which resulted from the analysis of the video episode.

At the end of the structured talk the moderator asks every participant to express their own new insights after analysing the video episode and what kind of new information they got about the mathematical abilities of the child and the possibilities to support the child.

3. THE FIRST STRUCTURED TALK ABOUT AJDIN AND MRS. WHITE

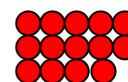
The MathKiD project started in August 2007 and five teachers from two different primary schools are participating. One group consists of three teachers, the other of two teachers. Each of the three teachers conducted one to three diagnostic talks with grade 1 or 2 pupils before the first structured talk in November 2007. The first informal talk was in October 2007 and is not audio taped.

The structured talk is the first meeting of the three teachers with the project leader to analyse a video episode and the belonging transcript under the analytic perspective “understanding of the child in the observed situation”.

Content of the video episode Ajdin and Mrs. White

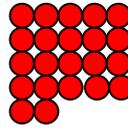
The content of the chosen video episode is the talk between Ajdin (grade 2) and Mrs. White about a pattern of coins at the beginning of the second grade. On one side the coins are red and on the other side they are blue. They are playing the game “Collecting coins” (Hengartner, Hirt, Wälti, & Lupsingen, 2006, pp. 27-30). In this game you throw your dice and move forward the shown number on the playing field. On special fields, where you see a structured or unstructured amount of coins, you can win coins. The goal of the game is to structure the won coins in a way that you always find out very easily and quickly how many coins you already won and to be able to compare your coins with the amount of coins your partner won.

Ajdin and Mrs. White play the game “Collecting coins” the second time. At the beginning Mrs. White told Ajdin that he should display his coins so that they would not have to count a lot to find out who has already won more coins. They have already talked about 13 minutes. Mrs. White won 14 coins and she structured them in $5+5+4$.



Ajdin is winning his first 6 coins and he structures them like that: 

Mrs. White wins 5 more coins. Ajdin tells her that she now has 19 coins and she structures it like $5+5+5+4$. She first asks him how he saw this and then how he calculated it. He tells her that $14+5=19$, because $4+5=9$. After that Mrs. White wins 3 coins and structures them like that $5+5+5+5+2$:



Ajdin wins four coins and structures the coins like that:  Mrs. White says that it is a “strange” pattern and asks what he thinks about it. He first tells her $3+4=7$ and $7+3=10$ and later he says $3+3=6$ and $6+4=10$ while pointing on the lines of his pattern.

Epistemological analysis of the video episode Ajdin and Mrs. White

For the interpretation it is important to notice that “Collecting coins” is on the one hand a game and on the other it is dealing with mathematical contents. The arrangement of the coins is different for Mrs. White and Ajdin. She refers to five and ten as the base of our counting system when arranging her coins. She is not changing her pattern after winning some more coins. She continues her pattern (Nührenbörger & Steinbring, 2008).

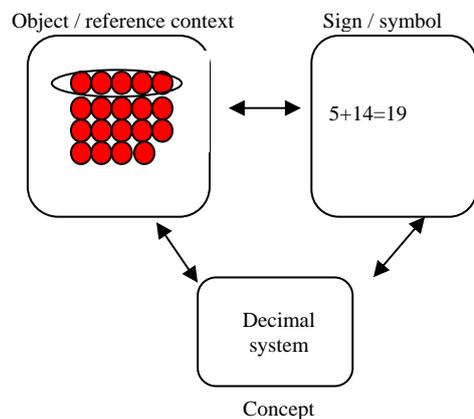


Fig. 1: Epistemological Triangle Mrs. W

Ajdin’s first pattern would be called triangle number. He is “continuing” his pattern to the second pattern. There is no (geometric) label for this pattern like square or triangle or something else. It is not clear in which way he would continue his second pattern. The second pattern seems so complex for Ajdin that he gives two different calculations as interpretations: first $3+4=7$ and $7+3=10$ and later $3+3=6$ and $6+4=10$. With the calculations Ajdin does not explain his actions when arranging the coins to the first pattern. The second calculation explains the pattern in a symmetric way, but Mrs. White is not dealing with it.

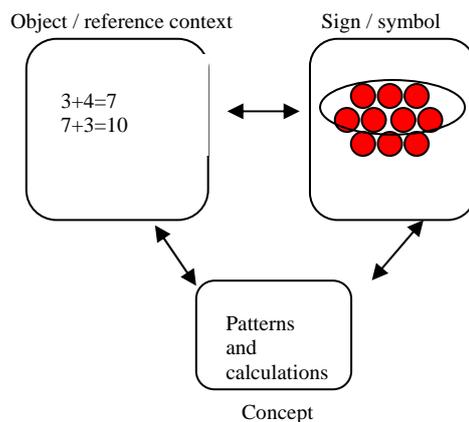


Fig. 2: Epistemological Triangle Ajdin task 1

Mrs. White uses the term “strange pattern” for his second pattern. Perhaps she uses it, because in her thinking her pattern is mathematically correct and not comparable with the pattern of Ajdin. For Mrs. White it is probably important to be able to “see” the amount of coins quickly and for Ajdin it is important to find an easy calculation for the pattern.

The moderator wants to discuss with the teachers about the different patterns of Ajdin and about the term “strange pattern”, which Mrs. White used.

Content of the structured talk about the video episode Ajdin and Mrs. White

The whole structured talk lasted 2 h and 15 min. Two different episodes were selected dealing with the first and the second pattern of Ajdin.

Content of the first episode of the structured talk

In the first episode the moderator tells the teachers that the first pattern of Ajdin is still a pattern even if it is not structured in rows of five or ten coins. This is meant as a stimulus for the others to discuss this statement. The participants are not discussing the first pattern. Through a statement of Mrs. White all the participants discuss the continuation from the first to the second pattern of Ajdin. The teachers discuss their own different interpretations of continuing the first pattern if they had won four additional coins.

Analysis of the first episode of the structured talk

The first episode deals with the continuation from the first to the second pattern of Ajdin. The teachers talk about patterns as a mathematical content and the working process of Ajdin. They do not differentiate between these two topics.

Each teacher talks about the cases in different readings, as specified below.

Mrs. White talks more than half of the time and dominates the discussion. She explains her understanding of patterns and what she believes how Ajdin is thinking. Probably Mrs. White has the feeling that she has to justify and to defend her actions in the diagnostic talk. On the one hand she is telling about her own thinking (“I would have” / “I put” / “for example I would” / “I would do”) and on the other hand it is presumable that she tries to get a sense of Ajdin’s statements (“I don’t know what he” / “I think” / “I believe” / “I find this unexpected” / “I can imagine”) (line 65 ff.). She describes her working process when she builds patterns, which is mainly based on her experiences. In this episode Mrs. White **narrates** and **evaluates** the continuation from the first to the second pattern of Ajdin (l. 69).

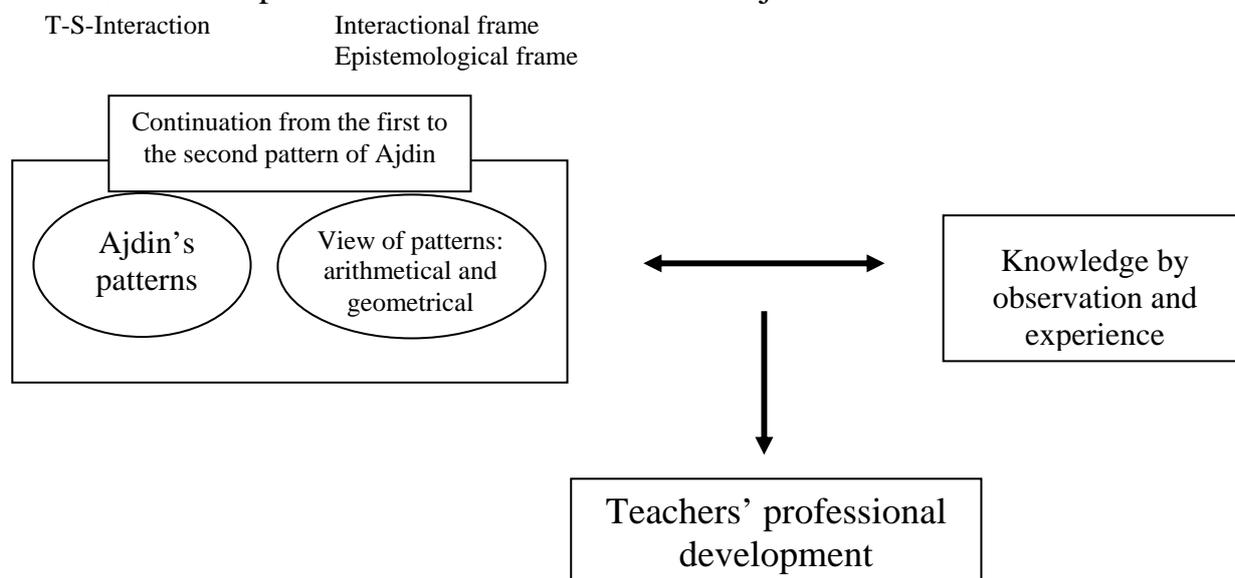
Mr. Peter talks about the structure of Ajdin’s first pattern, which Ajdin loses in the eyes of Mr. Peter when he creates the second pattern. Mr. Peter assumes that Ajdin followed the sequence of natural numbers in his first pattern (l. 71, 73, 75). Mr. Peter **evaluates** the situation in this episode.

Mrs. Dieter reacts to the stimulus of the moderator (l. 77, 79) by creating a pattern different from Ajdin’s second pattern. She neither refers to the transcript nor the

episode. She connects the pattern with geometrical shapes like a square (l. 83, 85, 87, 91, 96, 98). Her statement seems like an insertion. Mrs. White rejects Mrs. Dieter's statement and therefore Mrs. Dieter tries to justify her thinking (l. 101, 112). At the end she refers to the transcript when she talks about Ajdin seeing six coins at once (l. 114). Mrs. Dieter briefly **narrates** the situation at the end. The other time she does not refer to the episode.

In this episode Mrs. Otto shortly **paraphrases** that Ajdin counted the six coins when he won them (l. 115, 117). She refers to the transcript.

The moderator gives a stimulus to think about Ajdin's first pattern if it is a pattern (l. 64) and how each of the participants would put the four coins Ajdin won to his first pattern (l. 77). Then she tries to understand the statements of the teachers and demands further information. In line 104 she refers to the rule of the game that says that you have to structure your won coins, but not in a specific or given way. The moderator tries to initiate that the teachers develop different interpretations of continuing the first pattern to the second pattern of Ajdin.



Discussion of the first episode of the structured talk

If we look at the readings of the teachers we can see that they react more *biased – spontaneous* (narrate, evaluate) than *open – reflected* (paraphrase, interpret).

If we look at the generation of case knowledge we can see that the teachers use their knowledge by observation and experience they have developed. For example Mrs. White refers to her remedial teaching (l. 74) as knowledge by experience. The teachers are not interpreting the given material in detail, the video episode and the belonging transcript. They do not refer to the statements of the other participants and therefore they do not generate knowledge by transfer and interrelation.

Content of the second episode of the structured talk

In the second episode the participants discuss from where Ajdin got the first pattern. Was it his own idea or did he see this pattern on the playing field? One teacher says

that Mrs. White could have asked him why he structured the pattern like this. Mrs. White says that she could ask him but his answer would not help her to know from where he got his first pattern. Then they talk about the change from the first to the second pattern. The teachers tell their own different interpretations of the second pattern. They think about how to foster the mathematical abilities of Ajdin. They believe that you only have to support children with low-level competencies. They are convinced that they do not have to support him, but to foster over the usual level. In line 320 the moderator refers to the diagnostic-talk-transcript and says that Ajdin interprets his second pattern in a second way and one teacher states that Ajdin re-interprets his second pattern when he gives another calculation.

Analysis of the second episode of the structured talk

The second episode deals with the development of several cases. They talk about the origin of the first pattern of Ajdin and again about the continuation from the first to the second pattern of Ajdin. They discuss about patterns as a mathematical content and the working process of Ajdin. Furthermore they think if they have to support Ajdin even if he is not a low achiever.

First we will look at each teacher. Each of them talks about the cases in different readings again.

Mrs. White talks more than one third of the time and like in the first episode she tells what she thinks about the patterns and what she believes how Ajdin is thinking. Probably Mrs. White has the feeling that she has to justify and to defend her actions in the diagnostic talk. It seems like that because she dominates these two episodes. She uses “I” very often differently. We already described this in the analysis of the first episode. It seems that she thinks she knows what Ajdin wanted to do. She express that she can demand explanations of Ajdin, but they will not help her understanding what Ajdin thought (l. 254, 256). Most of the time in this episode Mrs. White **evaluates** the working process of Ajdin when he builds his patterns (l. 238, 240, 242). She decides that Ajdin needs no supporting, so she also evaluates the process (l. 313) and tries to finish the discussion in this episode.

Mr. Peter talks again about the first pattern of Ajdin. He seems to be convinced that he knows how Ajdin saw his pattern. For him the only view is following the sequence of natural numbers (l. 235, 290 ff.). He refers to the transcript when he evaluates the working process of Ajdin. At the end he describes that Ajdin finds two different calculations for the second pattern. Mr. Peter **evaluates** and **narrates** in this episode.

After the moderator repeats the statement of Mrs. Dieter (l. 279) she is the only one who reacts and she explains her statement (l. 280 ff.) how she looks on the second pattern of Ajdin. Her statement seems like an insertion because nobody refers to her. It seems that only Mrs. Dieter tries to answer to the stimulus of the moderator. Mrs. Dieter **narrates** in this episode.

In this episode Mrs. Otto reacts to the statement of Mrs. White and suggests her to ask Ajdin what he thinks about his patterns. She refers to the transcript when Mrs. White says “pattern”. She reflects about the term “pattern” and the interpretation of it (l. 257 ff.). Later she points out that one can also support children who show a good performance (l. 316, 318). Mrs. Otto **paraphrases** and **interprets** in this episode.

The moderator gives feedback to the statements of the teachers with “mhm”. In line 279 she points to the continuation from the first to the second pattern and takes up the statement from Mrs. Dieter (l. 273). Later she refers to the transcript and explains that Ajdin has two different interpretations of his second pattern (l. 320 ff.). Most of the time she listens to the conversation.

Discussion of the second episode of the structured talk

If we look at the readings of the teachers we can see that all the four teachers stick to their roles. They react more *biased – spontaneous* (narrate, evaluate) than *open – reflected* (paraphrase, interpret) apart from Mrs. Otto. In this second episode Mrs. White and Mr. Peter discuss a lot, but the others are also active, but not talking that much.

If we look at the generation of case knowledge we can see that the teachers use their knowledge by observation. The teachers refer more to the transcript than in the first episode, but they rarely use knowledge by transfer and interrelation.

Comparison between the first and the second episode of the structured talk

We can see that in both episodes the teachers use almost the same readings and generate almost the same case knowledge. Only the moderator reacts more restrained in the second episode. It seems that the moderator helps the teachers to refer again to the transcript. But sometimes it seems that the teachers give the moderator the role of an inspector whom they have to answer to, especially Mrs. Dieter.

4. CONCLUSIONS AND OUTLOOK

We found out that in this first structured talk the teachers react more *biased – spontaneous* (narrate, evaluate) than *open – reflected* (paraphrase, interpret) and use mainly knowledge by observation and experience and rarely knowledge by transfer and interrelation. Probably the teachers develop a more *open – reflected* view over the course of three structured talks in one year. And perhaps they get used to this kind of discussion and interpretation as a result they refer more to the statements of their colleagues to generate knowledge by transfer and interrelation.

The influence of the moderator seems to remind the teachers to focus their attention on the transcript and to initiate reflection processes about the statements of the other participants. We have to look for more evidence what impact the moderator has on the course of the structured talks and the case knowledge the teachers develop. We also can compare the influence of the cautious moderator (“Malin”, first part of this paper) and the active moderator (“MathKiD”) on the course of the structured talks.

After one structured talk we can draw no consequences and we cannot describe lasting changes in the readings and case knowledge the teachers develop. We will investigate and describe the development over the three structured talks. At the end we will look at video graphed lessons from the beginning and the end of the project MathKiD and will investigate if the structured talks had an impact on the teaching of each participant and on their professional development. Furthermore we will reflect if the participants want to continue the collegial reflections in their school without a moderator intended of the cautious moderator (first part of this paper).

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Transcripts can be ordered from the authors.

INTERNET-BASED DIALOGUE: A BASIS FOR REFLECTION IN AN IN-SERVICE MATHEMATICS TEACHER EDUCATION PROGRAM^[1]

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In this paper, the asynchronous interactions of two groups of mathematics teachers in an internet-based in-service course are analyzed. During the interactions, teachers are solving a mathematical modeling activity designed to stimulate the teachers' reflections on the modeling process. In one of groups these kinds of reflections occurred frequently while they were absent in other group. The analyses reveal clear differences in the communicative characteristics of the interactions in the two groups. Some of the characteristics of the first group are argued to be important factors favoring the emergence of the teachers' reflections on the modeling process.

INTRODUCTION

In this work, the asynchronous interactions of two groups of mathematics teachers in an internet-based in-service course are analyzed. The teachers are involved in an internet-based mathematics education in-service program for teachers from different Latin American countries. The acronym for this program is PROME-CICATA, and this is an educational program sponsored by the Instituto Politécnico Nacional of México, one of the largest public universities in Mexico. I am interested in finding ways of encouraging “rich” interactions and reflections among the teachers enrolled in the PROME mathematics education program. That is why I am trying to determine when an interaction can be regarded as “rich” or not, and what characterise communication in such rich interactions.

FRAMEWORK

The concept of communication is central for this work and particularly the computer-mediated communication (CMC). There are very clear differences between the everyday communication (or face-to-face) and the CMC. Although in both types of communication some kind of information (such as thoughts and feelings) is exchanged among individuals, the CMC does not require people staying in the same place or at the same moment of time. Communication may be atemporal to some extent and free of geographic barriers. Everyday communication is primarily verbal, but the CMC fosters written communication, which can be recorded, stored and accessed by people during conversation. This creates a record of ideas and comments that can serve as a reference or *collective memory* (de Vries, Lund & Baker, 2002) for the communication process. The expression and representation of ideas, and particularly mathematical ones, can be enhanced in CMC by the use of technological

tools such as software and video. The ideas can become entities with physical properties (such as a spreadsheet file in which somebody expresses a hypothesis based on graphical and arithmetical information represented in the file) which can be stored, handled and distributed.

The characteristics of the CMC influence the nature and dynamics of the interactions that I am analyzing in this study. The data analysis is based on the Inquiry co-operation model (IC-Model) of Alrø & Skovsmose (2002). This model was developed based on the observation of students, collectively solving mathematical open-ended activities. The model, strongly rooted in the critical mathematics education approach, argues that in order to have a fruitful interaction, it must be based on mutual respect, on the willingness to make public our ideas and subject them to scrutiny, as well as in a real interest to listen and analyse our interlocutor's ideas. The IC-Model is constituted by a set of communicative characteristics. According to this theoretical approach, an interaction as the previously described should have several of these communicative characteristics. In fact when these characteristics are present in an interaction, it is regarded as a special kind of interaction called *dialogue*, which possesses the potential to serve as a basis for critical learning and reflection. The communicative characteristics that define a dialogue are *getting in contact*, *locating*, *identifying*, *advocating*, *thinking aloud*, *reformulating*, *challenging* and *evaluating*; and they could be succinctly defined as follows:

Getting in contact basically refers to the act of paying attention to the ideas expressed by our partners in an interaction. The act of *locating* takes place when you discover an idea or a way of doing that you did not know or were not aware of before. It is a process of examining possibilities and trying things out. *Identifying* is a clarifier act in the sense that appears when you explore or try to explain an idea or perspective with the intention of making it clear to all the members of the interaction (including yourself). *Advocating* appears when you present your ideas or positions and you justify them with arguments. An advocating an also implies a willingness to revisit and discuss your own ideas or positions. To *think aloud* simply means to express in public your thoughts, ideas and feelings during the interaction process. *Reformulating* means repeating some idea but with different words or in other terms, usually to try to make it clear to your interlocutors. When we question a perspective or when we try to push it toward another direction to explore new possibilities, it is said that this is a *challenging* act. An *evaluative* act appears when we examine, criticize or correct an idea or proposal from others or ourselves.

In the communicative approach of Alrø & Skovsmose (2002), the concepts of dialogue and reflection are linked. First, reflection is defined as follows: "Reflection means considering at a conscious level one's thoughts, feelings and actions" (p. 184), but the dialogical interactions are also conceived as a basis for reflection: "We find that reflections are part of a dialogue. In particular we find elements of reflection in

dialogic acts like locating, thinking aloud, identifying, advocating, etc. This means that we do not follow the Piagetian line, seeing reflections as carried out by an individual. We consider reflections referring to ‘shared considerations’ and we see dialogue as including processes of reflection”

In the context of research on mathematics teacher education, reflection plays a key role. In her recent review, Judith T. Sowder says that several studies identify reflection as a crucial element in furthering teachers’ professional development (see Sowder, 2007, p. 198).

METHODOLOGY

In this section I refer to different aspects of the production and collection process of data, namely, the mathematical activity applied, the selected population, and the collection and presentation of data.

The selected population and the research goal.

The data that I will present were taken from one of the courses of the PROME program. The course was taught between March and April 2008. The course was an introduction to the teaching and learning of mathematical modeling. The teachers who participated in this course are in-service teachers working in different educational levels, from elementary to university level. This course was part of their academic obligations in order to get a master’s degree in mathematics education.

I present here the analysis of the asynchronous interactions produced in two groups of teachers while working collectively with a mathematical modeling task. I use the term ‘asynchronous interactions’ to specify that the sort of communication that takes place into this interaction is asynchronous. An asynchronous communication is the one that is carried out mainly by means of an exchange of written messages between two or more people (very often located in different geographical positions), but the answers or reactions that the participants get are not immediate, for example, you can raise a question or an observation and get the feedback or reactions to it several minutes or hours after. The asynchronous discussions usually last several days, allowing the participants to have more time to formulate their opinions and to reflect on comments and opinions expressed by the other participants. It is even possible to consult external sources in order to enrich and clarify a discussion in an asynchronous communication. The email messages and the discussion forums are some examples of asynchronous communication.

The activity lasted six days and although both groups of teachers solved the mathematical activity, only in one group emerged some meta-reflections about the modeling process, which were expected to be produced through the activity and the interaction. In other words, I will show an interaction that is “rich” in terms of the reflections produced and another that it is not rich, and, through the application of IC-

Model, I will try to identify the differences in the communicative characteristics that are present in each of those interactions. That is the purpose of the research.

The mathematical activity

The mathematical activity was taken from Lesh & Caylor (2007), but it was slightly modified to fit the purposes of the course. The context of the activity is a paper airplane contest in which four planes were involved, and where each of these planes were thrown by three different pilots five times each. The activity includes two tables (see tables 1 and 2) containing numerical values generated during one of the tests. Table 1 shows the landing points for each launch, represented by ordered pairs (x, y) ; Table 2 shows data such as distance from target, length of throw and air time for those launches. In this test the three pilots flew the four paper planes. Each time the pilot was placed at the point $(0, -80)$ on the floor, and their aim was to launch the planes so that the plane come as close as possible to the point $(0, 0)$, which was marked with an X.

A non-explicit purpose of this activity was that teachers will experience a portion of a mathematical modeling process, enabling them to see that in an mathematical activity as such, it is possible to have several possible and valid answers (or models), depending on the assumptions and considerations in which the model is based. To support the emergence of multiple approaches and answers to the activity, I decided to replace the original request “[to explain] how they could use this data and data from future contests to measure and make judgments about the accuracy of the paper airplanes”, for a more general question, namely: “Which one is the best airplane?”. Any model that answered the previous question should be based on the definition or concept that the modeler holds about what does it means to be “the best airplane”. This is where I expected to have a variety of definitions/concepts, and as a consequence, a variety of possible answers to the question.

	Plane 1			Plane 2			Plane 3			Plane 4		
	Flight	X	Y									
Pilot 1	1	45	-78	1	-37	-30	1	-42	-4	1	15	66
	2	7	-78	2	-48	-9	2	45	5	2	32	-46
	3	55	-42	3	28	26	3	35	-7	3	5	32
	4	-14	-46	4	12	-35	4	-13	5	4	14	-21
	5	21	-29	5	-19	26	5	4	-11	5	27	-5
Pilot 2	1	-12	26	1	-11	13	1	-4	12	1	-13	22
	2	-40	-20	2	-14	-36	2	24	9	2	-3	-21
	3	-38	22	3	31	13	3	-9	34	3	-11	-51
	4	-61	15	4	-43	10	4	-39	29	4	-12	56
	5	-48	61	5	-14	50	5	40	21	5	5	-69
Pilot 3	1	42	71	1	-9	6	1	24	41	1	53	-62
	2	61	38	2	-27	-13	2	-3	52	2	52	54
	3	43	27	3	17	-17	3	-31	54	3	48	-26
	4	18	50	4	14	47	4	5	64	4	45	25
	5	15	6	5	-36	41	5	36	63	5	25	40

Table 1: Where did the plane land?

Flight	Plane 1			Plane 2			Plane 3			Plane 4			
		Distance from target (inches)	Lenght of throw (inches)	Air time (seconds)	Distance from target (inches)	Lenght of throw (inches)	Air time (seconds)	Distance from target (inches)	Lenght of throw (inches)	Air time (seconds)	Distance from target (inches)	Lenght of throw (inches)	Air time (seconds)
Pilot 1	1	90	45	0.66	47.6	62.2	0.52	42.2	86.8	0.59	67.7	146.8	2.22
	2	78.3	7.2	0.58	48.8	85.7	0.91	45.3	96.2	0.7	56	47.7	1.42
	3	69.2	66.9	0.76	38.2	109.6	1.05	35.7	81	0.55	32.4	112.1	1.91
	4	48.1	36.8	0.51	37	46.6	0.51	13.9	86	0.61	25.2	60.6	1.51
	5	35.8	55.2	0.65	32.2	107.7	0.99	11.7	69.1	0.54	27.5	79.7	1.58
Pilot 2	1	28.6	106.7	0.94	17	93.6	0.95	12.6	92.1	0.73	25.6	102.8	1.88
	2	44.7	72.1	0.81	38.6	46.2	0.54	25.6	92.2	0.7	21.2	59.1	1.4
	3	43.9	108.8	0.93	33.6	98	0.92	35.2	114.4	0.77	52.2	31	1.35
	4	62.8	112.9	0.9	44.1	99.7	0.99	48.6	115.8	0.8	57.3	136.5	2.35
	5	77.6	148.9	1.28	51.9	130.8	1.1	45.2	108.6	0.78	69.2	12.1	0.97
Pilot 3	1	82.5	156.7	1.34	10.8	86.5	0.88	47.5	123.4	0.8	81.6	56	1.36
	2	71.9	132.8	1.14	30	72.2	0.73	52.1	132	0.88	75	143.7	2.39
	3	50.8	115.3	1.01	24	65.3	0.55	62.3	137.5	0.92	54.6	72.2	1.46
	4	53.1	131.2	1.21	49	127.8	1.08	64.2	144.1	0.89	51.5	114.2	2.18
	5	16.2	87.3	0.85	54.6	126.2	1.1	72.6	147.5	0.94	47.2	122.6	2.09

Table 2: Distance, time and flight sequence data for each pilot and airplane.

The activity was uploaded as a *pdf* file on the web-based educational space where all participants of the course could access it. Teachers were organized into groups of three or four members and each of those groups were assigned to a discussion forum where the activity was collectively solved.

Data collecting and data presentation

As I mentioned before, one of the characteristics of the computer mediated communication is that it can be easily recorded, stored and shared. This feature represents a significant advantage for educational research, because the need of making transcriptions disappears. In my work for instance, I am studying some of the written asynchronous discussions produced in an internet-based educational program. Those discussions are permanently recorded and accessible on the internet-based workspace, ready to be analyzed. These asynchronous discussions may be composed of dozens of utterances. Due to the space available, it will not be possible to present the complete interactions, but only those sections that I consider most significant and illustrative. I will use bracketed ellipsis [...] to denote the omission of certain segments of text; this edition was made for the sake of brevity and to increase the readability of the data. The data that I will present has been translated from Spanish into English; moreover, the original names of the teachers have been replaced to protect their identity.

To start the analysis of an asynchronous discussion, I order all its utterances in a chronological way. From this arrangement, I try to locate those sections in which two or more participants are involved in a discussion of a particular topic. Each of these sections is broken down into individual utterances, trying to 'label' them with some of the communicative characteristics that define the communication IC-Model, according to the content of the utterance and its role within the whole discussion. Let me consider utterance (1) as an example (see 'Results' section below): This is not an *evaluative* or *challenging* act, nor is *getting into contact* with someone else because

Juan is not criticizing, questioning or being referred to the ideas of another person. He is not *reformulating* because this is the first time that he presents these ideas. Juan says “I think the most important is the proximity to the target”, but he did not present any argument to be able to classify the act as an *advocating* one. The utterance could be classified as a *thinking aloud* act, but because Juan is raising different ways of tackling the problem, I have classified it as a *locating* act. A similar analysis was done with every utterance. In some cases it is difficult to carry out the categorization since the differences between some communicative acts of IC-Model are not entirely clear for some utterances.

RESULTS

Data analysis – Group A

The working group A was composed of two teachers from Argentina (Juan and Susana) and one mexican teacher (Horacio). The interaction begins with some *thinking aloud* acts where the teachers begin to make public some of their initial ideas on how to address the problem. For instance, Susana suggests that they should find a way to use the three variables contained in Table 2 (distance, length and time). Juan answered to Susana in (1):

1 *Topic: Re: The first message*

From: Juan

Date: Thursday, the 3rd of April 2008, 11:40

Colleagues. One possible option is to work with some type of weighted mean for the 3 considered variables (length of throw, distance from target and air time). I think the most important is the proximity to the target. Another option is to think on the deviation from the target (because definitely it is a measure of the dispersion) what do you think?

In (1) Juan is *locating*, I mean, he is examining different ways of facing the problem and trying things out. He is doing a specific suggestion on how to relate the three selected variables. He proposes to use a weighted mean where “proximity to the target” is the most important variable.

2 *Topic: Re: The first message*

From: Susana

Date: Thursday, the 3rd of April 2008, 13:05

Flight partners: I was planning to ask you if you have thought in a linear regression, but I read your proposal of the weighted mean. We just have to decide about the importance assigned to each variable. Since the target is point (0,0) I would give 40% to distance from target, and 30% for the other two, if you agree. [...] Susana

3 *Topic: Re: The first message*

From: Juan

Date: Thursday, the 3rd of April 2008, 19:06

Fellows. I have been outlining a sketch of the things worked so far and I expressed it on this first draft that I am attaching. [...] Best wishes. Juan

In (2), Susana mentions the possibility of using a linear regression, but this possibility was not further explored because she simply leaves this alternative and without any question she adheres herself to the proposal of the weighted mean. Without a clear argumentation, Susana proposed the weight for each element of the weighted mean. In turn, Juan in (3) contributes to *not locate* Susana's idea of linear regression. In his utterance he completely ignores the timid suggestion of Susana and he only “hear” the proposal of the weights. In a file attached to his utterance (3), Juan *identifies* or clarifies in mathematical terms his perspective on the weighted mean. In this file he defines the concept of “performance” that is used to determine which one is the best airplane. The plane that gets the higher performance will be the winner. This concept is defined as follows: $Performance = 0.4x + 0.3y + 0.3z$

Where:

x = the arithmetic mean of the distances from target

y = the arithmetic mean of the lengths of throw

z = the arithmetic mean of the air times

Juan never questioned the weights suggested by Susana. He never asks which were the assumptions that Susana considered in order to establish those values, he just includes the values in his own proposal. In general, the interaction between Susana and Juan could be described as uncritical. They experienced a “smooth” interaction where they did not question nor evaluate the proposals from the other. An example of this is in the performance formula. Neither Susana nor Juan noted that this model favoured the airplanes having a landing fare away from the target. On the other hand, Juan’s attitude was not the most appropriate to establish a *dialogue*, apparently Juan was more interested in delivering the solution of the task on time, that in paying attention to the proposals of his colleagues. For example, although the asynchronous discussion forum lasted until the 6th of April, Juan showed in (5) a strong rejection attitude towards other proposals to his colleague Horacio (see (4)):

4 *Topic: Re: The first message*

From: Horacio

Date: Friday, the 4th of April 2008, 11:10

Susana, Juan. I am sorry but my time is very limited. I will try to communicate with you later on. Best regards. Horacio

5 *Topic: Re: The first message*

From: Juan

Date: Friday, the 4th of April 2008, 11:26

Horacio. We are against the clock, this activity started on tuesday and there is 1 day left...I think you will have to accommodate yourself to the things that Susana and I were working on...there is no time to make any modification... Do you agree? What do you think?

Thus, even though group A was able to successfully solve the mathematical modelling task (i.e. to establish a model to select the best airplane), the interaction inside the team was characterized by a poor exchange of perspectives and ideas on how to address the mathematical task.

Data analysis – Group C

The group C had three members, but almost all the interaction took place between an Argentinean teacher (Nora) and a Mexican one (Maria). Since the beginning of the interaction, Norma and Maria were *locating* different ways of tackling the problem, but always *maintaining the contact* between them, namely, listening to the proposals of the other, taking them into consideration and evaluating them. At one point, based on Maria's suggestion about excluding the pilots of the analysis, Norma proposed in (6) a new way to find the best paper airplane:

6 *Topic: Some issues*

From: Norma

Date: Saturday, the 5th of April 2008, 06:17

[...] We could choose the ten shots that are closer to the origin, and then see which of those planes did it in more time and with the biggest length, what you think? [...]

7 *Topic: Re: Some issues*

From: Maria

Date: Saturday, the 5th of April 2008, 21:44

[...] I propose to choose the other way around, let's say that the best planes are the ones who entered into a circle with center (0.0) and a fixed radio, and then to take the ones who did it in less time [...] you said more time... but are we judging the fastest or the longest stay in the air[?]?... both cases are possible to judge [...] in a model it should be fixed the aspects to take into account and the rest are discarded because it is a model. I think that the idea of the radio is more close to the kind of things that are considered in the accuracy competitions as in archery. Maria

8 *Topic: Re: Some issues*

From: Maria

Date: Saturday, the 5th of April 2008, 22:32

Colleagues: I am writing you because I think that a good size for the radio could be 20 because it is one fourth of the distance from the point of departure to the target point. With this we only have six throws with three planes, I mean, the fourth plane does not participate, it does not surpass the first filter, then we can evaluate the next point.... and if we estimate the maximum speed [...] It would be like the thing that I am sending you ...What do you say? [...] I will wait for your criticism

In (7) Maria is *challenging* Norma's proposal by suggesting replacing the ten shots criterion with the radio criterion. I think this intervention is particularly valuable because explicitly brings into the discussion the need to establish the criteria, assumptions or variables to consider for building a mathematical model. Her next

sentence sums up this point: “[I]n a model it should be fixed the aspects to take into account and the rest are discarded because it is a model”. This is the kind of meta-reflection that I was looking to produce through the activity.

Maria’s utterance (8) includes a spreadsheet file that illustrates with more detail the ideas presented in (8) and (8). She concludes that the winner is the plane number 3. As a reaction, Norma in (9) *evaluates* the proposal of Maria, and qualifies as arbitrary the choice of a radio with longitude 20. Norma agrees with Maria about using the proximity to the target as a first filter for selecting the best plane, but she suggested to use the mean of the distances from target instead of the radio proposed in (7) and (8).

9 *Topic: Re: Some issues*

From: Norma

Date: Sunday, the 6th of April 2008, 12:19

Girls, Maria: The radio that you mention is a bit arbitrary, why do not we take advantage of the fact that we already have the mean of the distances from target, and then to select the planes that were above that mean??? [...]

10 *Topic: Re: Some issues*

From: Norma

Date: Sunday, the 6th of April 2008, 13:03

Well, here you have what I made according to the previous observation about the radio. But I would also mention that I love your conclusions, Maria.

If you agree, let’s vote; choose one of the three options, or choose all of them because for me all of them are ok. I mean, they are all equally valuable and correct. There are as many answers as aspects and ways of evaluating we have agreed previously.

In (10) Norma attached a file showing her new calculations, in which the winner is the plane number 4. Despite she is *advocating* a different model and getting a different winner, Norma recognizes the validity of the model suggested by her colleague Maria, in fact I think that this recognition is the basis for issuing the comment made by Norma in (10), a comment linked to another reflection implicitly sought for the modeling activity: the recognition that there may be different valid answers or mathematical models to answer the same question. It may be noted that the discussion has reached an interesting point: the participants in the discussion have been able to *locate* different ways (or models) that can serve as a mean to answer the original question which one is the best airplane? Moreover, apparently they have recognized as valid each of those models, then ... what model to choose?

This discussion continued even addressing issues of *responsibility* (see Alrø & Skovsmose, 2002, p. 217). At one point Maria asked, “[I]f the owner of the plane 3 shows up, with what criteria would we justify that we do not chose the early drafts in which he would win and instead we took the other one[?]”. No doubt, this was a rich interaction in terms of the reflections achieved by the teachers.

CONCLUSIONS

The analysis of the interactions through the IC-Model shows that there are some differences in the communicative characteristics that are present in the interactions of groups A and C. For example, the interaction within the A team can be described as uncritical because there is a lack of communicative acts such as *challenging* or *evaluating*; additionally they did not seize the opportunities to find additional ways to address the problem (see for example the utterances sequences (2)-(3) and (4)-(5)).

In the team C, participants were able to *locate* several ways to tackle the problem. There was a general interest in hearing (or *keep the contact*) and evaluate the proposals of the other, and they were able to recognize the existence of multiple perspectives to solve the problem.

I argue that members of team C team were able to establish a *dialogue* that fostered the emergence and recognition of multiple perspectives to solve the problem. I think that the existence of this dialogue encouraged the emergence of meta-reflections about the modeling process.

It is necessary to continue working in a more explicit characterization of the concept of reflection. It is also necessary to discuss how the characteristics that are specific to the internet-based communication affect the emergence of reflections. Methodologically speaking it is necessary to find appropriate tools to detect or to point out when a reflection takes place in an online setting, but particularly in an asynchronous interaction.

NOTES

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THE USE OF ALGEBRAIC LANGUAGE IN MATHEMATICAL MODELLING AND PROVING IN THE PERSPECTIVE OF HABERMAS' THEORY OF RATIONALITY

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In this paper we consider the use of algebraic language in modelling and proving. We will show how a specific model of rational behaviour derived from Habermas' elaboration allows to describe and interpret several kinds of students' difficulties and mistakes in a comprehensive way, provides the teacher with useful indications for the teaching of algebraic language and suggests further research developments.

Key-words: Habermas, rationality, algebraic language, modelling, proving

INTRODUCTION

Habermas' work has attracted the interest of many scholars in the domain of Sciences of Education (see the review of the translation into English of *Truth and Justification* by Tere Sorde Marti, 2004). We think that at least one of his constructs, that of "rational behaviour", is of specific interest for mathematics education, if we want to analyse complex mathematical activities (like conjecturing, proving, modelling) in a comprehensive way and to deal with them not only as school subjects and sets of tasks, but also as ways of experiencing mathematics as one of the components of western rationality. In a long term research perspective, we think that Habermas' construct is a promising analytic instrument in mathematics education if we want to connect the individual and the social by taking into account the epistemic requirements of "mathematical truth" in a given cultural context and the ways of discovering, ascertaining and communicating it by means of suitable linguistic tools. Indeed, according to Habermas' definition (see Habermas, 2003, Ch. 2), a rational behaviour in a discursive practice can be characterized according to three inter-related criteria of rationality: *epistemic* rationality (inherent in the conscious control of the validity of statements and inferences that link statements together within a shared system of knowledge, or theory); *teleological* rationality (inherent in the conscious choice and use of tools and strategies to achieve the goal of the activity); *communicative* rationality (inherent in the conscious choice and use of communication means within a given community, in order to achieve the aim of communication).

In our previous research we have dealt with an adaptation of Habermas' construct of rational behaviour in the case of conjecturing and proving (see Boero, 2006; Morselli, 2007; Morselli & Boero, 2009 - to appear). In this paper we focus our interest on the use of algebraic language in proving and modelling. Algebraic language will be intended in its ordinary meaning of that system of signs and transformation rules, which is taught in secondary school as a tool to generalize arithmetic properties, to develop analytic geometry and to model non-mathematical situations (in physics, economics, etc.). In particular, for what concerns modelling (according to Norman'

broad definition: see Norman, 1993, and Dapueto & Parenti, 1999, for a specific elaboration in the case of mathematics) algebraic language can play two kinds of roles: a tool for proving through modelling within mathematics (e.g. when proving theorems of elementary number theory) - *internal modelling*; or a tool for dealing with extra-mathematical situations (in particular to express relations between variables in physics or economy, and/or to solve applied mathematical problems) - *external modelling*.

Our interest for considering the use of algebraic language in the perspective of Habermas' definition of rational behaviour depends on the fact that our previous research (Boero, 2006; Morselli, 2007) suggests that some of the students' main difficulties in conjecturing and proving depend on specific aspects (already pointed out in literature) of the use of algebraic language, which make it a complex and demanding matter for students. In particular, we refer to: the need of checking the validity of algebraic formalizations and transformations; the correct and purposeful interpretation of algebraic expressions in a given context of use; the goal-oriented character of the choice of formalisms and of the direction of transformations; the restrictions that come from the needs of following taught communication rules, which may contradict private rules of use or interfere with them.

In this paper, we will try to show how framing the use of algebraic language in the perspective of Habermas' theory of rationality: first, provides the researcher with an efficient tool to describe and interpret in a comprehensive way some of the main difficulties met by students at any school level when using algebraic language; second, provides the teacher with some useful indications for the teaching of algebraic language; third, suggests new research developments, in particular those concerning the interplay between epistemic rationality and teleological rationality in the use of algebraic language, and those related to the role of verbal language as a crucial tool for a rational behaviour in the use of algebraic language, thus potentially adding new arguments to the elaboration presented in Boero, Douek & Ferrari (2008) and concerning the specific functions of verbal language in mathematical activities.

ADAPTATION OF HABERMAS' CONSTRUCT OF RATIONAL BEHAVIOUR TO THE CASE OF THE USE OF ALGEBRAIC LANGUAGE

The aim of this section is to match Habermas' construct of rational behaviour to the specificity of the use of algebraic language in modelling and proving.

Epistemic rationality

It consists in:

- *modelling requirements*, concerning coherency between the algebraic model and the modelled situation: control of the correctness of algebraic formalizations (be they *internal* to mathematics - like in the case of the algebraic treatment of arithmetic or geometrical problems; or *external* - like in the case of the algebraic modelling of physical situations) and interpretation of algebraic expressions;

- *systemic requirements* in the use of algebraic language and methods. In particular, these requirements concern the manipulation rules (syntactic rules of transformation) of the system of signs usually called algebraic language, as well as the correct application of methods to solve equations and inequalities.

Teleological rationality

It consists in the conscious choice and finalization of algebraic formalizations, transformations and interpretations that are useful to the aims of the activity. It includes also the correct, conscious management of the writer-interpreter dynamics (Boero, 2001): the author may write an algebraic expression under an intention and, after, interpret it in a different goal-oriented way, by discovering new possibilities in the written expression.

Communicative rationality

In the case of algebraic language we need to consider not only the communication with others (explanation of the solving processes, justification of the performed choices, etc.) but also the communication with oneself (in order to activate the writer-interpreter dynamics). Communicative rationality requires the user to follow not only community norms concerning standard notations, but also criteria for easy reading and manipulation of algebraic expressions.

Some comments

The previous requirements define a model of “rational behaviour” in the use of algebraic language in modelling and proving.

We are aware of the existence of several analytical tools to deal with the teaching and learning of algebraic language. In the case of most of them, the researcher adopts a specific point of view, performs in-depth analyses according to it, but usually does not take into consideration the connections between the different aspects of the use of algebraic language and suggests only partial indications for its teaching. In our opinion, Arcavi's work on Symbol sense (Arcavi, 1994; 2005) offers the most comprehensive perspective for the use of algebraic language. With different wordings, it includes concerns for teleological rationality and some aspects of epistemic rationality. Comparing our approach with Arcavi's elaboration, we may say that we add the communicative dimension of rationality. We will see how it will allow us to account for: the possible tension between private rules of communication in the intra-personal dialogue, and standard rules; and the interplay between verbal language and algebraic language. Moreover we will see how our distinctions between the epistemic dimension and the teleological dimension, and between the modelling and the systemic requirements of epistemic rationality allow to deal with the tensions and the difficulties that can derive from their coordination.

In order to justify a new analytic tool in Mathematics Education it is necessary to show how it can be useful in describing and interpreting students' behaviour, and/or in orienting and supporting teachers' educational choices, and/or in suggesting new

research developments. The aim of the following Sections is to provide evidences for all the three mentioned aspects of the use of the adapted Habermas' model.

DESCRIPTION AND INTERPRETATION OF STUDENTS' BEHAVIORS

The following examples are derived from a wide corpus of students' individual written productions and transcripts of *a posteriori* interviews, collected for other research purposes in the last fifteen years by the Genoa research team in Mathematics Education. In particular, we will consider four categories of students:

- (a) 9th grade students who are approaching the use of algebraic language in proving;
- (b) 11th grade students who are learning to use the algebraic language in modelling;
- (c) students who are attending university courses to become primary school teachers;
- (d) students who are attending the third year of the university course in Mathematics.

A common feature for all the considered cases is that the individual tasks require not only the solution, but also the explanation of the strategies followed to solve the problem. Each individual task was followed by *a posteriori* interviews. However, while in the cases (c) and (d) the explanation of the strategies is inherent in the didactical contract already established with the teacher for the whole course, in the cases (a) and (b) such explanation is only an occasional request.

EXAMPLE 1: 9th grade class

The class (22 students) was following the traditional teaching of algebraic language in Italy: transformation of progressively more complex algebraic expressions aimed at « simplification ». In order to prepare students to the task proposed by the researcher, two examples of “proof with letters” had been presented by the teacher; one of them included the algebraic representation of even and odd numbers.

THE TASK: “Prove with letters that the sum of two consecutive odd numbers is divisible by 4”.

Here we report some recurrent solutions (in parentheses the number of students who performed such a solution; note that “dispari” means “odd” in Italian)

- E1 (4 students): $d+d=2d$

In this case, we can observe how the *systemic requirements of epistemic rationality* are satisfied (algebraic transformation works well), while the *modelling requirements* fail to be satisfied (the same letter is used for two different numbers).

- E2 (8 students): $d+d+2=2d+2$

In this case, both the *systemic and the modelling requirements of epistemic rationality* are satisfied, but the requirements inherent in *teleological rationality* are not satisfied: students do not realize that the chosen representation does not allow to move towards the goal to achieve (because the letter d does not represent in a transparent way the fact that d is an odd number) and do not change it.

- E3 (5 students): $d=2n+1+dc=2n+1+2n+1+2=4n+4$ (or similar sequences)

We can infer from the context (and also from some a-posteriori comments by the students) that "dc" means "**dispari consecutivi**" (consecutive odd numbers).

In this case *epistemic rationality* fails in the first and in the second equality, but *teleological rationality* works well: the flow of thought is intentionally aimed at the solution of the problem; algebraic transformations are used as a calculation device to produce the conclusion (divisibility by 4).

EXAMPLE 2: University entrance, primary school teachers' preparation

The following task had been preceded by the same task of the Example 1, performed under the guide of the teacher. 58 students performed the activity.

THE TASK: Prove in general that the product of two consecutive even numbers is divisible by 8

Very frequently (about 55% of cases) students performed a long chain of transformations, with no outcome, like in the following example:

- E4: $2n(2n+2)=4n^2+4n=4(n^2+n)=4n(n+1)=4n^2+4n=n(4n+4)$

In this case, we see how both requirements of *epistemic rationality* are satisfied: *modelling requirements* (concerning the algebraic modelling of odd numbers and even numbers); and *systemic requirements* (correct algebraic transformations). The difficulty is inherent in the lack of an interpretation of formulas led by the goal to achieve, thus in *teleological rationality*. The student gets lost, even if the interpretation of the fourth expression would have provided the divisibility of $n(n+1)$ by 2 because one of the two consecutive numbers n and $n+1$ must be even. We can also observe how (in spite of the didactic contract) in general no substantial verbal comment precedes or follows the sequence of transformations (sometimes we find only a few words: "I use formulas"; "I see nothing").

In the following case, both *modelling* and *systemic requirements* are not satisfied: the same letter is used for two consecutive even numbers (note that "**pari**" means "even" in Italian) and the algebraic transformation is affected by a mistake.

- E5: $p*p=2p^2$, divisible by 8 because p is divisible by 2 and thus p^2 is divisible by 4.

The student seems to work under the pressure of the aim to achieve: having foreseen that the multiplication by 2 may be a tool to solve the problem, she tries to justify it by considering the juxtaposition of two copies of p that generates "2". Indeed in the interview the student said that she had made the reasoning " p is divisible by 2 and thus p^2 is divisible by 4" before completing the expression. In this case we can see how *teleological rationality* prevailed on *epistemic rationality* and hindered it.

We have also found cases like the following one:

- E6: $p*(p+2)=p^2+2p=8k$ because $p^2+2p=8$ if $p=2$

Also in this case, from the *a posteriori* interview we infer that probably the lacks in *epistemic rationality* depend on the dominance of *teleological rationality* without sufficient epistemic control:

I have seen that in the case $p=2$ things worked well, so I have thought that putting a multiple $8k$ of 8 in the general formula would have arranged the situation.

EXAMPLE 3: The bomb problem

TASK: A helicopter is standing upon a target. A bomb is left to fall. Twenty seconds after, the sound of the explosion reaches the helicopter. What is the relative height of the helicopter over the ground?

The problem was proposed to groups of third year mathematics students in seven consecutive years, and to two groups of 11th grade students (high school, scientific - oriented curriculum). According to the school levels, some reminds were provided (or not) about the fact that the falling of the bomb happens according to the laws of the uniformly accelerated motion, while the sound moves at the constant speed of 340 m/s. However no formula was suggested.

The problem is a typical applied mathematical problem, whose solution needs an *external modelling* process. In terms of *teleological rationality*, the goal to achieve should result in the choice of an appropriate algebraic model of the situation, in solving the second degree equation derived from the algebraic model, and in choosing the good solution (the positive one).

The first difficulty students meet is inherent in the time coordination of the two movements: it is necessary to enter somewhere in the model the information that the whole time for the bomb to reach the ground and for the sound of the explosion to reach the helicopter is 20 seconds. The second difficulty is inherent in the space coordination of the two movements: the space covered by the falling bomb is the same covered by the sound when it moves from the ground to the helicopter.

Let us consider some students' behaviours.

Most students are able to write the two formulas:

- E7: $s=0,5 gt^2$, $s=340 t$

They are standard formulas learnt in Italian high school in grades 10th or 11th, in physics courses. About 25% of the high school students and 20% of the university students stick to those formulas without moving further. From their comments we infer that in some cases the use of the same letters for space and time in the two algebraic expressions generates a conflict that they are not able to overcome. We can see how general expressions that are correct for each of the two movements (if considered separately) result in a bad model for the whole phenomenon. *Teleological rationality* should have driven formalization under the control of *epistemic rationality*; such control should have put into evidence the lack of the *modelling requirements* of *epistemic rationality*, thus suggesting a change in the formalization.

In the reality for those students such an interplay between *epistemic rationality* and *teleological rationality* did not work.

In other cases (about 10% of both samples) the coordination of the two times was lacking, and the idea of coordinating the spaces (together with the formalization of both movements with the same letters) brought to the equation:

- E8: $0,5gt^2 = 340t$

with two solutions $t=0$, $t=68$ that some students were unable to interpret and use (because 68 is out of the range given by the text of the problem). But other students found the height of the helicopter by multiplying 340×68 ; the fact that the result is out of the reach of a helicopter did not provoke any critical reaction or re-thinking, probably because it is normal that school problems are unrealistic!

One part of the students who introduced the third equation $t_b + t_s = 20$ added it to the first two equations without changing the name of the variable (t).

Less than 60% of students of both samples wrote a good model for the whole phenomenon:

$$t_b + t_s = 20$$

$$h = 0,5gt_b^2 = 340t_s$$

and moved to a second degree equation by substituting $t_s = 20 - t_b$ or $t_b = 20 - t_s$ in the equation: $0,5gt_b^2 = 340t_s$

Many mistakes occurred during the solution of the equation (mainly due to the management of big numbers). Once two solutions were got (one positive and the other negative), in most cases the choice of the positive solution was declared but not motivated. *A posteriori* comments reveal that the fact that a negative solution is unacceptable (given that the other solution is positive!) was assumed as an evidence, without any physical motivation.

In terms of *epistemic rationality*, three kinds of difficulties arose; they were inherent: first, in the control that the chosen algebraic model was a good model for the physical situation; second, in the control of the solving process of an equation with unusual complexity of calculations (big numbers); third (once the valid equation - a second degree equation - was written and solved), in the motivation of the chosen solution.

In terms of *communicative rationality*, we can observe how (in spite of the request of explaining the steps of reasoning) very few students of both samples were able to justify the crucial steps of the solving process. How is it possible to interpret this kind of difficulty? In some cases the steps were derived from a gradual adaptation of the equations to the need of getting a “realistic” solution. In other cases the equations were written as if the idea of coordination of the spaces and times of the phenomenon was supported by an intuition, but no wording followed. *A posteriori* interviews revealed that most students who had been unable to justify their choices were sure about their method only afterwards, when checking the positive solution and finding

that it was “realistic”, thus putting into evidence a lack in *teleological rationality* (lack of consciousness about the performed modelling choices). However a number of solutions was quite realistic, even if got through a bad system. Many authors of the correct solution were not able to explain (during the comparison of solving processes) why the other solutions were mistaken. This suggests that lacks in *communicative rationality* (as concerns verbal justification of the validity of the performed modelisation) can reveal lacks in *teleological rationality* (motivation of choices with reference to the aim to achieve) and even in *epistemic rationality* (control of the validity of the steps of reasoning). This conclusion can be reinforced if we consider the fact that almost all students who were able to provide a verbal justification for their modelisation were also able to explain why the other solutions were not acceptable (even if results were realistic).

DISCUSSION

As remarked in the second section, the usefulness of a new analytical tool in mathematics education must be proved through the *actual* and the *potential research advances* and the *educational implications* that it allows to get.

Research advances

In the frame of our adaptation of Habermas' construct, the distinction between *epistemic rationality* and *teleological rationality* allows to describe, analyse and interpret some difficulties (already pointed out in Arcavi's work), which depend on the students' prevailing concern for rote algebraic transformations performed according to *systemic requirements* of *epistemic rationality* against the needs inherent in *teleological rationality* (see E4). Moreover, the distinction in our model between *modelling requirements* and *systemic requirements* of *epistemic rationality* offers the opportunity of studying the interplay between the *modelling requirements* and the requirements of *teleological rationality* (see E7); we have also seen that formalization and/or interpretations may be correct but not goal-oriented (like in E2 and E4), or incorrect but goal-oriented (like in E5, E6 and E8).

Together with the other dimensions of rationality, *communicative rationality* allows to describe and interpret possible conflicts between the private and the standard rules of use of algebraic language, and the ways student try to integrate them in a goal-oriented activity (see E3).

At present, we are engaged in establishing how the requirements of the three components of rationality intervene in the phases of production and interpretation of algebraic expressions.

Further research work should be addressed to establish what mechanisms (meta-cognitive and meta-mathematical reflections based on the use of verbal language? See Morselli, 2007) can ensure the control of *epistemic rationality* and the intentional, full development of *teleological rationality* in a well integrated way. With reference to this problem, taking into account *communicative rationality* (in its

intra-personal dimension, possibly revealed through suitable explanation tasks and/or interviews) suggests a research development concerning the role of verbal language (in its mathematical register: see Boero, Douek & Ferrari, 2008, p.265) in the complex relationships between *epistemic*, *teleological* and *communicative rationality*. In particular, previous analyses (see E3, E4 and Example 3) suggest not only that the request (related to *communicative rationality*) of justifying the performed choices can reveal important lacks in *teleological rationality*, but also that the development of a kind of personal “verbal space of actions” can be relevant for a successful development of the activity (even if algebraic written traces are not satisfactory from the *systemic-epistemic rationality* point of view, like in the case E3). The respective role of the space of verbal actions and of the space of algebraic manipulations should be investigated on the *teleological rationality* axe. Here Duval's elaboration about the productive interplay between different registers in mathematical activities might be borrowed to better understand and frame what students do (see Duval, 1995). Also the results by Mac Gregor & Price (1999) could help highlighting the relations, as emerged from our data, between the production of verbal justifications and the effective use of algebraic language to achieve the goal of the activity.

Educational implications

We think that the analyses performed in the previous section can provide teachers as well as teachers' educators with a set of indications on how to perform educational choices and classroom actions to teach algebraic language as an important tool for modelling and proving. Some of those indications are not new in mathematics education; we think that the novelty brought by Habermas' perspective consists in the coherent and systematic character of the whole set of indications.

First of all, the performed analyses suggest to balance (at the students' eyes, according to the didactical contract in the classroom) the relative importance (in relationship with the goal to achieve) of:

- production and interpretation of algebraic expressions, vs algebraic transformations;
- flexible, goal-oriented direction of algebraic transformations, vs rote algebraic transformations aimed at “simplification” of algebraic expressions.

These indications are in contrast with the present situation in Italy and in many other countries: teachers' classroom work is mainly focused on algebraic transformations aimed at “simplification” of algebraic expressions, and most simplifications are performed by elimination of parentheses, thus suggesting a mono-directional way of performing algebraic transformations. At the students' eyes, the importance of the formalization and interpretation processes is highly underestimated. The fact that algebraic expressions are given as objects to “simplify” (and not as objects to build, to transform according to the aim to achieve, and to interpret during and after the transformation process in order to understand if the chosen path is effective and correct or not) has bad consequences on students' *epistemic rationality* and *teleological rationality*. As we have seen, many mistakes occur in the phase of

formalization (against the *modelling requirements*), and even when the produced expressions are correct, frequently students are not able to use intentionally them to achieve the goal of the activity (against the *teleological rationality requirements*).

A promising indication coming from our analyses concerns the need of a constant meta-mathematical reflection (performed through the use of verbal language) on the nature of the actions to perform and on the solving process during its evolution. At present, the only reflective activity in school concerns checking the correct application of the rules of syntactic transformation of algebraic expressions (thus only one component of rational behaviour - namely, the *systemic requirements of epistemic rationality* - is partly engaged).

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OBJECTS AS PARTICIPANTS IN CLASSROOM INTERACTION

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In this article an object-integrating approach to interaction in the mathematics classroom is proposed. Accordingly, not only human beings, but also non-human objects are considered as participants in the course of action. Symbolic interactionism and Actor-Network-Theory both serve as a theoretical basis for the development of the object-integrating approach to classroom interaction outlined in this article.

Keywords: objects, classroom interaction, Symbolic interactionism, ANT, analysis

INTRODUCTION

Research on teaching and learning processes in the mathematics classroom focuses on different aspects. Mathematical language, or communication in a broader sense, are possible points of interest. In this article I take an interactionistic perspective on processes of teaching and learning. I investigate classroom interaction as it is developed by its participants. My current interest is on the role of objects in such interactional processes. How do they affect the proceeding of interactional learning processes in primary education? My concern is the development of an object-integrating approach to interaction in the mathematics classroom.

OBJECTS AND CLASSROOM INTERACTION

The ‘discovery’ of the mere existence of objects in the mathematics classroom is rather innocuous. Besides, the observation that objects have an influence on interaction in mathematics primary education is not new either. Moreover, systematic implementation of objects such as books, paper and pencil, blackboards, calculators, cubes or dice in teaching and learning activities is a commonly shared practice. It gains wide acceptance amongst researchers as well as amongst primary teachers. Undoubtedly, objects play a role in the course of mathematical teaching and learning. But how can one describe the objects’ role in the course of classroom interaction theoretically? Interactionistic perspectives on primary mathematics education traditionally focus on students and teachers (see e.g. Mehan, 1979; Cobb & Bauersfeld, 1995). These persons are the actors developing the interactional process. However, no special attention is paid to non-human objects, and no interactionistic thought is given to them. Thus, there remains uncertainty concerning things and their role within the interactional development. Subsequently I am going to outline a theoretical approach to interaction in which objects have “agency” (Latour 2005, p. 63) as well. Proposing this object-integrating approach to classroom interaction, I draw on the framework of symbolic interactionism (Blumer, 1986) and on Latour’s Actor-Network-Theory (ANT) (Latour, 2005). Referring to ANT I go beyond the more common idea of interpreting objects as tools or instruments in human’s hands.

Nor do I concentrate on mediated thinking or objectification (Radford 2006). Instead, I accept objects as participants in classroom interaction. Thus, Latour's theory serves as an impetus for a radical change in studying mathematical learning processes. While the suggested object-integrating approach is not yet a fully developed theory, I suggest it as a thought-provoking impulse.

Symbolic Interactionism

Blumer (1986) gives an outline of the nature of symbolic interactionism, calling in three premises. The first premise is that "human beings act toward things on the basis of the meanings the things have for them." (ibid., p. 2). Here, Blumer's use of the term 'thing' differs fundamentally from the understanding of 'things' throughout the rest of this article. It is as broad and overarching as possible. Blumer defines: "Such things include everything that the human being may note in his world – physical objects [...], other human beings [...], institutions [...], guiding ideals [...], activities of others [...] and such situations as an individual encounters in his daily life." (ibid., p. 2). In contrast, I apply the everyday-term 'thing' with regard to ANT in a much closer form. I use it as a colloquial and sensitizing version of the term 'object', taken as short for non-human physical object.

The second premise refers to the source of meaning. Meaning is not intrinsic to the thing. Nor is it a psychical accretion like a sensation, memory, or feeling brought into play in connection with perceiving the thing. Instead, "symbolic interactionism sees meaning as arising in the process of interaction between people. The meaning of a thing for a person grows out of the ways in which other persons act toward the person with regard to the thing." (ibid., p. 4). Thus, the meaning of things is formed in the context of social interaction. It is seen as a social product.

The meaning of a thing is derived by the person from the interactional process. But meaning is not an already established application to a thing. It is nothing that has to be arisen from the thing itself. In contrast, the use of meaning by the actor occurs through a process of interpretation. And this leads to the third fundamental premise put forward by Blumer: "The meanings are handled in, and modified through, an interpretative process." (ibid., p.5). Thus, interpretation becomes a matter of handling meanings. It is considered as a formative process in which meanings are used and revised as instruments for the guidance and formation of action.

Analysing interaction in the mathematics classroom on the basis of the framework of symbolic interactionism is a matter of interpretation. It is an interpretative effort to reconstruct, as in the case of my research work, *processes* of meaning making. How is meaning formed and negotiated in the process of interaction? How do actors collectively create mathematical meaning? In order to investigate the process of meaning making, every single *action* is interpreted extensively in the sequence of emergence. The analyst tries to generate as many alternative interpretations as possible. Thus, he or she opens up the range of potential ways of understanding and construing the action. In order to get hold of the process of *inter-acting*, actions are

considered to be related to each other. They are interpreted as turns to previous actions. Analysing turn by turn the process of meaning making can be reconstructed.

Actor-Network-Theory (ANT)

Latour (2005) poses the question who and what participates in the course of action. He criticises the established definition of action: If action is limited a priori to “what ‘intentional’, ‘meaningful’ humans do” (ibid., p. 71), objects have no chance to come into play. Instead, he recommends a broader understanding of action and agency. He defines that “*any thing* that does modify a state of affairs by making a difference is an actor” (ibid., p. 71). In doing so, he equips objects just as well as humans with agency. All actors, human or not, are “*participants* in the course of action” (ibid., p. 71). Thus Latour extends and modifies the list of actors assembled as participants fundamentally. He gives several reasons why ANT accepts objects “as full-blown actor entities” (ibid., p. 69). One is that the social world will “retain a sort of provisional, unstable, and chaotic aspect” if it was made of local face-to-face interaction. However, such temporary and fugacious interactions can become far-reaching and durable. Latour calls the “steely quality” (ibid., p. 68) of things to account for this durability and extension. What is new is, that objects are highlighted as actors that might “authorize, allow, afford, encourage, permit, suggest, influence, block, render possible, forbid, and so on” (ibid., p. 72). Latour does not give privilege; human as well as non-human participants in the course of interaction have agency. Latour refrains from imposing “some spurious asymmetry among human intentional action and a material world of cause relations” (ibid., p. 76). He denies loading things into social ties. Objects do not serve as a “backdrop for human action” (ibid., p. 72). Neither do they determine the interactional process; they are not the causes of action. But he does not propose some sort of equality either (ibid., p. 63; p. 76). Instead, he emphasises the varieties and differences in modes of action (ibid., p. 74ff.).

Doing research on mathematical education from an interactionistic perspective, the merge of ANT and symbolic interactionism might be a fruitful effort. Latour considers objects as actors contributing to the process of interaction in different modes of action. They participate in the process of meaning making, even though they have different options open. Concerning methodology, Latour preaches to “follow the actors” (Latour, 2005, p. 156) and “describe” (ibid., p. 144; p. 149). Blumer emphasizes that non-human objects as well as human activities have no intrinsic meaning. They do not carry an established meaning that has to be revealed. Meanings are formed in the process of interaction. Meaning making, according to Blumer, is a matter of interpretation. Symbolic interactionism serves as a point of reference for interpretative research trying to reconstruct the process of meaning making. Merging symbolic interactionism and Latour’s approach might help to bring the consuetudinary excluded objects into the course of interaction. It might contribute to the development of an object-integrating theory of learning in mathematical

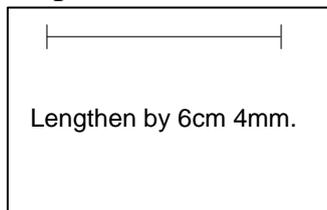
classroom interaction. Latour states with regard to interaction, that “the number and type of ‘actions’ and the span of their ‘inter’ relations has been vastly underestimated. Stretch any given inter-action and, sure enough, it becomes an actor-network” (2005, p. 202). But how do you investigate interactional processes if you consider objects as full-blown actors? How do you deal with the modified list of participants and with the increased modes of action? In the following paragraph, I propose an object-integrating approach on classroom interaction.

OBJECT-INTEGRATING APPROACH TO CLASSROOM INTERACTION

Empirically grounded development of an object-integrating theory of learning in mathematical classroom interaction includes the development of analytic tools, analysis of numerous scenes, and the comparison of interpretations to various scenes. Below, methodological thoughts are discussed as a basis for analysis of object-related classroom interaction and accordingly as a contribution to the development of an object-integrating theory of learning. To exemplify the methodological points of interest, a short episode taken from a third year German primary class is introduced (first published in Fetzer, 2007).

Example

In this scene the task is to lengthen a graphically given straight segment by 6cm 4mm (compare fig.). First the children work on the problem on their own. They are asked to put written notes on their problem solving process. Afterwards some children present their approaches on the blackboard. Sonja is the first to explain her proceeding. The teacher requests those students that “can’t follow anymore” to “ask what’s going on”. Sonja selects Sabina as next speaker. She says: “Somehow I don’t get it.” This last utterance will be the focus of investigation.



Person	Aktivität	Activity
Sonja	<i>Steht an der Tafel, schaut zur Lehrerin</i>	<i>Stands at the blackboard, looks towards the teacher</i>
Teacher	Die andern- da sind viele gewesen da kann ich mir vorstellen die komm jetzt schon nicht mehr mit- da müsst ihr auch mal fragen was da los iss- aber wenn die nich meinn sie müssten fragen erklär weiter-	The others- there have been many I can imagine who can't follow anymore- you have to ask what's going on then- but if they don't bother asking keep on explaining-
Sonja Sabina	<i>Schaut in die Klasse Sabina-</i> Ich kapier des irgendwie net-	<i>Looks towards the class Sabina</i> Somehow I don't get it-

First, I will give a ‘traditional’ analysis of the scene focussing on the verbal activities of the human participants. This brief analysis may serve as a basis for the subsequent theoretical and methodological discussion.

An extensional analysis of Sabina’s utterance in the last line of the transcript opens up a wide range of possible ways of understanding. Here only a small selection is

given. By stating “somehow I don’t get it”, Sabina perhaps intends to express that she could not follow Sonja’s explanation. On the one hand this could be a statement referring to herself and her own learning process. On the other hand her utterance could be understood as a statement concerning Sonja’s performance. In the context of the latter interpretation, Sabina would indicate that Sonja’s explanation was not comprehensible. Alternatively one might understand her utterance as an expression of her troubles in solving the given task. If so, her difficulties would not relate to Sonja’s explanation, but to the task itself. Eventually her utterance might be interpreted as a contribution to the classroom interaction in order to demonstrate alertness. In this case, the mathematical substance of her contribution could be minimal.

Who could Sabina possibly refer to? The turn-by-turn analysis basically reveals two alternatives: Sabina’s utterance could be understood either as a turn on Sonja, or alternatively as a turn on the teacher. Following the first interpretation, Sonja addresses Sabina and picks her as the next speaker. Sabina gets active and paraphrases the teacher by translating “can’t follow anymore” into “somehow I don’t get it”. In the context of this interpretation, Sabina would invest hardly any mathematical effort. According to the second understanding, Sonja might just as well get active as a turn on the teacher’s invitation “You have to ask what’s going on”. Again her utterance might be understood as a paraphrasing of the teacher’s “can’t follow anymore” (see above). Following this interpretation, not much mathematical content can be attested to her utterance. An alternative understanding would suggest that Sabina indeed could not follow Sonja’s explanation. She then actually belongs to those who were addressed by the teacher and were invited to get active. Again, Sabina takes the turn offered by the teacher. In the context of this latter understanding the mathematical content attributed to her utterance would be (slightly) increased.

On actors

According to an object-integrating approach to classroom interaction, not only humans but also objects have agency. This modified understanding of who and what acts in mathematical interaction entails a modified way of transcribing as demonstrated below.

Actor	Aktivität	Activity
Board	$5+6=11$ $4+7=11$ <i>[Tafelanschrieb bleibt während der gesamten Szene unverändert und sichtbar]</i>	$5+6=11$ $4+7=11$ <i>[Notes on the blackboard remain untouched and visible throughout the whole scene]</i>
Sonja	<i>Steht an der Tafel, schaut zur Lehrerin</i>	<i>Stands at the blackboard, looks towards the teacher</i>

Teacher	Die andern- da sind viele gewesen da kann ich mir vorstellen die komm jetzt schon nicht mehr mit- da müsst ihr auch mal fragen was da los iss- aber wenn die nich meinn sie müssten fragen erklär weiter-	The others- there have been many I can imagine who can't follow anymore- you have to ask what's going on then- but if they don't bother asking keep on explaining-
Sonja Sabina	<i>Schaut in die Klasse</i> Sabina- Ich kapier des irgendwie net-	<i>Looks towards the class</i> Sabina- Somehow I don't get it-

The first column indicates the interacting participants. It is captioned with 'actor' because the term 'person' solely refers to human beings and excludes other participants. The second and third columns give the actions in English and in German, differentiating verbal (regular font) and non-verbal actions (*italic font*). In contrast to 'conventional' transcripts, activities of objects are included as well. In the illustrating scene, for example, the notes on the blackboard are highlighted in grey.

Who and what participates in the given scene? Sonja, the blackboard, the teacher, and Sabina are actors in the scene. Besides, the children have their own written work at hand. Accordingly, Sonja's and/or Sabina's written approach might just as well enter into account. Working with an object-integrating approach to learning processes casts a different light on the selection of participants. The identification of the actors becomes more difficult for two reasons. Firstly, the fact that objects enter into account does not as a matter of course show in the restricted lines of a transcript. The reason is the time-spreading quality of things. Some-*thing* lying on the table like Sabina's written work or written on the blackboard as in the given example might not be mentioned in the specific scene selected for analysis. Nevertheless, board and written work might become participants within the course of action. Secondly, indicating participants in object-related classroom interaction is not a matter of fact, but a matter of interpretation. Some objects may be appraised as participants in one interpretation, but remain unconnected to the course of interaction in another interpretation. Regarding the interpretation on Sabina's utterance given above neither the board nor Sabina's work get connected to the interaction. However, understandings that take the blackboard as well as Sabina's work as actors can be reconstructed, if an object-integrating approach is applied. As a consequence, the selection of the actors of a given scene can always be no more than a *pre-selection*. Supplementary nominations of participants are likely to become necessary within the process of analysing. Accordingly, the pre-selection of participants should accept a wide range of possible actors. Concerning the example, Sabina's work should be at hand for analysis.

The selection of actors is one crucial point in implementing an object-integrating approach to classroom interaction. Another striking aspect is the matter of sequence and time span of participating. Who and what assembles as participants in the course of action might change very quickly. Especially non-human objects may enter into account one moment and recede into the background an instant later. They appear

associable with one another only momentarily. Analysis of interactional processes focuses on visible actions and the process of interweaving. Consequently, children, teachers or things can become the ‘object’ of analysis just as long as they leave an observable trace. If no trace is produced, no information is offered to the observer. If humans as well as things remain ‘silent’, they are no actors anymore. They remain unaccountable (Latour, 2005, p. 79). The written work on the table is not an actor. But Sabina and her notes might weave together and both become active participants in the interactional process as soon as Sabina picks up her sheet or has a glance on her notes. With Sabina and her written work entering the course of interaction, they may be captured by analysis. Interaction analysis based on the framework of symbolic interactionism takes a micro perspective and proceeds sequentially. Thus, intermittent existence and rapidly changing assembling of participants may be captured appropriately. But in the context of an object-integrating approach, durability and lasting time spans have to be considered as well. The blackboard might show Sonja’s notes for quite a while. Consequently it is a potential actor for a certain length of time. This abiding participating could be indicated in the transcript, for example, by implementing an additional column.

On modes of action

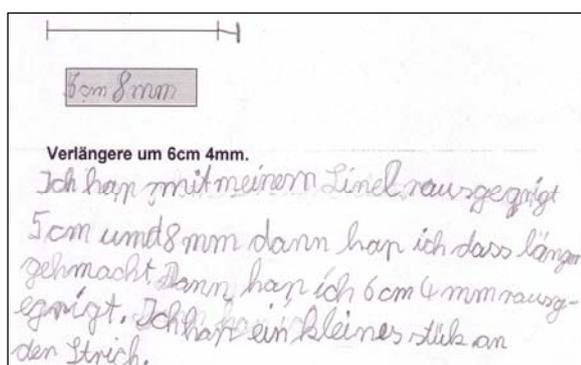
Investigating processes of teaching and learning in mathematics education actions are analysed in their order of emergence (see above). The analyst generates as many sensible interpretations to the given action as possible in order to expand the range of potential understandings. Reconstructing the process of meaning making in the context of ANT widens the spectrum and modes of actions under investigation. Both, human and non-human actions have to be analysed. However, analysing non-human’s actions on the first glance appears to be a bold venture. How can an object’s agency be interpreted? In order to investigate the object’s agency one may firstly explore the object itself ‘nakedly’. What does this object tell the analyst, what does it remind him of? What might it express, suggest, allow, forbid, enable, etc.? This mode of analysis compares to a methodical dodge often applied in analysing human action: the variation of the interactional context. The action is taken out of the given context and conveyed into another. This is an established proceeding in interaction analysis in the theoretical framework of symbolic interactionism. What is new is to implement the variation of the context to objects and their activities. This analytic move raises the analyst’s awareness and sharpens his or her analytic senses when it comes to interpret the object’s actions. This is possible as soon as objects get visibly connected to other participants in the course of action. Once they become associated with one another, their action might be captured by analysis. With Sabina glancing on her notes, the written work becomes a participant in the interaction. It is no longer a sheet of paper on the table, but a tangible link between now and earlier. It is a concrete backing of argumentation or a means of distraction. As an actor, the written work in front of Sabina might demonstrate alertness, or it might assign her to be the current speaker. The assumption that objects have agency, too, widens the range of observable

actions. Consequently, the analysis of the sequentially emerging action must be implemented to human as well as to non-human actors' activities.

Interesting enough an object may well be there unaltered or untouched for a couple of minutes or half an hour. In the selected example this applies for both, the blackboard and for the written work(s). Their 'steely quality' persists, although objects just momentarily enter into account, and become active only from time to time. In the context of the traditional analysis of interaction we are used to focus on actions as momentary affairs producing visible or otherwise perceivable traces only here and now. Objects prompt the analyst to open the perspective. The potentially long lasting effect of an object's activity on classroom interaction has to be considered. The blackboard *is* there. Any participant might refer to the notes any time within the interaction. Thus the notes on the board become participants.

These theoretical thoughts have an impact on the method of analysis in the context of an object-integrating approach to classroom interaction. To illustrate the effects on the analysing procedure, the investigation presented above is adopted and supplemented accordingly. Subsequently, the blackboard and Sabina's work are explored.

On the blackboard there are two number problems. Both are additions, both sums are eleven. Due to a lack of space, again, only a selection of possible interpretations is given. The two lines seem to refer to an arithmetic problem. They might for instance be related to each other by the mathematical strategy of inverse changing of summands. Assuming that Sonja's notes are related to the given task on measuring and calculating lengths, the two sums might be read as operations with numerical values omitting the units (cm and mm). In this case, the two sums could be interpreted as short versions of $5\text{cm}+6\text{cm}=11\text{cm}$ and $4\text{mm}+7\text{mm}=11\text{mm}$. From a mathematician's point of view, this interpretation would give the written sums the touch of side notes. Taking a (weak) student's perspective, these two lines could be seen as the extract or the fundament of the problem: Plain numerical values, assorted by different values. One rather complex calculation with units is reduced to two simple arithmetic problems that can be managed easily. Anyhow, the blackboard displays an arithmetic problem. The geometric element of the graphically given straight segment does not show anymore.



Below the task (Lengthen by 6cm 4mm) Sabina's work says: "I found out with my ruler 5cm and 8mm then I have lengthened that Then I found out 6cm 4mm. I had a little bit to the line." (See fig.). Her work shows a rather geometric approach based on the idea of adding *up to* 6cm 4mm (instead of *lengthen by*). The little figure on the right hand side can be interpreted as the answer to

the given task; it is the missing bit to the requested length. The written text proves this interpretation valuable. The ruler is assigned to be the clue to the solving process. First, it serves to find out the length of the given line. Afterwards, it shows the gap between given and requested length.

On turns

In order to reconstruct the process of meaning making in mathematical classroom interaction according to symbolic interactionism, actions are understood as turns on previous actions. As soon as objects are accepted as actors in the ongoing course of interaction, not only the concept of ‘action’ has to be adopted (see above). The concept of ‘turn’ as originally introduced by Sacks (1996) has to be re-thought as well. In his book “Lectures on Conversation” he works on the subject of turn-taking and introduces the adjacency relationship if utterances are related to each other as turns (Vol. II, part 1, p. 41ff.). This utterance-based understanding of ‘turn’ does not meet the demands of interactions. It is not only verbal, but rather all sorts of activities that might be related to each other as turns. The teacher’s utterance might be interpreted as a turn on Sonja’s look at her. Sabina’s “Somehow I don’t get it” might be a turn on the written notes on the blackboard or her working sheet. As a consequence, in the context of an object-integrating approach to classroom interaction, I use the term ‘turn’ in a broader sense: Actions are interpreted as turns, if they are closely related to previous actions. The underlying concept of ‘action’ is closely linked to ANT. It includes different modes of actions carried out either by human beings or by objects. If the concept of action and turn is extended in this way, analysis on the basis of the framework of symbolic interactionism will serve as an appropriate method to reconstruct object-related classroom interaction. Objects and things will be integrated into the course of interaction again. To me, re-thinking the concept of turn is the decisive approach in investigating object-orientated classroom interaction. It is the adopted understanding of turn that helps to trace object’s activities. On the level of turns objects leave observable marks and become visibly connected to one another. Human as well as non-human actors get involved as soon as it comes to think about possible relations between actions as turns.

Analysis on the basis of the adopted concept of turn may work as presented below. Again I refer to the example “Somehow I don’t get it.” In addition to the interpretations suggested above, I now propose an interpretation taking Sabina’s action as a turn on her own written work. Sonja presented her arithmetic proceeding to the task, based on the idea of adding two specific lengths. She did it in a convincing way, and Sabina could follow well. Consequently, she remains silent when the teacher asks those, who got in trouble, to become active. However, looking onto her written work causes confusion. Two different approaches, yet both convincing, show neither conformance nor consensus. The ideas of lengthen *up to* on the one hand and lengthen *by* on the other hand seem incommensurate. The geometrical and the arithmetic approach simply won’t merge. According to this

interpretation, the utterance “Somehow I don’t get it” appears to be a mathematically spoken reasonable statement. The last line of the transcript can be interpreted as a mathematically substantial statement. Its mathematical relevance is closely connected to the two objects, blackboard and written work.

ANALYSING OBJECT-RELATED CLASSROOM INTERACTION

Based on the presented outline of an approach to object-integrating interaction in the mathematic classroom, I will eventually point out some key points concerning the related method of analysis.

The identification of the actors in the scene to be investigated is an interpretative act. Thus, assembling of the list of participants is a pre-selection. In order to leave space for a wide spectrum of alternative interpretations, the list of (potential) participants should not be prematurely limited.

In order to maximize the range of possible interpretations to an observable action, the analytic dodge of variation of the context might be called on. This applies both for human as well as non-human actions.

Actions are related to each other as turns. On the one hand, actions are interpreted as turns on previous human-actors’ actions. On the other hand, actions are explicitly related to non-human actions that may be perceived in distinct ways. How could a certain action be interpreted if it was a turn on an object-participant’s action? Performing such an object-integrating turn-by-turn analysis prevents from accidental neglect or premature exclusion of objects as actors. However, the list of participants might need reassembling or supplementation in the context of this analytic move.

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THE EXISTENCE OF MATHEMATICAL OBJECTS IN THE CLASSROOM DISCOURSE

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In this paper we are interested in the understanding of how the classroom discourse helps to develop the students' comprehension of the non ostensive mathematical objects as objects that have "existence". First, we examine the role of the objectual metaphor in the understanding of the mathematical entities as "objects with existence", as well as in some of the conflicts that the use of this type of metaphor can provoke in the students' interpretations. Second, we examine the mathematics discourse from the perspective of the ostensives representing non ostensives that do not exist.

INTRODUCTION

In this report we present some findings from our current research on the role of objectual metaphors in the interpretation of the existence of non ostensive mathematical objects within the classroom discourse. We illustrate these findings with a reinterpretation of data from Acevedo (2008). In particular we analyze certain remarks of different teachers that have in common the use of metaphors in their teaching practices. In that study, the fourth author presented an analysis of some teachers' discourses while teaching the graphic representation of functions in Spanish high schools. The focus was on the teachers' discourses and practices when interacting with the students in certain lessons. The main data was gathered by means of video and audio tapes, together with written tests, students' work and filed notes.

We organize the report from theory to example in order to deal with language and communication issues in mathematics classrooms from a semiotic point of view. We begin by briefly reviewing part of the literature on metaphors and presenting the notions of image schema and conceptual metaphor, which are drawn on the theories of the embodied cognition. When introducing some findings, we show how the use of metaphorical expressions of the objectual metaphors in the teachers' discourses leads the students to understand the mathematical entities like "objects with existence". Finally, we show how the mathematics discourse on ostensives representing non ostensives that do not exist and on the identification of mathematical objects with some of its representations, leads the students to separately interpret the mathematical objects and its ostensive representations.

IMAGE SCHEMAS AND METAPHORICAL PROJECTIONS

In recent years, several authors (see, for instance, Bolite, Acevedo & Font, 2006; Lakoff & Núñez, 2000; Núñez, Edwards & Matos, 1999; Pimm, 1981, 1987;

Presmeg, 1997) have pointed to the role of metaphors in the teaching and learning of mathematics, and some of them have reflected on the embodied cognition theory. Sriraman and English (2005), in their survey of theoretical frameworks that have been used in mathematics education research, talk about the importance of the embodied cognition theory. On the other hand, the discursive emergence of mathematical objects is interpreted as a research focus within that theory. Sfard (2000, p. 322) has stressed some of the metaphorical questions concerning the existence of the mathematical objects:

To begin with, let me make clear that the statement on the existence of some special beings (that we call mathematical objects) implicit in all these questions is essentially metaphorical.

We argue that the use of objectual metaphors in the mathematics classroom discourse leads to talk about the existence of mathematical objects. Our notion of objectual metaphor is highly related to the notions of image schema and metaphorical projections (Johnson, 1987; Lakoff & Johnson, 1980). The image schemas are basic schemas, in the middle of the images and the propositional schemas that help to construct the abstract reasoning by means of metaphorical projections. These schemas are constituted by multiple corporal experiences experimented by the subject. Some of these experiences share characteristics that are incorporated within the image schema. Both the experiences and the shared characteristics are a consequence of situations that have been physically and repeatedly lived.

Lakoff and Núñez (2000) claim that the cognitive structure for the advanced mathematical thinking shares the conceptual structure of the non mathematical daily life thinking. The metaphorical projection is the main cognitive mechanism that permits to structure the abstract mathematical entities by means of corporal experiences. We interpret the metaphor as the comprehension of an object, thing or domain in terms of another one. The metaphors create a conceptual relationship between an initial or source domain and a final or target domain, while properties from the first to the second domain are projected. In relation to the mathematics, Lakoff and Núñez distinguish two types of conceptual metaphors:

- Grounding metaphors: they relate a target domain within the mathematics to a source domain outside them.
- Linking metaphors: they maintain the source and the target domains within the mathematics and exchange properties among different mathematical fields.

Within the group of grounding metaphors, there is the ontological type, where we find the objectual metaphor. The objectual metaphor is a conceptual metaphor that has its origins in our experiences with physical objects and permits the interpretation of events, activities, emotions, ideas... as if they were real entities with properties. This type of metaphor is combined with other ontological classical metaphors such as that of the “container” and that of the “part-and-whole”. The combination of these types leads to the interpretation of ideas, concepts... as entities that are part of other

entities and are conformed by them. This interpretation is clear in the axioms of existence and link, as they are mentioned in a classical Spanish textbook on Geometry (Puig Adam, 1965, p. 4):

Ax. 1.1. We recognize the existence of infinite entities called <points> whose set will be called <space>.

Ax. 1.2. The points of the space are considered grouped in partial sets of infinite points called <planes> and those from each plane in other partial sets of infinite points called <lines>.

METAPHORICAL EXPRESSIONS OF OBJECTUAL METAPHOR

We consider it necessary to make a distinction between the metaphorical expressions and the conceptual metaphors, as highly interrelated but different ideas. This distinction permits to establish generalizations that, otherwise, would remain invisible. The metaphorical expressions may be grouped into conceptual metaphors, and seen as isolated, they can be thought of as individual cases of particular conceptual metaphors.

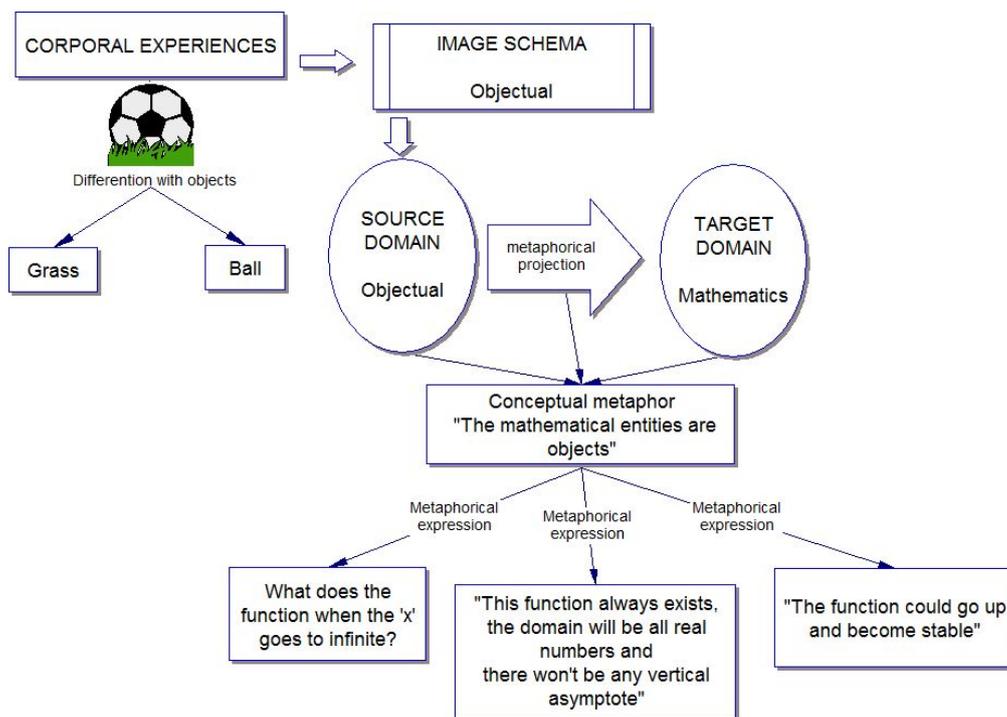


Figure 1. A representation of the objectual metaphor

The conceptual metaphor “The mathematical entities are physical objects” is a grounding ontological metaphor. Figure 1 (Acevedo, 2008, p. 138) illustrates the metaphorical projection with the different metaphorical expressions that appear when using this conceptual metaphor in a mathematics classroom where the graphical representation of functions is being taught to students in high school. Figure 1 shows our experiences in the world of things and the interpretation of the physical objects as separated from this world context; these experiences generate the “objectual” image

schema that become the source domain that is projected into the world of the mathematical objects. Table 1 refers to the source and target domains that intervene in the interpretation of this metaphor.

<i>“The mathematical entities are physical objects”</i>	
Source domain: Image schema	Target domain: Mathematical entities
Physical object	Mathematical object
Properties of the physical object	Properties of the mathematical object

Table 1. Domains of the metaphorical projection

THE OBJECTUAL METAPHOR IN THE TEACHERS’ DISCOURSE

The objectual metaphor is always present in the teachers’ discourse because here the mathematical entities are presented as “objects with properties” that can be physically represented (on the board, with manipulatives, with gestures, etc.). In Acevedo (2008), metaphorical expressions of the objectual metaphor occur when the mathematics teacher refers to the graphic of a function as an object with physical properties. When he talks about the application of mathematical operations in order to obtain the first derivative of a function, the teacher uses verbal expressions and gestures that suggest the possibility of manipulating mathematical objects as if they were things with a physical entity (Acevedo, 2008, p. 137):

Teacher1: The derivative of the numerator, no! You multiply by the denominator as it is, minus the numerator multiplied by the derivative of the denominator. Ok. Now you divide it by the denominator... square, it is. (...) This is the first derivative. Now, what’s next? To operate, to manipulate... What’s left?

The use of the objectual metaphor facilitates the transition from the ostensive representation of the object –written on the board, drawn with the computer, etc.– to an ideal and non ostensive object. Hence, the use of this type of metaphor leads to talk in terms of the “existence” of the mathematical objects. This use may lead the students to interpret that the mathematical objects exist within the mathematical discourse (internal existence) and, sometimes, may lead them to interpret that they exist like chairs and trees do (external existence, physical or real). In Acevedo (2008, pp. 136-137), we first find a classroom discussion on the domain of the logarithm function and later a discussion on the domain of the square root function, during the instruction of the graphical representation of functions. Here the “existence” is considered within the language game of the mathematical discourse, in comparison to the former teacher’s comments on the existence of the first derivative of a function:

Teacher2: The domain goes from zero to infinite because logarithms of *negative numbers do not exist*, logarithm of *minus one does not exist*. Shall we say with the zero included?

Teacher2: Not the negative... because *the square root of a negative number does not exist*. We could also say the real numbers without the negatives, or even easier, all the positive numbers, we can write it like this, with an interval, from the zero to the infinite, now the zero is included.

If the teacher is not careful enough with the way of using (or not using) the verb “exist” in his discourse, the students in this class may not remain within an “internal existence” position. Instead, they may change the “language game” (Wittgenstein, 1953) and assume the “external existence” of the mathematical objects. In the following paragraph, a third different teacher explains the graphical representation of functions to the students in the class and explicitly mentions the idea of existence, although he does so in a rather controversial way (Acevedo, 2008, p. 137):

Teacher3: Then...this function always exists, the domain will be all real numbers and there won't be any vertical asymptote.

We observe a deviation in the “expected” use of the word “exists” within the language game of the mathematics discourse. It would be reasonable to affirm that the images of the values in the domain exist or are defined. When attributing the existence to the whole function instead of talking about its images, the teacher is making a use of the word “exists” that can lead to the understanding of the function as a “real” object with properties, like a chair or a person. Moreover, by doing so, the teacher can promote the movement from the mathematical internal existence of the object to a physical external existence.

DIFFERENTIATION BETWEEN OSTENSIVES AND NON OSTENSIVES

We draw on the theoretical distinction between ostensive and non ostensive objects as established by the onto-semiotic approach to mathematics education (Godino, Batanero & Font, 2007, p. 131):

Ostensive–non-ostensive Mathematical objects (both at personal or institutional levels) are, in general, non-perceptible. However, they are used in public practices through their associated ostensives (notations, symbols, graphs, etc.). The distinction between ostensive and non-ostensive is relative to the language game in which they take part. Ostensive objects can also be thought, imagined by a subject or be implicit in the mathematical discourse (for example, the multiplication sign in algebraic notation).

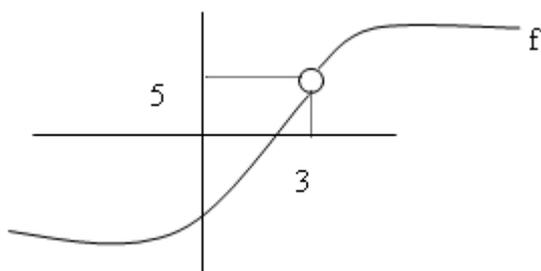
In the mathematics discourse, it is possible to talk about ostensives representing non ostensives that do not exist. For example, we can say that $f'(a)$ does not exist because the graphic of $f(x)$ has a pointed form in $x = a$. This gives us another example of the semiotic and discursive complexity of the classroom discourse when referring to the

existence of mathematical objects. In Acevedo (2008, p. 320) we find the following remark made by a teacher in his classroom discourse:

Teacher4: As you can see, the one-sided limits are not the same and then the limit does not exist... or the limit is infinite, I mean it is more or less infinite.

In García (2008, appendix 2, p. 8), we find a teacher who uses a discourse with ostensives ($f(3)$) that represent non ostensives that do not exist. He does not say that they do not exist but literally says that “we cannot have them”. The instances from García’s research were obtained in a similar methodological setting –in regular high school classrooms focused on functions and graphs–, to that constructed for the study that was developed by Acevedo.

Teacher 5: [...] Let’s imagine this function:



What is the domain of f ? [He answers on the board $\mathbb{R} - \{3\}$]. And $f(3)$? Don’t make the mistake of saying five, because it is not in the domain and we cannot have an image. We are not worried about $f(3)$, but about going as closer as possible to three, before and after the three. Attention, where are the images? Now I don’t have a formula.

Students: Near the five.

Teacher 5: And now if I get closer to three on the right, where are its images?

Students: Over the five.

Teacher 5: Yes we can say limit of $f(x)$ when x goes to three.

Students: But $f(3)$ does not exist.

Student: But the asymptote does not touch it either.

Teacher 5: It is curious but $\lim_{x \rightarrow 3} f(x) = 5$ [on the board]. It is not defined in three but its limit does exist. That limit exists without having the analytical expression and without having $f(3)$.

In order to talk about the existence of certain non ostensives, we have to use a discourse with ostensives constituted in accordance to the “grammar” that regulate the construction of the well-established formulas. This type of discourse is frequently used by many students, as the following remark shows (Acevedo, 2008, p. 368):

Student: Then you do the same here, well you first put the zero here because it is... you look for it, it is the number that you have obtained and the derivative is zero. Then in minus one and in one, you also have to write a zero, but as you have vertical asymptotes you can say that the derivative does not exist, neither does it exist the function. Then you do it with minus one and zero and you get a negative, with the same procedure, and then with the zero and the one you get a positive. As it is positive, it means that you have a minimum here because you have this drawing and it is a minimum.

The use of ostensives that represent non ostensives that do not exist may create confusions in the students' thinking, although it also can turn into philosophical implicit reflections for them. This is the case with a student (Acevedo, 2008, p. 213) that makes a distinction between "to be" and "to exist". He misunderstands the vertical asymptote and makes a mistake:

Teacher5: Could you explain a bit more about the vertical asymptote?

Student: I understand that the vertical asymptote is the value that does not exist in the function.

The existence of well-established ostensives that represent non ostensives that do not exist facilitates the consideration of the non ostensive object as something different from the ostensive that represents it. Duval's work (2008) has pointed to the importance of the different representations and transformations between representations in the students' understanding of the mathematical object as something different from its representation.

Many textbooks of mathematics, implicitly or explicitly make the students observe that an object has many different representations and it is needed to distinguish the object from its representation. In a popular Catalan textbook (Barceló et al., 2002, p. 89), for instance, the following is written:

In all the activities made, you have been able to observe the different ways of expressing a function: as a statement, as a table of values, as a formula and as a graphic. You always have to remember these four forms of representation and know how to go from one to another.

However, these textbooks frequently tend to identify the mathematical object with one of its representations. In the same Catalan textbook (Barceló et al., 2002, p. 90), it is said "Given the function $f(x) = 1/x \dots$ " The explanation is that the representation is identified with the object or differentiated from it depending on the purpose. Peirce (1978, §2.273) mentions this idea in his work:

To stand for, that is, to be in such a relation to another that for certain purposes it is treated by some mind as if it were that other. Thus a spokesman, deputy, attorney, agent, vicar, diagram, symptom, counter, description, concept, premise, testimony, all represent something else, in their several ways, to minds who consider them in that way.

In the mathematical practices, we constantly identify the object with its representations and, on the other hand, we make a distinction between the object itself and some of its representations. The rules of this language game, where the objectual metaphor is crucial, may be difficult to learn for some students. When we deal with physical objects, we can differentiate the sign from the object (for instance, the word “watch” and the physical object “watch”). The objectual metaphor as it is used in the mathematics discourse permits to transfer this differentiation to the mathematical objects and, therefore, we also differentiate the “representation” from the “mathematical object”. Moreover, the type of discourse that we produce within the mathematics classroom, leads us to infer the “existence” of the object as something independent from its representation. This situation let us conclude about the existence of a mathematical object that can be represented by means of different “representations”.

FINAL REMARKS

In this report we have argued that the objectual metaphor plays a central role in the pedagogical process in the classroom, where teachers (and, consequently, the students) talk about mathematical objects and physical entities. We have shown how the use of metaphorical expressions of objectual metaphors in the mathematics classroom discourse leads the students to interpret the mathematical entities like “objects with existence”. On the other hand, the mathematics discourse about ostensives representing non ostensives that do not exist and about the identification (differentiation) of the mathematical object with one of its representations leads the students to interpret the mathematical objects as being different from its ostensive representations. As a consequence, the classroom discourse helps to develop the students’ comprehension of the non ostensive mathematical objects as objects that have “existence”.

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MATHEMATICAL ACTIVITY IN A MULTI-SEMIOTIC ENVIRONMENT

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Abstract: Different semiotic systems provide different sets of resources for the construction of mathematical meanings. In this paper, we argue that a multi-semiotic environment not only affords rich potential for developing mathematical concepts but may also affect more fundamentally the goals of student activity. We present a multimodal analysis of an episode from a teaching experiment with software that allows students to construct animated models using equations. In the course of this short episode, the students made use of drawing and gesture as well as mathematical and everyday speech in ways that transformed the purpose of their activity from drawing a static pattern to constructing an animation, changing the mathematical problem from using velocities to determine the direction of motion to considering how to stop a moving object.

INTRODUCTION

The study of mathematical language and other sign systems has developed in recent years with increasing recognition of the importance of a variety of specialised mathematical systems, including graphical and diagrammatic forms as well as linguistic and symbolic (Alshwaikh, 2008; O'Halloran, 2005), and of interaction between the various systems (Duval, 2006) in the development of mathematical discourse. Moreover, where mathematical communication takes place in face-to-face contexts, body language and gesture also play a part (see, for example, Bjuland, Cestari, & Borgersen, 2007; Radford & Bardini, 2007). The development of new modes of representation through the medium of new technologies has generated further interest in this area by opening up possibilities for dynamic forms and for interactions between systems (such as graphs and algebraic equations) in ways that were previously inaccessible.

From a social semiotic perspective (see Morgan, 2006), each semiotic system provides a different range of meaning potentials (Kress & van Leeuwen, 2001). For example, as O'Halloran argues, visual modes such as graphs allow representation of 'graduations of different phenomena' rather than the limited categorical distinctions available through language or algebraic symbolism, while dynamic modes additionally allow the representation of temporal and spatial variation (2005, p.132). Such different potentials have been exploited in the design of interactive learning environments (for example, Yerushalmy, 2005) and research from various theoretical perspectives has focused on the kinds of mathematical meanings constructed by students working with such novel representations, especially in the contexts of use of dynamic geometry (for example, Falcade, Laborde, & Mariotti, 2007).

In this paper we report a teaching experiment, involving a multi-semiotic interactive learning environment, MoPiX, produced as part of the ReMath project [i]. This environment and the associated pedagogical plan were designed to provide multiple linked representations to support students' development of concepts of velocity and acceleration [ii] by allowing them to experience and connect formal symbolic definitions and dynamic animations. We report elsewhere how the semiotic resources provided by this environment appear to support students' development of ways of operating with velocity and acceleration compatible with their formal definitions and with Newtonian laws of motion (Morgan & Alshwaikh, 2008, 2009). Here, however, we discuss the influence of the multi-modal environment on the process of problem solving, presenting an example of an episode in which interaction with the various available semiotic systems transformed the goals of the activity.

A MULTI-SEMIOTIC ENVIRONMENT

The interactive learning environment of MoPiX allows users to construct animated models and investigate their behaviour. It is conceived as a constructionist toolkit (Strohecker & Slaughter, 2000), providing fundamental elements (in this case objects, represented by shapes such as squares or circles, and equations) with which students can build models, form and investigate hypotheses by activating their constructions and observing their behaviour. The environment of MoPiX is essentially multi-semiotic, linking symbolic representations (equations) using a variation of standard mathematical notation, with animated models and graphs. In addition, the planned pedagogy of the teaching experiment, the social environment of the classroom and the nature of the technology (individual tablet PCs) were intended to encourage use of a range of modes of communication, including talk, gesture, various paper-and-pencil representations and the electronic sharing of constructions through the ReMath portal [iii]. The variety of semiotic systems provides a range of meaning potentials and hence rich opportunities for users to construct meanings for the mathematical objects and concepts represented.

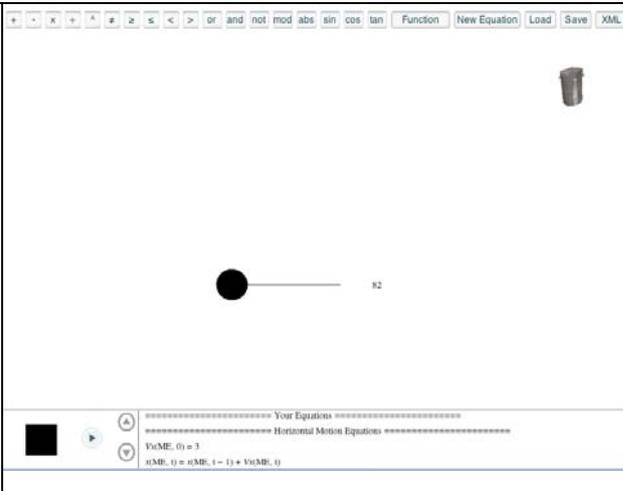
<p>$x(\text{object_1},t)=x(\text{object_1},t-1)+Vx(\text{object_1},t)$ <i>x-coordinate of the circle (object_1) is augmented by the value of Vx as time (t) increases</i></p> <p>$Vx(\text{object_1},0)=3$ <i>variable Vx, assigned an initial value of 3 (when time=0), may be considered the velocity of the circle</i></p> <p>$Vx(\text{object_1},t)=Vx(\text{object_1},t-1)+Ax(\text{object_1},t)$ <i>Vx (velocity) is augmented by the value of Ax as time (t) increases</i></p> <p>$Ax(\text{object_1},t)=-0.1$ <i>variable Ax, in this case assigned a value of -0.1, may be considered the acceleration of the circle</i></p>	 <p>The screenshot shows the MoPiX interface. On the left, there is a control panel with buttons for navigation (back, forward, search, etc.) and a menu with options like 'or', 'and', 'not', 'mod', 'abs', 'sin', 'cos', 'tan', 'Function', 'New Equation', 'Load', 'Save', and 'XML'. The main area displays a horizontal motion diagram with a black circle on a line. Below the diagram, there is a status bar showing 'Your Equations' and 'Horizontal Motion Equations' with a list of equations: $VxME, 0) = 3$ and $xMB, 1) = xMB, 1 - 1) + VxME, 1)$.</p>
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Figure 1: A set of equations defining horizontal motion

A MoPiX object is caused to move by applying a set of parametric equations defining how its position will change over time. For example, the set of equations shown in Figure 1 would cause object_1 (the circle in the screen shot) to move in the horizontal direction with an initial velocity of 3 and constant acceleration -0.1 [iv]. Horizontal and vertical components of motion are defined separately. The notation thus draws attention to vector concepts of velocity and acceleration, while the form of the equations embodies the definitions of velocity as change in position and acceleration as change in velocity. Equations may be taken from a library of basic equations, edited or authored directly and applied to objects. Once equations have been added to one or more objects, the model may be played and each object in the model will move according to its own set of equations. (It is also possible to apply equations defining interactions between two or more objects.) Visual feedback from the animated model allows students to test their hypotheses about the functioning of the equations they have used. They may then continue their investigations: editing the sets of equations and adding new objects to their model.

THE TEACHING EXPERIMENT

A pedagogic plan was devised, in collaboration with teachers in a London tertiary college, with the educational goal of developing understanding of ideas of velocity, acceleration and force. A group of seven students (aged 17-18 years) volunteered to participate in the study, which took place during 10 weekly one-and-a-half hour sessions outside the normal curriculum. The participants were all enrolled in an Advanced level mathematics course. They had not previously studied the mathematics of motion (though some had studied physics) and, though all were familiar with the formal definitions of velocity and acceleration as rates of change, a pre-course paper-and-pencil questionnaire revealed that they nevertheless relied on informal non-Newtonian intuitions in order to describe and explain motion. Participation in the study was presented to the students as extra preparation for the Applied Mathematics (Mechanics) module that they were to start the following term.

The intended pedagogy was founded on constructionist principles, providing students with access to the means of manipulating the elements of the MoPiX microworld while posing challenges that would encourage them to experiment, shaping their own goals and hypotheses. The episode we consider in this paper is taken from the second session. During the first half of this session, the students had been given a worksheet with a sequence of tasks introducing them to the equations needed to create straight line motion, to the idea that the direction of motion is determined by a combination of velocities in the horizontal and vertical directions and to the equations for drawing a trace of the motion of an object. Having done the set tasks, they experimented in a playful way with these and a range of other equations taken from the MoPiX equation library, creating multi-coloured objects moving in various ways, not only in straight lines. They then had their attention drawn to the next task on the worksheet: '*As a group, plan a design formed by several lines.*' In designing this challenge, it was

anticipated that students would make use of the combination of horizontal and vertical motions to make objects move in different directions drawing straight lines with different gradients, thus developing their appreciation of relationships between components of motion in two dimensions.

DATA ANALYSIS

During the teaching experiment we gathered data in the form of video and audio records of pairs of students, together with any incidental paper-and-pencil work. In addition we administered paper-and-pencil pre- and post-questionnaires. Our broad research aim was to investigate how students would make use of the semiotic resources offered by MoPiX and the broader classroom environment in the course of their work on tasks related to motion. We were particularly interested to see what contribution the various resources might make to students ways speaking about and operating with ideas of velocity and acceleration.

Extracts of video were identified as ‘of interest’ and were transcribed. In accordance with our research focus on multiple semiotic resources, extracts chosen for transcription included, in particular, those where several modes of communication were being used together. We consider the form of transcription to be part of the analytic process as a preparation for the multi-semiotic analysis needed to address our research questions. The use made of each mode of communication was thus recorded in a separate column of a spreadsheet, allowing both horizontal (a snapshot of all simultaneous semiotic activity at each ‘moment’) and vertical (an overview of semiotic activity within a particular mode through the whole period of the extract) examination of the data. The transcript was divided into ‘moments’ of communication that were considered to have some meaningful coherence; this division was a pragmatic consideration with no explicit theoretical basis.

Our approach to analysis involved both the application of a priori categories and the iterative definition and refinement of categories derived from the data. In the episode discussed below, we discuss the interaction between mode of communication (an a priori categorisation) and the goal of the students’ activity (a strand of analysis arising from our exploration of the data). The episode is a five-minute extract from about half way through the second session, focusing on two male students, Baz and Vin as they start to work on the design task.

CM if two of you think about a pattern maybe with some parallel lines and perpendicular lines and a number of lines to make some sort of a pattern on the screen. Yeah? And design that in advance and then one of you does some of the lines, the other does the other set of lines and then you combine the two to make the whole pattern. Yeah? So you might want to do some pencil paper work first. think about your design

Vin Do you have a pen?

Baz Just use the computer

Vin Yeah.. in Paint [*this refers to the Paint drawing programme on the PC*]

- Baz } [*laughing together*] yeeeah.. Paint
 Vin }
 Vin Bring it over
 [... *about a minute trying to find the Paint programme on an unfamiliar PC*]
 Baz Here we go. All right so we can do the horizontal lines and vertical lines.
 Vin Can't we do the diagonal ones
 Baz We can do squiggly lines, but
 Vin Like in our thing, if it has a formula, then it's not going to be random is it
 Baz Yeah exactly
 Vin Do a log [*i.e. logarithmic function*] actually you can't do log because it'll get kind of mad because it'll go on for ever
 Baz You can have different colours right [*both laugh*] so make it like a firework so it goes like that and then you could have vertical ones like that and diagonal ones and another horizontal, I mean vertical one going even further up
 Vin like a sparkler
 Baz yeah but we need it to start from here and then these start after this one and then .. I don't know how that'll work

We originally identified the extract for detailed transcription and analysis because it seemed interesting for two reasons. In the first place, the students chose to make use of the *Paint* programme on their PCs, thus providing us with an opportunity to consider how they were making use of the various modes of communication available to them. Secondly, the mathematical nature of the problem they were working on and the focus of their MoPiX programming task changed through the course of the episode.

Strand 1: Mode.

This strand of analysis was identified as a fundamental component of our social semiotic theoretical framework and of importance in addressing our research questions. It was initially defined by a priori categories. Each moment was first coded according to the mode or modes in use. The initial categories used were:

- spoken language (subdivided into everyday/ mathematics/ MoPiX registers)
- written language (natural language/ conventional mathematics notation/ MoPiX notation)
- drawing (outcome of MoPiX animation/ aid to problem solution)
- gesture (pointing/ mimicking MoPiX motion/ other)
- MoPiX equations (library/ authored/ complete models)

During the coding process, however, it became clear that this categorisation was not sufficient by itself to capture the ways in which the meanings produced during the extract were realised using the available semiotic resources. In particular, the functional relationship between the various modes used in any moment appeared significant. For example, Baz, creating the initial design, used simultaneous words and drawing (see Table 1). The initial causal connection 'so' made by Baz between

the possibility of using *colours* and the decision to make the design ‘like a firework’ draws attention to the significance of the semiotic potential of the available technology. Both the *Paint* programme the students had chosen to use instead of paper-and-pencil and MoPiX itself afford easy application of a range of colours. It seems that the availability of colour as a resource suggests representational possibilities that might not have been chosen when working with traditional tools.

<i>spoken language</i>	<i>drawing (in Paint)</i>
Baz You can have different colours right [both laugh] so make it like a firework	
so it goes like that	draws vertical bottom to middle twice
and then you could have vertical ones like that	horizontal middle to left; horizontal middle to right twice
and diagonal ones	3 diagonals: middle to NW; middle to NE; middle to SW
and another horizontal, I mean vertical one going even further up	vertical middle to top

Table 1: Interaction of speech and drawing

There is a direct congruence between Baz’s words (*spoken -mathematics*) and his *drawing*; as he speaks the word ‘vertical’, he draws vertical lines (although he initially confuses vertical and horizontal). In addition, however, the motion of drawing (*gesture*) mimics the imagined motion of the firework (*spoken -everyday*) thus combining use of the static meaning potential of the descriptive language - vertical, horizontal, diagonal - and the completed drawing (displaying the outcome of the intended MoPiX animation) with the dynamic meaning potential of gesture.

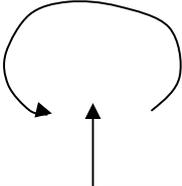
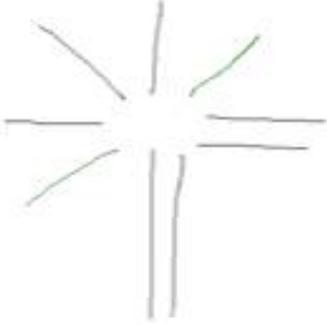
<i>spoken language</i>	<i>gesture</i>	<i>drawing</i>
Vin like a sparkler		
Baz yeah but we need it to start from here and then these start after this one and then I don't know how that'll work	slide-pointing bottom to middle, then slide-pointing anticlockwise circle around the perimeter of the whole shape 	

Table 2: Interaction of speech, gesture and drawing

In the next moment (see Table 2), Vin echoes Baz's original everyday discourse identification of the design as a firework, now specifying it more concretely as a sparkler, then Baz uses *gesture* to interact with the now complete *drawing*, simultaneously verbalising the process needed to construct the design with moving objects (*spoken -MoPiX*). In this case, the students use the drawing mode as readers, producing new meanings for the drawing through their use of spoken language and gesture. The spoken language naming of the design as firework/ sparkler here provides a holistic (everyday) image of the outcome of the design, while Baz's simultaneous use of language and gesture affords a dynamic representation of the development of the animated design over time.

Strand 2 Goal of the design activity: static versus dynamic outcome

In order to capture the complexity of the relationships between modes in use in any moment, the coding was developed to take account of the changing nature of the design activity. This strand of analysis was developed after initial examination of the whole extract, emerging as a theme from the data. It was observed that the ways in which the participants talked about their pattern included attending both to the properties of the lines drawn as traces of the MoPiX animation (a static outcome) and to the properties of the motion itself (a dynamic outcome). At the beginning of the chosen extract, the task is introduced by the teacher/researcher, using what we have now characterised as a static representation of the goal of the task:

think about a pattern maybe with some parallel lines and perpendicular lines and a number of lines to make some sort of a pattern on the screen.

This static goal is taken up initially as the students discuss the types of lines they might make using MoPiX (horizontal, vertical, squiggly, defined by a formula). By the end of the episode, however, the focus of the activity is related to the motion of objects needed to construct the pattern. This focus was not the anticipated task of coordinating horizontal and vertical components of motion in order to draw lines with particular gradients. Rather, the students identified an important new goal that influenced the progress of their work through the remainder of the session: to find a way of stopping a moving object. This proved a substantial problem for them as its solution demanded a more analytic use of MoPiX equations than they had developed up to that point, in particular the use of equations specifying values of velocity or acceleration at a given time.

The question thus arises as to why this change from static to dynamic goal may have occurred. We coded references in any mode to the pattern or to components of the pattern as static or dynamic, identifying for each reference the mode and the indicators used to apply the code. Through this process of coding it became apparent that significant moments in the students' developing image of their pattern occurred as they moved between different modes of representation (see Table 3). In particular, the naming of the pattern as a 'firework' (apparently influenced by the articulated recognition of the possibility of using colour in their design), and interaction using

gesture with the drawing of their design introduced new semiotic resources with meaning potentials that highlighted dynamic aspects of the design.

(i)	The original MoPiX programming challenge focuses on the direction of lines: “parallel”, “perpendicular”.	written and spoken language - mathematics	static
(ii)	Vin discusses the need for mathematical formulae to define MoPiX motion.	spoken language - mathematics; MoPiX programming	static
(iii)	Vin introduces of the idea of using a formula involving ‘log’ and the dynamic idea that it will ‘go on forever’, perhaps invoked by a concept image of a logarithmic graph (note O’Halloran’s (2005) identification of the dynamic meaning potential of mathematical graphs).	spoken language - mathematics; imagined graph?	static - dynamic
(iv)	The use of Paint or perhaps the use of MoPiX enables the suggestion to use different colours.		
(v)	This suggestion then seems to trigger the naming of the design as a “firework”.	spoken language - everyday; imagined dynamic object	dynamic
(vi)	The firework idea is realised in Paint.		
(vii)	Interaction with this drawing through gesture introduces a temporal aspect.	drawing; gesture	dynamic
(viii)	This temporal aspect is taken up immediately by Baz's verbal description of the motion "we need to start from here and then these start after this one"	drawing; gesture; spoken language - MoPiX	dynamic
(ix)	The MoPiX programming challenge then becomes the problem of how to make motion stop.	MoPiX programming	dynamic

Table 3: Change from static to dynamic

CONCLUSIONS AND DISCUSSION

The analysis we have offered here has focused on the multiple modes of communication used by these two students. Not only does each mode have its own set of meaning potentials but the different modes also interact, providing further potential. The complex interaction of use of language, drawing, gesture and MoPiX programming thus contributes to the construction of new meanings in the communication between the two students. The new semiotic resources provided by

MoPiX play relatively little explicit part in the episode we have considered. Nevertheless, we would argue that they play an influential role in shaping the students' activity, not only because the overt goal of the task involved use of MoPiX but also because the students were influenced by their recent use of MoPiX and their awareness of its potential. Moreover, the technological environment and the students' familiarity with its capabilities enabled them to choose to use Paint and its colour resources rather than traditional monochrome paper-and-pencil.

The resources afforded by gesture have been identified as significant in the move from a static to a dynamic goal. We consider here not only the pointing gestures accompanying the deictic spoken language seen in Table 2 but also the bodily movement implicit in the act of drawing in Table 1. This draws attention to the duality of the *drawing* mode: it is both a product - the outcome or picture - and the process by which the outcome is produced. In different moments it thus has both static and dynamic meaning potential and may play an important part in shifting focus between the two types of meaning.

However, the change from a static to a dynamic focus for the students' problem solving activity was not solely a product of the multi-semiotic environment. The nature of the pedagogic discourse of the classroom also played an important role. In particular, the students had enough agency within the classroom to enable them to make decisions about their own activity. In the first place, they were able to decide to ignore the teacher/researcher's suggestion to use paper-and-pencil, choosing to use Paint instead. Further, they were able to follow their own interests in designing their firework, thus enabling the change in the focus of their attention. Indeed, at a later stage in the same lesson, the teacher/researcher worked with this pair to help them solve the MoPiX programming problem of making a moving object stop, using techniques whose introduction had been planned for a later lesson.

Our analysis of this episode illustrates the very complex space of communication and learning and, we hope, contributes to Kress's call for development of theory of learning from a social semiotic perspective (Kress, 2008). The focus of students' attention and the direction of their learning are shaped by the multi-modal resources available and the interactions between them. However, this takes place within a learning environment that affords and/or constrains students' agency and their ability to change the direction of their activity in ways that will be considered legitimate.

NOTES

ⁱ ReMath (Representing Mathematics with Digital Technologies) funded by the European Commission FP6, project no. IST4-26751.

ⁱⁱ MoPiX also has potential to be used in many other areas of mathematical modelling.

ⁱⁱⁱ MoPiX version 1 is available at <http://remath.cti.gr>; version 2.0 is under development at <http://modelling4all.nsms.ox.ac.uk/>

^{iv} Units are non-standard and not identified explicitly in the notation.

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ENGAGING EVERYDAY LANGUAGE TO ENHANCE COMPREHENSION OF FRACTION MULTIPLICATION

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Dedicated to the memory of the Cypriot teacher Georgia Kyriakidou

Using as analytic frames the Pirie-Kieren model and theoretical constructs on the role language and communication could play in the process of learning, I attempt to sketch the pathway of understanding of a sixth-grade student (Avgusta) while she is attempting to make sense of fraction multiplication. The viewing of mathematical understanding as a dynamic process proved supportive in enabling me to identify the role language could play both at any level and in the growth between levels of Avgusta's understanding. Occasioning learners to fold back to everyday language in order to collect the spontaneous interpretation of the word "of" and combine it with the scientific notation of multiplication could awaken learners' awareness that the interpretation of multiplication involves finding or taking a part of a part of a whole.

INTRODUCTION

The story to be recounted here evolves in a public elementary school in Cyprus, where I work as a full-time teacher. It is part of a two-year research studying the complexities of learning to compute fractions as revealed from the use of a novel peda-cultural tool. Though in Cypriot culture school mathematics textbooks introduce the concept of fraction with images of partitioned rectangles and circles, they make little or no use of diagrams when they show students the way to compute.

During the first year of the study I was the teacher of a fifth grade class (10 boys & 12 girls) and had to address all subjects' objectives set by the curriculum. Once a week, I took the role of a teacher-researcher and taught students how to learn fractions through manipulating diagrams. To be consistent and learn from my experiences I revisited my group of students a year later and conducted individual interviews in order to collect some retrospective evidence about the nature of their learning. It is the purpose of this paper to zoom in on one of those interviews and describe how one girl, Avgusta, could derive meaning in multiplication of fractions. Worthy of consideration is that in sixth grade my ex-students had been exposed to a different teacher's instructional mode which gave no emphasis on diagrams as a learning tool.

This study is of interest because it refers to an educational culture unused to use diagrams to compute fractions and more used to show and tell than to getting learners to make sense by using the diagrams as mediating tools. Its contribution lies in

offering Avgusta's learning as grist for the learning and development of other pupils, beyond the local boundaries of the particular school.

THEORETICAL BACKGROUND

The role of language in learning and particularly the social role of other people in the development and use of language was explicitly stressed by Vygotsky when he emphasized the importance of getting students talking about their thinking in order to help them make sense of, or construct, mathematical meaning. Vygotsky also observed that there are differences between what pupils can achieve working alone and what they can achieve when assisted by someone more experienced, such as a teacher. He captured this in a phrase which in English is usually rendered by "zone of proximal development" (Vygotsky, 1978). This term suggests that the teacher wants to support awareness that is imminent but not yet available to learners and not do those things which learners can do, since this will only raise dependency. Bruner (as cited in Wood et al., 1976) while presenting Vygotsky's ideas in English, made use of the metaphor "scaffolding" to refer to the assistance that a teacher some time may offer, which can be gradually withdrawn as students are able to function independently. The critical part of scaffolding is its removal or fading because when the support has not been removed, pupils may become dependent upon the teacher or any employed pedagogical tool (Love & Mason, 1992).

Zack (2006) appears in synch with Vygotsky's and Bruner's observations when she claims that because "students use sophisticated reasoning but may not see the power in the reasoning they are doing", it might be useful if teachers could "revisit what students have said, and connect their talk with the ways in which a mathematician would express those ideas" (p. 211). Linking everyday and scientific ways of knowing in order to support learners' imminent awareness is, according to Zack (1999), a much more challenging task than most researchers have appreciated.

The Pirie-Kieren theory and its associated model [Figure 1] is a well-established and recognized tool for listening and looking at growing understanding as it is happening. Growth in understanding is seen as a dynamical and active process involving a continual movement between different layers or ways of thinking, with no implication of a linear ladder-like system. These layers, which are intentionally represented in the form of eight nested circles so that the accent is put on the embedded nature of understanding, are named Primitive Knowing, Image Making, Image Having, Property Noticing, Formalising, Observing, Structuring and Inventising. A critical feature of this theory is the act of returning to an inner layer, or re-visiting and re-working existing understandings and ideas for a mathematical concept. This act is called "folding back" (Pirie & Kieren, 1989). A slightly differentiated form but equally important to folding back is that of "collecting". Its major difference from folding back is that, in collecting, the inner level activity does not involve a modification (or thickening) of the individual's earlier understandings. Instead, learners' efforts are concentrated on finding and recalling what they know

they need to solve a task. They are consciously aware that this knowledge exists but their understanding is not sufficient for the automatic recall of it (Pirie & Martin, 2000).

METHOD AND METHODOLOGY

Avgusta, 12 years old when the interview was conducted, was one of the twenty two students participating in the study. I have chosen to present here selected pieces of her responses to a scenario on multiplication [Table 1], as well as explanations of these responses. By choosing particular moments and voicing them through a temporal sequence, I aim to convey not only a succession of Avgusta's learning experiences but also how she experienced this succession. What counts is not only the content and structure of the practice itself but also the ways in which it is talked about, perceived and assimilated by the learner.

When the principal of the school entered the classroom and asked the children what they were doing, they replied that they were learning how to multiply fractions. Then the principal asked who could come up to the board and show to her how to find the product $\frac{2}{3} \times \frac{1}{2}$ without performing any calculations but using only the area models. Orestes wrote the following on the board but the principal did not seem satisfied. If Orestes asked for your help, what would you say to him?

$$\frac{\overset{3}{1}}{2} + \frac{\overset{2}{2}}{3} = \frac{3}{6} + \frac{4}{6} = \frac{7}{6}$$



$$= \frac{7}{6}$$

Table 1: Interview scenario

Using as analytic frames the Pirie-Kieren model for the growth of one's understanding, theoretical constructs on the role language and communication could play in the process of learning, as well as personal reflections on pedagogy, I shall attempt to map the growth of Avgusta's understanding. Throughout the analysis, my specific goal is to explore her thinking "in-change" and how this is accomplished and shared. In other words, how shifts in Avgusta's thinking occur and in what ways such shifts in thinking supported her understanding of the meaning of multiplication.

Taking the position with Doerr and Tripp (1999), I argue that shifts in thinking could be described in terms of an initial interpretation of the task situation and a later interpretation that stands in opposition to the initial interpretation. It is sensible to assume that somewhere between the two interpretations there will be evidence of

what precipitated the change in Avgusta's thinking. For this reason, attention will be cautiously focused on the sequence of events between initial and later interpretations, as well as on identifying those characteristics that illuminate the growing understanding of Avgusta throughout the interview.

INTERVIEW FINDINGS

The conversation I had with Avgusta about the multiplication scenario [Table 1] is the focus of this section. The quoted transcript has been intentionally split into three parts each of which has a distinct subheading. This division is absolutely artificial and it does not imply any linearity in the girl's growth of understanding. Rather, it is meant simply to organize structurally the data and facilitate the development of discussion later on.

Avgusta's tenacious-but-futile struggle to recall and apply a half-remembered algorithm in order to shed meaning to the procedure of multiplying fractions

What really strikes me here is Avgusta's "trapped" awareness of the falsehood of her actions.

507 Interviewer: Would you like to write down what Orestes [Table 1 - scenario on multiplication] should have done?

508 Avgusta: Yes.

[Avgusta is drawing the first and second figure of sheet 5. See Table 2 below, read left to right, up to down direction].

Sheet 5	Sheet 6	Sheet 7

Table 2: Avgusta's handwritten notes

- 509 Interviewer: What are you doing now?
- 510 Avgusta: The two thirds. He takes the two. Then... times one half. He takes the one and then we reverse them. No, I did it wrong.
- 511 Interviewer: Why?
- 512 Avgusta: I should have done it like that, a line.
- 513 Interviewer: How about doing it below to see what you mean?
[Avgusta is drawing the third and fourth figure of sheet 5 – Table 2]
- 514 Avgusta: Like this.
- 515 Interviewer: Yes?
- 516 Avgusta: We will reverse them and...we will reverse them.
- 517 Interviewer: Why?
- 518 Avgusta: To find...to find the same number of small boxes...to do them common fractions.
- 519 Interviewer: Okay, you could do whatever you think Avgusta and we will see.
[Avgusta is drawing the fifth and sixth fig of sheet 5 – Table 2]
- 520 Avgusta: We will reverse them.
- 521 Interviewer: Okay.
- 522 Avgusta: The two thirds...we will bring the one half...one minute...this one and then we will do times....We will reverse the one half and...
- 523 Interviewer: And what do we have now?
- 524 Avgusta: The small squares are now the same.
- 525 Interviewer: Yes?
- 526 Avgusta: But we have...
- 527 Interviewer: What do you have there?
- 528 Avgusta: Four sixths and here three sixths.
- 529 Interviewer: Yes.
- 530 Avgusta: And it becomes twelve sixths [She writes it at the bottom of sheet 5 – Table 2]
- 531 Interviewer: So is this your answer?
- 532 Avgusta: I think it is wrong.
- 533 Interviewer: Why do you think so?
- 534 Avgusta: [pause]
- 535 Interviewer: Would you like to tell me why do you think it is wrong?
- 536 Avgusta: But I don't know sir.

An invocative intervention aimed to occasion the link between everyday language and multiplication notation

The point that merits attention here is that Avgusta's folding back to everyday language could open the door for her to notice fractional symbols from a lens, which in turn could affect her way of thinking.

569 Interviewer: Okay. Now I would like to ask you something else. What does “times” mean? For instance, when we say one half times one hundred, what does that mean? You may write it down if you want.

[Avgusta is writing on the top of sheet 6 – Table 2]

570 Avgusta: We will multiply one half times one hundred.

571 Interviewer: Yes. Could you not say “we multiply”? How about our everyday language? Will you say one half times? Or, do we use any other word?

572 Avgusta: The word of?

573 Interviewer: How about saying it to see what you mean?

574 Avgusta: One half of one hundred.

575 Interviewer: That is? What does it mean? One half of one hundred is what?

576 Avgusta: Fifty.

577 Interviewer: Could you tell me Avgusta what does one half mean?

578 Avgusta: They are two and we are taking the one.

579 Interviewer: Nice. If I had one fourth, what does that mean?

580 Avgusta: There are four and I take one of them.

Educating awareness through encountering conflicting results and detecting the origin of the conflict

After Avgusta had been exposed to the foregoing intervention, she worked on the examples $\frac{1}{3} \times \frac{2}{5}$ [Table 2 – sheet 6] and $\frac{2}{6} \times \frac{1}{5}$ [Table 2 – sheet 7]. Lines 720-759 are indicative of what had been exchanged between me and Avgusta later on. Of great importance here is the gradual refinement of the girl’s awareness of what it means to multiply two fractions, and the restructuring of ill-defined algorithmic knowledge.

720 Interviewer: Which way from the two, do you think, could help a child to understand what multiplication means? If you show him that you should multiply the... But, first, Avgusta do you know how we could multiply two fractions?

721 Avgusta: Yes, don’t we do them common fractions?

722 Interviewer: Could you show me the example two thirds of one half, with the way of area models?

[Avgusta is drawing the second figure of sheet 7 – Table 2]

723 Avgusta: We will do the one half, we will take the one and then we will divide it in three...vertical ones and we will take the two.

724 Interviewer: Would you like to shade again what are you going to take?

725 Avgusta: These here [She shades again the two left small squares of the top row of the second figure of sheet 7 – Table 2].

726 Interviewer: Could you now tell me which your result is?

727 Avgusta: Two sixths.

- 728 Interviewer: Right. Earlier Avgusta we had this example again, it was on sheet 5 [Table 2]...and you found what?
- 729 Avgusta: Twelve sixths.
- 730 Interviewer: You found twelve sixths and now you found two sixths. Which of the two is the correct one? Earlier you said that when we multiply we do the fractions common ones, didn't you?
- 731 Avgusta: Yes.
- 732 Interviewer: Here [He points to sheet 5 – Table 2] you did common fractions, didn't you?
- 733 Avgusta: Yes.
- 734 Interviewer: You did two thirds, four sixths, and one half, three sixths. And what did you do then?
- 735 Avgusta: I did it times.
- 736 Interviewer: Could you explain a bit more?
- 737 Avgusta: I did four sixths times three sixths.
- 738 Interviewer: And how much did you find?
- 739 Avgusta: Twelve sixths.
- 740 Interviewer: How did you find twelve?
- 741 Avgusta: Four times three.
- 742 Interviewer: And how about six?
- 743 Avgusta: Because the denominators are...
- 744 Interviewer: But here [He points to sheet 7 – Table 2] how much did you find?
- 745 Avgusta: Two sixths.
- 746 Interviewer: Which of the two is the correct one?
- 747 Avgusta: This one, the two sixths.
- 748 Interviewer: Could you tell me why?
- 749 Avgusta: [pause]
- 750 Interviewer: You saw it here Avgusta, didn't you? Whereas there [He points to sheet 5 – Table 2]?
- 751 Avgusta: I didn't see it.
- 752 Interviewer: What should you have done here [He refers to sheet 5 – Table 2], do you think?
- 753 Avgusta: The same with this one [She points to sheet 7 – Table 2].
- 754 Interviewer: So, how do we multiply Avgusta? Do you see here [He points to sheet 5 – Table 2]? There was something wrong. When we multiply two fractions, we multiply the numerators...
- 755 Avgusta: And the denominators.

DISCUSSION

Avgusta's main difficulty seems to be a dependence on a half remembered algorithm. The way she manipulates the rectangles she drew [Table 2 – sheet 5], her rapid but purposeful shift from solely vertical to both vertical and horizontal type of partitioning [lines 507-518], as well as the multiplying of the numerators of the newly

formed common fractions [lines 527-530], all could suggest that her understanding of multiplication is compartmentally drawn upon a vague memory of the standard change-into-common-denominators rule.

The ability to produce a partition of a partition in the service of finding the product of $\frac{2}{3} \times \frac{1}{2}$ might not be straightforward to Avgusta because it entails the composition of the operator “ $\frac{2}{3}$ of” and the operator “ $\frac{1}{2}$ of”. This idea is complex because it is removed from the whole number knowledge that learners could employ when first introduced to a single operator, such as “ $\frac{1}{2}$ of”.

In lines 532-536 Avgusta is observed to express concerns about the correctness of her actions but is failing to exemplify the origin of this uncertainty, at least in the short term. This could indicate that after using diagrams, Avgusta pauses and reflects by considering what it is that the results tell her. It is possible that while checking against her intuitions that the results seem to be reasonable and roughly what she expects, the girl encountered an internal conflict which, in turn, generated doubt. Avgusta’s assertion that she knows that something went wrong [line 532] but does not know what [line 536], catches my attention and opens the possibility that I could provide for her some cognitive “scaffolding” (Wood et al., 1976) to support, and perhaps transform that state. There was a sense of her having, and being aware that she has the necessary understandings but that these are just not immediately accessible.

One of my enduring questions, thus, while interviewing Avgusta [lines 569-580] was in regard to the role I could play in pulling to the forefront of her mind the “Primitive Knowing” (Pirie & Kieren, 1989) that was going to be the basis for locating the source of perplexity. My intention was to encourage the girl to keep in touch with her personal way of knowing mathematics and sustain a back and forth movement, not unidirectional, between that understanding and the conventions of the culture. It is for this reason I occasioned [lines 569-580] Avgusta to “fold back” (Pirie & Kieren, 1989) to everyday language, “collect” (Pirie & Martin, 2000) the spontaneous interpretation of the word “of” and combine it with the scientific notation of multiplication. This invocative intervention resulted in the student returning to an inner, more localized layer of understanding, which, in turn, seems to have given rise to a succession of “Image Making” activities (Martin, 2008). The handwritten notes on sheets 6 and 7 [Table 2] are indicative of the replacement of faded images of multiplication by meaningful diagrammatic illustrations linking recursive area partitioning with the respective symbolic notation.

It is of great importance to stress here that it is the response of Avgusta to the particular intervention that determined the actual nature of it, namely, to occasion folding back to existing understanding, searching for, finding and then remembering this understanding (Martin, 2008). If the girl did not assign herself the everyday meaning of the word “of” to “x” or “times” [lines 569-576], it is ambiguous whether Avgusta would awaken her awareness that the interpretation of multiplication

involves finding or taking a part of a part of a whole. Standard multiplication symbols appear, hence, not mere marks on paper for her but become manageable and confidence-inspiring so as to be used in further manipulation.

After successfully re-collecting the image she needed and through experiencing a series of Image Making activities [Table 2, sheets 5-7], the last of which was centered on the same example she worked on at the very beginning, Avgusta noticed a conflict between the two images she had constructed for the product of $\frac{2}{3} \times \frac{1}{2}$. This discerned contradiction [lines 728-747] between $\frac{12}{6}$ [Table 2 – sheet 5] and $\frac{2}{6}$ [Table 2 – sheet 7] is likely what occasioned Avgusta to reject her initial way of using diagrams and revise her existing Formalizing level of understanding by re-structuring the procedure of multiplying two fractions [lines 748-755]. Figure 1 is an attempt to illustrate by means of the Pirie-Kieren onion model (Pirie & Kieren, 1989) the pathway of Avgusta's growth of understanding. Based on my observations, this is seen to grow in a non-linear way: from the Primitive Knowing layer to the Image Making and Image Having layers. Then, evidence exists of folding back to the Primitive Knowing in order to collect an earlier understanding to use it anew at the Image Making layer. Avgusta seems to reach the Formalizing layer having first gone through the Image Having and Property Noticing layers.

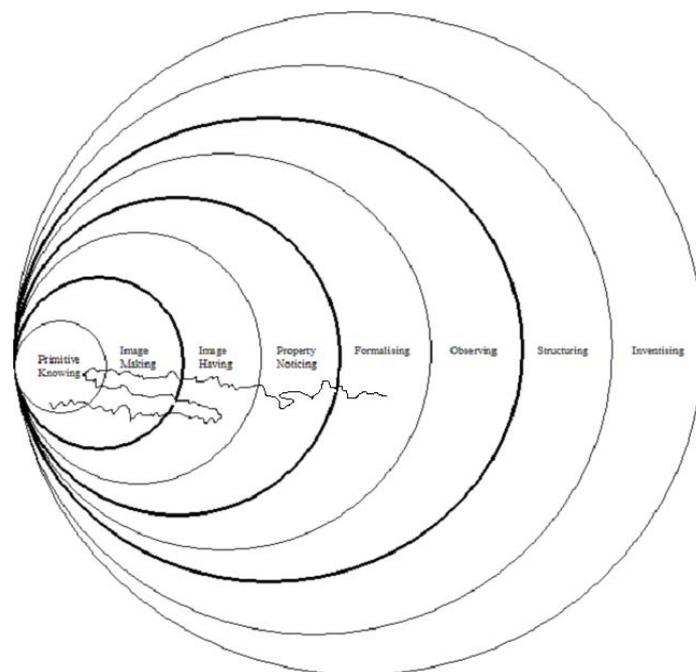


Figure 1: Avgusta's growth of understanding

The case of Avgusta comes to question the generalization of the assumption that once the meaning of a mathematical concept has been discussed, explained, formally articulated in class and students have at one time proven fluent with the corresponding algorithm, then the learning of this concept has been accomplished and a degree of readiness has been achieved for more sophisticated ones (Rasmussen et al., 2004). The fact that Avgusta struggled with the idea of fraction multiplication that

had been taught to it while in fifth grade, neither speaks of a teacher's nor of a learner's failure per se. Rather, it points to the need for teachers to occasion students to re-encounter ideas that they already have, in a different light or in relation to unfamiliar circumstances.

The viewing of mathematical understanding as a dynamic process proved in the current study supportive in enabling me as a teacher-researcher to identify the roles language and thought could play both at any level and in the growth between levels of Avgusta's understanding. If, as in the case of Avgusta, the student needs to activate a link between everyday language and mathematical notation, then in order to allow that student to progress in making sense, occasioning –not imposing– an awareness as to what to collect could be of assistance.

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TENSIONS BETWEEN AN EVERYDAY SOLUTION AND A SCHOOL SOLUTION TO A MEASURING PROBLEM

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This paper reports on an empirical study from a mathematics lesson in a Norwegian 4th grade classroom. The pupils are making batter for waffles, and the mathematical challenges are mainly connected to measuring. The paper will focus on the process of determining the correct amount of milk for the batter and furthermore on the tension that can be observed in the discursive practice as a result of the pupils' and the teacher's conflicting goals.

THE CLASSROOM SITUATION

This study is done in a group of 20 4th grade pupils in a Norwegian primary school in a mathematics lesson. During the lesson the pupils come in groups of five to the kitchen area in the back of the classroom where they make batter for waffles that are going to be prepared later the same day and eaten by themselves and the rest of the 4th graders at the school. Each group is supposed to make an equal amount of batter based on a recipe that is written on a poster. Before starting the actual work with the batter each group had a discussion where the task was to find out how much of each ingredient they would need in order to make three times as much as indicated on the recipe. The teacher expressed to me that her main mathematical focus with the waffle making was the discussion about the three folding. I will not report on this discussion but I will go into the part of the working process where the pupils are actually going to measure out 15 dl of milk. The milk comes in boxes marked “1/4 liter”, and the pupils have measuring beakers available that can take 1 litre. The beakers are transparent, with a scale reading “1 dl, 2 dl, 9 dl, 1 lit” from bottom to top. Each group has to determine the number of boxes needed to get the correct amount of milk.

THEORETICAL BACKGROUND

The notion of a complex mediated act goes back to Vygotsky (e.g. 1978) and has led to the idea of sociocultural artefacts that mediate between stimulus and response. Such artefacts can take many forms and they shape the action in essential ways (Wertsch, 1991). In mathematics the tools are often signs and symbols that represent an abstract mathematical concept, and the signs and symbols also often refer to a context or a specific object. A sign typically has two functions, a semiotic function – something that stands for something else – and an epistemologic function as the sign contains knowledge about that what it stands for (Steinbring, 2005).

One of the pioneers of semiotics is the American mathematician and philosopher Charles Sanders Peirce (1839-1914). He defines the terms involved in his triadic model of semiosis in the following way.

A sign is a thing which serves to convey knowledge of some other thing, which it is said to stand for or represent. This thing is called the object of the sign; the idea in the mind that the sign excites, which is a mental sign of the same object, is called an interpretant of the sign. (Peirce, 1998, p. 13, emphasis in original)

Peirce describes three kinds of signs (or representamens), *icons*, *indices* and *symbols* referring to three ways the representamen is related to its object. An icon stands for its object by likeness, an index stands for its object by some real connection with it or because it makes one think about the object, whereas a symbol is only connected to the object it represents by habit or by convention (Peirce, 1998, pp. 13-17, 272-275).

Presmeg (2005) turns the triadic model of semiosis into a nested model. This nestedness is based on the idea that the totality of the triad (representamen, object and interpretant) becomes reified (Sfard, 1991) as a new object to which one can assign a representamen and an interpretant. This gives a nested chaining of signs which can serve as a model to describe processes leading to more general or more abstract situations.

An important justification for mathematics in school is often the alleged usefulness of mathematics in other subjects and in situations outside of the school. It has been questioned whether it is possible to use a school subject such as mathematics outside of its own domain, and in this context it has been found fruitful to investigate the boundaries between the in-school and out-of school practices (Evans, 1999).

On areas where an overlap between in-school and out-of-school practices occurs it could be expected that there is some tension between the motives and goals lying in the school mathematics and the specific out-of school activity. To analyse this tension I will use the framework from activity theory. Leont'ev writes that *activity* is energised by a *motive*, and that “[t]here can be no activity without a motive” (Leont'ev, 1979, p. 59). Further he talks about the components of the activity as *actions* – processes that are subordinated to certain *goals*. On the third level there are the *operations* – the means by which the action is carried out. It is possible to carry out the same action by means of various operations, which means that the chosen operation “is defined not by the goal itself, but by the objective circumstances under which it is carried out” (Leont'ev, p. 63). Hence, the choice of operation may depend on the specific *conditions* in the given situation. It is henceforth possible to envisage one particular action but different operations that may be chosen depending on whether one is situated within a school practice or within an out-of school practice. According to Leont'ev the activity is driven by a motive, and the actions are directed towards certain goals. An important point is that each activity answers to a specific need of the active agent. “It moves towards the object of this need, and it terminates when it satisfies it” (Leont'ev, p. 59).

METHOD

I have been collaborating with all the teachers in grades 1-4 at this particular school for two years. This collaboration has involved working with the teachers in workshop activities, discussing in small groups and observing in classroom situations. When observing in the classrooms I have videotaped the activities going on. On some occasions parts of the videotapes have been shown and discussed with the teachers afterwards. Prior to the episode reported on here the teachers and I had been working with aspects of multiplication and division in a sequence of several workshops. We had agreed that on two given days in February I was going to videotape a session from each of the four grades 1-4. Each teacher, or group of teachers, was free to design the activities in accordance with the normal progression in the class. The only constraint was that it should have something to do with multiplication and division, or preliminary work leading up to these concepts. I did not partake in designing the lessons.

In the grade four class, which is the focus of this paper, the mathematics lesson was scheduled for two hours. I stayed in the kitchen area all the time, and with a hand held video camera I tried to capture as much as possible of the activity going on. During the lesson I was mostly passive but as can be seen from the excerpts of the dialogue I sometimes posed questions to the pupils.

THE HANDLING OF THE MEASURING PROBLEM IN EACH GROUP

Group 1

One measuring beaker is filled with flour, and Ellie is mixing flour and eggs. Lucy (the teacher) asks what they think is a good idea to do to avoid lumps, and they agree to start adding milk. James and Jessica fetch one box of milk each, and they agree that altogether they need 15 dl. Jessica looks at the box on which is written “1/4 liter”.

- 1.1 Jessica: This is one four litre
- 1.2 James: One four litre
- 1.3 Jessica: Yes, so we take one of these first. One whole of these
- 1.4 Lucy: How are you thinking now?
- 1.5 James: Have no idea
- 1.6 Jessica: Yes, it should be five
- 1.7 James: Yes, fifteen so now you must. We just say that this is one and a half
- 1.8 Jessica: It is one comma¹ five. No, we are supposed to take ... like this
- 1.9 Lucy: Emily, what do you think?
- 1.10 James: Now it will be two comma eight, now it is two comma eight if we take
- 1.11 Ellie: You are supposed to measure in the other decilitre measure

Jessica starts by looking at the text “1/4 liter” on the box but she and James do not have a clear sense of what this means and how it relates to the 15 dl that they know

they are supposed to have. In utterance 1.10 James states that the two boxes they have will be “two comma eight” which indicates that one box would be “one comma four”. It is not clear which unit this relates to, and it is also not clear what is the meaning of the words (two comma eight) that are spoken out. The teacher perceives what the pupils are saying as not correct and asks them what they are thinking. When they do not give a satisfactory answer she turns to Emily (#1.9) but she does not react to the question. Ellie comes to rescue by pointing to the existence of one more measuring beaker (#1.11). The existence of the second measuring beaker makes the meaning of “two comma eight” or “1/4 liter” redundant. After this Jessica and James are no longer interested in how much there is in one box, and the conversation that follows is about practical solutions, for example how to avoid lumps. The teacher also seems to be mainly interested in the practical solutions at this point.

After having put in the first litre of milk Jessica and James start to measure out another 5 dl. Jessica pours in one box, looks at the scale and says “three decilitres”. She does not seem to make any connection between the sign on the scale (level of milk being close to 3 dl) and the sign 1/4 liter on the box. Then she gets another box and gives it to Emily who asks “How much is it we need?” Jessica answers: “We had ten before and then we need fifteen.” Up to now I have not contributed to the discussion at all but at this point I ask a question which seems to shift the focus somewhat for the rest of the lesson.

- 1.12 Frode: How many decilitres are there in one of these? (Jessica looks at the box)
- 1.13 Lucy: How many decilitres are there in one box?
- 1.14 Jessica: It is one comma four litres. (Emily pours in the content of the box. Jessica looks at the scale.)

I suggest that they keep track of how many boxes they have used. They figure this out by counting the empty boxes but make no connection to the number of decilitres. I do not push this any further but Lucy repeats the question about how many decilitres there are in one box, and James answers:

- 1.15 James: One comma four
- 1.16 Lucy: One comma four?
- 1.17 James: One comma four litres.
- 1.18 Jessica: Yes, but she asked about decilitres.
- 1.19 Lucy: Is it more than one litre?
- 1.20 Ellie: No, it isn't. It is less. This isn't even half a litre.

As in the beginning of the episode 1/4 is read as “one comma four”, this time with the emphasis “litres”. Jessica realises that the question was about decilitres, and on Lucy's expressed doubt whether it could be more than one litre (#1.19), Ellie gives a practical estimate, stating that it is indeed less than half a litre (#1.20). After this I end the conversation on this topic suggesting that it might be better that they work on the batter.

The pupils in Group 1 make notice of the sign $\frac{1}{4}$ liter but they never develop a meaning of it. They also have no real need to find out what the sign means because they solve the practical task using the measuring beaker. The pupils answer the question about how many boxes they have used but they do not make any connection between the number of boxes and the number of decilitres.

Group 2

Also this group starts by looking at the milk box and the pupils pay attention to the text $\frac{1}{4}$ liter.

- 2.1 Chloe: One (looking at the box)
- 2.2 Chris: slash four, what does that mean?
- 2.3 Chloe: Four and a half
- 2.4 Chris: Four and a half
- 2.5 Chloe: And we need fifteen.

The teacher asks the same question as to the previous group about how much is in one box.

- 2.6 Chris: Four and a half
- 2.7 Lucy: Four and a half?
- 2.8 Chris: Decilitres. No, litres.
- 2.9 Lucy: Is it four and a half litres in here?
- 2.10 Chris: No, decilitres.

The answer is first given in terms of the number words only (four and a half), and when Lucy wants them to be more precise they hesitate a little between decilitres and litres but stick to litres (#2.8). To this Lucy expresses astonishment (#2.9), and Chris changes to decilitres. Lucy is still not satisfied, and she takes Chris and Matthew to the board at the other end of the room. Lucy writes $\frac{1}{4}$ on the board. She also draws a circle that she partitions into four equal sectors, and she fills one of the sectors. This evokes the concept “one fourth” in the children. Lucy links this to “one fourth of a litre” and asks how many of these go into one litre. This evolves into a discussion that moves between various issues; how many decilitres in one litre, how many boxes in one litre, how many decilitres in total, and how many boxes in total.

Group 3

Joseph and Thomas find the crate with the milkboxes and Joseph starts by asking how much one box is. Thomas says that it is a quarter of a litre. At first Thomas will not engage in Joseph’s thinking when he wants to find out how many boxes they need. Joseph asks Lucy if he may use the measuring beaker. Lucy encourages him to try without it and after a brief discussion he accepts this.

- 3.1 Joseph: Ohh. A quarter of a litre, that is ... a quarter ... ten decilitres is one litre. We have to have three of these then, then it will be. Five of these I think ... no not five. How much should we, Thomas, if we take three

of these, no four, then it is one litre and we want fifteen decilitres, and that is, and ten decilitres that is one litre. But how many more than four do we have to take then?

- 3.2 Thomas: Then we have to take four, and then we have to take ... two
 3.3 Joseph: Then we have two, and ten decilitres here. And then it is fifteen.
 3.4 Thomas: Yes.
 3.5 Joseph: Lucy, is this correct?

In turn 3.5 Joseph asks the teacher for reassurance of the solution, and then she makes him explain his reasoning. Joseph explains that four boxes equal one litre, and that two more boxes are two quarters which is equal to a half. Joseph and Thomas now state that they have one and a half litre which is the same as fifteen decilitres.

Group 4

Group 4 starts in the same way as Group 1 by pouring milk into the beaker. When they cannot find 15 on the beaker they decide that they have to split, and they choose to measure 9 dl first and 6 dl afterwards. They do not pay any attention to the number of boxes they use or to what is written on the boxes. When fetching the sixth box Katie says “it could be that it will be enough”. Grace looks at the scale saying “no, it is ... it is exactly enough”. Katie replies “yes, exactly. Good.” Lucy asks how many boxes they have used. Katie counts them and answers “six”. Again Lucy asks the pupils to figure out how many boxes they need without using the measuring beaker. The following dialogue takes place.

- 4.1 Grace: Put in three milkboxes ... no six
 4.2 Lucy: Yes, but why?
 4.3 Grace: (...)
 4.4 Lucy: Yes, because you know that now
 4.5 Grace: Yes.
 4.6 Lucy: Yes, but if you hadn't known
 4.7 Adam: Then we could have imagined having one like this (pointing to the measuring beaker)
 4.8 Grace: Then I could have walked home to get one

Lucy pushes them further and Katie asks how much is in one box. They come up with some suggestions, and I suggest that maybe something is written on it. They look at the box.

- 4.9 Hollie: There, one comma five.
 4.10 Katie: No, one comma
 4.11 Grace: Comma, this is a slash. One slash four litres.
 4.12 Lucy: What does that mean?
 4.13 Hollie: Haven't a clue.

Adam suggests “one fourth”, Lucy completes this to “one fourth of a litre” and goes on to ask how many they would need to get one litre. The pupils suggest that they need four fourths, and Lucy asks how many boxes that will be. They agree that this

will be four, and Lucy points to the original problem to explain why they need two more to get the correct amount of milk.

4.14 Lucy: Why do you need two more then?

4.15 Grace: To get six, no

4.16 Adam: To get three times as much

4.17 Grace: To get fifteen – fifteen decilitres

4.18 Lucy: Mmmm

4.19 Adam: Can we put in the flour now?

Lucy is pushing the issue further and wants to know how many decilitres there are in four boxes which she states to be equal to one litre. In the dialogue that follows answers like “four fourths”, “four decilitres”, and “four litres” can be heard. At the end Lucy holds up one box at a time and they count one fourth, two fourths, three fourths and four fourths. Lucy states that four fourths is one whole. The pupils add “litre” and Katie says “plus two more is one half”.

DISCUSSION OF THE EPISODES

The semiotic issues

Central to the task is the sign or symbol $\frac{1}{4}$ liter printed on the milk boxes. The pupils read the sign in various ways (one comma four, one slash four, four and a half) but many of them do not have a clear meaning linked to it. Groups 1 and 4 solve the measuring task completely by using a measuring beaker holding 1 litre. For these groups it is irrelevant to know the meaning of $\frac{1}{4}$ liter to solve the task. They relate to the fact that they need 15 dl of milk and by using the measuring beaker as a mediating tool (Vygotsky, 1978) they are able to get the correct quantity. When the teacher asks these two groups to figure out how many boxes they would need without using the measuring beaker they are facing a difficult problem. I interpret the teacher here to be working with $\frac{1}{4}$ liter as the representamen and the amount of milk in the box as the object. The teacher’s interpretant is that this is a fourth of a litre and that four boxes are needed to get one litre. The pupils are working within another triad where the representamen is the scale on the measuring beaker, an indexical sign pointing to the quantity of milk in the beaker as the object. The interpretant is the concept “fifteen decilitres” or “one and a half litre”, which they know that they need. I see the problem as having to do with creating a link between these two semiotic triads. As it is the symbolic sign $\frac{1}{4}$ liter is not seen as a representamen for the semiotic triad involving the measuring beaker. Since the pupils do not have a clear meaning of what $\frac{1}{4}$ liter means, the sign might just be an index connected to the box. In Group 3 the situation is quite different. The pupils make the connection between the sign $\frac{1}{4}$ liter and the amount of milk, and as a result they are able to identify $4 + 2$ boxes with one and a half litre.

In Group 2 the teacher physically moves from the kitchen part of the classroom to the opposite end where the blackboard is. She writes $\frac{1}{4}$ on the blackboard and also draws a circle partitioned in four sectors, filling one of them. Here the interpretant ‘one fourth’ is evoked in the pupils, and the teacher and the pupils seem to be working within the same semiotic triad, situated in a school practice. However, the sign $\frac{1}{4}$ is not seen as a representamen for the triad in which 1/4 liter is the sign, and therefore the link to the actual measuring problem is also missing in this case.

The sign $\frac{1}{4}$ is a symbol, clearly embedded in the school practice. The scale on the measuring beaker is an index, firmly based in the everyday practice. The sign 1/4 liter could be seen as a symbol representing the amount of milk in one box but for some of the pupils it might seem as if it is an index by its connection to the box, or a symbol with no interpretant. Based on this I identify three semiotic triads; the first where the scale is the sign, the second where 1/4 liter is the sign, and the third where $\frac{1}{4}$ is the sign. The everyday solution to the measuring problem is to pour milk into the measuring beaker until the indexical sign (the scale) points to 15 dl (seen as 1 litre + 5 dl, or 9 dl + 6 dl). The school solution could for example be to establish the relation $6 \cdot \frac{1}{4} = 1,5$ (litres) or $6 \cdot 2,5 = 15$ (decilitres). I have showed various attempts to create connections between these two practices. Based on Presmeg’s (2005) model I suggest that a nested chaining of the semiotic triads described above could establish a connection between the practices, and I have showed that lack of connection can be explained by lack of connection between the semiotic triads.

The discursive practice

Seen as a task from school mathematics the measuring problem could be formulated as follows. “Each milk box holds $\frac{1}{4}$ litre of milk. How many boxes are needed to get 15 decilitres of milk?” All four groups were able to find a solution to the practical problem of getting the right amount of milk, so indirectly they also know how many boxes of milk they need. Therefore they have all found the solution to the question in the imaginary school task, albeit not in a school like manner. I perceive the main *motive* for this lesson to be to produce batter for the waffles, and this determines the direction of the *activity* in the lesson. The activity consists of a number of different *actions* that can be linked to specific *goals*. Some of these actions can be carried out in a number of different ways, using different *operations*. The choice of operations depends on the *conditions* that are there at any given time (Leont’ev, 1979). My main objective in this section is to analyse the teacher’s and the pupils’ goals and actions in the lesson. My interpretation is that there is some tension between the teacher’s and the pupils’ goals, and that this tension is due to the fact that the lesson is operating on the border between a school practice and an everyday practice.

In Group 1 it seems that both teacher and pupils share the same goals in the beginning. The pupils (Jessica and James) have the idea to try to figure out how many boxes of milk they will need (#1.1-1.11). The teacher sees that their idea will not work and she tries to guide them or bring in Emily to help (#1.4 and 1.9) but when Ellie (#1.11) points to the fact that there is one more measuring beaker the teacher just lets them go on with the measuring without going any further into their interpretation of $\frac{1}{4}$ liter. The measuring beaker is the only tool they rely on to get the correct amount of milk. When I pose the question about how many decilitres there are in one box (#1.12), the situation changes somewhat. This question seems to bring in new goals that guide the teacher's action and in turn influences the pupils' goals. The teacher becomes more concerned about the mathematical content of the situation (e.g. #1.13). The fact that her attention to the mathematics appears after my question leads me to characterise her new goals as 'seeing the mathematics' and 'satisfying me'. The pupils do not relate this question to the work they are doing so their new goal can be expressed as 'answering the questions' or maybe 'satisfying the teacher'. They stick to reading $\frac{1}{4}$ as "one comma four" (#1.15), emphasising "litres" (#1.17). Ellie is aware that there is not more than one litre in one box, "[t]his isn't even half a litre" (#1.20), indicating a lack of meaning to "one comma four".

In Group 2 the process with the milk starts with the pupils reading on the box "one slash four" (#2.1-2.2) which they suggest means "four and a half" (#2.3), but they are not quite sure whether it is litres or decilitres (#2.8). With this group the teacher to a much larger extent goes into the role of the mathematics teacher, and she literally crosses the boundaries between practices by walking over to the blackboard at the other end of the room. In a funnelling pattern of interaction (Bauersfeld, 1988, p. 36) the teacher leads the group to a conclusion about how many boxes are needed.

Group 4 solves the whole measuring problem using the measuring beaker, thereby reaching their goal. It is only on the teacher's request that the number of boxes being used is brought into the picture. The pupils give an answer, because that is what is expected of them as pupils, but without enthusiasm. They have reached their goal, and they have no need to use any more energy on this. Each activity, here the measuring of the milk, answers to a specific need of the active agent, here getting the correct amount of milk for the batter, and when this need is satisfied the activity stops (Leont'ev, 1979). The answers of the pupils (some examples are shown in turns 4.14 to 4.19) indicate little interest. The numbers that come up can be connected to certain incidents throughout the process but not necessarily corresponding to the questions that the teacher asks. For example in turn 4.15 when Grace answers "to get six", she applies the fact that they used six boxes, which she already knows, but this is not in line with the hypothetical situation that the teacher has constructed. Towards the end the teacher leads the pupils via the question about how many boxes they need to get one litre. Even this evokes answers that indicate that the pupils do not engage in the problem.

I have shown that by operating on the border between practices, the mediating tools from the non-mathematical practice offer alternative possibilities for solving a task. The teacher, being pulled between the two practices, is seen to struggle in order to keep the pupils' motivation to solve the task in the mathematical context when they already have solved it in the practical context.

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¹ In Norwegian the sign for the decimal point is a comma. Since this sign is central in the interpretation of the dialogues I am using, I will keep the word 'comma', and I will also for example use the notation 1,5 instead of 1.5 which would be the standard English notation. Also when I directly refer to the text on the milk box I will use the Norwegian word 'liter' instead of 'litre'.

LINGUISTIC ACCOMPLISHMENT OF THE LEARNING-TEACHING PROCESS IN PRIMARY MATHEMATICS INSTRUCTION

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The linguistic accomplishment of a mathematics lesson, based on an illustrative example of an everyday lesson in a Hamburg fourth grade class, was analyzed via the person instructing. The linguistic accomplishment of instruction, for the purpose of analysis and with the help of qualitative procedures of interpretative classroom research of German mathematics education (Krummheuer/Naujok 1999), was analyzed on the basis of three hierarchical levels, developed from an existing theory. The results of these analyses grant on the one hand a hypothesis of the learning opportunities for a multilingual pupil body in German classes. On the other hand the results in the sense of local theory genesis can be integrated into a theory concept, which the author designates Implicit Pedagogy.

1 Introduction

If one looks into the classrooms of German schools, one notes that the pupil body is increasingly becoming shaped by multilingualism and various cultural backgrounds; currently, almost a third of all pupils in the German educational system hold a migrant background. Despite the increasingly linguistic and cultural diversity in German schools, instruction seems to be only slightly flexible and adapted to the needs of the diverse pupil population. Students with a migrant background or students who grow up in a semi-illiterate environment perform worse, according to the findings of international and national scholastic achievement tests, in comparison to their classmates who grow up in a monolingual German environment (compare the results of PISA 2000 and 2003, as well as IGLU 2003). It appears to be indisputable, that the origin of this poor performance is in a not insignificant manner to be found in an insufficient mastery of the language of instruction. However, these differences in the mentioned studies are often gladly categorized as unchangeable via school and their cause legitimized by the socio-economic background and/or language of the family. The goal of the article at hand is thus to demonstrate the underlying reasons for the poorer performance of students with a migrant background and/or who grew up in a semi-illiterate environment. The achieved results will then be subsequently explained with the assistance of theoretical approaches and in this manner demonstrate possible consequences or potential for change. On the basis of this, further studies may be able to develop concrete possibilities of how to fit instruction better to students affected by lingual-cultural plurality.

Linguistic accomplishment of instruction constitutes a substantial aspect of the adjustment of instruction to suit the needs of multilingual pupil bodies. In accordance with some approaches in the field of mathematics education, language and communicative competence both have a special significance for the learning of

mathematic content. Above all, Maier (compare e.g. 2006, 2004 and 1986) was concerned with research in the field of language and mathematics within German-speaking countries. Maier (2006) justified, that language holds a special relevance in Mathematics instruction, as objects in Mathematics, “... do not have a material nature and thereby are not accessible through the senses” (p.137, translated by the author). This consequently accounts for the significant focus of Maier’s works on the observation of technical terminology in Mathematics instruction. In the international community there are several authors who can be mentioned, who concern themselves with the relevance of language in the learning of mathematics. In the following, it should be initially referred back to Pimm (1987) who understood Mathematics as a social activity that is structurally and closely connected with verbal communication. From this, Pimm introduces the metaphor “Mathematics is a language?” (ibid, p.XiV) as a question of whether Mathematics could be evaluated not in the sense of a natural language, but as its own style of language. He compares, for this purpose, teachers as a role model of a “native speaker” of Mathematics and other people, for whom Mathematics appears to be incomprehensible, as per a foreign language, to which they are not empowered (ibid, p.Xiii).

The empiric material of the underlying research consists of transcripts from video recordings of an everyday primary lesson. The video recordings took place over a time period of four months in three classes of the fourth grade in two Hamburg primary schools with an approximate 80% migration contingent amongst its pupils.

In section 2 of this article, the analytical findings of the analysis of interactions within a selected instructional episode will be presented. In connection, a methodologic indexing of the procedure of the underlying research will be taken as preparation of further analysis. The selected episode will be used in section 3 as an illustrative example to demonstrate how lingual accomplishment of primary mathematics instruction lends itself to be described and analyzed with the here-accepted theoretical perspective. To this, three hierarchical levels are developed from this theory, by which the linguistic accomplishment of the lesson in the selected episode will be deeply analyzed. In section 4, the possible outcomes will be described, that yield from the results of the analysis to learning opportunities for pupils in German primary school classes. Furthermore, the results will be presented for the purpose of local theory development in a theoretical concept developed by the author from the entire research.

2 An Episode from the Lesson Sequence “LCM”

In the following a short transcribed episode of an everyday primary school mathematics lesson during the introduction of a new mathematic concept will be looked at.

2.1 Prehistory and Transcript of the Lesson Episode

At the beginning of the scene “LCM” Ms. Teichmann along with 25 female and male pupils, 17 of which have a migration background, are situated in the classroom. In

this lesson the introduction of a new mathematic concept should take place: the LCM- the Least Common Multiple.

It is Wednesday morning Ms. Teichmann asks initially what the abbreviation LCM stands for. Thereafter she allows the multiples to be calculated. Finally she draws two circles on the board, that she divides into four and three segments respectively, with an addition symbol between them and an equals sign. She marks for each circle one of the segments in pink. While one pupil very quietly says, “ $1/3$ plus $1/4$,” Ms. Teichmann asks the pupils which equation stands on the board. The pupils begin to guess and first give the answer, “1 plus 1,” or, “2,” and then somewhat later label the segment with $1/3$ and $1/4$. The teacher notes this in the drawing on the board and adjusts the fractions from $1/3$ and $1/4$ to $4/12$ and $3/12$. Several pupils offer many creative solutions for their addition, such as for example “ $2/7$ ”. In closing, her generalization of the procedure follows.

241	16:30	<L:	right/ you may not- add a large piece of pizza [points to the left circle]
242		>L:	and a small one and a smaller -.one together [points to the right circle]
243		L:	that is not equal right/
244		<L:	you must practically...
245			chop them into such pieces that they are equal\
246		<L:	[makes a chopping motion with her hand]
247		>L:	..right/these pieces are equal\ [points to the left circle]
248		<L:	[points to the right circle] These pieces as well\
249			only here it is less\ right/ here there are only three-
250		>L:	and here there are four pieces. [Points to the left circle]
251		S:	ah now I understand it
252	16:57	L:	and for that reason one need this\, if you at all want to (add) fractions-
253			so that you can add together such pieces of cake together\
254			right/one can not simply
255			say three and four is seven and from above
256			we will take two and then I have two sevenths\
257	17:11		Two sevenths is something completely different
258			no that doesn't work\

2.2 Concise Analysis of the Interaction

At the end of the episode the teacher attempts to show the pupils a generalization of the addition of fractions. She uses for this purpose the everyday example of the division of a pizza, respectively cake and makes the division of them visual through gestures. Hereby both levels of the illustration on the basis of the everyday and the generalization of the rules of fractional arithmetic meld together. This is shown in the statement by Ms. Teichmann in <252-258>. The reference to “LCM” seems to have been completely lost, or left as implicit. Alone the, “...and for that reason one needs this...” in <252> from Ms. Teichmann gives us the idea that there is still a reference to the “LCM”, since one needs an “LCM” in order to find the least common denominator for the addition of the two fractions. Ms. Teichmann does not further explain this connection. Also the final generalization by hand of the cake example <252-258> can barely be accounted for as a further clarification of the procedure, since Ms. Teichmann says that one may not simply add three and four together and means thereby apparently the denominators of one third and one fourth. Through the selected example, however, pupils did indeed have to add three and four in order to

ascertain the solution of the task – though, on the level of the numerator. They added $\frac{3}{12}$ and $\frac{4}{12}$. Moreover, the addition of the numbers three and four are everyday tasks for primary pupils in basic arithmetic operation. Why one may no longer carry out this arithmetic remains unexplained. Since one cannot assume, that the pupils are competent to differentiate between numerators and denominators, one can classify the statement of the teacher as contradictory. Consequently, pupils in the end of this episode were merely able to solve an addition task, which they were already capable of solving before and whose correctness would now be put into question.

2.3 Methodology

After having summarized the analysis of the scene, I would like to offer as preparation of further analysis a few explanatory notes to the methodological situating of the underlying research. The underlying research to this article is qualitatively oriented and grounded in interpretative classroom research. More exactly: in the domain of the interactionist view of interpretative classroom research in the field of mathematics education. Through the analysis of the units of interaction in the videotaped instructional episodes, I oriented myself to a reconstructive-interpretative methodology and on a central element of the research style of Grounded Theory- the methodic approach of comparative analysis. The goal of interpretative classroom research is to pursue a local theory genesis through “understanding” of interactions of individuals in concrete instructional practice. The scope of this concept theory is related to the interpretative classroom research, however, to be decidedly restrained, since this is in many areas mostly globally and universally connoted. The theoretical results of research of such a reconstructive-interpretative procedure present hypothetical outcomes, which do not follow the claims of the development of globalizing and universalizing theoretical approaches (compare Krummheuer/Naujok 1999, p. 105). These hypotheses stay arrested to the fact, that they are directly connected to the respective context of the researched field of study and are thereby rich in empirical elements and feature inner consistency. A universality of underlying results does not lend itself to be understood as, “is always applicable,” rather may be related to only a limited scope of classes, who are taught and will learn under similar conditions.

3 The analysis of the linguistic accomplishment

Here subsequently follows the analysis of the linguistic accomplishment of the instruction on the basis of the selected instructional episode on three hierarchical levels.

3.1 Technical terminology versus everyday language

Since objects of Mathematics are according to Maier (1986, p.137) of an abstract nature, the introduction of new mathematic concepts allows for particular attention to the technical language of Mathematics, as objects of Mathematics can ultimately be handled and represented only on a linguistic-symbolic level (compare *ibid*, p. 137). The question, which should be answered in the following sections, is how these technical terms of Mathematics are introduced into the analyzed lesson. To this, Maier (2004) refers to the fact that in the technical language of Mathematics, as well as in

other technical languages, there is a problem of ambiguity within the technical language, since it interferes with the everyday language of the pupils (compare *ibid*, p. 153). The problem of ambiguity within the technical language of Mathematics, according to Maier (2004, p.153), carries a significant relevance in the verbal actions of teachers. Maier writes, that teacher language moves in a stress-ratio between technical linguistic “*Hypertrophy*” and accordingly “*Hypotrophy*”ⁱ. The goal should be, according to Maier, to have the teaching language, which moves on a scale between these two extreme points of Hypertrophy and Hypotrophy, positioned “in the middle.” Thus a necessary technical language development of the pupil body can be assured and on the other side the pupils can be given the opportunity to comprehend mathematic phenomena with their own language. In which forms the usage of mathematic concepts let themselves be differentiated from the usage of everyday language concepts in instruction follows as next in the first level of hierarchisation.

The analysis of the selected episode

In the underlying episode the teacher attempts to give a generalization for the addition of fractions. She stresses here the relevance of LCM for the addition of fractions in line <252> in saying, “*and for that reason you need this.*” In this statement she uses the place holder “*in addition*” and “*this*” instead of the technical terminology. In her entire generalization she uses a multiplicity of everyday language concepts such as, “*a piece of pizza*” <241>, “*pieces*” (of a pizza or cake) <245, 247, 248, 250>, “*chopping*” <245>, “*pieces of cake*” <253>. From the terminology she used, the following language can be found in everyday language as well as in technical language: “*to add together*” <241-242, 253>, “*not equal*” <243>, “*equal*” <245, 247>, “*less*” <249>. Only the expressions of “*fractions*” <252> und “*two sevenths*” <256, 257> suggest, on the other hand, technical linguistic terminology. With this analysis in mind, the procedure of the above-mentioned teacher would surely be described, according to Maier, more in terms of technical Hypotrophy, since the teacher through the generalization of the procedure, where the greatest level of abstraction could have been conjectured, reverted only minimally back to technical terminology.

According to the statements of Maier one could reason, that such a procedure enables pupils to describe mathematic phenomena with their own language, but also endangers the development of technical language. Since, however, these attempts to explain multiplicity are through everyday language concepts and the usage of place-holders, the general principle remains implicitly hidden (see section 2.2) and it is doubtful, that pupils are in a position to shift into their own language to describe this mathematic phenomena.

3.2 The embedding of mathematic concepts in a mathematics register

The second level of analysis of the linguistic accomplishment of instruction via the teacher by the introduction of a new mathematic concept lends itself to a reference of the statements of Pimm (1987). Pimm compares teachers as a role model of a “*native speaker*” of Mathematics (*ibid*, p. Xiii) and other people, for whom Mathematics ap-

appears to be incomprehensible, as per a foreign language, to which they are not empowered (ibid, p.2). In this context, Pimm (1987) is speaking of a “*mathematics register*” (p. 74). With the term register, Pimm is referring to Halliday (1975). Halliday understands a register as an assemblage of meanings that are intended for a particular function of language, that together with the words and structures are able to express these meanings. Halliday subsequently talks of the mathematics register only when a situation is concerned with meaning, that is related to the language of Mathematics, and when the language must express something for a mathematical purpose. Mathematics register in this sense can be understood as not merely consisting of terminology and that the development of this register is also not merely a process to which new words can be added (Halliday 1975, p. 65). The task of the pupils to learn mathematical concepts in their lessons contains, according to Pimm (1987), more a deeper learning of linguistic competence than is the case by Maier (e.g. 2004). In Maier’s approach the focus lies on the acquisition of technical linguistic competence through a well-balanced application of technical linguistic terminology and everyday language concepts in the linguistic accomplishment of instruction via the teacher. Pimm (1987, p.76) sees the task of pupils, however, as to become proficient in a mathematics register and in this way to be able to act verbally like a native speaker of Mathematics. The second level of hierarchisation of the linguistic accomplishment of instruction falls into what extent the newly learned mathematic concepts in the researched lesson were integrated into a mathematics register or if they were to be introduced and regarded as isolated units.

The analysis of the selected episode

In the selected episode the teacher appears to attempt to explain the mathematic concept “LCM” in connection with the addition of fractions. In the beginning of this episode the teacher produced for this purpose a reference to the concept of multiples in allowing pupils to calculate them. According to the theoretical perspective of Pimm (1987) the attempt by the teacher to reconstruct the concept of “LCM” only allows itself to be incorporated, not as an isolated conceptual unit, but through its connection with other mathematic concepts in a mathematics register. According to Pimm, it should be the goal to make pupils competent native speakers of Mathematics. In the introduction by the teacher, however, there was no time point in the entire scene in which the mathematic concepts of denominator, numerator, fractions, fraction strokes, or multiples were verbally and content-wise clarified in the official classroom discourse. They remain implicit and are integrated without reflection in the already familiar calculation routines. Even the teacher herself seldom uses the concepts to be learned actively, such as is shown in the first analysis, rather reverts back predominantly to the everyday language concepts. Pupils must extract the meanings of the new concepts by themselves from the illustration on the board. Pupils are then additionally given only the possibility to calculate the multiple as an active manner in which to solely understand the meaning of the concept of a multiple. That pupils are able to extract the concepts, without a verbal contextual explanation of the concepts by the teacher seems questionable. For example, in the analysis at the beginning of

the scene there were alternatives for interpretation, in which the pupils interpreted the fraction stroke as minus sign. Pupils must extract the subject with this *implicit procedural method* from their everyday background or from that which they already know from their lessons and will thus be able to take no decisive steps in the direction of becoming a native speaker of Mathematics.

3.3 The embedding of the mathematic concepts in a formal language register

The third level of analysis of linguistic accomplishment of instruction unfolds from the reference of the theoretical explanations of Bernstein (1977), Gogolin (2006), and Zevenbergen (2001). According to Gogolin (2006), pupils in German schools are submitted to the normative standard, that they are receptively and productively in command of the cultivated linguistic variations in class. This language of school- described by Gogolin as “*Bildungssprache*”ⁱⁱ (ibid, p.82 ff., according to the concept of “*Cognitive Academic Language Proficiency*”, Cummins 1979)- has on a structural level more in common with the rules of written linguistic communication. It is in large part inconsistent with the characteristics of the everyday verbal communication of many pupils.

Bernstein (1977) and Zevenbergen (2001) target, with their discussion of the language of instruction, the children from the working and middle class for differentiation. According to them, the linguistic abilities of formal language that are required in schools set a line of demarcation in everyday language, that is more in accordance to the abilities of the middle class, than to those of the working class. This formal language of instruction stands out through its precise grammatical structure and syntax as well as through its complex sentence structure. Through proficiency in this formal language, pupils develop - those in the middle class in particular - a sensibility in regards to the structure of objects and the structure of language, that helps them to solve problems in life and in school in a relevant and goal-oriented manner. Successfully receptive in “*being (a) part (of)*” and productive as in “*taking part (in)*” (Markowitz 1986, p.9, translated by the author) a linguistic discourse of instruction is something that is only possible for pupils, according to the above-mentioned authors, when they have competence in the formal language or the *Bildungssprache* of instruction. In this way it is possible for them to understand abstract concepts independent of concrete context and to be able to transfer them into written decontextualized form. In the third level of hierarchisation of the linguistic accomplishment of primary mathematics instruction there follows the question, to what extent, and how pupils are introduced during instruction to a formal *Bildungssprache*.

The analysis of selected episode

In her attempt to make a generalization, the teacher says in <241-242> “*Right/ you may not add a small piece of pizza and a small one and smaller one together*” <241 – 242>. She also uses the comparative form of the adjective “small” for this purpose, but does not go into the “Least Common Multiple” more explicitly. However, it is not self-explanatory that all pupils- most especially those who have grown up

multilingual- are familiar with the comparative forms of adjectives in the German language. It is not self-explanatory that pupils will be able to differentiate between “Small Common Multiple” and “Least Common Multiple”. This interpretation is supported by analysis of previous episodes, in which pupils used the incorrect comparative form when attempting to use the term “Least Common Multiple”. Another correlation to this can be seen in the procedure at the beginning of the scene where the teacher allowed the pupils to calculate multiples. At no point in time did the teacher explain the connection between the terms “multiple” and “Least Common Multiple”. In this way it is made difficult for students to be able to recognize that the “Least Common Multiple” is really a subset of all “multiples”. It is not attempted on the part of the teacher to integrate the new concept into a related text. Hereby the question may be asked if and how the students should be empowered to understand such abstract concepts independent of concrete examples and to be able to transfer them into written form.

Summary of the analysis of the linguistic accomplishment of instruction

In the underlying research of this article there were 15 different episodes in total which were analyzedⁱⁱⁱ. These episodes with the help of comparative analysis were systematically compared. The comparison thereby of the three hierarchical levels of the linguistic accomplishment of instruction resulted in the following structure characteristics:

In the case of the first level, the application of technical terminology or everyday language by the teacher in instruction, allows no structural commonalities to be reconstructed. A unified procedure by the usage of mathematics register and everyday language does not seem to make a difference in the episode. The teachers use either predominantly everyday language concepts or several new and unexplained mathematic concepts. Unlike the first level, the results of the analysis of the other two levels behave in a different way. The implicitness of learning content, as a phenomenon in the introduction of a new mathematic concept, allows itself to be reconstructed as the common basic structural characteristic of the linguistic accomplishment of instruction via the teacher. The implicitness of the learning content defeats itself by the usage of different mathematics and formal linguistic registers. In this introduction of new mathematic concepts one can reconstruct through mathematics register, that the meanings of the concepts, just as the content references between the new mathematic concepts to be learned or the already known everyday language concepts is not made clear or only implicitly. The meanings or connections are not explicitly taken up in the instructional discourse and find thus no consideration in the classroom discourse. The meaning or the reference are explicitly assimilated by the teacher into the instructional discourse and thus find no consideration in the interaction of the classroom discourse. The formulated goal of Pimm (1987, Xiii; see Ch. 2.4) that students should learn to speak Mathematics like a native speaker, will be difficult for students to achieve, as the native speaker of Mathematics - the teacher - does not exemplify this active speaking themselves. A

similar picture shows itself in the way the teachers commit themselves to linguistic particularities of formal linguistic register. Also here there is an implicitness that rules the teaching. The teacher only refers back to the grammatical structure implicitly, in which the mathematical concept is embedded, or to that which characterizes the meaning carrying elements. With which linguistic methods the complex and abstract mathematic concept, in the sense of the conceptual writing, is expressed to a connected text is left, as regards content or implicitness, in the end of the attempted explanations, unconnected. An integrated embedding of the mathematical concept in a *Bildungssprache* is not noticeable.

4 Implicit Pedagogy and its consequences

In the basis of the research the reconstructed procedures of the teacher in the linguistic accomplishment of the lesson alone was with mathematics teaching approaches not enough to explain, and for this reason further pedagogical, sociological and linguistic approaches were expanded into the theory genesis (compare Bourne 2003; Bernstein 1996; Walkerdine 1984). Through this opening of the theoretical framework of the underlying research, there allows for the procedure of the teacher to be conceptualized under the concept of “*Implicit Pedagogy*” (compare “*Implizite Pädagogik*” Schütte 2009). This displays itself in the introduction of new mathematical concepts, in the manner, that decisive aspects of meaning negotiating of the individuals and the thereby possible constructions of enduring, non-situational bodies of knowledge for the individuals, remain concealed. One such Implicit Pedagogy is attached to the main idea, that students alone on the basis of the abilities they bring along with them can unlock meanings. Not the lesson, the qualifications of the teachers, nor their efforts can bring a deciding influence on the possible educational success of students in school, but rather, and above all else, the abilities that the children have brought with them decides this. The linguistic accomplishment of the instruction via the teacher, that follows such fundamental ideas, would not appear to make enough adjustments to the existing relationships of linguistic-cultural plurality in the classroom, since the procedure as it stood only served to reproduce existing social relationships in the educational system. The consequence of such an implicit procedure by the teacher can be, for example, that the comprehensive development of the relevance of the new concepts to be learned, on the side of the students, can be hindered. On the other hand it is a possible consequence that the students could be hindered by, or could refuse to participate in, a formal linguistic educational discourse in their lessons. Additionally, the opportunity is taken away from them to participate actively, that means productively, in the lesson, and through this accomplish the lesson. This happens for the main reason that the teacher, through her primarily implicit procedure, presents no model for her students to follow in her interactions with the formal linguistic *Bildungssprache*.

ⁱ The excessive use of almost “*pure technical language*” (ibid) by teachers and instructional media is viewed by Maier (2004, p.153) as technical linguistic hypertrophy. The excessive use of almost

“*pure colloquial language*“ (ibid) by teachers and instructional media is characterized by Maier (2004, p. 153) as technical linguistic hypotrophy.

ⁱⁱ *Formal linguistic instructional language* (translated by the author).

ⁱⁱⁱ This episode under consideration deals primarily with a shortened extract from the original episode, since for reasons of space limitations no analysis of the entire episode was possible. The detailed analysis of this episode can be found in Schütte (2009).

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MATHEMATICAL COGNITIVE PROCESSES BETWEEN THE POLES OF MATHEMATICAL TECHNICAL TERMINOLOGY AND THE VERBAL EXPRESSIONS OF PUPILS

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Verbal expressions by students in mathematical conversational situations provide insight into the individual mathematical imagination and express what patterns and contexts children recognize in mathematical problems. Children just starting school utilize means of expression of their mathematical ideas that go from everyday speech descriptions to detailed action sequences. They already use technical facets, even though their repertoire of mathematical language of instruction has to be considered initially as tentative. In our article, by dint of methods of qualitative analysis, we want to present initial descriptions in terms of the identified capability of mathematical expression of pupils just starting school, based on a conversational situation about a combinatorial problem.

Keywords: mathematical cognitive process, mathematical language, mathematics in elementary school, combinatorics, mathematical concepts

INTRODUCTION

The mathematical cognitive process is initiated extrinsically and/or intrinsically by tangible problems or questions the young learner encounters in various contexts. This process is of a discursive nature. Furthermore the mathematical problems are expressed in manifold linguistic forms. In the process of understanding, individual prior knowledge, mathematical concepts and strategies are activated by the learner. According to the learner's estimation the activated strategies promise the most probable possibility for a solution.

Within the framework of our research, we wish to focus on linguistic activities within the mathematical cognitive process that significantly mold this very process: mathematical content is conveyed by dint of language; mathematics is talked and written about. This approach is needed to broaden the perception of language from purely verbal expressions to other activities such as gesture, body language and facial expression, as well as bringing mathematical facets into written form and presenting them. In addition, it is important to take into account what cognitive grasp, from their perspective, the respective protagonists have in terms of handling mathematical problems. It is also interesting which patterns of action are consciously or unconsciously activated in terms of the situation. Individual interpretations, concepts and models of the mathematical content, which is always restricted to context, and also social aspects (communication patterns, specific language of instruction, structure of the interactive negotiation process, teaching and learning patterns and cultural conditions) and the personal image of mathematics are especially pivotal and demand detailed consideration.

“The emerging of mathematical knowledge is fundamentally taking place in the context of social construction and individual interpretation processes. [...] it is constructed by means of social activities and individual interpretations.” (Steinbring 2005, p. 7)

In the present paper we would like to give an outline of the provisional state of knowledge resulting from our activity in the field of ‘The learning of mathematics and language’. Translating the mathematical content of a problem into technical terms is related to the mathematical language of young learners, in particular their mathematical concepts and individual conceptions, which are reconstructed based on verbal activities. We expect that the detailed consideration of the children’s verbal expression will afford us with insights into what they view as the problem’s mathematics. This identification of mathematical and individual concepts is to be deepened in the future *inter alia* by the interactional view of mathematical negotiation processes mentioned above. In doing so, we wish to focus on ‘mathematical language’ in the broadest sense of the term, that is, constituting all forms of expression accompanying the mathematical cognitive process. In our opinion learners of mathematics, especially young learners, approximate more and more to technical mathematics-orientated language in their process of learning mathematics. This “speaking mathematically” (Pimm 1989) is more than just learning vocabulary and using these words in the right linguistic form. Linked to that is the notion of utilizing this knowledge to design processes of teaching and learning. If we demonstrate our considerations in the following and represent our thoughts by means of an example, we focus at first verbal expressions and unique actions in the mathematical cognitive process that are examined as unique expressions for now. We will present an exemplary conversational situation of first-graders concerning a combinatorial problem. Our research perspective is guided by the super ordinate question about mathematical language and a potential mathematical language development in the process of learning mathematics on the part of young learners. In the present paper we want to focus on the following embedded questions:

What language do the here described pupils have at their disposal when handling a combinatorial problem in the conversational situation being presented?

What individual conceptions and mathematical (‘target-consistent’) concepts can be surmised behind the described children’s verbal-linguistic activities in the examination with a combinatorial problem?

What patterns of actions are activated or what conceptions about ‘to do mathematics’ in the examination with an explicit structured combinatorial problem can be reconstructed by means of verbal activities?

The data and considerations used have emerged from our research within the framework of an initial exploratory pilot study. We conducted this exploratory study with a focus on designing and interpreting situations which could be analyzed in the view of mathematical concept development and the linguistic means of expression in discourse situations. This study could contain useful information and serves as a trial of

such situations. It is embedded in the context of a current developed longitudinal study to investigate early steps in mathematics learning (related to the Centre for Research on Individual Development and Adaptive Education of Children at Risk (IDeA), a centre of DIPF (German Institute for International Educational Research) and the Goethe university, Frankfurt/Main in cooperation with the Sigmund-Freud-Institute, Frankfurt/Main).

THEORETICAL FRAMEWORK – MATHEMATICAL LANGUAGE ACTIVITIES OF CHILDREN IN ELEMENTARY SCHOOL

At the beginning of their time in school, young, monolingual, linguistically inconspicuous learners have at their disposal a fundamental passive and active vocabulary. Their language acquisition in the unique grammatical sub-systems can be termed basic. Now what becomes relevant in terms of language is the growth of special communication and action patterns to be ascribed to the institution of the school, such as the acquisition of a certain language of instruction (cf. “cognitive academic language proficiency,” according to Cummins 2000 after Gellert 2008, p. 140). For mathematics lessons in particular, a vocabulary and a specific language have to be acquired in which symbols are employed or terms from everyday speech adopt a different meaning (like ‘equal,’ ‘less,’ ‘greater’). Negotiation processes in the social context have to be mastered linguistically within the learning process so as to understand mathematical teaching contents and be capable of participation. Verbal expressions are thus embedded in the interaction process in which they are uttered. The process of analysis documented here represents an initial approach to a form of analysis yet to be developed, which would permit one to make statements about the applied forms of language in the context of mathematical cognitive processes. Beside that, the analytical method to be developed could be interlocked with other approaches like interaction, argumentation and participation analysis (Brandt & Krummheuer 2000; Krummheuer 2007).

The approach presented here in an initial outline bears a certain resemblance in several parts to Steinbring’s (2005, 2006) epistemological approach. In the epistemological triangle developed by Steinbring, the interactively constructed mathematical knowledge is of central importance. This knowledge, which is again based on pre-existing conceptual ideas, is generated by creating relations between the signs being utilized and reference context. In our approach the children used signs in the form of verbal, gestural and also written expressions to communicate their meaning or interpretation of the given mathematical content. In doing so, they needed to revert to their pre-existing conceptual ideas. Their expressions or signifier could only refer to the reference context or signified, whereas a common interpretation of this mathematical content has to be negotiated in interaction.

The question is how these mathematical pre-existing conceptual ideas and knowledge in Steinbring’s approach can be described. The point of departure of our analysis is the problem’s so-called mathematical content. While handling the ‘mathematical content,’ we try to describe the mathematical concepts or mental models (here in the

meaning of Prediger 2008) that are of import for solving the problem. Mathematical concepts or mental models, according to Prediger (2008), are contrasted with the personal conceptions of the individual who is learning, which are reconstructed here by means of pupils' expressions. These individual "students' conceptions" (Prediger 2008, p. 6) which are comparable with Steinbring's pre-existing conceptual ideas (Steinbring 2006, 140), sum up the conceptions of the individuals who are learning, which could be developed up to now to handle similar mathematical problems. Any other structurally similar mathematical problem will re-activate these "individual models," which are then confirmed in the situation or may lead to irritations and potential expansions of these individual models. Mathematical experts and novices alike use individual mathematical models to be able to approach the abstract and immaterial mathematical objects and develop mental images for them: "[...] mathematical concepts are sometimes envisioned by help of 'mental pictures' [...]. Visualization [...] makes abstract ideas more tangible, [...] almost as if they were material entities." (Sfard 1991, 6) Should a discrepancy arise between the individual model and the 'mathematical concept' relevant to the problem and prove to be too large to overcome, this may create learning opportunities that can be utilized more or less beneficially.

RELEVANT MATHEMATICAL CONCEPTS IN SOLVING COMBINATORIAL PROBLEMS

Combinatorics involves the determination of the number of elements of finite sets. The point is to select elements from a given total (basic set) and re-combine and rearrange them according to specific criteria (cf. Krauter 2005/2006). The description "combining selected elements" refers to the formation of new combinations of sets. The description "arranging selected elements" focuses on the order and thus on the formation of variations (cf. Selter & Spiegel 2004, 291). Again the determination of the number, of the sets or lists that arise this way, will be of importance. Thus, combinatorics centers around counting. Although here we are moving in the context of discrete mathematics and hence in the range of countability, this will frequently take on a theoretical character and provoke mathematical methods that go beyond the act of counting. These arithmetical "counting methods" are documented as formulas that in a compressed form describe the appropriate algorithm. In addition to the formulas, instructions are described having the function of activating inner images with the learner. These images help to translate familiar situations into the unknown mathematical problem and encourage the utilization of a suitable formula (for instance, without regard to order and without replacement).

The conversational situation that our analysis is based on is a part of an explorative study in which a total of eight first-graders were under examination. We selected this particular situation because its progress is comparable with all other videotaped and transliterated situations. Furthermore we choose such situation with a combinatorial problem, because this requires from the pupils counting and manipulation with sequences in practice. For the explorative study we developed mathematical problems

of different mathematical areas, e. g. combinatorics, and then presented one problem to a student-duad in a conversational situation. The setting which is important in the following descriptions was hence set up as follows: The researcher presents a combinatorial problem to two first-graders. The pupils had the joint task of solving the combinatorial problem. In the progress of the situation, the researcher simply joins the conversation of the children in an appropriate way. As material at their disposal the children had paper, pencils and a bag full of candies.

Problem: Emma has two red cherry candies and six green apple candies in her bag. She pulls four times from her bag and gives the candies that have been pulled to her brother Tom. What candies can Tom get? Find all the options that are not identical!

The problem describes precisely how the desired subsets – consisting of four elements – are to be generated. Four pullings in a row are to take place. Replacement does not make sense, as the generated subset is to be given away. This makes it quite explicit that one element of the initial set cannot be pulled more than once. Thus, the problem describes the combinatorial figure of pulling without replacement (a total of four pullings) of k elements from n . The second criterion of order is irrelevant to the problem (cf. set concept). Thus, the act can be translated into a pulling all at once, that is, without replacement and without regard to order (cf. Kütting & Sauer 2008, p. 93).

Cardinal number concept / set concept

The point of departure for the problem is an n -element set ($n = 8$), which is comprised of two subsets with the element numbers $r = 2$ and $g = 6$. In tangible terms, the problem is about the set of eight candies that differ in color (two subsets). In this way the cardinality of set or the subset comes to the fore. There are eight candies which consist of six green apple candies and two red cherry candies. Within these subsets, there exists no possible differentiation; hence no specific sequences that would be distinguishable are imaginable. For the subsets of four candies that are to be created anew, as well, the only thing that can be said is that each subset consists of candies that might be different in taste. A specific sequence is neither necessary nor would it make sense in the chosen everyday situation. Thus, all combinations of four candies that are distinguishable from one another have to be found from a set of eight candies.

Selection concept / combinatorial concept

Initially, all possible cases of distinguishable combinations according to the given assumptions of the problem have to be considered: With $k = 4$ pullings 0, 1 or 2 red candies and correspondingly 4, 3 or 2 green candies can be pulled. The following k -element sets are possible: $\{g, g, g, g\}$; $\{g, g, g, r\}$; $\{g, g, r, r\}$. The number of possible outcomes of the experiment could be found by a lexicographical counting of the combinations, following the formula of hypergeometric distribution (cf. Kersting & Wakolbinger 2008, p. 28) or by dint of a tree diagram. With the latter method, the doubles that are generated have to be discarded.

What is important for this concept is that there be combinations of selection distinguishable from one another that are created in a specific way, namely without replacement. In addition, a selection of candies may occur consisting of only one kind, since there are only two of the other kinds in the initial set. Moreover, fictitious combinations are generated mentally, of which only one will actually occur (cf. randomness concept). For that reason the initial situation (eight candies in the bag) has to be restored after each pulling, although there must be no replacements for each four-time pulling. For the discovery of all possibilities, it is advisable to compare the combinations that have been found and written down, thus eliminating doubles. Hence, this approach provokes a certain kind of documentation, since the process of pulling has to be repeated until all the various combinations have been discovered. Furthermore, written documentations often indicate a certain order, which in this context is unimportant, though.

Randomness concept / combinatorial concept

Which of all the possible combinations will occur cannot be definitively predicted. All imaginable possibilities can be pulled, but the pulling does not lead automatically to all the different combinations. It is possible that the same combination is pulled several times. Hence, a situation has to be considered that will only possibly occur. With the facet of the randomness concept that is relevant here, it is less the probability of particular combinations than the determination of all possible events that is in the foreground. The combination of the four candies that have been pulled is random. The missing combinations have to be added by thought experiment.

TECHNICAL TERMINOLOGY – MATHEMATICAL COGNITIVE PROCESS – PUPILS' EXPRESSIONS

Mathematical cognitive processes take place between the poles of mathematical and individually formed concepts. Mathematical as well as individual concepts are expressed in signs in form of the respective language culture (mathematical technical language, mathematical language of instruction, mathematical everyday speech). In this paper, we define the mathematical technical language as a language, which is used in the conversation between mathematical experts with a focus on formalization in verbal and written contexts in support of an agreed form of communication over a particular issue. The mathematical technical language is hence the result of many discursive negotiation processes that lead to a formal presentation. The mathematical everyday speech displays a discursive, processual character and serves more for individual formation of concepts and the approach to mathematical concepts.

Using the example presented above, figures 1 and 2 (see below) illustrate mathematical and individual pre-concepts, which, at best and naturally individually formed, approach one another. Verbal orientated signs that would be used by an expert (e.g. mathematician) are listed in the category of mathematical technical language and expresses mathematical knowledge which is adequate for the given problem. This mathematical knowledge and the expression of it also emerged in discursive negotia-

tion processes and in build a relation between signs and reference context and aim at an agreed form of communication – language culture among mathematicians (Morgan 1998). The pupils’ expressions specific to the situation are listed in the right column and are conceptually oral as well.

ANALYSIS

At first glance, the language of the pupils is molded by phrases taken from everyday speech and child-like action patterns like “which should I take [using a counting-out rhyme]” as well as by terms from the text of the posed problem. In Steinbring’s words you can reconstruct out of these expressions the children’s given pre-existing conceptual ideas or in Prediger’s words their individual concepts. These conceptions are tried to communicate by dint of signs or signifiers which should convey the children’s interpretation of the meaningful mathematical content.

mathematical technical language	mathematical concepts	individual mathematical concepts	verbal expressions by students
	mathematical cognitive process		
There is a finite set, called A, with n elements. This set is composed of two disjoint sets, called A1 and A2, with r respectively g elements. From the finite set A should be removed a set of k elements, in this case four elements.	set concept / cardinal number concept		Student 1: Emma has six candies Student 1: yes two cherry candies and em six apple candies Student 1: she wants to give her brother four of them
	formally abstract conceptions of sets	additive conception, tangible objects – candies	Student 2: and we must pull them Student 1: Yes, two plus two
There are four different combinatorial cases. In view of this task the case can be described as pulling without replacement and without regard to order. You can use the formula of n over k and adhere to the lamination.	selection concept / combinatorial concept		Student 1: how much possibilities Student 1: which should I take [using a counting-out rhyme with one hand in the sac]
	‘inner action images’: for instance, urn model, 0-1 sequence	playful action	Student 1: at first we must always pull them and later then we have to lay all of them back into the sac

Figure 1: Mathematical cognitive process, exemplified by the set concept / cardinal number concept and the selection concept

The technical-language part described here uses phrases that are more typically mathematical: “There is a finite set, called A, [...]”. It can be determined that the pupils utilize terms like “pull” or “replacement,” which they probably connect with their everyday conception of pulling situations (pulling lottery tickets, for instance). In the situation that is presented, the pupils skillfully focused mathematical facets: not the taste or type of candies (cherry, apple) but the number, the color as a differentiator, and the possible combinations under the given assumptions constitute the focus of their consideration. “At first we must always pull them and later then we have to lay all of them back into the bag,” is the description of the combinatorial figure of pulling without replacement and, in addition, something actually in contrast to that: the restoration of the initial situation after pulling four times. Here, the close connection of

context and mathematical conception (→urn model) – intended by the text of the problem – is presumably taking hold.

In terms of the technical language, mathematical terms are used also as typical formulations like “as pulling without replacement and without regard to order” for modeling, which are applied in a way relevant to the problem. The students are still in the process of model discovery, which is displayed in such comments about possible combinations: “Ah we can’t red, red, red, red we can’t because there are only two red,” which presents an interactive verbal negotiation of this cognitive process and suggests mathematical concepts that are still developing but are already target-consistent and are moving within the domain relevant to combinatorics. The production of relations between signs and reference context here therefore generate new mathematical knowledge. While the children at the beginning of the situation seem to utilize more operational and process-oriented dynamic concepts (they pull, put down, count by dint of a counting-out rhyme), they use in the proceeding of the situation more and more also structural descriptions: “We have red, red, green [...]” Sfard (1991, p. 5) said, that seeing both “[...] a process and [...] an object is indispensable for a deep understanding of mathematics [...].”

mathematical technical language	mathematical concepts	individual mathematical concepts	verbal expressions by students
	mathematical cognitive process		
[in continuation of figure 1] With the different given sets taking into account, you can hence say, that r is less than k, which is less than g, which is finally less than n. Accordingly you can suggest, that k could be equal g or g plus r but never just r.	selection concept / combinatorial concept	Negotiation, what combinations are possible?  selection concept / combinatorial concept	Student 1: [Looking at the subscription]: we have red, red, green / red, green, green, red / green, green, green, red / green, green, green, green Student 2: we can also take red, red red/ Student 1: Ah we can't red, red, red, red we can't because there are only two red Student 2: I would say I take three red and a green but it doesn't work because we have only two red
The result set could be defined as gggg, gggr or ggrr. You have to use the classic propability formula and divide the number of well cases by all possible cases.	randomness concept		Student 2: Don't look while pulling Student 1: perhaps we will pull the same Student 1: [By pulling]: please, please, please
	stochastic model: classic probability	random access by dint of desire, luck, bad luck	Student 1: please no red, no red please Student 1: [By pulling]: if we get a green

Figure 2: Mathematical cognitive process, exemplified by the selection concept

The pupils’ randomness concept is molded by child-like pre-conceptions where events are attempted to be ‘wished’ to come true, which becomes implicit in expressions like “please no red, no red please.” Mathematically, randomness becomes comprehensible by dint of the formula about classic probability. Nonetheless, the students already have at their disposal the skill that is crucial for handling combinatorial problems: being capable of mental imagining the configurations of possible combinations

and they can also communicate this by dint of verbal signs: “Perhaps we will pull the same.” It becomes manifest that the pupils have a concept of mathematizing at their disposal. Certainly in part guided by the setting, the students activate action patterns, which focus on those facets of the problem that are relevant to combinatorics and express mathematical thought processes verbally. Individual conceptions converge with mathematical concepts.

CONCLUSIONS

With our initial attempts at analysis, preliminary insights in the mathematical utterances of first-graders can be described. Concerning our introductory questions we can summarize the following conclusions:

1. In view of the presented analysis of this exemplary situation we suggest that there is first evidence that children, who just starting school obviously have at their disposal manifold forms of expression in terms of mathematical problems. They convey these forms of expression by dint of everyday speech as well as of first technical language, e. g. in using mathematical terms like “possibilities” or abstract from the given context in using “red, red, red” rather than the concrete objects (here: candies). Terms belonging to combinatorics are utilized in a meaningful and productive way during the process of handling the problem and suggest mathematical concepts that have been already acquired or are developing.
2. In reference to the problem’s core question, language is dominant for action steps that are in need of explanation, or when considering an action result (here the combinations of candies that have been pulled). Concepts are verbalized that have to be tested or that only develop in – and through – the process of verbalization. In doing so, the individual mental concepts converge with mathematical concepts, which can be partially considered as already acquired.
3. The young learners in the presented situation utilize process-oriented and structural concepts, which indicate they are focusing on what doing mathematics means to them in the context of the specific combinatorial problem.

These initial conclusions have to be examined in further research to follow, in other mathematical areas or different problem arrangements, for instance. Moreover, it is essential to approach the analytical procedures mentioned above and, for one, to examine more closely the construction of mathematical knowledge in the focus of interaction. In our further investigations we want to deepen this analysis and adopt it to other comparable situations in which children solve problems in different mathematical areas. In this context we plan to investigate the mathematical development in the age of kindergarten children in a longitudinal study (a study inside IDEA, in front explained). This could enable us to describe over the period in which the children visit the kindergarten the development of mathematical thinking. The project of research is applied as a cooperation study with researchers of language acquisition, which should enable us to investigate in particular the coherency of mathematical development and language acquisition. Furthermore it is possible to broaden the perception of language from purely verbal expressions to other activities such as gesture or

body language as well as written and presented mathematical facets and also focus on interaction processes for an implication of a social perspective.

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CERME 6 – WORKING GROUP 7

TECHNOLOGIES AND RESOURCES IN MATHEMATICAL EDUCATION

TABLE OF CONTENTS

Introduction.....	1046
<i>Ghislaine Gueudet, Rosa Maria Bottino, Giampaolo Chiappini, Stephen Hegedus, Hans-Georg Weigand</i>	
Realisation of mers (multiple external representations) and melrs (multiple equivalent linked representations) in elementary mathematics software	1050
<i>Silke Ladel, Ulrich Kortenkamp</i>	
The impact of technological tools in the teaching and learning of integral calculus.....	1060
<i>Alejandro Lois, Liliana Milevicich</i>	
Using technology in the teaching and learning of box plots.....	1070
<i>Ulrich Kortenkamp, Katrin Rolka</i>	
Dynamical exploration of two-variable functions using virtual reality	1081
<i>Thierry Dana-Picard, Yehuda Badihi, David Zeitoun, Oren David Israeli</i>	
Designing a simulator in building trades and using it in vocational education	1091
<i>Annie Bessot, Colette Laborde</i>	
Collaborative design of mathematical activities for learning in an outdoor setting	1101
<i>Per Nilsson, Håkan Sollervall, Marcelo Milrad</i>	
Student development process of designing and implementing exploratory and learning objects	1111
<i>Chantal Buteau, Eric Muller</i>	
How can digital artefacts enhance mathematical analysis teaching and learning.....	1121
<i>Dionysis I. Diakoumopoulos</i>	
A learning environment to support mathematical generalisation in the classroom	1131
<i>Eirini Geraniou, Manolis Mavrikis, Celia Hoyles, Richard Noss</i>	
Establishing a longitudinal efficacy study using SimCalc MathWorlds®	1141
<i>Stephen Hegedus, Luis Moreno, Sara Dalton, Arden Brookstein</i>	

Interoperable Interactive Geometry for Europe – First technological and educational results and future challenges of the Intergeo project.....	1150
<i>Ulrich Kortenkamp, Axel M. Blessing, Christian Dohrmann, Yves Kreis, Paul Libbrecht, Christian Mercat</i>	
Quality process for dynamic geometry resources: the Intergeo project.....	1161
<i>Jana Trgalova, Ana Paula Jahn, Sophie Soury-Lavergne</i>	
New didactical phenomena prompted by TI-Nspire specificities The mathematical component of the instrumentation process.....	1171
<i>Michèle Artigue, Caroline Bardini</i>	
Issues in integrating cas in post-secondary education: a literature review	1181
<i>Chantal Buteau, Zsolt Lavicza, Daniel Jarvis, Neil Marshall</i>	
The long-term project “Integration of symbolic calculator in mathematics lessons” The case of calculus	1191
<i>Hans-Georg Weigand, Ewald Bichler</i>	
Enhancing functional thinking using the computer for representational transfer.....	1201
<i>Andrea Hoffkamp</i>	
The Robot Race: understanding proportionality as a function with robots in mathematics class	1211
<i>Elsa Fernandes, Eduardo Fermé, Rui Oliveira</i>	
Internet and mathematical activity within the frame of “Sub14”	1221
<i>Hélia Jacinto, Nélia Amado, Susana Carreira</i>	
A resource to spread math research problems in the classroom	1231
<i>Gilles Aldon, Viviane Durand-Guerrier</i>	
The synergy of students’ use of paper-and-pencil techniques and dynamic geometry software: a case study	1241
<i>Núria Iranzo, Josep Maria Fortuny</i>	
Students’ utilization schemes of pantographs for geometrical transformations: a first classification	1250
<i>Francesca Martignone, Samuele Antonini</i>	
The utilization of mathematics textbooks as instruments for learning	1260
<i>Sebastian Rezat</i>	
Teachers’ beliefs about the adoption of new technologies in the mathematics curriculum	1270
<i>Marilena Chrysostomou, Nicholas Mousoulides</i>	
Systemic innovations of mathematics education with dynamic worksheets as catalysts	1280
<i>Volker Ulm</i>	

A didactic engineering for teachers education courses in mathematics using ICT	1290
<i>Fabien Emprin</i>	
Geometers' sketchpad software for non-thesis graduate students: a case study in Turkey	1300
<i>Berna Cantürk-Günhan, Deniz Özen</i>	
Leading teachers to perceive and use technologies as resources for the construction of mathematical meanings	1310
<i>Eleonora Faggiano</i>	
The teacher's use of ICT tools in the classroom after a semiotic mediation approach.....	1320
<i>Mirko Maracci, Maria Alessandra Mariotti</i>	
Establishing didactical praxeologies: teachers using digital tools in upper secondary mathematics classrooms	1330
<i>Mary Billington</i>	
Dynamic geometry software: the teacher's role in facilitating instrumental genesis	1340
<i>Nicola Bretscher</i>	
Instrumental orchestration: theory and practice.....	1349
<i>Paul Drijvers, Michiel Doorman, Peter Boon, Sjef van Gisbergen</i>	
Teaching Resources and teachers' professional development: towards a documental approach of didactics	1359
<i>Ghislaine Gueudet, Luc Trouche</i>	
An investigative lesson with dynamic geometry: a case study of key structuring features of technology integration in classroom practice.....	1369
<i>Kenneth Ruthven</i>	
Methods and tools to face research fragmentation in technology enhanced mathematics education.....	1379
<i>Rosa Maria Bottino, Michele Cerulli</i>	
The design of new digital artefacts as key factor to innovate the teaching and learning of algebra: the case of Alnuset	1389
<i>Giampaolo Chiappini, Bettina Pedemonte</i>	
Casyopée in the classroom: two different theory-driven pedagogical approaches.....	1399
<i>Mirko Maracci, Claire Cazes, Fabrice Vandebrouck, Maria Alessandra Mariotti</i>	
Navigation in geographical space	1409
<i>Christos Markopoulos, Chronis Kynigos, Efi Alexopoulou, Alexandra Koukiou</i>	
Making sense of structural aspects of equations by using algebraic-like formalism.....	1419
<i>Foteini Moustaki, Giorgos Psycharis, Chronis Kynigos</i>	
Relationship between design and usage of educational software: the case of Aplusix	1429
<i>Jana Trgalova, Hamid Chaachoua</i>	

INTRODUCTION

TECHNOLOGIES AND RESOURCES IN MATHEMATICAL EDUCATION

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INTRODUCTION

Technologies in mathematical education has been a theme present at CERME from the first edition. The available technologies have evolved a lot during these years. At CERME 5 conference, the conclusions of the technology Working Group (Kynigos et al. 2007), as well as Artigue's and Ruthven's interventions (Artigue 2007, Ruthven 2007), signal perspective evolutions towards more comprehensive studies, in several respects. Drawing on these previous works, CERME 6 WG7 intended to go further in the directions they have indicated.

An important issue, accounting for the introduction of the word "resources" in the name of a group which was previously called "tools and technologies in mathematical didactics", is the need for considering technologies within a range of resources available for the students, the teachers, teacher's trainers etc. These agents can draw on software, computers, interactive whiteboards, online resources, but also on more traditional geometry tools, textbooks, etc. Various kinds of digital material are now extensively used, and they can be viewed as belonging to a wider set of curriculum material (Remillard 2005) and teaching resources (Adler 2000). The papers in WG7 concern different kinds of resources, still with a specific focus on digital material. Another specific focus of WG7 is on theoretical approaches. Design issues need to focus on integration and impact, especially in the use of innovative technology. This entails the development of approaches framing research on fidelity, efficacy, and effective integration (Hegedus & Lesh, 2008). These approaches have been discussed in the group, and several issues linked with the articulation between research and development have been raised, as it is presented below.

The work in the group was organized into three parallel sessions, corresponding to three themes summarized below; specific slots were devoted to the presentation of the work done within three projects co-funded by the European Community, whose participants are represented in WG7: the Telma European Research Team, the Remath project, and the Intergeo project. The whole group was nevertheless gathered for the first session, with two important activities: the identification of questions considered important for the group's work by the participants (figure 1); and the "plenary" address of Jean-Baptiste Lagrange on the results of the ICMI17 study (Hoyles & Lagrange, to appear). The following trends in research and salient elements presented by Jean-Baptiste Lagrange were extensively present in the group's discussions:

- The integration and synthesis of previously fragmented theories and the development of broad approaches;
- The consideration of the design of tools and curricula as a major issue for mathematical education;
- The development of teacher-oriented research studies with a specific consideration on methodological issues such as the consideration of "ordinary" teachers by researchers

1. The word "dynamic" permeates mathematics teaching and learning activities which involve technology. Where and when is it appropriate?
2. How is it possible to restructure the maths curriculum to take advantage of new technologies to generate mathematical thinking?
3. How can we assess the applicability and the effectiveness of current theories?
4. Do we need specific theoretical tools or approaches to study the different ways in which teaching can be carried out using technologies?
5. What kind of professional development could support pre/in-service teachers to integrate new technologies in their classroom practice?
6. How can "old" and "new" resources interact each other? For example, how is it possible to incorporate e-technologies in textbooks?
7. If we take seriously into account a semiotic perspective considering the evolution of ICT, what new is offered in terms of creative power of semiotic means?
8. Can the European projects presented at WG7 contribute to create a "general theory" of teaching and learning with ICT which can be useful in different European countries?

Figure 1. Examples of questions raised by WG7 participants

THEMES AND PROJECTS IN WG7

Design, articulation of design and use

The work within the « design » theme extensively dealt with the link between design and learning. In particular, the question on the way in which mathematical knowledge can be modified according to different environments was raised. Changes induced by visualization (Ladel & Kortenkamp, Lois & Milevicich, Kortenkamp & Rolka), virtual reality and simulations (Dana-Picard et al., Bessot & Laborde) opportunities were discussed. Such changes are linked with the specific software tools considered and related features and their analysis is central for the design process. Tools can modify knowledge and learning as well, as it clearly happens for outdoor activities (Nilsson et al.). A broad view on the technology involved, but also on the appropriate associated mathematical tasks is essential (Buteau & Muller, Diakoumopoulos). Amongst the possibilities offered by technology and likely to affect learning and mathematical knowledge, the collective dimensions deserve a specific attention. Software connectivity features, enabling students' collaboration, modify their participation in the mathematical work (Geraniou et al., Hegedus et al.). Questioning collective dimensions also includes a focus on the link between designers and users, and the possible interventions of the users within the design process. This is one of the aspects tackled by the Inter2geo project that dealt with the interoperability of digital geometry systems in Europe (Kortenkamp et al., Trgalova et al.). Beyond the interoperability and indexation issues, this project produced resources for teachers. These resources were tested by users, and users feed-back was included in the design process, with a quality objective. The question of quality, that is the way to assess and validate design and use of technology, was central in the group's discussions. There is still a need for methods to evaluate efficacy of a given technology as far as the learning, engagement and motivation of the students. Even if the projects presented within the group were all grounded in research and linked to given theoretical framework, many questions remain open on the way in which research results can operatively contribute to successful design and use of educational technology.

Technologies, tools and students mathematical activity

Papers presented under this theme concerned a wide range of tools and technologies: online resources, software tools, as well as more traditional tools, such as textbooks. Amongst these technologies, Computer Algebra Systems were considered in several papers (Artigue & Bardini, Buteau et al., Weigand & Bichler). But even with a specific interest on CAS, the research works presented in WG7 consider in fact new complex artefacts, articulating CAS and graphing tools in particular, and raised the problem of designing resources to scaffold the use of these artefacts.

Another direction in which the papers showed richness and variety is the theoretical frameworks they draw upon. On the one hand, general theories of mathematical education were considered: functional thinking (Hoffkamp), situated learning, activity theory (Fernandes et al., Jacinto et al.), theory of didactic situations (Aldon & Durand-Guerrier). These theories were used to enlighten specific aspects of learning with technologies: the idea of functional dependency, the mediation provided by an artefact, the didactic contract. On the other hand, several papers refer to specific theories such as that of instrumental approach (Iranzo & Fortuny, Martigone & Antonini, Rezat). This framework, specially designed to study teaching and learning phenomena involving technology, proposes a genesis perspective on learning with technology. It leads to analyses of learning phenomena in terms of schemes. In WG7, precise classification of schemes and of operational invariants were discussed. Such discussions can contribute to a further progress of the instrumental approach framework.

Interactions between resources and teachers' professional practice

The acknowledgment of difficulties linked with the integration of technology in classrooms, identified in previous CERME conferences (Kynigos et al. 2007, Drijvers et al. 2006) was still present in CERME 6 WG7 together with the acknowledgment of the key role played by teachers. The need for investigating teachers' beliefs about technology adoption (Chrysostomou & Mousoulides) was recognized as well as the need for conceptualising systemic innovations of educational systems (Ulm). (Emprin, Cantürk-Günhan & Ozen, Faggiano) discussed the importance of setting up pre-service and in-service teachers' training programs taking into account, for in-service teachers, their pre-established repertoires of resources. An evolution from specific studies of individual teachers' practice to investigations of general integration issues was observed, thus moving a step towards theoretical evolutions. As the matter of fact, for example, teacher's use of ICT was examined with a semiotic mediation perspective in (Maracci & Mariotti), while some authors addressed the development of the instrumental approach to study the role of the teacher, drawing on the notion of instrumental orchestration (Trouche 2004), and introducing the consideration of teachers' instrumental genesis (Billington, Bretscher, Drijvers et al.). The acknowledgment of the variety of resources involved in the teacher's activity as well as the need to take into account the whole classroom context, led some authors to develop holistic approaches, such as a documentary approach to didactics (Gueudet & Trouche) and key structuring features of technology integration in the classroom practice (Ruthven).

Delicate methodological issues are attached to the implementation of these theoretical developments, in particular to the question of the "ordinary teacher", which remains open.

TELMA/Remath projects

The topic of the articulation of different theoretical frames is central in the TELMA and Remath European projects. In an effort for overcoming the national specificities, these projects developed a cross-experimentation methodology: The key idea around which this methodology was built was the design and the implementation by each team involved in one of these projects of experiments, carried out in real classroom settings, making use of an ICT-based tool developed by another team (Bottino et al., 2009). They also designed meta-tools, in particular scenarios for researchers and for teachers, and proposed developing an integrated theoretical framework (Bottino & Cerulli; Chiappini & Pedemonte; Maracci et al., Markopoulos et al., Moustaki et al., Trgalova & Chaachoua). In fact the three themes of WG7 are present within these large projects, which opened promising methodological and theoretical directions for research.

CONCLUSION

What do we retain from the work in WG7? The three themes proposed for the contributions, oriented towards design, students and teachers were from the beginning presented as articulated. The design loops integrate more and more the users, students or teachers. The interactions between students and teachers in class are a focus of attention for the researchers. The articulations between different kinds of resources were also extensively discussed, confirming the need for a broad point of view on resources. Research presented in WG7 is focused on technology, but technology does not mean here a precise delimited tool; it includes meta-tools and complex sets of resources. Reflecting on this evolving meaning of technology can be a direction for the work in future CERME conferences.

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REALISATION OF MERS (MULTIPLE EXTERN REPRESENTATIONS) AND MELRS (MULTIPLE EQUIVALENT LINKED REPRESENTATIONS) IN ELEMENTARY MATHEMATICS SOFTWARE

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Assumptions of multiple mental representations lead to the presumption of an enhanced mathematical learning, especially of the process of internalization, due to MERS (Ainsworth 1999) and MELRS (Harrop 2003). So far, most educational software for mathematics at the primary level aims to help children to automatize mathematical operations, whereby symbolical representations are dominating. However, what is missing is software and principles for its design that support the process of internalization and the learning of external representations and their meaning themselves – in primary school these are in particular symbols. This paper summarizes the current state of research and presents a prototype that aims to the above-mentioned purpose.

INTRODUCTION

In this article we describe the theory and new achievements of a prototypical educational software for primary school arithmetic. After developing the guiding principles that are based on multimedia learning models, we present DOPPELMOPPEL¹, a learning module for doubling, halving and decomposing in first grade.

THE COGNITIVE THEORY OF MULTIMEDIA LEARNING (CTML)

In the 1970s and 80s it was assumed that comprehension is limited to the processing of categorical knowledge that is represented propositionally. Nowadays, most authors assume the presence of multiple mental representation systems (cp. Engelkamp & Zimmer 2006; Schnotz 2002; Mayer 2005) – mainly because of neuro-psychological research findings. With regard to multimedia learning the Cognitive Theory of Multimedia Learning (CTML) of Mayer is to emphasize (Fig. 1).

¹ see <http://kortenkamps.net/material/doppelmoppel> for the software

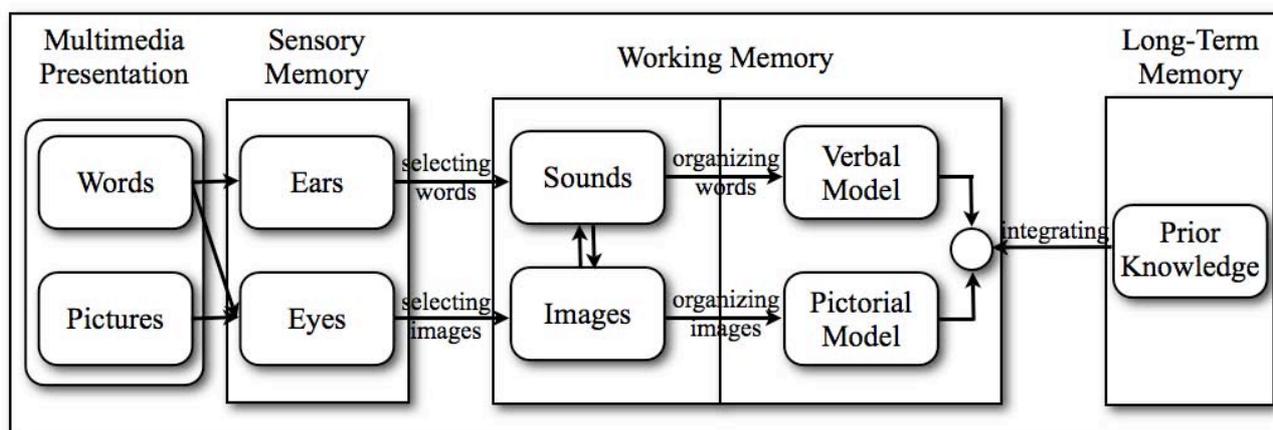


Figure 1: The Cognitive Theory of Multimedia Learning (CTML) of Mayer

Mayer (2005) acts on the assumption of two channels, one for visually represented material and one for auditory represented material. The differentiation between the visual/pictorial channel and the auditory/verbal channel is of importance only with respect to the working memory. Here humans are limited in the amount of information that can be processed through each channel at a time. Besides the working memory Mayer assumes two further types: the sensory memory and the long-term memory. Furthermore, according to Mayer humans are actively engaged in cognitive processing. For meaningful learning the learner has to engage in five cognitive processes:

- (1) Selecting relevant words for processing in verbal working memory
- (2) Selecting relevant images for processing in visual working memory
- (3) Organizing selected words into a verbal model
- (4) Organizing selected images into a pictorial model
- (5) Integrating the verbal and pictorial representations, both with each other and with prior knowledge (Mayer 2005, 38)

Concerning the process of internalization the CTML is of particular importance. The comprehension of a mathematical operation is not developed unless a child has the ability to build mental connections between the different forms of representation. According to Aebli (1987) for that purpose every new and more symbolical external representation must be connected as closely as possible to the preceding concrete one. This connection takes place on the second stage of the process of mathematical learning where the transfer from concrete acting over more abstract, iconic and particularly static representations to the numeral form takes place (Fig. 2). A chance in the use of computers in primary school is seen in supporting the process of internalization by the use of MELRs. This is the main motivation for the research on how the knowledge about MERs and MELRs in elementary mathematics and educational software is actually used and how it can be used in the future.

TO THE REALISATION OF MERS AND MELRS IN ELEMENTARY MATHEMATICS SOFTWARE

Despite the fact that computers can be used to link representations very closely, it is hardly made use of in current educational software packages. Software that offers MERs and MELRs with the aim to support the process of internalization is very rare. This is also the reason why tasks are mainly represented in a symbolic form (Fig. 2).

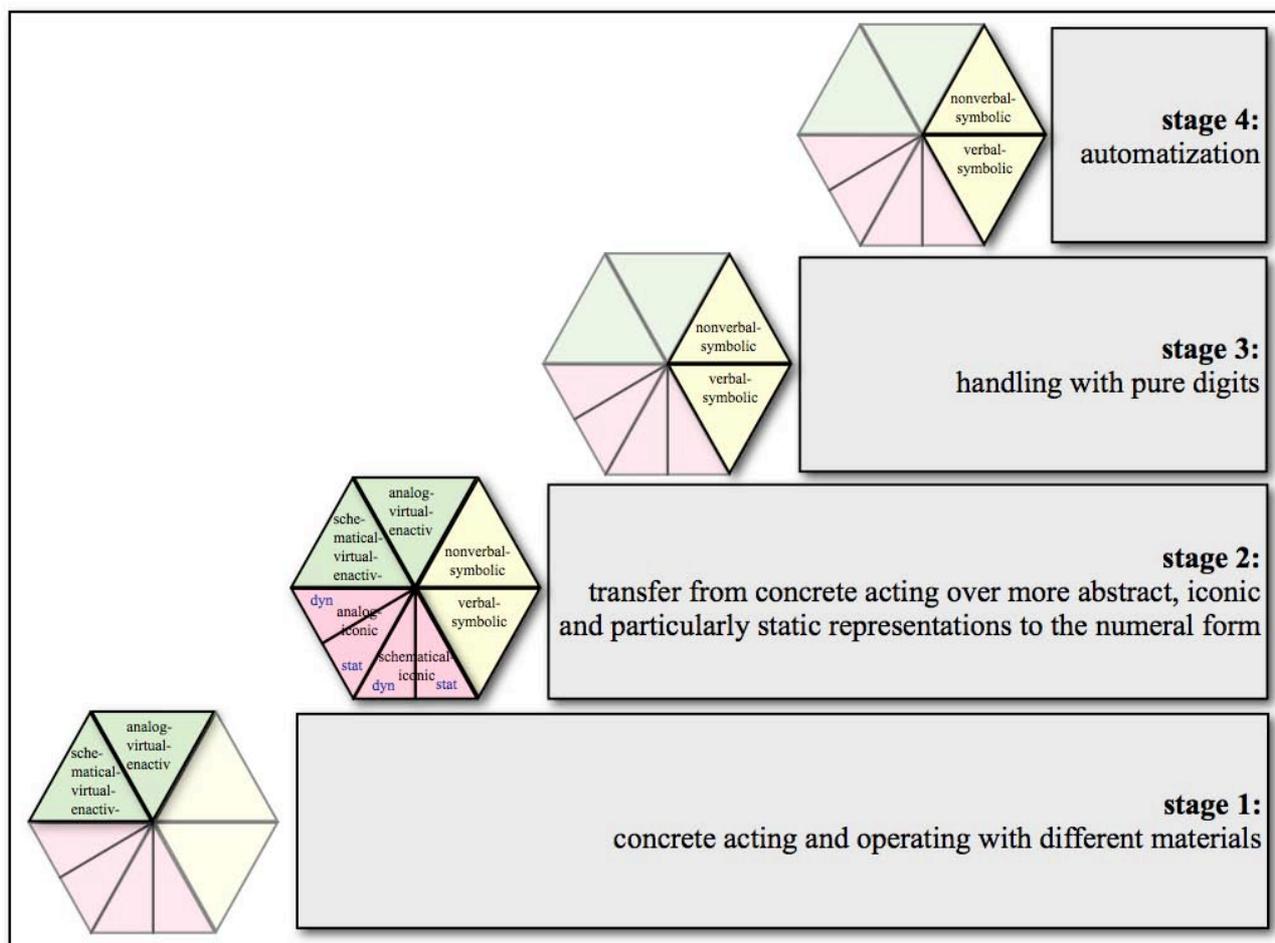


Figure 2: Forms of external representations combined with the four stages of the process of mathematical learning

Nevertheless, most software offers *help* in form of visualizations and thereby goes backward to the second stage. This is realised in different ways, which is why a study of current software was done with regard to the following aspects:

- Which forms of external representations are combined (MERs) and how are they designed?
- Does the software offer a linking of equivalent representations (MELRs) and how is the design of these links?

After this analyse, a total of sixty 1st- and 2nd-grade-children at the age of six to eight years were monitored in view of their handling of certain software (BLITZRECHNEN 1/2, MATHEMATIKUS 1/2, FÖRDERPYRAMIDE 1/2). Beside

this own exploration – which will not be elaborated at this point - there is only a small number of studies that concentrates on MERs and MELRs on elementary mathematics software. In 1989, Thompson developed a program called BLOCKS MICROWORLD in which he combined Dienes blocks with nonverbal-symbolic information. Intention was the support of the instruction of decimal numeration (kindergarten), the addition, subtraction and division of integers (1st – 4th grade) as well as the support of operations with decimal numbers (Thompson 1992, 2). Compared to activities with “real things”, there were no physical restrictions in the activities with the virtual objects to denote. Furthermore the program highlighted the effects of chances in the nonverbal-symbolic representation to the virtual-enactive representation and reverse. In his study with twenty 4th-grade-children Thompson could show that the development of notations has been more meaningful to those students who worked with the computer setting compared to the paper-pencil-setting. The association between symbols and activities was established much better by those children than by the others.

Two further studies that examined multi-representational software for elementary mathematics are by Ainsworth, Bibby and Wood (1997 & 2002). The aim of COPPERS is to provide a better understanding of multiple results in coin problems. Ainsworth et al. could find out, that already six-years-old children do have the ability to use MERs effectively. The aim of the second program CENTS was the support of nine- to twelve-years-old children in learning basic knowledge of skills in successful estimation. There were different types of MERs to work with. In all three test groups a significant enhancement was seen. The knowledge of the representations themselves as well as the mental linking of the representations by the children were a necessary requirement. The fact that a lot of pupils weren't able to connect the iconic with the symbolic representation told Ainsworth et al. (1997, 102) that the translation between two forms of representations must be as transparent as possible.

The opinions about an automatic linking of multiple forms of representations vary very much. Harrop (2003) considers that links between multiple equivalent representations facilitate the transfer and thus lead to an enhanced understanding. However, such an automatic translation is seen very controversial. Notwithstanding this, it is precisely the automatism that presents one of the main roles of new technologies in the process of mathematical learning (cf. Kaput 1989). It states a substantial cognitive advantage that is based on the fact that the cognitive load will be reduced by what the student can concentrate on his activities with the different forms of representations and their effects. An alternative solution between those two extremes – the immediate automatic transfer on the one hand and its non-existence on the other hand – is to make the possibility to get an automatic transfer shown to a decision of the learner.

PRINCIPLES FOR DESIGNING MERS

The initial point and justification of multimedia learning is the so-called multimedia principle (cf. Mayer 2005, 31). It says that a MER generates a deeper understanding than a single representation in form of a text. The reason for this is rooted in the different conceptual processes for text and pictures. In being so, the kind of the combined design is of essential importance for a successful learning. The compliance of diverse principles can lead to an enhanced cognitive capacity. Thus Ayres & Sweller (2005) could find a *split-attention-effect* if redundant information is represented in two different ways because the learner has to integrate it mentally. For this more working space capacity is required, and this amount could be reduced if the integration were already be done externally. Mayer (2005) diversifies and formulates besides his *spatial contiguity principle* the *temporal contiguity principle*. According to this principle, information has not only to be represented in close adjacency but also close in time. If information is also redundant, the elimination of the redundancy can lead to an enhanced learning (*redundancy-effect*). The *modality principle* unlike the split-attention principle does not integrate two external visual representations but changes one of it into an auditory one. Hence an overload of the visual working memory can be avoided.

In addition to the modality principle Mayer recommends the segmenting principle as well as the pretraining principle to enhance essential processes in multimedia learning. As a result of the *segmenting principle* multimedia information is presented stepwise depending on the user so that the tempo is decelerated. Thus the learner has more time for cognitive processing. The *pretraining principle* states that less cognitive effort will be needed if an eventual overload of the working memory is prevented in advance through the acquisition of previous knowledge. Finally, the abundance of the *signaling principle* allows a deeper learning due to the highlighting of currently essential information. Extraneous material will be ignored so that more cognitive capacity is available and can be used for the essential information.

In elementary instruction the children first of all have to learn the meaning of symbolic representations and how to link them with the corresponding activities. So the above-described principles cannot be adopted one-to-one. Based on an empirical examination of the handling of six- to eight-years-old pupils with MERS and MELRs in chosen software, we could identify new principles and the above-described ones could be adapted, so that their compliance supports the process of internalization. These principles are demonstrated and realized in the following example of the prototype DOPPELMOPPEL.

THE PROTOTYPE DOPPELMOPPEL

Didactical concept and tools

The function of the ME(L)Rs in DOPPELMOPPEL is the construction of a deeper understanding through abstraction and relations (fig. 3). The prototype was built

using the Geometry software Cinderella (Richter-Gebert & Kortenkamp 2006) and can be included into web pages as a Java applet.

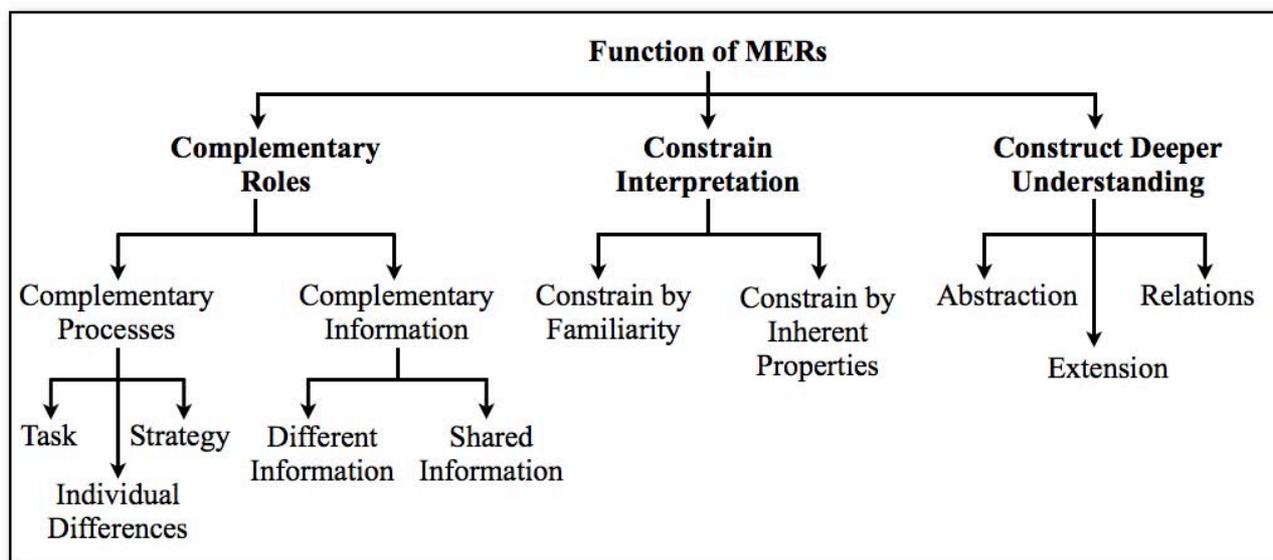


Figure 3: Functions of MERs according to Ainsworth (1999)

Using the example of doubling and halving the children shall – in terms of internalization – link their activities with the corresponding nonverbal-symbolic representation and they shall figure out those symbols as a log of their doing. The mathematical topic of doubling and halving was chosen because it is a basic strategy for solving addition and subtraction tasks. In addition, DOPPELMOPPEL offers to do segmentations in common use.

The main concern of the prototype is to offer a manifold choice of forms of representations and their linking in particular (MELRs). Two principles that lead the development are the *constant background principle* and the *constant position principle*. The first one claims a non-alteration of the design of the background but an always-constant one. Furthermore the position of the different forms of representations should always be fixed and visible from the very beginning so that they don't constrict each other.

DOPPELMOPPEL provides the children with the opportunity to work in many different forms of representations. On the one hand there is a zone in which the children can work **virtual-enactive**. Quantities are represented through circular pads in two colours (red and blue). To enable a fast representation (*easy construction principle*) and to avoid “calculating by counting” there are also stacks of five next to the single pads. According to our reading direction the five pads are laid out horizontally. The elimination of pads happens through an intuitive throw-away gesture from the “desk” or, if all should be cleaned, with the aid of the broom button. A total of maximal 100 pads fit on the table (10x10). The possible activities of doubling, halving and segmenting are done via the two tools on the right and the left hand side of the desk (fig. 4).

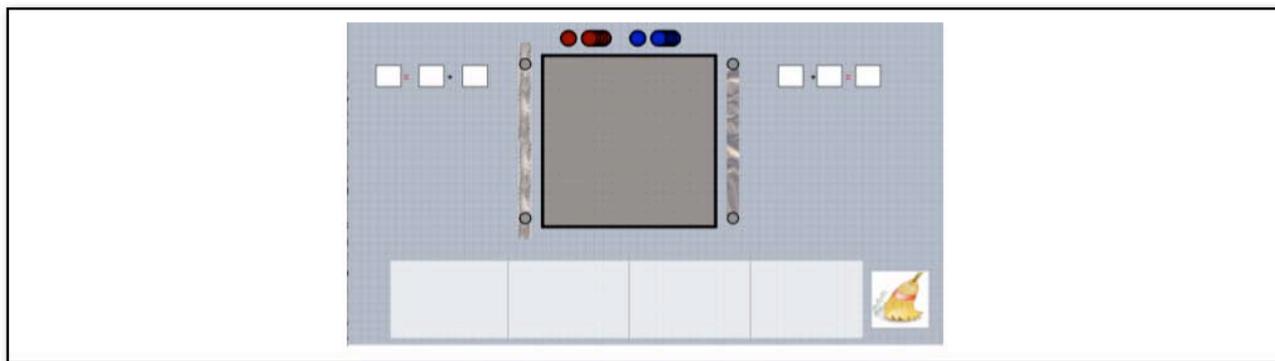


Figure 4: Screenshot of the prototype DOPPELMOPPEL

The doubling-tool (to the right) acts like a mirror and doubles the laid quantities. The saw (to the left) divides the pads and moves them apart. Both visualisations are only shown for a short time after clicking on the tools. Afterwards, the children only see the initial situation and have to imagine the final situation (mirrored resp. divided) themselves. The pupils can use the mouse to drag the circular points on the doubling-tool and the saw to move them into any position. A special feature of the saw is that it also can halve pads. At this point the program is responsive to the fact that already six-years-olds know the concept of halves because of the common use in everyday life.

The children can do **nonverbal-symbolic** inputs themselves in the two tables on the right and the left hand side. The left table enables inputs in the form $_ = _ + _$, the right one in the form $_ + _ = _$. The table on the right is only intended for doubling and halving tasks. That's why the respectively other summand appears automatically after the input of one. In the table on the left any addition task can be entered.

If the pupils don't fill in the equation completely they have the possibility to get their input shown in a **schematic-iconic** representation. Depending on the entered figures, the pads appears in that way that the children can't read the solution directly by means of their colour. The doubling-tool respectively the saw are placed according to the equation so that the children – like in the virtual-enactive representation – are able to act with the tools (fig. 5).

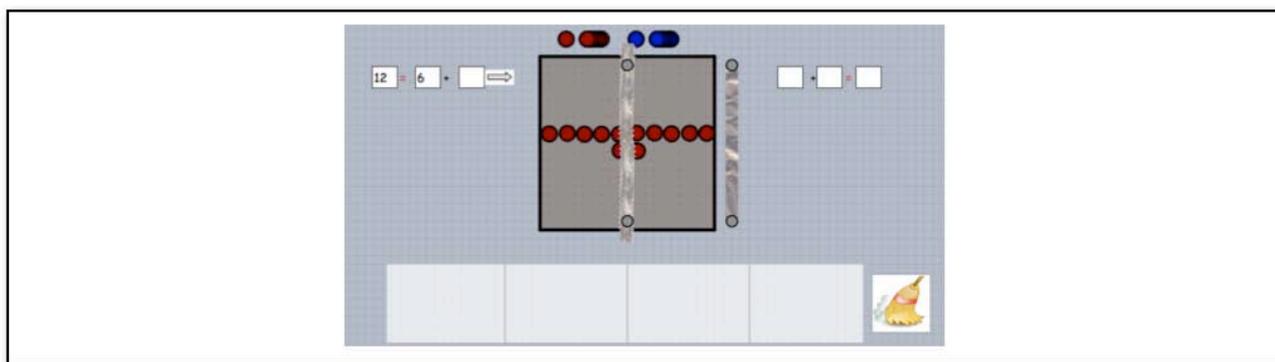


Figure 5: Schematic-ionic representation of a task

According to the *signaling principle* an arrow is highlighted when the pupils enter numbers in the free boxes. A click on this arrow initiates the **intermodal transfer**. A similar arrow appears below the desk after every activity done by the children (click on the doubling-tool respectively the saw). Here, the pupils have the possibility to let the software perform the intermodal transfer from the virtual-enactive and the schematic-ionic representation to the nonverbal-symbolic one. This is another special feature of DOPPELMOPPEL that is rarely found in current educational software. If external representations are linked, the linking is mostly restricted to the contrary direction. Depending on the activity the equation appears again in the form $_ = _ + _$ or $_ + _ = _$. Those equations aren't separated consciously, however a coloured differentiation of the equal and the addition sign (as in the tables above) point to pay attention.

Besides the forms of representations there are two more functions available. Both – the broom to clean the desk and the exclamation mark for checking answers – take some time in order to encourage considerate working and to avoid a trial-and-error-effect. If the equation is false the program differentiates on the type of error. In case of an off-by-one answer or other minor mistake the boxes are coloured orange otherwise red. If the equation is correct a new box appears below.

This prototype doesn't already respond to *modalities* but the concept already incorporates auditory elements.

Testing of DOPPELMOPPEL

For the testing of DOPPELMOPPEL four versions of the prototype were created. Two of those feature multiple representations; the other two only offer single representations. One of the multiple representations provides an additional linking, that is an intermodal transfer in both directions (fig. 6).

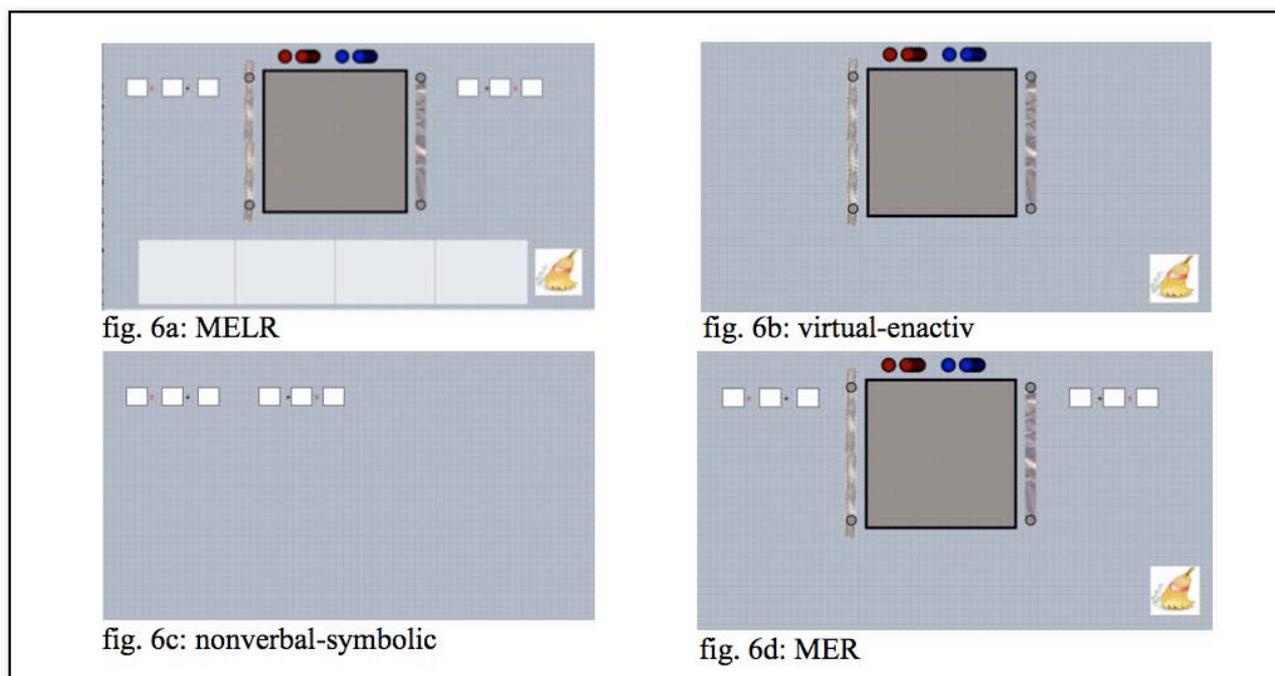


Figure 6: 4 versions of the prototype

The dedication of those four versions is to make sure that it is neither the medium computer nor the method of instruction that causes results of the testing.

28 pupils of a 1st class worked about 20 minutes per five terms with the program. During their work there was one student assistant who observed and took care of two children. In addition, the activities of the children were recorded with a screencorder-software. Furthermore a pre- and a posttest were done.

To the current point of time the data interpretation is still in progress but first results should be available to the end of January.

CONCLUSION

Educational software that is based on the primacy of educational theory, as claimed by Krauthausen and others, has to take both mathematics and multimedia theory into account. Carefully crafted software however, is very expensive in production. We hope to be able to show with our prototype that this investment is justified.

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THE IMPACT OF TECHNOLOGICAL TOOLS IN THE TEACHING AND LEARNING OF INTEGRAL CALCULUS

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There is still a tendency to see that mathematics is not visual. At University education, it's evident in several ways. One of them, is an algebraic and reductionist approach to the teaching of calculus.

In order to improve educational practices, we designed an empirical research for the teaching and learning of integral calculus with technological tools as facilitator resources of the process of teaching and learning: the use of predesigned software that enables to get the conceptualization in a visual and numeric way, and the using of a virtual platform for complementary activities and new forms of collaboration between students, and between teachers and students.

KEY WORDS

Predesigned software – virtual environments – registers of representation - social infrastructure - epistemological infrastructure

INTRODUCTION

The ideas, concepts and methods of mathematics presents a visual content wealth, which can be geometrically and intuitively represented, and their use is very important, both in the tasks of filing and handling of such concepts and methods, and for the resolution of problems.

Experts have visual images, intuitive way of knowing the concepts and methods of great value and effectiveness in their creative work. Through them, experts are able to relate, most versatile and varied, often very complex, constellation of facts and results of their theory and, through such significant networks, they are able to choose from, so natural and effortless, most effective ways of solving the problems they face (Guzman, 1996). Viewing, in the context of teaching and learning of mathematics at the university, has to do with the ability to create wealthy images that individuals can handle mentally, can pass through different representations of the concept and, if necessary, can provide the mathematic ideas on a paper or computer screen (Duval, 2004). The creative work of mathematicians of all times has had “the visualization” as its main source of inspiration, and this has played an important role in the development of ideas and concepts of the infinitesimal calculus.

However, there is a tendency to believe that mathematics is not visual. At university education, it's evident, particularly through an algebraic and reductionist approach of the teaching of calculus. One of the didactic phenomena which is considered essential in the teaching of Mathematical Analysis, is the “*algebrización*”, that is: the algebraic treatment of differential and integral calculation. Artigue (in Contreras, 2000) expresses this fact in terms of an algebraic and reductionist approach of the calculation which is based on the algebraic operations with limits, differential and integral calculus, but it treats the thinking and the specific techniques of analysis in a

simplistic way, such as the idea of instantaneous rate of change, or the study of the results of these reasons of change.

We believe that the problems with Mathematical Analysis learning, in the first year of college, have to do with this context. These difficulties are associated with the formalism in dealing with the concepts and the lack of association with a geometric approach. Anthony Orton has worked for a long time about the difficulties in learning calculus. His research work at the University of Leeds confirmed that students had difficulty in learning the concepts of calculus: the idea of exchange rate, the notion of a derivative as a limit, the idea of area as the limit of a sum (Orton, 1979). Cornu (1981) arrived at similar conclusions regarding the idea of "unattainable limit" and Schwarzenberger and Tall (1978) regarding the idea of "very near". Ervynck (1981) not only documented the difficulties of the students in understanding the concept of limit but he also remarked the importance of viewing the processes by successive approximations. In this sense, we can see that usual graphs met in textbooks of calculus have two problems: they are static, which can not convey the dynamic nature of many of the concepts, and also they have a limited number of examples, usually one or two, which leads to develop, in students, a narrow image of the concept in question. (Tall and Sheath, 1983). In this sense, taking into account our previous exploratory research (Milevicich, 2008), we can say that students can not understand the concept of definite integral of a function as the area under the curve, because they do not visualize how to build this area as a sum, usually known as Riemann Sum.

In terms of the educational processes, it should be noted that teachers usually introduce the concept of integral in a narrative way, avoiding the real purpose, which is to obtain more precise approximations. A simplistic approach to the concept is usually done, disconnected from integral calculus applications, which hinders the understanding of students, and consequently, the resolution of problems relating to calculation of areas, length of curves, volume of solids of revolution, and those dealing with applications to the engineering work, pressure, hydrostatic force and center of mass.

JUSTIFICATION

Innovation in educational processes including the use of multimedia means demands not only on teachers' professionalism but also new activity managing. Research work is currently being carried out at different universities aiming to find out what use teachers make of these tools and the specific competencies that they have to acquire for making effective use of them. From a didactic point of view, the usage of multimedia in teaching-learning process, presumably, should increase students motivation, and, in that sense, we ask ourselves: What should be the goals of education aimed at improving the university today? and How can we make it easier through the use of technological tools? The answers to these questions are not clear for us. Students, nowadays, have more and more information than they can process, so that one of the functions of the university education would be to provide them with cognitive and conceptual tools, to help them to select the most relevant information.

University Students should try to get skill and develop attitudes that enable them to select, process, analyse and draw conclusions. This change in the goals represents a departure from traditional learning. In this sense, the use of a predesigned software in the classroom, designed within the group research, can be a teaching facilitator resource of the process of teaching and learning:

- to convey the dynamic nature of a concept from the visualization,
- to coordinate different registers of representation of a concept,
- for the creation of personalized media best suited to the pedagogical requirements of the proposal.

RESEARCH CHARACTERISTICS

Population and sample

The population is made up of Engineering students from Technological University and the specimen is a Electrical Engineering commission of about 30 students. Regarding the characteristics of the population, some considerations can be made about their previous knowledge of integral calculus. Some students come from the Mechanic School of a known automotive Company and others, from a technical electricians school. Based on a detailed analysis of library materials used by teachers in these institutions, and the students' writings, we infer that integrals are taught as the reverse process of derivation, with the focus on the algebraic aspects. These students study the concept of integral associated with a primitive, practice various methods of integration, transcribe or solve hundreds of exercises in order to calculate integrals, and some of them even achieve a considerable level of skill in the use of tricks and recipes that help to be more effective in getting results. Another group of students come from near schools where geometric concepts are little, essentially the calculating of areas studied during primary and middle school. However, the largest group, is made up of students studying Mathematical Analysis for the second or third time. Some of them have completed the course in previous years but failed in the exams. It may be that those students have some ideas about integral calculus and its applications, or not. It is possible that those ideas interfere with the getting of new knowledge or hinder it (Bachelard, 1938), primarily on those students who associate the integral exclusively to algebraic processes. That is why it was very important to carry out a diagnostic test (pretest) that would allow exploration on the previous skills and students ideas about definite integral and thus, categorize according to the following levels of the independent variable:

Level 1: associate the concept of integral to the primitive of a function and calculates easy integrals.

Level 2: associate the concept of integral to the primitive of a function, calculates easy integrals and links the concept with the area under the curve.

Level 3: associate the concept of integral to the primitive of a function and links the concept with the area under the curve.

Level 4: has no specific pre knowledge associated with the topic.

Focus

The general purposes of our research work were:

to determine if students understand the concept of integral through the implementation of a proposal that would allow its teaching in an approaching process, using different systems of representation, according to the processes man has followed in his establishment of mathematical ideas,

to analyze, in a reflective learning context, the ways in which students solve problems related to integral calculus,

and the specific purposes were:

to categorize the students, involved in the experience, according to his integral calculation preconceptions, at the beginning of the intervention,

to implement a proposal that provided, on the one hand, the use of different systems of representation in the development of individual and group activities, and on the other, to promote conjecture, experiment, formalization, demonstration, synthesis, categorization, retrospective analysis, extrapolation and argumentation, with the help of specific software, and feedback on students' early productions so they could reflect on their own mistakes,

to review progress achieved after the implementation of the didactic proposal,

to analyze the impact of using a virtual platform for complementary activities.

Methodology

The design is pre-experimental type of pretest - treatment - posttest with a single group. The independent variables in this study are: the design of teaching and pre knowledge of students on the definite integral. The dependent variable is: the academic performance.

Regarding these previous knowledge, a pretest at the beginning of the intervention allowed to place each student in one of the preset categories. After 8 weeks of intervention, a posttest allowed to determine the levels of progress made in learning the concepts of integral calculus in relation to the results obtained in the past three years cohorts (2003, 2004 and 2005). In addition, an interview at the end of the experience was implemented, in order to gather qualitative information.

In order to improve educational practices, we designed a proposal for teaching and learning integral calculus according to the proposal of using a pre designed software as indicated in the goals. In this sense:

We designed a software package allowing the boarding of integral calculus from the concept of definite integral associated with the area under the curve, from a geometric point of view.

We selected the problems students should solve, in a way, that their approach would allow to establish a bridge between conceptualization of integration and problems related to engineering. In that sense, the use of the computer allowed to have a very wide range of problems, where the choice was not conditioned by the difficulty of algebraic calculus.

The students used pre designed software for:

- a) The successive approximations to the area under a curve, considering left and right points on each of the subintervals. The software allows to select the function, the interval and the number of subdivisions. (See Figure 1).
- b) The successive approximations to the area under a curve through the graph of the series which represents the sum of the approach rectangles (See Graphic 1) and the table of values (See Table 1).
- c) The visualization of the area between two curves, it also allows to determine the points of intersection.
- d) The representation of the solid of revolution on different axes when rotating a predetermined area. (See Figure 2)
- e) The numerical and graphical representation (through table of values) of the area under the curve of an improper integral.

It was designed a set of activities with the purpose students conjecture, experience, analyze retrospectively, extrapolate, argue, ask their peers and their teachers, discuss their own mistakes and evaluate their performance. Assessment techniques were redesigned, so that the analysis of students productions would provide feedback about their mistakes.

We incorporated a Virtual Campus using Moodle supporting design, as an additional element, in order to keep continuity between two spaced weekly meetings. According to Misfeldt and Sanne (2007), communication on mathematical issues is difficult using computers and a weekly meeting is insufficient. In response to this problem, we used the virtual campus for communication, flexibility and cooperation, but the use of it was not a learning objective in itself. Instead, we used it to publish texts and exercises guides and also, students made active use of the forum for discussion groups.

We also had in mind that the challenges in creating an online learning environment might be different when working with mathematics than in other topics (see also: Misfeldt et. al, 2007 & Duval, 2006). Many of the signs that goes into building mathematical discourse is not available on a standard keyboard, and the way that mathematical communication often is supported by many registers and modalities that are used simultaneously, as writing and drawing various representations on the blackboard or paper is also not available. Students, using the Virtual Campus, had the possibility to upload files showing the solving process and using every symbol they needed.

Implementation of the proposal

Students were distributed in small groups no more than three, who worked in several sub-projects. Each of them included a significant number of problems.

Subproject No. 1: The concept of integral.

Subproject No. 2: Fundamental theorem of Calculus.

Subproject No. 3: Improper integrals.

Subproject No. 4: Area between curves.

Subproject No. 5: Applications of Integral Calculus.

Guidelines for systematic work for each of the meetings were made. In the first part, it was discussed the progress and difficulties of the previous practice, where the essential purpose was to ensure that students analyze their own mistakes, and the second part, teachers and students worked on new concepts at the computer laboratory. The first part of each meeting was guided by the teacher, but an assistant teaching and an observer teacher were present in the class. The second half had the same staff and an extra assistant teaching.

The assessment took place during the whole experience through:

- weekly productions of students reflected in their electronic folders and notebooks. These ones allow cells to keep comments, observations, etc.; very valuable material in assessing the level of understanding achieved by students.
- students interaction in classes and into working groups.
- Students participation in the discussion forums of the virtual campus.

In that sense, spreadsheets were used for monitoring activities, which proved to be an effective tool to assess different aspects relevant to student's performance. Summary notes taken by the observer teacher along the 8 weeks allowed us to infer the change of attitude in an important group of these students. From the initial population, made up of 30 students, 24 of them showed increased commitment to the development of activities.

Some of these activities were:

Subproject 1: Evaluate the following integrals by interpreting each in terms of areas

$$a) \int_1^3 e^x dx \quad b) \int_0^3 (x-1)dx$$

Case a: because $f(x)=e^x$ is positive the integral represents the area. It can be calculated as a limit of sums and a computer algebra system can be used to evaluate the expression.

Case b: The integral cannot be interpreted as an area because f takes in both positive and negative values. But students should realize that the difference of areas works.

Subproject 3: Sketch the region and find its area (if it is possible)

$$a) S=\{(x,y)/ 0 \leq x \leq \pi, 0 \leq y \leq \tan(x)\sec(x)\}$$

$$b) S=\{(x,y)/ x \geq 0, 0 \leq y \leq e^{-x^2}\}$$

Case a: Probably students confuse the integral with an ordinary one. They should warn that there is an asymptote at $x= \pi/2$ and it must be calculated in terms of limits.

At this point students must bear in mind that whenever they meet the symbol $\int_a^b f(x)dx$

they must decide, by looking at the function f on $[a,b]$, whether it is an ordinary definite integral or an improper integral.

Case b: The integral is convergent but it cannot be evaluated directly because the antiderivative is not an elementary function. It is important students look for a way to solve the problem and although it is impossible to find the exact value, they can know whether it is convergent or divergent using the Comparison Test for Improper Integrals.

Both examples above show activities where students need to find out solutions and get conclusions without teacher telling them.

RESULTS

The pretest was done by 30 students, the results allowed us to locate them as follows: 15 at Level 1, 1 at Level 2 and 14 at level 4. It should be noted that those who came from technical schools had achieved a considerable level of skill in the calculation of integrals but they didn't know about the links with the concept of the area. The posttest consisted of 6 problems related to the sub projects students had worked on, each of which was formed by several items. It was provided to the 24 students remaining at the end of the experience, and took place at the computer laboratory, where students usually worked. In general, the level of effectiveness was above 50%, except in the case where they were asked to determine the area between two curves and then the volume to rotate around different axes. The difficulty was to get the solid of revolution from a shift in the rotation axis. Although the students had no difficulty in getting the solid geometrically, they could not get an algebraic expression for it.

In a comparison with the three previous year cohorts, it was possible to emphasize the following differences:

- a) There were no important difficulties in linking the concepts of derivative and integral.
- b) An important group of students (83% of them) successfully used Fundamental Theorem of Calculus.
- c) In general, there were no difficulties in algebraic developments, however it is possible to associate the lack of such obstacles to the use of the computer. All of students tested, could associate the concept of solid revolution with the concept of integral, and even more, they were able to correctly identify the area to rotate.
- d) The 74% of the students tested could identify improper integrals, but only 43% of them, correctly, applied the properties.
- e) Most of the students tested succeeded in establishing a bridge between the conceptualization of integration and problems related to engineering: 89% of them correctly solved problems relating to applications for work, hydrostatic pressure and force.

The written interviews at the close of the experience reflects the importance that students attribute to the use of virtual campus as an additional resource: most of students were very keen on having prompt responses from the teacher when asking questions in the forum and the help offered by other students.

One of the questions was:

“How did teachers interventions at the forum helped, when you had difficulties in the development of practices? (A: they were decisive, B: they helped me to understand, C: they were not decisive. I managed without them, D: they did not contribute at all. Please explain your choice).”

12 students selected A , 8 pupils selected B, 4 students selected C and D was not selected.

Some of the explanations given by students were:

Student a: “...*They helped me because teachers answered quickly and clearly*”

Student b: “...*Excellent, clear and concise answers that helped with the resolution of the problems.*”

Student c: “...*There were many situations where I managed to solve a problem just reading the doubts of my fellow students. I have not done a lot of questions at the forum because someone asked my doubt before me...*”

It is worth mentioning that there were no substantial differences between the students belonging to different categories, according to the pretest. An analysis of results in relation to the initial categorization, suggests that pre conditioned ideas did not influenced the acquisition of new knowledge. There were no significant differences among the largest groups of students ranked in levels 1 and 4.

CONCLUSIONS

The failure of the students in understanding the concepts of calculation, more generally, and the definite integral, in particular, is one of the most worrying problems in the learning of Mathematical Analysis, in the first year of Engineering, as this hinders the understanding and resolution of problems of application. The way to search for the causality of this failure led us to raise the need for a change in the point of view. This is a change in the processes and representations through which students learn, in this case, the concept of integral.

Focusing our attention on the problem how students can understand more deeply the concepts using tools and technology, we can conclude that the recent evolution of digital materials leads to devote a specific interest to the change of activities induced by virtual learning environments which allow new forms of collaboration between students, and between teachers and students. Besides, the use of the computer is a valuable strategy with the aim of achieving significant learning. While learning the concept of definite integral, the computer facilitates making the important amount of calculations and displays the successive approximations, contributing to the concept of area under the curve. In that sense, the use of a predesigned package software allowed students to view the alignment between the smaller and smaller geometric rectangles and curvilinear area to be determined.

The carrying out of the activities required the use of the predesigned package software, specifically adapted to the needs of the experience. Students had to make numerous graphs, edit their guesses, propose new solutions, test, and analyze retrospectively the achieved results. Dynamic graph was valued for making student work with figures easier, faster and more accurate, and consequently for removing drawing demands which distract them from the key point of a problem. Various aspects of making properties apprehensible to students through dynamic manipulation were expressed in CERME V Plenaries: “*When a dynamic figure is dragged, students can see it changing and see what happens, so that properties become obvious and students see them immediately*” (Ruthven,2007: 56). In that sense, technology is seen

as supporting teaching approaches based on guiding students to discover properties for themselves. We agree on suggesting that teachers might guide students towards an intended mathematical conclusion, but students could find out how it works without us telling them so that they could feel they are discovering for themselves and could get a better understanding.

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APPENDICES

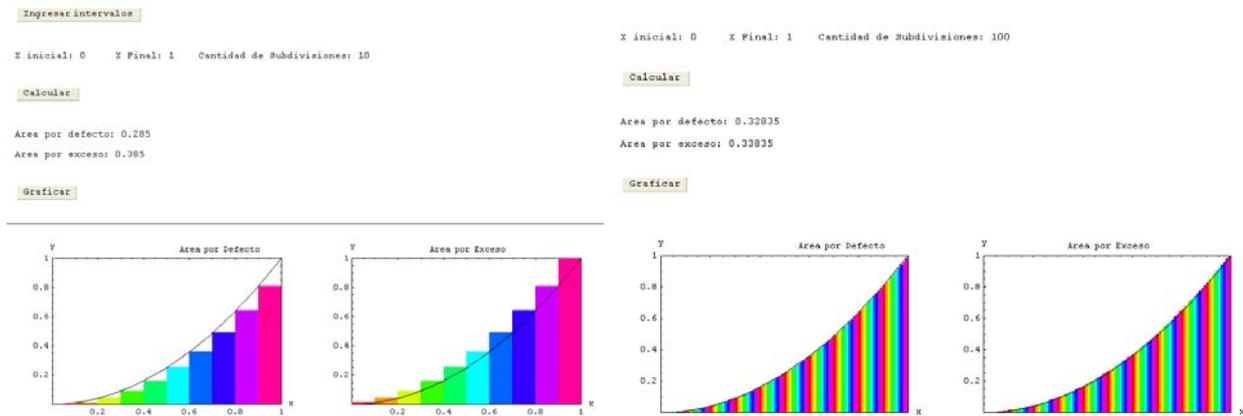


Figure 1. Capture screen from the predesigned software about conceptualization of definite integral. Estimation of the area of $y=x^2$ using 10 subdivisions and 100 subdivisions, $0 \leq x \leq 1$

number of subdivisions	default sums	excess sums
4	0,219	0,467
10	0,285	0,385
20	0,308	0,358
30	0,316	0,35
40	0,321	0,346
50	0,323	0,343
60	0,325	0,342
70	0,326	0,34
80	0,327	0,339
90	0,328	0,339
100	0,327	0,337

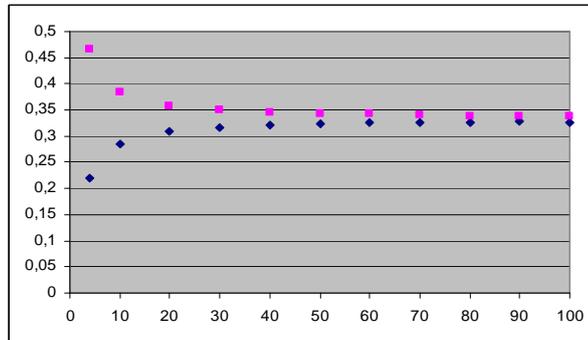


Table 1. Sums for different subintervals increasingly small under the curve $y= x^2$ on the interval $[0,1]$

Graphic 1. the series which represents the sum of the approach rectangles, default sums are in blue and excess sums are in pink.

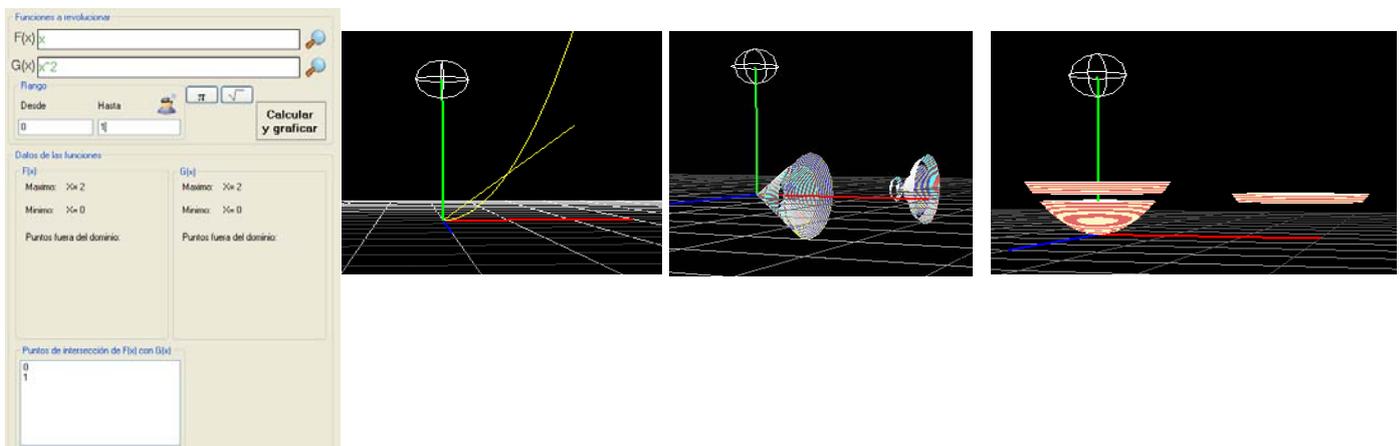


Figure 3. Captured screen from the predesigned software about Solid of revolution. Area between the functions $y=x$ and $y=x^2$, and the solid of revolution that is generated to rotate on the x-axis and the y- axis.

USING TECHNOLOGY IN THE TEACHING AND LEARNING OF BOX PLOTS

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Box plots (or box-and-whisker-plots) can be used as a powerful tool for visualising sets of data values. Nevertheless, the information conveyed in the representation of a box plot is restricted to certain aspects. In this paper, we discuss both the potential and limitations of box plots. We also present a design for an empirical study in which the use of a variety of tasks explicitly addresses this duality. The activities used in the study are based on an interactive box plot applet that surpasses the currently available tools and offers new ways of experiencing box plots.

MOTIVATION

Recently, the mathematics curricula of many parts of the world were revised in order to include more statistics and data analysis. In the literature, one can find an extensive discussion about this idea under the notion of “statistical literacy” (Wallman, 1993; Watson & Callingham, 2003). This reflects the growing importance of the ability to understand and interpret data that has been collected or is being presented by others. The NCTM (2000) standards, for example, state, “To reason statistically--which is essential to be an informed citizen, employee, and consumer--students need to learn about data analysis and related aspects of probability.” The global availability of data through the Internet makes it easy to access and process huge data sets. For these, it is important that students have the skills and tools to summarise and compare the data, also by using the computer.

In this paper, we focus on *box plots* as a means to visualize statistical data. Box plots are used not only in textbooks, but are also available in graphing calculators. In order to use statistical information properly, the students have to develop a clear concept of what the information means, no matter whether it is given numerically or, in this case, visually.

The situation described also applies to Germany where some states have incorporated a larger amount of statistics and data analysis into the mathematics curriculum. Our personal experience with teacher students teaching in 8th grade (14-year-olds) has shown that both teachers and learners tend to ignore the mathematical concepts behind the statistical analysis and fall back to recipes that enable them to solve the standard exercises from the text books. In a similar way, Bakker, Biehler and Konold (2004) point out that some of the features inherent to box plots raise difficulties in young students’ understanding and use of them. As a remedy, we developed a series of activities that should enable students to develop a clear understanding of the statistical terms. The ultimate goal of the activities is that students can not only draw box plots for given data, but also interpret box plots that describe real world situations.

THEORETICAL BACKGROUND

Box plots are part of the field of Exploratory Data Analysis where data is explored with graphical techniques. Exploratory Data Analysis is concerned with uncovering patterns in all kinds of data. A box plot (or box-and-whisker-plot) is a relatively simple way of organizing and displaying numerical data using the following five values: the minimum value, lower quartile¹, median², upper quartile, and maximum value. Considering a set of data values like, for example, 52, 32, 29, 30, 35, 17, 42, 63, these five values are easy to calculate: minimum value = 17, lower quartile = 29.5, median = 33.5, upper quartile = 47, and maximum value = 63.

Using these five numbers, the related box plot can be constructed on a vertical (which we use in the following description) or horizontal scale (which is used in Fig. 1) by (a) drawing a box that reaches from the lower quartile to the upper quartile, (b) drawing a horizontal line through the box where the median is located, (c) drawing a vertical line from the lower quartile (the lower end of the box) to the minimum value, (d) drawing a vertical line from the upper quartile (the upper end of the box) to the maximum value, and finally (e) marking minimum and maximum with horizontal lines. Figure 1 shows the box plot corresponding to the data above, created with a box plot applet provided by CSERD.

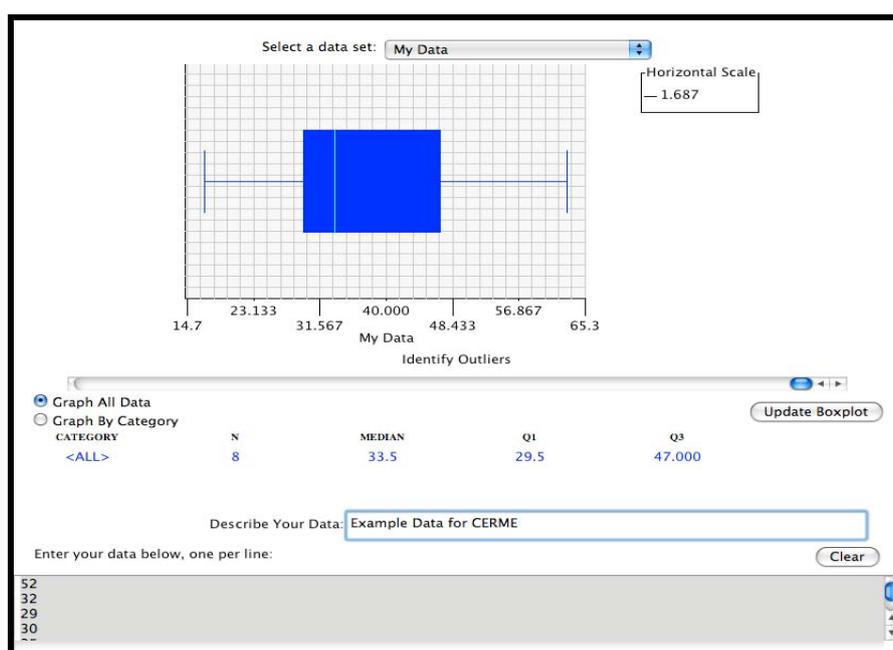


Figure 1: Box plot created online for the sample data in this article

At the same time, box plots contain *more* and *less* information. On the one hand, the representation of a box plot communicates certain information at a glance: The median and the quartiles can easily be recognized which is not the case for the

¹ As there is no universal definition of a quartile, we dedicated a whole subsection of this article to this issue. Also, the original box plot uses the lower and upper hinge instead of the quartiles.

² The median can be defined as the number separating the lower half of a data set from the higher half in the sense that at least 50% of the values are smaller than or equal to the median.

original set of data values. Moreover, the line indicating the median illustrates the centre of the data, the width of the box demonstrates the spread of the central half of the data, and the length of the two lines next to the box show the spread of the lower and upper quarters of the data. This enables skilled people to interpret the box plot and draw conclusions about the underlying distribution. Various authors have declared that box plots are particularly useful for easily comparing two or more sets of data values (e.g. Kader & Perry, 1996; Mullenex, 1990). In order to illustrate this idea, compare two data sets where the minimum and maximum values as well as the arithmetic mean are equal and reveal no hint of how to draw conclusions about the values as shown in Figure 2.

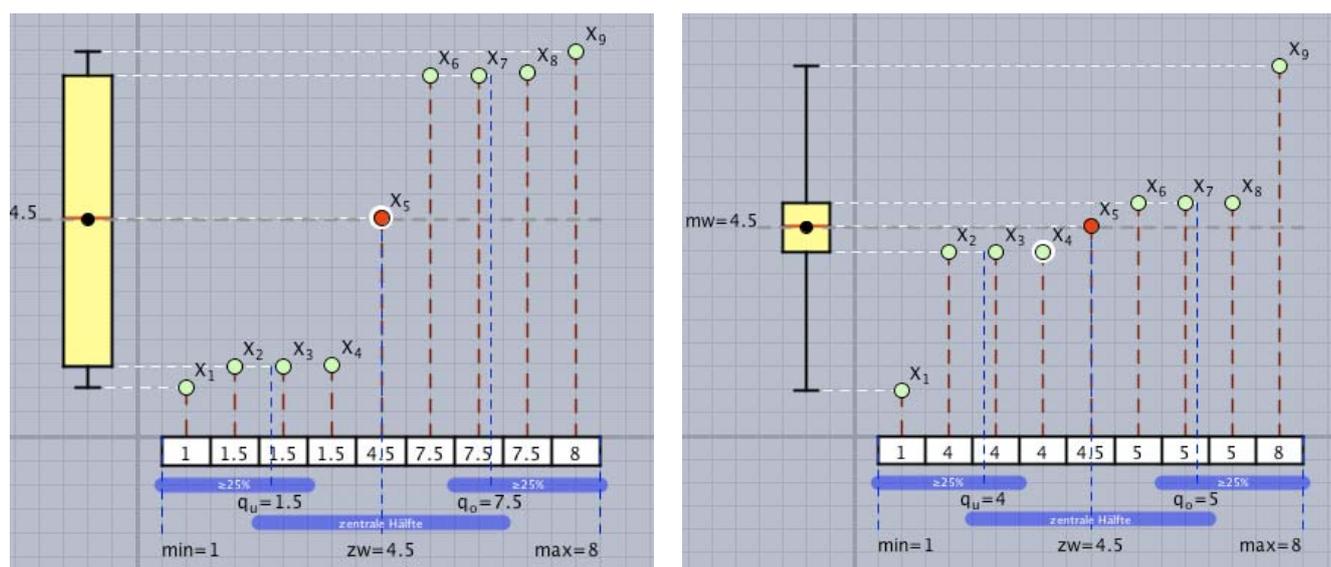


Figure 2: Two box plots with different interquartile ranges

It is obvious that in the second case, the box is much smaller than in the first one, indicating that the spread of the central half of the data is lesser. We use this technique extensively in the exercises that are part of the teaching unit.

On the other hand, the box plot representation is reduced to just five key values and the underlying individual values are not apparent any more – one considerable reason for students' difficulties with this kind of graphical representation (Bakker, Biehler & Konold, 2004). In addition, box plots – compared to many other graphical representations like, for example, histograms – do not display frequencies but rather densities (Bakker, Biehler & Konold, 2004). This means, the smaller a particular area is, the more values are contained in it.

A Useful Quartile Definition

There is no universal definition of a quartile; actually, there are at least five different definitions in use (Weisstein 2008). The situation is even worse for software packages. According to Hyndman and Fan (1996) even within a single software package several definitions might be used concurrently. A visualization sometimes uses a different definition than a numerical calculation. One reason for this is that the

original concept of box plots as introduced by Tukey (1977) used the *hinges* of a data set instead of the quartiles, which are different in one of four cases. Unsurprisingly, the concept of a quartile is obscure to most students and even teachers.

School textbooks in Germany usually do not give an exact definition of quartiles, but combine a colloquial description with a recipe to calculate the quartiles. All definitions are not based on the desired result (i.e., “the first quartile is a value such that at least 25% of the values are less or equal, and at least 75% of the values are greater or equal”), but on a specified way to calculate them (i.e. “the first quartile is the value that is placed at position $(n+1)/4$ if this is an integer, else...” or similar). Unfortunately, these recipes are incompatible with the QUARTILE function as provided by Excel, which is the most common tool for data analysis in German schools, besides the availability of special purpose educational tools for statistical analysis like, e.g., Fathom (Key Curriculum Press, 2008). The documentation of the QUARTILE function in Excel³ is similar to the text book definitions of quartiles: it lacks a formal definition or explanation of the desired properties, and focuses on examples instead. It is not possible to explain the results of Excel on that basis.⁴

Most of the critique above only applies to small data sets. With larger amounts of data the actual definition used is not as significant as with less than, say, 20 values. Still, these data sets are the ones that are accessible to hands-on manipulation in the classroom.

For our study, we chose a definition that is both easy to understand and easy to use. A lower quartile⁵ of a set of values is a number q_u such that at least 25% of all values are less than or equal to q_u , and at least 75% of all values are larger than or equal to q_u . In many cases, this number is a value of the data set, but we do not restrict quartiles to be chosen from the values. The definition for the upper quartile q_o is analogous. Using 50% instead of 25% and 75% we can also use it to define the median. All definitions are valid even if some values occur several times.

Finding the Median and Quartiles

A very useful and action-oriented way to *find* the median and quartiles is the following one:⁶ Order all values in increasing order, and write them down in a row of equal-sized boxes. The strip of ordered values may look like this (for 8 values):

³ We used the German version of Excel 2004 on Mac OS X. There are explanations of the formulas used available, for example, in learn:line NRW at <http://www.learn-line.nrw.de/angebote/eda/medio/tipps/excel-quartile.htm>. Excel uses a weighted arithmetic mean for the quartiles.

⁴ Büchter and Henn (2005) provide a definition of quartiles that is precise and matches the expectation that the lower and upper quartile are the smallest values that cut off at least 25% of the values.

⁵ We are using the standard German notation here, instead of Q_1 and Q_3 for lower and upper quartile.

⁶ A student teacher, Simone Seibold, came up with this method during her traineeship in school.

and Sumner (2006) categorized the possible types of interaction in mathematics software. Again, the activities found on the web so far do not obey these rules.

Data Cycle

Biehler (1997) suggests a “Cycle of solving real problems with statistics”, similar to the typical modelling cycle (Fig. 3 left). However, we suggest that in our case another model is more suited. The typical way to work with data and data analysis in school can be described in a “data cycle” (Fig. 3 right), where data is created by, e.g. measurements in the real world, this data is processed to create a representation of it, the representation can be used for interpretation, and this should be connected to the original data. From top to bottom there is less information (in the information-theoretic sense), but more structure. On the left we work with the real world, that is concretely, on the right we work with a mathematized version of it, that is abstractly.

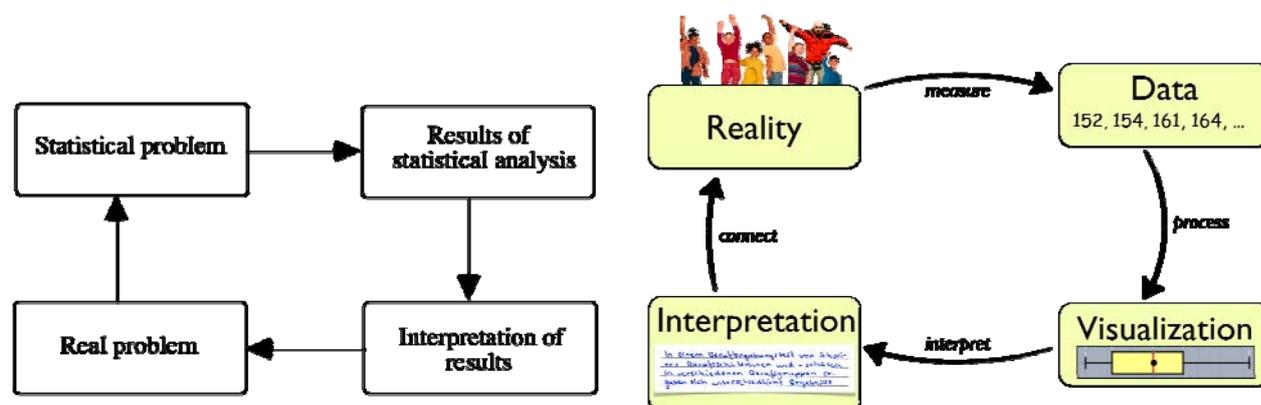


Figure 3: Problem solving cycle by Biehler (1997) on the left, and our proposed data cycle on the right

DESIGN OF THE ACTIVITIES

The design of the study is used in order to answer our main research question: *To what extent are students able to interpret box plots related to real world situations if they work with them interactively on abstract data sets?* Based on the theoretical analysis given above we therefore designed a set of exercises that enables the students to experience both the power and the restrictions of box plots. In all exercises students use the same interactive applet.⁷ The applet is embedded into a plain web page and can be used without prior installations using a standard Internet browser. Using this applet, students can view and manipulate data with up to 22 values (the limit is not due to technical reasons, but given by the screen size). They can add or remove data, change data by dragging the associated data point with the

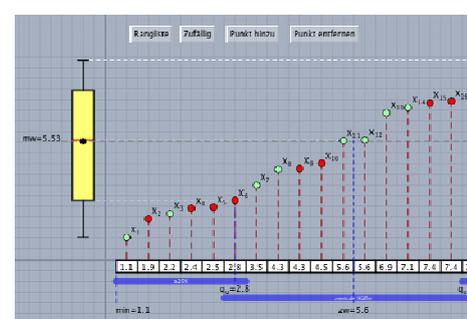
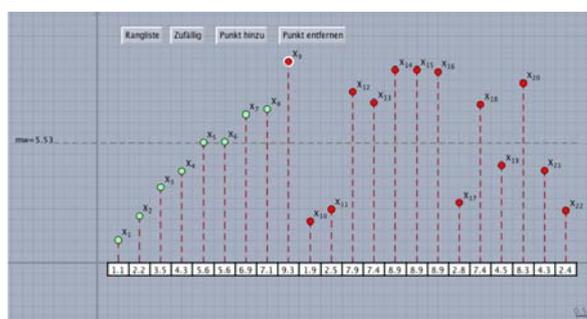
⁷ See <http://kortenkamp.net/material/stochastik/Quartile.html>. The applet is based on Cinderella (Richter-Gebert & Kortenkamp 2006). In our box plot visualization we do not use outliers, as these are not used in the standard textbooks, either.

mouse vertically, and re-order values by dragging the points in direction of the x -axis. Points that have been added by the students are shown in red, others that were given are depicted in green.

According to Bakker, Biehler and Konold (2004), it is helpful for students if individual cases can be recognized within the box plot representation. This is granted in the applet that we use in our study. All data is visible at all times. While the students are manipulating the data, the current mean value is displayed both numerically and by a dashed horizontal line. The values that correspond to the data points are shown numerically in a white box below each point (Fig. 4 left).

If the values are ordered ascending the applet adds more statistical information to the visualization. To the left of the values the corresponding *box plot* showing the minimum, maximum, quartiles and median, is drawn. Those are connected through dashed lines with the corresponding “creases” and the values that are shown below the data. The blue bars mark the lower and upper quarters of the values as well as the central half (Fig. 4 right).

Figure 4: Applet with unordered values on the left, and ordered values on the right



Exploratory Exercises

Assuming that the students cannot

master the interpretation step if they already fail at processing the data, we designed a set of exercises that aim at connecting the visualized data and the concepts behind them with the original data. Using the applet, students can easily process data dynamically, while modifying it, with an immediate update of the visualization. The exercises focus on modifying data sets in order to change or preserve the measures of variation: (a) Change *only* the arithmetic mean by changing values, (b) Change *only* the minimum or maximum by changing values, (c) Change *only* the length of the whiskers, (d) Change *only* the size of the box (the interquartile range), (e) Add values without changing the box plot, (f) Remove values without changing the box plot, (g) Try to move the arithmetic mean outside of the box, and (h) Try to move the median outside of the box.

Our primary goal is that students understand that box plots are a compact visualization of five (or six, depending on the plot) statistical measures, which in turn describe the distribution of values in a data set. Based on these measures it is possible to draw conclusion about the original set. Students should be able to find as many conclusions as possible, while not over-interpreting the measures. The activities force the students to create data sets that differ only in certain aspects, while showing an interactive visualization of the data and the measures.

For example, while experimenting with (d) students will see that for a distribution with smaller box (i.e. a smaller interquartile range) the values in the central half are more densely distributed than for a distribution with a larger box. Also, common misconceptions like a correspondence between the size of the box and the number of values in the data set are addressed. Adding or removing values does not necessarily change any of the measures of variation.

SUBJECTS AND METHODS

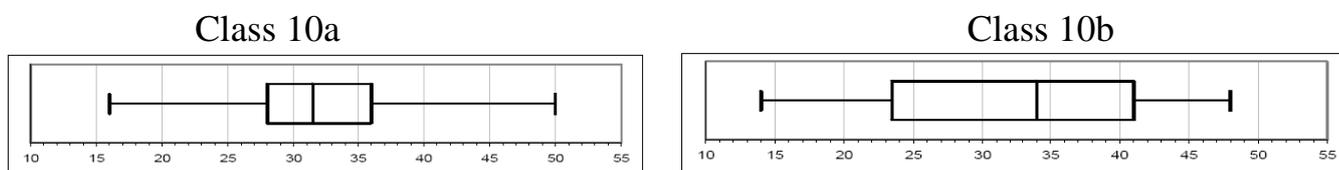
In line with the recommendations formulated by a group of stochastic educators in Germany (Arbeitskreis Stochastik, 2003), the participants in our study are aged at least 15 years. We conducted preliminary tests with the material in schools in two German states, Baden-Württemberg and North Rhine-Westphalia.

In Baden-Württemberg, we worked with 28 students in grade 9 at the “Realschule” level. They already received some training with box plots, but not with interpretation, in grade 8. In order to let them recall the basics they all received a hand-out about medians, quartiles, and box plots. First, they worked for 20 minutes in pairs with the applet and were asked to answer the exploratory exercises as given in the last paragraph in writing. Next, they were asked to analyze a series of box plots on another (paper) work sheet and interpret them in writing. Their answers were collected for further analysis.

In North Rhine-Westphalia, three students of grade 11 were involved in an interview-like situation where they had the possibility to explore the applet and work on the above presented exercises related to box plots. Beforehand, they had also received a hand-out providing an overview of medians, quartiles, and box plots. Subsequent to the exploration of the applet, they were given two interpretation tasks that they answered in written form.

EXAMPLE OF AN INTERPRETATION TASK

In class 10a, there are 30 students, in class 10b 29. In both classes, the same test was written. The two box plots are based on the scores achieved by the students:



- Describe as detailed as possible which information you can extract from the two box plots and compare them with each other.
- Which class wrote the better test? Justify your answer.
- Give examples for scores of the 30 students from class 10a that fit the given box plot and explain your procedure.

FIRST RESULTS

We only report on the results from one of the three students who took part in the interview-based exploration of the applet and then answered the interpretation task presented above. At first, the student describes the two box plots by simply listing the five key values respectively. This observation is in line with results reported on in the literature, and also our observations with the other student group in Baden-Württemberg. However, he does not remain at this merely descriptive level and formulates the following statement:

In class 10a, a good portion of the students are located in the centre, whereas the points in class 10b are more distributed. However, here the higher points are more pronounced.

Being sympathetic to the student's answer, one could conclude that he has understood some basic principles of the box plot representation. However, in order to get more information about his competencies without construing too much, he was later asked by e-mail to clarify this answer. These are his additional explanations:

The set of students is divided into four parts by the median and the two quartiles. In class 10a, the two middle areas are particularly small. This means that particularly many students are located there. In class 10b, the four areas are about the same size. This means that the students are distributed equally regarding to the score. The rightmost area in class 10b is considerably smaller than the one in class 10a. This means that the students in this area have achieved particularly high scores.

The additional explanations illustrate that the student has mastered some of the difficulties and challenges related to box plots that are described in the literature (Bakker, Biehler & Konold, 2004). He realizes that a box plot consists of four areas that approximately contain 25% of the data respectively. Moreover, he is able to formulate the relationship between the size of the particular areas and the density of the values contained in them.

CONCLUSION

We agree with the NCTM (2000) standards that students should also be able to create and use graphical representations of data in form of box plots as well as discuss and understand the correspondence between data sets and their graphical representations. The applet presented in this paper and employed in our study does not need any further software packages and therefore provides a basic but powerful tool for students in order to explore the potential and limitations of box plots. The applet is definitely easy to implement in the classroom. However, at the moment we cannot say too much about the effects on the interpretation competencies of the students who worked with the applet in a classroom situation. For the interview-like individual exploration our results show that the work with the applet can support the ability of students to analyze and interpret box plots. Currently, we are concerned with using the promising experiences based on the interview-like situations in order to make the applet also accessible to the work in the classroom.

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DYNAMICAL EXPLORATION OF TWO-VARIABLE FUNCTIONS USING VIRTUAL REALITY¹

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We present the rationale of an ongoing project, aimed at the development of a Virtual Reality assistant learning of limits, continuity, and other properties in multivariable Calculus. The Mathematics for which this development is intended is described briefly, together with the psychological and pedagogical elements of the project. What is Virtual Reality is explained and details are given about its application to the specific field. We emphasize the fact that this new technological device is suitable for self-teaching and individual practice, as well as for the better storing and retrieving of the acquired knowledge, and for identifying its traces whenever it is relevant for further advanced learning.

BACKGROUND

The institution and its pedagogical situation

The Jerusalem College of Technology (JCT) is a High-Tech Engineering School. During the Spring Term of first year, a course in Advanced Calculus is given, mostly devoted to functions of two, three or more real variables. A problem for many students is a low ability to "see" in three-dimensional space, with negative consequences on their conceptualization of notions such as limits, continuity, differentiability. Another bias appears with double and triple integrals, as a good perception of the integration domain is necessary to decide how to use the classical techniques of integration. Sik-Lányi et al. (2003) claim that space perception is not a congenital faculty of human being. They built a Virtual Reality environment for improving space perception among 15-16 years old students. With the same concern we address a particular problem of space perception with older students, using the same digital technology.

Berry and Nyman (2003) show students' problems when switching between symbolic representation and graphical representation of a 1-variable function and of its first derivative. They say that "with the availability of technology (graphical calculators, data logging equipment, computer algebra systems), there is the opportunity to free the student from the drudgery of algebraic manipulation and calculation by supporting the learning of fundamental ideas". Tall (1991) notes that the computer "is able to accept input in a variety of ways, and translate it's flexibly into other modes of representation, including verbal, symbolic, iconic, numerical, procedural. It therefore gives mathematical education the opportunity to adjust the balance between various

¹ The first author is supported by Israel Foundation Grant 1340/05. The development is supported by a grant from the Israel Inter-University Center for e-Learning, number 5768/01.

modes of communication and thought that have previously been biased toward the symbolic and the sequential".

Until now, various technologies have been introduced as a tentative remedy to problems encountered with three-dimensional perception. Nevertheless, problems still remain. Numerous technologies have been introduced for the sake of visualization. Arcavi (2003) classifies the roles of visualization as a) support and illustration of essential symbolic results, b) provider of a possible way of resolving conflicts between (correct) symbolic solutions and (incorrect) intuitions, and c) a help to re-engage with and recover conceptual underpinnings which may easily be bypassed by formal solutions.

In the present paper, we focus on functions of two real variables, plotting and analyzing their graphs, considering especially the b) component in Arcavi's classification. A problem may appear inherent to all kinds of support: a graphical representation may be incorrect, either because of non appropriate choices of the user or because of the constraints of the technology (Dana-Picard et al. 2007). In order to overcome this problem we turn our attention towards another technology: Virtual Reality (VR). This technology is extensively used for training pilots or other professionals. Jang et al. (2007) discuss the usage of VR related to representation of anatomy, clearly a 3D situation too. But as far as the authors know, it has been implemented yet neither for Mathematics Education in general, nor for the Mathematics Education of Engineers. In this paper, we present the rationale for the authors to start the development of a VR assistant to learning Mathematics. We describe an environment where the learner is not passive and has some freedom to choose his/her actions. A VR environment offers cognitive assessment, spatial abilities, executive and dynamical functions which are not present in more traditional environments.

Representations of a mathematical object

Among the characters articulated in mathematics teaching cognitive aspects:

- Multiple representations of the same objects: textual (i.e. narrative) presentations, literal formulas, graphical representations, tables of numerical values, etc. These presentations may either be redundant or leave empty holes. Note that every presentation has to be accompanied by a narrative presentation for embodying the rule and for the sake of completing the given description of a rule. Mathematics educators generally agree that multiple representations are important for the understanding of the mathematical meaning of a given notion (Sierpiska 1992).
- When using together multiple representations in order to give a concrete appearance of composite consequences of the rule under consideration, it can be necessary to perform a transfer between an abstract concept and concrete representations. For example, Gagatsis et al. (2004) present a hierarchy among the possible representations of a function, calling tables as a prototype for

enabling students to handle symbolic forms, and graphical representations as a prototype for understanding the tabular and verbal forms of functions (for a study of prototypes, see Schwarz and Hershkowitz 1999).

- The more numerous the rule's implications (in Physics, Biology, Engineering, Finance, etc.), the more important is the requirement of creative skills (e.g. interpolations, extrapolations, which the learner will have to apply). Here the teacher will generally try and guide the learner with examples, graphical representations, and animations.
- The more fundamental the rule, the more important for the learner to store it, to internalize it and its consequences for a long duration. This will enable him/her to build more advanced rules. More than that, the learner needs ways to extract the knowledge and to find its traces whenever it is relevant for further learning (Barnett et al, 2005).
- Regarding a mathematical rule with geometrical implications and representations, its complete mastering requires from the learner, according to the Gestalt conception, a permanent transfer from one kind of representation to another kind (see Hartmann and Poffenberger, 2007). On the one hand, it is necessary to understand how a change in the parameters of the rule influences the representation. On the other hand, abstraction skills enable to conjecture the rule from the graphical representation and to modify the parameters in the formula according to the changes in the graphical representation. This is the rationale for the usage of software for dynamical geometry.

The graphical representation has been made using either Maple 9.5 or the free downloadable software DPgraph (www.dpgraph.com). Because of the dynamic character of a VR device, we do not include screenshots. Suitable presentations can be found at URL: http://ndp.jct.ac.il/companion_files/VR/home.html.

LIMITATIONS AND CONSTRAINTS ON THE CONVENTIONAL REPRESENTATION TOOLS

Real functions of two real variables may have various representations: symbolic (with an explicit analytic expression $f(x, y) = \dots$), graphical (the graph of the function, i.e. a surface in 3D-space), numerical (a table of values), not necessary all of them at the same time. This last kind of representation is generally not easy to use in classroom; the plot command of a CAS uses an algorithm which provides numerical data, and the command translates this numerical data into a graphical representation. Generally the higher level command is used, and the user does not ask for a display of the numerical output. The VR device that we develop uses this numerical output to create a *terrain* (a landscape) over which the student will "fly" to discover the specific properties of the function, either isolated or non-isolated singularities, asymptotic behaviour, etc.

It happens that a symbolic expression is unaffordable. This creates a need, central for teaching, for suitable tools to illustrate the function and make it more concrete. An example is given by Maple's **deplot** command for plotting the solution of a Differential Equation without having computed an analytic solution; of course this command uses numerical methods. Within this frame, educators meet frequently obstacles for their students to achieve a profound and complete understanding of the behaviour of such functions. Examples of the limitations have been studied by Kidron and Dana-Picard (2006), Dana-Picard et al. (2007) and others. The student's understanding of the behaviour of a given function depends on the representations which have been employed.

Dana-Picard et al (2008) show that the choice of coordinates has a great influence on the quality of the plot produced by a Computer Algebra System (CAS). Compare the plots of $f(x,y)=1/(x^2+y^2-1)$, displayed in Figures 1 and 2. Cartesian coordinates have been used for Figure 1 and polar coordinates for Figure 2. The discontinuity at every point of the unit circle is either not apparent or exaggerated. Moreover Figure 1b shows a kind of waves which should not be there.

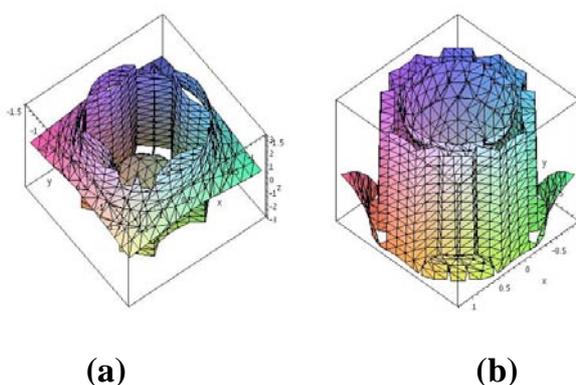


Figure 1: Plots of a 2-variable function, with Cartesian coordinates

The choice of suitable coordinates is not the sole problem for getting a correct plot. Figure 2a shows that our discussion on "correct coordinates" is not the ultimate issue, and even with these coordinates, other choices influence the accuracy of the graph, whence the student's understanding of the situation. In Figure 2a the discontinuities are totally hidden, as a result of the interpolation grid chosen by the software. This issue is discussed by Zeitoun et al. (2008).

A "wrong" choice of coordinates may *hide* important properties of the function, but may *show* irrelevant problems, whence numerous problems with the figure and its adequacy to the study. A central issue is to decide what "correct coordinates" are and what a "wrong choice" is. It has also an influence on the possible symbolic proof of the properties of the function. A couple of students have been asked why they have hard time with such problems; they answered that the reason is a lack of basic understanding of the behaviour of the represented mathematical object (no matter whether the representation is symbolic, numerical, or graphical). A problem can arise

when checking that data of two different kinds actually represent the same function. Experience must be accumulated by the learners.

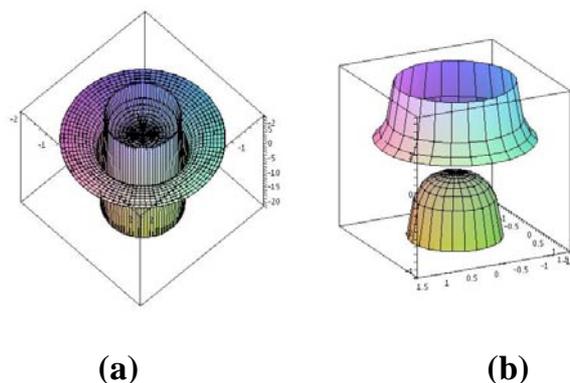


Figure 2: Plots of a 2-variable function, with polar coordinates

Moreover, the students may receive a proof of a certain property using an abstract-symbolic representation of the mathematical object under study. Despite the proof's precision, it happens that the student needs a more concrete presentation. In a practice group of 25 students, the teacher chose the function defined by $f(x, y) = 1/(x^2 + y^2 - 1)$ and showed plots like those displayed in Figure 1. Two thirds of the students saw immediately that the function has a lot of discontinuities (intuitively, without giving a proof), but could not explain immediately what is wrong with Figure 1.

The graph of a 2-real variable function is a surface in 3-dimensional space. A function of three real variables can be represented by level surfaces. Excepted at certain points, this is the same mathematical situation as before, because of the Implicit Function Theorem. At the beginning of the course, about 70% of our students have problems with surface drawing. A lack of intuition follows, for example concerning the existence of discontinuities. This may incite the student to make successive trials, i.e. to multiply *technical* tasks not always relying on real mathematical *thinking*. Afterwards a symbolic proof is required, and maybe a graphical representation will be needed to give the "final accord".

Graphical features of a Computer Algebra System are used to enhance visual skills of our students, hopefully their manual drawing skills. With higher CAS skills, an animation of level surfaces can help to visualize graphically a 3-variable function. We meet two obstacles:

- The dynamical features of a CAS are somehow limited. In many occurrences, it is possible to program animations, and/or to rotate the plot, but not more.
- A CAS cannot plot the graph of a function in a neighbourhood of a singular point. In this paper we focus on limits and discontinuities. The CAS either does not plot anything near the problematic point (Figure 3b) or plots something not so close to the real mathematical situation (Figure 3b: where do these needles come from?). Note that this occurs already with 1-variable functions, but with 2-variable functions the problem is more striking.

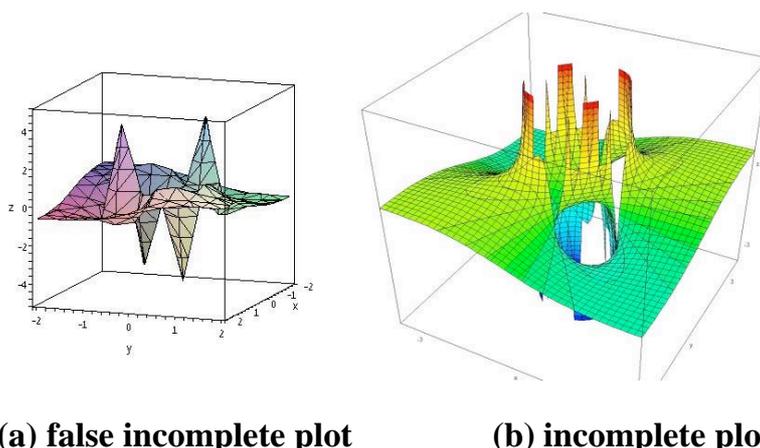


Figure 3: Two Problematic plots for $f(x, y) = \frac{xy}{x^2 + y^2 - 1}$.

VIRTUAL REALITY

What's that?

The technology called **Virtual Reality** (VR) is a computer-based physical synthetic environment. It provides the user with an illusion of being inside an environment different from the one he/she is actually. This technology enables the building of a model of a "computerized real world" together with interactive motion inside this world. The VR technology gives the user a feeling that he/she an integral "part of the picture", yielding him/her *Presence*, *Orientation*, and even *Immersion* into the scenario he/she is exposed. After a short time he behaves like it's the real world.

The goals: VR-concretization and its added value

A CAS is not a cure-all for the lack of mathematical understanding when dealing with discontinuities of multi-variable functions. A more advanced, more dynamical concretization is given by a VR environment. It is an additional support to Mathematics teaching completing the classical computerized environments, beyond the traditional representations (symbolic, tabular-numerical, and graphical). Actually VR provides an integration of computer modes previously separate (Tall 1991):

- Input is not limited to sequential entry of data using a keyboard. Devices such as a joystick are also used.
- A working session and its output mix together the iconic, the graphical and the procedural modes.

When reacting to the student's commands, the VR device computes anew all the parameters of a new view of the situation. The student takes a walk in a landscape which is actually part of the graph of the function he/she studies. At any time, VR simulates only part of the graph, the discontinuity is never reached, but it is possible to get arbitrarily close to it. The VR may provide the student what is missing in

his/her 3-dimensional puzzle, by eliminating the white areas appearing in CAS plots, such as Figure 3a. It is intended to provide him/her a real picture of how the function he/she studies behaves.

A VR environment provides compensation to the limitations and the constraints of the imaging devices already in use (CAS and plotters). It presents an image of a real world and gives a direct 3-dimensional perception of this world, as if the user was really located in it. The higher the quality of the VR environment, the more powerful the impression received from this imaginary world's imitation of the real world.

In our starting project, the simulation provided by VR is intended to improve the students' understanding of continuity and discontinuity, and afterwards give also a better understanding of differentiability of a multi-variable function. Among other affordances, the VR simulation cancels problems of discontinuity related to graphs because of its local and dynamical features.

COGNITIVE CHARACTERISTICS AND SIMULATION FEATURES OF A VR ENVIRONMENT

The final rules may be represented in a concrete fashion by interaction with the environment and by showing to the learner the limitations of the rules, as they appear in a (almost) static environment generally yielded by a CAS. Non graphical representations of functions, such as numerical representations, cannot show continuity and discontinuity. This comes from the discrete nature of these representations, a feature still present in the computerized plots.

The new knowledge afforded by the learner is a consequence of his/her own efforts to explore the situation. His/her ability to change location, to have a walk on the graph, will lead him/her to internalize in a better way the mathematical meaning of continuity and discontinuity. An added value is to help him/her to understand the meaning of changing parameters in the geometric representation. This added value is made possible by the *live experience* of the behaviour of the function, no matter if the transitions are discrete or continuous (according to changes in the variables or in the parameters). The mental ability to feel changes, their sharpness, their acuteness, comes from the immersion into the topography in which the learner moves.

This added value is still more important when the function under study encodes a concrete situation, in Physics, Engineering, Finance, etc. The interactive experience enables the learner to translate the rules to which the function obeys, to find analogues of these rules for other concrete situations. The concrete sensations provided by VR improve the learner's understanding of interpolation and extrapolation, and to translate this understanding into the graphical situation (see also Dana-Picard et al., 2007). The more immersive features of the mathematical knowledge that are incorporated into VR representation for the learner, the faster he/she will find the traces of it whenever it is relevant for further learning. Besides, the more immersive features are incorporated into VR knowledge representation the

greater the longevity of preserving the acquired knowledge. This means a slower extinction of it in the memory system (Chen, et.al. 2002).

Interactivity improves the learning experience. Numerous studies show that the more deeply lively experienced the learning process the more internalized its results (Ausburn and Ausburn, 2004; Barnett et al, 2005). The internalization is assessed by an improved conservation of the knowledge, i.e. a slower decrease of the knowledge as a function of elapsed time. Therefore, a Virtual Reality assisted learning process yields a better assimilation of the mathematical notions than with more conventional simulations devices, as it provides this live sensorial experience. This is a more than a realization of the request expressed by a student involved in a research made by Habre (2001); this student wished to be able to rotate surfaces in different directions. A Computer Algebra Systems does this already. VR meets a further requirement of this student, namely to have "a physical model that you can feel in your hands".

According to the brain mapping, the numerical representation of functions is acquired by the left hemisphere of the brain, and the space-live experienced acquisition in a learning process is devoted to the right hemisphere. The transfer from the symbolic rule to a 3D representation and vice-versa requires transfer between two brain lobes with different functionalities. Concerning conceptualization, especially when it must be applied to a concrete domain, there exists a mental difficulty to "move" from one lobe to the other (in terms of longer reacting time, or of completeness of the process). An interactive environment where functional parameter changes are allowed, and where the environment changes can be sensitively experienced, enables a faster building of bridges between the different registers of representation, symbolic, numerical, and graphical.

Finally, the usage of a VR assistant to learning is purely individual. The teacher can show a movie, but it is only an approximation of the requested simulation. The student's senses are involved in the process, the hand on the joystick, the eyes and the ears in the helmet, etc. Therefore the VR device should take in the learning computerized environment a place different from the place of other instruments.

OUR VR DEVICE AND FUTURE RESEARCH

The digital device described above is now in its final steps of initial development. The user can fly over (or walk along) the terrain, i.e. over the graph of the given function. The details of the graphs, the possible discontinuities, are made more and more visible. This effect is not obtained by regular zooming, as this operation only inflates the size of the cells of the interpolation grid. For new details to appear the data has to be computed anew and only part of the surroundings is displayed.

Furthermore, a VR environment seems to contribute an added value by representing more holistic characteristics of the mathematical knowledge. Among the main contributions are the dynamics or flow traits. A more integrated one is the ability to understand its place in the whole mathematical or physical context it is playing with.

In cognitive terms it means that by VR environment, the teacher should provide to the student a more accurate mental model of the mathematical knowledge, including the applicable images of it (Croasdell et al, 2003).

In particular, the dynamical properties of a VR device and their appeal to various sensitive perceptions (vision, audition, etc.) induce also the need of the integration of the hand into the educative schemes. As Eisenberg (2002) says, the hand is not a peripheral device, but is as important as the brain. He discusses the issue of the importance of physical approximations to purely abstract concepts, rejected by Plato's point of view. Here we use the hand totally coordinated with vision and sensorial perception.

As noted by Artigue (2007), "The increasing interest for the affordances of digital technologies in terms of representations have gone along with the increasing sensitivity paid to the semiotic dimension of mathematical knowledge in mathematics education and to the correlative importance given to the analysis of semiotic mediations". In this perspective, a preliminary double blind research is on its way, with two groups of JCT students. We intend to report on the results in a subsequent paper.

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DESIGNING A SIMULATOR IN BUILDING TRADES AND USING IT IN VOCATIONAL EDUCATION

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This paper deals with the design, the production and the uses of a simulator for the activity of marking out on building sites from reading a marked plan. The main design principle of this simulator lies in that it is not meant for reproducing accurately the real context of the activity but it should offer the possibility of posing problems of the work situation through a prior conceptual analysis of the professional activity.

What is a reading-marking out activity in a building work? Most of building tasks are based on reading plans for marking out on the building site. We call this kind of tasks, reading-marking out tasks. In a building site, setting out elements takes into account what will be set out later. For example, when a floor is to be laid down, the marking out of the floor must leave holes for water pipes and electric cables. Setting out a wall must plan location for windows and doors by marking out their contour. Such marking out is called “boxing out”. Generally speaking, a boxing out is a formwork placed in the middle of a structure before casting concrete, used to set aside an area in which additional equipment can be added at a later date. This task of reading information from a plan to mark out contours and boxing out on the building site is usual for workers in building trades.

Two types of controls can be distinguished in the marking out of boxing out:

- controls coming from reading information on the plan
- subsequent and effective controls at the moment of putting the additional elements (pragmatic controls),

the first type of controls being oriented towards the second type of controls.

The first type of controls is the focus of our attention. In absence of pragmatic control, only controls guided by knowledge about space and instruments can take place. The activity of setting out boxing out can allow researchers to observe conceptualisation and help them answer questions such as: what is the nature of knowledge involved in this activity? How is such knowledge organized and what relationship does it have with the artefacts available on the building site?

The observation of students of a vocational school gave evidence of a discrepancy between procedures of students and of professionals in this reading-marking out activity on building site from reading a plan. Two types of analysis were carried out in order to better know this discrepancy and to understand the reasons: an analysis of the geometry in action underlying the students’ activity in reading marking out tasks in workshop and an analysis of the transposition of the professional activity in vocational education was needed. The first analysis is presented in Bessot & Laborde (2005). The second analysis focused on the place and status of reading-marking out

activities in vocational education, in particular when preparing students to a certification of qualified workers for building trades (in French : Brevet d'Enseignement Professionnel). It was carried out and showed that the reading marking out situation that constitutes an indivisible entity in the professional practice is divided or almost absent from the vocational education institution (Metzler 2006). A simulator is for us a means of designing situations restoring the unity of reading marking out activity in the three teaching places of French vocational education in which knowledge about space is part of the learning aim: in the mathematics teaching (in particular geometry), in the teaching of construction, in the teaching of practice in workshop.

According to a key design choice, the simulator was meant as an *open-ended environment offering the possibility of constructing didactic situations* based on problems previously identified in the analysis of professional situations.

1. FONDAMENTAL PROBLEMS INVOLVED IN READING-MARKING OUT PROFESSIONAL SITUATIONS

Previous research on different types of space (Bessot & Vérillon 1993, Brousseau 1983, Berthelot & Salin 1992, Samurçay 1984) as well as the analysis of professional practices (Bessot & Laborde 2005) allowed us to identify three types of problems related to the invariants specific to reading-marking out situation. The two first types are related to mesospace, the third type to the instruments of the building site.

The first type of problems is the problem of locating the local space in which marking out takes place within the mesospace of the building site. Two types of space are involved: the local spaces in which marking out the lines is achieved, and the global space of moves that allows the worker to move from one local working space to another one.

Locating the local space requires coordinating three frames of reference (Samurçay *ibid.*):

- the frame of reference attached to the subject (egocentric reference frame)
- the frame of reference of the lines marked on the building site (allocentric reference frame) to construct from fixed existing objects of the mesospace that may also be lines already marked on the building site
- the frame of reference of the plan that is the dimension system.

The second problem related to mesospace deals with the coordination of local spaces (Brousseau *ibid.*) that may be distant from each other. This coordination is needed in the process of obtaining the expected global set of marked lines of mesospace.

The third problem is related to the use of instruments: transferring measures requires taking into account the features of the instruments.

2. CHOICES FOR SIMULATING MESOSPACE

In order to decouple the problem of local marking out from the one of moving and orienting were created two different windows: the first window allows the worker to have access to various local spaces but never to the entire space; the second one provides access to the visual field of the worker within the global space and his/her move in this global space. In the second window (global space) one can only move, in the first one, one can mark out by means of instruments and one can move without a general view (through the scrolling bars). Here are presented the features of these two windows.

Window simulating the local space for marking out

This window simulating the visual field of the worker with real dimensions 1,50 m by 1,10 m is the screen of the computer providing a representation of the real visual field on a scale of 1 to 5 (Fig. 3).

One can perform measurement and marking out with the simulated instruments (see below). This window is located within the global space for marking out which is not visually totally accessible. One can move in the global space from one local space to another one by using the scrolling bars of the window (Fig. 1) but with only a partial view at each moment making difficult the linking up of local spaces.

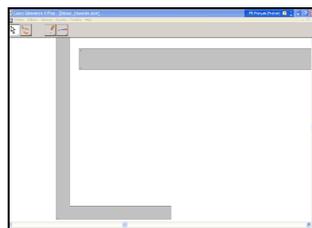


Fig. 1: Window simulating the marking out local space

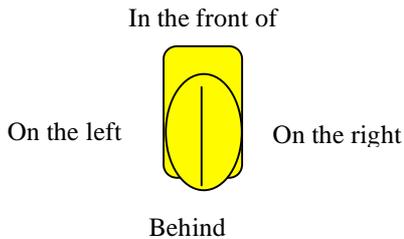
We wanted to simulate the change of viewpoint when the worker is moving away from or closer to the lines marked on the site. Zoom out (Zoom-) and zoom in (Zoom+) possibilities have been set up to simulate these moves, moving away and moving closer. Zoom facilities are limited in order to avoid a global view of the space for marking out. In addition, it is not possible to perform marking out when the zoom tool is active but it is possible to move the instruments. At any time, it is possible to come back to marking out by pressing the key “Zoom 0”. This zooming possibility makes easier an accurate reading of the marks of the measuring tape and the move from one marking out local space to another one at a small distance.

Window simulating the global space

In order to locate the current marking out local space within the whole space, it is possible at any time to have access to the simulation of the global space by pressing F9 key. The window global space is simulated by a squared vignette with a 7,5 cm long side representing a real squared space with a 5m long side (Fig. 3 et 4).

When opening the window, a yellow hard hat appears that represents the worker with its visual field represented by a rectangle. This rectangle is the image at scale of the screen (marking out local space). When opening the window, the yellow hard hat is always oriented vertically below the rectangle (Fig. 3, 4 et 7).

It was chosen to simulate the moves of the worker (yellow hard hat) and not its position (Fig. 6 et 7). Two moves are possible: shifts and rotations which are multiples of a quarter turn. Shifts are performed by directly moving the rectangle through the mouse. Rotations are egocentric and are performed by pressing one of the three buttons « > », « < », « »: to get the marking out



local space on the right of the worker press button « > », on the left of the worker press button « < », behind the worker press button « ». When back to the local space (Fig. 6), the worker sees the lines oriented as resulting from the move performed in the global space window.

In this way the decision of moving and the effect of the move on the visual field are decoupled. If from the marking out local space one comes back to the global view (F9 key), when opening the window, the yellow hard hat is always below the rectangle representing the local space (Fig. 7). Without a fixed frame of reference, the change of position cannot be inferred from the position of the yellow hard hat with respect to the fixed border of the screen.

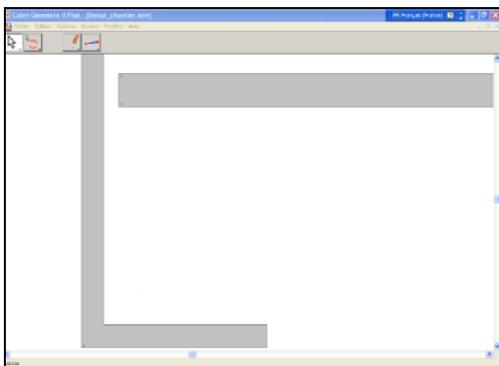


Fig. 2: Window «marking out local space»

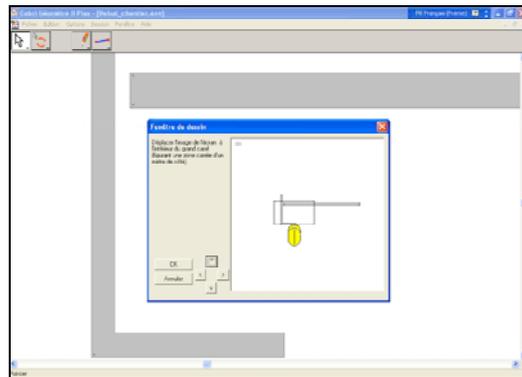


Fig. 3: Window global space in the screen (after pressing F9 key)

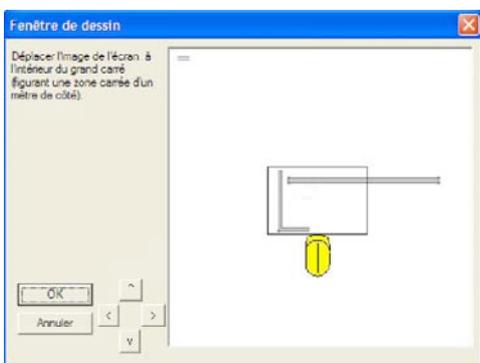


Fig. 4: Local space in the global space window

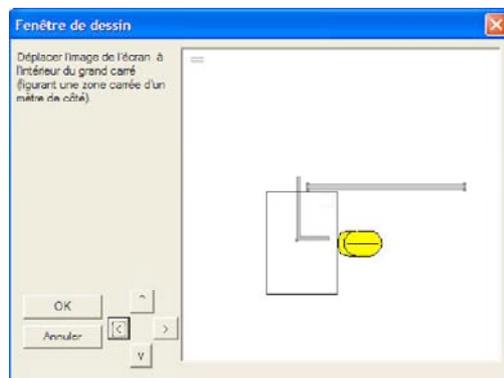
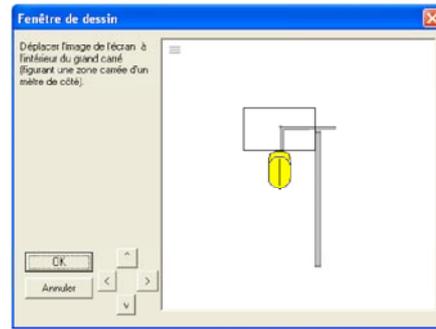


Fig. 5: After pressing button « < »



**Fig. 6: After pressing « OK »
back to local space**

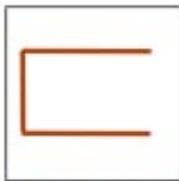


**Fig. 7: After pressing « F9 »
back to global space**

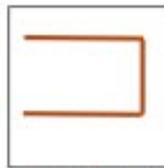
3. CHOICES FOR SIMULATING OBJECTS

Choices for simulating the prefabrication table

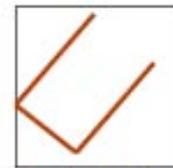
The prefabrication table in which the slab is poured, is simulated by three rectangles with same width 0,05m joined in an U shape: the table is 4m long and 2,5m wide. When opening the simulator, the borders of the table may have various directions with respect the borders of the screen: parallel to the screen borders (see Fig. 8) or not (see Fig. 10). The U shape can be oriented in various directions (see Fig. 8 and 9).



**Fig. 8: Prefabrication
table parallel to the
screen borders**



**Fig. 9: Prefabrication table
parallel to the screen
borders, in another
orientation**



**Fig. 10: Prefabrication
table non parallel to the
screen borders**

The table is not totally visible in the local space although as fixed object of this space, it can serve as frame of reference of the mesospace for locating lines in coordination with the plan. The table is only totally visible in the global space window (key F9).

Choices for simulating the use of instruments

The choices for simulating instruments deal with their aspect, their accessibility, their moves and their use. We decided that all instruments should look like real instruments. In particular their dimensions are proportional to real dimensions. The 2,5m long ruler and the 3m long tape even partly unwound stick out beyond the visual field (see Fig. 11 and 12).

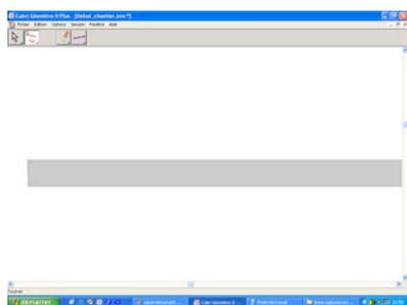


Fig. 11: The ruler cannot be totally seen

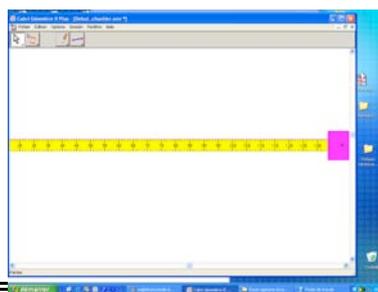


Fig. 12: Apart of the measuring tape

Marking out instruments, namely the pen and the blue line are permanently visible as icons at the top of the screen.

Instruments for measuring and transferring geometric properties read from the plan (setsquare, ruler and tape) are put at the beginning in three boxes labelled with their names, which are simulated by rectangles located in a corner of the global space accessible by moving in this space. Once an instrument is out by clicking on its box, the worker may have to move to find it again in his/her visual field (resorting to the global space window or to zoom) and to shift it in the screen (local space) to the adequate location in order to perform a marking out.

The materiality of the instruments was not preserved in that simulated instruments can overlap. However seeking to make the edge of an instrument coinciding with the prefabrication table or with the edge of another instrument partly replaces this materiality. However note that the simulated tape is also retractable as in reality in a pink squared case.

4. CONCLUSION ABOUT THE DESIGN OF THE SIMULATOR

One of the important contributions of simulators lies in the possibility of being freed of the constraints of reality, like the irreversibility of some actions or the time passage.

It is clear that the simulator transforms the relationships of the worker with space. But what is lost in fidelity can be gained in terms of problems and control. Indeed, in the use of the simulator, separating local and global spaces requires from the subject to make the decision of seeking information in the global space. To this end the subject leaves the local space in order to be and move in the global space, and then must come back in order to perform marking out. These conscious back and forth moves do not occur in reality. As a result of this separation, the subject is certainly faced with a coordination problem of frames of reference of the two spaces.

The additional action of back and forth moves between the two spaces is tedious, it transforms the reading marking out strategies and favours predictions to decrease the number of back and forth moves. But it gives rise to observations for the subject and

the educator and consequently can become an object of a reflexive work analysing strategies in real and simulated situations.

Another contribution of the simulator is the possibility of controlled variation offered to the educator. The same simulator can give rise to different uses in vocational education. The educator has the command of the type of use and of tasks given to the students. An example of a didactical situation is briefly presented below.

5. EXAMPLE OF A DIDACTICAL SITUATION MAKING USE OF THE SIMULATOR

The situation reported here raises the problem of continuing a marking out already done without transmitting to the worker information on what has been set out. This situation simulates a usual professional problem. Solving this problem requires that the worker identifies the local space within the global space by coordinating various frames of reference including the frame of reference of the plan.

Instructions

The plan of slab 1 with three boxings out is given (Fig.13) to the students.

- 1) Open the file “slab 1”
- 2) As visible, the contour of slab 1 and one boxing out have already been marked.
- 3) Mark out the two other boxings out of slab 1.

Here below is given the plan of slab 1 provided to the students as well as the windows local space and global space.

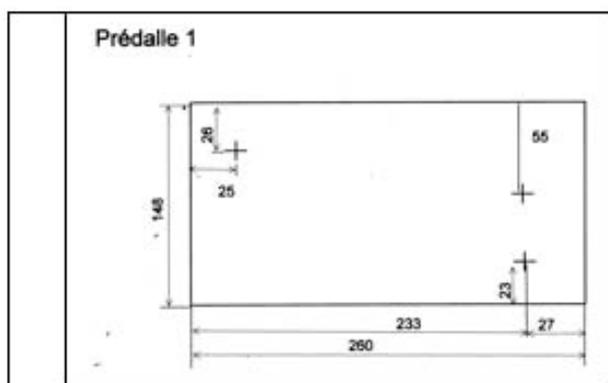


Fig. 13: Plan of slab 1

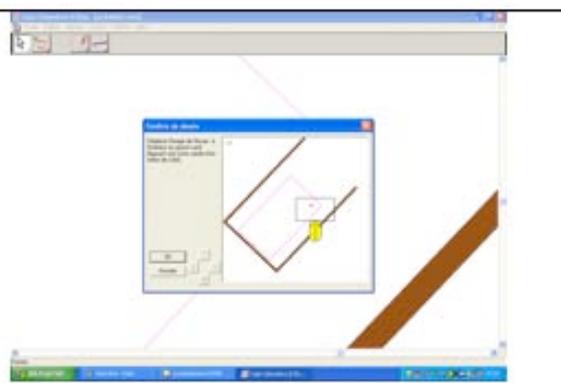


Fig. 14: The two windows

The plan is oriented by the orientation of the writing (from left to right and from top to bottom) and consequently imposes a position for reading. It is represented in this position on Figure 13. When opening the file “slab 1”, part of the prefabrication table, part of the lines and the boxing out R(25, 26) are visible in the local space (Fig.15).

In figure 14, it is visible that the slab is rotated through 180° with respect to the frame of reference of the plan.

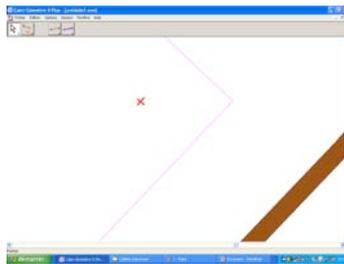


Fig.15: At the opening of file slab 1

A priori analysis of the situation

In the marking out activity, the worker's aim is to reproduce in the mesospace the image of the drawing of the fabrication plan. The continuation of the marking out requires interpreting the boxing out already marked in mesospace as the image of a boxing out of the plan.

Two cases are possible:

- Either the plan and its (*unfinished*) image in the working local space have a similar orientation and the boxing out is erroneously considered as R(27,23)
- Or measures are taken in order to identify the already drawn boxing out with a boxing out of the plan.

The choice of the dimensions of boxings out in slab 1 is deliberate. The distances to the border of the two boxings out R(25 ; 26) and R(27 ; 23) are visually close, favouring thus the mistake of the first case in absence of the professional gesture of taking information on what has already set out.

Incorrect interpretation of the already marked boxing out without measuring : R(27 ; 23)

Two other boxings out must be marked. Here is only considered the case of boxing out R(27 ; 55) as the only one likely to lead to feedback. Two procedures for marking out R(27 ; 55) are possible:

- Either through an alignment with R(27 ; 23) by resorting to the only measure 55 : *no feedback*.
- Or by resorting to two measures 27 and 55 without making use of the alignment. Once the marking out is done, *the absence of alignment of the two marked boxings out provides feedback that leads to reject the interpretation of the existing boxing out as R(27 ; 23)*. This leads to the second case which is analyzed below.

Correct interpretation of the already marked boxing out through checking by measuring: R(25 ; 26)

The coordination between the plan and its unfinished image can be achieved in two ways.

- *Real or mental half turn of the plan of slab 1*

The plan is rotated through 180° effectively or in thought to superimpose the image on the screen with the rotated plan: the marking out is performed with a prefabrication table in the position “open on the right, closed on the left”.

- *Move in the mesospace through resorting to the global space window.*

To keep the prefabrication plan in its privileged position and make it coinciding with its image on the screen, it is possible to use F9 key to get access to the global space in order to simulate a half turn in this space: the table is then in the position “open on the left, closed on the right”. When back in the local space, the boxing out already marked is the image of R(25 ; 26). Boxings out can be marked in the same position as they are on the plan.

The situation is aimed to provide multiple opportunities in which checking measures of marked objects in mesospace (prefabrication table, lines) lead to an economy in marking out. Checking is a critical gesture of building trade as claimed by the educators in vocational education.

***A posteriori* analysis of the situation**

As displayed in table 1, only 3 pairs out of 5 resort to measuring on the marking out, in order to identify the boxing out.

Interpreting the already marked boxing out	without measuring	with measuring	
	R(27,23) <i>Pairs 1 and 2</i>	R(20,21) then R(27,23) <i>Pair 6</i>	R(25,26) <i>Pairs 4 and 5</i>

Table 1: Checking procedures of already marked boxing out

Let us analyze the checking procedures of the three pairs 4, 5 and 6.

Pair 4 made two checks by measuring the dimensions of the slab and the dimension of the already marked boxing out (26 cm) which is sufficient for identifying the boxing out.

Pair 5 checked only one measure (26 cm) and did a half turn of the plan to make the screen matching the plan.

Pair 6 drew surprising conclusions: the already marked boxing out is first considered as not in the plan, then as the erroneous boxing out R (27, 23). Verbal interactions among students V and N of this pair allow us to understand those successive conclusions. As pairs 1 and 2, V immediately identifies the already marked boxing out as R(27,23). But N insists on measuring. Then he measures one of the dimensions of the boxing out and obtains 20 cm as a result of a wrong use of the measuring tape: the distance is measured by making coinciding the centre of the boxing out with the border of the case of the measuring tape (with width 5 cm in real size). He then measures the second dimension in the same way and obtains 21cm. Surprised not to find any boxing out of the plan, he resumes each measuring twice or three times.

V: it fits nothing. It means that it is already marked, then we must mark out the three others. We make one more, that's it.

N doubts that there can exist 4 boxings out and asks questions about the use of the measuring tape to observer O. He admits that he never used a measuring tape!

N: the end of the tape, is it at the black mark (corresponding to the clip of the real tape) or at the other end?

O: it is at the black mark as on a real tape... do you know, don't you?

N: No, I don't know, I never used a tape.

V: Didn't you?

The doubt about correct using of the tape as well as the cost of its use in the simulator lead them to give up checking the correspondence between measures and dimensions on the plan. They come back to the first opinion of V, i.e. identifying the already marked boxing out as R(27,23).

The simulator made possible to face the students with the usual professional problem of continuing a marking out, which is a fundamental issue of the professional activity, as claimed by the teachers. The simulator revealed that even at the end of the vocational training, almost half the students do not resort to checking and among those who checked, the use of instruments may cause difficulties. This checking professional gesture is not available to all students at the end of the school year.

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COLLABORATIVE DESIGN OF MATHEMATICAL ACTIVITIES FOR LEARNING IN AN OUTDOOR SETTING

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In recent years, teaching mathematics in an outdoor setting has become popular among teachers, as it seems to offer alternative ways to motivate children's learning. These new learning possibilities pose crucial questions regarding the nature of how mathematical activities should be designed for outdoors settings. In this paper we describe our current work related to the design and implementation of mathematical activities in this particular environment in which a specific mathematical content was used as the central component in the design. We illustrate our collaborative design approach and the results from observations of two activities. Our initial results provide us with valuable insights that can help to better understand how to design and implement this kind of educational activities.

INTRODUCTION

A recent trend in Swedish elementary schools is an increasing interest to teach mathematics in an outdoor setting. Teachers believe that this particular approach motivates the children more than solving problems in textbooks, thus offering new ways to introduce and work with mathematical concepts (Lövgren, 2007). Teaching mathematics in an outdoor setting usually refers to school children solving practical problems using whichever forms of mathematics they find appropriate (Molander, Hedberg, Bucht, Wejdmark, Lättman-Mash, 2007). The approach presented in this article is somewhat different. The paper describes our initial efforts with regard to an ongoing project in which a specific mathematical content within the field of geometry was used as the central component in the design of mathematical activities in an outdoor setting.

Our project involves a development team consisting of schoolteachers, university teachers and researchers, who collaborate to develop mathematical activities with the purpose of supporting students' processes of learning. The mathematical activity described in this paper was developed during a period of eight months, counting from the first meeting of the development team and until the completion of the activities involving students. The methodological approach used for developing the mathematical activity will be the central focus of our discussions.

Even if outdoors teaching of mathematics has got an increasing interest among teachers and teacher educators in recent years, we found few published materials with reference to outdoor environments in the research field of mathematics education. For instance, we found no results when searching on *outdoor*, *outdoors* or *embodied* in titles or keywords in *Educational Studies in Mathematics*, *Journal for research in Mathematics Education* and *The Journal of Mathematical Behaviour*. When we

searched on the term *physical*, some results showed up. However, in a brief check on research methodologies adopted in these studies, no one was centred on an outdoor activity.

Against this background, the current (ongoing) project aims at investigating different possibilities to support students' processes of learning by designing mathematical activities for an outdoor setting. This approach does not aim at replacing traditional mathematics teaching. It should rather be interpreted as a complementary method to be used at the discretion of the mathematics teacher in combination with other teaching methods. In this paper, we particularly aim at discussing our method of design in connection to the principles of Design experiments (Cobb, Confrey, diSessa, Lehrer & Schauble, 2003). Throughout the discussions presented in this paper, special attention is paid to the constitution and the working conditions of the development team.

The rest of paper is organized as follows; in the next section we present the mathematical tasks that guided our design and activity while the subsequent section gives a brief overview on the concept of design experiments. The preceding sections illustrate the results from observations of two activities followed by discussions on the notions of group and individual mathematical understanding and practices. The last two sections conclude this article by providing a description of current and coming directions of our work together with a discussion about future challenges.

DEVELOPMENT OF ACTIVITIES

In this section we describe, both the content of the proposed activities as well as the approach taken while designing the different tasks. The driving force in the design process has been experience-based suggestions from the schoolteachers. Each meeting of the development team has involved four to six teachers and two to three university researchers. The first meeting of the development team focused on identifying mathematical content and learning objectives for an outdoor activity suitable for beginners at lower secondary school. We soon agreed to focus on geometry. Aspects that were discussed dealt with the problems students have on understanding geometrical concepts such as area and perimeter. An early idea was to produce a series of activities showing progression from length to area and then to volume, using physical objects close to the school yard. The university representatives suggested utilizing non-standard measurements (sticks, steps and squares) to be used in relation to triangles, rectangles and polygons defined by trees or within the school soccer field. The school teachers instead suggested to focus on four aspects of the selected domain, namely the following learning objectives; comparison of figures, making own estimates, constructing figures with given measures and, specifically, discovering that a doubling of lengths makes the area four times larger.

It was decided that the university teachers should work on designing a task incorporating as many as possible of the agreed suggestions and present it to the

whole group after the summer 2007. The proposed mathematical task, as described in figure 1, aimed at having the students construct the following sequence of figures using ropes and metal hooks to be fastened in the ground.

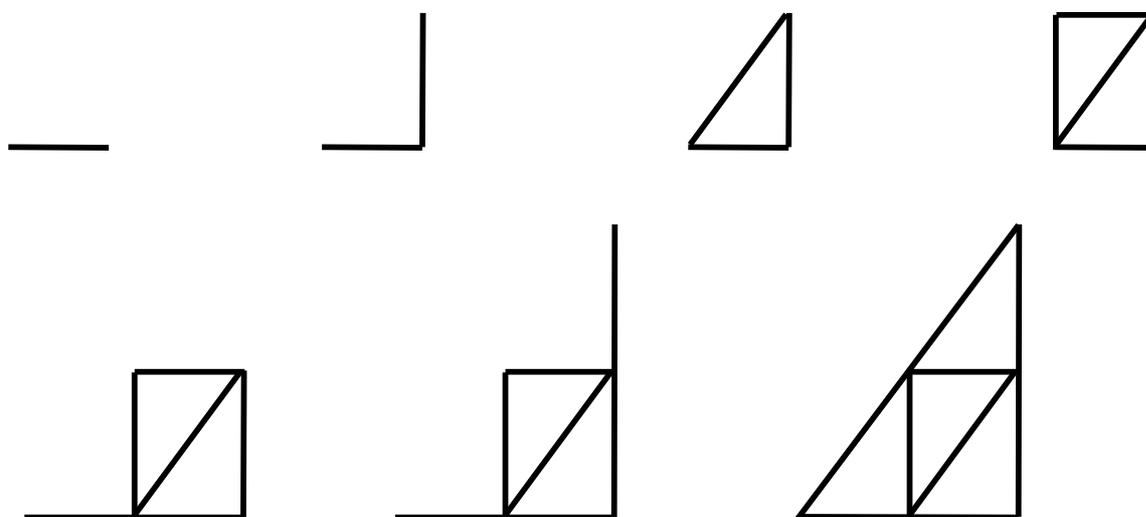


Figure 1: Intended sequence of figures to be constructed by the students.

Shortly after the summer, Växjö University hosted Professor Matthias Ludwig from *Pädagogische Hochschule Weingarten* in Germany, who offered to give two one-hour lectures at our department. One of these discussed outdoor geometrical tasks and tools used in connection with the tasks. Inspired by his lecture we decided to suggest construction of two tools; one for producing a right angle and one for measuring arbitrary angles, both based on making judgments by eyesight. The planned right angle tool consisted of a wooden square with markers at the middle of each side, as shown to the left in Figure 2.

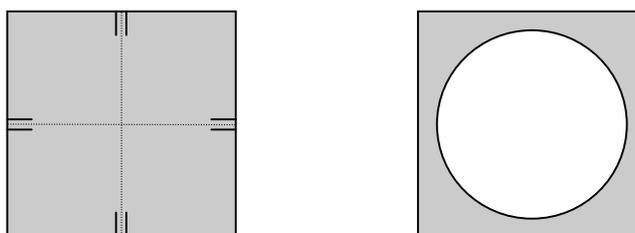


Figure 2: Ludwig's tool to the left, our tool to the right.

The woodwork teacher at the school prepared a number of square boards and also prepared a number of round boards intended for use in another activity. The square shaped tool could also be used to represent a square meter since its side was exactly one meter. However, we identified several disadvantages of this tool with respect to the intended task: it could not be used while placed on the ground, it was quite heavy, and the handling required several people operating close to the tool. We later chose the tool shown to the right in the figure above, which was actually what was left over after the round boards had been cut out. This second tool had several advantages. It

could be used while placed directly on the ground, it was easy to carry due to the hole in the middle, and could be used at a distance. The right angle was aimed at the sides of the tool.

In the first proposal, the lengths for the catheti (that were to be doubled during the task) were 3 meters and 4 meters. In the construction, metal hooks and flag lines were used. While trying out the task on the (grass-covered) school yard we all agreed that larger measures were needed, to give the students a better overview of the construction and to give them reason to move within the figure. The first suggestion was to double the lengths to 6 meters and 8 meters, but we also agreed to avoid an exact measure for the hypotenuse and ended up choosing 5 meters and 8 meters as lengths for the catheti.

The task was communicated to the students through written instructions on paper. The first page of the instructions described the tools the students were supposed to bring to the school yard (3 flag lines, 6 metal hooks, roll-out length measure, right-angle tool, paper and pen). Three separate tasks were described on the following three pages.

Each task was divided into three subtasks in the same way (construct a figure, determine perimeter, determine area). This was done for several reasons. Since the students were not used to this kind of activity, we wanted to restrict the content in each subtask. We also wanted to encourage the students to discuss their conclusions on each subtask as a group, especially to verify that the construction was made according to the descriptions as we suspected that they otherwise might focus only on calculations. Also, since the written instructions were not supported by figures, we found it reasonable to restrict each subtask in order not to make it too difficult for the students to interpret the task. Our aim was to let the students work on the tasks without the support from the teacher; thereby inviting them to take on different roles and take more own initiatives than they were used to in their usual mathematics classroom. Another important aspect was that the tasks should allow for applying different solution strategies, such as measuring, calculation, and comparison.

DESIGN EXPERIMENTS

The methodology used in this project is founded on the principles of Design experiments (Cobb et al., 2003). Cobb and colleagues (2003) summarize Design experiments (DE) in five crosscutting features. The first feature, *develop theories*, concerns understanding processes of learning and the means that are designed to support that learning. The second feature, which concerns *control*, may be seen as the focus of the current project: “The intent is to investigate the possibilities for educational improvement by bringing about new forms of learning in order to study them” (Cobb et al., 2003, p. 10). To develop theories about learning processes, and to try to exert control of such processes, implies the need for *prospective* and *reflective* analyses. Prospective and reflective work is the third feature of DE. On the prospective side, our designs have been implemented with a hypothesized learning

process in mind. The activity has been carried out with students and the following reflective work has been based on observations of students' actions. The prospective and reflective aspects come together in a fourth characteristic of DE, *iterative design*. Iterations are carried out with the modification and development of explaining learning and the means of supporting learning. The project so far has included only two iterations which have been based on informal observations with a rather weak theoretical base. Our strategy has been to let the preliminary informal observations guide us toward relevant learning theories to support later iterations. The fifth feature refers to the *pragmatic roots* of DE. As school teachers take active part in the design process, we feel confident that the activities are relevant for teachers' practice.

OBSERVATIONS FROM TWO ACTIVITIES

Two activities involving students have been carried out in the project. The two activities included two different groups of four students (14-15 years old). The activities were neither videotaped nor audiotaped. Instead, two researchers and two teachers observed the activities. The researchers were the same both times.

During the activities, the students were very eager to start working with the lines and hooks. We feel that the division of each task into subtasks made it possible for them to interpret the subtask while arranging lines and hooks. On a few occasions, when they were getting lost in the construction, we had to intervene and ask them to read the instructions again. We also observed that some of the students had problems handling the instruction papers. These problems concern locating and returning to the instructions after they have been left on the ground, as well as documenting answers to the questions.

One specific observation concerned the change in social behaviour. One of the teachers commented on a female student who was busy constructing sides by pulling flag lines:

Look at her. She seldom takes initiatives in the classroom; she is very quiet and rarely shows interest. Here she is, pulling flag lines, talking to her classmates and really enjoying what she is doing.

Another notable observation can be seen as relating to gender issues. In a group of two boys and two girls, the boys were trying to solve the problem of extending the catheti, seemingly ignoring the girls. As the boys got stuck, one of the girls walked up to the (female) teacher and whispered her solution. The teacher encouraged her to talk to the boys, and the whole group ended up producing the intended construction.

One specific topic of discussion concerning mathematics emerged in our follow-up meetings. To recall, one of our intention with the design was to encourage different solution strategies, such as measuring, calculation and comparison. What was noticed however, was that measuring took a rather dominant role in the activity. Moreover, since the students were not familiar with the Pythagorean Theorem we did not expect them to calculate the hypotenuse of the first triangle, in order to determine its

perimeter. However, when the students were asked to determine the perimeter of the larger triangle, i.e. after the catheti of the first triangle being doubled, they also now measured the hypotenuse. None of the students reflected on or argued that also the hypotenuse was doubled. The students did not even reflect on this after the three sides were measured. The data they used for determining the perimeter was the measured data.

During the first activity, the students quickly turned to calculating the area of the larger triangle by the rule; base times altitude divided by two. No attempt was made to compare the larger triangle with the smaller triangle, even if the construction supported looking four smaller triangles within the larger (see Figure 1). In the instructions for the second activity, we therefore explicitly asked the students if they could find out from the constructions any relation between the area of the larger triangle and the smaller triangle. After some discussion and guidance the students at least articulated that the area of the larger triangle was four times the area of the first triangle. However, we were not comfortable that the activity did not by itself provoke the students to involve principles and relations in their discussions.

We observed that the students solved the tasks rather pragmatically and routinely, in terms of measuring and applying rules for calculation. However, we do not have evidence that the students' behaviour depended on conceptual limitations. In the follow-up discussions within the development team we identified possible explanations in terms of the design of the activity and the students' history of being part of a certain educational system. Therefore, to develop the activity and to understand students' actions and potential, we have reached a point where we find it necessary to deepen the theoretical approach of our work, taking into account analytical constructs on several levels of interaction. In the next section we describe principles of the emergent perspective (Cobb et al., 2001), which we find suitable for our purposes.

CONCEPTUALIZING GROUP AND INDIVIDUAL MATHEMATICAL UNDERSTANDING

In Cobb, Stephan, McClain and Gravemeijer (2001) terms, the evolution of mathematical learning in classrooms constitutes of *social* as well as *psychological* structures of behaviour and reasoning. Within the social structure, they identify three analytical categories: *Classroom social norms*, *Sociomathematical norms* and *Classroom mathematical practices*. Examples of Classroom social norms can be for instance; that students collaborate to solve problems, that meaningful activity is valued more than correct answers, and that partners should reach consensus as they work on activities. With reference to our observations, Classroom social norms may have been in play when the quiet girl had to be encouraged by her teacher to communicate with her team members. *Sociomathematical* norms are defined as social constructs specific to mathematics. These are the norms in play when explanations and justifications are made acceptable (Hershkowitz and Schwarz, 1999). When

applying the analytical construct of *classroom mathematical practices* the analytical lens is closer to a certain instructional activities. It concerns regularities of the collective engagement in a specific situation in terms of symbolizing, arguing and validating.

A student may experience a study activity in different ways, as compared to the teacher's and to other students' interpretations (Wistedt, 1987; Iversen and Nilsson, 2007). The psychological perspective concerns the nature of individual students' reasoning. It brings attention to the diversity in students' ways of interpreting and acting in mathematical activities (Cobb et al., 2001).

It is crucial to understand that the relation between the social and the psychological perspective is considered to be reflexive (Cobb et al., 2001): "...neither perspective exists without the other in that each perspective constitutes the background against which mathematical activity is interpreted from the other perspective" (p. 122).

An implicit assumption of the current project has been that an unfamiliar teaching arrangement might encourage students to act beyond previously established Classroom social and Sociomathematical norms, with the possibility that these new actions may be more mathematically productive than their correlates of ordinary classrooms. The initial results of our observations, specifically the two separate incidents involving girls, support this assumption.

THE ORGANIZATION OF MATHEMATICAL PRACTICES

Weber, Maher, Powell, and Lee (2008) summarize some important ways in which discussions may establish opportunities for the learning of mathematics. Discussion can objectify students' experiences, thereby making these experiences the subject of analysis, encourage students to take a more reflective stance on their mathematical reasoning, require students to consolidate their thinking by verbalizing their thoughts, and help students learn to communicate mathematically and participate in a wider range of mathematical argumentation. Weber et al., (2008) also contend that group discussion can facilitate learning by inviting students to be explicit both about the ways in which they make new claims from previously established facts and about the standards they are using in deciding whether an argument is acceptable. Challenges from classmates can encourage students to debate whether a particular method of argumentation is appropriate and provide students with the opportunity either to justify their methods when their reasoning is sound or revise or abandon their methods when their reasoning is flawed.

In the organization of group discussions, Cobb et al., (2001) distinguish between three specific structures: taken-as-shared purposes, taken-as-shared ways of reasoning with tools and symbols, and taken-as-shared forms of mathematical argumentation. A taken-as-shared purpose is what the students and the teachers are trying to achieve together mathematically. The second structure is concerned with the ways in which tools and symbols are used and given taken-as-shared meanings. To account for

taken-as-shared forms of argumentation Toulmin's (1969) analytical model of argumentation has proven useful (Cobb et al., 2001). According to Toulmin (1969), an argumentation consists of at least three core components: *the claim*, *the data*, and *the warrant*. When a speaker makes a claim he or she may be challenged to present evidence or data to support that claim. The data typically consist of facts that lead to the conclusion that is made. If a listener does not understand why the data justify the conclusion that was drawn she may challenge the presenter to clarify why the data led to the conclusion. When this type of challenge is made and a presenter clarifies the role of the data in making her claim the presenter is providing a warrant. A warrant can of course be questioned, thus obligating the presenter backing up the warrant.

DISCUSSION ON OUR METHOD OF DESIGN

Our choice of method has been influenced by the constitution and working conditions of the development team. The main focus has been on collaborative development of the mathematical activity. The project emphasizes the potential benefits of collaborative development in close interaction with stakeholders. There has been a very open climate of discussion where teachers' knowledge and experiences have been given equal attention as input from the researchers. The teachers have been very active providing ideas and reacting on suggestions from the researchers, both during physical meetings and through e-mail communication. We argue that this way of collaboration differs from the approach usually used by DE practitioners. In DE, theories are usually introduced in early stage of the design process (diSessa & Cobb, 2004). From the observations of two activities, we have been identified a need for supporting theories. The interpretative frameworks outlined above will enable us to strengthen our design and to better understand our observations. However, we have found it fruitful to use an experienced based approach. No theories have been explicitly communicated during the initial work of the development team. Particularly, we believe that introducing abstract theories early in the discussions would have reduced the teachers' interest and possibilities to communicate empirically grounded ideas, thereby limiting the pragmatic root of the project. Our approach may therefore serve as a reasonable model for others, who wish to engage in collaborative activities in order to enhance school teaching. On account of this, we suggest that researchers in collaboration with teachers should take seriously the role of theories, particularly when to introduce them in the project at hand.

We suggest a balance between theories and practice, where practice takes on a rather dominant role in the early work. As the project and iterations proceed, the role of theories may be increased in order to enhance control of the learning activity. The analytical categories argued by Cobb et al., (2001), and Toulmin's (1969) model of argumentation, offer instruments both for supporting the design process and for serving as tools for analysis of observed actions.

Finally, one can question the validity of our approach in relation to the pedagogical implementation and learning outcomes of these activities but the main point here is not assess the effectiveness of the learning materials neither the mathematical content but instead to explore how to design and organize the flow of pedagogical activities in an outdoor learning setting. Our initial impressions indicate that this kind of learning activities seem to encourage discussions and new collaboration patterns, thus promoting deeper understanding among students. Therefore, we believe that a major challenge for the mathematics education community is to create new possibilities for learners to understand complex mathematical concepts, as well as to develop new analytical tools and theories in order to facilitate our understanding on how learning takes place under these new circumstances.

FUTURE EFFORTS

Based on the discussions presented in this paper, the following suggestions appear to be relevant for the design of the next iteration. The design of the next activity should take into consideration how:

- collective understanding can be provoked by encouraging students to make claims and be explicit about the warrants on which the claims rest,
- collective discussion can capitalize on individual variations (implying that the activity should encourage a variation in reasoning and solution strategies),
- norms and structures of mathematical practices may support or limit students' behaviour.

The last aspect specifically refers to the observation of how measuring took on a rather dominant role in the activities, narrowing the students' conceptual structures. On account of these guidelines we suggest to follow up the described activity with a second activity, where the students are not allowed to use a measuring tool. Instead they start with a triangle with given perimeter and given area and whose sides are not known. The triangle will be marked with flag lines and the students will be asked to continue the construction of the same pattern as in the previous construction and will be asked to determine the perimeter and the area of the larger triangle. We conjecture that such a setup will provoke the students to reflect on conceptual aspects, by comparing features of the triangles. Another suggestion is to let the students choose their own measures and construct a triangle which will be extended to a rectangle, with the aim that they discover the connection between the areas of the two figures.

An obvious next step of the project is to investigate how the described outdoor activity can be followed up in the regular classroom. Earlier mentioned shortcomings concerning students' documentation may be overcome by using mobile technologies. According to Spikol and Milrad (2008), mobile technologies offer the potential for a new phase in the evolution of technology-enhanced learning, marked by a continuity of the learning experience across different learning contexts. In particular, we propose to let students use mobile technology in order both to communicate the tasks

and to support the documentation of their solutions. Moreover, offering the students possibilities to videotape and taking pictures during the activity will support them in recalling and sharing experiences when they return to their regular classroom. We believe moreover that interesting applications may be developed in additional fields such as arithmetic and statistics, and even in algebra and functions. Our ambition is to invite students from the teacher training program at our university, so they can participate in widening our design approach to the above mentioned fields.

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STUDENT DEVELOPMENT PROCESS OF DESIGNING AND IMPLEMENTING EXPLORATORY AND LEARNING OBJECTS

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In 2001 a core undergraduate program, called Mathematics Integrated with Computers and Applications (MICA) was introduced in the Department of Mathematics at Brock University, Canada. In this program that integrates evolving technologies, students complete major projects that require the design and implementation of 'Exploratory and Learning Objects' (ELO). In this paper, we propose schematic representations and descriptions of the student development process as s/he completes an ELO project. We highlight the important role that ELO interfaces play in this development process.

Keywords: Exploratory and Learning Objects (ELO); student development process; students designing and implementing ELO; university mathematics education.

INTRODUCTION

There have been a number of publications (Muller, 1991, 2001; Muller & Buteau, 2006; Buteau & Muller, 2006; Pead et al, 2007; Muller et al., forthcoming) about the long-term implementation of evolving technology use in undergraduate mathematics education at Brock University (Canada) that started in the early 80s. The most recent development is the 2001 implementation of the core undergraduate mathematics program called *Mathematics Integrated with Computers and Applications* (MICA). Two of the program aims are to (1) develop mathematical concepts hand in hand with computers and applications; and (2) encourage student creativity and intellectual independence (Brock Teaching, 2001). Three innovative core courses, called *MICA I, II, III*, were implemented in addition to a review of all traditional courses to incorporate the MICA program aims. Results of a 2006 MICA student survey and an enrolment analysis covering the years 2001 to 2006 are reported in Ben-El-Mechaiekh et al. (2007). Highlights include

Students overall rated the use of technology in their mathematics courses as positively beneficial (77.74% of responses; 79.36% when restricted to mathematics majors). (p.10)

and, furthermore,

... students overwhelmingly rated the use of technology in [MICA] courses as [positively] beneficial (91.13% of responses) (p.9)

In this paper proposal, we focus on one of the major student activities in the MICA courses, namely their designing, implementing (VB.net, Maple, C++), and using of interactive and dynamic computer-based environments, called Exploratory and

Learning Objects (ELO). By Exploratory Object (EO) and Learning Object (LO), we mean the following.

An Exploratory Object is an interactive and dynamic computer-based model or tool that capitalizes on visualization and is developed to explore a mathematical concept or conjecture, or a real-world situation

and,

A Learning Object is an interactive and dynamic computer-based environment that engages a learner through a game or activity and that guides him/her in a stepwise development towards an understanding of a mathematical concept. (Muller et al., forthcoming, p.5)

To illustrate these objects, we provide without comment three examples of original student ELO projects that can be accessed at (MICA Student Projects website, n.d.): (1) *Structure of the Hailstone Sequences* EO by first-year student Colin Phipps for the investigation of a mathematical conjecture; (2) *Running in the Rain* EO by second-year students Matthew Lillie and Kylie Maheu for the investigation of a real-world situation; and (3) *Exploring the Pythagorean Theorem* LO by first-year student Lindsay Claes for the learning of a school mathematical concept.

In previous publications, we have elaborated how the MICA I course is designed to progressively bring the students to acquire the skills and understanding required for the development of ELOs (Muller & Buteau, forthcoming). In brief, as the course progresses, our students are guided through each step in the development process of ELOs that we describe in the next section of this paper. We have also explained that this requires a significant change in the teaching paradigm of faculty involved in these courses, and motivates a change in attitude in the students about learning and doing mathematics with technology at the university level (Muller et al., forthcoming). And also, we have argued that learning activities in the MICA program accelerates students' growth towards independence in doing mathematics (Buteau & Muller, 2006).

In this paper we propose a first attempt at defining a structure for the student development process in their activity of designing, implementing, and using an ELO. These final MICA projects are completed individually or in pairs selecting a topic of their own choice. We also briefly discuss the role of interfaces in the student development of an ELO. As in the past, we, as mathematicians in a mathematics department, look forward to receiving constructive feedback from mathematics educators. We hope that the presentation of our innovative student learning activities, as part of the systemic integration of technology in our university mathematics curriculum, will instigate educational research questions on learning mathematics with use of technology in tertiary education.

STUDENT DEVELOPMENT PROCESS OF EXPLORATORY AND LEARNING OBJECTS

In what follows, we suggest schematic representations of the development process for ELOs. Even though the schematic representations are worded generally, in their descriptions we focus on students in MICA courses.

Development Process of an Exploratory Object to Investigate a Conjecture

We propose the following diagram (Figure 1) to illustrate this development process.

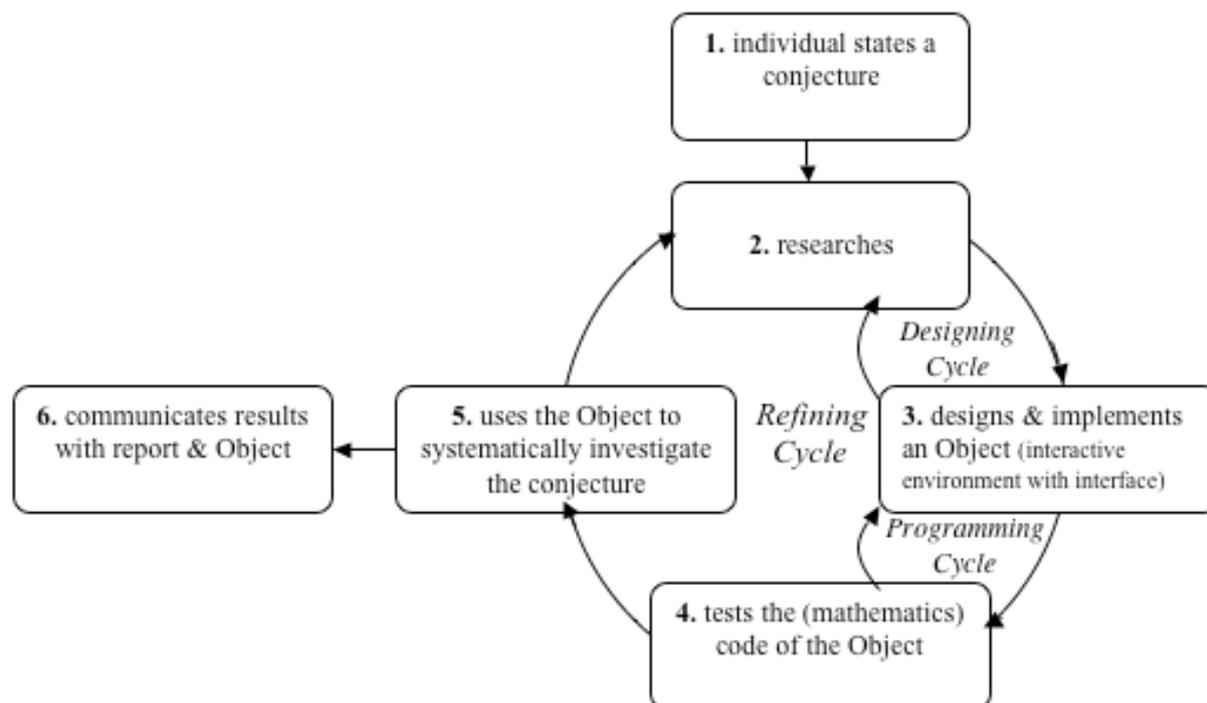


Figure 1. Development process of an Exploratory Object for the purpose of investigating a conjecture.

Here is a description of each step in the diagram.

1. Student states a conjecture, and may discuss it with the instructor; some of the more independent students wait until step 3 to discuss their project.

2. Student researches the conjecture using library and Internet resources, and may refine his/her conjecture. In conjunction with step 3, student identifies the mathematics, such as variables, parameters, etc., and is involved in a *Designing Cycle*.

3. With his/her understanding of the conjecture, student starts designing and implementing (i.e., coding) an interactive environment (i.e., program with interface) with a view to testing the conjecture. Student organizes the interface to make parameters accessible and to display diverse representations of results. As the interface plays such an important role in EO, we discuss it further in the next section.

4. Student selects, in a step-wise fashion, simple and more complex cases to test that the mathematics is correctly encoded and that the interface is fully functional. Together with step 3, the code testing and revising involve the student in a *Programming Cycle*.

5. At this step, student now returns to focus on his/her conjecture and uses the Object to systematically investigate it. Following the results of the investigation, the student may decide to refine the Object, e.g., introducing new parameters, etc., and be involved in a *Refining Cycle* (with steps 2, 3, and 4).

6. Student produces a report of his/her results and submits it with the Object. The report includes a statement of the conjecture, the mathematical background (from step 2), results of the exploration including an interpretation of the data and graphs (from step 5), a discussion, and a conclusion. This is somewhat similar to a science laboratory report. Building on this analogy, the Object is the laboratory itself. In other words, student submits his/her self-designed 'virtual laboratory' for the investigation of a self-stated conjecture together with his/her laboratory report.

Development Process of an Exploratory Object to Investigate a Real-World Situation

We propose the following diagram (Figure 2) to illustrate this development process.

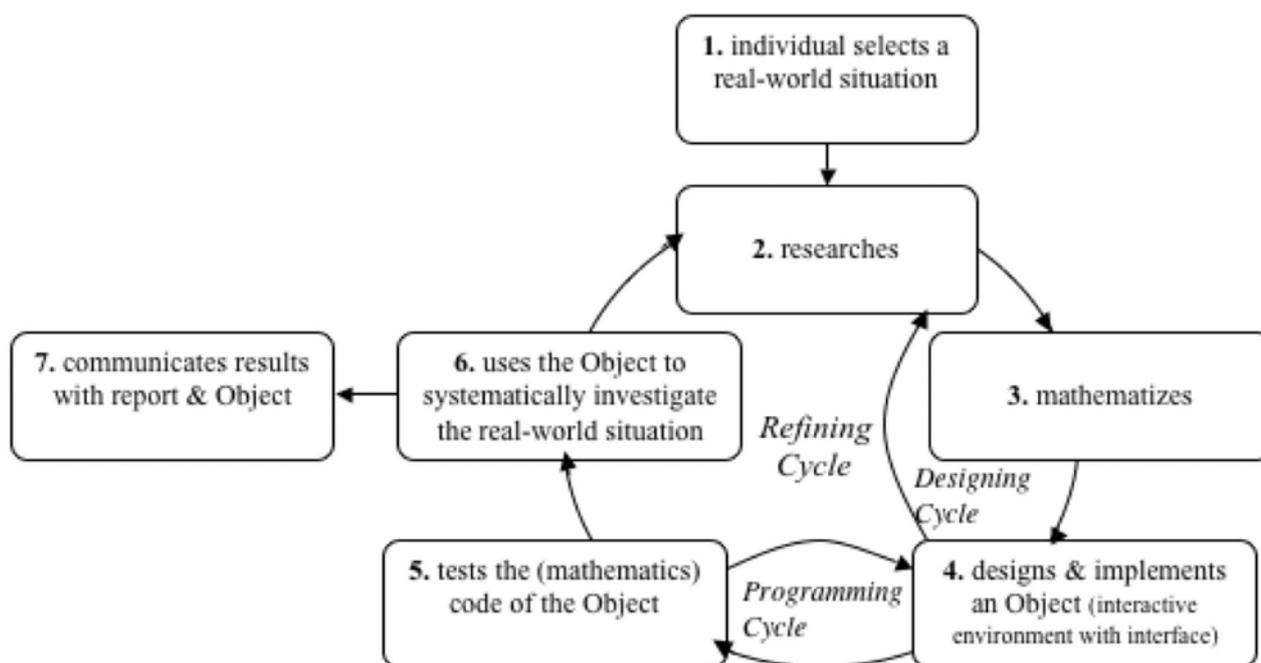


Figure 2: Development process of an Exploratory Object for the purpose of investigating a real-world situation.

Here is a description of each step in the diagram.

1. Student selects a real-world situation of particular interest, and may discuss it with the instructor; some of the more independent students wait to discuss their project until step 3 or 4.
2. Student researches the real-world situation using library and Internet resources, and may restrict or modify the scope of the real-world situation. In conjunction with steps 3 and 4, student identifies the mathematics, such as variables, parameters, etc., and is involved in a *Designing Cycle*.
3. Student develops a mathematical model of the real-world situation using the variables and parameters selected in step 2 and in the majority of cases, consults the instructor.
4. With his/her understanding of the model, student starts designing and implementing (i.e., coding) an interactive environment (i.e., program with interface) with a view to investigating the real-world situation. Student organizes the interface to make the model parameters accessible and to display diverse representations of solutions. As the interface plays such an important role in EO, we discuss it further in the next section.
5. Student selects, in a step-wise fashion, simple and more complex cases to test that the mathematical model is correctly encoded and that the interface is fully functional. Together with step 4, the code testing and revising involve the student in a *Programming Cycle*.
6. At this step, student now returns to focus on his/her real-world situation and uses the Object to systematically investigate it. Following the results of the investigation, the student may decide to refine the model and the Object, e.g., introducing or deleting, new parameters and variables, new conditions, etc., and may be involved in a *Refining Cycle* (with steps 2, 3, 4 and 5).
7. Student produces a report of his/her results and submits it with the Object. The report includes a description of the real-world situation, a development of the mathematical model (from step 3), results of the exploration (from step 6) including an interpretation of the data and graphs, a discussion, and a conclusion. This is somewhat similar to a science laboratory report. Building on this analogy, the Object is the laboratory itself. In other words, student submits his/her self-designed 'virtual laboratory' for the investigation of a self-selected real-world situation together with his/her laboratory report.

Development Process of a Learning Object of a Mathematical Concept

We propose the following diagram (Figure 3) to illustrate this development process.

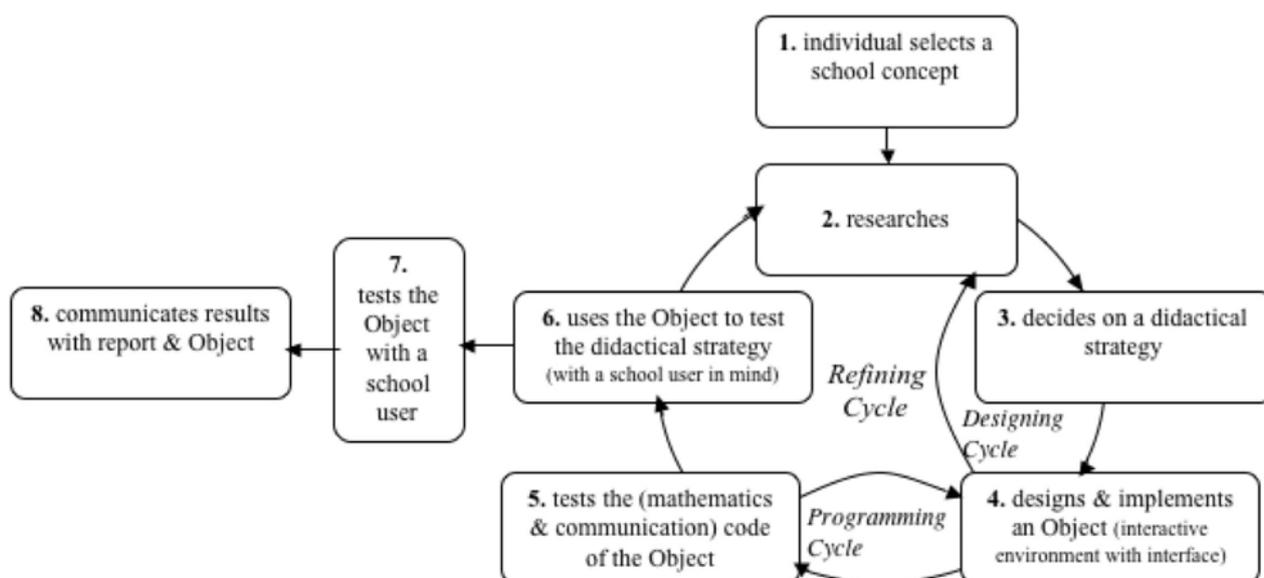


Figure 3: Development process of a Learning Object of a mathematical concept.

Here is a description of each step in the diagram.

1. Student selects a school concept.

2. Using library and Internet, student looks at resources about the concept and its teaching. In particular, student identifies when in the school curriculum the concept is taught, reviewed and expanded, what previous mathematical understanding, general knowledge and reading capabilities can be assumed, etc. In conjunction with steps 3 and 4, student identifies and develops the mathematics didactical features that could be used for his/her Object, and is involved in a *Designing Cycle*.

3. Based on the information gathered in step 2, student selects a didactical strategy for a fictive school pupil learning of the concept that may include developing a game or activity to engage the learner, breaking down the concept, setting up a testing procedure, etc. Student may discuss the strategy with the instructor or wait until the next step.

4. Student starts designing and implementing (i.e., coding) an interactive environment (i.e., program with interface) with a view to implement the didactical strategy. Student structures a self-contained interface realizing that the fictive school pupil will be using the LO independently. As the interface plays such an important role in LO, we discuss it further in the next section.

5. Student tests that the interface (communication, navigation, etc.) is fully functional and tests with simple and more complex cases that the mathematics is correctly encoded. Together with step 4, the code testing and revising involve the student in a *Programming Cycle*.

6. At this step, student now returns to focus on his/her didactical strategy and works through the Object with a school pupil in mind. Following the results of this investigation, the student may decide to refine the Object, e.g., changing the sequence

of activities, improving the clarity of communication, etc., and may be involved in a *Refining Cycle* (with steps 3, 4, and 5).

7. Student tests his/her Object by observing a school pupil, at appropriate grade level, working with the Object. In some cases, student returns to the refining cycle and revises the Object.

8. Student produces a report of his/her results and submits it with the Object. The report includes the didactical purpose, the target audience, the mathematical background of the target audience, a brief account of the school pupil experience (step 7), and a discussion. This report is somewhat similar to a lesson plan, including a post-lesson reflection, though without a description of the lesson. Building on this analogy, the Object is the lesson itself. Thus, student submits his/her lesson plan of a self-selected mathematical concept in which the written description of the lesson is replaced by an 'interactive self-directed lesson (with a virtual learner)', i.e., by the Object.

ROLE OF THE INTERFACE IN THE DEVELOPMENT PROCESS OF EXPLORATORY AND LEARNING OBJECTS

The interface provides interactivity and (dynamic) visualization. In the Development Process of ELOs (Figures 1, 2, and 3), the student creates an interface in the Designing Cycle with the aim of using it for his/her mathematical or didactical investigation (step 5 in Figure 1 and step 6 in Figures 2 and 3).

During the Designing Cycle of an Exploratory Object, the potentiality of interactivity encourages the student to make explicit the parameters that could play a role in the investigation of his/her conjecture or real-world situation in such a way that they are accessible from the interface. The potentiality of visualization urges the student to decide on the representations to be displayed in his/her interface so as to best support his/her investigation.

At the step in the Development Process when the student uses the Object for his/her investigation (step 5 in Figure 1 and step 6 in Figure 2), both interactivity and visualization aspects of the interface play a role in the student's systematic investigation. The latter can be seen as a dialogue between the student and the computer, though the discussion is fully controlled by the student. During the systematic investigation, the student sets a question by fixing values to parameters (interactivity), the computer answers the question (visualization), and the dialogue continues in that way unless the student concludes that the answers are not satisfactory to meet his/her goal and decides to refine the Object (Refining Cycle). In other words, the student is in an *intelligent partnership* (Jones, 1996) with technology.

The interface plays a central role in Learning Objects but which is different than in the Exploratory Objects. A Learning Object is designed for other users to use by themselves, i.e., without the Object designer who is the student in our case. Thus the

navigation in the interface should be very clear and easy. The interface should also provide, at any time, motivation for the intended users to go to a next step in the Object. As such, the visual presentation and the wording should be adapted to the intended users:

For Learning Objects students [are] reminded constantly that they are designing interfaces for people who are not experts and that they need to take into account such issues as the user's age, educational level, gender, cultural background, experience with computers, motivation, disabilities, etc. (Muller et al., forthcoming, p.12)

Also, students should

... break away from the linearity of the written tradition in order to take full advantage of the technological paradigm. (Muller et al., forthcoming, p.12)

In step 8 of the Development Process of the LO (Figure 3), we introduced an analogy where the Object is a 'lesson with a virtual learner'. Using this analogy, the interface's potentiality of interactivity encourages the student during the Designing Cycle to develop an active 'lesson', i.e., a lesson that is interactive, with the intended fictive pupil. The interface's potentiality of visualization facilitates the development of transparent communication of the 'lesson' flow and makes it possible for the student to test his/her 'lesson' (steps 6 and 7 in Figure 3). In other words, we suggest that these two potentialities allow the student to develop a 'guided intelligent partnership' between a fictive pupil and technology.

REFLECTIONS

Diagrams shown in Figures 1, 2, and 3 clearly indicate our view that the student mathematics learning experience through the designing and implementing of an ELO is richer than what is experienced through activities of only programming mathematics. The interface plays a major role through its interactivity and visualization potentialities as it provides students with an opportunity to be involved in an 'intelligent or guided intelligent partnership' with the technology.

In a recent collaborative project between a local elementary school, *École Nouvel Horizon*, and our Department of Mathematics, MICA student Sarah Camilleri was involved as part of her Honour's project in the development of *Fractions Fantastiques/Fantasy Fractions Learning Objects* (Camilleri, 2007; Buteau et al., 2008a and b; MICA Student Project website, n.d.). In this development, she worked with a Grade 5 class, the teacher, and the school principal. It is worthwhile to explore the ways in which individuals took different roles and responsibilities in the Development Process (Figure 3).

Sarah and the teacher selected the fraction concept (step 1), and Sarah researched it (step 2). The teacher taught fractions to the class and presented the collaborative project. In the *Designing Cycle*, guided by the teacher and the principal, the Grade 5 pupils developed the dynamic mathematics lessons, interactive mathematics games,

story line of the Object, its characters, etc., and provided drawings and written materials to communicate their ideas to Sarah who had to select and adapt some of them for programming purposes. The pupil design work was achieved in class discussions and in smaller groups of two or three. Within the *Programming Cycle* Sarah took the responsibility of faithfully implementing the pupils' design which also involved the digitizing of the pupils' drawings. The *Refining Cycle* involved Sarah and the teacher for testing the functionality of the Learning Object and checking the faithful integration of the pupils' ideas. *Fractions Fantastiques* Learning Object was presented by Sarah to the Grade 5 class and each pupil received a CD-ROM copy of *their* Learning Object (step 8).

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HOW CAN DIGITAL ARTEFACTS ENHANCE MATHEMATICAL ANALYSIS TEACHING AND LEARNING

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Digital technologies seem to be still very promising to fruitfully support the construction of mathematical knowledge. Far more interesting is the way to incorporate them into the design of a learning environment framed by certain institutional constraints. Through this study we present some reflections and ideas arising from the dialectic interplay between the environment and the students in their effort to formulate a calculus theorem and construct its proof. Related teaching and learning phenomena providing information on instrumental genesis processes are primarily discussed.

INTRODUCTION

Elementary pre-calculus is at the heart of the syllabus at secondary level mathematics education and the entry-point to undergraduate mathematics as well. Many research studies witnessing students' problems to attain a satisfactory level of conceptualisation have been held on this field (for example, see Artigue, 1999). This fact is related to mathematically superficial strategies (Lithner, 2004) implemented by traditional procedure-oriented teaching practices. We claim that these practices are generated by both teachers' attitudes and institutional constraints implicitly or explicitly imposed by textbooks and curricular objectives (Ferrini-Mundy & Graham, 1991). Even at the university level, this situation results in detecting serious difficulties on behalf of the students when faced with non-algorithmic type demands which entail reasoning and conceptual understanding (Gonzales-Martin & Camacho, 2004).

On the other hand, the development of mathematics has always been dependent upon the material and symbolic tools available for mathematical computations (Artigue, 2002). Current research on mathematics education regarding the relationships between curriculum, classroom practices and software applications (Lagrange, 2005) offers the ground to address and develop questions concerning technology's fitting into learners' actual social and material environments, the problems users have that technology can remedy, and, furthermore, ways of conceptualizing the design of innovative learning tools as emergent from dialectics between designers and learners-users of those tools.

The learning environment is supported by a Dynamic Geometry software (DGS) enhanced by a function-graphing editor to help Mathematical Analysis teaching at the level of 12th grade.

The produced didactic sequence covers the introduction of global and local extrema definitions, Fermat's theorem (stationary points) with its proof, the mean value theorem, monotonicity definitions and the derivative sign/monotonicity theorem along with the proof and its applications. Selection of the exact targeted mathematical material on the field of differential calculus, as well as further elaboration of the activities, were attempted with the intention to form a rational succession of concepts to a coherent local unity of mathematical knowledge, mainly including introduction of definitions, formulation of theorems and construction of proofs. From this still on-going research, we present here some elements derived only from an activity concerning the teaching and learning of Fermat's theorem formulation and proof on the field of differential calculus.

THEORETICAL FRAMEWORK

Complexity and close interweaving of cognitive, institutional, operational and instrumental aspects obliged us to adopt a multidimensional approach (Lagrange et al, 2003) in order to design the learning environment and study the teaching/learning phenomena produced.

According to Duval (2002), construction of mathematical knowledge is strongly attached to the manipulation of different semiotic representations. This term refers to productions made up of the use of signs and formed within a semiotic register which has its own constraints of meaning and function. More specifically he defines a "*register of semiotic representation*" as a system of representations by signs that allows the three fundamental activities tied to the processes of using signs: the formation of a representation, its treatment within the same register, its conversion to another register. Interaction between different registers is considered to be of great importance and necessity to achieve understanding of a mathematical concept. Under this aspect, our tools were designed with the intention to mobilise and flexibly articulate semiotic representations within the numerical, the algebraic and the graphical register, so that to generate mathematical conjectures.

Very special and idiomorphic conditions existing within the local educational culture of Greek 12th grade students obliged us to take into consideration the notion of didactical transposition (Chevallard, 1991). At this level, a huge amount of institutional pressure results in the development of an "exam-oriented mentality" on behalf of the students as well as their families, which promotes a procedure-oriented attitude towards the mathematical knowledge in context. Candidates' needs to be prepared for a final national university-entrance examination at the end of the year result, finally, in an implicit (or even sometimes explicit!) meta-didactical attitude leading them to ignore or reject conceptual approaches not strongly attached to exam demands. Through this perspective, we were obliged to take into account and reinforce both the epistemic and the pragmatic value (Artigue, 2002) of the mathematical knowledge to be taught without any decrease or discount of any of them, in the economy of the available didactical time. Relating this idea to the tools' design, we considered the possibility to teach basic mathematical concepts within a

reasonable amount of learning time, and in ways compatible to both its institutional dimension and the transition to advanced mathematical thinking.

The theory of didactic situations (Brousseau, 1998) helped us conceive the whole learning environment (milieu) as a source of contradictions, difficulties, and disequilibria stimulating the student (on his own responsibility to control it) to learn by means of adaptations to this environment. At this point, we took also into account activity theory (originated in socio-cultural approaches and mediation theories rooted in Vygotski, 1934) to assign to the environment a character sometimes antagonistic to the subject (as pointed by TDS) but also sometimes cooperative and oriented to an educational aim, guided by distinctive didactical intentions.

In order to best incorporate digital artefacts in our didactical engineering, we considered the potential technology offers for linking semiotic registers within the frame introduced by the instrumental approach (Rabardel, 1995, Artigue, 2002, Trouche, 2004). A cultural tool or artefact, designed to mediate mathematical activity and communication within a socio-cultural context, differs from the corresponding instrument into which this artefact can be transformed. The artefact, as the final result, encompasses a psychological component; a construction by the subject, in a community of practice, on the basis of the given artifact by means of social schemes. This transformation is developed through an instrumentation process directed towards and shaping the subject's conceptual work within the constraints of the artifact and an instrumentalisation process directed towards and shaping the artifact itself. Both constitute a bidirectional dialectic and sometimes unexpectedly complex process called instrumental genesis (Artigue, 2002). Concerning tool design, we tried to keep simplicity and friendliness to the user, in the sense that their implementation demands, as far as possible, a short process of appropriation by the user and an easy way to be transformed into mathematical instruments to be utilised in the context of the activities. The necessity of any technical support by the teacher was also minimised as far as possible.

The crucial question to answer through our research is whether a design philosophy under the norms mentioned above has the potential to determine a set of effective digital learning tools, pre-constructed on the dynamic software, which can be easily transformed to learning instruments successfully integrated into the teaching of important calculus concepts at the level of theorem formulating and proof. By the term *successfully integrated* we mean that, firstly, they can make visible phenomena previously invisible, secondly, they can potentially generate innovative approaches to important mathematical concepts, and, thirdly, they shape and better our understanding of some productive or problematic dimensions of the computer transposition regarding the mathematical knowledge accessed through the instrument's mediation.

METHODOLOGY

The activity (of total duration 90 min) was developed in two different schools in groups of 12th grade students (10 in one group and 5 in the other) during the month of February, 2008. The main differences between the students of the different schools were identified on the socio-cultural and financial background of the corresponding families (we did not address any comparison issue in our research goals) and as well to the fact that comparatively more students belonging to a certain school had a facility for using mathematics software, being exposed several times in the past at different kinds of technology enhanced approaches. For the latter we did not find enough evidence to support the idea that different software cultures of the students have great impact on their attitude and capabilities of manipulating the pre-constructed software tools induced by our activities.

The informatics laboratory of every school was used and the pupils were at couples situated in a PC-environment. This time the researcher played the role of the teacher as an orchestrator of in-class situations. A Teacher-Analysis sheet has also been developed to provide necessary details so that other teachers can handle the in-class orchestration.

At the beginning, a worksheet was given to the students to work with and at the end of the session they received a corresponding post-assessment sheet including several T/F type questions of mathematical nature, which they returned back completed next day. The whole didactic sequence (consisting of four Sessions) was recorded by a voice-recorder and, the whole sequence being completed, a post-questionnaire was passed to the students in order to collect and save some of the instrumental marks being left on them through the entire approach. Finally, four students (two for each group) were interviewed to explicitly clarify their answers at this questionnaire concerning the instrumented actions performed and the students' attitude towards mathematics teaching before and after the whole experience.

The way of obtaining results-serving the a posteriori analysis-from the raw input data has to be explained here. The whole content referring to the 2nd Activity (Fermat's Theorem: Stationary Points) has been divided (according to the conceptual meaning development) into 12 *Episodes* and each one of them potentially to one up to four *Phases*. Next, for every one of the 24 *Phases* produced, we used the transcribed outcomes of the recorded class discourse, along with the written notes and answers of the students on the worksheet, to produce some discrete entities of information we called *Events*. An *Event* in this terminology is characterised and differentiated by components of mathematical or didactical or instrumental nature which can probably coexist. The study and analysis of these *Events* provided our a posteriori analysis with the material to compare the results composed up to this point to the analysis of the students' answers to the corresponding post-assessment sheet-being sorted out and analysed separately. Finally, we took into consideration the students' answers on the final post-questionnaire as well as the transcribed explicitation interviews in order to enhance our vision and come up to some final conclusions.

LEARNING ENVIRONMENT

Concerning the tools' design (and being sensitive to the complexity of instrumental genesis processes), we tried to reduce, at least, the complexity of the interface. We tried also to keep tools' implementation strongly attached to the mathematical needs emerging within the predefined context. The learning environment regarding the whole activity was, thus, perceived with the intentions to:

- a) Mobilise students' interest to estimate local extrema departing from a real problem,
- b) Make up a link with the students' previous knowledge on the subject of local extrema and the limit concept,
- c) Stimulate the students to construct the targeted mathematical knowledge by mobilising different registers of representation (graphic, numerical, symbolic, and verbal) for the same concept and favouring representational interconnections between them,
- d) Use the in-class discourse to generate an activity space favouring students' effective instrumental processes,
- e) Support conjecturing, conceptualisation, and institutionalization,
- f) Insert certain examples or counter-examples when necessary (Gonzales-Martin & Camacho, 2004).

We focus especially on the activity designed to introduce the concept of Fermat's theorem. As far as the students were concerned, our specific didactical aims were: to conjecture the theorem, construct its formal statement and proof realising the absolute necessity of its presuppositions and its application range, to perceive that the reverse form of the theorem is not valid, and, finally, to apply it in calculating the local extrema of the function-given by a formula-induced by the problem.

In the following we describe and analyse some selected *Events* drawn out of two different *Episodes*. The material that will be presented is coming from a blend of actual events produced by both groups of students, whose comments and actions have been complementing each other over the flow of the activity.

Remark: The term S-Tools refers to the specific on-Screen pre-constructed tools on the software.

Episode A: Introduction to the *Line $y=k$* , *IntersectionPoints*, *Magnification* S-Tools and applications in approximating local extrema positions on the function graph.

Tool Description: The students were prompted to open *Line $y=k$* and *IntersectionPoints* S-Tools. The first one draws a horizontal parametric line, whose position can be controlled by the active parameter k (a number in yellow background on the screen that can be modified by the user, see Image 1). If this line has some common points with the function graph then the second S-Tool *IntersectionPoints* draws these intersection points and provides their x -coordinates. Furthermore, a technique permitting the students to change the decimal length and the digits of any active parameter was explained to them by the teacher.

The following question given by the corresponding worksheet came to stimulate students to S-Tools utilisation:

Q1: Could you find or estimate points of local extrema for function P ?

Aim Description: The main intention of the constructed situation was to encourage students to explore and use the S-Tools in order to estimate several intervals of x -axe that could enclose positions of internal local extrema and to get approximate values for these positions by shortening the length of the corresponding intervals. Moreover, they had to identify the kind of local extremum (maximum or minimum) and perceive which of them are internal to the interval.

Events: The teacher asked the students to change the active parameter k and see what happens. Some of them could not understand the changes on the counters of intersection points coordinates and that was clarified by the related discussion in the class community. Then, the students were asked to use these tools to numerically estimate the local extrema positions (Question Q1) on the graph the better they could

(Image 1). Some students could not cope with changing the decimal length and the values of several digits so they were given additional technical instruction for that. The teacher asked them to find an interval including the abscissa of a local maximum (this was done very easily) and then to try to shorten this interval by means of the tool. This was not so easily done by every pupil but remarks made by several students and on-screen indications gave good results.

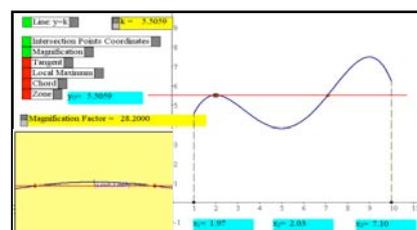


Image 1

Interesting events identified on behalf of the students were:

- During exploring with decimal digits many students observed two intersection points approaching each other and, finally, coinciding to only one but the indications on the corresponding counters were different.

- Six of them noticed that they could see intersection points on the screen but the indications on the coordinate counters did not attest such an existence.

Concerning these two events, the teacher's proposition was to use the *Magnification S-Tool*.

Tool Description: This S-Tool could be used to magnify a selected region around a point which can be displaced anywhere on the graph and is controlled by the *Point-Abscissa* and the *Magnification Factor*.

Subsequently, the students were asked to use the same process to estimate the values of every local extremum they could perceive on the graph.

Remarks: Students' written answers on the worksheet revealed that the whole class succeeded at the qualitative level (number of local extrema, approximate position and characterisation). However, at the numerical level, only a small part (26% of them) tried to test in the extreme the instrument's potentialities and even less (6,6%) achieved at exhausting them-providing the values asked at 3rd or 4th decimal digit accuracy as we had anticipated. A technical weakness versus time disposal has been estimated as a possible reason for that.

Results: This first contact with the notion of approximation opened up the ground for a further in-class discussion. The discourse came up to the point that the tool is able

to provide visual images of a certain validity only as an indication generator (which in certain cases can be of great importance for the mathematical knowledge) but not always to produce an arithmetic value in absolute accuracy. The teacher reinforced this situation by asking what would happen if the extremum in search had the real value $\frac{2}{3}$ or $\sqrt{2}$. This fact conducted the discussion to bring into light the inherent inadequacy of every computing system to represent infinite decimal numbers in a complete way. So the students realised that, through this attempt, and also in general, they could obtain only relative accuracy for the local extrema values. The necessity of devising new mathematical tools that could probably provide absolute accuracy for these values came in the discourse.

Episode B: Introduction to the tangent: Relating line $y=k$ when passing through an internal local extremum to the function graph – Derivability

Next Question Q2 had the intention to sensitise students' attention and make them focus to what is going on locally at the area near an internal local extremum point.

Q2: When line $y=k$ is passing through an internal local extremum point on the graph, how is this line related to the curve at an area near this point?

Description: Within this *Episode* the students were asked to express their thoughts regarding the visual relation between the line $y=k$ when passing through an internal local extremum point on the graph and the curve itself near the extremum point. The first attempt was made on normal view and the second by means of the *Magnification S-Tool* (Image 2). Subsequently,

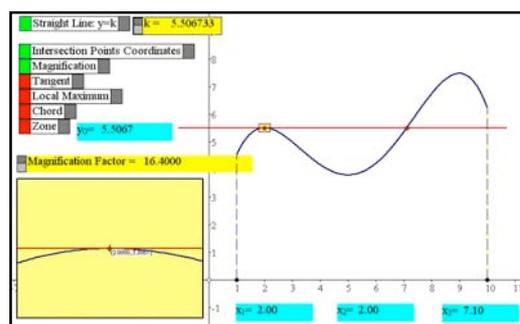


Image 2

at the third phase of the *Episode* a new subroutine program file was invoked, where the students could alternatively observe under magnification the behaviour of functions $y = x^2$ and $y = \text{abs}(x)$ in the neighbourhood of $x = 0$ (Image 3). This was done by changing only the function formula through a menu of the file. The necessary technique was shortly explained by the teacher.

Events: The class discourse developed at this *Phase* helped many of the students to communicate their thoughts and formulate them in an intelligible way. They came up with the visualisation of the inequality relations $f(x) \leq k$ or $f(x) \geq k$ near the local extremum. Relating this fact to the image produced by the function graph and the horizontal line, they could easily conjecture that this line when passing through a local extremum point on the graph “leaves the whole curve on one side” or “does not cut it” at the area near this point.

Remarks: Analysis of students' written answers on the worksheet showed that the big majority of them (80%) succeeded in perceiving the visual relation between the curve

and the line and, moreover, 26,6% of them were able to connect it with the corresponding symbol relation. 26,6% of the students proceeded to conjecture that, at this case, this horizontal line should be a tangent of the graph, whereas even fewer (13,3%) mentioned that there was only one common point of the line and the curve at the area near the local extremum.

To the question of the teacher if these two conditions (namely existing of a single common point and “not cutting” in the area near a local extremum) are able to assure the existence of a local extremum, confusion arose and the community could not provide a clear answer. This event, along with the term tangent mentioned earlier, was used as a bridge to the discussion of next question:

Q3: At the area near the extremum point, can you observe any additional relation between the curve and the line $y=k$ when the latter is passing through this point?

Remarks: Class discourse concerning this question resulted in the assertion on behalf of the students that under magnification the curve tends to become a horizontal line or to coincide with it. Moreover, there were some more students stating in a clear way the conjecture that the horizontal line when passing through a local extremum point on the graph *keeps the position of a tangent of the graph at this point*. This conjecture provided the bridge through which the teacher introduced the issue of the existence of the tangent at such a point. Additionally, as a natural consequence of the previous discussion, the subroutine file was used to support students’ exploring and help them visualise the difference between the function graphs of $y=x^2$ and $y=abs(x)$ on point $x=0$ under magnification (Image 3) and relate it to the derivability of the function at this point. Most of the students’ expressions were for example “*Oh, there’s an angle there!*” or “*... in this case we have a peak point ...*” etc.

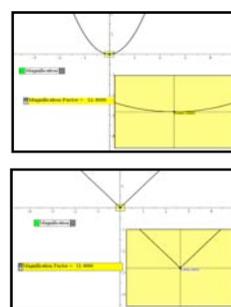


Image 3

DISCUSSION AND PRELIMINARY RESULTS

In this paper, we tried to describe a few situations concerning only the instrumental dimension of our research. The *Episodes* presented above contribute, as a first step, to Fermat’s theorem construction departing from an intuitive approach. This goal is achieved by exploring and visualizing the local extrema positions related to the premises of the theorem.

As it has been pointed by Guin and Trouche (1999), students’ answers were strongly dependent on the environment:

At a first attempt, many students tried to configure the artefact regarding the needs of the specific work: screen view adaptation by transposition of toolboxes and active parameters configuration (i.e. changing the decimal length and the values of certain digits of parameter k). These facts confirm, on their behalf, an effort to adapt the artefact to the demands of the specific task induced by the first question Q1 (Estimation of local extrema values) and we consider that as a step to the direction of instrumentalisation in the evolution of instrumental genesis processes (Rabardel,

1995, Trouche, 2004). As instrumentation processes had intently been designed and anticipated not to be very complex, soon after, we could observe automaticity towards certain instrumented action schemes to the execution of necessary tasks (i.e. utilising active parameters).

We point to an internal constraint (Trouche, 2004) of the instrument, which is related to computer's inherent deficiency in providing absolute preciseness through computations, regarding infinite decimal numbers. This issue was discussed with the students during several activities and, finally, was used as an entry to the discussion concerning the notion of approximation. Additionally, a common feeling was developed pointing out that computers will not solve all the mathematical questions inserted. This fact was also used to encourage students to develop their knowledge so as to overcome these limitations.

Students' answering to questions of the post-assessment sheet regarding the statement of Fermat's theorem or its applications within only the graphic register showed that the great majority of them (86,6%) could cope very good at this level. However more complex questions relating this register to the algebraic one have been far too difficult for the students, proving that more work is necessary to be done at this level.

Analysis of students' answers to the final post-questionnaire testified a generally positive attitude towards "*this way of teaching*". For example, to the question: "*Could you identify any positive or negative points through this series of activities you have been attending?*", some of their answers were: "*We could discover and see by ourselves most of the things on the screen...*" or "*By the aid of the computer we could really see and work on the stuff we treat usually in the class*", or "*It was easy-going because, we first, ... we didn't realize that we had made the proof of the theorem and only at the end we got the typical statement*" or "*It was very helpful to recollect the images on the screen, but the problem was that we didn't solve many exercises!*" etc. Of course, more work and analysis need to be done on this issue in order to obtain some reliable results.

Due to the lack of space, we did not address issues concerning the ways through which the rest of our theoretical perspectives shape our research. However, some results seem to deepen our reflection. They show the potential of such a learning environment design to produce didactical phenomena giving an illumination to both problematic and productive aspects of the mathematical knowledge developed through the educational use of digital technologies.

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A LEARNING ENVIRONMENT TO SUPPORT MATHEMATICAL GENERALISATION IN THE CLASSROOM

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This paper discusses classroom dynamics and pedagogical strategies that support teaching mathematical generalisation through activities embedding a specially-designed microworld. A prototype of our microworld was used during several one-to-one and classroom studies. The preliminary analysis of the data have allowed us to see the implications of designing and evaluating this specific technological tool in the classroom as well as the teachers' and the students' requirements. These studies feed into the design of the intelligent support that we envisage the system will be able to offer to all students and the teacher. In particular, they helped us identify which aspects of teachers' interventions could be delegated to our system and what types of information would be useful for supporting teachers.

KEYWORDS: Mathematical Generalisation, Microworlds, Classroom Practices, Teachers, Intelligent Support

INTRODUCTION

It seems that there is a growing diversity of computer-assisted material and tools for mathematics classrooms. Even though this proliferation of digital tools and new technologies has broadened the instructional material available for teachers, they are still rather insignificant to classroom practice and their use is far from regular (Artigue, 2002, Mullis et al., 2004, Ruthven, 2008). This suggests a challenge for mathematics educators to develop complete, consistent and coherent systems that not only assist students, but also support teachers' practice in the classroom.

The aim of the MiGen¹ project is to design and implement a system with teachers that meets their as well as students' requirements. We are developing an intelligent exploratory learning environment for supporting students in making mathematical generalisations. In more detail, our focus has been on the difficulties, first students face in their efforts to generalise and second teachers face in their efforts to support students appropriately during lessons with 20-30 students. For our initial investigations, we restricted the domain of mathematical generalisation to the generation and analysis of patterns. Activities with patterns often appear in the UK mathematics curriculum and have been identified as motivating for students (see Moss & Beatty, 2006). They also comprise a good domain for generalisation, since they allow students to come up with different constructions for the same pattern, find the corresponding rules and realise their equivalence.

Our aim is to develop a system that provides the means to understand the idea of generalisation, but also the vocabulary to express it, while supporting rather than supplementing the teacher. The system is intended to provide feedback to the teacher

about their students' progress and, where the system's 'intelligence' is unable to help students, to prioritise the students in critical need of the teacher's assistance.

The core of our system² is a microworld, called the eXpresser (described briefly in the next section), in which students can construct and analyse general patterns using a carefully designed interface. In order to build the microworld, our team³ started with a first prototype (Pearce et. al, 2008). Using an iterative design process, and in order to investigate the effectiveness of our approach, we carried out a number of studies with individual students or pairs of students, each time using the feedback we obtained to build the next prototype. This process resulted in the evolution of the prototype and its subsequent evaluation in classroom.

This paper, after a brief discussion of our methodology, presents the preliminary data analysis of the classroom studies that not only support the next version of the microworld, but also feed into the design of the intelligent support that we envisage the system will be able to provide. Our focus here is on the teachers' pedagogical strategies and the students' needs for support and assistance during their interactions with the microworld. This analysis is followed by a discussion of the teachers' interventions that could be delegated to the 'intelligent' system and what types of information would be useful for supporting teachers and therefore necessary for the development of the intelligent support components of our and other similar systems.

A microworld for patterns – the eXpresser

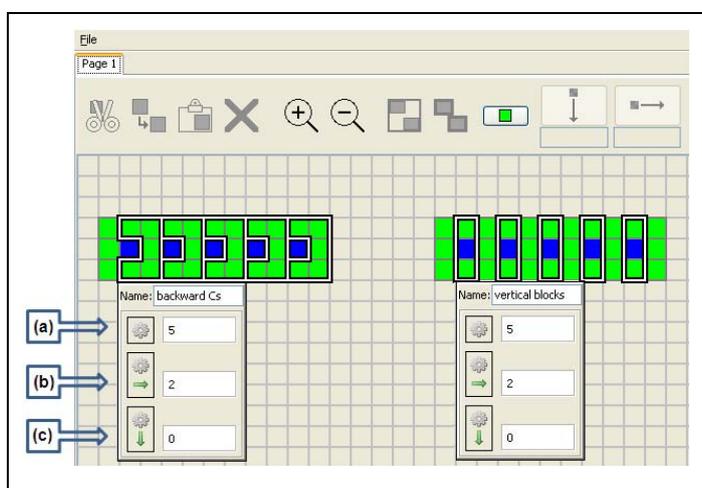


Figure 1. The interface of eXpresser with two different constructions of the same pattern. The left one is made out of a vertical block of 3 squares and 5 'backward C-s' and the right one of alternating vertical blocks.

First, we present briefly the main features of the eXpresser. We emphasise that at the stage of the study, attention was focused largely on the features key to our research goals. So, the following description of the system is by no means complete. In addition, its design has evolved significantly through studies such as the ones described in this paper. The interested reader is referred to Noss et al. (2008), where the system's rationale and design principles are described in detail.

In eXpresser, students can construct patterns based on a 'unit of repetition' that consists of square tiles. These patterns can be combined to form complex patterns, i.e. a group of patterns. A pattern's property box (depicted in Figure 1) shows three numeric attributes that characterise the pattern⁴. The first specifies the *element count* (number of repetitions) of this pattern (a). The icon with the right arrow (b) specifies *how far*

to the right each shape should be from its predecessor and, similarly, the icon with the down arrow (c) specifies *how far down* a shape should be.

A requirement of our constructivist approach was to allow students to construct patterns in a variety of ways (Figure 1). Additionally, an important design feature is the ability to 'build with n' (see Noss et al., 2008), i.e. to use independent variables of the task to create relationships between patterns.

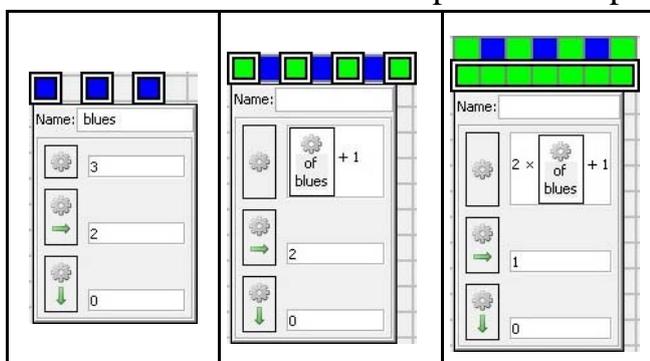


Figure 2. Another way to construct the pattern in Figure 1. To relate the middle row with the first pattern (named “blues”), the number of repetitions should be one more than the number of repetitions of “blues”. For the bottom row it should be twice more plus one. These relationships are specified iconically.

This feature not only provides students additional ways to construct patterns but we hypothesised that it enables students to realise what are the independent variables and use them to express relationships. To overcome difficulties that students face with symbolic variables the microworld employs what we call ‘icon-variables’, which are pictorial representations of an attribute of their construction. We have illustrated in previous work (Geraniou et al., 2008), that these ‘icon-variables’ provide a way to identify a general concept that is easier for young learners to comprehend. An example of expressing such relationships is depicted in Figure 2.

METHODOLOGY

Our own previous work and studies by Underwood et al. (1996) and Pelgrum (2001), for example, concerning the adoption of educational software in classrooms emphasise the importance of teachers’ involvement in the whole design process of computer-based environments. Therefore, several meetings with the teacher were held before each classroom session so that they were familiarised with the prototype, agreed and made input to the lesson plans and in order to clearly state the teacher’s, the students’ as well as the researchers’ objectives.

The overall methodological approach is that of ‘design experiment’, as described by Cobb et al. (2003). One of our goals during these sessions was to inform our system’s design and evaluate the effectiveness of our pedagogical and technical approach. We aimed at investigating the classroom dynamics by looking at individual students’ interactions with the microworld, the collaboration among pairs or groups of students as well as the teachers and researchers’ intervention strategies.

We investigated the use of eXpresser in several one-to-one and classroom sessions with year 7 students (aged 11-12 years old). Particularly for the classroom sessions, two researchers played the role of teaching assistants and another was observing and

keeping detailed notes regarding the researchers' and the teacher's interventions. The sessions were recorded on video and later analysed and annotated with the help of the written observations. Based on these, we were able to get information regarding the time and duration of the interventions, the type of feedback given, the students' reactions and immediate progress after the interventions. Therefore, our goals in the study reported in this paper were to identify not only the students' ability to collaborate successfully and articulate the rules underpinning their generalisation of the patterns but particularly when and how the teacher or the researchers intervened.

However, to maintain the essence of exploratory learning, research suggests a teacher's role should be that of a 'technical assistant', a 'collaborator' (Heid et al., 1990), a 'competent guide' (Leron, 1985) or a 'facilitator' (Hoyles & Sutherland, 1989). Our aim was to achieve the right balance between students' autonomy and responsibility over their mathematical work and teachers' and researchers' efforts to scaffold and support their interactions. The teacher and the researchers set out to adopt this role by following a specific intervention philosophy that adhered to our framework of interventions (Mavrikis et al., 2008), which was based on our previous work with Logo and dynamic geometry environments. This framework was extended after the analysis of the data and is presented in the 'Classroom Dynamics' section. Our aim was to avoid imposing our (or the teacher's) views or ways of thinking, but instead allowing students to express their viewpoints and assist them by demonstrating the tools they could use: for example, by directing their attention, organising their working space and monitoring their work.

CLASSROOM SCENARIO

We illustrate here a classroom scenario carried out with a year 7 class with 18 high-attaining students. Students were introduced to the microworld through a familiarisation process, during which the teacher introduced all the key features to construct a simple pattern and students followed his actions on their laptops.

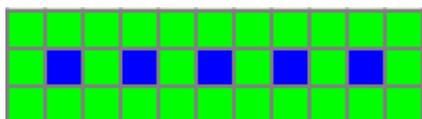


Figure 3. The activity: Find a rule for calculating the number of green (light) tiles for any chosen number of blue (dark) ones.

Students were then presented with the task in Figure 3. The pattern was shown dynamically on the whiteboard; its size changed randomly showing a different instance of the pattern each time. This made it impossible for students to count the number of tiles

while allowing them to 'see' variant and invariant parts of the pattern. We hypothesised that a dynamically presented task would discourage 'pattern-spotting', which focuses on the numeric aspect of specific instances of the pattern, and counting, which encourages constructing specific cases of the pattern. It also provided a rationale for the need of a general rule that provides the number of tiles for *any* instance of the pattern.

Students were given the freedom to construct the pattern in their own way, using the system's features they had been shown earlier. They were asked to write on a hand-out how they constructed the given pattern and then discuss in pairs their constructions and the methods they followed. They also worked collaboratively to find a rule that gives the number of green tiles for any chosen number of blue ones. Students' next challenge was to find different ways to replicate the pattern and describe them on the hand-out explicitly, so as their partner could understand it. After discussing with their partner, if they had come up with the same constructions, they were expected to try to see whether there were any other ways and find all the rules that represented their constructions and write them down. Finally, the teacher initiated a discussion, where students were asked to present their rules to the rest of the class. Rich arguments were developed and students challenged each other to justify the generality of their construction and the rules they have developed.

During this classroom study many interesting issues regarding the classroom dynamics were identified that informed our further design of the microworld and the overall system and the next phase of the research.

CLASSROOM-DYNAMICS

As expected, to ensure the success and effectiveness of students' interactions with the eXpresser, there was a need for significant support from the teacher and the researchers. As discussed already, we had agreed a specific intervention philosophy with the teacher. The analysis of the data (video recordings and written observations) revealed further strategies and extended our previous framework of interventions (Mavrikis et al., 2008). The revised framework is presented in Table 1.

- | |
|---|
| <ul style="list-style-type: none"> • Reminding students of the microworld's affordances • Supporting processes of mathematical exploration <ul style="list-style-type: none"> ➢ <i>Supporting students to work towards explicit goals</i> ➢ <i>Helping students to organise their working environment</i> ➢ <i>Directing students' attention</i> ➢ <i>Provoking cognitive conflict</i> ➢ <i>Providing additional challenges</i> • Supporting collaboration <ul style="list-style-type: none"> ➢ <i>Students as 'teaching assistants'</i> ➢ <i>Group allocations</i> ➢ <i>Encourage productive discussion (group or classroom)</i> • Ensuring task-engagement and promoting motivation |
|---|

Table 1. Types of interventions observed during our studies

Below we pull out some illustrative episodes under each category.

Reminding students of the microworld's affordances

As facilitators the teacher and the researchers (referred to as 'facilitators' for the rest of the paper) managed to support students' interactions and explorations by reminding them of various features of the system that assisted students' immediate

goals. This intervention acted sometimes as a prompt and other times as an offer of assistance. If the facilitator sensed a student was working towards a direction where they could be assisted by a specific tool, they would point it out to their students. This teaching strategy might have proved rather common as for some students the one lesson spent on familiarisation with the system seemed not enough.

Supporting processes of mathematical exploration

We often needed to support the students' problem-solving strategies. For example, we noted that students tended to forget their overall goal. Students seemed to get lost in details and got carried away with various constructions ('drawings'), which, even though offering students more experience of the system's features and affordances, it sometimes led them in the wrong direction. One of the downsides of any microworld is that students' actions can become disconnected from the mathematical aspects under exploration. Even though, the system's affordances were carefully designed to support students' thinking processes, they were not always naturally adopted by them. Therefore, when needed, we provided a reminder of their goals or helped them re-establish them by asking questions like "What are you trying to do?" or "What will you do next?" (*supporting students' work towards explicit goals*).

Another aspect of problem-solving skills (particularly when working in microworlds) that some students seemed to lack was being able to come up with *an organised working environment*. We occasionally advised students to delete shapes that were irrelevant to the solution or change the location of a shape so that they could concentrate on ones that could prove useful. It was evident that students who worked effectively and reached their goals were the ones that organised their working space and therefore supported their perception of the task in hand.

Directing students' attention was a necessary pedagogic strategy. We prompted students to notice invariants or other details which are important for their investigations without giving away the answer. For example, we asked questions such as "Did you notice what happened when you increased the length of this pattern?" or "when you changed this property of your pattern?". These pointed out certain facts that students might have missed out or ignored, but also exposed possible misconceptions and misinterpretations. If students were focusing on or manipulating unnecessary elements of their construction, the facilitators provided hints towards more constructive aspects. If students' responses revealed any misconceptions, then such a prompt acted as an intervention for *provoking cognitive conflict*. There were cases where the cognitive conflict was not obvious to the students directly and further explanations were required from the facilitators. These normally involved giving counter-examples to provoke students' understanding and challenge their thinking processes. Besides this intervention we used another strategy, referred to as "messing-up", used in our previous work in dynamic geometry (Healy et al., 1994). This strategy challenged students to construct a pattern that is impervious to changes of values to the various parameters of the tasks. Students tended to construct patterns

with specific values and had their constructions ‘messed-up’ when the facilitators suggested: “What happens when you change this to say 7 (a different value to the student’s chosen one)?”. This strategy gave a rationale for students to make their constructions general by encouraging them to think beyond the specific case. In other cases where students seemed to have reached a satisfactory general construction, the facilitators intervened by *providing additional challenges*. For example, “Could you find another way of constructing the pattern?”.

Supporting collaboration

Students who achieved a seemingly general construction and found a rule (general or not, representing their construction or not), often failed to find different ways of constructing the pattern. Our approach in these circumstances was to introduce them to the collaborative aspect of the activity, in which they had to discuss, justify and defend the generality of their constructions and their rules to their partners. We envisaged that learners’ general ways of thinking would be enhanced by the sharing of their different perspectives. Accompanied by the facilitators’ or fellow students’ assistance, students could appreciate the equivalence of their approaches and possibly adopt a more flexible way of thinking. In this study, the rationale behind collaboration was to give students an incentive to enrich their perception and understanding of the given pattern, to find more ways of constructing it and begin to appreciate their equivalence mathematically. The allocation of students to groups aimed at ensuring the best possible collaboration (*group allocations*). Ensuring though that discussions carried out within the groups were fruitful was not an easy task. The first step towards this goal was grouping the students in a way that promoted participation from all members of the group while discouraging students from dominating a discussion (*encourage productive discussion*).

On some occasions, the facilitators, particularly the teacher who has better insights into his students’ competence, encouraged students to take the role of a ‘*teaching assistant*’ and help others who were less successful in their constructions. This intervention boosted students’ confidence, but also gave them an opportunity to reflect upon their actions and an incentive to explain their perspective.

Ensuring engagement and promoting motivation

Finally, although the activities and the system affordances were designed to assure engagement as well as promote students’ motivation, there were various occasions (e.g. being stuck or ‘playing’ by drawing random shapes) when the facilitators’ intervention was required. Our vision was to give the right rationale for students to solve the task and praise their efforts. These studies supported our view that avoiding tedious activities that were pointless in the students’ eyes, not only reduces the risk of off-task behaviour, but also sustains a productive atmosphere for students.

TOWARDS AN INTELLIGENT SYSTEM IN THE CLASSROOM

The interventions that were discussed above require an intensive one-to-one interaction with the students who require help. However, it is unrealistic to expect teachers in classrooms to be able to adhere to the demanding role of facilitators, keeping track of all students' actions while allowing them to explore and have the freedom to choose their immediate goals. As mentioned above, there are multiple ways of constructing a pattern and therefore multiple ways of expressing general solutions for such activities. It is at this point that the value of a system that can provide information to the teacher becomes apparent.

One of the most practical issues regarding students' interactions in such environments is that despite the familiarisation process, there is a need to remind students of certain features or even prompt them to use those which could prove useful for their chosen strategy. Therefore, it should be possible to identify (based on students' actions) which tasks of the familiarisation activity they should repeat. An intelligent system could highlight tools relevant to their current actions or offer a quick demonstration directly taken from their familiarisation activity. Furthermore, it could repeat their previous successful interactions relevant to the current activity.

In terms of the teachers' responsibility to attend to and help all the students in a classroom our studies highlighted the difficulty to prioritise which student to help. It is inevitable, therefore, sometimes to offer support to students who do not need it as much as others or even leave some students unattended due to the time constraints of a lesson. Moreover, it is possible for students to misunderstand certain concepts and leave a lesson with a false sense of achievement. Of course, it is difficult for an intelligent system to detect this accurately. However, it is possible to draw the teacher's attention to students potentially in need. By providing therefore information regarding students' progress at various times during a lesson as well as alerting them of likely misconceptions, it becomes possible for the teacher to spend their time and effort efficiently.

Besides these teachers' difficulties, there are situations when, despite having carefully-planned lessons, teachers are required to take immediate and effective decisions during lessons to accommodate their students' needs. For example, noticing when students are having difficulty with certain tasks or providing extension work are interventions which could be delegated to our system, allowing more time for teachers to provide essential help. Moreover, the collaborative component of an activity could be supported by the system by recommending effective groupings of students and allowing them to co-construct patterns whilst reducing dominance and promoting successful collaboration. The system could inform the teacher about the dynamics of different groups and alert them of possible concerns regarding the groups' progress as well as suggest more productive groupings (e.g. group students with different constructions but equivalent general expressions).

In addition, although we acknowledge the strong dependency between motivation, engagement and the design of the activities, it was evident that some students were at

points disengaged. Even if off-task behaviour can sometimes lead to fruitful outcomes and intrigue students' thinking processes towards a direction, there is a need in automatically detecting such behaviour and informing the teacher. It then becomes the teacher's responsibility to decide how and whether to intervene.

The aforementioned suggestions for intelligent support could ease the use of an exploratory environment like the eXpresser in the classroom. It is often the case that such systems end up being used as a tool just to demonstrate certain mathematical concepts because of similar difficulties faced in classroom as those we reported here. Moreover, although quite a few 'intelligent' tutoring systems have been designed to provide support and personalised feedback to students and are starting to be integrated in classroom (Forbus et al., 2001), they usually scaffold the students with predetermined solution methods and by definition restrict students' reaching their own generalisations. Our team's challenge is to build a system that provides students the freedom to explore, make mistakes, get immediate feedback on their actions while assisting teachers in their difficult role in the classroom and therefore enable the successful teaching and learning of the idea of mathematical generalisation.

NOTES

1. See <http://www.migen.org/> for details. Funded by the TLRP, e-Learning Phase-II; Award no: RES-139-25-0381.
2. Our system comprises of two additional components, the eGeneraliser, which aims to provide students with personalised feedback and support during their interactions with the microworld, and the eCollaborator, which aims to foster an online learning community that supports teachers in offering their students constructions and analyses to view, compare, critique and build on.
3. We would like to acknowledge the rest of our research team and particularly Sergio Gutierrez, Ken Kahn and Darren Pearce who are working on the development of the MiGen system.
4. Each attribute has an associated icon tentatively depicted as cogs "to indicate the inner machinery of a pattern". As the design of eXpresser is evolving our team is evaluating the appropriateness of these icons.

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ESTABLISHING A LONGITUDINAL EFFICACY STUDY USING SIMCALC MATHWORLDS®

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We describe the construct of a 4-year longitudinal efficacy study implementing dynamic mathematics software and wireless networks in Algebra 1 and 2 classrooms. We focus on student learning and motivation over time, and issues of effective implementation in establishing a longitudinal study.

INTRODUCTION: BACKGROUND TO DYNAMIC MATHEMATICS

New forms of mathematics technology (e.g., dynamic geometry) can provide *executable representations*—representations that transform the mathematics made by students into a more tangible and exciting phenomenon (Moreno-Armella, Hegedus & Kaput, 2008). In particular, we have designed and used SimCalc MathWorlds® to transform students' mathematical constructs into fascinating motion phenomena. Second, networks can intimately and rapidly *link private cognitive efforts to public social displays*. Consequently, students can each be assigned a specific mathematical goal (e.g., playing the part of a single moving character by making a graph with certain mathematical characteristics), which instantly links to public social display (e.g., the parade constituted by all characters moving simultaneously). This approach shifts the types of critical thinking that are possible in mathematics classrooms and transforms the role of instructional technology by integrating it into the social and cognitive dimensions of the classroom.

Our connected approach to classroom learning highlights the potential of classroom response systems to achieve a transformation of the classroom-learning environment. Similarly other investigators have expanded their approaches to include devices that allow aggregation of mathematical objects submitted by students. (Stroup, Ares & Humford, 2005).

SITUATED NEED

Our proposed work addresses three essential needs: (i) the Algebra Problem (RAND, 2002), (ii) the related problem of student motivation and alienation in the nation's schools, especially urban secondary schools (National Research Council, 2003), and (iii) the widely acknowledged unfulfilled promise of technology in education, especially mathematics education (e.g., Cuban, 2001).

An important analysis by the National Academies Institute of Medicine (National Research Council, 2003) of student motivation at the high school level reveals in painful detail what most high school teachers (and parents) know only too well: that

student motivation in high schools, and even more acutely in urban high schools, is an urgent and complex national problem. The report also recommends that high school courses and instructional methods need to be redesigned in ways that will increase adolescent engagement and learning.

Ethnographical studies of high school students (Davidson & Phelan, 1999; Phelan, Davidson, & Yu, 1998) reveal a world of alienation with strongly negative responses to standard practices (Meece, 1991) and strong sensitivity to interactions with teachers and their strategies (Davidson, 1999; Johnson, Crosnoe & Elder, 2001; Skinner & Belmont, 1993; Turner, Thorpe, & Meyer, 1998). Negative responses, particularly as they are intimately connected with self image and sense of personal efficacy, can be deeply debilitating, both in terms of performance variables (Abu-Hilal, 2000) as well as in the ability to use help when it is available (Harter, 1992; Newman & Goldin, 1990; Ryan & Pintrich, 1997). See the comprehensive reviews by Brophy (1998), Newmann (1992), Pintrich & Schunk (1996), and Stipek (2002). On the other hand, students exhibit consistently positive responses to alternative modes of instruction and content (Ames, 1992; Boaler, 2002; Mitchell, 1993), particularly those that build upon intrinsic instead of external motivation (Linnenbrink & Pintrich, 2000).

The literature on motivation in education and social situations in general has focused on intrinsic and extrinsic motivation with a great deal of debate (Sansone & Harackiewicz, 2000). Intrinsic motivation reflects the propensity for humans to engage in activities that interest them. Extrinsic motivation, such as rewards, can have an undermining effect and decrease intrinsic motivation, i.e., the reason why the person chose to want to do the activity in the first place (Deci, 1971). Yet both intrinsic and extrinsic motivation, as a key feature of participation in mathematics classrooms, have appeared to be an orthogonal field of inquiry to the development and instruction of content, with motivation hesitantly intersecting with education in the form of “motivational strategies,” incentivizing students to learn mathematics because it is “fun” or “applicable” to *their* life, through relevant contexts, e.g., sports or vocations.

Relevance, unfortunately, is a somewhat indirect means to link motivation and mathematics—the link between immediate cognitive effort and later applications that may seem improbable to students. There is a more direct alternative. Students can become motivated because they want to participate more fully in what their classroom is doing now. The alternative, thus, is to link motivation and mathematics through *participation*.

We advocate two radically new forms of participatory activity in technology-enhanced environments:

1. **Mathematical Performances.** These activities emphasize individual student creations, small group constructions, or constructions that involve coordinated

interactions across groups that are then uploaded and displayed, with some narration by the originator(s).

2. Participatory Aggregation to a Common Public Display. These activities involve systematic variation, either within small groups, across groups, or both, where students produce functions that are uploaded and then systematically displayed and discussed to reveal patterns, elicit generalizations, expose or contextualize special cases, and help raise student attention from individual objects to families of objects.

These activities aim at enhancing mathematical literacy, debate and coherent argumentation—all fundamental mathematical skills. The central point is that each requires and rewards students for cognitive engagement in producing tangible phenomena that are simultaneously phenomenologically exciting and mathematically enlightening. This happens not at some future time when mathematics can be applied to a career or personal goal; instead these activities draw students in and sustain their interest because they are exciting and enlightening in the moment, in the classroom. These activities create an intrinsic motivation context with a socio-cultural view to “motivation in context” (Hickey, 2003) that is also intrinsically mathematical, accomplishing a much more intimate intertwining of motivation and mathematics that can be typically accomplished in existing classrooms.

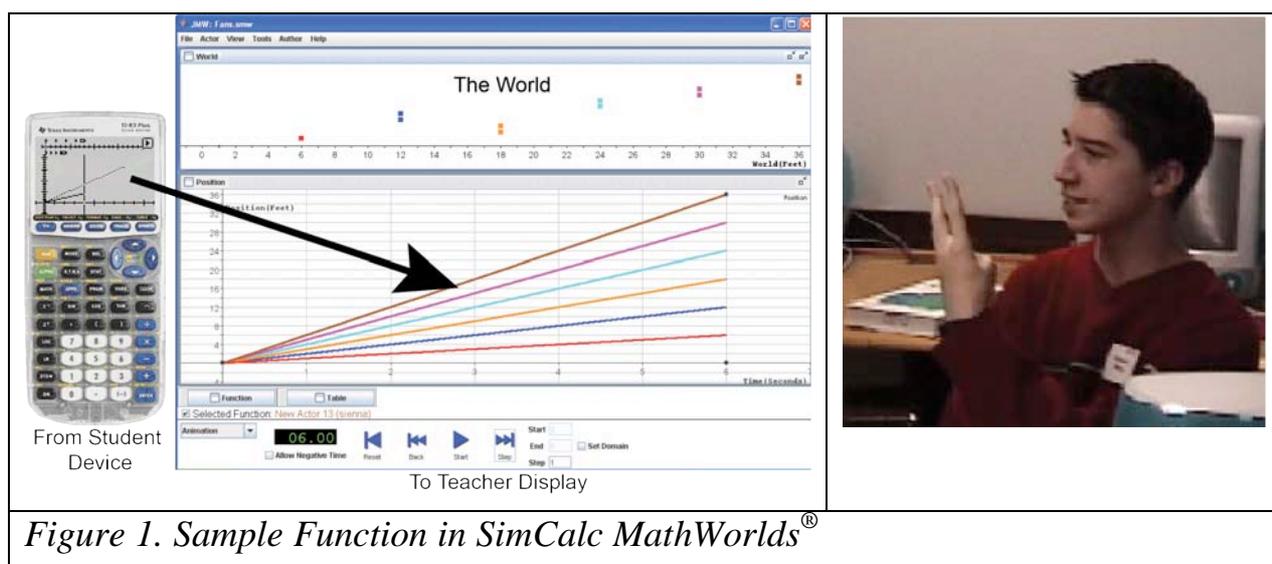
PRIOR WORK

SimCalc MathWorlds[®] creates an environment where students can be part of a *family of functions*, and their work contributes to the mathematical variation across this mathematical object. Consider this simple activity, which exemplifies a wider set of activity structures. Students are in numbered groups. Students must create a motion (algebraically or graphically) that goes at a speed equal to their group number for 6 seconds. So, Group 1 creates the same function, $Y=(1)X$, Group 2, $Y=(2)X$, etc. When the functions are aggregated across the network via our software, students’ work becomes contextualized into a family of functions described algebraically by $Y=MX$ (see Figure 1 below). Students are creating a variation of slope and in doing so this can help each student focus on their own personal contribution within a set of functions.

At the heart of SimCalc is a pedagogical tool to manage classroom flow. This tool allows teachers to control who is connected to the teacher computer using a simple user interface, and choose when to “freeze” the network and aggregate students’ work or allow students to send a number of tries via the TI-Navigator[™]. In addition, teachers have control over which set of contributions (e.g., Group 1’s functions) and which representational perspectives (e.g., tables, graphs, motions) to show or hide. Thus, the management tool encapsulates a significant set of pedagogical strategies

supported by question types in existing curriculum materials to satisfy a variety of pedagogical needs, focus students' attention depending on their progress, and promote discussion, reasoning and generalization in a progressive way at the public level.

In our prior research, students build meaning about the overall shape of the graphs and have demonstrated gestures and metaphorical responses in front of the class when working on this activity. For example, in two entirely different schools, students have raised their hand with fingers stretched out (see Figure 1 below), and said it would look like a “fan.” In this socially-rich context, students appear to develop meaning through verbal and physical expressions, which we observe as a highly powerful way of students engaging and developing mathematical understanding at a whole group level. Various forms of formative assessment can be said to be evident as each student's work emerges in a public display, and representations can be “executed” (Moreno-Armella & Block, 2002) to test, confirm or refute ideas. These forms of reflection, enabled through particular question-types and classroom dialogue focused on the dynamic representations, can be attributed to students learning and resonate with established research on formative assessment (Black & William, 1998; Boston, 2002).



Over the past ten years, over the course of three consecutive research and development projects (NSF ROLE: REC-0087771; REC-0337710; REC-9619102) and related projects at TERC (NSF REC-9353507), the SimCalc project has examined the integration of the Mathematics of Change and Variation (MCV) as a core approach to algebra-intensive learning. This work has led to a Goal 3 IERI-funded study (NSF REC-0437861), led by SRI International, focusing directly on large-scale implementability and teacher professional development in TX, and a recently funded IES Goal 2 project in the high school grades (IES Goal 2 # R305B070430) focusing on longitudinal impact of our curriculum and software products distributed by Texas Instruments on their popular graphing calculators in

combination with a commercially available wireless network (TI-Navigator™ Learning system).

The Scale-Up pilot work employed a set of SimCalc resources in a delayed-treatment design. Teachers were initially randomly assigned to one of two groups. An ANOVA of difference scores (again teacher nested within condition) was significant [$F(1,282)=178.0$, $p<0.0001$]. The effect size for the gain in the group that used SimCalc is 1.08. In our main study, which is a randomized controlled trial in which 95 7th-grade mathematics teachers were randomly assigned to implement a 3-week SimCalc curriculum unit following training, our analyses show an effect size of 0.84 (Roschelle, Tatar, Shectman et al., 2007).

Prior work has documented statistically significant evidence for impact of SimCalc materials in connected “networked” environments with computers and calculators (Hegedus & Kaput, 2004) under multiple quasi-experimental interventions across grades 8-10 and college students demonstrating statistically significant increases ($p<0.001$) in student mean scores (effect=1.6) but with an even higher effect on the at-risk 9th grade population (effect=1.9). A major finding of our work was that critically important skills such as graphical interpretation were improved, i.e., cognitive transfer was evident. Recent studies show similar statistically significant results in terms of student learning and shifting attitudes towards learning mathematics in connected environments (Hegedus, Kaput, Dalton et al., 2007). We have also analyzed the changing participation structures using frameworks from linguistic anthropology (Duranti, 1997; Goffman, 1981). Our work has described new categories of participation in terms of gesture and language (Hegedus, Dalton, Cambridge et al., 2006) new forms of identity (Hegedus & Penuel, 2008), and theoretical advances in dynamic media and wireless networks (Hegedus & Moreno-Armella, 2008; Moreno-Armella et al., 2008).

DESIGN ASPECTS OF EFFICACY WORK

In this context, our research program (funded by the US Department of Education, IES Goal 2 # R305B070430) builds on prior work to examine this problem. It is focused on outcomes in terms of both grade-level learning gains and longitudinal measures that relate to students’ progress and motivation in mathematics across the grades in Algebra 1 and 2 classrooms.

SimCalc combines two innovative technological ingredients to address core mathematical ideas: Software that addresses content issues through dynamic representations and, wireless networks that enhance student participation in the classroom. We have begun to develop materials that fuse these two important ingredients in mathematically meaningful ways and developed new curriculum materials to replace core mathematical units in Algebra 1 (8-12 weeks) and Algebra 2 (4-8 weeks) at high school. We are measuring the impact of implementing these

materials on student learning (high-stakes State examinations in Massachusetts (MA), USA) and investigating whether one or multiple involvements in this type of learning environment over the course of their high school years affects their motivation to continue studying mathematics effectively and enter STEM-career trajectories.

Our work is conducted in eight school districts in MA offering a wide variety of settings in terms of performance on State exams and Socio-Economic Status (SES). Our treatment interventions are in 9th and 11th grade classrooms (Algebra 1 then 2) but we will also track some students when they are in 10th and 12th grade collecting simple questionnaire data. Our study is a small-scale cluster randomized experiment where we cluster at the classroom level, randomly assigning two classrooms in each school to treatment in our main studies (total of 28 classrooms and a. 500 students in each main study).

We are using two instruments comprised of standardized test items to measure student's mathematical ability and problem-solving skills before and after each intervention. We are also collecting survey data on student's attitude before during and after the intervention. We are administering these tests and surveys at similar times (with respect to curriculum topics covered) in treatment and control classrooms. Video data from periodic classroom visits are being analyzed using participation frameworks from prior work and triangulated with variations in student survey data on attitude.

We are using suitable statistical methods to assess gain relative to the control groups, and between-cluster variation using mixed-Hierarchical Linear Modeling. We are also collecting survey and classroom observation data to assess changes in attitudes and participation, and daily logs by teachers to monitor fidelity of implementation.

We have completed our first year of 4 years work with our first cohort of students that we will track for the duration of their high school career and will present initial findings from our pilot study and challenges we have addressed in sampling and establishing a longitudinal program of research. We focus on results from factor analyses of our survey instruments on student and teacher attitude and correlations with student learning. Following a minimal effect size in our pilot study, we aim to present findings for improving effective implementation from analyses of teacher daily logs and classroom video.

Such methodologies build a comprehensive program for evaluating how prior findings (briefly highlighted above) can scale to larger implementations whilst being cognizant of issues of fidelity. Our ongoing work and preliminary analyses report of the potential effect on outcome measures such as student learning and motivation.

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INTEROPERABLE INTERACTIVE GEOMETRY FOR EUROPE – FIRST TECHNOLOGICAL AND EDUCATIONAL RESULTS AND FUTURE CHALLENGES OF THE INTERGEO PROJECT

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In this overview article we describe the manifold achievements and challenges of Intergeo¹, a project co-funded within the eContentplus programme² of the European Union.

THE INTERGEO PROJECT

The Intergeo project started in October 2007 and will be funded until September 2010. Its main concern is the propagation of Interactive or Dynamic Geometry Software.

Goals

Interactive Geometry is a way to improve mathematics education by using computers and Dynamic Geometry Software (DGS) and there are many advantages in comparison to “classical” geometry without DGS. Figures can e.g. be easily manipulated [see e.g. Roth 2008] and thus virtually be brought to life, comparable to what movies mean to images or to what interactive computer games mean to motion pictures.

It is therefore not amazing that Interactive Geometry obtains more and more attention in many educational institutions. Around 25 per cent of the countries within the EU refer explicitly to DGS in their national curricula or guidelines and roughly 40 per cent refer to ICT in general. And although the remaining countries do not mention ICT, some of them recommend the use of DGS in schools [Hendriks et al. 2008].

Still, the adoption of DGS at school is often difficult. Despite the fact that a lot of DGS class material exists, Interactive Geometry is still not used in classrooms regularly. Many teachers do not seem to know about the new possibilities, or they do not have access to the software and/or resources.

The Intergeo Project has identified the three following major barriers, that have a negative impact on the use of Interactive Geometry in classrooms [Intergeo Project 2007]:

¹ <http://inter2geo.eu>

² http://ec.europa.eu/information_society/activities/econtentplus/index_en.htm

- **Missing search facilities**
Though many resources exist, there remains the problem of finding and accessing them. If the files were put on the internet by their developers, they are virtually scattered all over the web and it is extremely hard to retrieve them by using search engines like Google.
- **Lack of interoperability**
There are many different programmes for Interactive Geometry on the market and each software has its own proprietary file format. Thus, finding a file does not automatically mean that it can be used – it must be a file for the specific software that is used.
- **Missing quality information**
And even if a teacher finds a file and the file works with her DGS, it may still be unsuitable for the use in class due to a lack of quality. Lacking quality can be software-sided in the way the figures are constructed or missing (or even wrong) mathematical background.

The aims of Intergeo are to dispose of the problems stated. In other words, Intergeo will

- enable users to easily find the resources they are looking for,
- provide the materials in a format that can be used with different DGS systems, and
- ensure classroom quality.

All three facets will be dealt with in the following chapters in extenso.

Furthermore, Intergeo attends to a topic that is mostly neglected but of high importance nonetheless: the question of copyright.

Consortium

The Intergeo Consortium, the founding partners of the Project, assembles software producers, mathematicians, and mathematics educators: Pädagogische Hochschule Schwäbisch Gmünd (D), Université Montpellier II (F), Deutsches Forschungszentrum für künstliche Intelligenz DFKI (D), Cabrilog S.A.S. (F), Universität Bayreuth (D), Université du Luxembourg (LUX), Universidad de Cantabria (ES), TU Eindhoven (NL), Maths for More (ES), and Jihočeská Univerzita v Českých Budějovicích (CZ). As the common interest of all partners is the propagation of sensible use of Interactive Geometry in the classroom, it was possible to collect both commercial, semi-commercial and free software packages. This is one of the key ingredients of the project: By building upon the joint knowledge and expertise of all parties, we hope to be able to address the needs of the teaching community.

Participation of External Partners

The participation of External Partners, as Associate Partners, Country Representatives, and User Representatives justifies the basis for assuring the sustainability of the projects' goals as mentioned above. Furthermore, gathering partners, as software developers, teachers, and persons at school administration level enables the development of a Europe-wide network that is indispensable for obtaining the projects' major achievements.

Since the project start in October 2007, several key actors in interactive geometry throughout Europe, including software producers, mathematics educators, governmental bodies, and innovative users that can provide additional content or serve as test users for the first content iterations were acquired.

Associate Partners

The role of Associate Partners implicates a variety of tasks and expectations, as the adoption of the common file format for their software, the provision of significant content to the Project, the development of ontologies, and the conduction of classroom tests. The project could successfully find several important Associate Partners, see [Intergeo Project 2008] and the following table.

Table: List of Associate Partners

Nr.	Country	Name	Nr.	Country	Name
1	Austria / USA	Markus Hohenwarter (GeoGebra)	15	Germany	Andreas Göbel (Archimedes Geo3D)
2	Brazil	Leônidas de Oliveira Brandão (iGeom)	16	Germany	Reinhard Oldenburg
3	Canada / Spain	Philippe R. Richard, Josep Maria Fortuny (geogebraTUTOR)	17	Germany	Andreas Meier
4	Canada	Jérémie Farret (3D Geom)	18	Germany	Roland Mechling (DynaGeo)
5	Croatia	Sime Suljic (Normala)	19	Italy	Giovanni Artico (CRDM)
6	France	Cyrille Desmoulins	20	Luxembourg	Daniel Weiler
7	France	Odile Bénassy (OFSET)	21	México	Julio Prado Saavedra (GeoDin)
8	France	François Pirsch (JMath3D)	22	Portugal	Arsélio Martins
9	France	The Sesamath association	23	Portugal	José Francisco Rodrigues (CMAF)
10	France	EducTice - INRP / Luc Trouche	24	Slovakia	Dusan Vallo
11	France	IUFM - Jacques Gressier (Geometrix)	25	United Kingdom	Albert Baeumel
12	Germany	Jürgen Roth (Universität Würzburg)	26	United Kingdom	Nicolas van Labeke (Calques 3D)
13	Germany	Heinz Schumann	27	United States	Joshua Marks (Curriki)
14	Germany	René Grothmann (C.a.R. / Z.u.L.)			

Country Representatives

For each EU country a Country Representative serves as a contact person in their respective country. They come from ministries of education, preferably, and enable the Project to easily contact the relevant persons at school administration level. Based on these contacts, the project develops ways to map curricula into the ontology for geometry that suits all countries of the EU. The project could successfully find several Country Representatives, and a list is available at [Intergeo Project 2008].

User Representatives

User Representatives, as teachers and software partners, build the basis for the sustainability of the project. They are a contact point with their associations, in order to support the relationship with potential Intergeo-users [Intergeo Project 2008].

- Selected teachers ease experimentations in the classroom of educational content gathered by the project, promote the use of the Intergeo-platform and the philosophy of resource sharing and quality control.
- Selected Software-partners promote the uploading of content to the Intergeo-platform.

Among others, the selection of external partners will be performed at several local user meetings during the project period. The local user meetings have a central role in gathering the community of practice. They intend to help providing a complete European coverage:

- The Local User Meetings present Intergeo to the users: The need of a common file format for interoperability, the need of a web platform to share resources, the need of the ontology and the curriculum mapping to share resources across all European countries.
- The Local User Meetings are a good way to reach power users and engage them into the project to improve the projects' dissemination.
- Local User Meetings identify suitable schools for the Quality Assessment.

MAJOR ACHIEVEMENTS

Content Collection

The consortium promised to offer a significant amount of content for use in the database. Before the project started in Oct. 2007 we identified more than 3000 interactive resources to be used. All these and more³ have been collected through the Intergeo platform by September 2008, first as *traces*, and now being converted to real assets that are searchable and tagged with meta-data. The available content ranges through all ages and educational levels, and also mathematical topics and competences. See <http://i2geo.net> to access and use the content.

Copyright/Licence issues

A major issue with content re-use and exchange is the handling of intellectual property rights. This affects not only the copying of resources, but also the modification and the classroom use. Without being able to process the data, it is also impossible to offer the added value of cross-curriculum search, for example.

Thus, all content that is added to the Intergeo portal has a clear license, usually of the creative commons type allowing for modification and free (non-commercial) use. See <http://creativecommons.org> for details.

³ On September 30th, 2008, there was a total amount of 3525 traces available.

Theoretical Foundation For Cross-Curriculum Categorization and Search

Interactive geometry has one quality that makes it very particular among learning resources: it is often multilingual. This led us naturally to propose a search tool for interactive geometry resources that is not just a textual search engine but a *cross curriculum* search engine.

A simple scenario can explain the objective of cross-curriculum search: a teacher in Spain contributes a Cabri construction which is about the intercepting lines theorem (the *Teorema de Tales*) and measuring segment lengths; a teacher in Scotland looks for a construction which speaks about the *enlargement* transformation, *segment lengths*, and the competency to *recognize proportionalities*. They should match: the Scottish teacher should find the Cabri construction of the Spanish teacher (and be able to convert it to his preferred geometry system). No current retrieval system can afford such a matching process: there is no common word between the annotation and the query.

For cross-curriculum matching to work, a language of annotations is needed that encompasses the concepts of all curriculum standards and that relates them. Careful observation of the current curriculum standards (see [Laborde et al. 2008]) has shown that topics, expressed as a hierarchy, and competencies are the two main type of ingredients that are needed. To this end the Intergeo project has built an ontology of topics, competencies, and educational levels called GeoSkills. This OWL ontology [McGuinness et al. 2004] has been structured and is now being populated by a systematic walk through the national curriculum standards; a report of this encoding is at [Laborde et al. 2008]; completeness for several school-years has been reached in French, English, and Spanish curriculum standards. Because the edition of an ontology using a generic tool can be difficult, a dedicated web-based tool is under work which will make it possible for the complete German, Spanish, Czech, and Dutch curriculum standards to be encoded by the Intergeo partners and its associates.

For the match to happen, the input of topics or competencies has to be cared for. We use the auto-completion paradigm for this purpose: the (textual) names of each topic and competency are searched for in this process and the user can thus choose the appropriate node with sufficient evidence, maybe browsing a presentation of the topics and competencies. An alternative approach proposed is to browse curriculum standards, being

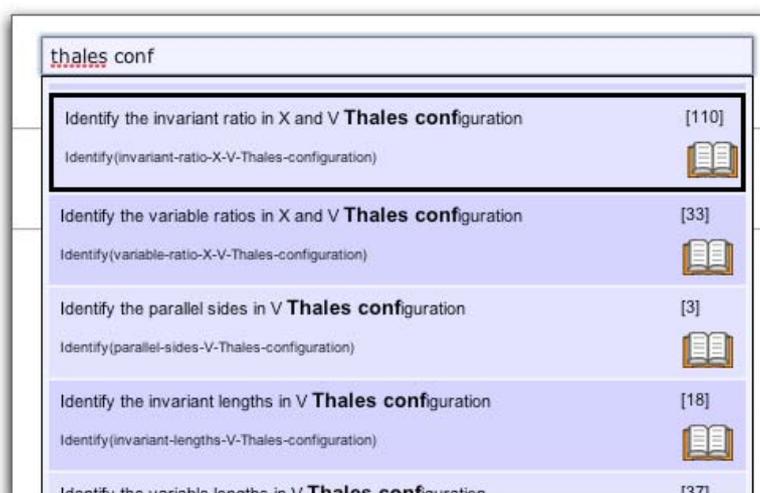


Figure 1: The skills textbox

documents that teachers potentially know well, in order to click a paragraph to choose the underlying topics and competencies.

Quality Assessment Framework

A Quality Assessment Framework for the Intergeo project was set up based on a questionnaire filled freely by the teachers themselves [Mercat et al. 2008]. This assessment has two different aims:

- To rank the resources so that, in response to a query, "good" resources are ranked before "bad" resources, at equal relevance with respect to the query.
- To help improve resources by identifying criteria to work upon in order for the author to revise his resource according to the user's input.

The questionnaire is both easy and deep; it can provide a light 2 minutes assessment as well as a deep pedagogical insight of the content. This is achieved by a top-down approach: The quick way just asks for 8 broad statements that can be answered on a scale from "I agree" to "I disagree":

- I found easily the resource, the audience, competencies and themes are adequate
- The figure is technically sound and easy to use
- The content is mathematically sound and usable in the classroom
- Interactivity is coherent and valid
- Interactive geometry adds value to the learning experience
- This activity helps me teach mathematics
- I know how to implement this activity
- I found easily a way to use this activity in my curriculum progression

These broad questions can be opened up by the reviewer to give more detailed feedback on issues of interest for him, such as "Dragging around, you can illustrate, identify or conjecture invariant properties" in the "Interactive geometry adds value to the learning experience" section.

Of course a thorough questionnaire is weighted more than a quick reply in the averaging of the different answers. The questionnaire is to be taken twice, as an a priori evaluation, before the actual course, and as an a posteriori evaluation, after the teaching has taken place. This second variant is being more weighted than the first one.

Different users are weighted differently as well: seasoned teachers with a lot of good activity, or recognised pedagogical experts, will have a high weight: their reviews are taken into account more than the average new user. Negative behaviour like steady bashing or eulogy will, on the contrary, lower user's weight. We are thinking as well

about a social weight: teachers could flag some of their colleagues as "leaders", users whose past choices they liked, because they are teaching at the same level for example, and the weight of these leaders would increase.

The I2Geo Platform

The central place of exchange of interactive geometry constructions is a web-platform; the i2geo.net platform is becoming a server where anyone with interest to interactive geometry can come to search for it and to share it.

The i2geo.net platform is based on Curriki, an XWiki-extension tuned for the purpose of sharing learning resources: strong metadata scheme, quality monitoring system and self-regulated groups. Being based on a wiki platform, Curriki offers an online editing and inclusion facility and thus also makes collaborative content construction possible.

The i2geo platform has three major adaptations compared to the tools provided by Curriki: the search and annotation tools, the review system, and the support for interactive geometry media.

The i2geo search and annotation tool uses the GeoSkills ontology described above: this allows the trained topics and competencies, the required ones, and the educational levels to be all entered using the input methods described above (auto-completion and pick-from-document).

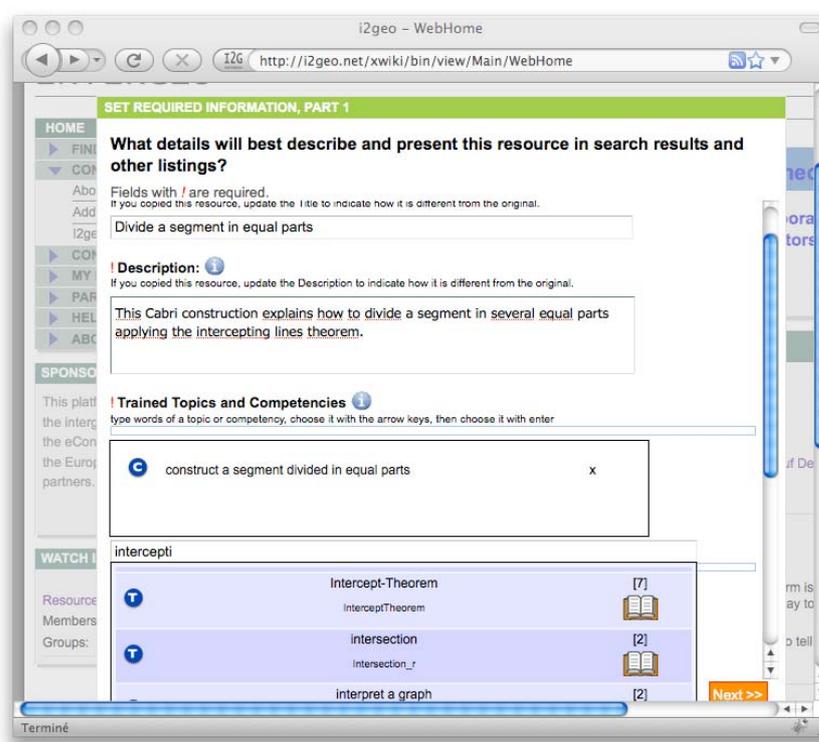


Figure 2: Editing metadata

Such elaborate methods are needed if one wants to honour the rich set of educational levels in Europe and the diversity of curriculum standards sketched in [Laborde et al. 2008].

The i2geo search tool uses the GeoSkills ontology as well: queries for any concept are generalized to neighbouring concepts which thus allows the match of the intercepting-lines-theorem when queried for the concept of enlargement.

The i2geo platform is under active development and can be experimented with on <http://i2geo.net>. Its current development focus is the input of metadata annotated resources and the review system described in the previous sections. The services

specialized to the geometry resources, enabling easy upload, preview, and embedding of interactive geometry resources will be provided later.

A Common File Format

A wide variety of Dynamic Geometric Systems (DGS) exist nowadays. Before this project, each system used incompatible proprietary file formats to store its data. Thus, most of the DGS makers have joined the project to provide a common file format that will be adopted either in the core of the systems or just as a way to interchange content.

The Intergeo file format aims to be the convergence of the common features of the current DGS together with the vision of future developments and the opinion of external experts. Its final version based on modern technologies and planned to be extensible – to capture the flavour of the different DGS – could serve as a standard in the DGS industry.

The specification of the first version of the Intergeo file format has been released by the end of July as deliverable D3.3 [Hendricks et al. 2008] after intensive collaboration between DGS software developers and experts. At present, the file format is restricted to the geometry in the plane, although it does not seem difficult to extend it, in the future, to the space. Besides it specifies only a restricted subset of possible geometric elements, which however lead to an agreement on the structure and basic composition of the format.

The general framework was clear from the outset: to design a semantically rich format that could be interpreted by at least all DGS in the consortium. One main design decision in this respect consists of the choice of constructions, as opposed to constraints, because in general, it is very difficult to give any particular solution for a set of constraints. Besides constraints of a strictly classical geometric nature do not say anything about the dynamic behaviour of a figure. A natural way to shed light on both of these problems is a more precise specification of how the objects depend on each other, stipulating first which objects are free and then proceeding step by step. Such a specification is called a construction. This decision implies less interoperability with constraint-based systems, since some of their resources will not be encodable into this format. But it ensures that construction-based DGS – the majority of the existing systems – will be able to interpret the resources.

As stated in the Description of Work, OpenMath Content Dictionaries are used to specify the symbols – the main ingredients used to describe a construction – of the file format. The XML schema can be generated automatically with some knowledge of how the atoms are expressed in XML. The complete list of official symbols defined so far can be found at <http://svn.activemath.org/intergeo/Drafts/Format/>.

As soon as version 1 of the file format got more concrete, some software developers started to investigate its practical usage by integrating it (partially) into their software. It was possible to move simple content between several of the packages in

the project. For more information on the file format we refer to [Hendriks et. al 2008], which also lists the relevant URLS to see the progress.

NEXT STEPS AND CHALLENGES

Metadata Collection

With the arrival of the first curriculum-aware beta version of the i2geo.net platform we are now able to attach metadata to the existing content. This includes information about the authors, but also about the intended audience for a resource, the skills and competences that can be acquired through the resource, the prerequisites, and, of course, the topic – categorized according to the ontology.

While some of this information can be extracted automatically, there is still need for a lot of manual intervention. At the same time, the curricula available on the platform have to be revised and extended to accommodate all the content.

Quality Testing

The partners in the Quality Assurance work package will conduct small-scale experimentations in the classroom during the period January-April 2009. Teachers, whether alone or in homogeneous teams, will

- Use the platform in order to identify content suitable for their course,
- First fill an a priori questionnaire,
- Teach the resource in the classroom,
- And finally report on its use by updating the a posteriori questionnaire.

We will have to agree on a *modus operandi*, recruit volunteers, especially among the teachers that were contacted during the users meetings, instruct them and have them conduct the experimentations.

Then these assessments will be analyzed. The analysis will be used to iteratively improve the quality assessment framework according to the users' feedback on usability and relevance of the different items and of the online platform.

It is a primary concern that all resources receive at least basic testing. Thus, we will check the overall coverage in the project and, if necessary, identify resources to be tested.

As the quality assessment primarily aims to make it possible to improve ranking and quality of the resources, we can use this as a performance indicator. For this, the changes in ranking due to the quality evaluation will be measured. Additionally, selected examples will be analysed in order to understand whether authors can infer improvements of their resources.

Via interviews with selected authors we try to understand how they perceived quality assessment and how we can improve its perception as positive, constructive and scientific more than negative, useless and personal.

In the final year of the project, mass scale experimentations will take place. More countries and more parts of the curriculum shall be covered.

File format

As for version 1 of the file format some decisions that should be made with the help of other developers of DGS have been postponed, those experts are invited to join the discussion and propose solutions or give remarks, see [Hendriks 2008]. Thus, substantial modifications of this specification are expected to solve all practical issues that might arise.

Better Visibility

The ultimate goal and a measure of success is the visibility of the Intergeo platform in Europe as a whole. After the first year was devoted to setting up the technical prerequisites and administrative processes, as well as clearly describing how we can measure and improve the standards for successful interactive resources, we can now offer a *usable* platform with substantial content. We now have to make the platform more visible and raise interest within the didactical community, the teachers, and the governments throughout Europe.

Today, the websites of the individual software packages from the project still have much more visits a day than the i2geo.net portal. So a first step will be to announce the portal on the websites of the software packages and on the websites of (associate) partners using banners and an i2g-compliance badge that shows the compatibility of the software with the i2g file format.

CONCLUSIONS AND CALL FOR PARTICIPATION

In this article, we can only highlight the basic structure of the project. We invite everybody to visit the project website at <http://inter2geo.eu>, submit their own content on <http://i2geo.net>, join as an Associate Partner or become a User or Country Representative.

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QUALITY PROCESS FOR DYNAMIC GEOMETRY RESOURCES: THE INTERGEO PROJECT

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In this contribution, we present the European project Intergéo whose aims are first to develop a common language for a description of geometric figures that will ensure interoperability of the main existing dynamic geometry systems, and second, to gather and to make available pedagogical resources of a good quality. This text focuses on the quality process for dynamic geometry resources aiming at their perpetual improvement.

Keywords: *pedagogical resource, quality of a resource, dynamic geometry, teacher training*

INTRODUCTION

This contribution concerns the issue of integration of ICT tools into teachers' practices and the means of supporting it. One of the keys is to provide teachers with pedagogical resources helping them to develop new activities for their pupils. However, we now know that the availability of resources is not sufficient. On the one hand, the abundance of resources makes difficult to find appropriate and quality resources (Guin and Trouche 2008, Mahé and Noël 2006). On the other hand, the availability of resources does not solve the problem of their appropriation by the teachers, which requires an evolution of teachers' competencies and their conceptions about the role of technology in teaching and learning mathematics (Chaachoua 2004). This leads to consider the issue of teachers training. Numerous research works pointed out the efficiency of training based on co-design of pedagogical resources (Krainer 2003, Miyakawa and Winsløw 2007). Various training actions have been developed in France based on this principle, e.g., SFODEM and Pairform@nce (Gueudet et al. 2008). In Brazil, AProvaME project aimed to study the effects of a collaborative design of resources involving ICT tools by the teachers on their conceptions about the notion of proof and its teaching, as well as about the role of technology in mathematics learning (Jahn et al. 2007).

THE INTERGEO PROJECT

Despite the availability and accessibility of ICT tools, and despite the recommendations in the curricula to use technology in France and in Brazil, teachers are reluctant to use these technologies (Artigue 2002). In the case of dynamic geometry systems (DGS) several reasons explain this resistance. The most important is certainly the shift in considering mathematical activity and teacher profession caused by the introduction of ICT into mathematics classroom (Lagrange and Hoyles 2006). However, other obstacles to using DGS by the teachers can not be neglected. First, the complexity of choice of a reliable and easy to use DGS among a number of

existing systems, and the resulting constraints on the choice of resources that must match the chosen DGS. Next, it is hard to find pedagogical resources appropriate to a specific educational context. This can be attributed to a great amount of resources available on the Internet, but mostly to the lack of metadata, providing an accurate description of the resource content. Moreover, available resources do not often have the required quality to be used in a classroom. The difficulty for a teacher to evaluate quality and adequacy of a resource to her/his specific context is an obstacle to the ICT integration. For this reason, tools for indexing resources, as well as evaluating their quality appear essential.

These considerations lead to 3 goals of Intergeo project (www.inter2geo.eu/fr): (1) interoperability of the main existing DGS, (2) sharing pedagogical resources, and (3) quality assessment process of resources discussed in this paper.

THEORETICAL BACKGROUND

Notion of pedagogical resource

First, it is important to clarify what we mean by pedagogical resource. Indeed, Noël (2007) points out that the issue of resource evaluation relies on the definition of what is a pedagogical resource. Nevertheless, according to the author, in spite of numerous efforts, the definition of pedagogical resource remains vague and rather broad in its scope. The most often used one is drawn from LOM standards (2002): “... *any entity, digital or non-digital, that may be used for learning, education or training*” (p.5). Flamand (2004) specifies that in order to enhance learning, a Learning Object has to possess intrinsically a pedagogical intention. Thus, for the purposes of Intergeo project, we will consider as resources those “entities” (dynamic geometry figures, texts...) for which pedagogical intention is specified.

In addition, we share Trouche and Guin’s (2006) point of view, which, referring to the instrumental approach (Rabardel 1995), considers a pedagogical resource as an artefact that needs to be transformed into an instrument by a teacher in the process of its use in her/his class. For the authors, usage of a resource is a condition for its existence. Resources are therefore living entities in evolution through their usages. In this perspective, the quality assessment process of Intergeo DG resources aims at enabling their perpetual improvement.

Quality assessment process

The quality of a resource depends on its intrinsic characteristics, as well as on its adequacy to the context in which it will be used. A given resource can be “good” in one context and “poor” in another. Thus clarifying its educational goals and the school context in which its use is intended is also essential in determining and improving the quality of the resource.

Mahé and Noël (2006) constituted an evaluation typology based on a detailed analysis of evaluation means set up by various web sites offering pedagogical resources: a priori evaluation by the adherence institution; validation of resource conformity to a deposited content; peer-review by expert teachers; user evaluation;

cross-evaluation both by peers and users. The quality assessment in Intergeo project regarding DG resources consists of an evaluation by users and a peer review of a number of resources by a group of teachers supervised by math education researchers based on a priori analysis, use in a class, and a posteriori analysis of the resources. This process corresponds to the 5th type of evaluation mentioned above, rarely encountered according to the authors.

Mahé and Noël (ibid.) bring to light critical aspects of a resource to take into account in the evaluation process: technical aspect, content, design aspect and metadata. Criteria we have set up for the quality assessment process of DG resources draw from these categories, as well as from theoretical frameworks suitable for resource analysis: (1) *didactic theories*, namely Brousseau's theory of didactic situations offering tools for analysing pupil's activity and teacher's role, and Chevallard's anthropological theory allowing to address issues of resource adequacy to institutional expectations, and (2) *instrumental approach* (Rabardel 1995) providing a framework for instrumented activity analysis.

USER EVALUATION OF THE QUALITY OF A RESOURCE

Our elaboration of a questionnaire for DG resource quality evaluation by users started by listing characteristics or elements of a resource related to its mathematical, didactical and pedagogical quality. We attempted to obtain a list as complete as possible. These characteristics were classified into 9 classes considered as relevant indicators of the resource quality: metadata, technical aspect, mathematical dimension of the content, instrumental dimension of the content, potentialities of DG, didactical implementation, pedagogical implementation, integration of the resource into a teaching sequence, usage reports. In what follows, we give an overview of criteria related to four classes referring to mathematical and didactical value of a resource.

Mathematical dimension of the content of a resource

There is no doubt that, for a resource to be usable in a school context, its content has to be mathematically correct. Adequacy of the content with the curricula allows the evaluation of the resource utility. Finally, mathematical activities need to be in adequacy with the declared educational goals.

<i>Criterion</i>	<i>Question</i>
Validity	Are the activities in the resource correct from a mathematical point of view?
Adequacy to the curriculum	Are the activities in adequacy with curricular and institutional constraints?
Adequacy to declared goals	Are the activities in adequacy with the declared educational goals?

Table 1. Mathematical dimension of the content of a DG resource

Instrumental dimension of the content of a resource

When a resource includes a DG file, it is necessary to check the coherence between the proposed activity and the geometric figure. In addition, the figure should behave

as expected. Particular attention should be paid to the handling of limit cases and of numerical values such as measures of lengths and angles. Indeed, the dynamic diagram should behave according to mathematical theories and didactical goals. If special functionalities, such as macro-constructions, are used, a description of their operating mode will make easier the appropriation of the resource by a teacher.

<i>Criterion</i>	<i>Question</i>
Adequacy of diagrams	Do the dynamic diagrams correspond to the proposed activities?
Behaviour of diagrams	Do the dynamic diagrams behave as expected in the activity?
Management of limit cases	Is the management of limit cases in the dynamic diagrams acceptable from the mathematical point of view?
Management of numerical values	Is the management of numerical values acceptable in the sense that it does not hinder mathematical aims of the activity?
Special functionalities	If the diagrams rely on special functionalities (e.g., macro-construction), is their operating mode clearly described?

Table 2. Instrumental dimension of the content of a DG resource

Potentialities of dynamic geometry

Numerous researches on DG put forward its potentialities and their contribution to the learning of geometry (Laborde 2002, Lins 2003, Tapan 2006). Criteria in this class aim first at evaluating how these potentialities are exploited in the resource, and more specifically to what extent DG contributes to improve learning activities comparing to paper and pencil environment. Second, its contribution to the achievement of educational goals is also analysed. This class comprises two criteria: (1) specific features of DG offering an added value to the resource, (2) role and use of drag mode, drawing on diversity of DG potentialities highlighted by research works (Laborde 2002, Healy 2000, Mariotti 2000). Even if a resource cannot benefit from each of them, we consider a resource that does not take any advantage of DG is of poor quality. Our hypothesis is that teachers perceive DG mainly as enabling to drag points to make pupils observing invariant properties (Tapan 2006).

<i>Criterion</i>	<i>Question</i>
Elements contributing to the added value of DG in the resource	Is DG a visual amplifier improving graphical quality and accuracy of diagrams?
	Is DG used to obtain easily and quickly many cases of a same figure?
	Does DG provide an experimental field for the learner's activity?
	Do the feedbacks enable students validate their constructions by themselves?
	DG offers a possibility to articulate different representations of a same mathematical problem. Is this possibility used in the resource?
	Does DG allow students to overcome the spatio-graphical characteristics of a diagram to focus on its geometrical properties?
	Is the activity specific to DG, i.e., it would be meaningless without it?
	Does the use of DG in the activity contribute to achieve the educational goals?
Use and role of the drag mode in the resource	Is dragging used to illustrate a geometrical property, i.e., students are encouraged to drag elements and observe a given property that is invariant while dragging?
	Is dragging used to conjecture geometrical relationships, i.e. the point is to observe whether a supposed property is invariant while dragging elements?
	Is dragging used to study different cases of the diagram?
	Is dragging used to obtain a specific configuration satisfying given conditions?

	Is dragging used to identify dependencies between objects?
	Is dragging used to illustrate link between hypotheses and conclusion in a theorem, i.e., the point is to momentarily satisfy hypotheses by dragging elements (soft construction) and consider obtained properties as necessary consequences?
	Is dragging used to explore trajectories of geometrical elements (locus, trace)?
	Is the use of dragging explicitly mentioned in the instructions for students?

Table 3. Potentialities of dynamic geometry

Didactical implementation of the resource

Trouche (2005) points out that a successful integration of ICT requires a specific organization of pupil-computer interactions, which he calls “class orchestration”. The author emphasises the importance of instrumental processes management in relation with learning mathematics. For this reason, we are convinced that a quality resource should provide a kind of assistance related to the class orchestration by means of elements concerning mathematics learning management with technology, which would help the teacher organize favourable learning conditions. We propose the criteria and questions, reported in table 4, addressing the issue of didactical implementation of a resource.

<i>Criterion</i>	<i>Question</i>
Mathematical learning management	Do the students get involved easily in the proposed activity?
	Does the activity let enough initiative to students to choose their strategies?
	Does the resource describe students’ possible strategies and answers?
	Does the resource provide information about teacher reactions to students’ errors?
	Does the resource provide information about the teacher interventions at the beginning of the activity with the students?
	Does the resource provide information about the teacher interventions making the students’ strategies evolve?
	Does the resource provide information about the teacher interventions during the phase of synthesis?
	Does the resource provide information about the validation phases?
Instrumented activities management	Does the resource discuss main characteristics of the activity, their effects on students’ behaviours and other possible choices?
	Does the resource provide information about feedback from the software?
	Do the dynamic diagrams provide feedback enabling the student to progress in solving the given tasks?
	Does the resource provide information about the possible teacher interventions regarding instrumental aspects of the activity?

Table 4. Didactical implementation of a resource

The resulting questionnaire comprises 9 classes with 59 questions altogether. It deals with a great variety of aspects of a quality DG resource and should be comprehensive. However, the questions are not homogenous from the point of view of expertise required to understand and to be able to provide a sound answer to each question. It can be expected that all users will not evaluate all aspects of a resource, but they will rather focus at those that correspond to their own expertise and their own representation of what is a quality resource. Nevertheless, the quality of a

resource will take account of all evaluators; therefore we expect that each aspect will be evaluated by some of the users.

Given the length of the questionnaire, it seemed necessary to start by proposing a lighter version to users focusing on a few large questions (one per class) addressing globally each aspect of the resource. At the same time, the user will have the possibility to deepen her/his answer by answering more precise questions related to aspects s/he will wish to analyse further, according to her/his expertise. Moreover, s/he will be given opportunity to go back to the evaluation repeatedly. Note that the process of resource ranking (under development) will take account of the user's declared expertise and assign a weight to each provided answer accordingly.

Since the end-users of the questionnaire are teachers, we wished to test relevance and clarity of the questions. For this purpose, we organized a pilot experimentation with a group of teachers using a simplified version of the questionnaire. The experiment and some results are described in what follows.

EXPERIMENTATION

Some elements of the initial questionnaire available in (Mercat et al. 2008) have been tested in Brazil, within an in-service teacher training "Geometry" module. Our goal was to analyse the relevance of evaluation criteria we defined, as well as to understand what a quality resource is for the teachers. A few more open questions were added aiming at identifying elements of a resource the teachers consider as helpful in order to appropriate and use the resource in their classes. A DG resource has also been designed to control some of its aspects for the experiment purposes and to be relevant for a teacher training.

Presentation of the resource and of the questionnaire

The resource addresses the "quadrilaterals" topic and makes use of Cabri-geometry. It is constituted of a student worksheet, a teacher document and three DG files: two dynamic figures (cf. Fig. 1) and one macro-construction.

The teacher document provides a description of the resource: topic, school level, educational goals, prerequisites and required material. It also provides a brief presentation of the suggested organization of the sessions: classroom setting and roles of teacher and students.

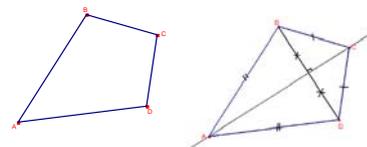


Figure 1. Dynamic figures composing the resource

The first mathematical activity, whose aim is to introduce a special type of a quadrilateral, an isosceles kite, draws from the idea of a "black box" specific to DG environments. It consists in reproducing a geometrical figure that behaves in the same way as a given model. Students are expected to explore the model in order to identify relationships between its elements, then to reconstruct the kite and validate their construction by using the macro-construction. In the resource, the exploration phase is partly guided to lead the students to characterize a kite by means of a maximum of

its properties (related to its sides, angles and diagonals). Indeed, the activities are intended for 12-14 year old students and the instructors consider inappropriate to let them completely responsible of exploring the figure and identifying properties and relationships linking its elements. In the second activity, the students are invited to explore the figure and to conjecture a possibility to obtain other types of quadrilaterals (square, rhombus, non squared rectangle) from the kite. In both activities, the drag mode is essential to explore given dynamic diagrams.

For the purpose of the experiment, we selected and adapted several questions from the Intergeo questionnaire (cf. Fig. 2), namely those concerned with mathematical and instrumental quality of the resource, potentialities of DG and didactical implementation of the resource. The questions regarding DG are intentionally open aiming at highlighting which elements the teachers spontaneously mention as contributing to the added-value of DG in the resource.

<p>Regarding the pupil's worksheet:</p> <ol style="list-style-type: none"> 1. Are the texts of the activities: <ol style="list-style-type: none"> a. Clearly formulated? b. Mathematically correct? 2. Are mathematics tasks proposed to pupils easily identifiable? 3. Are DG figures provided with the resource: <ol style="list-style-type: none"> a. Easily accessible? b. In adequacy with the proposed activities? c. Do the figures behave as expected? 4. Would you use this kind of resource with your pupils? With or without modifications? Which ones? <p>Regarding the dynamic geometry:</p> <ol style="list-style-type: none"> 1. What do you think about the role of dynamic geometry in the resource? Do you think this sequence could be proposed in paper-and-pencil environment? 2. If not, what aspects justify the use of the software? 3. Does the use of drag mode contribute to the achievement of the declared educational goals? <p>Regarding the teacher document:</p> <ol style="list-style-type: none"> 1. What elements of the teacher documents do you consider as essential and the most relevant for a pedagogical use of the resource? 2. Do you think this document is sufficient for an easy appropriation of the proposed teaching sequence? Do you miss some information? Which? 3. Would you like to have other types of material or documents? Which ones? 4. Suppose that some of your colleagues have tested the resource in their class. What information do you consider important to be shared their experience?
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Figure 2. Questionnaire for resource evaluation used in the experiment

Written answers provided by the teachers were one kind of data we gathered. These were completed by field notes of an observer recording relevant elements of exchanges among teachers.

Experimentation and first results

The experimentation consisted in one 2h30 training session for 22 secondary mathematics teachers, who had, in average, six years of experience in teaching and most were “beginners” in DG. The training session was organized in three phases: solving activities from the student worksheet, a priori analysis of these activities, and analysis of the resource guided by the questionnaire (cf. Fig. 2). In what follows, we describe the phase 3 and present the first results.

In the teacher document, the participants particularly appreciated the brief description of the sequence considered as a kind of the resource “*visit card*”, as well as the synthetic description of the sequence organisation: “*very well like that, one gets directly every essential information*”; “*one understands immediately how to organise the sequence*”.

As regards the student worksheet, the teachers have found the tasks easily identifiable, mathematically correct and clearly formulated. A special attention was paid to the vocabulary with the intention to make the wording of activities accessible to pupils. The teachers used these worksheets also to understand the sequence organisation and its progression: “*student sheets allow us to understand well the whole sequence and to spot contents and objectives*”; “*Student sheets are very well designed. [...], one sees clearly the sequence progression: observation of sides, symmetry between vertices and angles. Then, the construction is proposed and finally the study of some cases [...]*”.

Regarding elements helpful for resource appropriation but missing in the resource, the teachers expressed a need to understand how the macro had been constructed and how it works. They would also have liked to have more information about the teacher’s role: what interventions and when, particularly during the institutionalisation phases; how to assist students’ work. Some teachers pointed out that a document with reports of use, containing expected solutions and answers, but also possible students’ difficulties accompanied with advices how to cope with them (e.g., student worksheet with commentaries for a teacher) would be helpful for a better appropriation of the resource.

Regarding DG, all teachers find unquestionable its contribution in the resource: “*activities specific to Cabri*”; “*the software is essential*”; “*impossible without Cabri*”. This is not surprising since the resource was designed for. The teachers state more precisely that “*the software favours checking of properties*”; “*without drag mode and possibility to modify diagrams, properties wouldn’t be visualized*”. They spontaneously mention that dragging enables manipulating the figure and thus identifying its properties; checking properties; obtaining easily many different cases of a same figure; constructing figures easily, quickly and more precisely; making conjectures.

It is important to note that the teachers formulated all these criteria spontaneously, but they admitted that they would not have been able to do it without the framework of the questionnaire and without having done previously an a priori analysis of the resource. The questionnaire helped them focus on important aspects of the resource and they were able to provide a deeper analysis than expected. Thereof, the criteria set up for the evaluation questionnaire seem to be understandable by teachers, but what’s more, they helped them analyse the quality of the resource. Thus, the questionnaire is not only a tool for characterizing the quality of a resource and for highlighting aspects to be improved, but it can also be used to train users’ awareness

of positive and negative aspects of a resource and in this way develop their professional skills enabling them to use it efficiently with their pupils.

CONCLUSION

The results from the experimentation show the importance of training teachers to resource analysis. Indeed, the questionnaire helped the teachers focus on important aspects of the resource to look. These aspects were rarely taken into account before the training session. Among those, there is the teacher document containing information about the implementation of the resource and the added value of DG, in particular the role of drag mode.

On the other hand, the quality assessment process will lead to an improvement of a quality of resources, both at the metadata level highlighting information allowing an easier spotting of relevant and quality resources and at the level of the resource itself. Indeed, the quality criteria may be considered as a grid allowing to improve certain aspects of resources or to design new resources satisfying these criteria from the very beginning. Thus, this process can eventually give rise to a model that would act as a guide for resource designers by pointing necessary elements and helping make them explicit in an understandable and accessible way for potential users.

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NEW DIDACTICAL PHENOMENA PROMPTED BY TI-NSPIRE SPECIFICITIES – THE MATHEMATICAL COMPONENT OF THE INSTRUMENTATION PROCESS

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Relying on the collective work carried out in the e-CoLab project concerning the experimentation of the new calculator TI-nspire, we address the issue of the relationships between the development of mathematical knowledge and instrumental genesis. By analyzing the design of some resources, we first show the importance given to these relationships by the teachers involved in the project. We then approach the same issue from the student's perspective, using some illustrative examples of the intertwining of these two developments framed by the teachers' didactical choices.

INTRODUCTION

Educational research focusing on the way digital technologies impact, could or should impact on learning and teaching processes in mathematics has accumulated over the last two decades as attested for instance by the on-going ICMI Study on this theme. Questions and approaches have moved as far as research understood better the ways in which the computer transposition of knowledge (Balacheff, 1994) affects mathematical objects and the possible interaction with these, the changes introduced by digital technologies in the semiotic systems involved in mathematical activities and their functioning, and the influence of such characteristics on learning processes (Arzarello, 2007). They have also moved due to the technological evolution itself, such as the increased potential offered by technology to access mathematical objects through a network of inter-connected and interactive representations, or to develop collaborative work (Borba & Villareal, 2004). Increased technological power, nevertheless, generally goes along with increased complexity and distance from usual teaching and learning environments, and researchers have become more and more sensitive to the processes of instrumentalization and instrumentation that drive the transformation of a given digital artefact into an instrument of the mathematical work (Guin, Ruthven, & Trouche, 2004). They have revealed their underestimated complexity, and the diversity of the facets of such instrumental genesis both on the student and teacher side (Vandebrouck, 2008).

This contribution situates within this global perspective. It emerges from a national project of experimentation of the new TI-nspire in which we are involved. This artefact is quite innovative but also rather complex and distant from standard calculators, even from the symbolic ones. This makes the didactical phenomena and issues associated with its instrumentalization and instrumentation especially problematic and visible. In this contribution, we pay particular attention to the interaction between the development of mathematical knowledge and of instrumental genesis, analyzing how the teachers involved in the project manage it and how

students experience it. Through a few illustrative examples, we point out some phenomena which seem insightful from this point of view, before concluding with more general considerations.

PRELIMINARY CONSIDERATIONS

Let us first briefly present the TI-*nspire* and its main innovative characteristics, then the French project e-CoLab and also the theoretical frame and methodology of the study.

A new tool

TI-*nspire* CAS (Computer Algebra System) is the latest symbolic ‘calculator’ from Texas Instruments. At first sight it undoubtedly looks like a highly refined calculator, but also just a calculator. However, it is a very novel machine for several reasons:

- Its nature: the calculator exists as a “nomad” unit of the TI-*nspire* CAS software which can be installed on any computer station;
- Its directory, file organiser activities and page structure, each file consisting of one or more activities containing one or more pages. Each page is linked to a workspace corresponding to an application: Calculator, Graphs & Geometry, Lists & Spreadsheet, Mathematics Editor, Data and Statistics;
- The selection and navigation system allowing a directory to be reorganised, pages to be copied and/or removed and to be transferred from one activity to another, moving between pages during the work on a given problem;
- Connection between the graphical and geometrical environments via the Graphs & Geometry application, the ability to animate points on geometrical objects and graphical representations, to move lines and parabolae and deform parabolae;
- The dynamic connection between the Graphs & Geometry and Lists & Spreadsheet applications through the creation of variables and data capture and the ability to use the variables created in any of the pages and applications of an activity.

When presented with the TI-*nspire*, we assumed that these developments could offer new possibilities for students’ learning as well as teachers’ actions. They could foster increased interactions between mathematical areas and/or semiotic representations. They could also enrich the experimentation and simulation methods, and enable storage of far more usable records of pupils’ mathematics activity. However, we also hypothesized that the profoundly new nature of this calculator and its complexity would raise significant and partially new instrumentation problems both for students and teachers and that making use of the new potentials on offer would require specific constructions, and not simply an adaptation of the strategies which have been successful with other calculators.

Excerpts both from students’ interviews and teachers’ questionnaires carried out/handed out at the end of the first year of experiment support our hypotheses:

“At first it was difficult, honestly, I couldn’t use it... now it’s OK, but at first it was hard to understand... the teacher, other students helped us and the sheet we got helped us out... how to save, use the spreadsheet, things like that...” Student’s interview

“In my opinion the richness of mathematical activities thanks to the connection between the several registers is the key benefit [...] The difficulty will be the teacher’s workload to prepare such activities so to render students autonomous.” Teacher’s questionnaire

“There are still a few students for whom mathematics poses a big problem and for whom the apprenticeship of the calculator still remains arduous. These students find it hard to dissociate things and tend to think that the obstacles they face are inherent to the tool rather than to the mathematics themselves.” Teacher’s questionnaire

Context of the research

This study took place in the frame of a two-year French project: e-CoLab (Collaborative mathematics Laboratory experiment) [1]. It was based on a partnership between the INRP and three IREM: Lyon, Montpellier and Paris. It involved six 10th grade classes, all of the pupils of which were provided with the TI-*nspire* CAS calculator. The students kept their calculators throughout the whole school year and were allowed to take them home. The groups on the 3 sites were composed of the pilot class teachers, IREM facilitators and university researchers. They met regularly on site although the exchange also continued distantly through a common workspace on the *EducMath* site, which allowed work memories to be shared and common tools (questionnaires, resources, etc.) to be designed.

All pilot teachers had a strong mathematical background but the expertise in using ICT varied from one to another. In the 1st year of the project, teachers and students were equipped with a prototype of the TI-*nspire* they had never worked with before. However, the willingness to articulate mathematical with instrumental knowledge was shared by all teachers, despite the work they later on admitted it required:

“We have to devote an important amount of time to the instrumentation. This requires teachers to invest quite some time in order to design the activities, especially if they want to associate the teaching of mathematical concepts.” Teacher’s questionnaire

Theoretical framework

Two theoretical streams guide our analyses. The first one is related to the *instrumental approach* introduced by Rabardel (1997). For Rabardel, the human being plays a key role in the process of conceiving, creating, modifying and using instruments. Throughout this process, he also personally evolves as he acclimatizes to the instruments, both with regard to his behaviour as well as to his knowledge. In this sense, an instrument does not emerge spontaneously; it is rather the outcome of a twofold process involved when one “meets” an instrument: the instrumentation and the instrumentalization. Rabardel’s ideas have been widely used in mathematics education in the last decade, first in the context of CAS (cf. Guin, Ruthven & Trouche, 2004 for a first synthesis) then extended to other technologies as

spreadsheets and dynamic geometry software, and more recently on-line resources. Recent works such as the French GUPTEn project have also used the concept of *instrumental genesis* for making sense of the teachers' uses of ICT (Bueno-Ravel & Gueudet (2008).

We are also sensitive to the semiotic aspects of students' activities. Not only are we taking into account Duval's theory of semiotic representation (Duval, 1995) and the notions attached to it (semiotic registers of representation and conversion between registers), but more globally the diversity of highly intertwined semiotic systems involved in mathematical activity including gestures, glances, speech and signs, *i.e.* the "semiotic bundle" (Arzarello, 2007). In particular, when examining students' activity, we pay specific attention to the embodied and kinesthetic dimension of it (Nemirovsky & Borba, 2004) via the pointer movement or students' gestures.

Methodology

We are interested in the students' instrumental genesis of the TI-*nspire* and in particular in considering the role mathematical knowledge plays in this genesis. Such analysis cannot be done without taking into account the characteristics of the tasks proposed to students and the underlying didactical intentions. Our methodology thus combines the analysis of task design as it appears in the resources produced by the e-CoLab group, and the unfolding of students' activity.

The analysis of students' activity relies on screen captures of students' activities made with the software Hypercam. HyperCam, already used in other research involving the study of students' use of computer technology (see for e.g. Casyopée, Gélis & Lagrange (2007)), enables us to capture the action from a Windows screen (e.g. 10 frames/sec) and saves it to an AVI movie file. Sound from a system microphone has also been recorded and some of the activities have been video-taped.

When relevant, we also back up our analysis by relying on students' or teachers' interviews/questionnaires carried out independently from the activities.

TEACHERS' INSTRUMENTATION – DIDACTICAL INTENTIONS

Didactical intentions

The pilot teachers involved in the experiment cannot be said to be "ordinary teachers". All of them have been involved, in one way or another, in the IREM's network, thus they were all somehow sensitive to didactical considerations and shared a fairly common pedagogical background. The relative success of the project was in part due to this familiarity, as one teacher acknowledged: "It is easier to communalize if we share the same pedagogical principles."

In particular, the willingness of intertwining mathematical content with instrumental knowledge was commonly held and despite the hard work that it meant, the joint work was perceived as a true added value as teachers seemed to work in harmony:

“We have to carry the instrumentalization and the mathematical learning in parallel. Activities are not evident to think of and take time to design. The help from others make us gain time and provide us with new ideas.” Teacher’s questionnaire

Imprint on resources

Around 25 resources were designed during the two years of the project. There are two kinds of resources: those created essentially to familiarize pupils with the new technological instrument (presentation of the artifact and introduction of some of its potentials), and those constructed around (and we should add “for”) the mathematics activity itself [2]. In what follows, we mainly focus on the resources that support the teaching/learning of mathematical concepts and examine how teachers managed to articulate mathematical concepts with instrumental constituents.

The didactical intentions previously mentioned are clearly visible when examining the resources teachers designed, showing that these were built *from* the mathematical component yet at the same time *planning* a progressive instrumentation.

The *Descartes* resource is very enlightening in this sense. Teachers who have designed it acknowledged it appeared to be useful as an introduction into the dynamic geometry of the calculator, articulated with an application of the main geometrical notions and theorems introduced in Junior High School. It also offered the advantage of linking the work which had just been performed on numbers and geometry.

In this resource, several geometrical constructions are involved, enabling products and quotients of lengths to be produced and also the square root of a given length to be constructed. For the first construction proposed, the geometrical figure is given to the pupils together with displays of the measurements required to confirm experimentally that it does provide the stated product (fig. 1). The pupils simply had to use the pointer to move the mobile points and test the validity of the construction. Secondly, for the quotient, the figure provided only contained the support for the rays [BD) and (BE). The pupils were required to complete the construction and were guided stepwise in the successive use of basic tools as “point on”, “segment”, “intersection point”, “measurement” and “calculation”. Thirdly, they were asked to adapt the construction to calculate the inverse of a length. Finally for the square root they had the Descartes figure and were required to organise the construction themselves. Instructions were simply given for the two new tools: “midpoint” and “circle”.

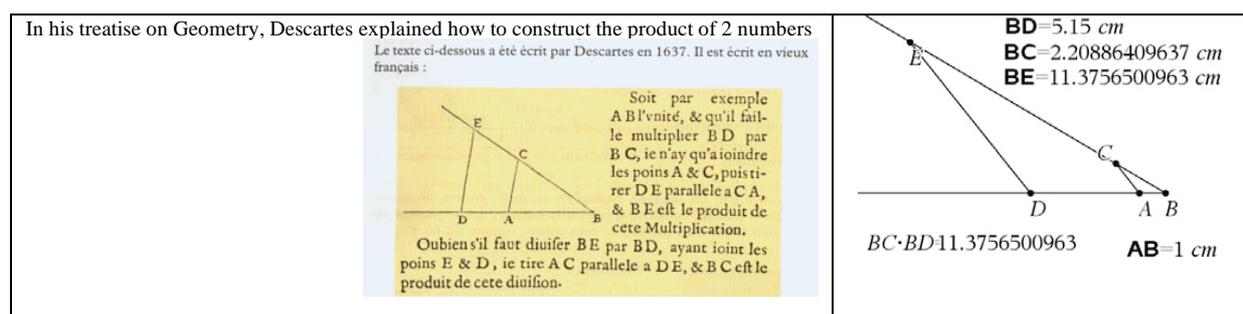


Figure 1. First part of the *Descartes* resource (extracted from the pupil sheet and the associated tns file)

In what concerns the resource *Equal areas*, the mathematical support is an algebraic problem with geometrical roots; it consists in finding a length OM such that two given areas are equal (fig. 2). The expressions of the two areas as functions of OM are 1st and 2nd degree polynomials and the problem has a single solution with an irrational value. This therefore falls outside the scope of the equations which the observed students are able to solve independently. In the first version of the resource, their work was guided by a sheet with the following stages: geometrical exploration and 1st estimate of this solution, refining the exploration with a spreadsheet to give the required value within a tolerance of 0.005, the use of CAS to obtain an exact solution, and finally the production of the corresponding algebraic proof by paper/pencil.

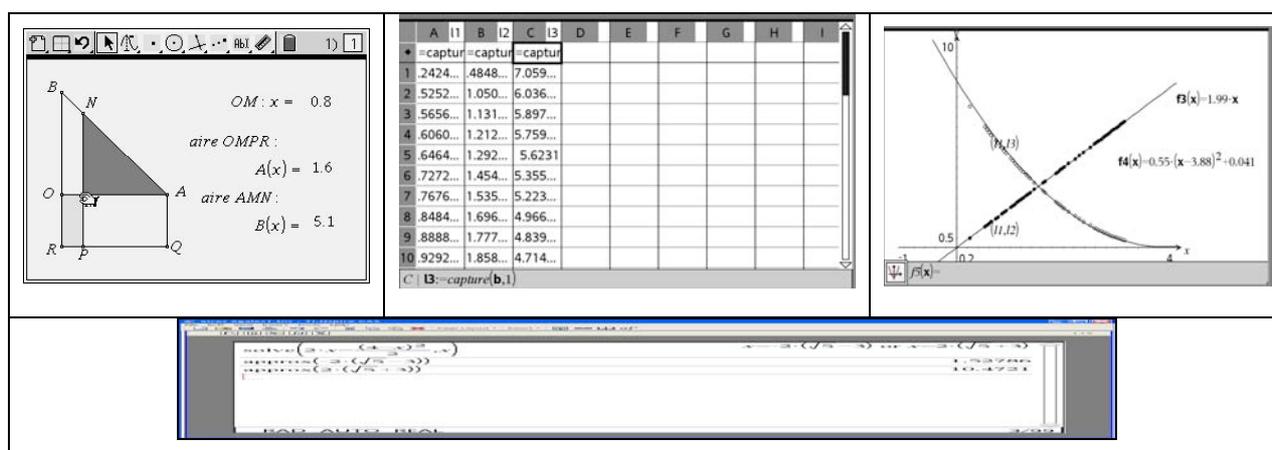


Figure 2. Exploring progressively the problem of Equal Areas using different applications

Experimentations led to the development of successive scenarios where more and more autonomy was given to the students in the solving of this problem, yet still requiring the use of several applications, discussing the exact or approximate nature of the solutions obtained, and the global coherence of the work.

MERGING MATHEMATICS AND INSTRUMENT – STUDENTS' VIEWPOINT

Our analysis will rely on the experimentation of two particular resources already mentioned (*Descartes* and *Equal areas*) for the following reasons: they have been designed with an evident attention to both mathematical and instrumental concerns, but take place at different moments of students' learning trajectory and have different mathematical and instrumental aims. *Descartes* has been proposed early in the school year; it aims at introducing the dynamic geometry of TI-*nspire* while revisiting some main geometrical notions of junior high school, and connecting these with numbers and operations. *Equal areas* was given to students several months later, at the end of the teaching of generalities about functions. It aims at the solving of a functional problem from diverse perspectives, and at discussing the coherence and complementarities of the results that these perspectives provide. It also aims at informing us about the state of students' instrumental genesis after 6 months of use of the TI-*nspire*.

Students and the *Descartes* resource

Two sessions and some homework were associated with this resource in the experimentation, and an interesting contrast was observed between the two sessions. The smooth running of the first session evidenced that a first level of instrumentalization of the dynamic geometry of the TI-*nspire* was easily achieved in this precise context. The successive difficulties met in the second session illustrated both the limits of this first instrumentalization and the tight interaction existing between mathematics and instrumentation. In what concerns the instrumentalization, we could mention students who inadvertently created a point that could superimpose on the points of the construction and invalidate measurements; the fact that they could not handle short segments on the calculator, or that they had not understood how to “seize” length variables in the geometry window for computing with them...

Regarding the interaction between mathematics and instrumentation, one difficulty appears to be especially visible in this situation: measures and computations in the geometry application are dealt with in approximate mode. Thus, when testing the validity of the construction proposed by Descartes for the quotient for instance, the students did not get exactly what they expected and were puzzled. Very interesting classroom discussions emerged from this situation which attest the intertwining of mathematical and instrumental issues. Students had limited familiarity with the tool, and had to understand that exact calculations are restricted to the Calculation application. The problem nevertheless was not solved just by giving this technical information, showing that this was not enough for making sense of such information, rather related to the idea of number itself, the distinction between a number and its diverse possible representations, the notions of exact and approximate calculations.

Students and the *Equal area* resource

As already explained, this resource is quite different from the previous one and students had been using the TI-*nspire* for more than 6 months. It has been experimented several times with different scenarios, and the analysis of the data collected is still ongoing. Some instrumentalization difficulties were still observed, even when students worked with an improved version of the artifact. These often concerned the spreadsheet application, less frequently used, but the main difficulties involved tightly intertwined mathematics and instrumental issues as in the previous example. We will illustrate this point by the use of a spreadsheet for finding and refining intervals including the solution.

Students used the spreadsheet application after a geometrical exploration of the problem. This convinced them of the existence and uniqueness of the solution, provided its approximate value and showed that the geometrical application could not provide exactly equal values for the two areas. The use of the spreadsheet application generally raised a lot of difficulties linked to the syntax for defining the content of the successive columns, for refining the step taking into account the existing limitation in the number of lines available. Students often tried to refer to spreadsheet files used in

previous problems to solve them. Some could be helpful (another functional problem), some were problematic (a probabilistic situation recently studied). Choosing an appropriate file required an ability to see the similarities and differences between the mathematical problems at stake. Benefiting from an adequate file required the matching of the two mathematical situations, establishing correspondences between the data and variables involved, and understanding how these reflected in the syntax of the commands. The use of the generated tables, once obtained, also raised many difficulties. Students tried to get the same values for the two areas or to find the closest ones. This was not at all easy, and very few of them were spontaneously able to create a new column for the difference. Moreover, when asked to find an interval for the solution, they were unable to exploit the table in a successful way. The idea that the solution of the problem corresponded to an inversion in the order of the two areas, and that they had thus to look at the two successive lines showing this inversion for getting the limits of the interval asked for was not a natural idea. The screen copies and discussions between students or/and with the teacher of this episode clearly illustrates to what extent mathematics and instrumentation are intertwined.

In these two examples, we have focused on the mathematical/instrumental connection through the analysis of students' difficulties but the observations also show episodes where an original mathematical/instrumental synergy is at stake, made possible by the students' joint mathematical and instrumental progression. We will illustrate this by examining students' activity when working on the previous problem, but with greater autonomy. A group of two students had begun with a geometrical exploration, then defined the two functions expressing the areas and moved to a graphical exploration, selecting an appropriate window for the problem ($0 \leq x \leq 4$). They carried out this exploration cleverly, created the intersection point of the two curves to get its coordinates and found numerical values with only 6 decimals. This fact associated with the visual evidence of the intersection point convinced them that they had got the exact solution. They came back to the geometry page and checked that this solution was coherent with the approximate value with 2 decimals they had already got. They then moved to the calculation application (exact mode) and asked for the solution of the equation. They obtained 2 irrational values and were puzzled. The screen captures show several quick shifts between the graphic and calculation pages, before one of the boys decided to ask for an approximate value of the two solutions. Once obtained, they came back to the graph page, changed the window so to visualize the 2nd intersection point, seemed satisfied, went back to the geometry page and discarded the 2nd solution as non relevant. Once more, we cannot enter into more details, but the productive interplay here is evident. Let us just add that there has been an interesting collective discussion about the conviction of obtaining an exact solution in the graph page and the rationale underlying it. Linked with a deep mathematics discussion, the way TI-nspire manages approximations in the different applications and the way the user can fix the number of decimals was clarified.

For making sense of such synergies and instrumented practices, there is no doubt in our opinion that a semiotic approach limited to the identification of treatments inside a given semiotic register of representation or conversions between such registers is not fully adequate. What we observe indeed is a sophisticated interplay between different instruments belonging to the students' mathematical working space and a swing between these certainly supported by technological practices developed out of school. These are efficiently put at the service of mathematical activity and part of their efficiency also results from their kinesthetic characteristics.

Beyond that, there is no doubt that the work performed by the students in this task, through the diversity of perspectives developed around the same mathematical problem, and the small group and collective discussion raised about the potential and limits of these different perspectives and their global coherence, corresponds to a quality of mathematical activity hardly observed in most grade 10 classes.

CONCLUSION AND PERSPECTIVES

Due to its specific features which distinguish TI-*nspire* from other calculators and as it had been envisaged *a priori*, the introduction of this new tool was not without difficulty and required considerable initial work on the part of the teachers, both to allow rapid familiarisation on their part and those of the pupils but also to actualize the potentials offered by this new tool in mathematics activities. When examining both the design of the resources created by the pilot teachers and the work performed by students, as we have tried to show in this contribution, we grasp how delicate and somehow frail the harmony between the mathematical and instrumental activity is, and how the semiotic games underlying it are complex. We also see the impact of new kinds of instrumental distances (Haspekian & Artigue, 2007) and closeness that shaped teachers' and students' activities: on the one side, distance from more familiar mathematical tools and especially graphic and even symbolic calculators, on the other side closeness with technological artifacts on offer out of school (computers, iPods, etc...). These characteristics affect teachers and students differently, and individuals belonging to the same category differently, according to their personal characteristics and experience. They can have both positive and negative influences on teaching and learning processes and need to be better understood. For that purpose, beyond the theoretical constructs we have used in this study, we consider it interesting to extend the tool/object dialectics (Douady, 1986) to the instrumental component of the activities. By choosing to closely articulate mathematical and instrumental knowledge, the latter is inevitably introduced within a specific mathematical context. Reinvesting instrumental knowledge also requires students, even implicitly, to decontextualise and to a certain extent generalize what has been acquired.

NOTES

1. A more general overview of the project as well as other findings can be found elsewhere (see Aldon et al., 2008).
2. Some resources can be found at: <http://educmath.inrp.fr/Educmath/partenariat/partenariat-inrp-07-08/e-colab/>

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ISSUES IN INTEGRATING CAS IN POST-SECONDARY EDUCATION: A LITERATURE REVIEW

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We discuss preliminary results of a literature review pilot study regarding the use of CAS in higher education. Several issues surrounding technology integration emerged from our review and are described in detail in this paper. The brief report on the type of analysis and the integration scope in curriculum suggest that the multi-dimensional theoretical framework proposed by Lagrange et al. (2003) needs to be adapted for our focus on systemic technology integration in tertiary education.

INTRODUCTION

A growing number of international studies have shown that Computer Algebra System (CAS-based) instruction has the potential to positively affect the teaching and learning of mathematics at various levels of the education system, even though this has not been widely realized in schools and institutions (Artigue, 2002; Lavicza, 2006; Pierce & Stacey, 2004). In contrast to the large body of research focusing on technology usage that exists at the secondary school level, there is a definite lack of parallel research at the tertiary level. However, Lavicza (2008) highlights that university mathematicians use technology at least as much as school teachers, and that the innovative teaching practices involving technology that are already being implemented by mathematicians in their courses should be researched and documented. Further, Lavicza (2008) found that within the research literature there existed only a small number of papers dealing with mathematicians and university-level, technology-assisted teaching. In addition, most of these papers are concerned with innovative teaching practices, whereas few deal with educational research on teaching with technology. These findings coincide with school-focused technology studies conducted by Lagrange et al. (2003) and Laborde (2008).

We aim to point out that it is particularly important to pay more attention to university-level teaching, because universities face new challenges such as increased student enrollment in higher education, decline in students' mathematical preparedness, decreased interest toward STEM subjects, and the emergence of new technologies (Lavicza, 2008). Mathematicians must cope with these challenges on a daily basis and only a few studies have offered systematic review and developed recommendations in this area. Our project aims at both documenting university teaching practices involving technology, and formulating recommendations for individual and departmental change. Our research program also aims at raising the amount of attention paid to tertiary mathematics teaching from a research point of view and, from a more practical side, elaborating on specific issues and strategies for the systemic integration of technology in university mathematics courses.

METHOD DESIGN AND IMPLEMENTATION

Based on the above-mentioned Lavicza (2008) findings and recommendations, we designed a mixed methods research study which involves a systematic review of existing literature regarding CAS use at the tertiary level. The theoretical framework developed by Lagrange *et al.* (2003) involved several stages. They first reviewed a large number of papers in relevant journals and then categorized these papers into five “types.” Based on these types, they then selected a sub-corpus of papers dealing specifically with educational research papers focusing on technology use mainly in the secondary school. Through the careful analysis of this sub-corpus of papers, they further developed seven dimensions, each with key indicators, and then proceeded to identify and further analyze papers that best described each of these dimensions.

The theoretical framework of Lagrange *et al.* (2003) provided our research team with a helpful foundation from which to prepare for our own literature review which will involve approximately 1500 papers/theses. It was decided to implement a pilot study for this large literature review in order to begin to work with the Lagrange *et al.* framework and to determine if it would be sufficient for our purposes, or may be in need of certain modifications. In the summer of 2008, we therefore began our pilot study focusing on 326 contributions dealing with CAS use in secondary/tertiary education. These papers were drawn from two well-regarded journals, namely the *International Journal for Computers in Mathematical Learning* (issues since its beginning in 1996) and the *Educational Studies in Mathematics* (since 1990). We also selected proceedings from two technology-focused conferences, namely the *Computer Algebra in Mathematics Education* (since its first meeting in 1999) and the *International Conference on Technology in Collegiate Mathematics* (since 1994 with first electronic proceedings). A sub-corpus of 204 papers dealing specifically with CAS use at the post-secondary level was also identified to further focus the analysis.

While the descriptive categories found within the Lagrange *et al.* template were helpful, we began to notice that several other category/theme columns would be helpful at this stage of the instrument/template development (e.g., we added fields such as “computer/calculator,” “implementation scope,” and “implementation issues”). An important point to note here is that in contrast to the Lagrange study where the majority of papers were those describing educational research results, our selection of papers revealed a majority that focused on practitioner innovations with very few involving educational research. Thus, we realized that in order to develop our template for reviewing the large number (1500) of papers in the research study proper, we would have to separate the practitioner report type papers from the educational research papers, and further modify the template in both of these areas. In this paper we outline preliminary results of our ongoing pilot study, with a specific focus on a series of “issues of implementation” at the tertiary level of education.

RESULTS

The majority of the papers in the corpus are practice reports by practitioners (88%), whereas the remaining contributions are education research papers (10%) or letters to

journal editors (1%) (see Table 1). Among the practice reports, different types of contributions become apparent. Some (94) are merely examples of CAS usage. Other papers (41) are mostly examples of CAS but

feature reflections by the practitioner. A few (13) have the practitioners go further and include classroom data and perform some basic analysis. There are also papers (5) that focus on classroom surveys and a small set (7) that examines a specific issue in detail. The remaining contributions (23) are conference abstracts only. The analysis of the education research papers according to Lagrange et al.'s multi-dimensional framework (2003) is still in progress. In this paper, we focus our analysis mainly on practitioner reports.

In addition, nearly all papers are American (87%). The computer use is more evident (59%) than the use of graphical calculators (29%) or than the combined use of both computer and graphing calculators (10%). Furthermore, the most widely used CAS in the corpus is the graphing calculator (83 papers), followed by *Maple* (53) and *Mathematica* (43). *Derive* (21) and *Matlab* (11) are also common, as well as 27 papers dealing with other CAS. In what follows, we elaborate on one particular significant aspect of the study, namely “integration issues” that emerged from our review, and also briefly report on “integration scope.”

ISSUES OF CAS INTEGRATION

Education researchers and practitioners widely wrote about issues surrounding the use and implementation of CAS at post-secondary education (72 papers). With regard to practitioner reports, 56 papers identify some issues; of these there are 20 that go into considerable detail. These papers could be further divided into two categories: Seven of them deal with a specific problem relating to CAS (e.g., rounding error) and thirteen discuss various implementations of CAS while underlining the hurdles the

authors encountered. Of the sixteen issues identified in the corpus and summarized in Figure 1, we divide them into three categories: Technical (first four columns), cost-related (fifth column), and pedagogical (last 11).

There are four issues discussed in the literature dealing specifically with technological aspects: Lab availability (Lab), reliability of technical support (Tec), system requirements (Sys) and troubleshooting (TrS). These issues may not be independent from each other. For example, May (1999, p. 4) urges instructors to test out their *Maple* worksheets on the lab computers rather than their own workstations due to such machines having less

Table 1: Type of Contribution

Presentation of Examples	46%
Examples with practitioner reflections	20%
Classroom Study	6%
Classroom Survey	3%
Examinations of a specific issue	3%
Abstract only	11%

Education research papers	10%
Letters	1%

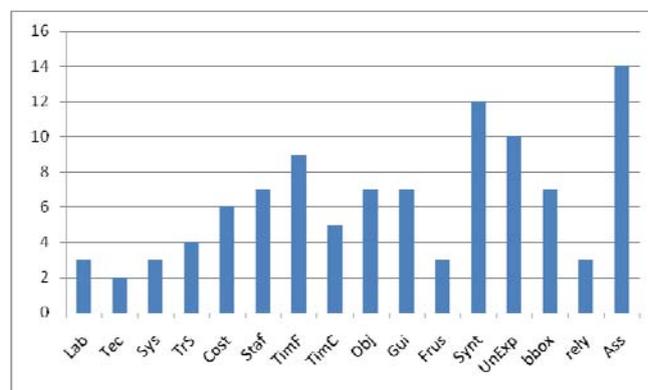


Figure 1: Issues in integrating CAS in university education

memory installed in them. Weida (1996, p. 3) notes that in troubleshooting, various hardware problems arise and his “experience and lots of calls to the Computer center” helps. An unexpected issue for him was the class interruption of students not enrolled in his class. While they would never think to disrupt a lecture, they would see nothing wrong with walking into his lab session to complete homework for other courses.

Many reports mention the issue of costs (Cost) incurred by integrating CAS into instructors’ courses, providing few further details beyond the existence of the financial obstacle. An exception occurs in one paper where the authors argue for a particular choice of open-source (free) technology (Hohenwarter et. al, 2007, p. 5).

Wu (1995) notes that besides the cost aspect, enacting calculus reform “requires more talent and training” (p. 1). This need for trained staff (staf) is mentioned in seven papers, often in conjunction with other issues. For example, to deal with technical difficulties during labs, Weida relies on his own experience to assist in troubleshooting (1996, p. 3). At the beginning of an attempt at CAS integration, Schurrer and Mitchell (1994, p. 1) wondered, “how they could go about motivating [sceptical mature faculty] to consider introducing the available technology and making the curricular changes this would require?”

Schurrer and Mitchel (pp. 1-2) further discuss the need for time for the faculty (TimF) to design courses and meaningful activities with technology. Their department required decisions on types of technology used and on what technology curriculum package had a “right mix.” They note that program-wide integration takes time. In their case at University of Iowa, it took seven years to implement (p. 3). Even after a curriculum change, additional time demands on faculty are reported by practitioners. Wrangler (1995, p. 8) notes that near constant improvement is needed in lab experiments and stresses that for faculty there is “no resting on laurels.” A closely related issue is the problem of time management in courses (TimC). Wrangler (p. 8) remarks that besides the time he spent outside of class, he had to take his students into the lab and walk them through basic commands. Many other practitioners, such as May (1999, p. 4), expresses similar sentiments. While this issue is discussed less frequently than time spent outside the classroom, practitioners report about both issues in conjunction (e.g., Wrangler p. 8).

CAS integration also affects classroom time management with respect to course content. Dogan-Dunlop (2003, p. 4) remarks that, “since class time was allocated for in-class demonstrations and discussions, detailed coverage of all the topics that were included in the syllabus was not possible.”

Another source of pressure on time management is the failure of students to achieve learning objectives (Obj). Krishnamani and Kimmons (1994, p. 4) note that students failed to learn material assigned in labs and they had to include it in later lectures.

One particular type of student error that clashes with learning objectives is the assumption on the part of students that their methodology is correct if their paper-and-pencil calculations match up with results obtained from the computer. As Cazes

et. al. (2006 p. 342) write, “a correct answer does not mean the method is correct or is the best one. Teachers and students must be aware of such... pitfalls.” Often students engaged in trial and error strategies, with students guessing the answer from feedback without making a proper mathematical argument (p. 347). Instructors sometimes failed to ensure that students found an “optimal” solution to a particular problem rather than just having a “correct” answer (pp. 342-343).

Pedagogical difficulties with learning objectives can place demands on faculty time not only inside but also outside of the lecture hall. Dogan-Dunlap (2003 p. 4) had to redesign his course and the use of CAS within it three different times because of such concerns. As previously discussed, there is an ongoing time commitment by faculty to improve their lecture and laboratory instruction and Dogan-Dunlap’s experiences show that student difficulties may greatly influence the nature of those changes.

Related to the learning objectives issue, that of guidance (Gui) also emerges from the review. Often practitioners show concerns as to how much help they should give their students without compromising learning objectives. Westhoff (1997) designed a student project for Multivariate Calculus on the lighting and shading of a 3-dimensional surface. He found that the difficulty in the project, due to its complexity, lays in determining how much he could tell his students (p. 6). Another area in which guidance becomes an issue is mentioned by Weida (1996). Noting that there is a “fine line between helping students... and ‘giving away’ the answers,” he remarks that such a problem is “particularly exacerbated at the end of a lab when the slower workers are running out of time” (pp. 3-4). Weida further presents the idea that careful scheduling could help alleviate this by ensuring that there isn’t a need to leave immediately after the lab.

Student frustration (Frus) is another issue related to learning objectives. Cazes et. al. (2006, p. 344) note that students would often seek help either online or via the instructor “after having encountered the first difficulty” rather than attempting to solve the problem on their own. Krishahamani and Kimmons (1994) took steps to reduce anxiety both in course design and in providing additional help for students. Several measures, including reduced expectations, more time for tests, increased extra credit problems and a homework hotline were implemented (p. 2). Clark and Hammer (2003, p. 3) had a project for first year calculus modeling a rollercoaster. They found that “students who were not as “good” at *Maple* struggled, found the project (and *Maple* syntax) frustrating and were just happy to produce one mathematical model.” This suggests possible relationship between student frustration and failure regarding activity learning objectives, and the CAS syntax issue.

Syntax (Synt) is the second most frequent concern for both practitioners and students. Cherkas (2003) found this to be a source of student dissatisfaction. He quotes a student complaining, “*Mathematica* would cause a lot of problems. If I make a mistake in the syntax, I couldn’t do my work” (p. 31).

Tiffany and Farley (2004) exclusively focus on common mistakes in *Maple*, emphasizing the hurdle for practitioners caused by syntax. Practitioners employ various schemes attempting to minimize this difficulty. Some such as May (1999) design interactive workbooks that eliminate the need for teaching syntax entirely. Others like Herwaarden and Gielen (2001, p. 2) provide *Maple* handouts with expected output to their students. Some emphasize a pallet-based CAS such as *Derive* (Weida, 1996, p. 1) because it is easier to learn and has, according to them, a more straightforward notation.

Another source of student frustration is the unexpected behaviour of CAS (UnExp) even when their reasoning is syntactically and mathematically correct. Sometimes this is merely the case of paper-and-pencil calculations not easily matching up with CAS output. CAS may employ an algorithm efficient for computation and not necessarily one that matches a hand technique. For example, Holm (2003, p. 2) found that an online integral calculator would (rather than using the substitution method for $\int x(3x^2-1)^7 dx$) simply expand the product and use the power rule. He notes that such cases provide an opportunity for learning, and that, referring to another classroom assignment, the more “savvy student would... expand $\frac{1}{48}(3x^2-1)^8$.” Unexpected behaviour of CAS also takes the form of errors by the computers themselves. Due to the nature of floating point arithmetic and in spite of correct input by the user, roundoff error can cause the output to be wrong (Leclerc, 1994, p. 1). To encourage her students to adapt, Wu (1995, p. 2) purposely designed a lab with roundoff error. LeClerc urges students to be instructed in the nature of floating point arithmetic so that they “will be able to detect when roundoff has corrupted a result and hopefully find better ways to formulate or evaluate the computation” (1994, p. 4).

The concept of the “black box” (bbox) is examined in seven papers. Though this issue tends to be explored in more detail in education research papers, practitioners comment on it as well. O’ Callaghan (1997, p. 3) writes that faculty at Southeastern Louisiana University expressed concern that “students would become button pushers rather than problem solvers.” The managed use of the black box as an opportunity for students to explore complex mathematics beyond their level is discussed in great detail in education research papers (e.g., Winsløw, 2003, p. 283). Practitioners do not emphasize this potential as much. However, Cherkas (2003, p. 234) notes that CAS allows practitioners “to teach at a higher level of mathematical sophistication than is possible without such technology.”

Closely related to the “black box” issue, is the fear that students become too reliant on the technology (rely). This, along with student frustration, is the least mentioned pedagogical issue. Cherkas reports on a student complaint that s/he could not do questions on tests because “Mathematica usually did them for me” (pp. 231-232). An over-reliance on technology may interfere with learning objectives. Considering this, Shelton (1995, p. 1) emphasizes her “top-down” approach and writes that “students can avoid the technology crutch and approach the goal of developing determination and mathematical maturity to perform mathematics without the technology.”

The last and most commonly examined issue encountered in the literature is that of assessment (Ass). Practitioners encounter problems in evaluation. Schlatter (1999) allowed for CAS use during his exam for his multivariate calculus course. Unfortunately, in a question designed to test student understanding of the divergence theorem, several students simply used the CAS capabilities to solve the integral in a “brute force” approach (pp. 8-9). A poorly designed assessment thus leads to a failure in learning objectives. Schlatter further writes that he expected “to spend more time during this semester... more carefully designing exam questions” (p. 8), pointing again to the issue of faculty time.

Interpreting CAS output is discussed frequently. Quesada and Maxwell (1994, p.207) never accept a decimal answer (even if correct) if there is a proper algebraic expression. Many papers that discuss mathematical projects stress the use of written reports (e.g. Westhoff, 1997, p. 1). Lehmann (2006, p. 3) writes in his assignment “the important part of this assignment is the thought you put into it, the analysis you do and the presentation of your solution, not the answers themselves.” Xu (1995, p. 1) found that students were finding derivatives of easy functions by hand on assignments, but using graphing calculators to solve the more difficult questions. To show students “that the calculator could not do everything for them” he found functions in the textbook that “were easy to handle by hand but could not be done easily on the calculator.”

CAS INTEGRATION SCOPE

Policy making regarding the curriculum in tertiary education is rather different than in school education. Hodgson and Muller (1992) mention that school mathematics curricula are in general developed by Ministries or Boards and implemented in the classroom by teachers, whereas tertiary mathematics curricula are developed and implemented by the same actors, i.e., faculty in departments of mathematics. However, change involving technology in tertiary curriculum, like in its secondary school counterpart, seems to remain very slow (Ruthven & Hennessy, 2002). Lavicza (2006) argues that due to academic freedom, “Mathematicians have better opportunities than school teachers to experiment with technology integration in their teaching”. This ad hoc basis is strongly reflected in our literature review. A large majority (67%) of the corpus restricted to practice reports discusses CAS usage with regards to one course, or in other words, CAS integration by one practitioner. While 16% has a scope that reaches across a series of courses (e.g. calculus courses), 11% discusses a CAS implementation with a grouping of courses (e.g. all first year courses). Only 6% discusses a program-wide implementation within a department.

CONCLUSIONS

There is a need to develop a framework for the review of literature on the use of CAS at tertiary education that will integrate specificities of university-level education and technology integration. A significantly stronger majority of papers in our study stemmed from practitioner use (88%) than in Lagrange et al.’s (2003) study (60%)

which stated, "Most of the [practitioner] papers lack sufficient data and analysis and we could not integrate them into the [detailed (statistical) analysis]" (p.242). Our selection of journals and conferences for our pilot study may have influenced the above percentage. Nevertheless, this reality will clearly influence the development of our analytical framework henceforth. Lagrange et al. (2003) further state,

[Practitioner] papers offer a wealth of ideas and propositions that are stimulating, but diffusion is problematic because they give little consideration to possible difficulties. Didactical research has to deal with more established uses of technology in order to gain insights that are better supported by experimentation and reflection. We have then to think of these two trends as complementary rather than in opposition. (p.256)

We aim at elaborating upon these complementary trends at the post-secondary level by both analyzing existing instructional practices and scrutinizing problematic issues within implementation. Lagrange et al. (2003) further state that the "integration into school institutions progresses very slowly compared with what could be expected from the literature" (pp. 237-8). This might be the case for school education, but apparently less so for tertiary education (Lavicza, 2008). The research literature about school mathematics and technology seems to pay less than adequate attention to the actual classroom implementation piece. The literature about tertiary mathematics and technology tends to inform us more about (individual) implementation than its didactical issues and benefits. This suggests that there may be a need for more education research focusing on the integration of technology in tertiary education. It also points, as suggested by Table 2, to the need of resources for departments of mathematics for systemic integration of technology in curriculum. At the recent ICME 11 conference, the results of a special survey highlighted concerns about the international trend of disinterest in university mathematics (ICME 11, n.d.). Departments of mathematics have a responsibility to question the current curriculum. We contend that part of this responsibility includes the careful consideration of the role and relevance of technology within that 21st-century curriculum and classroom.

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THE LONG-TERM PROJECT “INTEGRATION OF SYMBOLIC CALCULATOR IN MATHEMATICS LESSONS” – THE CASE OF CALCULUS

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A long term project (2003 – 2011) was started to test the use of symbolic calculators (SC) in grammar schools in Bavaria (Germany). The project was firstly done in grade 10. During the 2006/07 school year the project was implemented in grade 11. 732 students at 10 Bavarian grammar schools took part in an empirical investigation. The content taught was calculus: basic properties of functions, limits, continuity, derivatives, and applications of calculus. The evaluation of the project was intended give answers to the following questions: how basic mathematical skills (algebraic transformations, solving equations) changed; how the students used the SC, how they evaluated the use of the new tool. This article presents the results of this project for school year 2006/07.

1. BACKGROUND

In the past, many empirical investigations concerning the use of CAS or symbolic calculators (with CAS) in mathematics teaching have been published (see Guin, Ruthven and Trouche, 2005). The central results of these projects have meanwhile been confirmed by other investigations world wide. The use of a CAS brings a greater meaning to work with diagrams, reinforces experimental work, in which the assumptions were obtained through systematic testing and CAS appears to bring an increase in computer cooperative forms of work. The effects are primarily long term. It is therefore necessary to develop a namely educational concept to evaluate the changes in knowledge and abilities over a longer time period. However, many investigations in this area restrict themselves to the applications of the computer over “just” a few weeks (Schneider, 2000, Drijvers, 2003, Pierce and Stacey, 2004 and Guin et al, 2005) and do not show the long-term effects on the knowledge and ability of the students.

In the school year 2003/04 we started a long term project to test the use of symbolic calculators (SC) – the TI-Voyage 200 and the TI-Nspire – in grammar schools (Gymnasien) in Bavaria (Germany). The project was done in grade 10 and has been repeated in the following two school years with a greater number of classes and with – concerning the use of new technologies – inexperienced teachers. An overview of the empirical investigation and especially of the theoretical background of this project gives Weigand (2008). On account of the positive results of this project, the Bavarian Ministry decided to continue the project. The follow-up project was started in September 2006.

2. THE TEACHING PROJECT – GRADE 11

2.1 The participants

During the 2006/07 school year the project was implemented in grade 11. A total of 732 students at 10 Bavarian grammar schools took part in this project. 412 students in 16 classes acted as the “pilot classes”, working with Voyage 200 and/or TI-Nspire. Schools could apply for the participation in the project. The pilot schools have been chosen by the Bavarian Ministry. They are spread over the state. In addition, 320 participants from 11 classes – from the same schools as the pilot classes – formed a “control group” for the purposes of quantitative statistical investigation. The students had different previous experiences; some students had been exposed to the SC in the previous grade 10, but other students came into contact with these systems for the first time during this project.

2.2 The teachers

The project was mainly taught by teachers with little experience of tuition using computer algebra systems (CAS). The project teachers held two three-day meetings during which examples of possibilities and opportunities for SC use were discussed. The teachers jointly prepared a number of suggestions for a range of teaching units intended to highlight the possibilities of using SCs; during the year, the teachers were offered additional learning units¹ by the coordinator (Ewald Bichler). However, there was no uniform overall concept according to which teaching was to be organised in all classes. The personal experience, attitudes and circumstances at the individual schools were too different for this to be possible.

2.3 The learning contents

In grade 11, calculus is taught (in Germany). The content taught was subdivided into the following:

- basic properties of functions (symmetry, monotonicity, variations in function terms and their impact on graphs, ...)
- limits, continuity
- differentiability, derivation rules, derivation function(s)
- applications of differential calculus (“classical” functions discussion, extreme value problems)

2.4 Teaching methods with the SC

During the meetings with the teachers at the beginning and in the middle of the school year a theoretical frame of the use of the SC in the classroom was discussed

¹ One sort of learning units developed during the project is called “Minute Made Math”, more information on www.minute-made-math.com

with the teachers. Especially a short insight into the theory of instrumentation was presented and explained with examples (Artigue 2002, Trouche 2005).

Concerning the integration of the SC into the problem solving process we distinguished using the SC

- in the beginning of the problem solving process or a concept formation process (the SC as a “discoverer”),
- in the middle of the process (the SC as “solver”) and
- at the end of the process (the SC as a “controller”).

We also emphasized the “rule of three” while working with representations: If possible a problem or the solution of the problem should be represented on a symbolic, graphic and numeric level.

2.5 Research questions:

In the following we concentrate on a selection of the research questions (RQ) of the project:

RQ1. Can any differences be ascertained in terms of core mathematical abilities (substitutions, interpretation of graphs, solving equations, working with tables, and working with formulae) between the pilot and the control groups after one year?

RQ2. Can different effects of SC use be ascertained with “good”, “average” and “weak” students?²

RQ3. To what extent have students mastered the SC at the end of the year?

RQ4. In which phases of a problem solving activity do the students use the SC?

2.6 Test instruments

For the purpose of answering the 1st and 2nd questions we took a (classical) pre- and post-test-design – the tests using paper and pencil but no calculator – in pilot and control classes.³

For the purpose of answering the 3rd and 4th questions the pilot classes took a test using a SC in February 2007 and June 2007 in which they were asked to record their working methods with the SC in a questionnaire which they completed immediately *after* the test.

² The performance criteria used relate to the results of the pre-tests at the beginning of the school year.

³ See: www.dmuw.de/weigand/2009/CERME6/

3. EVALUATION OF PRE- AND POST-TESTS

3.1 The questions

The pre- and post-test-questions (PP-questions) can be divided into the following groups:

- Questions 1 and 2: doing “classical” simplification of terms
- Question 4 and 5: solving equations
- Question 5: understanding the concept of root functions
- Questions 6 – 8: seeing the correlation between graph and term
- Question 9: interpreting graphs

3.2 Comparison of results of pre- and post-tests

The post-test was the same as the pre-test. In the following diagram, the *differences* between the average scores achieved for each question in the pre- and post- tests for the pilot and the control group are shown. The “average performance increase” is therefore measured for each question.

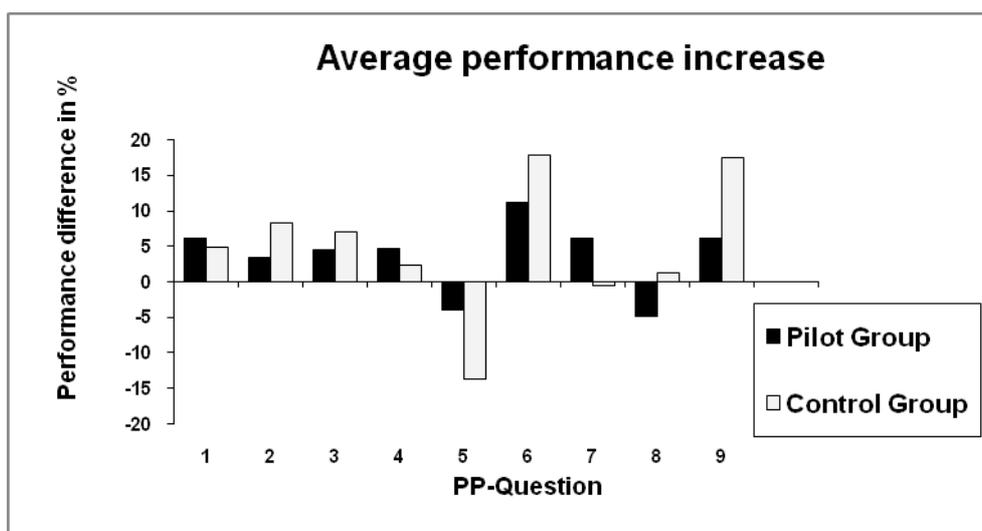


Figure 1: Average performance increase of the pilot and the control group

In PP-questions 5 and 7 the pilot classes' results are significantly better than those of the control groups (t-Test: PP 5: 0.01, PP 7: 0.02). However, in PP-questions 6 and 9 they are significantly worse (t-Test: PP 6: 0.01, PP 9: 0.01).

Overall there is not a significant difference in the average performance increase between the pilot and control classes. For the comparatively worse result of the pilot classes compared with the control classes (especially for questions PP 6 and PP 9), there are two possible hypotheses. On the one hand it could be due to the fact that the students in the pilot classes were no longer adequately challenged or motivated to

tackle this type of “traditional” question with enthusiasm, as they had tackled much more interesting questions during lessons – due to the SC. On the other hand the poor results of the pilot classes when determining functional equations from specified graphs (question 6) could be due to the fact that the students in the pilot classes had seen a large number of graphs – compared with the control group – during the course of the year and were therefore overtaxed by the diversity. However, the students in the control class have probably worked more often with the sine function graph which had been introduced in grade 10.

If, however, the range of performance increases is considered, an interesting picture emerges.

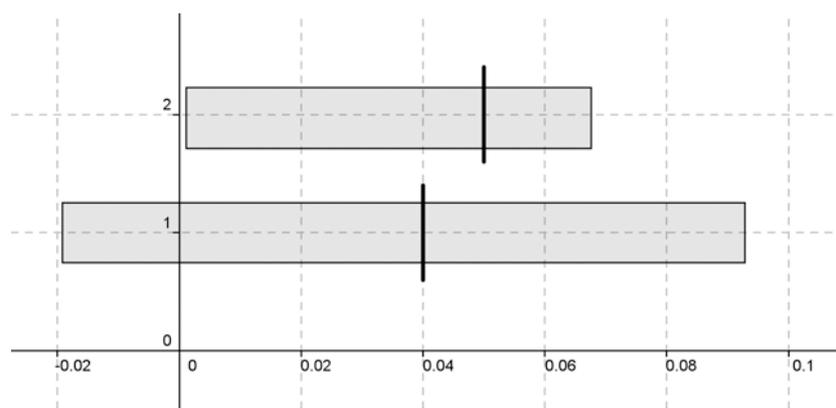


Figure 2: Average value and range of average performance increases in pilot (1) and control groups (2)

With an almost identical average value, it becomes apparent that the differences in performance are more varied with the students in the pilot classes than with the students in the control groups. Therefore, there are students in the pilot classes who benefit more from SC use than students in the control classes. However, there are also students whose results deteriorate compared with the initial test.

The test results can also be interpreted in a positive way for the pilot classes, as there are no differences in terms of classical technical and manual abilities and skills. However, this investigation has deflated hopes that the ability to interpret graphs and transfer between different forms of representation are automatically improved by the use of the SC.

3.3 Scores for “good”, “average” and “weak” test participants

In accordance with the results of the pre-test, we divided the test participants into “weak”, “average” and “good”.⁴ The following result is produced when the performances of these groups are compared in terms of pre- and post tests.

⁴ The “good” students form the upper performance quartile, the “weak” students the lower performance quartile, and the “average” students are represented by the two central performance quartiles.

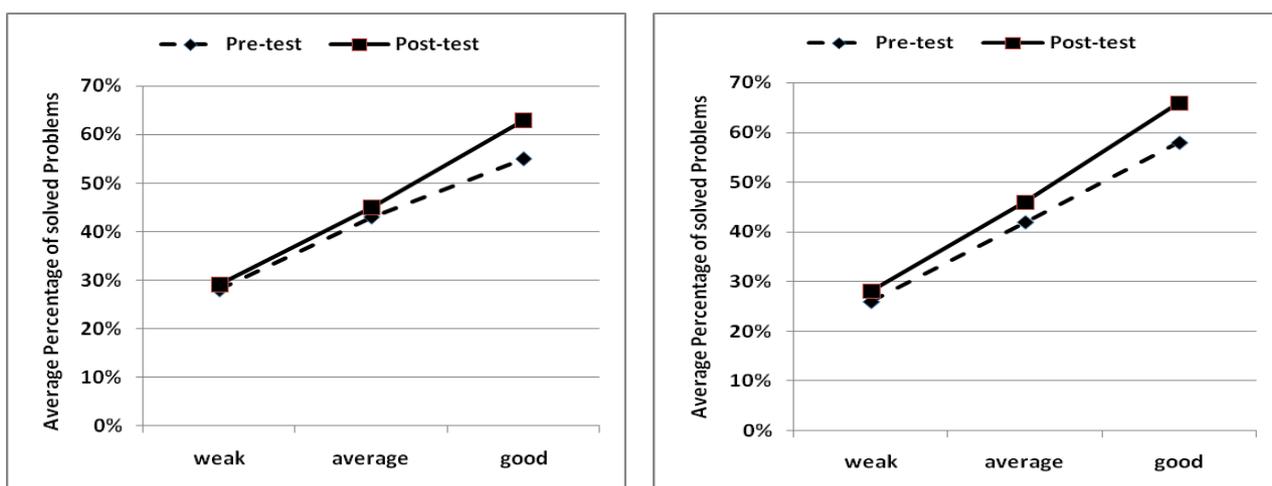


Figure 3: Performances of the pilot group (left) and the control group (right)

Compared with tests carried out in recent years in grade 10 (see Weigand 2008), different behaviour was demonstrated here. Whilst the “weak” students achieved a greater performance increase than the “average” and “good” students in grade 10, the “good” students – both in the control and pilot groups – improved more markedly (by 8 percentage points) than the “average” and “weak” students (by 3 percentage points and 1-2 percentage points respectively) in the grade 11 test.

The differences between the “weak” and “good” groups can be found in the understanding of concepts (question 5) and the transfer between different forms of representation (between graph and equation - questions 8 and 9)). The lack of performance increase in the case of weak students is attributable to the greater cognitive challenges posed by calculus, which may have taken some students to the limits of their capacities so that they were no longer able to follow lessons (“dropout effect”).

4. THE SYMBOLIC-CALCULATOR-TESTS (SC-TESTS)

4.1 Research questions

In February and in June the pilot classes took a test where they were allowed to use the SC. Use of the SC was optional for the students, i.e. they decided themselves whether or not they would use the calculator. The two tests consisted of four questions each.⁵ In order to establish how calculators were used, we applied a new investigation method: the students completed a questionnaire on SC-use immediately *after the test*, giving details of whether and how they used the calculator. This test was intended to answer the following questions:

- 1 How do students use the calculator?
- 2 In which phases of a problem solving process do the students use the calculator?

⁵ See: www.dmuw.de/weigand/2009/CERME6/

3 Which functionalities (symbolic – graphic – numerical) do the students use?

In addition, the teachers were presented with a questionnaire regarding the questions immediately *before the test*, in which they were intended to provide details of the difficulties expected in terms of the questions.

In the following, only a few spotlights of the results will be given.

4.2 Actual use of the SC

The following diagrams show how many students used the SC during the tests in February and in June – according to their own statements:

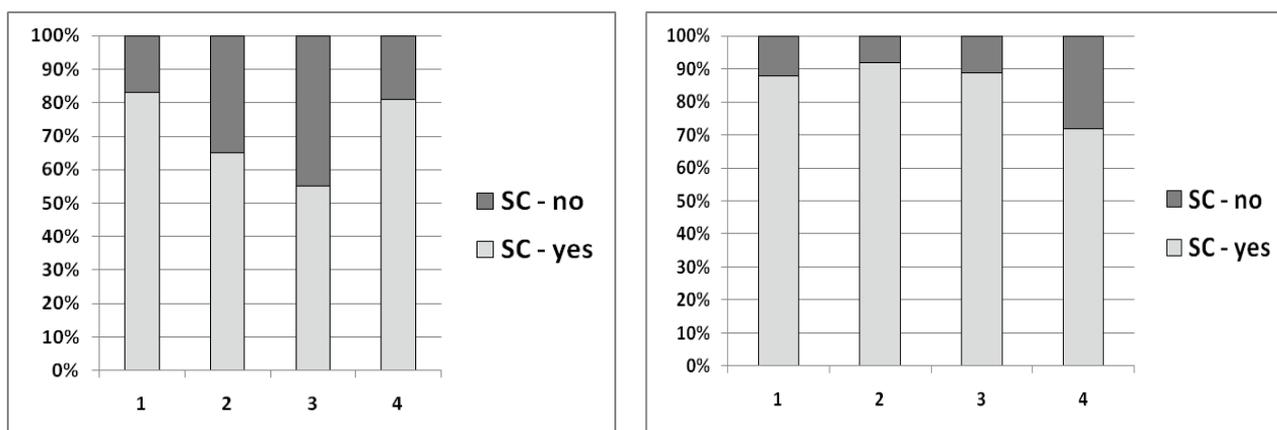


Figure 4: Results of the SC-test in February (left) and June (right) 2007

The difference between SC use in February and in June shows an increase in use of the calculator. More over, those students who used the SC in June when solving the questions scored significantly better than those who did not use it. We attribute this to the fact that it takes a full school year for students to acquire adequate confidence in the SC, as well as knowledge of the benefits of its use as a tool when solving problems, to be able to use these for the purpose of solving problems.

4.3 The SC-use during the problem solving process

The students also provided information in the questionnaire as to whether they used the SC in the beginning, during or at the end of the problem solving process.

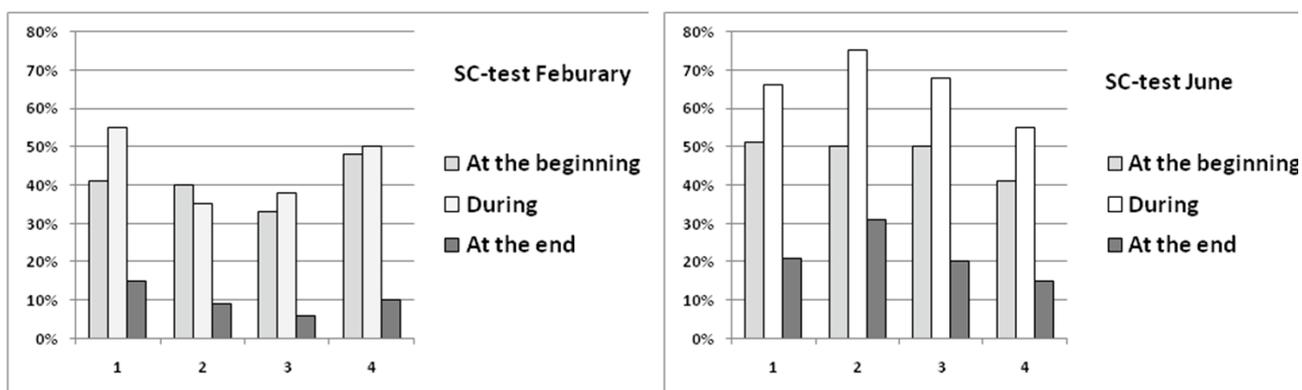


Figure 5: Use of the SC during the course of the solving process (according to statements made by the students themselves)

When students integrate the SC into their solving process, it is predominantly used at the beginning and during the solving process. If we compare the middle of the school year with the end, we can observe a clear increase in the frequency of positive responses to “during”. This allows us to conclude that the SC is more strongly integrated into the solving process by the students at the end of the school year. A slight increase can also be observed “at the end”, which makes us aware that the use of checking the solution is gaining in importance.

We also asked the students which representations they used while solving a problem with the SC. It appears that the students mainly use the symbolic and graphic possibilities of the SC. Numeric use is very limited. More over they are not familiar with the special advantages or disadvantages of the representations nor do they use the relationship between the different representations. The type of the used representation depends on the one hand very strongly on the way problems are given to the students. If it is asked for a “solution of an equation”, they mainly work on a symbolic level, if it is asked for an “intersection point of two graphs” they work on a graphic level. This shows that the SC is used in a very mechanical way, guided not by the type of problem but by the expressions used in the problem. On the other hand the type of use depends also very strongly on the classes and indicates the significance of the teacher and his or her didactic approach.

4.4 Teachers' predictions

Before each test was carried out, the teachers provided an assessment of the extent to which students would solve the problems. The question has been: “For each problem, a student gets 100 % of the marks for a completely right answer. What do you suggest will be the average score of marks your class gets for problem 1 (2, 3, 4)?” The results are as follows:

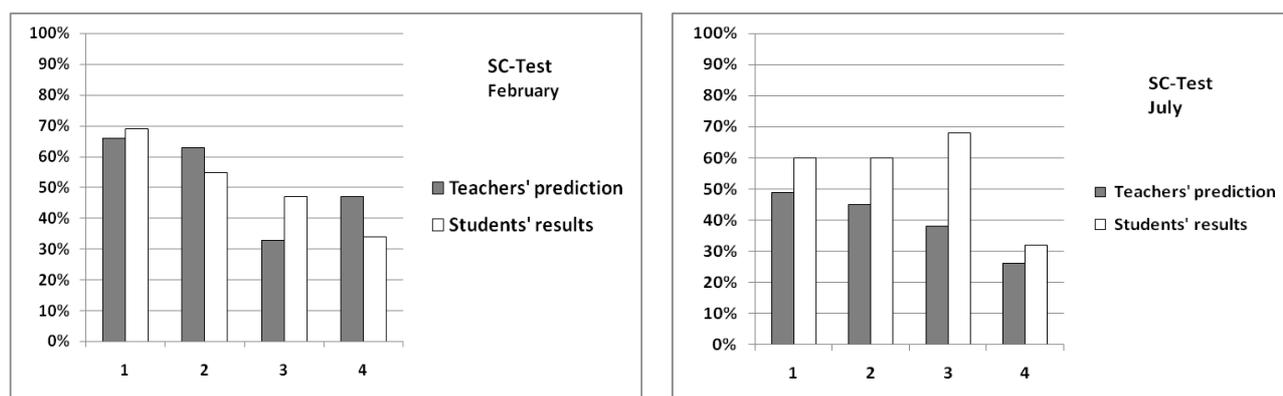


Figure 6: Comparison of teachers' predicted and student results in the SC tests

It is noticeable that the teachers underestimated the students in the June test.

5. Questions for the future

If we summarise the core results of this one-year school project there are some questions for up-coming investigations.

- **Methodology of pre- and post-tests.**

Hopes have not been fulfilled that students in the pilot classes would improve to a greater degree in terms of dealing with and interpreting graphs than students in the control classes. The hypothesis is that students in the pilot classes are not have been adequately challenged or motivated as the result of the largely traditional nature of the test problems. This raises the question whether the used pre- and post-test methodology is an adequate method to answer this question.

- **Polarisation.**

When working with new technologies, polarisation occurs in that some students benefit greatly from SC use, whereas for other students, SC use inhibits performance or even decreases performance. Two thirds of students are of the opinion that the SC was helpful and made them more secure and they classify lessons as “interesting”. Approximately one third of students do not share this view. Are there ways to get all students convinced of the benefits of the SC?

- **Calculator use.**

The reasons for non-use of the calculator are on the one hand the uncertainty of students regarding technical handling of the unit and on the other hand a lack of knowledge regarding use of the unit in a way which is appropriate for the particular problem. Is there a correlation between these two aspects?

- **Period of adjustment.**

The responses of the students confirm that familiarity with the new tool requires a very long process of getting used to it. It is surprising that it took almost a year to establish familiarity with this tool for students to use it in an adequate way. After one year of SC use, confidence in and familiarity with the SC grow. However there is still

a large group of students who experience technical difficulties when operating the SC. Will there be ways to shorten this period of adjustment?

- **Solution documentation.**

Students have problems how to record solutions when using the SC. Difficulties with the type and manner in which to document the solution decreased over the year, but still remain at a high level. This latter point will continue to be a permanent challenge when working with the SC, as there is no algorithmic solution for the procedure. Are there documentation rules for all or a special type of problems?

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ENHANCING FUNCTIONAL THINKING USING THE COMPUTER FOR REPRESENTATIONAL TRANSFER

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The area of functional thinking is complex and has many facets. There are several studies that describe the specific difficulties of functional thinking. They show that the main difficulties are the transfer between the various representations of functions, e.g. graph, words, table, real situation or formula, and the dynamic view of functional dependencies (process concept of a function). Interactive Geometry Software allows the visualization of the dynamic aspect of functional dependencies simultaneously in different representations and offers the opportunity to experiment with them. The author presents and discusses the potential of two interactive learning activities that focus on the dynamic aspect of functional thinking in a special way. Some preliminary results from a first adoption of the activities in class are presented. Resulting research questions and plans for further research are stated.

Keywords: Functional thinking, representational transfer, Interactive Geometry Software, Interactive learning activity, empirical study.

THEORETICAL BACKGROUND

Functional Thinking – Concept and Relevance

In Germany the term 'functional thinking' was first used in the 'Meraner Reform' of 1905. The 'education to functional thinking' was a special task of the reform. Functional thinking was meant in a broad sense: As a common way to think which affects the whole mathematics education (Krüger 2000). In the 60s and 70s the impact of functional thinking in the above sense on the mathematics curriculum in Germany was very low. Since the 80s it regains importance although not in the broad sense of the Meraner Reform. A common definition of functional thinking derives from Vollrath (1989): 'Functional thinking is the typical way to think when working with functions'. Functional thinking in this sense is strongly connected to the concept of function. In the German mathematics curriculum the 'idea of functional dependency' is one of five central competencies, which form the mathematics education (Kultusministerkonferenz 2003).

The concept of function and functional thinking includes many aspects and competencies: On one hand functional dependencies can be described and detected in several representational systems like graphs, words, real situations, tables or formulas. On the other hand the nature of functional dependencies has different characteristics (Vollrath 1989 or Dubinsky, Harel 1992): Functional dependency as a pointwise relation (horizontal, static aspect), functional dependency as a dynamic process (aspect of covariation and change, vertical aspect), Functions viewed as objects or as a whole.

There are many studies (e.g. Janvier 1978, Müller-Philipp 1994, Swan 1985, Kerslake 1981) describing the following main difficulties and misconceptions concerning functional thinking:

The interpretation of functional dependencies in different representations and the representational transfer is a main difficulty. Especially the interpretation of functional dependencies in situations and the transfer to e.g. the graphical representation and vice versa causes problems. For example: graphs are often interpreted as photographic images of real situations ('graph-as-image misconception'), which is mainly caused by the inability to interpret the functional dependency dynamically. Especially distance-time graphs are often interpreted as movement in the plane.

The above difficulties were affirmed by written tests the author gave to either 10th class students and to university students who just started their study on mathematics. Based on the problems in the test the interactive learning activities, which we describe below, were built. Figure 1 shows one of the problems (Schlöghofer 2000) from the tests.

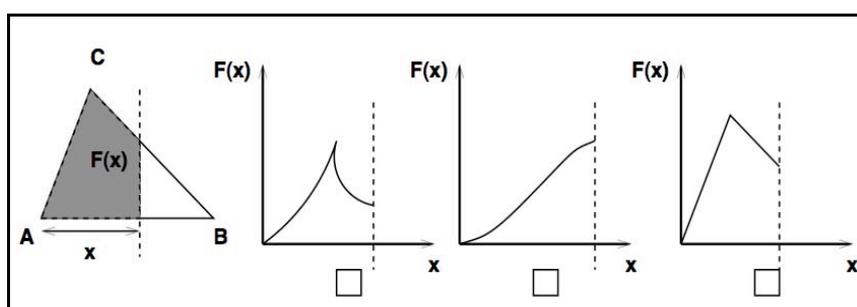


Fig. 1: The dashed line moves rightwards. $F(x)$ is the area of the grey part of the triangle dependent on the distance x . Which graph fits and why?

Only 66% of about 100 university students made their cross at the graph in the middle. Giving the problem to sixteen 9th and 10th grade high school students, resulted in only 37% correct answers. The main mistake was to put a cross at the graph on the right side. The reason for this choice was usually given by a statement like: The area [of the graph on the right side] is just like the area $F(x)$.

The chances of Interactive Geometry Software

When using the computer in classrooms on the topic functions one might think immediately of using Computer Algebra Systems (CAS). Most studies about the use of the computer when working with functions are about using CAS, e.g. Müller-Philipp (1994), Weigand (1999), Mayes (1994). While CAS is input/output based and gives back information and changes asynchronously, the use of Interactive Geometry Software (IGS) allows interactivity and gives immediate response. This difference will be used to emphasize the dynamic view of functional dependencies.

Especially the software Cinderella includes a functional programming language called *CindyScript*. This enables the teacher to create learning activities and own

teaching material like the ones described below by using standard tools (Kortenkamp 2007).

DESIGN OF THE ACTIVITIES AND CONCEPTUAL BASIS

Main research question

The learning activities are designed with regard to the following research question:

Is it possible to enhance the dynamic aspect of functional thinking by dynamically visualizing functional dependencies simultaneously in different representations and by giving the opportunity to experiment with them?

General design ideas and concept

We developed two interactive learning activities (joint work with Andreas Fest). The activities consist of single Java applets embedded into a webpage and can be used without prior installation with a standard Internet browser. The applets are built with the IGS Cinderella and are accessible by using the links on the webpage <http://www.math.tu-berlin.de/~hoffkamp>.

Figure 2 shows the typical design of a learning activity. Next to the applet there is a short instruction on how to use the applet and some work orders. The students are asked to investigate and describe the functional dependency between the distance A-D and the dark (if coloured: blue) area within the triangle.

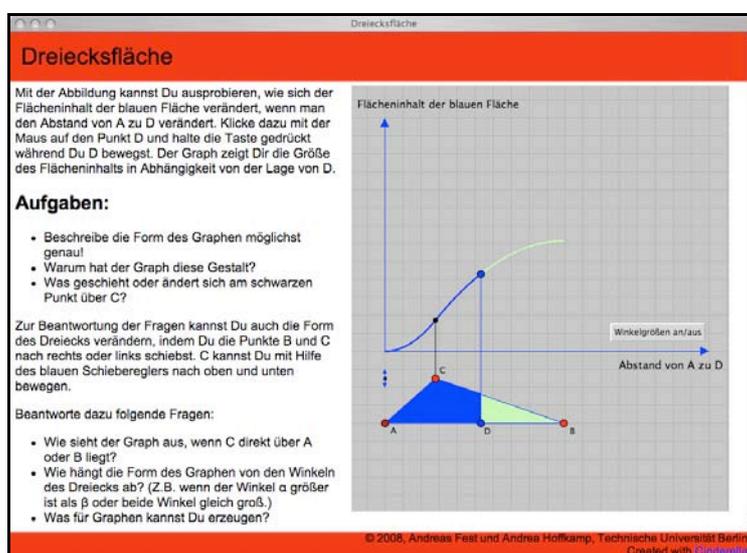


Fig. 2: Interactive activity 'Dreiecksfläche' ('Area of a triangle'). Moving D makes the dynamic aspect visual. Moving B and C changes the triangle and the function itself.

The learning activities have the following conceptual and theoretical ideas in common:

Connection situation-graph: The starting point is a figurative description of a functional dependency, which is simultaneously connected to a graphical representation. The graphical representation was chosen, because it relates to the

covariation aspect in a very eminent way. As analysed by von Hofe (1995) students are able to establish 'Grundvorstellungen' (GV) more easily when an imaginable situation is given. GV's are mental models connecting mathematical concepts, reality and mental concepts of students. Rich GV's of the functional dependencies are necessary to succeed in problem solving processes.

Language as mediator: The students are asked to verbalise their observations in their own words. Janvier (1978) emphasizes the role of the language as a mediator between the representations of the functional dependency and the mental conceptions of the students.

Active processing assumption: According to the cognitive theory of multimedia learning of Mayer (2005) humans are actively engaged in cognitive processing in order to construct a coherent mental representation. The activities are conceptualized as attempt to assist students in their model-building efforts. Therefore the activities allow to experiment with different representations of the functional dependencies. At the same time the actions of the user are limited to focus on the dynamic view of the functional dependencies.

Two levels of variation: The activities allow two levels of variation. First, one can vary within the given situation. This visualizes the dynamic aspect. To understand a dynamic situation one needs to construct an 'executable' mental model to achieve mental simulation. The idea is to support the mental simulation processes visually (Supplantation, Salomon 1994). Secondly, one can change the situation itself and watch the effects on the graph. We will call this **meta-variation**. Meta-variation allows the user to investigate covariation in several scenarios. It is variation within the function that maps the situation to the graph of the underlying functional dependency and changes the functional dependency itself. This leads to a more global view of the dependency. Therefore meta-variation refers to the object view of the function. To understand the covariation aspect one needs to find correlations between different points of the graph in order to describe changes. This requires a global view of the graph. For example the property 'strict monotony' of a graph is a global property and therefore refers to the object view of a functional dependency. But to describe it in terms of 'if $x > y$ then $f(x) > f(y)$ ' one has to understand the covariation of different points of the graph.

Low-overhead technology and practicability: To work with the interactive activities there is no special knowledge of the technology necessary. The activities make use of the students' experience with Internet browsing (actions like dragging, using links, using buttons etc.). The students (and the teachers) can work directly on the problems without special knowledge of the software and the software's mathematical background. This is important especially with regard to time economy.

Learning activity 'Die Reise' ('The journey')

Based on the conceptual ideas above the learning activity 'Die Reise' ('The journey') was developed. Like the activity 'Dreiecksfläche' it is adapted from a problem (Swan

1985) the author gave to university students and 10th grade students within a written test. After using 'Die Reise' in classroom within a first study the activity was worked over. Some results of the study are presented below. The activity in its current version consists of three parts. Part one is about the transfer situation-graph (Fig. 3): A car advances from Neubrandenburg (top of the map) to Cottbus (bottom of the map). The graph shows the corresponding distance-time graph for the journey.

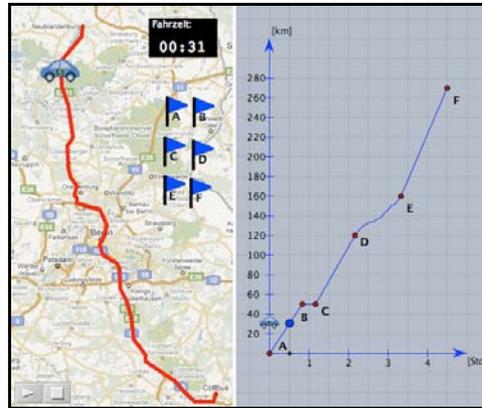


Fig. 3: Applet within the first part of 'Die Reise'. The point on the distance-time-graph is movable. The students are asked to mark the positions A-F with the flags on the map.

Part two of 'Die Reise' (without figure) refers to the first level of variation (visualization of the dynamic aspect in the given situation). It shows the distance-time graph of part one again together with the corresponding velocity-time graph. The work orders aim at interpreting the slopes in the distance-time graph in connection with the velocity-time graph.

Part three refers to the level of meta-variation (figure 4).

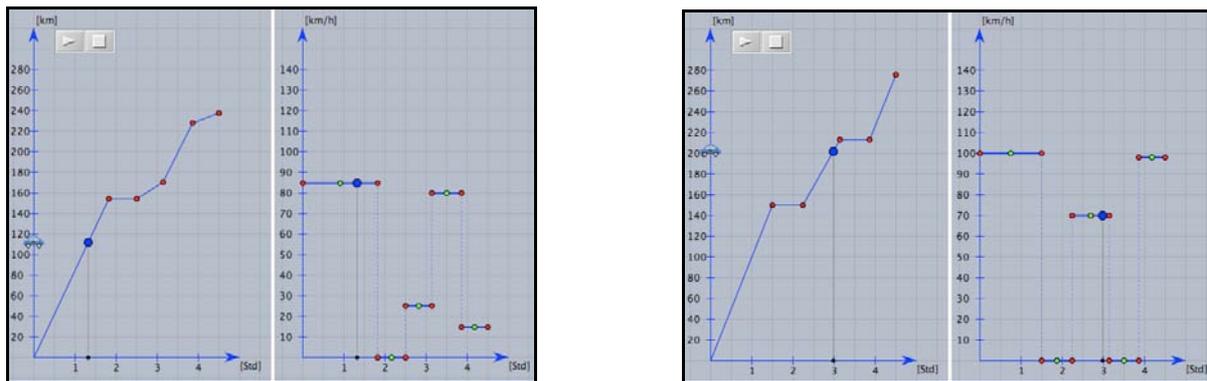


Fig. 4: Meta-variation in 'Die Reise'. Besides moving the points on both graphs, the bars in the velocity-time graph can be moved vertically and the width of the bars can be changed.

FIRST STUDIES

Setting and methods

The activities 'Dreiecksfläche' and 'Die Reise' were tested with 19 respectively 32 secondary school students of age 15-16 (10th class) in a block period of 2x45 minutes in each case. The students were not prepared to either the topic or the special use of technology. A worksheet was prepared which contained the Internet address of the interactive activity and some questions to work on. The students had to start on their own using the instructions of the worksheet. To provoke discussion and first reflection about the problems two or three students worked together. Afterwards the solutions were discussed in class. The results of the studies are based on student observations during their work with the computer, general impression of the discussion in the class, a short written test and a questionnaire. In addition the computer actions and student interactions of one student pair was recorded while working with the activity 'Die Reise'. All teaching material, tests and questionnaires can be found on www.math.tu-berlin.de/~hoffkamp.

The studies were conceived as preliminary studies with the following aims: Test the interactive activities and work them over for further studies, specify further research questions, create a study design for a larger study based on the experiences made.

Results and discussion

Computer-aided work and work with the activities in general:

The concept of low-overhead technology and practicability was successful. The students were able to work with the interactive activities without further instruction. This is also important concerning time economy, especially from the teachers' point of view.

The use of the computer had a very positive effect on the students' motivation. This is caused by many factors. For example the students appreciated to work autonomously in their own tempo following their own train of thoughts. They also highly appreciated that the computer takes over annoying actions like drawing or calculating. This is a crucial point especially for slow-writers and was observed when watching a recorded sequence of the students' working phase. The sequence shows that the order 'Draw a suitable distance-time graph' really blockaded and frustrated the student. The following student statements from the questionnaire confirm the above comments:

Question: Is there something special you like when working with the computer?

Answer 1: It is less monotonous and the lesson is organized differently. You learn by means of a different learning aid, which allows a better imagination. The studios atmosphere is more comfortable. You do not have to follow the group's train of thoughts.

Answer 2: The computer makes the calculations and I do not have to write so much, which means that it cannot be smeared and illegible.

Answer 3: That I can work independently (without teacher). One can use his own mistakes to come to the right result.

Statements like answer 3 were made several times. The students had the impression that they were able to use their mistakes in a productive way. Moreover the computer-aided work allowed for a better internal differentiation of the learner group. Slowly learning students asked the teacher for help more often than more advanced students, but they still worked independently for longer periods.

Effects on functional thinking:

By visualizing the representational transfer dynamically the students were forced to focus on the dynamic aspect of functional thinking and they seem to have established (more or less) adequate mental models integrating the dynamic view. Many student answers on the questionnaire aim at the aspect of 'dynamic visualization':

Question: What is different for you when you use the computer to work on mathematical problems?

Answer: Because of the visualization I am able to watch the problem from different perspectives and this makes it easier to solve it.

Question: Can you say what exactly you understood better by using the computer?

Answer 1: That I could see the problem.

Answer 2: How the graph changes when changing the triangle.

Answer 3: I liked this form of figurative illustration that was given directly when changes were made because it is easier to understand something by watching it.

Answer 4: The motion. When one graph moves although you use the other graph.

The second question was used to find out what aspects of the activities were considered by the students as showing them something new. In this sense many student answers aim at the level of meta-variation. As explicated above the level of meta-variation is connected with the object view of a function, a view, which is not (fully) attained in the age group the author is looking at (Sfard 1991). The student answers lead to the assumption that meta-variation makes the object view accessible for cognitive processes (in a may be implicit way) and could be a step towards the perception of a function as an object. This assumption is strengthened by the results of the tests. Figure 5 and 6 show some results from the written test.

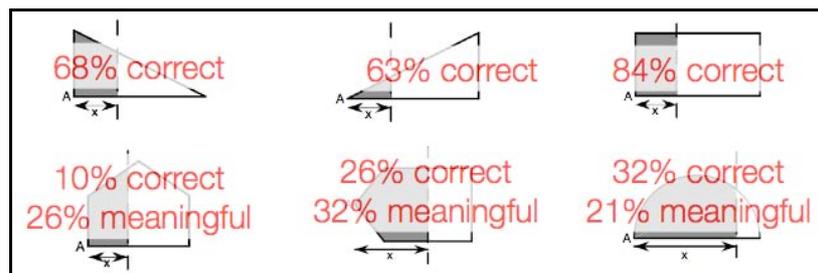


Fig. 5: The students had to sketch graphs describing the dependency between x and the grey area $F(x)$. The figure shows percentages of correct and meaningful graph sketches. An answer was 'meaningful' when the graph was strictly increasing, but e.g. left and right turn were mixed up.

As seen from figure 5 the students by majority seemed to have created an 'idea' of the dynamics of the functional dependency as far as the solution of problems like the one above is concerned, although it was still difficult to adapt the concept to other situations (here: other forms in line two of figure 5). However the students got aware that changes, variations, certain points (e.g. inflection points) and properties (e.g. symmetry, monotony) have a graphical correspondent, which gives qualitative information about the functional dependency.

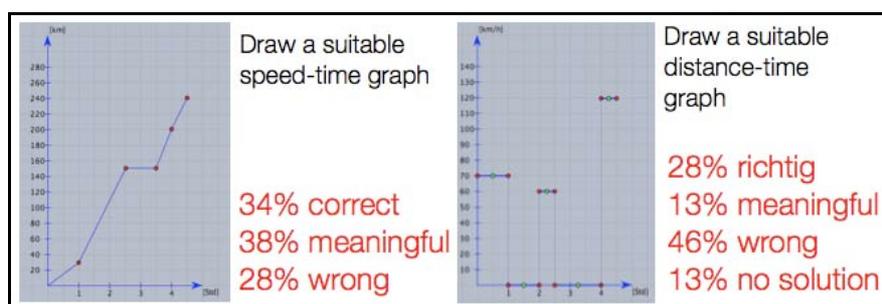


Fig. 6: The students had to draw suitable graphs to the given graphs above. Graphs were 'meaningful' when the graphs had 'correct shapes', but some slopes were done wrong.

Figure 6 shows some results of the post-test within 'Die Reise'. Most of the students were able to draw suitable speed-time graphs to given distance-time graphs. The other way round – from speed-time graphs to distance-time graphs – was more difficult. The results confirm the assumption that the potential of meta-variation in order to enhance the understanding of the dynamic aspect of the functional dependencies seems to be high. Furthermore the activities seem to allow an easy qualitative approach to concepts of calculus. In case of 'Die Reise' the applets visualize the physical intuition of the fundamental theorem of calculus. When discussing the question 'Can you see from the speed-time graph how far the journey is?' in class, the students finally ended up with an intuitive concept of integration.

The results from the preliminary studies lead to the following hypothesis: *Although the object view is more advanced, it facilitates the understanding of the covariation aspect and the establishment of mental models with regard to the dynamic view of functional dependencies.*

The class discussion of the results – which mainly consist of verbalisations of the properties of the functional dependency – ran pretty smooth. The students were highly engaged in making contributions to the discussion. But it was obvious that there was a high need for reflection of the students' train of thoughts since the student answers were mostly superficial. Concerning 'Die Reise' some test results showed,

that the 'graph-as-image misconception' is very persistent in the sense of interpreting distance-time graphs as movement in the plane. Based on these experiences the first two parts of the learning activity 'Die Reise' were modified to their current version.

OUTLOOK

The preliminary results of the first studies give valuable hints for the direction of further research. Basing on the conceptual ideas described above a third interactive activity will be developed and pretested. It is planned to conduct a larger qualitative study using the three activities. The leading question is how the work with functions within the activities affects functional thinking itself. The level of meta-variation is a central idea. It leads to the concepts of calculus and may be used as a qualitative approach to school-analysis in the context of propaedeutics.

The following research questions are of interest and will guide our future research:

Main question: Is it possible to enhance the dynamic aspect of functional thinking by dynamically visualizing functional dependencies simultaneously in different representations and by giving the opportunity to experiment with them?

Further questions:

- Do the students establish GVs concerning the dynamic view of functional dependencies? What sort of GVs do the students establish?
- Which elements of the applets have a positive effect on the dynamic view of functional dependencies?
- Is it possible to distinguish types of students who get along better or worse with the learning units?
- How do slow learners deal with the units compared to more advanced students?
- How can we use computer-based activities like these as diagnostic tools?

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THE ROBOT RACE:

UNDERSTANDING PROPORTIONALITY AS A FUNCTION WITH ROBOTS IN MATHEMATICS CLASS

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This paper presents and discusses the use of robots to help 8th grade students learn mathematics. An interpretative methodology was used and data analyses were supported by Situated Learning Theories and Activity Theory. These tools allowed the accurate description and analysis of student's practices in mathematics classes. The results indicate that the use of robots to study proportionality as a function aided and supported student learning.

INTRODUCTION

Educational systems the world over are investigating new and engaging mechanisms in order to better present complex concepts and challenging domains such as mathematics. The implementation and exploration of technologies in classrooms is a promising general approach. However, we should not neglect the real world where the actual students live – a world more and more dependent on technologies. Consequently, it is essential to combine computation aids and new educational aims with a redefinition of teaching processes and teachers role's in the classroom. It is in this context that the project DROIDE was initiated in 2005.

DROIDE²: “Robots as mediators of Mathematics and Informatics learning” is a project with three main objectives:

(1) to create problems in Mathematics Education/Informatics areas which are suitable to be solved using robots; (2) to implement problem solving using robotics at three points in the educational system: mathematics classes at K-9 and K-12 levels; Informatics in K-12 levels; Artificial Intelligence, Didactics of Mathematics and Didactics of Computer Science/Informatics at the university level; (3) to analyze and understand students' activity during problem solving using robots.

This paper discusses research on the second issue (the implementation of problem solving using robots in mathematics class) at the K-9 level. It addresses the following research problem: to describe, analyse and understand how students learn mathematics using robots as mediators of learning. It particularly focuses on the mathematical concept of proportionality as a function.

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THEORETICAL BACKGROUND

The research approach is derived from Situated Learning Theories (Lave & Wenger 1991, Wenger, 1998, Wenger et al, 2002) and Activity Theory (especially the 3rd generation introduced by Engeström, 2001). A key element of Situated Learning theories is the notion of a community of practice and the suggestion that learning is a situated phenomenon. In this paper, this viewpoint is used to reflect upon emergent learning within students' mathematical practices.

The Concept of Practice

According to Wenger, McDermott and Snyder (2002) practice³ is constituted of a set of “work plans, ideas, information, styles, stories and documents that are shared by community members” (p.29). Practice is the specific knowledge that the community develops, shares and maintains. Practice evolves as a collective product integrated in participants' work and the organisation of knowledge in ways that are useful and reflect the community's perspectives (Matos, 2005).

Wenger (2002) proposes three dimensions in which practice is the source of coherence in a community: mutual engagement, joint enterprise and shared repertoire. Mutual engagement is a sense of “doing things together”; the sharing of ideas and artefacts, with a common commitment to interaction between community members. Joint enterprise is having (and being mutually accountable for) a communal common goal, a procedure which rapidly becomes an integral part of practice (Matos, Mor, Noss and Santos, 2005). Shared repertoire refers to a set of agreed resources for discussions and negotiations. This includes artefacts, styles, tools, stories, actions, discourses, events and concepts.

The Concept of Mediation

Engeström (1999) conceptualizes an activity model formed by three elements – the subject, the object and the community – with mediation relations between them. In the context of this research, the mathematics classroom forms such an activity system. The subject is part of a collective; reflecting the fact that we do not act individually in the world. The subject is part of a system of social relations.

The concept of mediation has a central role in Activity Theory⁴. It is based on the presupposition that the subject does not act directly on the environment; that it has no direct access to the objects. The relation between subject and object is mediated by artefacts (Werstch, 1991); things constructed by individuals and maintaining a dialectic relation between people and activity (Werstch, 1991). To say that a tool or

³ The term practice is sometimes used as an antonym for theory, ideas, ideals, or talk. In Situated Learning theories that is not the idea. In Wenger's sense of practice, the term does not reflect a dichotomy between the practical and the theoretical, ideals and reality, or talking and doing. The paper extension does not allow the development of the idea of practice. For discussion of practice related with mathematics education see Fernandes (2004).

⁴ For a more general vision of Activity Theory see <http://pparticpar-t-act.wikispaces.com/>

artefact is mediator of learning means that it gives power to the process of transformation of objects; that it is a tool with which people think (Piteira, 2000).

This paper claims that robots can be artefacts, mediators of the learning of functions. The veracity of this claim is demonstrated in the following sections.

METHODOLOGY

The work reported in this paper was organised into three stages:

First stage – analysis of School Mathematics and Informatics curriculum; selection of didactical units where robotics can be used; creation of problems/tasks to be solved in Mathematics and Informatics classes.

Second stage – implementation of problems/tasks in Mathematics and Informatics classes; data collection through video recordings of students.

Third stage – analyses of student activity during learning with robots using interpretative methods introduced in Situated Learning Theories and Activity Theory. The unit of analysis was “(...) the activity of persons-acting in setting” (Lave, 1988, p.177).

LEARNING AS PARTICIPATION: ANALYSING STUDENTS MATHEMATICAL ACTIVITY WHEN USING ROBOTS

A brief description of mathematics class

In mathematics classes students worked in small groups. In the initial phase, the work involved construction of the robots and basic programming to solve simple tasks. This activity took place on a Windows® desktop environment and the students used a visual programming tool that ships with the robot kits. Subsequently, students used the robots to recognise and apply concepts in coordinate systems, to understand the meaning of function, to represent one function (proportionality) using an analytic expression and to intuitively relate a straight line slope with the proportionality constant, in functions such as $x \mapsto kx$.

General plan of work for functions unit

The first mathematical unit students worked on involved functions. Four sets of problems were prepared. **Problem set 1** presented examples and counterexamples of functions explaining things that take place in everyday situations. **Problem set 2** showed more complex graphs (beyond straight lines) and taught students to also recognize them as functions. In **problem set 3** it was intended that students learn proportionality as a function. The definition of proportionality emerged from the mathematical activity of students as they used robots. Finally, **problem set 4** was concerned with affin functions, such as y-intersect and slope. It also dealt with the

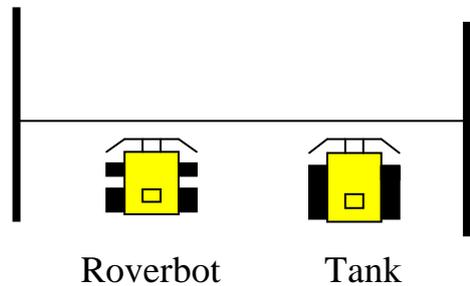
relation between the graphical appearance of these kinds of function those of proportionality shown earlier. This paper⁵ analyses students solving problem 3.

In the classroom

We will describe and analyse mathematical activity of two groups of students. One group consisted of four girls with similar mathematical levels and abilities (**C**, **La**, **Li** and **S**). When they started to work together, they had experienced considerable difficulty, even going so far as to repeatedly suggest that the problem could not be solved, at least individually. Eventually, they understood the problem could be solved if they teamed up and learned to work cooperatively. The other group featured three boys (**M**, **P** and **Ma**), in which one of them had a higher level of mathematical ability than the other two.

The class started with the teacher distributing materials to each group: one robot (either *Roverbot* or *Tank*), one laptop, one tape-measure and a worksheet including the following tasks⁶:

- I. Let's compare the two robots speed: Roverbot and Tank. Probably the first idea that occurs to you is to hold a robot race, to find out which is the quickest. However, that is not the best way to determine speed values and compare them accurately.



- a) Through experimentation of Roverbot (programming, test and registration of data) complete the following table:

Time(seconds)	1	3	6
Distance covered (cm)			

- (i) Calculate the quotient between distance covered and time. (ii) Do the values 'distance covered' and 'time' vary in direct proportion? Justify your answer. (iii) Which is the proportionality constant? In this situation what does the proportionality constant mean? (iv) Comment upon the following affirmation: "The correspondence between the distance covered by Roverbot and the time spent to cover that distance is a function."

⁵ For a more general discussion about mathematical activity of students using robots to learn functions see Fernandes, Fermé and Oliveira (2006, 2007) and Oliveira, Fernandes and Fermé (2008).

⁶ After the realization of several tests we verified that the time that the robot needs to reach the standard velocity as well as the braking time are negligible. So we can assume that, to the end of this question, time and distance covered vary in proportion.

Practice as meaning

According to Engeström (1999), in the structure of an activity we can identify subjects that act over objects, in a process of reciprocal transformations that culminates with the achievement of certain results.

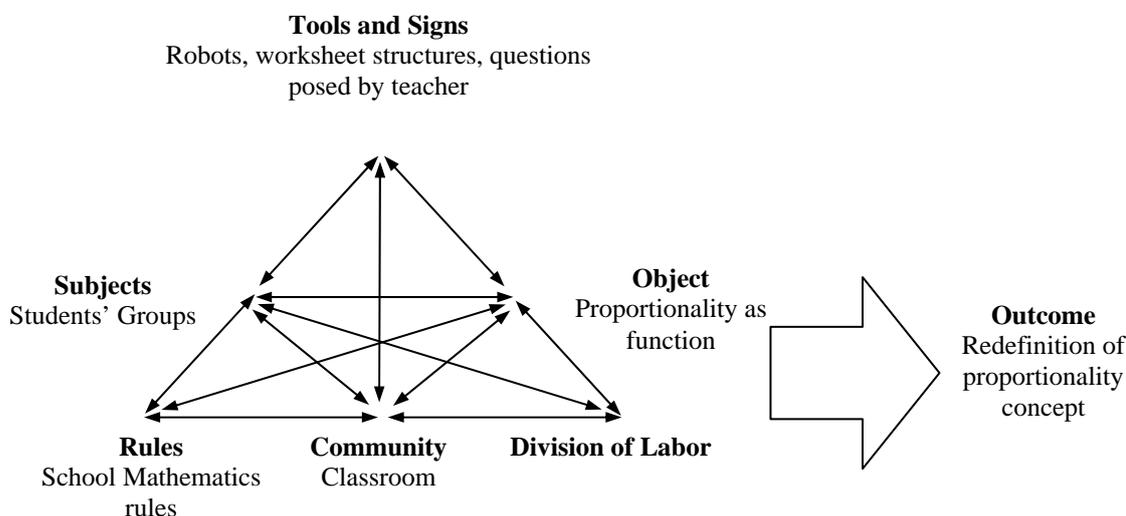


Figure 1 – School mathematics activity structure

Figure 1 shows activity during school mathematics class when robots were used to study proportionality as a function. In this case the term *subject* (figure 1) is collective and is represented by the different groups of students. The *community* is the class and its work methodology. The *object* is the ‘raw material’ at which the activity is directed and which is transformed (with the help of mediating instruments) as its outcome. In the situation considered here, the object is proportionality as a function and the instruments were the robots, the worksheet structure and the way the teacher posed questions to students. The episode described below shows how one of the groups solved the task described above.

Each student read the task. **C** programmed the Roverbot to move forward one second, then measured the distance covered. 33cm was recorded in the table. **S** followed the same process for 3 seconds and they registered 99cm. Then **C** programmed the robot to move forward 6 seconds. However, the desk on which they were working was too short for this last course. **Li** suggested they try out on the floor. This was done and 178 cm was recorded in the table. The students then began to discuss the results for the first time. They started to calculate the quotient between space covered and time, more or less the first times they speak. Their dialogue is shown below:



1. C: $33/1 = 33$ [data recorded on the worksheet].
2. C: $99/3 = 33$
3. Li: $178:6 = 29.6666$
4. S: It can't be. It has to be 33.
5. C: Let's programme and measure all again.



Something is wrong. [They repeat all the process and the values were again 33, 99 and 178].

6. S: But it can't be. It has to be 33 (referring to the value of the quotient between the two variables)
7. La: 33×6 is 198. Let's put 198 on the table.

They erased 178 on the table and wrote 198. Teacher came near to the group and saw 198 (but he had previously seen 178).

8. Teacher: Wasn't the result of measuring 178?
9. C: Yes, but $33/1$ is 33, $99/3$ is 33
10. La: So we changed 178 by 198 because 33 times 6 is 198.
11. S: Let's programme and measure all again.

Meanwhile another group calls teacher. They programmed again the robot to forward one second and then they measured the distance covered over the desk.

12. La: Oh! I know... We measured in two different places. We have to measure always on the floor.

The results obtained of measuring the distance covered were 30, 89 and 178 for 1, 3 and 6 seconds respectively.

13. The results of the quotient were 30, 29,(6) and 29,(6) respectively. Students accepted them as good and answered that time and distant covered are in direct proportion.

Wenger (1998) states that “meaning is a way of talking about our (changing) ability - individually or collectively – to experience our life and the world as meaningful” (p. 5). He describes meaning as a learning experience.

The concept of proportionality is studied in mathematics class from 5th grade onwards. It refers to a constant relationship between two variables and is usually discussed abstractly, such as in the example below:

Verify that there is no proportionality between the following variables.

a	13	26	39	52.08
b	1	2	3	4

Many times, in school mathematics, proportionality is discussed without context; only numbers matter and the emphasis is on the mathematical concept instead of in the meaning of mathematical concept. This process makes difficult the learning experience in Wenger (1998) sense.

In the episode presented above, the students believe that the variables time and distance should be in proportion. Analysing the episode we can not determine the origin of that belief. But we can conjecture that it comes from the presence of the robot (a car) or from the way the question is written in the worksheet (question iii). Although we are guessing at its source, it is clear that the idea of proportionality is meaningful for the students, as they choose to recapture their data several times in the face of results that violate this principle. Only when an inconsistency appears, do the students begin to discuss where they made a mistake and what to do in order to solve it. But the idea that time and distance should be in proportion is really meaningful for them. This can be seen when they changed the result (from 178 to 198) to ensure that the calculations adhere to the rule and neglecting the fact of the last quotients are not equal. In spite of the evidence of the measurements, students believed that values should be in proportion. This shows that the ‘dogmatic’ knowledge of direct proportionality is more *entrenched*⁷ than their confidence in their ability to successfully run experiments and, consequently, they neglected the evidence of the experiment.

The use of unusual artefacts in mathematics class (tape-measure, robots, laptops) associated to a methodology of work where students can stand up, measure, program the computer and experiment with data helped students to construct and rebuild meaning about the concept of proportionality.

From the perspective of activity theory, students groups acted on robots, which were mediators’ elements, between them and the object. The robots were a facilitator of activity that they empowered students during the process of object transformation.

In the second student group, students had a different experience. After programming the robot for 6 seconds they had the following discussion:

M: It’s 172 cm [referring to the space covered by the robot in 6 seconds].

P: 172?

M: 172 or 173.

⁷ The term entrenchment refers to Goodman (1954). He claims that the criterion to decide between two predicates (in our case, the rule and the evidence) is the degree of entrenchment of the predicates. The entrenchment of a predicate depends of the history of the past projections and their success or failure. In our case, the students have more history records where they must leave their proper ideas when confronted with the formal concepts (teacher knowledge, textbook).

P: But it can't be. It's not correct. It should be 180. And the other value should be 90 [referring to the space covered by the robot in 3 seconds].

Ma: Why?

P: I have done it in the calculator. If in one second the robot covers 30cm, I multiplied it by 3 and it's 90. And for 6 seconds it is 180.

M: But it's not correct. Aren't you seeing the tape measure? It's 173cm.

In this dialogue we can notice that one of the students of the group knows the scholarly notion of proportionality well and applies it to compare with the results of the experiment. He seems to trust more in the mathematical rules that he knows than in the evidence of the measurement experiment.

The two students groups reacted differently to the inconsistency between mathematical rules and the empirical evidence: one believed the values they obtained through measurement and considered that the values they obtained by approximation from the quotient were enough to guaranty the proportionality (as shown the episode above); the others calculated values after they knew the space covered by the robot in one second. Where does this difference in attitude (in the face of the same evidence) come from?

The division of labour (figure 1) refers both to the horizontal division, of the tasks between different members of the community, and the vertical, of power and status. The vertical division of labour is connected with the fact that, in the groups, there are students with more power than others (due to their superior performance in mathematics class, assessed through evaluation by their co-students) and these lead the search to solve the problem. Therefore, by analysing the horizontal division of labour we can say that it has emerged naturally between different students of the groups and represents the way how they organized their work in order to solve the problem proposed by teacher.

Finally the *rules* (figure 1) refer to the explicit or implicit regulation, to norms and conventions that constrain actions and interactions in the activity system. What students believe to be mathematics class, the way they see mathematical rules, the way they interpret the question put by the teacher and the worksheet structure (that is connected with the way they see mathematics class and mathematics) impose a certain form to the students' actions. As we have said before we have two different reactions to the inconsistency between correctness of mathematical rules and the inexactness of the empirical evidence – for one group the rules won and for other the empirical evidence.

FINAL CONSIDERATIONS

Robots helped students to renegotiate the meaning of proportionality that they had previously encountered (during seven years of school mathematics) as depending uniquely and exclusively of the quotient between two variables. The negotiation of the meaning evolves through the interaction of two process – participation and reification (Wenger, 1998). When concepts are presented to students as reified objects participation (in Wenger's sense) becomes difficult. Learning through experience, essentially negotiating meaning through participation, helps students' better grasp mathematical concepts. Most of the students in the study described here redefined the concept of proportionality as a function directly because of the work done in this mathematics class and the robots had an important role in this process (Fernandes *et al.*, 2006, 2007, Oliveira *et al.* 2008). Furthermore, as this result was not explicitly intended. Instead, it was an emergent aspect of the students' mathematical practice and study of functions. In the course of their experience with robots, students transitioned from the abstract perfection of mathematics (the definition of proportionality in school mathematics) to the practical reality (proportionality in action) of everyday experience.

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INTERNET AND MATHEMATICAL ACTIVITY WITHIN THE FRAME OF “SUB14”

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In this paper we analyze and discuss the use of ICT, particularly the Internet, in the context of a mathematical problem-solving competition named “Sub14”, promoted by the University of Algarve, Portugal. Our purpose is to understand the participants’ views regarding the mathematical activity and the role of the technology they’ve used along the competition. Main results revealed that the participants see the usage of Internet quite naturally and trivially. Regarding the mathematical and technological competences elicited by this competition, evidences were found that develops mathematical reasoning and communication, as well as it increases technological fluency based on the exploration of everyday ICT tools.

A GLIMPSE OVER THE MATHEMATICAL COMPETITION “SUB14”

Sub14 (www.fct.ualg.pt/matematica/5estrelas/sub14) is a mathematical problem-solving web based competition addressed to students attending 7th and 8th grades.

It comprises two stages. The *Qualifying* consists of twelve problems, one every two weeks, and takes place through the Internet. The Sub14 website is used to publish every new problem; it provides updated information and allows students to send their answers using a simplified text editor in which they can attach a file containing any work to present their solution. The participants may solve the problems working alone or in small teams and using their preferred methods and ways of reasoning. They have to send their solution and complete explanation through the website mailing device or using their personal e-mail account. Every answer is assessed by the organizing committee, who always replies to each participant with some constructive feedback about the given answer.

The word problems are selected according to criteria of diversity and involve several aspects of mathematical thinking not necessarily tied to school mathematics. Their aim is to foster mathematical reasoning, either on geometrical notions, numbers and patterns, and logical processes, among others. There is a concern on presenting problems that allow different strategies and also some that have multiple solutions.



In Iona’s class the students had to elect a delegate and a co-delegate. Each student wrote two names in a voting sheet by order: the first for the delegate and the second for the co-delegate. There are 13 students in the class. How many ways have a student to vote if his or her own name is allowed?

Fig. 1: A problem aiming to elicit the abilities of organizing and counting

The *Final* consists of a one-day tournament where the finalists solve five problems, individually, with paper and pencil, and explain their reasoning and methods. This *Final* also provides some recreational activities addressed both to contestants and accompanying persons, namely parents and teachers.



Joanna, Josephine and Julia are all very fond of sweets. As the summer approaches they decide to go on a diet. Their father has a large scales and they used it to weigh themselves in pairs.
 Joanna and Josephine together weighed 132 kg
 Josephine and Julia together weighed 151 kg
 Julia and Joanna together weighed 137 kg.
 What is the weight of each one?

Fig. 2: A problem from the Final on identifying and relating variables and numbers

Demanding a clear description of the reasoning, methods and procedures was a strong concern of the committee. Moreover, the feedback sent to each participant had an essentially formative role (Diego & Dias, 1996), aimed at stimulating self-correction and valuing students' own ideas. Every two weeks the Sub14 committee publishes a proposal of the solution of the previous problem, stressing the diversity of strategies that students could have applied. Hence, the committee selects noteworthy excerpts from student's solutions, whether due to the originality of their reasoning, their creativity or the interesting usage of technological tools.

A THEORETICAL FRAMEWORK

In this paper we are addressing a part of a larger study and consequently we refer to a few theoretical aspects of the overall framework. There are four main focuses in the theoretical approach: (a) looking at mathematics as a human activity, (b) taking problem solving as an environment to develop mathematical thinking and reasoning, (c) exploring the concept of being mathematically and technologically competent and finally (d) considering the role of home ICT in out of school mathematics learning.

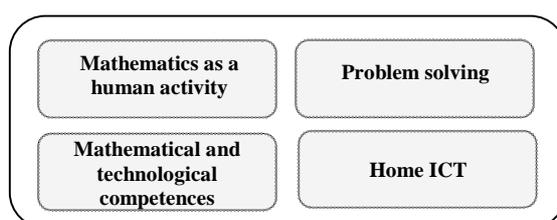


Fig. 3: Main conceptual elements of the theoretical framework

Mathematics as a human activity

Doing mathematics may be recognized as a human activity based upon a person's empirical knowledge, in search of a formalized understanding of the everyday problematic situations. From this point of view, Freudenthal (1973, 1983) states that human activity, which comprises empirical knowledge, guides oneself from the simple observation and interpretation of phenomena – horizontal mathematizing – to its abstract structuring and formalization – vertical mathematizing.

One of the criteria observed in launching a problem in Sub14 refers to the expectation that participating students will be able to activate their empirical knowledge and their experience to tackle mathematical problems. This perspective on mathematical activity is shared by many authors who emphasize the importance of exploring mathematical situations starting from common sense knowledge (Hersh, 1993, 1997; Ernest, 1993; Ness, 1993; Matos, 2005). As Schoenfeld (1994) claims, easiness in the use of mathematical tools, like abstraction, representation or symbolization, does not guarantee that a person is able to think mathematically. Rather mathematical thinking requires the development of a mathematical point of view and the competence to use tools for understanding.

This is the perspective that is present in Sub14 and which expresses the prevailing concept of mathematical activity arising from the perspective of Realistic Mathematics Education: bringing student's reality to the learning situation so that he/she is the one who does the mathematics, drawing on his/her knowledge and resources.

Mathematical knowledge and problem solving

Several authors from the field of mathematics education have proposed problem solving as a privileged activity “for students to strengthen, enlarge and deepen their mathematical knowledge” (Ponte et. al, 2007, p. 6).

This view on mathematical problem solving entails a conception of mathematical knowledge that is not reducible to proficiency on facts, rules, techniques, and computational skills, theorems or structures. It moves towards broader constructs that entails the notion of mathematical competence (Perrenoud, 1999; Abrantes, 2001) and problem solving as a source of mathematical knowledge. In solving a problem there are several cognitive processes that have to be triggered, either separately or jointly, in pursuing a particular goal: to understand, to analyze, to represent, to solve, to reflect and to communicate (PISA, 2003).

According to Schoenfeld (1992), the concept of mathematical problem can move between two edges: (i) something that needs to be done or requires an action and (ii) a question that causes perplexity or presents a challenge. The educational value of a problem increases towards the second pole where the solver has the possibility of coming across significant mathematical experiences. One of the purposes of mathematical problems should be to introduce and foster mathematical thinking or adopting a mathematical point of view, which impels the solver to mathematize: to model, to symbolize, to abstract, to represent and to use mathematical language and tools (Schoenfeld, 1992, 1994).

The formative aims of the problems proposed in Sub14 are essentially in line with the perspective of giving students the experience of mathematical thinking and also the opportunity to bring forth mathematical models and particular kinds of reasoning.

Communication, home technologies and learning

Considering that mathematics is a language that allows communicating your own ideas in an accurate and understandable way (Hoyles, 1985), Sub14 intends to develop that relevant communicational aspect, as stated in the current National Curriculum: “students must be able to communicate their own ideas and interpret someone else’s, to organize and clearly present their mathematical thinking” and “should be able to describe their mathematical understanding as well as the procedures they use” (Ponte et. al, 2007, p. 5). Conversely, the importance of developing the competence of mathematical communication draws on a strong connection between language and the processes that structures human thought, as it is referred by Hoyles (1985). Accordingly, language takes up two different roles in mathematical education: communicative, where students show the capacity to describe a situation or reasoning act; and cognitive, which may help to organize and structure thoughts and concepts. Hence, there is a multiplicity of capacities and competences, both mathematical and technological, which are triggered through the combination of facts and resources in order to solve each problem of the competition.

Technologies and particularly the Internet, which gave life to Sub14, had a somewhat “neutral” or “trivial” role since the main focus of students’ concerns was on the actual mathematical activity involved. Noss and Hoyles (1999) used the “window image” to emphasize this phenomenon: a window allows us to look beyond, and not only at the object itself. Although every new technology tends to draw attention to the tool itself, we soon need to “forget” the tool and concentrate on the potentialities it has to offer, namely on the learning and cognition field.

Using Lévy’s (1990) ideas, Borba and Villarreal (2005) claim that technology mediates the processes that are responsible for the rearrangement of human thinking. In fact, knowledge is not only produced by humans alone, but it’s an outcome of a symbiotic relationship between humans and technologies – which the authors entitled *humans-with-media*: “human beings are impregnated with technologies which transform their thinking processes and, simultaneously, these human beings are constantly changing technologies” (p. 22).

Indeed, human thought used to be defined as logical, linear and descriptive. Nowadays it is hastily changing into a *hypertextual thinking*, comprising many forms of expression that go beyond verbal or written forms, such as image, video or instant messaging. These social changes allow youngsters to develop a large number of competences, which grants them the skills and sophistication required to learn outside the school barriers.

Towards the conclusions of the ImpaCT2 project, that took place in Great Britain, Harrison (2006) asserted that the model used to measure the influence of new technologies on youngster’s school achievement was too simplistic and induced to settle on the absence of such influence. This author then proposed a new model that emphasized the importance of social contexts in which learning takes place. Harrison (2006) was thus able to conclude that learning at home must not be neglected, but be faced as a partner of the school curriculum.

Although knowledge gathered outside the school is frequently seen as worthless, it is clear that children are capable of watching a YouTube's video, talk to their friends through MSN, and also solve the Sub14 problems and express their thinking using an ordinary technological tool. These "digital natives" (Prensky, 2001, 2006) access information very fast, are able to process several tasks simultaneously, prefer working when connected to the Web and their achievement increases by frequent and immediate rewards.

METHODOLOGY

The purpose of this study was to identify and understand the participants' perceptions regarding the (i) mathematical activity, (ii) the competences involved and (iii) the role of the technological tools they've used along the competition.

A case study methodology reveals itself appropriated in cases where relevant behaviours can't be manipulated, but it is possible and appropriate to proceed to focused interviews, attempting to understand the surrounding reality (Yin, 1989). Since we intended to get diversity and interpret results, eleven participants were chosen intentionally, from the 120 finalists, hoping they would provide interesting data according to the research questions.

The field work began collecting data that would allow a complete understanding of the competition, in order to adjust the approach to the participants. Later on, we used other data collecting techniques: a *questionnaire* to the finalists, *video records* from the Final, *documental data* from participants (such as their solutions to the Sub14 problems, or their interactions with the Sub14 committee, using e-mail). That data allowed the planning of interviews to the eleven participants, as well as to their parents, aiming at collecting descriptive data, in their own language, hoping for an understanding on how they viewed certain aspects of Sub14 and of their involvement.

For the data analysis we used an interpretative perspective (Patton, 1990) and an inductive process (Merriam, 1988), based on content analysis. Thus, the objective was to understand the significance of the events from the interviewees' perspective, within the scope of the theoretical assumptions defined prior to the interviews.

THE INTERNET – THE SUB14 LIFE SUPPORT

The first evidence produced about students' perceptions on the problem solving environment was the fact that the Internet and the technologies used within Sub14 assumed, from the point of view of students, a neutral role in the development of their mathematical activity. However several aspects of their products and statements showed evidence of the importance and usefulness of different tools, behind their apparent indifference to technology if put in abstract terms. Therefore, we may state that the Internet undoubtedly is the technology that brings Sub14 to life; all the learning processes and the competences involved derive from the interaction provided and nourished by this tool.

Trivializing Technology

Resorting to the Internet and other technologies was seen as absolutely natural by some participants.

“As I see it, reasoning comes from the mind; therefore I think no technology will help us to really solve a problem.” [Bernardo]

Trivializing the role of the Internet and the technology involved in the competition can be found in the model proposed by Harrison (2006), which highlights the importance of the social context surrounding the learning process. These participants show all the characteristics of a digital native (Prensky, 2001), i.e., they start using computers at an early age, with a great variety of purposes, which can be related or not to school learning. Furthermore, these participants can also be considered as “humans-with-media”, or particularly, “humans-with-Internet”, according to the definitions proposed by Borba and Villarreal (2005), since their personality is being built, simultaneously, through the daily interaction with the Internet and other technologies.

The Role of Communication and Feedback

Essentially, the participants like the feedback sometimes provided immediately by the Sub14 committee, resulting from the analysis of their answers to each problem. The possibility of correcting little mistakes or even change the resolution completely, using the hints from the feedback, increase their self-esteem and motivation to remain in the competition. For the interviewed students, this is the characteristic that distinguish Sub14 from other similar competitions.

“This year I also participated in another competition. We send an answer to a problem, but they don’t reply to us, and the Sub14 committee keeps sending hints”. [Isabel]

As students pointed out there is someone who receives their answer to the problem, their questions or even their complaints.

“It’s not something that we send and no one will care about, they are always there.” [Lucia]

As mentioned above, the feedback is almost immediate and this is only possible due to the communicability that the Internet enables. The constant request for auto-correction forces the participants to reflect on their own reasoning and the mistakes given, stimulating them to submit a correct answer as quickly as possible. Some of them sent messages to Sub14 several times a day, until they get the confirmation that their answer was correct.

Another positive aspect of this bilateral communication is the request of a complete, coherent and clearly written explanation of the participant’s reasoning. This way, the feedback provided by the organizing team respects and nourishes the reasoning of each participant, as well as the processes used. We have even noticed a development on the correctness of the answers that the participants submitted throughout the

competition.

“In the beginning it was somehow strange. I wasn’t used to it. I’d put the calculations and that was it. But we had to present all our thinking. It was as if I had to write what I was thinking. Thus, I would think out loud and split it into parts. But from the 3rd or 4th problem I was already used to it.” [Isabel]

This feedback originated a change of attitudes in some participants within their mathematics classroom when facing assessment situations. The students themselves observed they took more care while answering to questions posed by the teachers, presenting all the necessary justifications and showing a greater predisposition to interpret a problematic situation, find a reasoning path or procedure in order to explain the solution in a convincing way.

“[...] I now pay more attention to little details that sometimes others don’t, and it reflects on the tests and on the problems that the teacher gives us, some of them really tricky... but now I am tuned!” [Lucia]

“Home Technologies”

The dynamic nature of the bidirectional communication can be felt in other aspects revealed by the participants. First off all, we note the usage of the Sub14 website: the participants use it frequently and think that the available information is important and useful, they like the design, the way it is organized and the fact that it is permanently updated:

“I like having an organized website (...) the ‘Press Conference’ page was always updated.” [Ana]

The purpose of posting submitted solutions was to show the methods used by some of the participants, hoping to improve their performance by the positive reinforcement of seeing their works and their names posted online.

“Yes! Sometimes I would go there to see if any of the posted solutions was mine! Once or twice I found my answer and I was very happy and shouted... ‘Daddy, daddy, come here!’” [Bernardo]

Bernardo’s enthusiasm, as well as many other participants’, supports the pedagogical and motivational aspects of the methodology adopted in Sub14. Not only it promotes the diversification of reasoning strategies and points out the several problem solving phases, but it also increases self-esteem and improves innovation and creativity as “special” answers are selected to be published online.

Moreover, the fact that Sub14 is a digital competition allowed the participants the opportunity of communicating their reasoning in an inventive way, since they could resort to any type of attachments, particular the ones they felt more comfortable with or the ones they found adequate to the problem itself. Therefore, the participants used mainly the text editor, MSWord, but they also used drawing and spreadsheet programs, like MSPaint and MSEXcel, all examples of home technology.

MSWord was used to compose text, organize information in tables, and insert images, automatic shapes, WordArt objects or Equation expressions. It was elected the favorite between the participants, since it is the one they better understand and constantly are asked to use for several school assignments.

“[Word] is the simplest to use, it’s the one that I have more confidence on to do school tasks, and I’m used to it. It’s the one I’m good at.” [Lucia]

Using images was a strategy that seven of the interviewee used. Nevertheless, some of them only inserted images that had something to do with the problem context, more like an illustration. In this case, we may consider that resorting to images had mainly an aesthetic function, as it didn’t help presenting or clarifying the reasoning and strategy used to solve the problem. However, other interviewees sketched their own images using MSPaint in order to improve the intelligibility of their thoughts:

“Anything that I thought that could help to improve the reasoning, I would draw it [in paper] and then I’d put it in the computer.” [Bernardo]

“We were playing with some straws and we reached the solution by trial and error. Then [we took some pictures] with the digital camera [and] put them in the computer so that we could send them.” [Alexandra]

In this way the image usage assumed, essentially, two roles in the answers of these participants. Firstly, it was merely a visual detail, which may be influenced by the type of work done in students’ school assignments. Secondly, the creation of images within the context of their interpretation of the problems is an evidence of their efforts on expressing their reasoning in the best possible way. Moreover, we can notice their awareness of the different representations that could materialize their reasoning and even some decision ability when facing the options they had at hand.

Two interviewees used Excel to present their answers. One of them used this tool to solve every Sub14 problem, showing however a narrow usage of the program as a means to organize the information and his answer. Seldom using the function “SUM”, he essentially resorted to tables and images, considering that the spreadsheet was better than a text editor. The referred simplicity seems to come from the fact that he has been exposed to this tool from an early age:

“Sometimes, when I was a kid – I got my first computer when I was six – I liked to get there [MS Excel] and do squares with the cells, paint them and that sort of things...” [Bernardo]

Another participant used the spreadsheet to solve five of the twelve problems, showing that he knew some of the advantages of this tool. Therefore, these participants were confident enough in using MSExcel, nonetheless not as a result of work within the school context, but rather of their domestic “findings”.

ANOTHER LOOK AT SUB14 AS A LEARNING ENVIRONMENT

Solving the Sub14 mathematical problems requires looking at a problem situation

from a mathematical perspective. This can be seen as a mathematizing process, since the participant is stimulated to express the way in which thinking was organized and progressed. In this competition, the participants found a place where they could freely communicate their ideas, had someone who listened and advised them, helping to make their mathematical thinking and expression become clearer. Moreover, when solving a problem, they faced the transition from convincing themselves to convincing the others (Mason, 2001). This led participants to develop their own understanding of the problem, promoting the usage of domestic technologies to communicate, thus adding competences that sometimes school neglects or forgets.

As a learning environment, although being external to the school context, Sub14 is aligned with school mathematics, and promotes a set of competences that fit within current mathematical education purposes and curricular targets. The fact that the competition occurs in a loose institutional context allows a greater family commitment and complicity with the participant's learning process, fostering the discussion on mathematical questions and problems outside the school environment, especially at home, maybe around dinner table.

Further work on this field shall include a future experience to investigate the possibility of allowing participants to communicate amongst them, within the website, bearing in mind the idea of a connected learning environment.

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A RESOURCE TO SPREAD MATH RESEARCH PROBLEMS IN THE CLASSROOM

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In this communication we intend to present a digital resource the aim of which is to give aid to teachers to use research problems in their classes; in a first part we are going to present the theoretical framework which was used by the team in the conception of the resource and the consequences on its model; we will present the results of a study dealing with the role and the impact of the resource used by teachers preparing lessons.

INTRODUCTION

Different works have shown the benefits of the use of research's problems [Polya, 1945, Schoenfeld, 1999, Brown and Walter, 2005, Harskamp and Suhre, 2007, Arsac et al., 1991, Arsac and Mante, 2007], in the construction of knowledge and both the interest of teachers and the difficulties to deal with in the classrooms; moreover, the institutional injunctions of using research problem are important in France and are going to take part in the final evaluation of the secondary school [Fort, 2007].

As far as we are concerned, and in the framework of the Piagetian psychological theory, we assume that the construction of knowledge has to go through an adjustment to the milieu as we will define it in the next section, and in this context, research problems are elements of the “material milieu” that teachers offer to learners.

We also assume that amongst all hindrances of generalization of research problems in the classroom, the following points are decisive:

1. the important part of the experimental dimension in problem solving clashes with the main representation of mathematics amongst maths teachers but also in the society;
2. the focus on heuristics and reasoning skills in maths research problem is in contradiction with the institutional constraints of teaching maths notions, particularly regarding French maths curricula;
3. difficulties for teachers to pick out in the students' activity the mathematics part of their work, and, as a result the notions which can be institutionalized;
4. the difficulties teachers have to assess such a work, the usual assessment modalities being not appropriate.

In this context, a team of teachers and researchers¹ from different institutions (IREM de Lyon, IUFM de Lyon, INRP and LEPS²), has worked on the construction of a numerical resource the aim of which is to give aid to maths teachers in order to use research problems in their teaching. In this paper, we will present the main theoretical frameworks used in the construction of this resource and will show, through the results of a particular study, the role this resource can play in the activity of teachers from the preparation of a lesson to the implementation in the classroom.

THE THEORETICAL CHOICES

This resource was written to be a part of the milieu of the teachers in the meaning Brousseau [Brousseau, 1986, Brousseau, 1997, Brousseau, 2004] and after him [Margolinas, 1995, Bloch, 1999, Bloch, 2005, Houdement, 2004] give to this concept. More precisely, learners learn through regulations of their links with their milieu. Going a bit deeper in this concept, Margolinas [Margolinas, 1995] described the structure of the milieu as a set of interlocked levels which can be described as follow:

Level	Teacher	Pupil	Situations	Milieux
3	Noosphere-T		Noospherian situation	Construction milieu
2	Builder-T		Construction situation	Milieu of project
1	Project-T	Reflexive pupil	Project situation	Didactical milieu
0	Teacher	Pupil	Didactical situation	Learning milieu
-1	Teacher in action	Learning pupil	Learning situation	Reference milieu
-2	Teacher observing	Pupil in action	Reference situation	Objective milieu
-3	Teacher organising	Objective pupil	Objective situation	Material milieu

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Table 1 Structuring of the milieu

In this table, the milieu of level n is the situation of level $n-1$ and is made up of the existing relationships between M , P and T . Using the symmetry of the table and, in our case, proposing to the teachers a situation, (in the acceptation of the didactical theory of situations) in which the a-didactical situations of action had as aim to allow teachers to construct, by themselves, the knowledge necessary to conduct a situation of problem research in the classroom [Peix and Tisseron, 1998] we speak of the material and objective milieu of the teachers. In this study, the resource appears to be a part of the material milieu of the teacher and the question is: is it possible, for a teacher, to use the resource to facilitate his tasks:

- organizing the material milieu of the pupils,
- understanding the objective milieu of the pupils and the links between their knowledge and conceptions
- choosing the pertinent notions to be institutionalized in the reference milieu of the pupils, and anticipating the conflicts between misconceptions and tools to solve...

Moreover, the theoretical framework of cognitive ergonomics through its concepts and methods allows us to study the competencies of the teacher in his interaction with the work system, and more particularly in the relationship between the prescribed tasks and his activity. Lastly, and in the field of using a numerical resource in professional tasks, the concepts of utility, usability and acceptability [Tricot et al., 2003] have been sounded out in two different ways:

- by an evaluation by inspection in order to construct and organize the resource,
- by an empirical evaluation in a professional situation.

Utility is “the question of whether the functionality of the system in principle can do what is needed” [Nielsen, 1993]

Usability can be defined as: “the capability to be used by humans easily and effectively” [Schackel, 1991], but also “the question of how well the users can use that functionality” [Nielsen, 1993]

Acceptability refers to the decision to use the artefact, and answers the questions: is this artefact compatible with the culture, the social values, global organisation in which the artefact has to be included.

PRESENTATION OF THE RESOURCE

Structure

It is possible to use this resource in different ways; theoretical texts about the experimental dimension in mathematics [Dias and Durand-Guerrier, 2005, Kuntz, 2007] can be read as well as different presentations made in conferences [Aldon, 2007, Aldon and Durand-Guerrier, 2007]. It is also possible to understand the sense

of the resource by reading a curriculum-vitae [Trouche and Guin, 2008] of the different steps and reflections of its building. The different situations are outlined using a common structure:

- Maths situation out of the classical literature on open problems developed in particular in IREM de Lyon (nowadays, there are seven maths situation):
 - Egyptian fractions: break down 1 into the sum of fractions of numerator 1.
 - Trapezoidal numbers: study of the sum of consecutive whole numbers.
 - The river: study of the shortest distance between two points with constraints.
 - The number of zeros of $n!$: study of the digits of $n!$ in different numeration systems.
 - The greatest product: study of the product of integers of fixed sums.
 - Polya's urns: study of the dynamic of the composition of an urn in a repeated experience.
 - Inaccessible intersection: find a line going through an inaccessible point.
- Maths objects that may be used to solve the given problem: for each of the situations, the a-priori analysis allows to extract the mathematics objects that are part of the mathematics situation and can be used in the process of resolution.
- Learning situation: how the maths situation has been transformed into a didactical situation? In this part of the resource, reports of real experiments can be read.
- References
- Synthesis: a ten pages synthesis of the situation allows teacher to familiarize themselves with the content of the section.
- Connected situations: how is it possible to protract the situation and what are the extensions in the maths researches nowadays?

Introduction of the resource

In order to confirm the hypothesis and to evaluate the utility, usability and acceptability of the resource, we built an experimentation with teachers from the first handover of the resource to the real experiment of a research problem in the classroom. In this section we are going to focus on the first handover in order to evaluate the usability of the resource.

The methodology of this part of the experimentation was built using an observation of teachers faced to a professional problem (preparing a lesson using research problem); the context was a training course with sixteen teachers involved. They

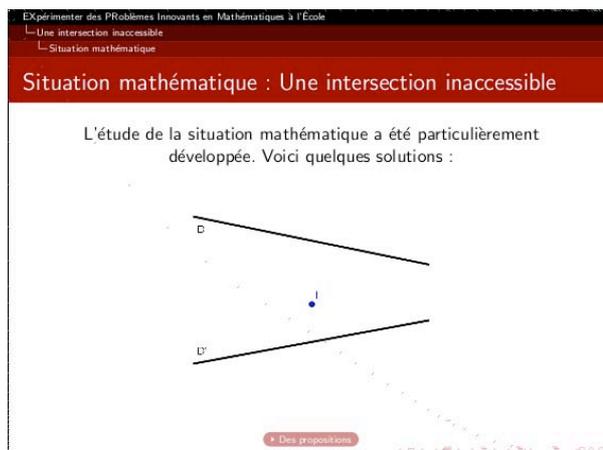
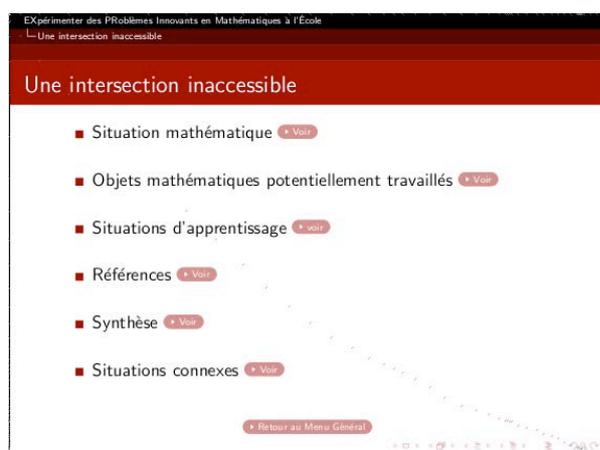
discovered the resource during this course as an artefact in the sense that the functioning of the resource has not been explained; the observer (the first author of this paper) recorded dialogues of two teachers and in the same time recorded the computer screen.

There is a confrontation, for the same person, between the position of expert (a teacher preparing a lesson, hence choosing objectives, a problem linked to these objectives, organising time of the lesson ...) and the position of beginner in two different ways: using research problem in his (her) preparation and using a new tool. The theoretical framework of the didactic situation theory gave us the possibility to observe the position of the resource in the milieu of the teacher and to observe why this resource gives a possibility to the teacher to have a look into the pupils' objective milieu as described above. The cognitive ergonomics framework gives us keys to analyse the activity of teachers in this professional situation. Moreover, the concept which is tested was the usability of the resource, using the following criterions [Tricot et al., 2003]:

- Possibility of learning the system
- Control of the errors
- Memorization of the functioning
- Efficiency
- Satisfaction

But also, and we will see why later, its acceptability, that is to say the degree of confidence the teachers have.

The first result that we can highlight is the very quick adaptability of the observed teachers in front of the resource. After à three minutes wandering, the teachers used the different path in the resource to find exactly what they want as it can be possible



to see when teachers changed from one situation to another. In the first time, the mouse hesitated on the screen, going from one button to the others before the click,

and progressively, the structure became clearer and the adequacy between the given objective and the browsing into the resource became safer:

After nine minutes:

Are you interested?

Yes

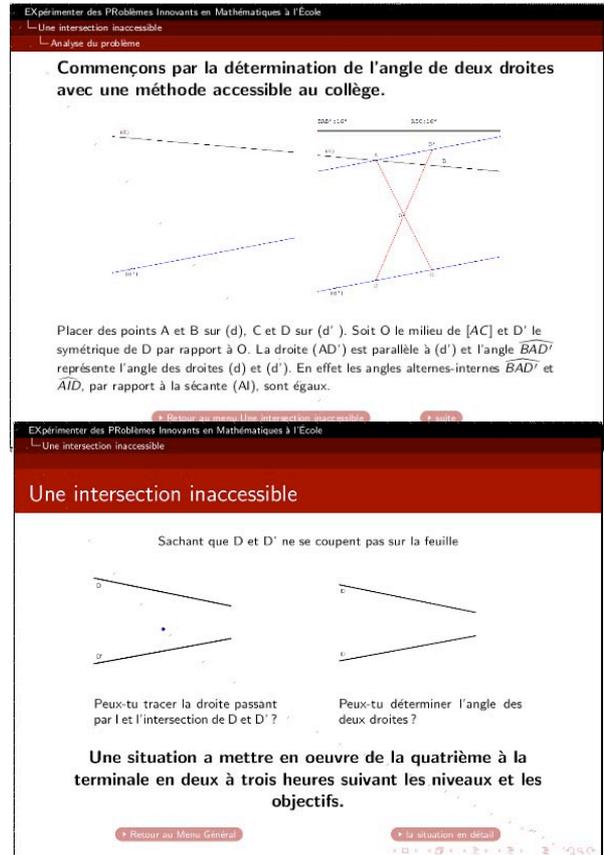
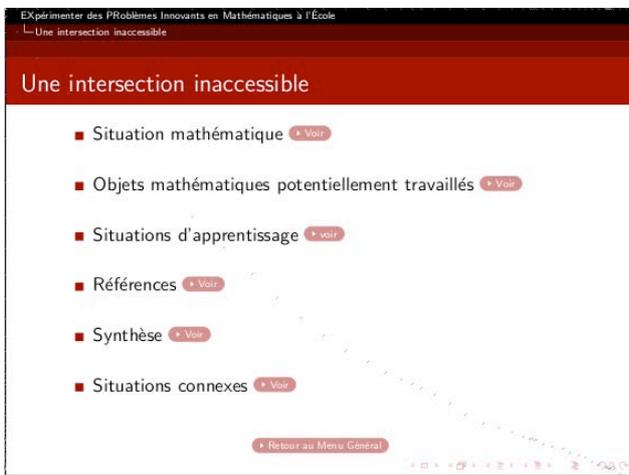
(click on “situation mathématique³”)

(two clicks and two screens in one second)

The mathematical situation... (they read)

Possible for our pupils (click, click)

I would like to see that (the mouse turn over the menu “possible maths objects...”)



³ Mathematical situation

The second important observation, linked to the concept of acceptability can be seen by the feeling of trust in the authors of the resource; at the beginning of the exploration, the two teachers click on the menu: theoretical framework, and after some seconds says:

“We are not going to read the whole text...”

And, later, in front of a situation, one of the teachers said:

“We are going to read what they say...”

These two brief sentences show us the growing of the confidence during the use of the resource and can be considered as a clue of the acceptability of the resource. The other observations and particularly the use of the resource to construct a real lesson confirm us in the feeling of the acceptability of the resource.

Realisation

In order to go on in the evaluation of the resource, a second experimentation has been built with the goal of testing the utility and the acceptability of the resource; we observed a research problem lesson focusing more particularly on the interactions between pupils and teacher during the situation of action. The teacher who was observed and interviewed, participated to the training course described above.

The chosen mathematical situation was the trapezoidal numbers and the question given to the pupils of a scientific eleventh class⁴ was:

What are the whole numbers which are sum of at least two consecutive integers?

Interrogating the two theoretical frameworks, the interview with the teacher allows us to bring to light utility and acceptability of the resource, but also to understand the position of this resource in the teacher's milieu.

Utility of the resource is in this case obvious, the teacher having prepared the lesson with the resource:

“Yes, yes I use it... I read all you wrote about this problem. Oh, yes, without the resource, I think I should not give this problem to my pupils, because I would have spent too much time to do this work... I would not do that!”

Regarding acceptability, a lot of clues allow us to consider, for this teacher and in this experimentation, the resource as acceptable, for example the feeling that the lesson created using the resource brought a new dimension to her course:

“I think I'll do that earlier next year, to create something in the class, precisely, this dynamic which makes the pupils actors, as I said to you, a pupil was speaking from the board, and I was at the back of the class, and the other pupils ask questions... I think it's a good way to involve pupils in the maths lessons, to put a lot of them in maths... For me, it's

⁴ Pupils are sixteen-seventeen years old

very confident, visibly they enjoy this time, and I think it's something important to insert pleasure in maths lessons, it's something which questions me, because it's so easy to do maths without pleasure!"

Moreover, the interview confirms the position of the resource in the objective milieu of the teacher in a posture of preparation of a lesson including a research problem.

In an other hand the observation of a group of pupils gives us interesting feedback about the mathematical objects students deal with and shows that the *a-priori* analysis of the resource corresponds to the reality of the class; for example, one of the mathematical object which was highlighted by the authors of the resource related to this problem was the powers of two. In other words, the hypothesis was that powers of two belong to the objective milieu of the pupils and, consequently are a field of experiencing; the confrontation of pupils with these objects allows them to change their position in the milieu and to bring with the help of the institutionalisation these objects in the reference milieu of the pupils:

F2: (using her calculator) two to the power five gives thirty two... Yes ; two to the power seven gives one hundred twenty eight

G: two hundred and fifty six, five hundred and twelve, thousand and twenty four, two thousands and twenty eight ...

F2: how do you calculate to obtain the results so quickly?

G: you multiply by two

F2: Ah yeah right!

In this small excerpt, the two definitions of the powers of two as an iterative or recursive process are called up and the link between these definitions is made by F2; it is possible to think that the recursive definition belongs now to her objective milieu and a necessary work must be done to institutionalize it in her reference milieu. The fact that this object was present in the resource allows the teacher to pay attention to this dialogue and to use it in her lesson:

CONCLUSION

The described engineering and the results of observations and interviews show the place of the resource in the milieu of the particular teachers involved in this experiment, and clearly show the utility, usability and acceptability of this resource. Regarding the didactical theory of situations, this experimentation shows that the resource emplaced in the material milieu of the teachers can be mobilised in their objective milieu and used in the setting up of research problem lessons in the classroom. The resource also allows teachers to launch themselves in the different milieu of the students and to understand the position of mathematical objects in these milieus, and consequently it facilitates the institutionalization.

However, new questions appear, in particular linked to the genesis of this resource and its transformation from an external resource possibly used by a teacher to a document available in his/her environment.

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THE SYNERGY OF STUDENTS' USE OF PAPER-AND-PENCIL TECHNIQUES AND DYNAMIC GEOMETRY SOFTWARE: A CASE STUDY

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This study is part of an ongoing research¹ on the interpretation of students' behaviors when solving plane geometry problems in Dynamic Geometry Software and paper-and-pencil media. Our theoretical framework is based on Rabardel's (2001) instrumental approach to tool use. We seek for synergy relationships between students' thinking and their use of techniques by exploring the influence of techniques on the resolution strategies. Our findings point to the existence of different acquisition degrees of geometrical abilities concerning the students' process of instrumentation when they work together in a computational and paper-and-pencil media. In this report we focus on the case of a student.

INTRODUCTION

We report research on the integration of computational technologies in mathematics teaching, in particular on the use of Dynamic Geometry Software (DGS) in the context of students' understanding of plane geometry through problem solving. We focus on the interpretation of students' behaviors when solving plane geometry problems by analyzing connections and synergy among techniques used in environments, DGS and paper-and-pencil, and geometrical thinking (Kieran & Drijvers, 2006). Many pedagogical environments have been created such as Cinderella, Geometer's Sketchpad, and Cabri Géomètre II. We focus on the use of GeoGebra because it is a free DGS that also provides basic features of Computer Algebra Software. As said by Hohenwarter and Preiner (2007), the software links synthetic geometric constructions (geometric window) to analytic equations, coordinate representations and graphs (algebraic window). Our aim is to analyze the relationships between secondary students' problem solving strategies in two environments: paper-and-pencil (P&P) and GeoGebra (GGB). Laborde (1992) claimed that a task solved using DGS may require different strategies to those required by the same task solved with P&P; this fact has an influence on the feedback provided to the student.

Our broadest research question aims at how the use of GGB in the resolution of plane geometry problems interacts with the students' paper-and-pencil skills and their conceptual understanding. We analyze and compare resolution processes in both environments, taking into account the interactions (student-content, student-teacher and student-GGB). In this report we focus on two research goals as being interpreted in the case of one student, Santi. We analyze this student's instrumentation process,

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and we compare his resolution strategies when using P&P and GGB within each problem. In the whole research we work with a total of fourteen individual cases from the same class group and establish some commonalities and differences among them.

THEORETICAL FRAMEWORK

We first draw on the instrumental approach (Rabardel, 2001). According to Kieran and Drijvers (2006), a theoretical framework that is fruitful for understanding the difficulties of effective use of technology, GGB in our case, is the perspective of instrumentation. The instrumental approach to tool use has been applied to the study of Computer Algebra Software into learning of mathematics and also to Dynamic Geometry Software. The instrumental approach distinguishes between an artifact and an instrument. Rabardel and V erillon (1995) claim the importance of stressing the difference between the artifact and the instrument. A machine or a technical system does not immediately constitute a tool for the subject; it becomes an instrument when the subject has been able to appropriate it for her/himself. This process of transformation of a tool into a meaningful instrument is called instrumental genesis. This process is complex and depends on the characteristics of the artifact, its constraints and affordances, and also on the knowledge of the user. The process of instrumental genesis has two dimensions, instrumentation and instrumentalization:

- Instrumentation is a process through which “the affordances and the constraints of the tool influence the students’ problem solving strategies and the corresponding emergent conceptions” (Kieran & Drijvers, 2006, p. 207). “This process goes on through the emergence and evolution of schemes while performing tasks” (Trouche, 2005, p. 148).
- Instrumentalization is a process through which “the student’s knowledge guides the way the tool is used and in a sense shapes the tool” (Kieran & Drijvers, op. cit., p. 207).

In our research, we select different problems for being solved first with P&P and then with the help of GGB. In order to analyze the connectivity and synergy between the students’ resolution strategies in both environments, the problems are to be somehow similar. The basic space of a problem is formed by the different paths for solving the problem. We transfer the similarity of the problems to the similarity of their basic spaces. For example, the problems considered in this article, share common strategies for reaching the solution such as equivalence of areas due to complementary dissection rules, application of formulas (area of a triangle), particularization, etc.

We plan to design an instructional sequence, focusing on a systematization of the interactions produced between artifacts (P&P, GGB), the mathematical actions and the didactical interactions. The theoretical framework is based on instrumental approach and activity theory (Kieran & Drijvers, 2006). We connect the activity theory as part of the “orchestration” (Trouche, 2004). The actions consist in different problem sequences to be proposed by the teacher to the students, to be solved in both

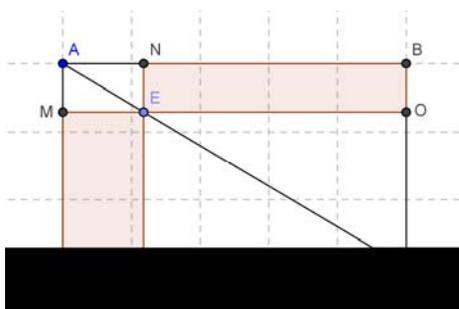
media. The teacher proposes different indications or new problems. For each problem, we prepare a document with pedagogical messages that provide differing levels of information, and we group them according to the phases of the solving processes which are being carried out: familiarization, planning, execution, etc. We classify the pedagogical messages, for each phase, in three levels. Level 0 contains suggestions that do not imply mathematical contents or procedures in the solving process. The messages of level 1 only convey the name of the implied mathematical contents or procedures. Level 2 provides more specific information on these contents or procedures. For the problems to be solved in a technological environment we also prepare contextual messages. These messages are related to the use of GGB. The teacher can help the students in case they have technical difficulties with GGB.

We also specify some terms that will be used in this study of students' GGB resolutions such as figure and drawing. We use these terms with their usual meaning in the context of the Dynamic Geometry Software (Laborde & Capponi 1994). We use this distinction between Figure and drawing in order to describe the way in which students interpret the representations generated on the computer.

CONTEXT AND METHOD

The study is conducted with a group of fourteen 16-year-old students from a regular class in a public high school in Spain. These students are used to working on Euclidean geometry in problem solving contexts. They have been previously taught GGB. The main source of data for this paper comes from the experimentation with two problems:

1. Rectangle problem: *Let E be any point on the diagonal of a rectangle $ABCD$ such as $AB = 8$ units and $AD = 6$ units. What relation is there between the areas of the shaded rectangles in the figure below?*



2. Triangle problem: *Let P be any point on the median $[AM]$ of a triangle ABC . What relation is there between the areas of the triangles APB and APC ?*

These problems have to do with comparing areas and distances in situations of plane geometry. They admit different solving strategies; they can be solved by mixing graphical and deductive issues, they are easily adaptable to the specific needs of each student, and they can be considered suitable for the use of GGB. For all the problems, we start by exploring the basic space of the problems in the P&P and GGB environments. After having identified the different resolution strategies and

conceptual contents of the problems, the focus is on analyzing the necessary knowledge to solve them. Finally, we prepare a document with the pedagogical and technical messages that provide differing levels of information.

All the activities with students are planned to take four sessions of one hour each with an average of two problems per session. The two problems above were developed in the first two lessons in which the students worked on their own. The inquiry-based approach to the lessons leads the students to assume the responsibility for the development of the task. The teacher fosters the students' autonomy by only intervening in certain moments and giving some messages, established a priori, concerning the resolution.

For the experimentation with each problem, the whole set of data is: a) the solving strategies in the written protocols (P&P and GGB); b) the audio and video-taped interactions within the classroom (student-teacher, student-content and student-GGB); and c) the GGB files. All these data were examined in order to inform about our research goals. The integration of data concerning these goals led us to the description of the students' process of instrumentation. For the description, different variables were considered, among them: the students' heuristic strategies (related to geometric properties, to the use of algebraic and measure tools or to the use of both...); the use of GGB (visualization, geometrical concepts, overcoming difficulties...); the obstacles encountered in each environment (conceptual, algebraic, visualization, technical obstacles...); etc.

For each case, we first analyze the P&P resolution with data coming from the tapes and the protocols. We consider the student's solving strategies and the use of mathematical contents. Then we analyze the GGB resolutions with data coming from the tapes and especially from those tapes that show the screen. We consider again the student's solving strategies, the use of mathematical contents and now we also pay attention to instrumented techniques and technical difficulties. After having developed these two types of analysis, we compare GGB and P&P resolutions by looking at the use of the two environments within each problem. To analyze the problem solving process, we also consider the phases of the problem solving process (Schoenfeld, 1985) as a whole in each group of problems (GGB and P&P).

THE CASE OF SANTI: An episode of exploration/analysis

The mathematical content of the problem was dealt with in courses prior to the one Santi is currently taking. Santi has procedural knowledge relating to the application of formulas for calculating the area of the Figure, and sufficient knowledge of the concepts associated with geometric constructions. He is a high-achieving student. Santi is asked to solve the first problem with P&P and the second problem with the help of GGB. In this section we summarize his problem solving process for both problems.

- Resolution of the rectangle problem (P&P):

In the resolution of the first problem, after reading the statement of the problem, Santi observes the figure and then he states that he does not have enough numerical data. The teacher suggests the student to consider a particular case (heuristic cognitive message of level 1 in the planning /execution phase). Santi reacts to this message, considering the particular case in which E is the midpoint of the diagonal and he conjectures that both areas should be equal. Then he tries to prove the conjecture for the particular case in which the length AE is 2 units. The student reaches a solution to the particular case by using trigonometry. He obtains the angles in the triangle EAN (Figure 1) and he calculates the measures of the sides, AN and AM. Finally he obtains the numerical value of both areas and he observes that he gets different values. Santi requests a message about the solution because he expected to obtain equal values. The teacher remarks that there is an algebraic mistake in his resolution and suggest Santi to review the process he has followed because there are algebraic mistakes (metacognitive message of level 1 in the verification phase). The student finds the mistake and obtains the equal values of both areas (Figure 1). He then tries to use the same strategy for the general case using the relation: $\frac{8}{6} = \frac{AN}{AM}$.

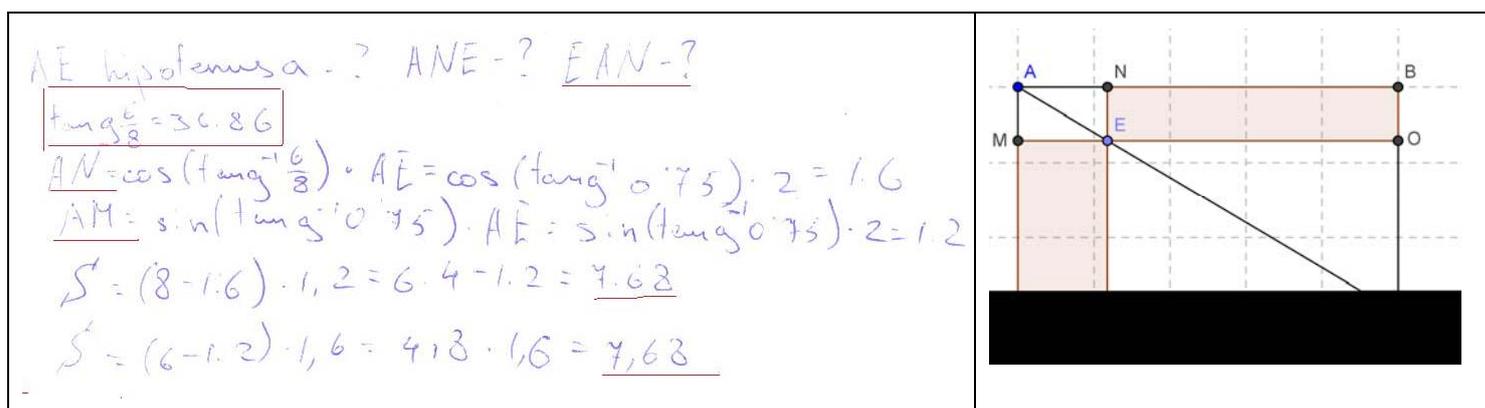


Figure 1: Resolution with paper and pencil of the first problem (Santi)

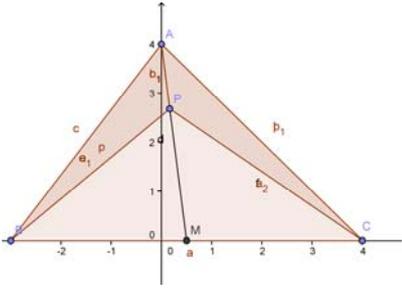
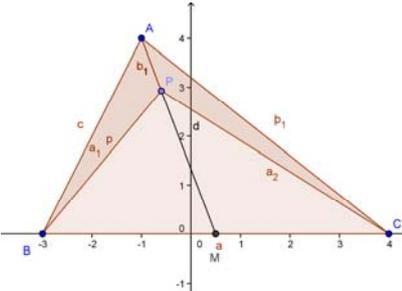
Santi bases his resolution strategy on applying trigonometry and he does not try to use the strategy based on comparing areas of congruent triangles (strategy based on equivalence of areas due to complementary dissection rules). The teacher proposes other problems to be solved with P&P and with GGB. In the following paragraph we consider one of these problems.

- Resolution of the triangle problem (GGB):

After reading the statement of the problem, Santi draws a graphic representation without coordinate axes before constructing the figure with GGB. The teacher observes that Santi has considered the point P in the side AC of the triangle instead of the median. The teacher gives Santi the following message: “*Try to understand the conditions of the problem*” (metacognitive message of level 0 in the familiarization phase). Santi constructs a new figure with GGB (Figure 2) and he observes the figure trying to find a solving path. Then he proposes a conjecture and asks the teacher for verification: “*the triangles APC and APB have a common side and the same area (he*

verifies this with the tool area of a polygon). How could I prove that these two triangles are equal [congruent]”? I have tried to prove that they have the same angles but I don't see it...”

We observe that Santi does not validate his conjecture with the help of GGB (using measure tools for instance). The teacher gives him a validation message of level 1 “Are you sure that these triangles are congruent!?” Santi reacts to this message changing the triangle ABC. He drags the vertex A (Figure 3) and he observes without measure tools that the triangles are different.

	
<p>Figure 2: Construction with GGB of the triangle ABC and its median. Santi uses the tool polygon to construct the triangles.</p>	<p>Figure 3: He moves the vertex A to obtain a general triangle. We observe that he tries to define vertices with coordinates that are integer numbers.</p>

The last graphic deduction marks the beginning of the search for a new strategy. He observes the figure, without dragging its elements. More than five minutes have gone without doing anything in the screen. Santi requests again the help of the teacher (Table 1, line 1) for the familiarization phase of the problem.

		Interactions
1	Santi	Is P any point in the segment AM? Isn't it the midpoint? [Santi tries to consider particular cases]
2	Teacher	P is any point in the median [AM]. The triangle ABC is also a general triangle [cognitive message of level 1 for the familiarization phase]
....	Santi	[Santi reacts to this message modifying the initial triangle. He drags again the vertices to obtain the triangle in Figure 3].
3	Santi	I think that I see it!...The triangles have a common side and the same height [the segments [BM] and [MC] (wrong deduction)]
4	Teacher	Are you sure about that?
5	Santi	[Santi reacts to this message observing the triangle without doing any action on the screen. Then he states:] No. These lines are not perpendicular! [(AM) and (BC)]. But, this was a good trial...

		Have they the same base? [he refers to the common side of both triangles]
6	Teacher	Yes

Table 1: How Santi tries a new solving path

For the first time, Santi tries to drag the vertices of the triangle trying to find invariants. While he drags the vertices he looks in the algebraic window for invariants. We observe here the simultaneous use of the algebraic window and the geometric window. He observes again that the triangles have the same area in all the cases and a common side. He tries to prove that the heights are equal but he wrongly considers that the side [BM] is the height of the triangle BAP (Figure 3). The teacher gives him a message of level 0 for the validation phase (Table 1, lines 3 to 6). Santi reacts to this message constructing with GGB the perpendicular line from the vertex B to the base of the triangle (Figure 4). He tries to follow the same strategy (proving that the heights have the same length) and he drags continuously the vertices A, B and C, changing the orientation of the triangle, and observing the constructed lines on the geometric window.

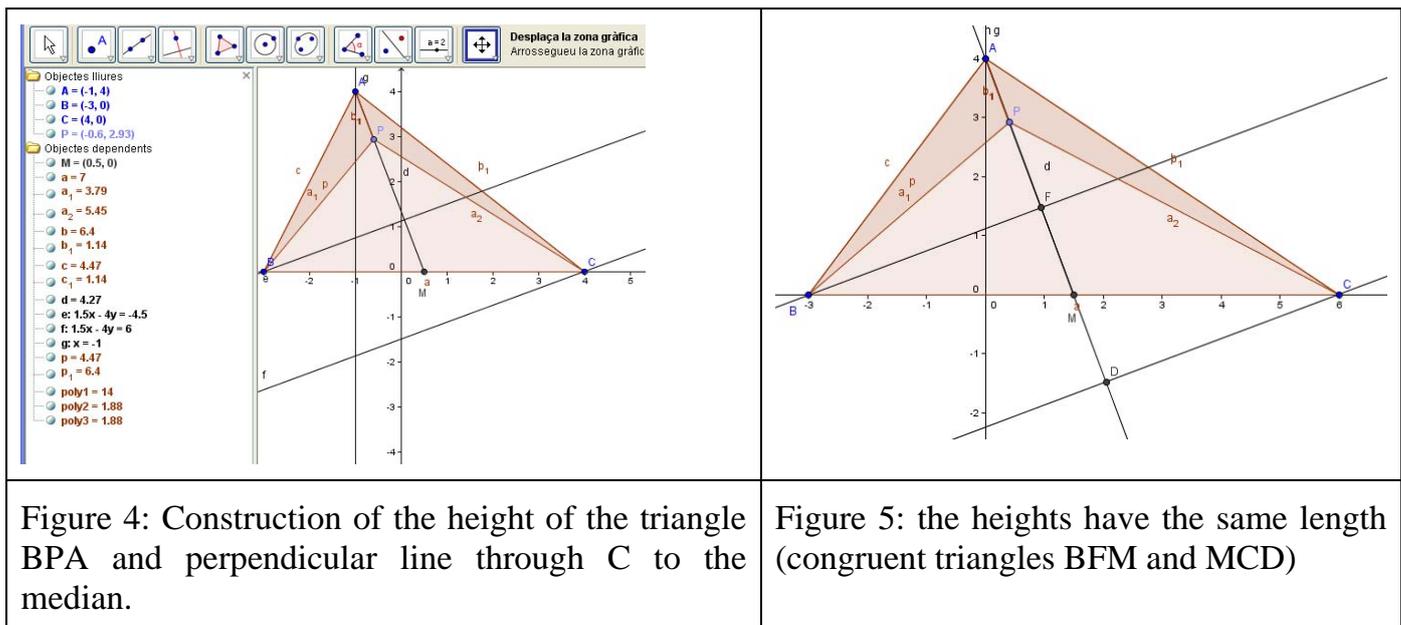


Figure 5: the heights have the same length (congruent triangles BFM and MCD)

In this time, he observes again the figure (Figure 4) without dragging. He is lost. This is the beginning of a new phase. We wonder if Santi had found a proof for his conjecture if he had constructed the heights of both triangles. Nevertheless, he does not construct the points F and D (Figure 5) and he abandons the solving strategy. Santi requests again the help of the teacher for the planning/execution phase and he states: “*Is it possible to solve the problem with trigonometry?*”. The teacher gives him a new message: “*Could you think of some way of breaking the triangle ABC into triangles and look for invariants with the help of GGB*” (cognitive message of level 2 for the planning phase). Santi reacts to the previous message of the teacher and starts a new exploration phase. He erases the perpendicular lines and drags continuously the

vertexes of the triangle ABC. He observes in the algebraic window the changing values looking for invariants. He extracts the inner triangles BPM and CPM which have the same area (Table 2, line 1) from the initial configuration. This observation will suggest him a new solving path based on comparing areas. He makes a new conjecture and requests the help of the teacher for validating his deductions (Table 2).

		Interactions
1	Santi	Are the triangles BPM and PMC equal? (Figure 2)
2	Teacher	What do you mean by equal?
3	Santi	The triangles have the same area
4	Teacher	Yes. You should justify this fact.
5	Santi	If I subtract two equal areas from two equal areas, do I get the same area?
6	Teacher	Yes
6	Santi	Ok! I justify this with paper and pencil.

Table 2: Strategy based on comparing areas

Finally Santi justifies his deductions with P&P, he proves that the median of a triangle divides the triangle into two triangles of same area. We wonder if the use of GGB helps Santi to find a strategy based on comparing areas.

FINAL REMARKS

We observe in this study that Santi appropriates the software in few sessions of class and he bases his constructions on geometric properties of the figures. He also combines the simultaneous use of the algebraic window and the geometric window and he tends to reason on the figure. We consider that the affordances of the software and teacher's orchestration have influenced Santi's resolution strategies. We have identified the following instrumented schemes: 'dragging combined with perceptual approach to find a counter-example' and 'dragging combined with perceptual approach to distinguish geometric properties of the figure (perpendicularity, congruence of triangles, equality of areas). In the ongoing research (longer teaching experiment) we have also observed some common heuristic strategies in both environments such as the strategy of supposing the problem solved and the strategy of particularization. We have also observed that Santi tends to use more algebraic strategies when he works only with P&P than when he works in a technological environment. Moreover he tends to produce more generic resolutions, independent of numerical values, fostered by a proposal of problems that accept these kinds of solving strategies. Nevertheless, given that students have different relationships with the use of GGB and the detailed study of Santi gives us some insight of a future classification of typologies in the instrumental genesis. In our broader research we try to follow the instrumental genesis for a group of fourteen students to observe different students' profiles. Future research should help to better understand the

process of appropriation of the software and to analyze the co-emergence, connectivity and synergy of computational and P&P techniques in order to promote argumentation abilities in secondary school geometry.

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STUDENTS' UTILIZATION SCHEMES OF PANTOGRAPHS FOR GEOMETRICAL TRANSFORMATIONS: A FIRST CLASSIFICATION[♦]

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The activities with the Mathematical Machines are very rich from educational and cognitive points of view. In particular, the use of pantographs has revealed educational potentialities for the acquisition of some important mathematical concepts and for the emergence of argumentation and proving processes, at any school level. In this paper, we propose a cognitive analysis of the processes involved in the manipulation of the mathematical machines, providing a first classification of utilization schemes of pantographs for geometrical transformations. This classification can be efficiently used to observe, describe and analyse cognitive processes involved in the exploration of mathematical properties incorporated in the machines.

Keywords: Mathematical Machines, utilization schemes, pantographs, geometrical transformations and cognitive processes.

INTRODUCTION

The Mathematical Machines Laboratory (MMLab: www.mmlab.unimore.it), at the Department of Mathematics in Modena (Italy), is a research centre for the teaching and learning of mathematics by means of instruments (Ayres, 2005; Maschietto, 2005). The name comes from the Mathematical Machines (working reconstruction of many mathematical instruments taken from the history of mathematics), the most important collection of the Laboratory. These machines concern geometry or arithmetic:

“a geometrical machine is a tool that forces a point to follow a trajectory or to be transformed according to a given law”...“an arithmetical machine is a tool that allows the user to perform at least one of the following actions: counting; making calculations; representing numbers” (Bartolini Bussi & Maschietto, 2008).

The MMLab research group carried out various activities with the Mathematical Machines, namely: laboratory sessions in the MMLab, long-term teaching experiments in classrooms, workshops at national and international conferences and also exhibitions (see chapters 2 and 5 of the forthcoming volume by Barbeau and

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Taylor, from ICMI Study n. 16) in collaboration with the members of the association “Macchine Matematiche” (<http://associazioni.monet.modena.it/macmatem>).

The laboratory sessions in the MMLab are designed in order to offer hands-on activities with mathematical machines for classes of students in secondary schools (an average of 1300-1500 Italian secondary students a year come with their mathematics teacher to experience hands-on mathematics laboratory), groups of university students, prospective and practicing school teachers (Bartolini Bussi & Maschietto, 2008). As the Mathematical Machines activities in school classrooms concerns, the MMLab research group organized different long-term teaching experiments in primary and secondary schools (Bartolini Bussi & Pergola, 1996; Bartolini Bussi, 2005; Bartolini Bussi, M. G., Mariotti M. A., Ferri F., 2005, Maschietto & Martignone, 2007).

All the activities quoted above are based on two fundamental components: the idea of the “mathematics laboratory”[1] and the didactical research on the use of tools in the teaching and learning of mathematics (Bartolini Bussi & Mariotti, 2007).

The MMLab researches aim at the development of different activities that should foster, through the use of the mathematical machines, the acquisition of some important mathematical concepts and the emergence of argumentation processes.

In order to implement the studies on MMLab laboratory activities, and to set up new teaching experiments, we consider important to carry out a cognitive analysis of the processes involved in the manipulation of the Mathematical Machines. The aim of our research is identifying Mathematical Machines utilization schemes and the connected exploration processes, providing a first classification. In the paper we shall present the first steps of this new research.

THEORETICAL FRAMEWORK

According to the educational goals that the activities with Mathematical Machines intend to realize, we investigate students cognitive processes involved in exploration of open-ended problems (in particular the problem of identifying the geometrical laws that explain how a machine works), in generation of conjectures and argumentations and in concept formation (for example: the concepts of geometrical transformations, of conic, of central perspective...). First of all, to analyse deeply these processes we propose a classification of Mathematical Machine utilization schemes [2]. This classification is suitable not only for describing the interactions between machines and subjects but also for analysing both their exploration and argumentative processes.

The processes through which a subject interacts with a machine have been studied by Rabardel in cognitive ergonomics: he grounded his research in constructivist epistemologies, primarily in *activity theories*, but also in the Piagetian and post-Piagetian developmental approach to the cognition-action dialectic (Rabardel, 1995; Béguin & Rabardel, 2000).

Rabardel proposed an original approach blending anthropocentric and technocentric approaches: as a matter of fact, in line with activity theory, he conceived the instruments as psychological and social realities and studied the instrument-mediated activity. According to Rabardel (1995) an *instrument* (to be distinguished from the material -or symbolic- object, the *artefact*) is defined as a hybrid entity made up of both artefact-type components and schematic components that are called *utilization schemes*.

“What we propose to call “utilization scheme” (Rabardel, 1995) is an active structure into which past experiences are incorporated and organized, in such a way that it becomes a reference for interpreting new data” (Béguin & Rabardel, 2000)

An artefact only becomes an instrument through the subject’s activity. This long and complex process (named *instrumental genesis*) can be articulated into two coordinated processes: *instrumentalisation*, concerning the individuation and the evolution of the different components of the artefact, drawing on the progressive recognition of its potentialities and constraints; *instrumentation*, concerning the elaboration and development of the utilization schemes (Béguin & Rabardel, 2000).

For the importance of these schemes, for their specificity in interacting with Mathematical Machine and for the limits that this paper has to respect, we focus here on utilization schemes in the case of pantographs.

METHODOLOGY

The method used for investigation was the clinical interview: subjects were asked to explore a machine and to express their thinking process aloud at the same time. In particular, after having explained to the student that the machines to be explored are pantographs for geometric transformations, we asked:

1. To define the mathematical law made locally by the articulated system.
2. In particular, to justify how the machine “forces a point to follow a trajectory or to be transformed according to a given law” and then to prove the existing relationship between the machine properties (structure, working...) and the mathematical law implemented.

The interviews were videotaped and the analysis is mainly based on the transcripts of the interviews. The interviews were analysed with special attention to verbal tracks and hands-on activities in order to detect mental processes developing during the exploration of the machines. Every protocol is analysed in a double perspective: as bearer of new information about possible exploration processes and as evidence for the existence of recurrent schemes.

The subjects were three pre-service teachers, two university students and one young researcher in mathematics. The choice to interview subjects which are familiar with (Euclidean) geometry and with problem-solving has allowed us to collect observations of complete machine exploration: namely, the generation of conjecture about the mathematical law implemented by the machine and, subsequently,

argumentation and proof of mathematical statements that can explain the functioning of the machine. Moreover, the subjects were new in working with this environment: in this way we could assume that they did not have an a priori specific knowledge about these machines.

The artefacts selected for this first research are machines concerning geometry, in particular pantographs: for the axial symmetry, for the central symmetry, for the translation, for the homothety and for the rotation. These machines establish a local correspondence between points of limited plan regions connecting them physically by an articulated system; they were built to incorporate some mathematical properties in such a way as to allow the implementation of a geometrical transformation (i.e. axial symmetry, central, translation, homothety, rotation).

CLASSIFICATION OF THE UTILIZATION SCHEMES

In this paper we present the first part of our research that aimed to introduce a classification of utilization schemes observed during the explorations of pantographs for geometrical transformations. The identified utilization schemes were divided into two large families: utilization schemes linked to the components of the articulated system (as the constraints, the measure of rods, the geometrical figures representing a configuration of rods, etc.) and utilization schemes linked to the machine movements. As regards the first family, we have identified the following utilization schemes: the research of fixed points, movable points (with different degrees of freedom), plotter points and straight path; the measure of rods length; the research of geometric figures representing the articulated system or some part of it; the construction of geometric figures that extend the articulated system components; the individuation of relationships between the recognized geometric figures; the analysis of the machine drawings.

As regards the utilization schemes linked to the machine movements [3], we distinguish between the movements aimed at finding particular configurations obtained stopping the action in specific moments and the continuous movements aimed to analyse invariants or changes. We summarize this classification in a table:

Linkage Movement that stops in	Movements description:
Generic Configurations	Movement that stops in a configuration which is considered representative of all configurations observed (that does not have "too special" features)

Particular Configurations	Movement that stops in a configuration that presents special features (i.e. right angles, rods positions...)
Limit Configurations	Movement that stops in configurations in which the geometric figures that represent the articulated system become degenerate
Limit zones	Movement that stops in the machine limit zones: i.e. the reachable plane points

Linkage Continuous movements	Movements description:
Wandering movement	Moving the articulated system randomly, without following a particular trajectory
Bounded movement (For example: Movements by fixing one point or one rod...)	Moving the articulated system, blocking particular points or rods
Guided movement	Moving the articulated system, forcing a point to follow a line or a specific figure
Movement of a particular configuration	Moving the articulated system, maintaining a particular configuration
Movements between limit configurations	Moving the articulated system so that it can successively assume the different "limit Configurations"
Movement of dependence	Moving (in a free, guided or bound way) a particular point and see what another particular point does
Movement in the action zones	Moving the articulated system in a such a way that all the possible parts of the plane are reached

A PROTOCOL

In this paragraph we present the first part of one clinical interview transcripts dealing with the exploration phase (i.e. the beginning of the machine exploration, before the identification of the geometrical transformation made by the machine), where we can identified some of the utilization schemes described in the previous paragraph [4]. The subject of the protocol, Anna, is a pre-service teacher graduated in mathematics and she explored the pantograph of Scheiner (see Fig. 1-2).

Anna: *(she touches a rod which seems to remain blocked) all motionless!...(she moves the articulated system) Ah, no, only a single fixed point ... I saw that leads are useful, and then... .. (opening and closing the linkage, she draws lines that converge in the fixed point) ... then (she turns the machine and she draws again “concentric lines”)...*

She starts controlling which part of the linkage is pivoted to the wood plane (*research of fixed points*) and then, in order to explore the linkage movements, she puts the leads in both plotter holes (*individuation of plotter points*) and draws curves produced by the linkage closing movement (*guided movements that end in a limit configuration*: see Fig. 3)

Anna: *I do not see anything then.....(she is looking the motionless machine and the curves drawn)...(she moves the linkage and she stops in a generic configuration) well, this is a parallelogram, I would say... That is... then, parallelogram, and in a vertex there is a lead... (with the ruler she measures two rods: in the fig. 2 CQ and CP)... are congruent (she points them out)*

The analysis of the drawn curves does not seem to help her to discover what transformation the machine makes, therefore she starts an analysis of the linkage structure (*research and individuation of a generic configuration and recognition of particular geometric figures in the linkage structure*): at first she identifies a parallelogram (see Fig.4), and then she focuses on other linkage rods (those parts that do not form the parallelogram). She recognizes the parallelogram without using the ruler (probably the visual perception of congruence has been supported by the previous exploration of movements during which the rods remained parallel). Differently, to discover the other characteristics of the linkage geometric structure, Anna feels the need to *measure the rods length*, so she discovers that there are two congruent rods (CQ and CP).

Anna: *... so this (she looks at the linkage and she uses two fingers to show the “virtual segment” PQ that completes the triangle PQC: see Fig. 5) is an isosceles triangle*

The identification of these congruent rods arouses the construction of a new geometric figure (an isosceles triangle) created completing, with an imaginary segment, the sequence of the congruent rods (*extending and individuation of geometric figures in the linkage structure*).

Anna: *but I will not see anything... but it doesn't say anything to me at this moment..... (she moves the machine, drawing always concentric lines) well they are always circumferences...(she is looking at the drawings) I do not understand if they are or not circumferences ...*

Also the exploration of linkage characteristics does not seem to help her, for this reason she comes back to the previous strategy: she starts again to draw lines that follow the machines closing movement (*guided movements that end in a limit*

configuration and analysis of these drawings), but, as before, she is not aware of the drawn lines characteristics; therefore, not knowing which properties designed curves have, she can not understand how they are transformed by the machines.

Anna: (*she makes a zigzag movement*) well, but it seems to me that they trace the same thing (*she makes the zigzag movement in another area of the paper*)... (*she points the zigzag drawing and she moves away the linkage*)... the leads then trace the same, the same image, it seems to me, but I dare say that (*she makes a gesture: see Fig.6*)...that it is reduced in scale.

Anna changes the *guided movements* (zigzag movements) and, this time, the *analysis of the drawings* leads to the recognition of the transformation (the homothety). Therefore it seems that what lets Anna to do the discovery of the transformation incorporated in the machine, is the drawings analysis more than the machine structure; but not all the drawings seem to be successfully: in fact each of them gives only partial information about the transformation. In particular, for Anna is determinant the choice to change the movement (and consequently, the drawing): as a matter of fact in the zigzag lines it can be seen that the correspondent segments are modified, while the angles are not (in the previous drawings these proprieties are “hidden”, while it came out the presence of a fixed point).

In conclusion, it is interesting to underline that also in a brief excerpt, it is possible to see the variety, the complexity of their relationships and, in particular, the plot of the different utilization schemes. After the individuation of the schemes, we can make a cognitive analysis of the exploration processes linked to these schemes. For example, we intend to examine closely how (and then why) Anna swings between two different strategies that remain separated (the drawing/analysis of lines and the study of linkage structure). This analysis brings important information for the understanding of subsequent processes: in fact, in the continuation of this protocol, the lack of interweaving of the information acquired through the different utilization schemes used, seems to be an obstacle in the Anna’s proof construction (about how the machine incorporates the transformation properties). This part of the research is still in progress, but the first results raise the hypothesis that successful strategies are those that maintain a tension and integration between the analysis of the articulated system proprieties, the drawings and the invariants of the movement.

CONCLUDING REMARKS

The studies on the interaction between a subject and a machine have to take into account an intriguing complexity because several components are involved. From a cognitive point of view and with educational goals, in this paper, we have presented a study to better understand the exploration of some geometrical machines: in particular, we have proposed a first classification of utilization schemes of pantograph for geometrical transformations and we have shown an analysis carried out through this classification. In this analysis we have underlined the importance of

the identification of the different schemes in describing the aspects of mathematical machines exploration.

Further researches are needed in two directions. On the one hand, we will study how these schemes are intertwined with the processes involved in conceptualisation, in argumentation and in proving; on the other hand, we will explore the evolution of the utilization schemes and its relationship with argumentation processes and subject's cultural resources.

Moreover, this study will be developed to offer teachers tools that could be efficient to set up activities with educational goals and to intervene in students' interactions with the machines, promoting those processes that are considered relevant for the activities with the mathematical machines.

NOTES

1. "A mathematics laboratory is a methodology, based on various and structured activities, aimed at the construction of meanings of mathematical objects. (...) The mathematics laboratory shows similarities with the concept of Renaissance workshops where apprentices learned by doing and watching what was being done, communicating with one another and with the experts" <http://umi.dm.unibo.it/italiano/Didattica/ICME10.pdf>.
2. In literature there are not previous cognitive studies of this type on mathematical machines. A classification of utilization schemes of instruments of different nature is proposed in Arzarello et al. (2002) where different modalities of dragging are discussed.
3. In addition to the linkage movements, there are also the movements of the machine wood base (on which the linkage is set): i.e. the rotations of the base that permit to look the machine from other points of view.
4. In these extracts there are not all the utilization schemes identified during our research. For the limit of this article we should not make an example for each of the utilization schemes previously listed.

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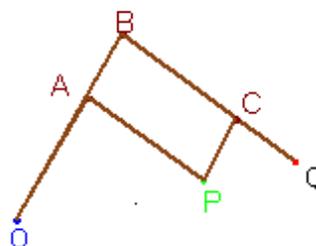
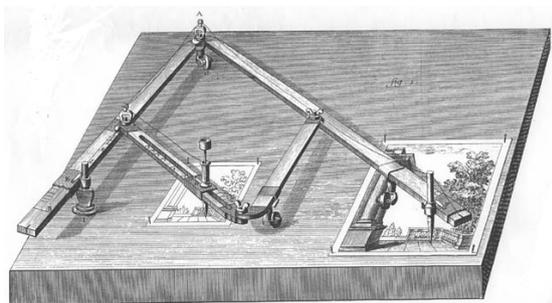


Fig 1: Encyclopédie ou dictionnaire raisonné des sciences, des arts et des métiers (1751-

Fig 2: An image from Scheiner pantograph graphic animation: Four bars are pivoted so

1780)

that they form a parallelogram $APCB$. The point O is pivoted on the plane. It is possible to prove that the points P , Q and O are in the same line and that P and Q are corresponding in the homothetic transformation of centre O and ratio BO/AO .

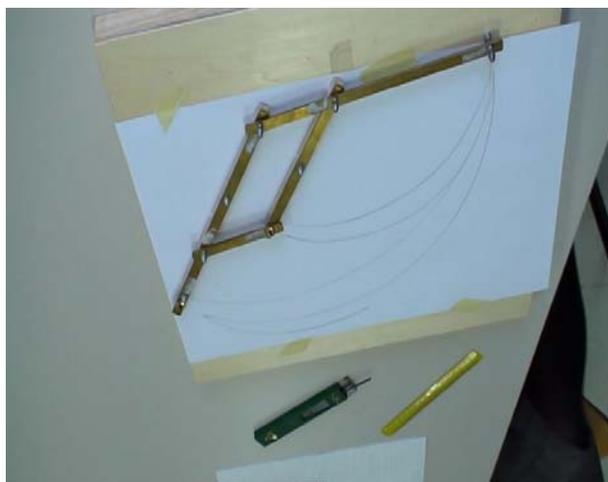


Fig. 3: Anna's drawings



Fig. 4: Anna identifies the parallelogram

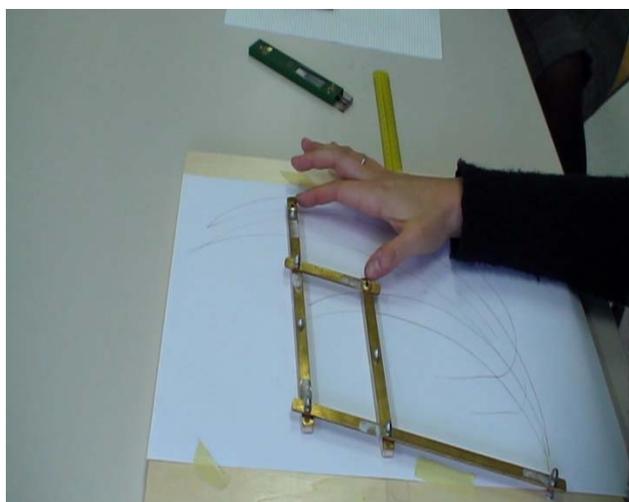


Fig. 5: Anna shows the isosceles triangle

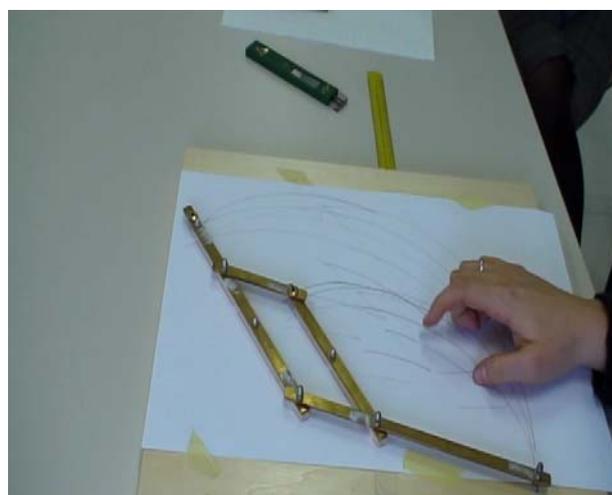


Fig. 6: Anna's gesture for indicating the "reduction in scale" of the zigzag lines

THE UTILIZATION OF MATHEMATICS TEXTBOOKS AS INSTRUMENTS FOR LEARNING

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The mathematics textbook is one of the most important resources for teaching and learning mathematics. Whereas a number of studies have examined the use of mathematics textbooks by teachers there is a dearth of research into the use of mathematics textbooks by students. In this paper results of an empirical investigation of the use of mathematics textbooks by students as an instrument for learning mathematics are presented. Firstly, a method to collect data on student's use of mathematics textbooks is introduced. It is explicated, that this method is capable to explore the actual use of the mathematics textbook by students, and a way of recording the use of the mathematics textbook whenever and wherever students use it. Secondly, results from the study are presented. The results outlined in this paper focus on typical self-directed uses of the mathematics textbook by students.

INTRODUCTION

Research in mathematics education has been concerned with the role of new technologies in the teaching and learning of mathematics from the very beginning computers and information technologies entered the mathematics classroom. In the first ICMI study the computer is even considered to be a new dimension in the mathematics classroom: "We now have a triangle, student-teacher-computer, where previously only a dual relationship existed" (Churchouse et al., 1984). But, this perspective disregards the fact that tools have always been incorporated in teaching and learning mathematics and thus the relationship in the mathematics classroom has never actually been dual. The mathematics textbook was and still is considered to be one of the most important tools in this context. According to Howson, new technologies have not affected its outstanding role: "despite the obvious powers of the new technology it must be accepted that its role in the vast majority of the world's classrooms pales into insignificance when compared with that of textbooks and other written materials." (Howson, 1995)

Valverde et al. (2002) believe that the structure of mathematics textbooks is likely to have an impact on actual classroom instruction. They argue, that the form and structure of textbooks advance a distinct pedagogical model and thus embody a plan for the particular succession of educational opportunities (cf. Valverde et al., 2002). The pedagogical model only becomes effective when the textbook is actually used. Therefore, mathematics textbooks should not be a subject to analysis detached from its use. It is an interactive part within the activities of teaching and learning mathematics In order to develop a better understanding of the role of the mathematics

textbooks within the activities of teaching and learning mathematics an activity theoretical model was developed (Rezat, 2006a):

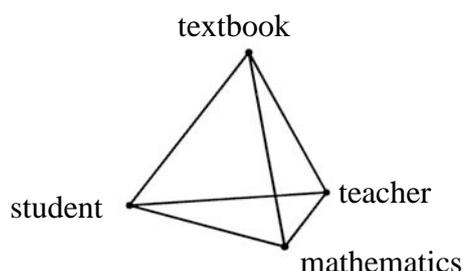


Fig. 1: Tetrahedron model of textbook use

This model is based on the fundamental model of didactical system: the ternary relationship between student, teacher, and mathematics (Chevallard, 1985). The mathematics textbook is implemented as an instrument at all three sides of the triangle: teachers use textbooks in the lesson and to prepare their lessons, by using the textbook in the lesson teachers also mediate textbook use to students, and finally students learn from textbooks. Thus, each triangle of the tetrahedron-model represents an activity system on its own. From an ergonomic perspective it is argued that artefacts have an impact on these activities, because on the one hand they offer particular ways of utilization and on the other hand the modalities of the artefacts impose constraints on their users (cf. Rabardel, 1995, 2002). Thus, the mathematics textbook has an impact on the activity of learning mathematics as a whole that is represented by the didactical triangle on the bottom of the tetrahedron.

Whereas a number of studies have examined the role of new technologies in terms of tool use (cf. Lerman, 2006) the role of the mathematics textbook as an instrument for teaching and learning has not gained much attention. So far, a number of studies have examined the use of mathematics textbooks by teachers (e.g. Bromme & Hömberg, 1981; Haggarty & Pepin, 2002; Hopf, 1980; Johansson, 2006; Pepin & Haggarty, 2001; Remillard, 2005; Woodward & Elliott, 1990) whereas there is a dearth of research into the use of mathematics textbooks by students (Love & Pimm, 1996). This is striking, because as pointed out by Kang and Kilpatrick (1992), textbook authors regard the student as the main reader of the textbook.

In order to develop a better understanding of the impact that textbooks have on learning mathematics a qualitative investigation was carried out in two German secondary schools that focused on how students use their mathematics textbooks.

METHOD AND RESEARCH DESIGN

The difficulty of obtaining data on students working from textbooks is one reason that Love and Pimm (1996) put forward in order to explain the dearth of research into student's use of texts. Therefore, developing an appropriate methodology to collect data on student's use of mathematics textbooks can be regarded as a major issue in this field.

First of all, the method of data-collection has to be in line with the situation of textbook use. In Germany, schools either provide mathematics textbooks to students for one year or students buy the books. Accordingly, students have access to their mathematics textbook at school and at home. From previous research there is evidence, that German teachers rely heavily on the textbook in the preparation of lessons and also during lessons. (Bromme & Hömberg, 1981; Hopf, 1980; Pepin & Haggarty, 2001).

The method to collect data on student's use of mathematics textbooks was developed within the framework of the activity theoretical model of textbook use. According to this model the use of mathematics textbooks is situated within an activity system constituted by the student, the teacher, the mathematics textbook, and mathematics itself. First of all, this implies that a method to investigate the use of mathematics textbooks by students has to take all four vertices of the tetrahedron-model into consideration.

In addition, three criteria were established for an appropriate methodology to collect data on student's use of mathematics textbooks:

1. The actual use of the mathematics textbook should be recorded in detail.
2. Biases caused by the researcher, by the situation or by social desirability should be minimized.
3. The use of the textbook should be recorded at any time and any place it is used.

Criterion 1 leads to the rejection of quantitative methods and of methods that are likely to reveal only verbalized uses of the textbook, e.g. interviews. Experimental settings and artificial situations are refused due to criterion 2. Approaches that are solely based on observation are discarded because of criterion 3.

The methodological framework that was developed according to the three criteria combines observation and a special type of questioning. First of all, the students were asked to highlight every part they used in the textbook. Additionally, they were asked to explain the reason why they used the part they highlighted in a small booklet by completing the sentence "I used the part I highlighted in the book, because ...". By assigning more than one comment to a highlighted book section the reuse of book sections becomes apparent. This method of data-collection was developed in order to get the most precise information about what the students actually use and why they use it by keeping the situation of textbook use as natural as possible. Nevertheless, highlighting sections in a textbook is not the natural way to use the textbooks and therefore a bias on the data cannot be totally excluded.

Provided that the students take their task seriously, this method enables to collect data on the use of the textbooks whenever and wherever students use it and therefore meets criterion 3.

In addition, the lessons were observed and field notes were taken. On the one hand the overall structure of the lesson was recorded in the field notes using a table

comprising three columns: time, activity/content and remarks. On the other hand all utterances concerning the textbook were transcribed literally. Furthermore, a focus was put on all utilizations of the textbook. Both, the use of the textbook by the students and by the teacher was taken into account. This is important for several reasons:

First of all, there is evidence from previous research that the teacher plays an integral part in mediating textbook use. Because of that, the teacher was included as a variable in the model of textbook use.

Secondly, the observation provides an insight into the way the teacher mediates textbook use in the classroom. It makes a difference if the students only use the textbook when they are told to by the teacher or if they use it of their own accord. This difference will become apparent through classroom observation.

Thirdly, the methodological triangulation provides a measure for the validity of the data. Collecting data on how the textbook has been used in the classroom makes it possible to compare the markings and comments of the students with the field notes. The degree of correspondence between these two sources relating to the use of the textbook in the classroom indicates how serious the students took their task.

While the method of highlighting and taking notes especially satisfies criterion 3 and at the same time aims at both, providing a precise record of the actual use of the textbook by students (criterion 1) as well as keeping biases low (criterion 2), the intention of the observation is threefold. On the one hand the idea is to lower biases that might be caused by the method of highlighting (criterion 2) and on the other the triangulation of two different data-sources provides a measure for the validity of the student's data.

In addition to the previously described methods interviews were conducted with selected students.

Data was collected for a period of three weeks in two 6th grade and two 12th grade classes in two German secondary schools. Within the German three partite school system, these schools are considered to be for high achieving students. All four classes were taught by different teachers.

The coding process followed the ideas of Grounded Theory by Strauss and Corbin (Strauss & Corbin, 1990). Categories were established in the process of analysing the data. Each highlighted section in the textbook was categorized according to the kind of block it belongs to (introductory tasks, exposition, worked example, kernels, exercises) (cf. Rezat, 2006b), the activity it was involved in, and finally whether the use of the section was mediated by the teacher or not.

In order to understand the role of the mathematics textbook as an instrument within the activity system represented by the tetrahedron model Rabardel's (1995, 2002) theory of the instrument was used. As Monaghan (2007) points out, this theory has proven fruitful to provide insights into the use of new technologies as instruments for

learning mathematics. According to Rabardel an instrument is a psychological entity that consists of an artefact component and a scheme component. In using the artefact with particular intentions the subject develops utilization schemes which are shaped by both, the artefact and the subject. Vergnaud (1998) suggests that schemes are characterized by two operational invariants: theorems-in-action and concepts-in-action. Since these two operational invariants are put forward in order to describe the representation of mathematical knowledge, it is not self-evident to apply them to knowledge related to the use of an artefact like the mathematics textbook. Therefore, it is suggested to generalize Vergnaud's notion of theorems-in-action and concepts-in-action to the notion of beliefs-in-action. As well as concepts-in-action beliefs are supposed to guide human behaviour by shaping what people perceive in any set of circumstances (Schoenfeld, 1998). Like theorems-in-action beliefs are propositions about the world that are thought to be true (Philipp, 2007). The appendix 'in-action' is supposed to underline that beliefs-in-action might be inferred from actions. They do not necessarily have to be expressed verbally. Because of its universality, the notion of beliefs-in-action offers an appropriate means to characterize operational invariants of utilization schemes linked to any artefact.

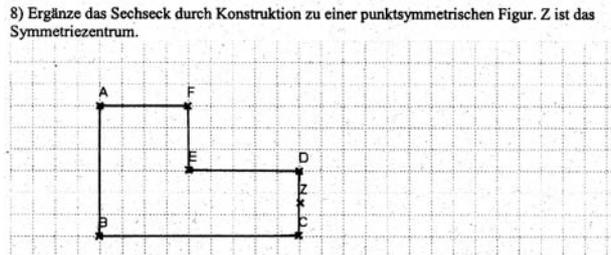
RESULTS

A first and a major result of the study is, that students do not only use the mathematics textbook when they are told to by the teacher. But, they also use the textbook self-directed. The following analysis focuses on utilizations of the mathematics textbook that students perform in addition to teacher mediated textbook use.

Students incorporate their mathematics textbook as an instrument into four activities: solving tasks and problems, consolidation, acquiring mathematical knowledge, and activities associated with interest in mathematics. From the data it was possible to reconstruct several individual utilization schemes of the mathematics textbook related to these activities. Comparing the individual schemes of different students related to the same activity revealed that some of the schemes were analogous in terms of the underlying beliefs-in-action. These schemes were generalized to utilization scheme types (UST). USTs are general in the way, that they allow to classify individual utilization schemes of the textbook into USTs and thus make individual utilizations comparable. Nevertheless, different students might show different USTs. The USTs are not general in the way that they are common to all students.

Solving tasks and problems is associated with activities where students utilize their mathematics textbook in order to get assistance with solving tasks and problems. Three different USTs were found related to this activity. It was observed that students repeatedly utilize specific blocks from the textbook as an assistance to solve tasks and problems. Worked examples and boxes with kernels were instrumentalized in most of the cases. This scheme could be traced back to the belief-in-action that a specific block from the textbook is useful in order to solve tasks and problems. It was also

observed that students choose sections from the textbook that show similarities to the task. For example, Oliver is working on the following task that is not from the textbook:



He looks for assistance in the textbook and reads a task in the textbook that is located next to an image, which is identical to the image in the task. From this behaviour it can be inferred that Oliver expects information concerning the image next to it. In his case, the information is not useful for solving the task, because it is a task itself.

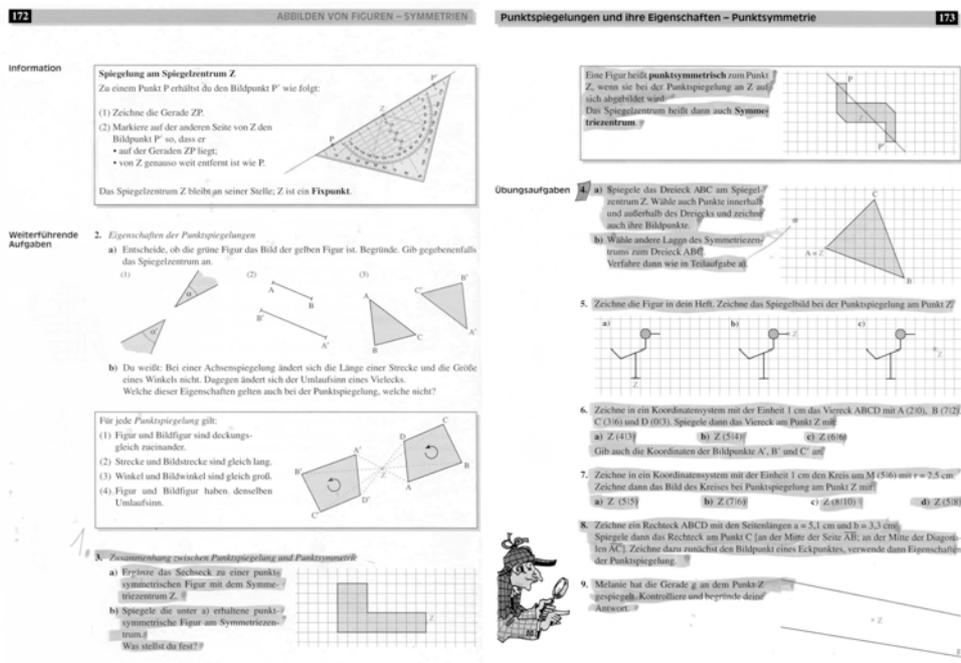


Fig. 2: Passage Oliver used from the textbook “because he was looking for something” (Griesel *et al.*, 2003)

In order to get assistance with solving tasks and problems it was also observed that students search an adequate heading in the book and start reading from there until they find useful information. From this behaviour it was inferred that these students expect useful information related to a subject at the beginning of a lesson in the textbook.

All three USTs reveal that students are looking for information in the book that can be directly applied to the task. The only difference is the way they are approaching the information. Hardly ever does it seem like students want to understand the mathematics first and then apply it to the task.

Consolidation is associated with all activities that students perform in order to improve their mathematical abilities related to subject matters that were already dealt with in the mathematics class. One UST of students using their mathematics textbook for consolidation is strongly related to teacher mediated exercises from the textbook. They either recapitulate tasks and exercises from the book that the teacher mediated or they pick tasks and exercises that are adjacent to teacher-mediated exercises. This was traced back to the belief-in-action that effective practising means to do tasks and exercises that are similar to teacher-mediated exercises. If students pick tasks that are adjacent to teacher mediated tasks this is also supported by the belief-in-action that adjacent tasks in the textbook are similar. The use of specific blocks for consolidation was also observed. One UST is that students only read the boxes with the kernels of several lessons in the textbook.

So far, consolidation seems to comprise learning rules, recapitulating teacher mediated tasks and solving tasks that are similar to teacher mediated tasks respectively. But, it was also observed that students either utilize special parts at the end of a unit that are designed especially for recalling and practising the main issues of the unit or they scan the section in the book relating to the actual topic in the mathematics class and read different parts of it in order to consolidate their understanding of the topic. Both UST are less dependent on teacher mediation and show more proficiency in the utilization of the textbook.

Whereas consolidation related to previously treated topics, acquisition of knowledge is associated with activities where students use parts of the book that have not been a matter in the mathematics class so far. The UST identified in this context is that students use parts from the proximate lesson in the textbook. This is supported by the belief-in-action that the chronological succession of topics in the mathematics class will follow the order of the textbook.

Students also used parts of their textbook because they thought they were interesting. These utilizations are associated with activities related to interest in mathematics. In this case the UST is connected to the use of images and other salient elements from the book. Students either only look at the images or they read passages that are next to images or other salient elements. Looking just at the pictures does not seem to be associated with learning mathematics though. This UST usually is observed in the context of other utilizations of the textbooks. It seems like this UST is not based on a belief-in-action, but that salient elements in the textbook catch the attention of the students while there utilizing it for another purpose.

CONCLUSIONS

The activities the mathematics textbook is involved in do not only give an insight into student's utilizations of mathematics textbooks, but they also give an idea of what learning mathematics is about for students. The USTs show that learning mathematics with the mathematics textbook comprises activities as solving tasks and problems, consolidating mathematical knowledge and skills, acquiring new contents.

The USTs show how the textbook is used as an instrument within these activities. Furthermore, these USTs reveal interesting insights into student's dispositions towards mathematics. Learning mathematics comprises mainly learning rules, applying rules and worked examples to tasks, and developing proficiency in tasks that are similar to teacher mediated tasks.

Consciousness about student's USTs could affect teacher's ways of implementing the mathematics textbook in the teaching process. Some USTs show that the use of mathematics textbooks by teachers in the classroom is an important reference for student's utilizations of the textbook. For example, the UST that is characterized by the utilization of tasks that are adjacent to teacher mediated tasks for consolidation is dependent on the mediation of tasks from the textbook by the teacher. Therefore, it is important that the teacher uses tasks from the textbook in order to support student's individual learning of mathematics. Another example is the anticipation of the next topic in the mathematics class by reading parts of the proximate lesson in the textbook. This UST shows that students believe that the course of the mathematics lessons will follow the order in the book. Accordingly, the textbook provides orientation for students, and it can therefore be considered important that teachers follow the succession of the topics in the book.

It was pointed out, that Valverde et al. (2002) argue that the structure of mathematics textbooks advances a distinct pedagogical model and is likely to have an impact on actual classroom instruction. From an ergonomical perspective it can be argued that the structure of the book also has an impact on the USTs of the students. This raises the question of how a textbook must be structured in order to promote desirable USTs.

Furthermore, this study provides evidence that Rabardels theory of the instrument is not only capable of conceptualizing human-computer-interaction, but is also applicable to non technological resources. The conceptualization of student-textbook-interaction on the basis of this theoretical framework provides interesting insights into different aspects of learning mathematics. The UST do not only provide a better understanding of student's utilizations of mathematics textbooks, but also reflect student's ways of learning mathematics. Furthermore, it can be inferred from student's USTs how the textbook is effectively used in the classroom by the teacher. Accordingly, a better understanding of student's utilizations of mathematics textbooks is a prerequisite for effective implementation of mathematics textbooks into teaching.

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TEACHERS' BELIEFS ABOUT THE ADOPTION OF NEW TECHNOLOGIES IN THE MATHEMATICS CURRICULUM

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The purpose of the present study was to examine elementary mathematics teachers' concerns in relation to the expected implementation of the new technology based mathematics curriculum in Cyprus. A questionnaire examining teachers' concerns towards this innovation was administered to seventy four elementary school teachers. Results provide evidence that the majority of teachers were positive towards the innovation. Results revealed the existence of four factors related to teachers' concerns and beliefs towards the innovation, namely the concerns about the nature of the curriculum, teachers' self-efficacy beliefs, concerns about the consequences on the organization of teaching, and concerns about the effectiveness of the curriculum.

INTRODUCTION AND THEORETICAL FRAMEWORK

Based on the premise that Information and Communication Technologies (ICT) can have a positive impact on mathematics teaching and students' learning outcomes, technology based activities have been implemented in mathematics curricula in a number of countries (Hennessy, Ruthven, & Brindley, 2005). This implementation is, however, not an easy yet straightforward task; a number of factors such as mathematics teachers' beliefs and concerns about the adoption of this innovation, facilities, in-service teachers' training, and available resources might influence the successful implementation of the innovation (Hennessy, et al., 2005).

Gibson (2001) argues that technology by itself will not and can not change schools. It is only when reflective and flexible educators integrate technology into effective learning environments, that the restructuring of the classroom practices will benefit all learners. The introduction and implementation of ICT in the teaching and learning of mathematics has not been successful in a number of cases in different countries (Hennessy, et al., 2005). As reported by the British Educational Communications and Technology Agency (2004), only few teachers succeed in integrating ICT into subject teaching in a fruitful and constructive way that can promote students' conceptual understandings and can stimulate higher-level thinking and reasoning. In most of the cases, teachers just use technology to do what they have always done, although in fact they often claim to have changed their teaching practice. Further, a number of teachers report that they do not feel comfortable with the integration of ICT in subject teaching, since their role was predetermined and designed by educational authorities and teachers feel that they face a lack of professional autonomy (Olson, 2000). Olson (2000) proposes that integrating new technologies challenges teachers and, thus,

requires innovators to understand and be engaged “in conversations with teachers about their work culture, the technologies that sustain it and the implications of new approaches for those technologies” (p.6).

Among the factors that have been identified as crucial for the successful integration of ICT in the mathematics curricula are teachers’ concerns and beliefs about this change (Van den Berg et al., 2000). To this end, a number of studies focused their research efforts on examining teachers’ concerns towards the adoption of ICT in general (Gibson, 2001) or towards an innovation in education (Hall & Hord, 2001). According to Hord and colleagues (1998), concerns can be described as the feelings, thoughts, and reactions individuals develop in regard to an innovation that is relevant to their job (Hord, Rutherford, Huling-Austin & Hall, 1998). In this framework, innovation concerns refer to a state of mental arousal resulting from the need to cope with new conditions in one’s work environment (Hord et al., 1998). Furthermore it is argued that teachers are also important as representatives of their students’ needs. In this respect, the opinions and views of teachers can be considered to be reflective of opinions and views from two major stakeholder groups instead of one, and this further underlines the importance of studying teachers’ concerns before and during implementing a new innovation in education (Hossain, 2000).

A model that has been widely used for the evaluation of the innovations in education is the Concerns-Based Adoption Model (CBAM) (Hord, et. al., 1998). This model can be used to identify how, for example, teachers (who feel that they will be affected by the new technology based curriculum in mathematics) will react to the implementation of the innovation (Christou et al., 2004). The CBAM includes three tools that are used for collecting data related to teachers’ concerns and beliefs. These tools include: (a) the levels of use questionnaire, (b) the innovation configurations, and (c) the stages of concerns questionnaire. The stages of concerns questionnaire was adopted, modified and used in the present study to measure elementary school teachers concerns and beliefs about the innovation of introducing a technology based mathematics curriculum (Hall & Hord, 2001). The stages of concerns questionnaire includes items for measuring teachers’ concerns towards seven stages of concern, namely the Awareness, Informational, Personal, Management, Consequences, Collaboration, and Refocusing stages.

Briefly, in the awareness stage teachers have little knowledge of the innovation and have no interest in taking any action. In the informational stage teachers express concerns regarding the nature of the innovation and the requirements for its implementation. In the personal stage teachers focus on the impact the innovation will have on them, while in the management stage their concerns begin to concentrate on methods for managing the innovation. In the consequences and collaboration stages their concerns focus on student learning and on their collaboration with their colleagues. Finally on the refocusing stage teachers evaluate the innovation and make suggestions for improvements related to the innovation and its implementation (Hord et al., 1998).

PURPOSE AND RESEARCH QUESTIONS

The purpose of the present study was to examine teachers' beliefs about an innovation that will soon take place in Cyprus, namely the adoption of a new mathematics curriculum. The new curriculum is expected to incorporate an inquire based approach and to integrate technological tools into the teaching and learning of mathematics. The study aimed at investigating how well prepared teachers feel about implementing the new curriculum and whether teachers are positive towards this innovation.

The research questions of the study were the following:

- (a) What beliefs do teachers have regarding the adoption of a mathematics curriculum that integrates technology?
- (b) Do teachers' beliefs differentiate in accordance to their teaching experience and their studies?
- (c) Do teachers feel capable to implement the new curriculum and if not, what do they reported that they need to be appropriately prepared?

METHODOLOGY

Participants

The participants in this study were 74 teachers from nine elementary schools in Cyprus. Schools were randomly selected from the district of Nicosia. One hundred questionnaires were mailed to schools and 74 were returned to researchers. Teachers were grouped according to their teaching experience and their studies, in three categories and in two categories, respectively. The numbers of teachers in each group are presented in Table 1.

Table1. *Teachers involved in the study by years of teaching experience and level of studies*

Studies	Teaching experience		
	1-5	6-15	>15
Postgraduate studies	16	13	9
Undergraduate studies	6	16	14
Total	22	29	23

Batteries

The questionnaire included 23 likert-scale items. Part of the items was adopted from previous stages of concerns questionnaires (e.g., Hall & Hord, 2001; Christou et al., 2004). Since these studies focused on teachers' adoption of innovations in general, the items were modified to serve the purposes of investigating teachers' concerns of the adoption of the innovation of using ICT in the teaching of mathematics. The 23 items were on a 7-point likert scale, from 1 (strongly disagree) to 7 (strongly agree); all responses were recorded so that higher numbers indicated greater agreement with the statement. The questionnaire also included two open-ended questions in which teachers were asked to report on: (a) what they need in order to feel confident and well prepared to implement the new technology-based mathematics curriculum, and (b) their beliefs and concerns in general about their new role in teaching mathematics after the implementation of the innovation.

The data were analyzed using the statistical package SPSS. An exploratory factor analysis and an multiple analysis of variance were conducted. Descriptive statistics were also used.

RESULTS

The exploratory factor analysis resulted in four factors, including the 21 items of the teachers' questionnaire. The following four factors arose: (a) Concerns/Beliefs about the nature of the new mathematics curriculum, (b) Teachers' self-efficacy beliefs, (c) Concerns about the consequences on the organization of teaching, and (d) Concerns/Beliefs about the effectiveness of the new curriculum. The loadings of each statement in the four factors are presented in Table 2.

Furthermore, teachers that participated in the study appeared to have positive beliefs about the nature of the proposed new curriculum ($\bar{x}=5,1$). Particularly, the majority of teachers reported that the new curriculum will put emphasis on pupils' way of thinking and their reasoning skills, on problem solving and on the enhancement of students' conceptual understanding. The mean score of the 'Self-efficacy beliefs' factor ($\bar{x}=4,1$) might claim that teachers feel quite confident and well prepared to use the new curriculum. Although the mean score can be considered quite large, it is important to underline that the majority of teachers reported that there is a strong need for in-service teachers' training before the implementation of the innovation.

Furthermore, it seems that teachers' beliefs concerning the consequences on the organization of teaching are also rather positive. The mean score ($\bar{x}=4,0$) reveals that many teachers who participated in this study believe that after the implementation of the curriculum the stress of the teacher regarding the organization of teaching will be reduced and that this innovation will relieve the teacher from a great deal of

Table 2: *Factor analysis results*

Statements	Factors			
	F1	F2	F3	F4
The adoption of the new curriculum will place sufficient emphasis on the development of pupils' thinking.	,831			
The use of the computer in mathematics develops pupils' mathematical thinking and reasoning skills.	,744			
The new curriculum that takes advantage of the computer in the teaching of mathematics promotes problem solving.	,730			
The use of computers promotes conceptual understanding in mathematics.	,704			
The new curriculum places emphasis on investigation.	,618			
The knowledge that students acquire through the use of computers is not superficial.	,572			
I do not feel confident about teaching mathematics with computers.		,808		
I do not face difficulties in teaching mathematics with computers.		,759		
The implementation of the new curriculum requires the use of methods that I am not familiar with. (recoded)		,723		
I do not need guidance to teach mathematics with the use of computers. (recoded)		,715		
I know how to use computers effectively in mathematics in the classes that I teach.		,541		
The computer based activities that will be included in the new curriculum will reduce teacher's preparation.			,856	
With the implementation of the new curriculum, teachers' stress about the organization of teaching will be reduced.			,846	
Pupils' homework will be reduced.			,578	
Teaching of mathematics with the use of computers will allow me to follow the progress of each pupil.				,775
The adoption of the new curriculum is a useful innovation.				,613
I believe that the adoption of the new curriculum will improve students' achievement.				,557
The integration of computers in mathematics teaching will result in major changes in the teaching of mathematics.				,418

preparation. They also reported that they expect that pupils' homework will be reduced and that the integration of technology will improve the organization/management of the classroom.

Similarly, the mean score for the fourth factor was also quite large ($\bar{x}=5,3$). Teachers appeared to be positive that the new curriculum will introduce major changes in the teaching of mathematics and that it will improve results. They also consider the mathematics curriculum that integrates technology as a useful innovation in primary education mathematics and as a means that will allow them to follow the progress of each pupil.

Table 3: *The four factor model mean scores*

Factors	Mean	SD
F1: Beliefs about the nature of the new mathematics curriculum	5,1	0,9
F2: Teachers' self-efficacy beliefs	4,1	1,2
F3: Concerns/Beliefs about the consequences on the organization of teaching	4,0	1,2
F4: Concerns/Beliefs about the effectiveness of the new curriculum	5,3	0,9

In order to investigate whether teachers' beliefs in four factors differentiate in accordance to the years of teaching experience and level of studies, a multivariate analysis of variance (MANOVA) was conducted, with the statements of teachers in four factors as dependent variables and years of teaching experience and studies as independent ones. The results of the multivariate analysis showed that there were significant differences between teachers beliefs across the years of teaching experience (Pillai's $F_{(2,64)} = 2,211$, $p < 0,05$). More concretely, the results indicated that there were statistically significant differences between the three groups only in the first factor, 'Beliefs about the nature of the new mathematics curriculum' ($F=5,667$, $p < 0,05$). It was found that the significant differences related to this factor appeared only between inexperienced teachers (years of teaching experience: 1-5) and experienced teachers (6-15) ($p < 0,05$) and between inexperienced teachers and teachers with more than 16 years of experience who probably possess administrative places (16+) ($p < 0,05$). As the years of experience increase the beliefs about the nature of the curriculum get higher. In the other three factors there were no significant differences between the three groups of teachers. The results of the multivariate

analysis indicate that there were no significant differences between teachers' beliefs in the four factors in relation to their level of studies (Pillai's $F_{(1,68)} = 0,661$, $p > 0,05$).

Of importance are also teachers' responses to a number of individual items of the questionnaire. The item with the highest mean score ($\bar{x}=6,1$) was the one that referred to the need for training courses. Specifically, the majority of teachers (60 teachers), agreed strongly (chose 7) or very much (chose 6), and only two teachers disagreed that training courses are necessary for the successful implementation of the technology based curriculum in mathematics. The items with the lowest mean score were the 'The knowledge that students acquire through the use of computers is superficial' ($\bar{x}=2,7$) and 'The adoption of the new curriculum for the integration of computers in the teaching of mathematics is a useless innovation' ($\bar{x}=2,1$). Teachers' responses to these items also showed that teachers consider the integration of technology in the teaching of mathematics as a useful innovation that will enforce learning, something that is in line with the high mean score ($\bar{x}=5,2$) which refers to the improvement of students' achievement after the implementation of the new curriculum. Their positive beliefs and willingness to integrate technology into teaching appears also from the high mean score ($\bar{x}=5,2$) of the item 'I would like to teach mathematics lessons using computers'.

Table 4: *Mean scores for questionnaire items*

Items	Mean	SD
The knowledge that students acquire through the use of computers is superficial.	2,7	1,2
Training courses for the integration of computers in the teaching of mathematics are necessary for teachers.	6,1	1,3
I would like to observe and participate in technology based mathematics lessons taught by more experienced teachers.	5,2	1,4
I believe that the adoption of the new mathematics curriculum that integrates technology into teaching will improve students' achievement.	5,2	1,1
The adoption of the new curriculum for the integration of computers in the teaching of mathematics is a useless innovation.	2,1	1,7

Teachers' need for training courses came also up from their answers in the first open-ended question. Fifty-five teachers answered this question and some of the answers consisted of a combination of different ideas. For this reason some of the teachers are included in the percentage of more than one category of answers. Forty-six teachers (83,6%) stated that they need 'Training courses for the integration of computers in the teaching of mathematics'. The second category that was pointed out by ten teachers (18%) was 'lesson plans and worksheets'. Also, ten teachers (18%) expressed that it is essential to become familiar with the software that will be used, before implementing the innovation, and eight teachers revealed their wish to attend courses that will be held by more experienced teachers. Six teachers stated that they need much guidance, three that they considered the co-operation with colleagues important and three that they need the appropriate infrastructure. The last four answers that were reported only by one teacher each, are the following: (a) training courses for the use of computers, (b) more hours devoted to the teaching of mathematics, (c) one coordinator in each school, and (d) adaptation of the books according to the purpose of the curriculum that integrates technology into teaching.

Regarding the second open-ended question, five categories of answers were identified from the 53 answers that were gathered. The majority of teachers (46 teachers-88.7%) stated that they feel that their role would be more like a facilitator during the learning process. Three teachers reported that their role will remain the same and two just mentioned that they will have a decisive role. Lastly, one teacher pointed out that his role will change; he will need to first develop more positive attitudes and knowledge towards the innovation and then transfer them to his students.

DISCUSSION

The purpose of this study was to examine teachers' beliefs and concerns regarding the expected innovation of integrating the new technology-based curriculum in mathematics at the elementary schools in Cyprus.

The questionnaire was used to provide a description of teachers' concerns and beliefs about the integration of the new technology-based mathematics curriculum, which shows that the great majority of teachers welcome the expected change in mathematics curriculum after the introduction of ICT and they seem to have positive beliefs in general and positive self-efficacy beliefs for teaching mathematics using ICT (Chamblee & Slough, 2002).

The present study showed that in general teachers welcome the introduction of ICT in mathematics education. According to the teachers that participated in the study, however, the majority underlined the importance of in-service and pre-service training on implementing ICT in the mathematics teaching. This is crucial for the successful implementation of the innovation as, according to teachers' answers, teacher role will be changed, new classroom dynamics will appear, and student learning in mathematics will be improved. The results of the study also revealed that

teachers believe that this innovation is important and can positively change the way mathematics are taught and student learning can be improved, but this is not an easy task; careful planning is needed and resources like software and lesson plans will help teachers in their new different role (Luehmann, 2002).

The results revealed that differences of beliefs across different groups of teachers in terms of teaching experience existed only for the first factor, namely the 'Beliefs about the nature of the new mathematics curriculum'. Specifically, teachers' beliefs about the nature of the curriculum differed between the inexperienced teachers and teachers with more than five years of experience. As teachers' experience increases, teachers feel that the new curriculum can place sufficient emphasis on the development of pupils' thinking and that the appropriate use of computers can assist students in further developing their mathematical thinking and reasoning skills. These teachers also reported that the integration of ICT in the teaching and learning of mathematics can assist teachers in teaching problem solving skills, an essential and core part of the mathematics curriculum.

The themes emerging from the analysis of teachers' beliefs and concerns about the expected integration of ICT in the mathematics curricula converge to offer a grounded model for the innovation. This model underlines the importance of teachers' training and knowledge on the various aspects that are related with the integration of ICT in mathematics. Furthermore, teachers appeared to be very positive about the innovation and that they expect that the role of ICT will assist the teaching and learning of mathematics. This result is very prominent and encouraging, considering that the majority of these teachers were not well informed about the innovation from educational authorities, but were rather themselves positive and they believe that the role of technology can positively influence the role and impact of school mathematics on student learning and problem solving abilities.

In the future, a longitudinal study could be conducted to examine the development of teachers' beliefs and concerns over the first steps of the innovation. Since teachers appear to have quite strong and positive beliefs and they expressed their willingness to adopt and use the new curriculum, a study on the development of their concerns and beliefs over a long period could provide more useful information for practitioners and researchers. To better examine the research questions that guided the present study, it is recommended that a comparative study could be conducted to examine the differences between pre-service and in-service teachers' concerns and beliefs towards the new technology based mathematics curriculum, and to identify how the more technology experiences pre-service teachers have might influence their concerns and beliefs about the innovation.

Teachers' beliefs and concerns are an important issue for the successful integration of the ICT in the mathematics curricula, and this study examined this issue in relation to elementary school teachers in Cyprus. It is expected that such explorations can suggest good practices for educational authorities and teacher educators. Finally, the findings discussed would provide avenue and references for future studies.

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SYSTEMIC INNOVATIONS OF MATHEMATICS EDUCATION WITH DYNAMIC WORKSHEETS AS CATALYSTS

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With reference to theories of cybernetics the paper proposes a general theoretical framework for initiatives aiming at systemic innovations of educational systems. It shows that it is essential to initiate incremental-evolutionary changes on the meta-level of beliefs and attitudes of the agents involved. For the theoretical foundation of concrete activities in mathematics education the didactic concept of learning environments is developed on the basis of constructivist notions of teaching and learning. Such learning environments may integrate dynamic mathematics for educational processes. So technology and especially dynamic worksheets can be considered as means and catalysts for improvements of mathematics education on system level.

Keywords: *systemic innovation, learning environment, dynamic mathematics*

INNOVATIONS IN COMPLEX SYSTEMS

There are many efforts to innovate educational systems – on regional, national and international levels – aiming at changes of teaching and learning. For understanding the structure of such initiatives a short glance at theories of cybernetics is useful.

Innovations

The OECD defines an *innovation* as the implementation of a new or significantly improved product, process or method (OECD, Eurostat, 2005, p. 46). Thus an innovation requires both an *invention* and the *implementation* of the new idea.

In the educational system we are in a situation where lots of concepts, methods and tools have been developed for substantial improvements of teaching and learning. Three examples:

- (1) There is a wide range of current pedagogical theories that emphasize self-organised, individual and cooperative inquiry-based learning.
- (2) There exists a huge amount of material for teaching and learning in a constructivist manner – available e.g. in electronic data bases or by print media.
- (3) A large variety of software and other tools for the integration of ICT in educational processes has been developed.

But for real innovations these promising theories and products have to be implemented in the educational system. Here implementation means a good deal more than diffusion or dissemination of material (papers, guidelines, software tools etc.). And implementation should reach the real agents in the school system, i.e. the

teachers and students, their thinking and their working. Let's remember the three examples from above:

- (1) Teachers should teach according to current pedagogical concepts.
- (2) The proposed new task culture should become standard in everyday lessons.
- (3) ICT should be used as a common tool for exploring mathematics.

So for substantial innovations we do not need further material. We need changes in teachers' and students' notions of educational processes, in their attitudes towards mathematics and in their beliefs concerning teaching and learning at school. Hence the crucial question is: How can substantial innovations in the complex system of mathematics education be initiated and maintained successfully?

Complex Systems

In theories of cybernetics a system is called "complex", if it can potentially be in so many states that nobody can cognitively grasp all possible states of the system and all possible transitions between the states (Malik, 1992; Vester, 1999). Examples are the biosphere, a national park, the economic system, mathematics education in Europe and even mathematics education at a concrete school.

Complex systems usually are networks of multiply connected components. One cannot change a component without influencing the character of the whole system. Furthermore real complex systems are in permanent exchange with their environment.

Maybe this characterization of complex systems seems a bit fuzzy. But, nevertheless, it is of considerable meaning. Let us regard the opposite: If a system is not complex, someone can overview all possible states of the system and all transitions between the states. So this person should be able to steer the system as an omnipotent monarch leading it to "good" states. In contrast, complex systems do not allow this way of steering.

Steering of Complex Systems

The fundamental problem of mankind dealing with complex systems is how to manage the complexity, how to steer complex systems successfully and how to find ways to sound states.

With reference to theories of cybernetics two dimensions of steering complex systems can be distinguished (Malik, 1992). The first one concerns the manner, the second one the target level of steering activities (see figure 1).

The method of *analytic-constructive* steering needs a controlling and governing authority that defines objectives for the system and determines ways for reaching the aims. Hierarchical-authoritarian systems are founded on this principle. However, fundamental problems are caused just by the complexity of the system. In complex systems no one has the chance to grasp all possible states of the system cognitively.

So the analytic-constructive approach postulates the availability of information about the system that cannot be reached in reality.

In contrast *incremental-evolutionary* steering is based on the assumption that changes in complex systems result from natural growing and developing processes. The steering activities try to influence these systemic processes. They accept the fact that complex systems cannot be steered entirely in all details and they aim at incremental changes in promising directions. The focus on little steps is essential, since revolutionary changes can have unpredictable consequences which may endanger the soundness or even the existence of the whole system.

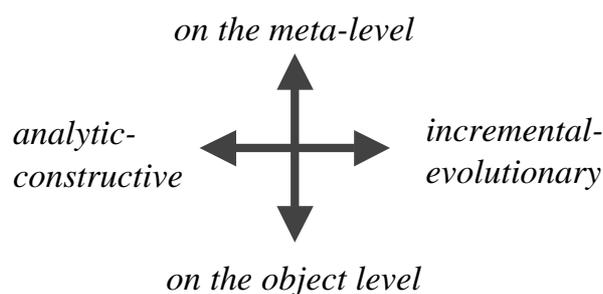


Figure 1: Steering of complex systems

The second dimension distinguishes between the object and the meta-level. The *object level* consists of all concrete objects of the system. In the school system such objects are e.g. teachers, students, books, computers, buildings etc. Changes on the object level take place if new books are bought or if a new computer lab is fitted out. Of course such changes are superficial without reaching the substantial structures of the system.

The *meta-level* comprehends e.g. organizational structures, social relationships, notions of the functions of the system etc. In the school system e.g. notions of the nature of the different subjects and beliefs concerning teaching and learning (e.g. Pehkonen, Törner 1996, Leder, Pehkonen, Törner 2002) are included.

Innovations in Complex Systems

How can substantial innovations in the complex system “mathematics education” be initiated successfully? The theory of cybernetics gives useful hints: Attempts of analytic-constructive steering will fail in the long term, since they ignore the complexity immanent in the system. Changes on the object level do not necessarily cause structural changes of the system. According to the theory of cybernetics it is much more promising to initiate *incremental-evolutionary changes on the meta-level* (see figure 2). They are in accord with the complexity of the system and do not endanger its existence. Nevertheless, they can cause substantial changes within the system by having effects on the meta-level, especially when they work cumulatively.

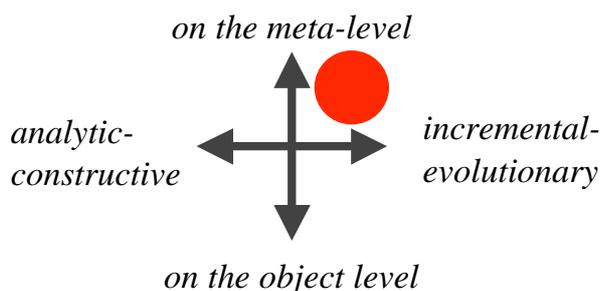


Figure 2: Innovations in complex systems

LEARNING ENVIRONMENTS WITH DYNAMIC WORKSHEETS

Aspects of Learning

Learning is a very complex phenomenon. Initiatives aiming at the development of mathematics education have to take in account the nature of learning. Let us have a very short glance at some fundamental aspects of learning (e.g. Reinmann-Rothmeier & Mandl, 1998; Haberlandt, 1997) which form a background for the latter:

- Learning is a *constructive* process. Knowledge and understanding cannot be simply transported from teachers to students. Cognitive psychology describes learning as a process of construction and modification of cognitive structures. From the view of neurobiology learning is the construction of neuronal networks. Connections between neurons develop and change.
- Learning is an *individual* process. Learning takes place inside the head of each learner. He creates his own knowledge and understanding by interpreting his personal perceptions on the basis of his individual prior knowledge and prior understanding.
- Learning is an *active* process. Cognitive activity means working with the content in mind, viewing it from different perspectives and relating it to the existing network of knowledge.
- Learning is a *self-organized* process. The learner is at least partially responsible for the organization of his individual learning processes. The degree of responsibility may vary in the phases of planning, realizing or reflecting learning processes.
- Learning is a *situative* process. It is influenced by the learning situation. A meaningful context or a pleasant atmosphere can foster learning processes, fear can hamper them.
- Learning is a *social* process. On the one hand the socio-cultural environment has great impact on educational processes. On the other hand learning in school is based on interpersonal cooperation and communication between students and teachers.

Concept of Learning Environments

Considering the aspects of learning noted above the following model seem adequate for teaching and learning processes in school:

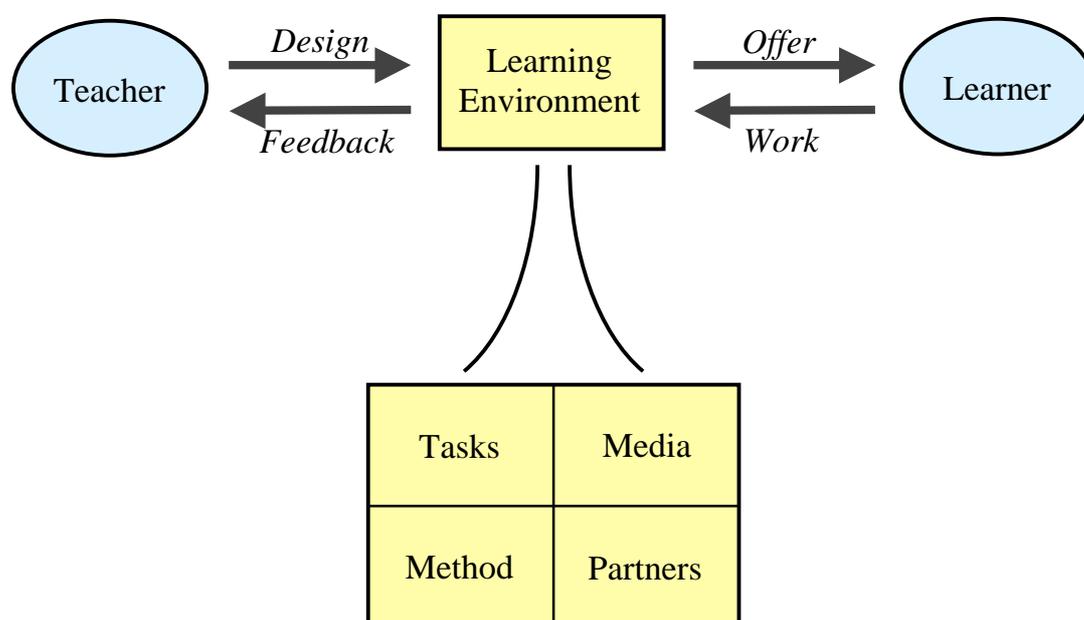


Figure 3: Working with learning environments, four components of learning environments

The *learning environment* is the essential link between the teacher and the learner. This notion includes the *tasks* for the learner's activities, the arrangement of *media* and the *method* for teaching and learning as well as the social situation with the teacher and other learners as *partners* for learning. It belongs to the teacher's field of responsibility to design the learning environment. So he offers a basis for the learner's work. This allows the teacher to get feedback about the learner as well as about the learning environment. This model is based on and extends the didactical concepts of "substantial learning environments" by Wittmann (1995, 2001) or "strong learning environments" by Dubs (1995).

The aspects of learning noted above imply fundamental consequences for the design of learning environments: Tasks should be problem-based with necessary openness for learning by discovery. They should offer meaningful contexts and view situations from multiple perspectives. The teaching methods should make the learners work actively, individually and self-organized. But not less important are the learners' communication and cooperation as well as discussions and presentations of ideas and results. Media can have several supporting functions for these processes.

Before we will discuss the relevance of this model for innovations in educational systems, we look at a specific kind of media which may carry general ideas to practice in school and serve as a catalyst for processes of change.

Dynamic Worksheets

The notion “dynamic mathematics” is currently used for software for dynamic geometry with an integrated computer algebra system, so that geometry, algebra and calculus are connected. When designing learning environments with dynamic mathematics, one faces the necessity to relate dynamic constructions to texts, e.g. for explanations or exercises for the students. For this purpose software for dynamic mathematics – like e.g. GEONExT or Geogebra – can be embedded in HTML-files. So dynamic constructions can be varied on the screen and are combined by the internet browser with texts, pictures, links and other web-elements. This kind of new media for mathematics education is called “dynamic worksheets“ (Baptist, 2004; Ehmman, Miller, 2006).

With respect to the model in figure 3 dynamic worksheets are strongly related to all four components of learning environments: Of course they serve as teaching and learning *media*. Since they include text, they may provide *tasks* and instructions for the students. So implicitly they influence the teaching *method* and the cooperation between the learning *partners* (see next section). Hence, when designing learning environments with dynamic worksheets one should carefully take account of all these components and their impact on students’ learning.

Figure 4 shows an example: The students are given a mathematical situation leading to an optimization problem. The text is combined with a dynamic construction which helps to understand the context. The rectangular can be moved while fitting exactly in the area between the parabola and the x-axis. The tasks help to structure the lesson according to the methodical concept described in the following section.

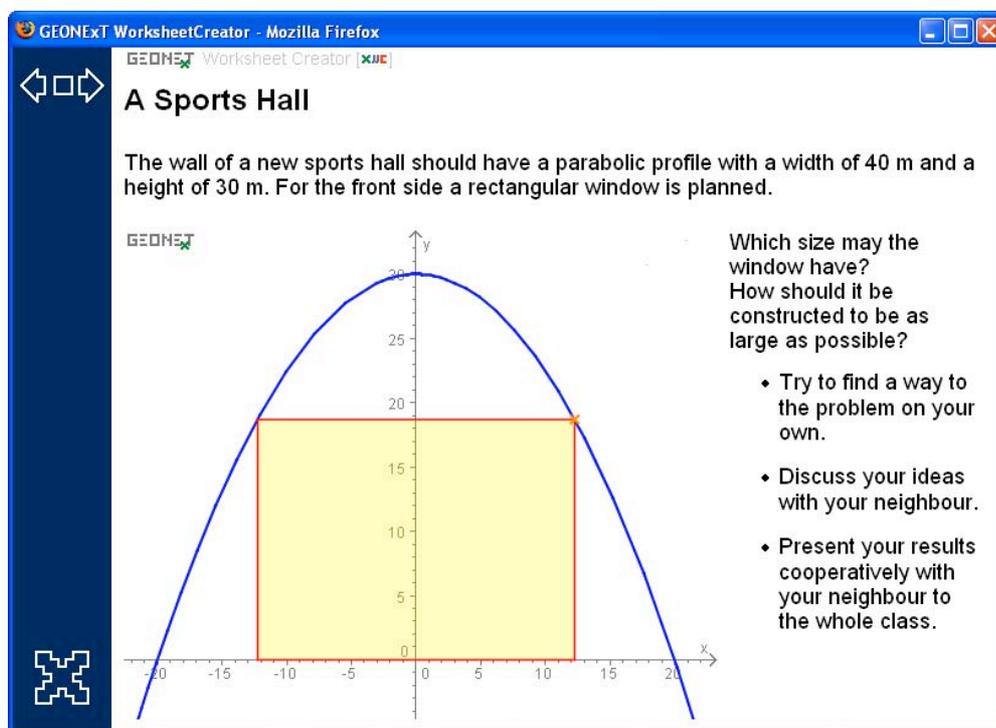


Figure 4: Screenshot of a dynamic worksheet

A Methodical Concept for Learning Environments with Dynamic Worksheets

The use of dynamic worksheets does not automatically improve mathematics education. It is crucial how these media are integrated in teaching and learning processes. If we want to initiate substantial changes on the meta-level of attitudes and beliefs concerning mathematics and mathematics education we have to organize lessons in a way that students work actively, individually, self-organized and cooperatively. They should experience that mathematics is a field for explorations and discoveries. And they should present and discuss their ideas and results cooperatively. Considering the aspects of learning noted above the following four phases structuring lessons with dynamic worksheets methodically are very natural:

1. Individual working: Learning is an individual, active and self-organized process. So at first the students work on their own. They are faced with the necessity to explore the content, to activate their prior knowledge, to develop ideas and to make discoveries. Learning environments with dynamic worksheets offer a framework for such activities and may support them.

2. Cooperation with partners: Learning is a social process. It is very natural that the students discuss their ideas with partners in small groups and that they work on the problems cooperatively. This communication helps to order thoughts and to get further ideas. Meanwhile the teacher may remain in the background or turn his attention to individuals.

3. Presentation of ideas: After having worked individually and in groups the students present their ideas and discuss them in the plenum. The different contributions reveal multiple aspects of the topic and help to view it from varying perspectives. Moreover the students train debating and presentation techniques.

4. Summary of results: Finally the students' results are summarized and possibly extended by the teacher. It is his task to introduce mathematical conventions and to consider curricular regulations. Since the students have already discovered the new content on their own paths, they can more likely integrate the teacher's explanations into their individual cognitive structures.

Table 1: Methodical concept

This methodical concept combines individual learning with working in small groups as well as in the plenum of the class in a very natural way. It is in close relationship to the methodical concepts "Think – Pair – Share" by Lyman (1981) or "I – You – We" by Gallin and Ruf (1998).

Learning by Writing: The Study Journal

The call for papers for working group 7 at CERME 6 emphasizes that technology in school should be considered within a wider range of resources for teaching and learning. Students should draw on ICT in combination with more traditional tools.

Accordingly, dynamic worksheets are only one element of rich learning environments. Especially pencil and paper do not lose relevance when student work with the computer. Noting down thoughts helps to order and arrange thoughts. Writing helps to develop understanding for new subject matters. Hence, when using dynamic worksheets students should regularly be asked to draw figures in their exercise book and to write down observations, conjectures, argumentations and personal statements. The exercise book gets the character of a personal “study journal” that accompanies students on their individual learning paths (Gallin, Ruf, 1998).

When designing dynamic worksheets for students’ self-responsible learning, one should be aware of the risk that students play with the media as with a computer game quite superficially and do not get to the deeper mathematical content. The regular request of working in the exercise book decelerates the process of clicking through the learning environment. So the students are forced to take their time which is indispensable for individual learning.

Finally, the notes in the study journal ensure that ideas and results are still available when the computer is switched off. They are a basis for further presentations, discussions and summaries in the plenum of class (Baptist, 2004).

INCREMENTAL-EVOLUTIONARY SYSTEMIC INNOVATIONS WITH DYNAMIC WORKSHEETS AS PARTS OF LEARNING ENVIRONMENTS

In their plenary talks at CERME 5 Ruthven and Artigue observed that current results of activities integrating ICT in school are rather disappointing on system level.

“Advocacy for new technology is part of a wider reform pattern which has had limited success in changing well established structures of schooling.” (Ruthven, 2007) “From the very beginning, digital technologies have been considered as a tool for educational change [...]. Unfortunately, the results are far from being those expected” (Artigue, 2007).

For substantial innovations in the educational system there is no lack of general ideas, pedagogical concepts or didactic tools – as discussed above. But there is a wide gap between theoretical knowledge and practice in school. So we have to develop strategies to bridge this gap.

Conclusion: A Pattern for Innovation Projects

Combining the theory of cybernetics and the concept of learning environments using dynamic worksheets we get a pragmatic, but also theory-based way of initiating innovations in school. Activities are most promising, if they focus on incremental-evolutionary changes on the meta-level of beliefs and attitudes of all agents involved. Learning environments with dynamic worksheets may serve as framework for learning processes of teachers *and* students. How can this be done concretely?

As a conclusion from all reflections above we sketch and propose a pattern for innovation projects for mathematics education. (It is realized e.g. by the current project “InnoMathEd – Innovations in Mathematics Education on European Level”, see <http://innomathed.eu>).

(1) The key persons for innovations in school are the teachers. Their beliefs, motivation and abilities are crucial for everyday teaching and learning in school. So regional networks of schools are established which form frameworks for teachers’ cooperative learning, exchange of experience and professional development.

(2) Universities are innovation centres for teacher education. They lead the school networks and provide regular and systematic in-service teacher education offers. This teaching and learning is designed according to the aspects of learning and the concept of learning environments described above. So the teachers get acquainted with these theories and concepts by making personal experiences in learning environments designed for them.

(3) Participating schools concentrate on one or a few areas of innovation, e.g. autonomous learning with dynamic worksheets, promoting student cooperation with dynamic worksheets or fostering key competences with dynamic worksheets. It is not promising to aim at total changes of mathematics education – because of the complexity of the system. However, such bounded fields of activity allow teachers to begin with substantial changes without the risk of losing their professional competence in class.

(4) The teachers get acquainted with general ideas and theories of teaching and learning as well as with techniques for constructing learning environments. To bridge the gap between theory and practice the teachers’ project activities are strongly related to their regular work at school. They develop learning environments for their students, they use, test and evaluate them in their classes and finally optimize them on the basis of all experiences. In this process they get guidance and coaching by the University leading the network.

(5) All learning environments which are tested, evaluated and optimized are collected in a data base and made available for public use.

(6) Teachers are given possibilities to exchange experiences with colleagues and to participate in teacher education offers on national and international level. Thus they understand that problems and necessities for development have systemic character and concern the fundamentals of mathematics education far beyond their own professional sphere. Moreover, they get ideas for innovation activities from a large community.

(7) Finally, further networks of teachers and schools are essential means for dissemination processes in the long term. Experienced teachers coach colleagues from schools starting with innovation activities.

This approach may be called “theory based and material driven”. On the basis of the theory of cybernetics and the theories of learning the teachers involved make incremental-evolutionary steps on the meta-level of beliefs and attitudes by designing and working with concrete learning environments for their classes.

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A DIDACTIC ENGINEERING FOR TEACHERS EDUCATION COURSES IN MATHEMATICS USING ICT

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A first part of our research led us to define a theoretical framework to analyse teachers' education courses and to make hypotheses to explain the lack of efficiency of teachers training (Emprin, 2008). This paper presents the continuation of this work. We use the methodology of didactic engineering, adapted to teachers' education, and a theoretical framework previously built to test our hypothesis. In a first part of this paper we will describe our theoretical framework and hypothesis about teachers training. In a second part we will develop the didactic engineering and its results.

TEACHERS EDUCATION COURSES ANALYSIS

The general question guiding this work is the difficulty for mathematics teachers to use ICT in their classrooms. Our choice is to focus on a particular factor explaining this difficulty: teachers' professional education; without denying the existence of other factors such as material problems, resources available etc. Several studies in France or wider as Empirica study of European Commission, TIMSS & PIRL of the Boston College and BECTA's reports indicate this explanatory factor. French political choices since 1970ⁱ show that a quantitative effort was made, our research thus relates to a qualitative problem of teachers' training.

A theoretical framework

First we chose to use a framework designed for the analysis of teaching practices and to specify it with teacher educators' practices: the two-fold approach. This framework, defined by Robert (1999), does not take into account specifics of the use of technology. This leads us to use, jointly with the two-fold approach, a framework making it possible to take into account this dimension as described in (Emprin, 2008). The instrumental approach developed by Rabardel (1995) appears to be relevant. This approach, which was already developed in the didactic of Mathematics ,for example in Trouche (2005), leads us to analyse instrumental geneses.

One difficulty is that teacher educators' practices can not be reduced to a teaching activity. A teachers' educator, in France, was most of the time a secondary school teacher, in many instances they keep on teaching to pupils. For this reason, like Abboud Blanchard (1994) specifies, the teacher trainer's previous practices as a teacher intervene in his practices as a teachers' educator.

We borrow the definitions of "activity" and "practices" from Robert & Rogalski (2002) which we must specify on various levels met during a teachers' education course:

This definition of “activity” is nearly similar to Rabardel’s notion of “productive activity” (Rabardel, 2005, p. 20). It contains actions but also statements, attitudes and unobservable aspects which influence actions.

The definition of “practices” we use is a reconstitution of the five components described in the two-fold approach. Robert & al. (2007) give the description we have translated here:

“We developed, taking into account the complexity of the practices, analyses capable of giving an account of what can be observed in class, which results from teacher’s homework and the unfolding, and factors which are external to the classroom but which weigh on practices, including those in the classroom, and eventually contribute to the teachers’ choices before *and* during the lesson. Indeed, practices in classroom are forced, beyond goals in terms of pupils’ acquisitions, by determinants related to teachers’ trade: institutional, social... Let us quote programs, timetables, schools, colleagues, class and its composition. Moreover, the practices have a personal anchoring which refers to the teacher as a singular individual, in terms of knowledge, picturing, experiments, trade’s idea and also conditions its choices. Our analyses start from class session in which we distinguish components, institutional, social, personal, meditative (related to the unfolding in the classroom and improvisations), cognitive (related to the prepared contents and expected unfolding), closely dependent for a given teacher, and having to be recomposed: it is necessary for us to think of the components together, and to estimate the compensation, balance, the compromises to include/understand and start to explain what is concerned. »

To build our framework of analysis we need to dissociate the various levels of activities and practice but also to see their interactions. Figure 1 makes it possible to describe these various levels.

The first level of activity is the one of the pupil. We note it A0 level. The pupil has a task to realize, and acts accordingly. He uses an instrument belonging to ICT. This level can thus be analyzed with the didactic of mathematics and the instrumental approach. The observation of the process of instrumentation/ instrumentalisation informs us about the instrumental geneses of the pupil and the instruments built.

The second level of activity is the one of the teacher whom we note at level A1. The tasks of the teacher consist of managing and organizing the activity of the pupils. He also organizes the instrumental geneses of the pupil. The two-fold approach enables us to analyze a first level of practices which we note P1 level.

The other two levels of activity are those which exist in teachers’ education courses. The activities of the trainees (who are thus teachers) during the training course, are noted as A2. They are organized by those of the teachers’ educator noted as A3. The Two-fold approach and the instrumental approach give us access to a second level of practices noted as P2, those of the teachers’ educators.

devoted to explanations (44 to 62%) and descriptions (35 to 52%) given by the teachers' educator, there is thus very few analysis or debate. In term of two-fold approach social, personal and institutional components of the practices are almost not approached. The mediative component of practices appears in the analysis of video or the narration of courses, but is not questioned. The cognitive dimension remains rather marginal. Our analysis also shows a possible drift of homology strategy: it is likely to introduce confusion between the various instrumental geneses, of pupil and teacher.

BUILDING OF A DIDACTIC ENGINEERING FOR TEACHERS EDUCATION COURSES

Hypothesis

We identify two complementary ideas explaining the lack of efficiency pointed previously. The first one results from the work of Ruthven & Hennessy (2002) and Lagrange & Dedeoglu (in press). These authors show a gap between teachers' needs and ICT potentialities presented by teachers' trainers. We also observe an absence. In France the "reflexive practitioner" of Schön (1994) and the "analysis of practices" developed by Altet (1994) or Perrenoud (2003) are two important models for teachers' education is thus remarkable that no allusion is made there in teachers' education courses to mathematics with ICT. That leads us to consider the introduction of a reflexive component in ordinary practices' analysis and to formulate four hypotheses taking into account the first part of our work:

- The analysis of real practices would make it possible to initiate a reflexive attitude in teachers (making it possible for the teacher to change their teaching practices)
- Leading trainees to analyze a real professional problem enables them to confront their representations, mobilize their knowledge (resulting from experience) and come to a consensus based on reasoning.
- An analysis of the professional practices taking into account several dimensions of practices (in terms of two-fold approach) and based on the analysis of the relationship between teaching practices and activity of the pupil, makes it possible for the trainees to mobilize their knowledge (resulting from experience and their theoretical knowledge).
- It is necessary to contribute, during teachers' training courses, to the professional instrumental geneses of teachers and to analyze the lessons in terms of instrumental needs and potential instrumental genesis of pupils.

In order to check these hypotheses we use the methodology of didactic engineering that we specify to teachers' education. This methodology defined in Artigue (2002) is based on the verifying of a priori hypothesis. Thus we need to define observable criteria linked to our hypothesis. We decline our four hypotheses in seven criteria:

- The trainees' ability to identify and define a problem.
- The formulation and the use, by the trainee, of knowledge coming from experience associated with theoretical knowledge to analyze the practice
- The implication of trainees' personal practices and of his own experience in the analysis.
- The trainees reach a consensus based on knowledge coming from experience and theory.
- During the session teachers' educator does not give any answers, any explanations. The knowledge is built by trainees and not given by the teachers' educator. We call that an a-didactical lesson referring to theory of didactical situations (Brousseau, 1998)
- The fact that the analysis makes it possible to take into account several dimensions of the practices
- It must then be possible to identify any trace of instrumental genesis making it possible for teachers to consider instrumented actions but also results on pupils' activity.

Our methodology leads us to conceive a scenario for teachers' education whose implementation will be analyzed by means of the theoretical framework built in the first part.

Scenario and analyzes

The scenario is inspired from Pouyanne & Robert (2004). It is based on the analysis of teaching practices by means of a video. Four periods are defined: an a priori analysis of the lesson (which has been recorded) where hypothesis about the effects of the teaching practices on pupils' activity are put forward; an analysis of the video and a comparison with the hypothesis; a search for alternatives based on the question "What would you do if you had to do such a lesson?"; and finally a debate around problems emerging during the first three period.

We implemented this scenario twice, in each one, videos show pupils using interactive geometry software (IGS): In the first training course eight grade pupils had to prove that perpendicular bisectors in a triangle converge. The second video show sixth grade pupils solving a problem (which is detailed below). We develop now this second session of teacher education.

In each teacher's education session, the scenario lasts about three hours. This part of the session has been recorded, transcribed and analysed. The analysis takes into account who is speaking, the type of speech (description, explanations, analysis) and its content.

An example of session

The lesson recorded for this teachers training is what we call in French “an open problem” referring to Arzac & Mante (2007). This type of problem is called “open” insofar as no specific solution is expected: what matters is pupils’ search.

Figure 2 gives the statement of the problem. Pupils are asked to say which one of [EG] or [AC] is longer.

During the first part of the work with trainees, the a priori analysis, we had to let them use the IGS. It is a first change in the scenario. It seems to be very difficult for teachers to analyse the problem without having a working time on the computer. This time is not a time of homology even if the trainees do what is expected from pupils.

During the analysis the trainees have a transcription of the discussion with the teacher who is in the video. She specifies what is at stake in this lesson: she wants pupils to develop their critical thinking and to show them not to trust their perception. The trainees identify three stakes: the drawing with the software, the location of the rectangles in the whole geometrical drawing and the property of the diagonals of a rectangle. They specify that they think that the situation cannot be done by the pupils. They propose teaching aids to make the situation feasible. They propose to reveal the radius of the circle, the other two diagonals of the rectangle. Another solution considered is to cut out the problem or to make a preliminary recall of the useful properties. In this stage there is thus an implication of the trainees who adapt the lesson since they try, to some extent, to make it feasible in their classrooms. This implication can be seen in the following example.

Trainee: that seems difficult to me in 6th grade also because I think that they will see that the diagonals have the same length but that they will not be able to justify it.

The viewing of the film reveals initially the need for dissociating the task of construction in the software from the remainder. Indeed the pupils encounter real difficulties to build the geometrical figure. The trainees realize that pupils need to build uses of the software. It is a part of the instrumental genesis. On the video, once geometrical construction has been carried out, the pupils try to conjecture. The trainees realize that pupils have the necessary knowledge to solve the problem but that they are not able to mobilize it.

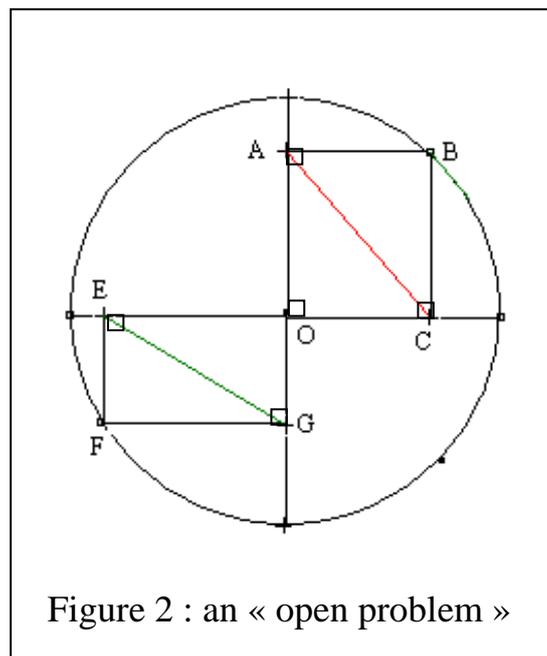


Figure 2 : an « open problem »

In the film, the pooling of pupils' works take place at the end of the lesson, whereas the pupils are still in front of the computers. It is quickly carried out by the teacher. The conclusions of the trainees are that it is necessary to take more time, to move the pupils away from the computers and to let them talk. There is thus a clear evolution in the trainees' mind. In the first part of the analysis they have doubts about the ability of the pupils to solve the problem and in the last part they say it is necessary to devote more time to the pooling of what pupils have found.

The search for alternatives contains the essential components of the analysis. The trainees reaffirm that it is necessary to dissociate the drawing on IGS from conjecture. Some even propose to remove the drawings' work. This work also allows a long discussion about the place of this problem in pupils' training. Before pupils know the property of the diagonals of the rectangle, the problem is centred on research whereas afterwards it acts more as a consolidation of knowledge. This also leads to discuss the place of observations in the geometrical trainings. A trainee proposes to use this problem to introduce the property of equality of the diagonals which disturbs another trainee who believes that observing properties is conflicting with the idea of mathematics. This trainee finally realizes that she does not have tools to give proof of the property to pupils of this level while at the same time the property is in the official programme. During these discussions the teachers' educator scarcely intervenes. Trainees are personally involved in the analysis:

Trainee: I do think that giving the instructions when the computers are "on" is always rather difficult; it is better to give instructions before turning the computers on.

In this example we can see that this trainee formulates a teaching knowledge, rather simple but which can now be used consciously by other trainees.

Most of the indicators can be observed for "many" trainees. Nevertheless, during a three hours session, a limited number of trainees can speak and consequently the internal evaluation of our methodology is only partial.

Finally, we noticed two changes in our scenario: the time of appropriation of the software was introduced during the analysis of the lesson and the final time of debates was removed. For the first change, the lack of acquaintance of the trainees with the artefact prevents them from making a real analysis. The second change is due to time devoted to debates during the session. The entire subject likely to be alluded to seems to have been discussed before. A last noticeable point is that trainees do not know other pieces of software which could be used in this lesson. The teachers' educator had to show different pieces of software as in the teachers' education courses we analysed in the first part of our work.

Conclusion on the didactic engineering of formation and continuation

The main results of this didactic engineering are linked with our criteria: it seems to be necessary to let the trainees use and try the artefact. It helps them to analyse the

lesson but it also seems to match with trainee expectations. It is possible to take into account several dimensions of the practices but in a smaller number than expected. The analysis of the video helps trainees to make cognitive and mediative components more explicit but the other components are more difficult to reach. The scenario built allows a reflexive analysis of the practices. Experience and theoretical knowledge is used to analyze the problem of introduction of the ICT. Instrumental geneses of the teachers and the pupils are really dissociated. The trainees considered what is necessary to pupil to use ICT in this lesson. They also found different options and they analysed the changes involved by these choices in term of learning or in lesson unfolding. For example ask pupils to draw the figure in the software helps them to use a proper vocabulary (because the software makes it compulsory) but it takes a long time and leads the teacher to reduce the time of conjecture.

Practices, in our didactic engineering, are shown in a video but it is possible to work on other types of practices such as real practices or simulated practices. Simulated practices make it possible for a whole group of trainees to work on the same teaching experience. The construction of such a simulator is the object of a work we initiated in 2007.

To conclude, the fact that teachers use experience knowledge to analyze practices with ICT makes it possible for us to consider the teachers' education course with ICT as a lever for teachers' education generally speaking. It seems to be easier to influence the way of teaching mathematics by influencing the way of teaching mathematics with ICT.

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i This very year the IPT plan began (Informatique Pour Tous) which could be translated in “data computing for everyone”. For example in 1985, it allowed the purchase of computers for 33.000 schools and represented 5.500.000

hours of training for teachers. For more information see Archambault, J.-P. (2005), 1985, vingt ans après... Une histoire de l'introduction des TIC dans le système éducatif français. Médialog (54).

GEOMETERS' SKETCHPAD SOFTWARE FOR NON-THESIS GRADUATE STUDENTS: A CASE STUDY IN TURKEY

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The purpose of this paper is to determine mathematics teachers' views about Geometers' Sketchpads Software (GSP) and to analyze the effects of training sessions on prospective teachers' ability to integrate instructional technology in the teaching of geometry. For that purpose, two graduate student teachers were selected; they investigated GSP activities. They followed training sessions about using GSP. The data come from interviews with them and GSP activities improved by them. The results of this study indicate that their awareness level about GSP was increased.

Keywords: *Teacher Education, Secondary Mathematics Education, Non-thesis Graduate Program, Integrating Technology, Geometers' Sketchpad Software.*

INTRODUCTION

Today's use of technology as a learning tool supplies the students with gaining the mathematics skills in their lessons. According to Newman (2000), the use of technology in learning arouses curiosity and thinking, and challenges students' intellectual abilities. Kerrigan (2002) state that using mathematics software promote students' higher order thinking skills, develop and maintain their computational skills. For this reason, teacher training is crucial in order to use technology in mathematics education.

Computers could be used in school for teaching geometry, and since then a lot of work has been done that discusses many aspects of using Dynamic Geometry Software (DGS) in education (Kortenkamp, 1999). In this study, it was concerned with DGS activities developed by non-thesis graduate student teachers. Non thesis graduate program in Turkey was opened for the purpose of educating future teachers. The secondary school (grade 9-11) mathematics teacher training program made up of two different programs. The Five-Year Integrated Programs (3.5+1.5) in Faculty of Education and Non Thesis Graduate Program (4+1.5) in Faculty of Science. Last 1.5 year part is the same for both 3.5+1.5 and 4+1.5 programs. Of these programs 3.5 and 4 year are spent on taking the mathematics courses and remainder years on pedagogical courses. After graduation, they can be secondary school mathematics teacher. This program is described in more detail in YOK (1998). The aim of this study was to investigate whether their views changed after the education process and to determine the outcomes about student teachers' proficiency.

THEORETICAL FRAMEWORK

In geometry, teachers are expected to provide “well-designed activities, appropriate tools, and teachers’ support, students can make and explore conjectures about geometry and can learn to reason carefully about geometric ideas from the earliest years of schooling” (NCTM, 2000). Mathematics teachers can help students compose their learning by using geometry sketching software. Geometer’s Sketchpad allows younger students to develop the concrete foundation to progress into more advanced levels of study (Key Curriculum Press, 2001).

Reys et al. (2006) point out young learners of mathematics need to

- experience hands-on (concrete) use of manipulative for geometry such as geoboards, pattern blocks and tangrams,
- connect the hands-on to visuals or semi concrete models such as drawings or use the sketching software on a computer,
- comprehend the abstract understanding of the concepts by seeing and operating with the picture or symbol of the mathematical concept (cited in Furner and Marinas, 2007).

GSP is an excellent tool for students to understand the properties of geometric shapes and to model for them mentally manipulating objects. GSP can also provide students to visualize the solid in their mind. In literature, McClintock, Jiang and July (2002) found GSP provides opportunities to have a distinct positive effect on students' learning of three dimensional geometry. In another study, Yu (2004) stated that the students’ concurrent construction of figurative, operative and relational prototypes was facilitated by dynamic geometric environment. That’s why, the knowledge about which DGS and DGS activities how prepared should be given the student teachers.

METHOD

Participants

Case study was used in this paper. This research was conducted during the spring term of 2007–2008 academic years in spring term. The study was conducted with two secondary school preservice teachers attending the 4+1.5 Integrated Secondary Mathematics Teacher Education Program at Dokuz Eylul University in Turkey. Of the ten students in this program there were two volunteers. In this process, they took the courses about mathematics content knowledge, pedagogical content knowledge and general pedagogical knowledge. All participants had basic computational skills but none of them knew how to use DGS.

Data Collection

The data were collected from interviews and the activities which are prepared by the student teachers. The interviews were semi-structured in nature. In the beginning of the research, the opinions of the participants towards GSP software are taken with semi-constructed interview form. Each interview took approximately 15-20 minutes and recorded with a tape. Then the participants attended a six-hour GSP training sessions which is given by the researchers. After the program, it was demanded that

the participants developed the GSP activities. Finally, the participants' opinions towards GSP software are taken again.

The Geometer's Sketchpad Training Sessions

The training sessions allowed the instructor to prepare the non-thesis graduate student teachers to enter their future mathematics classrooms not only knowledgeable about the capabilities of instructional technology, but also experienced enough to appropriately integrate their selected software. The GSP training sessions' content is given Table 1.

	Training Sessions	Topics	Duration
DAY 1	Introductory (Guided & Discussed)	<ul style="list-style-type: none"> major concepts of mathematics education the aim of the involved Software introduction to dynamic geometry environment with GSP introduction to tools and menus of the Software 	1 hour
	Constructing Geometrical Concepts (Guided & Discussed)	<ul style="list-style-type: none"> to construct basic concepts of geometry to transform the rotation, reflection, and dilation of the figures to construct regular and non-regular polygons, and its interiors to measure in geometry (length, distance, perimeter, area, circumference, arc angle, arc length, radius, etc.) to graph various functions and its derivative 	1 hour
DAY 2	Animation and Presentation (Guided & Discussed)	<ul style="list-style-type: none"> to use action and hide/show buttons to tabulate the data to prepare presentations 	2 hours
	Activity Planning (Guided & Individual)	<ul style="list-style-type: none"> to plan activities and practice it 	2 hours

Table 1: Training Sessions

DAY 1 included two sessions. Each session lasts an hour.

Introductory Session: The introductory session contained the major concepts of mathematics education, introduction to dynamic geometry environment with GSP and the aim of the involved Software.

In the beginning of the session, the participants discussed the major concepts - conceptual development, problem solving, modelling verbal problems, creative

thinking, analytical thinking etc.- in order to determine their readiness with researcher. Then, they argued the aim of the involved Software. Afterwards, the participants introduced Dynamic Geometry Environment, the menus, sub-menus and tools of the GSP Software. When the participants get information about tool box, text palette, file menu, edit menu, display menu, construct menu etc., the researcher advanced next session.

Constructing Geometrical Concepts: In this session, the participants find out how to construct the basic concepts of geometry; such as ray, line, segment, parallel line, perpendicular line, angle bisector, median of triangle, altitude of triangle, arc etc.

When the participants learned how to use the menus, sub-menus and tools, the researcher showed them some operations. The participants learned about constructing regular and non-regular polygons, and its interiors. After that, they learned to change the color and width of the lines and figures.

Then, they transformed the rotation, reflection, and dilation of the figures. Subsequently, they measured length, distance, perimeter, area, circumference, arc angle, arc length, radius, etc. with using GSP.

When they reached the graph menu, they defined coordinate system, chose grid form and they draw some graphs with GSP, such as sinus, cosinus, tangent, etc. Afterwards, they graphed various functions and its derivatives. During this session, the participants discussed the functions of GSP each other if it was necessary or it was forgotten.

DAY 2 comprised two sessions. Each session is made up of two hours.

Animation and Presentation: In this session, the participants found out text palette on advanced level. Next they learned motion controller, how to paste picture and then passed animation and hide/show buttons. They learned how to utilize animations and change it's speed. Then they learned to trace points, segments, rays and lines. Afterwards they focused on tabulate the data on tables in order to arrange them regularly.

After they learned animation and presentation clues, they started to organize page set-up and document options in order to prepare excellent presentations.

Activity Planning: This session includes all of the applications learned. The researchers wanted the participants to prepare activities. And they also wanted to apply all the operations learned in their activity. In the preparation period, if the participants needed to be supported, the researchers could be guiding them.

Data Analyses

In the interview, four open-ended questions were asked to the participants and the interview guide was used in this stage. During the interview, the questions like "What are the GSP aims in mathematics learning environment?" "Which students' skills are able to improve by GSP activities?", "What do you take into account while

the GSP activities are composed?” and “How can you assess the students with the GSP activities?” were answered by the students.

The evaluating criteria were determined in order to assess the activities improved by the student teachers. These criteria were adapted from Roblyer (2003).

1.	Connection to mathematics standards.
2.	Appropriate approach to mathematics topics with respect to grade, ability.
3.	Presence of conceptual development, problem solving/higher order thinking skills.
4.	Use of practical applications and interdisciplinary connections.
5.	Suitability of activities (interesting, motivating, clear, etc.)

Table 2: Evaluation Criteria adapted from Roblyer (2003)

RESULT

In this section, the analysis of data obtained from two preservice teachers' view transcripts and activities which they prepared are presented.

Handan's Case

Handan is working as an assistant teacher in private teaching institution for a year. During the pre-interview, four questions were asked her. She made explanations as follows:

- Researcher : What are the GSP aims in mathematics learning environment?
 Handan : It supplies the students with learning and visualizing in math lessons and preparing animations.
- Researcher : Which students' skills are able to improve by GSP activities?
 Handan : The students' spatial thinking skills are improved.
- Researcher : What do you take into account while the GSP activities are composed?
 Handan : It should be appropriate the students' cognitive level.
- Researcher : How can you assess the students with the GSP activities?
 Handan : I don't know because of lacking knowledge about GSP.

As can be seen in her statements, although she mentioned that she did not know GSP, she could be able to estimate its aims, skills to be improved and rules taken into account when the activities had done.

After training sessions, the researcher wanted her to prepare GSP activities whatever topics she wished. She chose the congruence as a subject of geometry instruction. Her activity is given Figure 1.

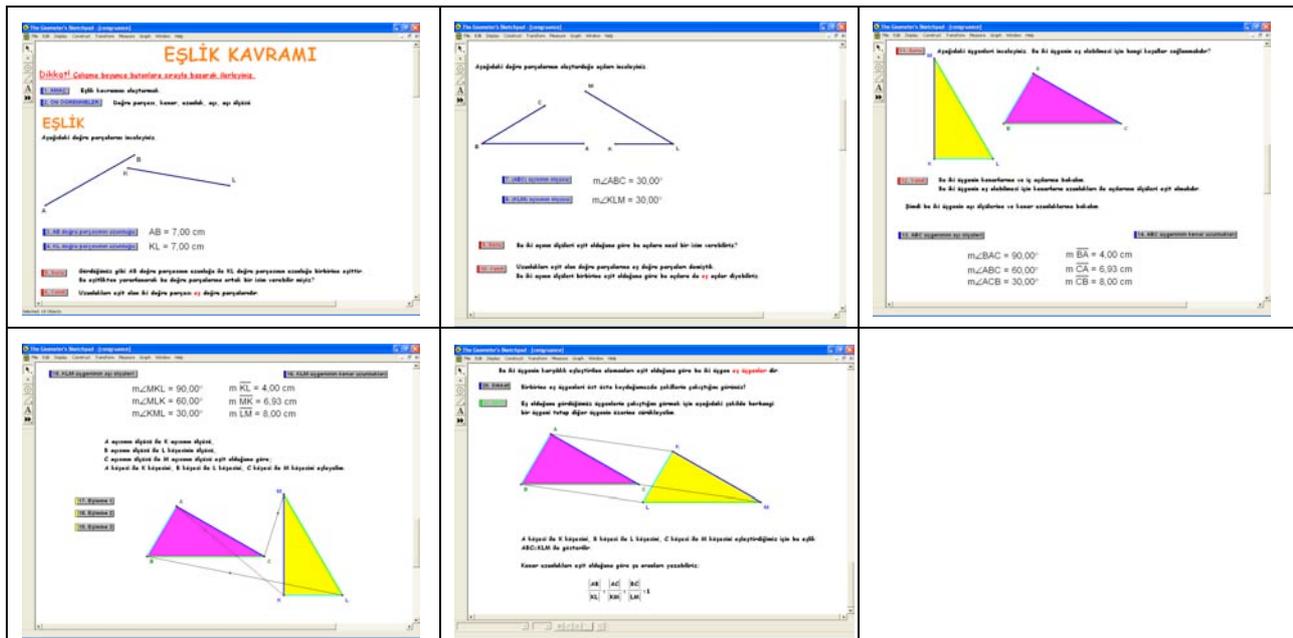


Figure 1: Handan’s Activity

The content of her activity was about congruence. She decided to plan her activity for constructing the concept of congruence. As regards to the activity, the student knows the aim of the subject (step 1) and the concepts related to the subject (step 2). Handan gave directions to the students in her activity, in general. Therefore the student follows the instructions and carries on step by step. Afterwards, she gave two segments as AB and KL. She demonstrated the length of AB and KL segments (step 3-4). In the next step of the activity, she wanted students to compare the length of AB segment with KL segment. She asked whether the students call a common name to these segments (step 5) and explained it simply (step 6). Subsequently, she gave two angles and its measurements (step 7-8). She told the angles have the same measurement (step 9) and asked what the common name of the angles is (step 10). Later she constructed two triangles (ABC and KLM) and asked the students in what conditions they are congruent (step 11). Later on she showed the conditions of the congruence (step 12) and measurements of the triangles (step 13-14-15-16). In following steps, she paired each corners of the triangles and animated them (step 17-18-19). Finally, she drew the students’ attention for the coincidence of triangles and demonstrated this (step 20-21).

When her activity arranged was assessed via the so-called evaluation criteria in Table 2, it was seen that the activity was connected to mathematics standards organized by Ministry of National Education (MNE) in Turkey, suited approach to mathematics topics -to explain congruence of triangle- with respect to 10th grade but it was too simple and like 8th grade level. It was provided conceptual development, also clear but not engaged the students in real life situations and interdisciplinary connections. It is useful for constructing the concept of congruence but not provide satisfactory knowledge. It wasn’t prepared for improving the students’ problem solving skills also. Handan utilized the mathematical language adequately. In respect of

technicality, the activity is good. Each step's button is made as hide/show button. The 17th and 19th steps' button have the same function, so one of them is needless. The activity hasn't got any other technical problem.

Afterwards she had done activity; the post-interview was carried out with her and it was given her comments as follows:

- Researcher : What are the GSP aims in mathematics learning environment?
 Handan : It provides the students learn geometrical concepts...their problem solving skills are improved and the concepts are visualized.
- Researcher : Which students' skills are able to improve by GSP activities?
 Handan : The students' spatial thinking.... and problem solving skills are improved.
- Researcher : What do you take into account while the GSP activities are composed?
 Handan : It should be interesting.... appropriate for the students' cognitive level and the students' opinions can be taken while the activities are prepared.
- Researcher : How can you assess the students with the GSP activities?
 Handan : The students can be able to do the applications involved in GSP and these are evaluated.

Considering her statements, it is seen that her views changed after training sessions and her activity. She has primarily information about GSP and she awakes of what taking into account while the GSP activities are composed.

Mualla's Case

Mualla is also working as an assistant teacher in private teaching institution for a year. In time of the pre-interview, she gave responses as follows:

- Researcher : What are the GSP aims in mathematics learning environment?
 Mualla : ...It constitutes long lasting learning in math lessons and provides the teachers and the student drawing figures, preparing animations.
- Researcher : Which students' skills are able to improve by GSP activities?
 Mualla : GSP improves the students' spatial thinking skills.
- Researcher : What do you take into account while the GSP activities are composed?
 Mualla : It should be interesting...
- Researcher : How can you assess the students with the GSP activities?
 Mualla : I don't know...

In the analysis of this interview, she determined which skills improved and what she pays attention during the GSP activities are composed. Besides it is seen that Mualla's responses are similar to the Handan's statements.

After training sessions, the researcher wanted her to prepare GSP activities whatever topics she wished. She chose the similarity as a subject of geometry instruction. The activity involved is given Figure 2.

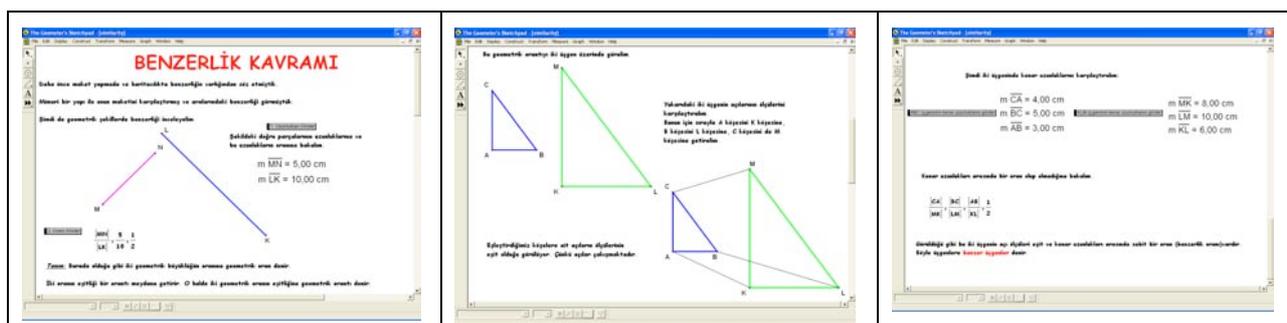


Figure 2: Mualla's Activity

Mualla's activity deals with similarity of triangles. She tried to carry out her activity for constructing the concept of similarity. According to her activity, she acknowledged that the students have little knowledge about the subject. Mualla generally gave directions to the students in her activity, as Handan did. However, her activity didn't similar to in terms of following the instructions step by step. In the beginning of the activity, she mentioned few real-life examples to the students about similarity and then she passed the similarity between geometrical concepts. She gave two segments, like Handan, and she compared the length of them under the first button. The second button shows the students the ratio of the lengths of the segments. After that, the definition -geometrical ratio and geometrical proportion- was given, and demonstrated. Then, she compared the measures of each angle of the triangles and mentioned the coincidence of each angle. Afterwards, she showed and compared the length of sides of the triangle and stated whether the sides of both triangles have a ratio or not. Lastly, she defined a stable ratio, as the ratio of similarity.

When her activity organized was assessed by means of the evaluation criteria in Table 2, it was seen that the activity was overlapped mathematics standards organized by MNE in Turkey, partly suited approach to mathematics topics -to explain similarity of triangle- with respect to 10th grade. It was provided conceptual development, but not connected to the students in real life situations and interdisciplinary connections. Her activity was clear and understandable but it was also towards 8th grade and too simple. It wasn't also provides sufficient knowledge. It wasn't prepared for improving the students' problem solving skills also. Mualla used the mathematical language few adequately. In respect of technicality, the activity is not bad. Each step's button was made as hide/show button, as Handan did. It didn't include enough animation and demonstration. Finally it was said that, the activity hasn't got any technical problem.

After she had done activity; her comments during the post-interview was given as follows:

- Researcher : What are the GSP aims in mathematics learning environment?
 Mualla : It provides the students learn geometrical concepts and problem solving, proof geometrical theorems. In addition to, it can be long lasting learning.

- Researcher : Which students' skills are able to improve by GSP activities?
 Mualla : The students' spatial thinking was improved.
 Researcher : What do you take into account while the GSP activities are composed?
 Mualla : It should be appropriate the students' cognitive level and the mathematics standards
 Researcher : How can you assess the students with the GSP activities?
 Mualla : It can be ask some question in GSP aiming at determining whether they learned the geometric concepts. We expect that the students reveal the relationships between geometric concepts.

As her statements, she increases information about GSP. It follows from her responses that her point of view enlarged after training sessions. She encouraged and determined carefully what she does with GSP in mathematics learning environment after she prepared activities herself.

DISCUSSION AND CONCLUSION

In this study, the data indicated that Dynamic Geometry Software (DGS) is important in geometry education. Generally speaking, Handan and Mualla learned some properties of GSP. At the end of the study, they realized how they can use GSP to prepare the activities. Handan gave detailed directives in her activity. She expected that the students to mention the concept of congruence; but this concept was given by her at the beginning of the study. In the other case, Mualla set out the similarity proportion when she prepared her activity. Both of them did not mention the kinds of congruence and similarity. They perhaps fostered the finding of these kinds by the students. As Key Curriculum Press (2001) mentioned, teachers can use GSP to create worksheets, exams, and reports by exporting GSP figures and measurements to spreadsheets, word processors, other drawing programs, and the Web. These results indicate that DGS is important in teacher education and DGS training must be present in non-thesis graduate education.

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LEADING TEACHERS TO PERCEIVE AND USE TECHNOLOGIES AS RESOURCES FOR THE CONSTRUCTION OF MATHEMATICAL MEANINGS

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This paper presents the early results of an on-going research project on the use of technology in the mathematics teaching and learning processes. A first aim of this project is to understand how deeply math teachers do perceive the opportunities technologies can bring about for change in pedagogical practice, in order to effectively use them for the students' construction of mathematical meanings. Secondly, the research aims at verify if teachers realise that, in order to successfully deal with perturbation introduced by technologies, they have to keep themselves continuously up-to-date and to acquire not only a specific knowledge about powerful tools, but also a new didactical and professional knowledge emerging from the deep changes in teaching, learning and epistemological phenomena.

INTRODUCTION

Due to the continuous spread of technology in the latest years, challenges and expectations in the everyday life, and in education in particular, have dramatically changed. Within this context of rapid technological change the world wide education system is challenged with providing increased educational opportunities. The use of Information and Communication Technology (ICT) in the classroom, however, seems to be, in the majority of cases, still based on a traditional transfer model characterised by a teacher-centred approach (see for example: Midoro, 2005).

But, according to Hoyles *et al.* (2006; p.301):

«...a learning situation had an economy, that is a specific organization of the many different components intervening in the classroom, and technology brings changes and specificities in this economy. For instance, technological tools have a deep impact on the “didactical contract”...».

That is, the technology-rich classroom is a complex reality that necessitates observation and intervention from a wide range of perspectives and bringing technology in teaching and learning adds complexity to an already complex process (Lagrange *et al.* 2003).

Moreover, as underlined by Mously *et al.* (2003; p.427),

«...technological advances bring about opportunities for change in pedagogical practice, but do not by themselves change essential aspects of teaching and learning ».

As research underlines (Bottino, 2000), indeed, innovative learning environments can result from the integration among educational and cognitive theories, technological opportunities, and teaching and learning needs. However, it is extremely important

for teachers to confront themselves with the necessity to understand how the potential offered by technology can help in the overcoming of the everyday didactical practice complex problems.

I believe that for technologies to be effectively used in classroom activities teachers need, not only to “accept” the presence of technologies in their teaching practice but also to see technologies as learning resources and not as ends in themselves. Moreover, learning activities involving technologies should be properly designed to build on and further develop mathematical concepts. Hence, an “adequate” preparation is essential for teachers to cope with technology-rich classrooms, so that using computers not merely consists on a matter of becoming familiar with a software.

This paper presents the early results of an on-going research project on the use of technology in the mathematics teaching and learning processes, investigating mathematics teachers’ perceptions of ICT and of their usefulness in promoting a meaningful learning.

A first aim of this project is to understand how deeply math teachers, both pre-service and in-service, do perceive the opportunities technologies can bring about for change in pedagogical practice in order to effectively use them for the students’ construction of mathematical meanings.

Secondly, the research aims at verify, whether or not, teachers realise that, in order to successfully deal with perturbation introduced by technologies, they have to keep themselves continuously up-to-date and to acquire not only a specific knowledge about powerful tools, but also a new resulting didactical and professional knowledge emerging from the deep changes in teaching, learning and epistemological phenomena.

THEORETICAL FRAMEWORK AND RELATED LITERATURE

Many researchers in the latest years are answering the challenge to provide educational opportunities by studying teaching and learning mathematics with tools (Lagrange et al., 2003).

Results of both empirical and theoretical studies have also led to the elaboration of the idea of “mathematics laboratory” as reported, for example, in an official Italian document prepared by the UMI (Union of Italian Mathematicians) committee for mathematics education (CIIM):

«A mathematics laboratory is not intended as opposed to a classroom, but rather as a methodology, based on various and structured activities, aimed to the construction of meanings of mathematical objects » (UMI-CIIM MIUR, 2004; p.32).

In this sense, a laboratory environment can be seen as a Renaissance workshop, in which the apprentices learned practicing and communicating with each other. In particular in the laboratory activities, the construction of meanings is strictly bound,

on one hand, to the use of tools, and on the other, to the interactions between people working together (without distinguishing between teacher and students).

According to this approach, and as in Fasano and Casella (2001), I believe that technological tools can assume a crucial role in supporting teaching and learning processes if they allow teachers to create suitable learning environments with the aim to promote the construction of meanings of mathematical objects. Moreover, in agreement with this point of view, I consider important to highlight that, again quoting the UMI-CIIM document (p.32):

«The meaning cannot be only in the tool per se, nor can it be uniquely in the interaction of student and tool. It lies in the aims for which a tool is used, in the schemes of use of the tool itself. The construction of meaning, moreover, requires also to think individually of mathematical objects and activities.»

Furthermore, as claimed by Laborde (2002; p.285),

«...whereas the expression integration of technology is used extensively in recommendations, curricula and reports of experimental teaching, the characterisation of this integration is left unelaborated.»

In particular, she underlines the idea that the introduction of technology in the complex teaching system produces a perturbation and, hence, for teacher to ensure a new equilibrium he/she needs to make adequate, non trivial choices. Integrating technology into teaching takes time for teachers because it takes time for them, first of all to understand that, and how, learning might occur in a technology-rich situations and, then, to become able to create appropriate learning situations. This point of view is based on the idea that a computational learning environment could promote the learners' construction of situated abstractions (Noss & Hoyles, 1996; Hölzl, 2001) and on the "instrumental approach" as developed by Vérillon and Rabardel (1995).

Within the instrumental approach, the expression "instrumental genesis" has been coined to indicate the time-consuming process during which a learner elaborates an instrument from an artefact: it is a complex process, at the same time individual and social, linked to the constraints and potential of the artefact and the characteristic of the learner. If, according to the instrumental approach, learners need to acquire non-obvious knowledge and awareness to benefit of a instrument's potential, I firmly believe that teachers need to take into account the student's instrumental genesis (Trouche, 2000).

Finally, I consider worthy of note the concept of "instrumental orchestration" proposed by Trouche (2003) aiming at tackling the didactic management of the instruments systems in order to conceive the integration of artifacts inside teaching institutions. In particular, he underlines that pre-service and in-service teacher training should take in account the complexity of this integration at three levels (Trouche, 2003; p.798):

- « - a mathematical one (new environments require a new set of mathematical problems);
 - a technological one (to understand the constraints and the potential of artifacts);
 - a psychological one (to understand and manage the instrumentation process and their variability). »

METHODS, CONTEXT AND PROCEDURE

The research I'm going to present consists in two main phases. The first has been carried out with a rather small group of in-service teachers at the University of Bari and a larger group of pre-service teacher at the University of Basilicata. The second involved another small group of pre-service teachers at the University of Bari.

Teachers belonging to the first group at the University of Bari were 16 high-school teachers. Although some of them already taught mathematics, on the whole they were qualified to teach related subject and they were attending a training program in order to get a formal qualification to teach mathematics.

At first, a preliminary anonymous questionnaire was submitted to them with the aim to know if they were able to see technologies as learning resources, as well as if they were available to continuously bring up-to-date in order to properly design and manage with technology-rich classroom activities. Key questions in the questionnaire included the following:

- 1 Do you think ICT could be useful for your teaching activities? Why?
- 2 Do you think that the use of ICT can somehow change the learning environment? And the way to teach? And the dynamics among actors in the teaching/learning situations?
- 3 Which difficulties do you think can be encountered when designing and developing a math lessons using somehow ICT?
- 4 As a teacher, do you think you need to have some didactical competences in order to properly use ICT? Eventually, which ones? And anyway, why?

Within the training program they attended, a thirty hours course was focused on didactical reflection aiming at helping student teachers to understand how to make the most of the use, in mathematics teaching and learning activities, of general tools such as spreadsheets, multimedia and Internet, as well as mathematics-specific educational software such as Cabri. In order to explain them that the changes produced by the introduction of a technological tool will not necessarily per se bring the students more directly to mathematical thinking, particular attention was devoted to stress the role of the a-didactical milieu in authentic learning situations, as in the known Brousseau's (1997) "theory of didactical situations". Furthermore, they were asked to analyse and discuss both successful and questionable examples of teaching/learning mathematics activities in which an important role has been played by the use of ICT.

At the end of the course student teachers designed a teaching/learning activity involving somehow the use of technology: in this way I intended to verify how deeply they have perceived the opportunity to effectively exploit the usage.

A further anonymous questionnaire, free from constraints, was later submitted with the aim to find out any signal for changes in their conceptions to have been occurred. Key questions in this further questionnaire were exactly the same.

Pre-service teachers involved in the research project at the University of Basilicata were a larger number (97). They were only asked to fill in the first questionnaire.

During the second phase, a group of 16 pre-service teachers at the University of Bari, instead, interacted with the researchers/educators in the same way of the first group of in-service teachers: to this further group of teachers a preliminary anonymous questionnaire was submitted; then, they were invited (during a thirty hours course) to reflect on didactical aspects of the use of technologies as well; at the end of the course they were asked to design a teaching/learning activity in which technology played an essential role; finally I analysed the extent of their changes in looking at the integration of technologies in the teaching/learning processes.

According to the results obtained during the first phase (that I'm going to present and discuss in the next paragraph), in the second phase I asked student teachers, not only to design a teaching/learning activity involving the use of technology, but also to put in action the activities they have designed, having as student sample their colleagues: in this way they proved themselves as "actors" in a technology-rich learning "milieu".

FINDINGS AND DISCUSSION

Findings from the first anonymous questionnaire revealed that in-service student teachers perceived that technology can bring support to their teaching (see Fig.1), but only as much as it is a motivating tool enabling students understanding per se (see Fig. 2).

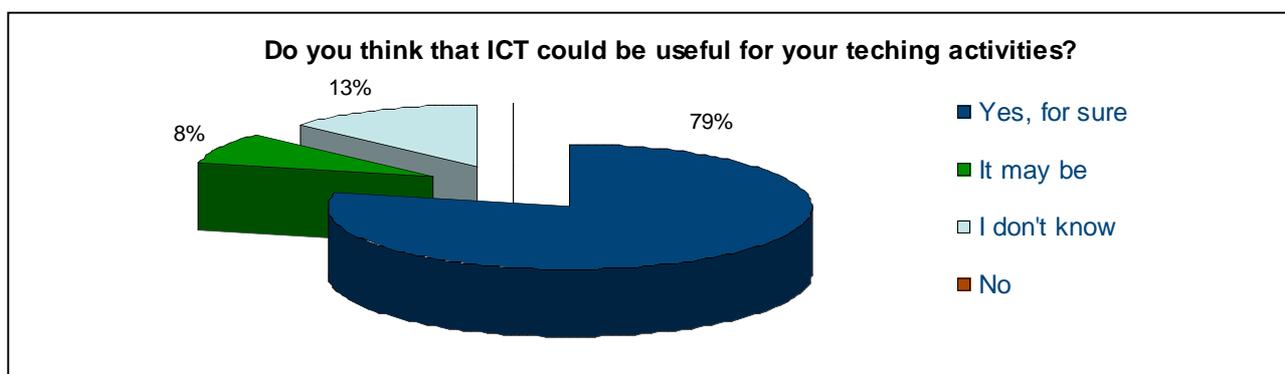


Figure1: The 79% of the in-service student teachers gave a positive (“Yes, for sure”) answer to question 1.

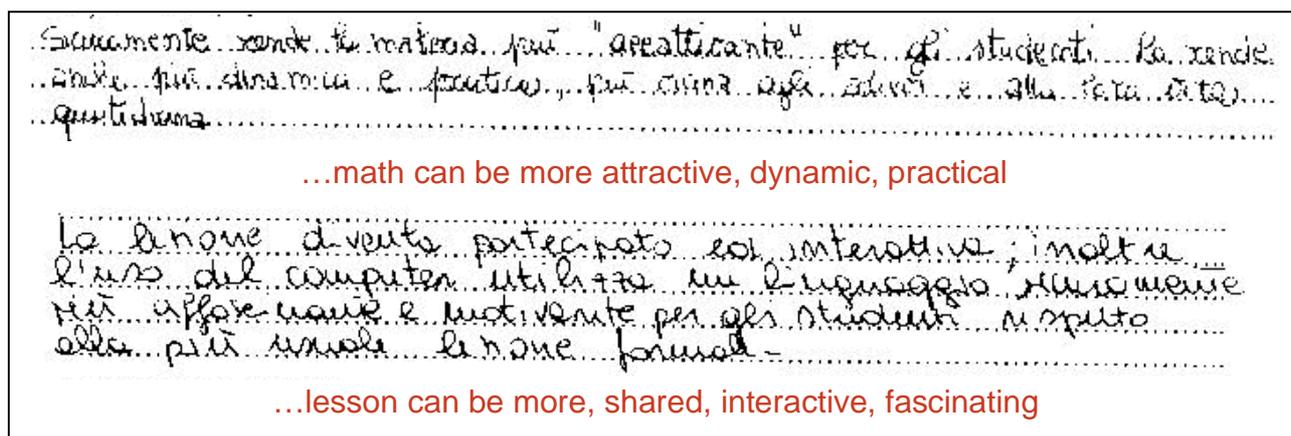


Figure2: Some in-service student teachers' answers to question 1: Do you think ICT could be useful for your teaching activities? Why?

Answers given by the pre-service teachers were, instead, a little bit more didactically oriented: some of them recognise that, if nothing else, the knowledge of the instrument functionality is probably not enough for a teacher to use it in an effective way in terms of construction of meanings by the students (see Fig. 3).

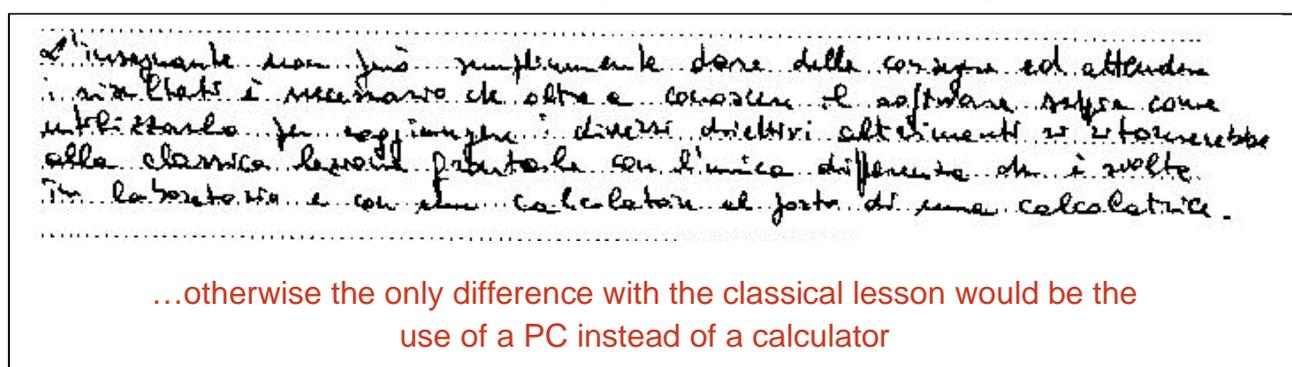


Figure3: A pre-service student teacher's answer to question 1.

None of the in-service teachers recognised that technology could bring a great support in creating new interesting and attractive learning environments. While, at least some interesting observation could be revealed among answers given (to question 2) by the pre-service teachers: some of them suggested the use of technological tools to allow students “collaboratively solve intriguing problems”.

Be aware of the opportunity to create a new “milieu” and change the “economy” of the solving process was, however, extremely far from their perception of the use of technology in mathematics teaching/learning activities, both for in-service and for pre-service teachers.

About the question 3, concerning the difficulties they think can be encountered when designing and developing a math lessons using somehow ICT, they mostly ascribed possible difficulties to the lack of an adequate number of PC and the technical problems that might occur, but also to the natural students' bent for distraction and relaxation, especially when facing a PC (see Fig. 4).

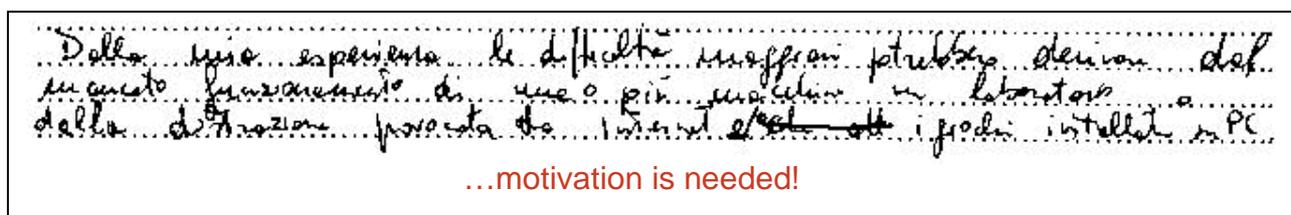


Figure4: Some student teachers consider new technology as a motivating tool that requires motivation.

As a consequence they did not feel the need to be skilled in using technology for their teaching and did not usually consider that their lack of skills presents them with any difficulties. And, although the 75% of the student teachers recognised (answering to question 4) the need to have some didactical competences in order to use new technology, what they asked to know about was, in most of the cases, just software functionalities (not potential, nor constrains): only some of the pre-service teachers also asked to know how to effectively integrate their use in the teaching practice.

Even though some of the activities that in-service teachers prepared at the end of the course revealed the willingness to attempt a new approach to the use of ICT, answers to the second anonymous questionnaire shown they still continued to find difficulty to be aware of the potential offered by ICT (see Fig. 5).

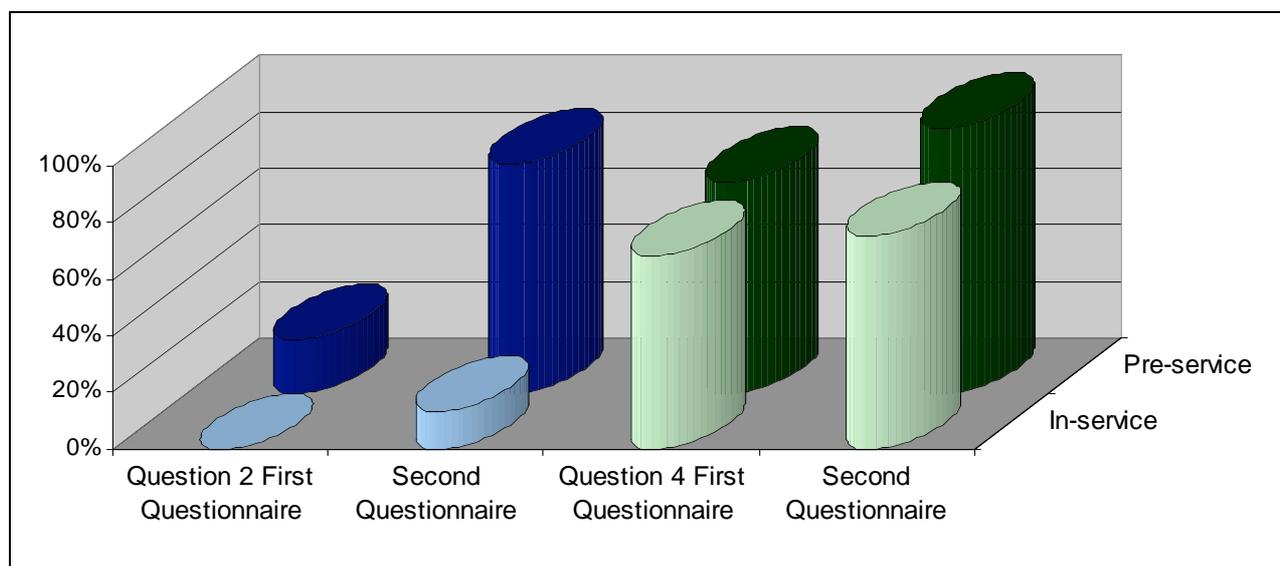


Figure5: Percentage of positive (“Yes, for sure”) answers given by both in-service and pre-service teachers respectively to the first and the second questionnaire to questions 2 and 4.

For this reasons, for the second phase of the project I planned to pay particular attention to promote teachers’ reflections on the opportunities offered by appropriate uses of technological tools in order to create new learning environment and, according to the idea of “mathematics laboratory”, to foster the construction of mathematical meanings.

Student teachers were invited not only to design a possible teaching/learning activity involving somehow the use of technology, but they were also involved in a “mise en situation” (as in the known Chevallard’s approach) during which they had the opportunity to assume the roles of the student, the teacher and a researcher/observer.

In this way, they faced with the complexity of the integration of technologies in classroom practice. Their comments at the end of the experience shown that they have developed an awareness of how the students’ instrumental genesis can take shape (psychological level). Moreover, answers to the second anonymous questionnaire revealed that they felt the need to understand the constraint and the potential of technologies (technological level) and to look for new mathematical problems (mathematical level).

EARLY CONCLUSIONS AND FUTURE WORKS

Discussion suggested by the researches in this field and by the analysis of this on-going experience led me to reflect on and to underline that an adequate preparation is essential for teachers to cope with technology-rich classrooms. In particular I believe that, only if teachers become aware of the potential usefulness and effectiveness of technologies as methodological resources (enable to foster the constructions of meaningful learning environment) they would recognise the need of an effective integration of them in the classroom activities and view new technologies as cultural tools that radically transform teaching and learning.

At the actual stage of this on-going research I can claim that, in my opinion, most of the teachers have difficulty to acquire the awareness of the potential of technology as a methodological resource. Hence, as educators, we also have to deal with the need to lead teachers to develop a more suitable and effective awareness of the usage of new technologies. Furthermore, I believe that the difficulty teachers have to acquire this awareness could be overcome giving teachers the opportunity to be subject of a “mise en situation”. In this way teachers can experience by themselves the difficulties students can encounter and have to overcome, the cognitive processes they can put in action and the attainment they can achieve. They also have the opportunity to understand and manage with the students’ instrumental genesis and to become more skilful and self-confident when deciding to exploit the potentials of technologies in mathematics education.

For future works I think in particular to go on with this idea, promoting further experiences of “mise en situation” according to the following stages:

- let teachers experience the importance of the relationship between the specific knowledge to be acquired by the students and the knowledge teacher possesses of it;
- let teachers experience the importance of the relationship between the specific knowledge to be acquired by the students and whatever students already know;
- let teachers experience the importance of the relationship between their knowledge and the students’ ones.

I suppose, indeed, that through these stages, teachers could experience by themselves the processes that come into play bringing technology in a teaching/learning situations. In particular, according to the early results of this study, I believe that in this way teachers do tackle with the obstacles encountered, the difficulties to be overcome, the cognitive and metacognitive processes carried out and the attainment that can be achieved.

To conclude, in the next future I aim to verify that, thanks to this methodology, not only they can cope with changes they could meet in a technology-rich learning situation but, reflecting on them, they can also become aware of how to better make use of technology as a resource to create an effective and meaningful learning environment.

Finally (also considering the explicit suggestions of the WG7 call for papers), I suppose that an interesting help to foster the development of teacher's instrumental genesis can be given by the use of Geoboards (Bradford, 1987). A Geoboard is a physical board (often used to explore basic concepts in plane geometry) with a certain number of nails half driven in, in a symmetrical square, (for example five-by-five array): stretching rubber bands around pegs, provide a context for a variety of mathematical investigation about concepts and objects such as area, perimeter, fractions, geometric properties of shapes and coordinate graphing.

Thus, I would like to let high school teachers operate with an unusual (at that level) context/tool like a Geoboard, and try to understand if, in this way, they can perceive teaching resources, both digital or not, as methodological resources: when teachers become aware that some resources can be effectively used for the construction of mathematical meanings they can start to successfully design and experiment new interesting learning activities.

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THE TEACHER'S USE OF ICT TOOLS IN THE CLASSROOM AFTER A SEMIOTIC MEDIATION APPROACH

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The issue of the teacher's role in exploiting the potentialities of ICT tools in classroom is more and more raising the interest of our community. We approach this issue from the Semiotic Mediation perspective, which assigns a crucial importance to the teacher in using ICT tools in the classroom. In the report we describe a Teaching Sequence centred on the use of the tool Casyopée and inspired by the Theory of Semiotic Mediation. Then we focus on the teachers' use of the tool with respect to the orchestration of collective activities and present an on-going analysis of her actions.

INTRODUCTION

Recent research points out a wide-spread sense of dissatisfaction with the degree of integration of technological tools in mathematic classrooms. Kynigos et al. observe that so far one did not succeed to exploit the ICT potential suggested by research in the 80s and the 90s and denounce that “the changes promised by the case study experiences have not really been noticed beyond the empirical evidence given by the studies themselves” (Kynigos et al. 2007, p.1332).

The acknowledgement of the existing gap between the research results on the use of technology in the mathematical learning and the little use of these technologies in the real classroom led recently to reconsider the importance of the teacher in a technology-rich learning environment, and to investigate ways of supporting teachers to use technological tools.

Those “teacher-centred” studies have been developed from different perspective and address different aspects, for instance: teacher education (Wilson, 2005), teachers' ideals and aspirations regarding the use of ICT (Ruthven, 2007), teacher's role in exploiting the potentialities of ICT tools in the classroom.

With that respect, as Trouche underlines, most studies refer to the importance of teachers' guide or assistance to students' activities with the technology (Trouche, 2005). Trouche himself emphasizes the need of taking into account the teacher's actions with ICT. For that purpose he introduces the notion of “instrumental orchestration”, that is the intentional systematic organization of both artefacts and humans (students, teachers...) of a learning environment for guiding the instrumental geneses for students (ibidem, p.126).

Within this approach the teacher is taken into account insofar as a guide for the constitution of mathematical instruments.

As we will argue in the next section, guiding the constitution of mathematical instruments does not exhaust the teacher's possible use of ICT. In fact ICT tools can

be used by the teacher (a) for developing shared meanings having an explicit formulation, de-contextualized with respect to the ICT tool itself and its actual, recognizable and acceptable in respect to mathematicians' community, and (b) for fostering students' consciousness-raising of those meanings. The Theory of Semiotic Mediation (Bartolini Bussi and Mariotti, 2008) takes charge of that dimension.

In this report, we present an analysis of the teacher's use of an ICT tool within the frame of the Theory of Semiotic Mediation. More precisely we focus on the teacher's promotion and management of collective discussions. But a systematic discussion of the role of the teacher or a classification of her possible actions is out of the goals of the present paper. The context is a teaching sequence, inspired by the Theory of Semiotic Mediation, and centred on the use of the tool Casyopée. Both the teaching sequence and the tool are presented in the next sections, after recalling some basic assumptions of the Theory of Semiotic Mediation.

THE THEORY OF SEMIOTIC MEDIATION

Assuming a Vygotskijan perspective Bartolini Bussi and Mariotti put into evidence that the use of an artefact for accomplishing a (mathematical) task in a social context may lead to the production of signs, which, on the one hand, are related to the actual use of the artefact (the so called **artefact-signs**), and, on the other one, may be related to the (mathematical) knowledge relevant to the use of the artefact and to the task. As obvious, this knowledge is expressed through a shared system of signs, the mathematical signs. The complex of relationships among use of the artefact, accomplishment of the task, artefact-signs and mathematical signs, is called the **semiotic potential** of the artefact with respect to the given task.

Hence, in a mathematics class context, when using an artefact for accomplishing a task, students can be led to produce signs which can be put in relationship with mathematical signs. But, as the authors clearly state, the construction of such relationship is not a spontaneous process for students. On the contrary it should be assumed as an explicit educational aim by the teacher. In fact the teacher can intentionally orient her/his own action towards the promotion of the evolution of signs expressing the relationship between the artefact and tasks into signs expressing the relationship between the artefact and knowledge.

According to the Theory of Semiotic Mediation, the evolution of students' personal signs towards the desired mathematical signs is fostered by iteration of **didactic cycles** (Fig.1) encompassing the following semiotic activities:

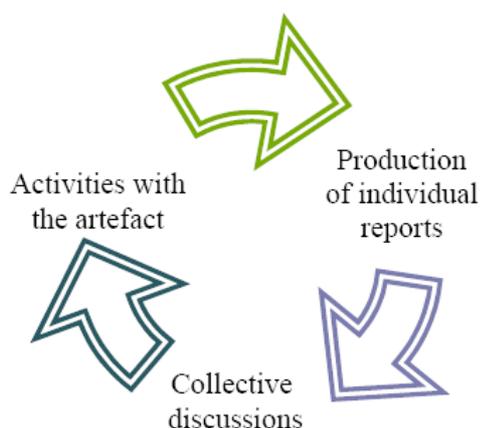


Fig. 1. Didactical Cycle

- activities with the artefact for accomplishing given tasks: students work in pair or small groups and are asked to produce common solutions. That entails the production of shared signs;
- students' individual production of reports on the class activity which entails personal and delayed rethinking about the activity with the artefact and individual production of signs;
- classroom collective discussion orchestrated by the teacher

The action of the teacher is crucial at each step of the didactic cycle. In fact the teacher has to design tasks which could favour the unfolding of the semiotic potential of the artefact, observe students' activity with the artefact, collect and analyse students' written solutions and home reports in particular posing attention to the signs which emerge in the solution, then, basing on her analysis of students written productions, she has to design and manage the classroom discussion in a way to foster the evolution towards the desired mathematical signs.

The Theory of Semiotic Mediation offers not only a frame for designing teaching interventions based on the use of ICT, but also a lens through which semiotic processes, which take place in the classroom, can be analysed (for a more exhaustive view, see Bartolini Bussi and Mariotti, 2008).

CASYOPÉE

Casyopée (Lagrange and Gelis 2008) is constituted by two main environments which can "communicate" and "interact" between them: an Algebraic Environment and a Dynamic Geometry Environment (though the designers' objective was not to develop a complete CAS or a complete DGE). Possible interactions between the two environments are supported through a third environment, the so called "Geometric Calculation". Without entering the details of Casyopée functioning, we can illustrate it through the following example.

If one has two variable geometrical objects in the DGE linked through a functional relationship (e.g. the side of a square and the square itself), Casyopée supports the user in associating algebraic variables to the geometrical variables and building an algebraic expression for the function (e.g. the function linking the measure of the length of the side, as independent variable, and the measure of the area of the square, as dependent variable). The generated algebraic variables and functions can be exported in the Algebraic Environment, and then explored and manipulated.

DESCRIPTION OF THE TEACHING EXPERIMENT

The Theory of Semiotic Mediation shaped both the design and the analysis of the teaching experiment carried out. In this chapter, we briefly describe the design.

Educational Goals of the designed teaching sequence.

The design of the teaching intervention started from the analysis of the semiotic potential of the tools of Casyopée. That analysis led us to identify two main educational goals: fostering the evolution of students' personal signs towards

1. the mathematical signs of function as co-variation and thus consolidate (or enrich) the meanings of function they have already appropriated, that entails also the notions of variable, domain of a variables...;
2. the mathematical meanings related to the processes characterizing the algebraic modelling of geometrical situation.

Description of the teaching sequence

According to our planning the whole teaching sequence is composed of 7 sessions which could be realized over 11 school hours.

The whole teaching sequence is structured in didactical cycles: activities with Casyopée alternate with class discussions, and at the end of each session students are required to produce reports on the class activity for homework.

The familiarization session is designed as a set of tasks and aims at providing students with an overview of Cayopée features and guiding students to observe and reflect upon the "effects" of their interaction with the tool itself, e.g.:

Could you choose a variable acceptable by Casyopée and click on the "validate" button? Describe how the window "Geometric Calculation" change did after clicking on the button. Which new button appeared?

Besides familiarization, the designed activities with Casyopée consist of coping with "complex" optimization problems formulated in a geometrical setting and posed in generic terms, e.g.:

Given a triangle, what is the maximum value of the area of a rectangle inscribed in the triangle? Find a rectangle whose area has the maximum possible value.

The aim is to elaborate on those problems so to reveal and unravel the complexity and put into evidence step by step the specific mathematical meanings at stake.

The diagram (Fig. 2) depicts the structure of the teaching sequence: the cyclic nature of the process, which develops in spirals, is rendered through the boxing of the cycles themselves.

Implementation and data collection

With some differences, the teaching sequence was implemented in 4 different classes (3 different teachers): two 13 grade classes and a 12 grade class of two Scientific High Schools, and a 13 grade class of Technical School with Scientific Curriculum.

Different kinds of data were collected: students' written productions; screen, audio and video recordings, and Casyopée log files. The analysis presented below is based on the verbatim transcription of the video recordings of the classroom discussions.

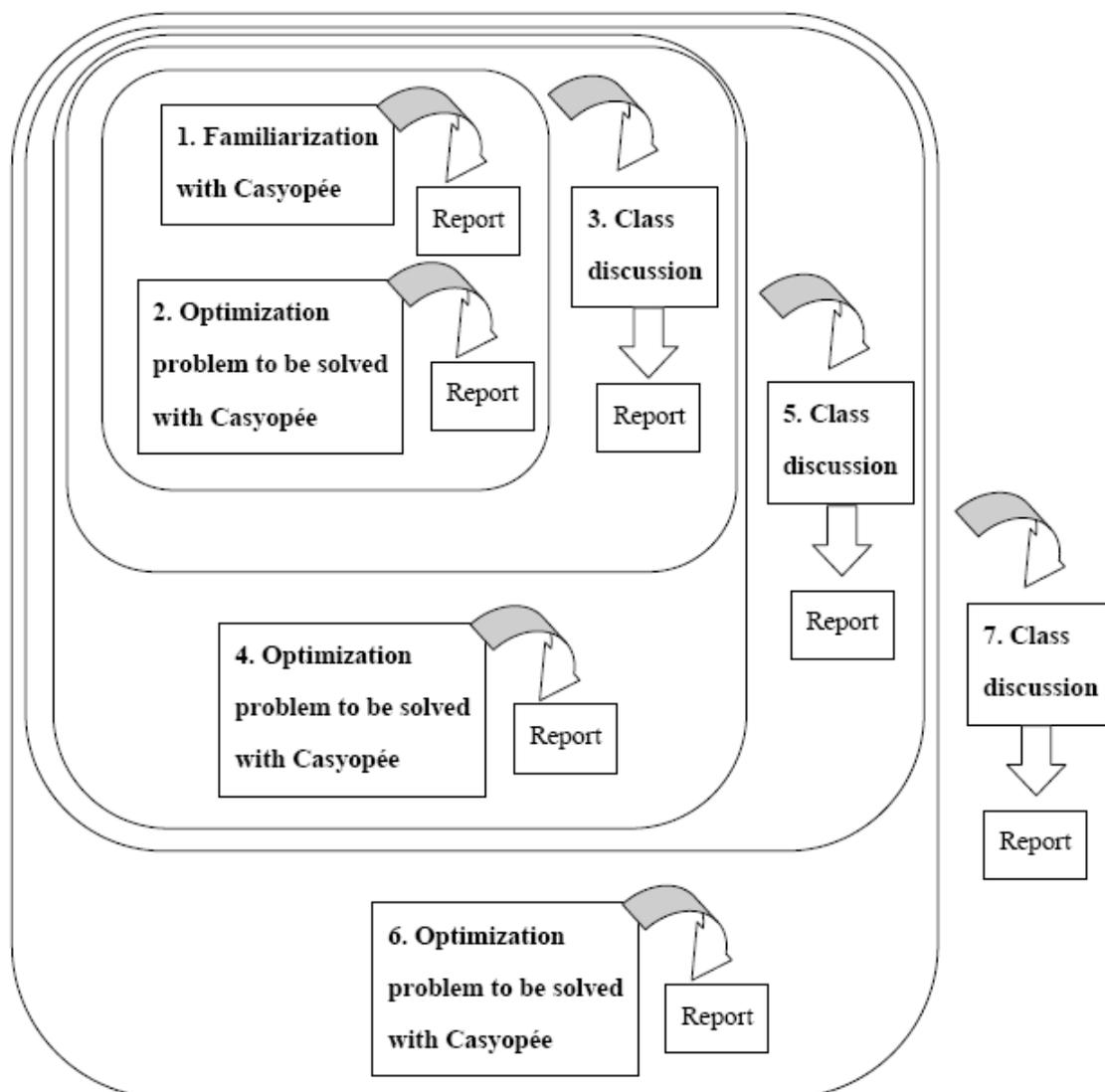


Fig. 2. Outline of the Teaching

ANALYSIS OF THE TEACHER'S ACTIONS

According to the theory of Semiotic Mediation, the teacher's action should aim at promoting the evolution of students' personal signs towards mathematical signs. Such evolution can be described in terms of *semiotic chains*, or chains of signification to use Walkerdine's terminology, that is:

“particular chain of relations of signification, in which the external reference is suppressed and yet held there by its place in a gradually shifting signifying chain.” (Walkerdine, 1990, p.121).

The following excerpt is drawn from the transcript of the class discussion held in the 5th session. It shows an example of how artefacts signs are produced in relation to the use of the artefact, and how they may evolve during the discussion. We first go quickly through the excerpt showing the evolution of signs, then we will analyse how the teacher contributes to this evolution.

1. Teacher A: “Which are the main points to approach this kind of problem? Which kind of problem did we deal with? [...] What is an important thing you should do now? To see the general aspects and apply them for solving possible more problems with or without the software, [...] the software guided you proposing specific points to focus on.[...]”
2. Cor: “[...] First of all we had to choose the triangle by giving coordinates”
[Students recall the steps to represent the geometrical situation within Casyopée DGE]
5. Luc: “But you have to choose a mobile point, first [...]”
6. Teacher A: “Does everybody agree?[...]How would you label this first part? [...]”
7. Students: “Setting up”
8. Teacher A: “Luc has just highlighted something [...] do you see anything similar between the two problems?”
9. Sam: “One has always to take a free point which varies, in this case, the areas considered [...]”
10. Teacher A: “Then we have a figure which is...”
11. Students: “Mobile.”
12. Teacher A: “Mobile, dynamical. Let us pass to the second phase. Andrea, which is the next phase? [...]”
13. And: “The observation of the figure would let us see... we need to study that figure and observe what the shift of the variable causes...”
14. Teacher A: “Ok, then? Everybody did that, isn’t it?”
15. Sil: “We computed the area of the triangle and of the parallelogram, we summed them, and by shifting the mobile point one observed as [the sum of the areas] varied [...]”

Focusing on students’ signs, one can notice:

- Elements of a collectively constructed *semiotic chain*, in which a connection is established between artefact signs (“mobile point”) and mathematical signs (“variable”). The elements of this semiotic chain are: “movable point” (item 5), “free point” (item 9), “variable” (item 13), and “movable point” (item 15). It is

worth noticing the two directions: from the artefact sign (“mobile point”) to the mathematical sign (“variable”) and vice versa. That semiotic chain shows: (a) students’ recognition that geometrical objects can be considered (can be treated, can act) as variables, and (b) the enrichment of students’ meanings of variable to include meanings related to “movement”.

- Elements of a collectively constructed *semiotic chain*, in which the meaning of function as a relation of co-variation of two variables emerges. The elements of this semiotic chain are: “a free point which varies [...] the areas” (item 9), “the shift of the variable causes” (item 13), “by shifting the movable point, one observed as [the sum of the areas] varied” (item 15).

Analysis of the Teachers’ orchestration of the discussion.

We reconsider the excerpt previously analysed from the point of view of the signs produced and used by students. Here we focus on how the teacher’s actions fuel the discussion, foster the production of artefacts signs in relation to the use of the artefact, and create the conditions for their evolution during the discussion.

1. Teacher A: “Which are the main points to approach this kind of problem? Which kind of problem did we deal with? [...] What is an important thing you should do now? To see the general aspects and apply them for solving possible more problems with or without the software, [...] the software guided you proposing specific points to focus on.[...]”

The teacher starts the discussion by making explicit its objectives: to arrive at a shared and de-contextualized formulation of the different mathematical notions at stake (“to see the general aspects and apply them for solving possible more problems with or without the software”).

In order to do that, the teacher asks students to recall the problem dealt with in the previous section and to report on the solutions they produced. She explicitly orients the discussion towards the specification of the main phases of the solution of the problem, asking students to look for similarities between the two problems addressed so far and between the strategies enacted to solve them.

While asking students to do that, the teacher suggests to refer to (or to remind) the use of the DDA. The suggestion to explicitly refer to the use of Casyopée facilitates the production and use of artefact-signs and the unfolding of the semiotic potential.

5. Luc: “But you have to choose a mobile point, first [...]”
- ...
8. Teacher A: “Luc has just highlighted something [...] do you see anything similar between the two problems?”
9. Sam: “One has always to take a free point which varies, in this case, the areas considered [...]”

Following the teacher’s request, students collectively report on their work with Casyopée. That leads to the production of the artefact sign “mobile point” (out of the

others) (item 5). The sign “mobile point” is clearly related to the task and the use of Casyopée for accomplishing it. At the same time it may be related to the mathematical knowledge at stake: the notion of variable. There are several possibilities for the subsequent development of the discussion: one could orient the discussion towards the distinction between mobile and variable, towards the specification of other variable elements, discussion towards the distinction between algebraic or numerical variable and geometrical variable, towards the recognition of the aspects of co-variation between the variable elements of the geometrical figure, towards the distinction between independent and dependent variable.

Certainly, the teacher’s intervention is needed both to drive the attention of the class towards the sign introduced by Luc and to orient the discussion. The teacher is aware of that and intentionally emphasizes Luc’s contribution to the discussion (item 8). At one time, she requires to generalize so to foster a de-contextualization from the specific problems faced and strategies enacted, and to provide the possibilities for the evolution of personal signs to initiate.

After the teacher’s intervention, Sam (item 9) echoes Luc’s words. But she uses the sign “free point” instead of “mobile point”, and introduces the consideration of other variable elements (“areas”) also emphasizing the existence of a link between them (“free point which varies [...] the areas”). Those are the first elements of the two semiotic chains described in the previous section.

10. Teacher A: “Then we have a figure which is...”
11. Students: “Mobile.”
12. Teacher A: “Mobile, dynamical. Let us pass to the second phase. Andrea, which is the next phase? [...]”
13. And: “The observation of the figure would let us see... we need to study that figure and observe what the shift of the variable causes...”

Sam’s contribution (item 9) ends with the reference to variable areas. That could prematurely move the discussion towards the consideration of algebraic or numerical aspects, without giving time to elaborate on variable and variation in the geometric setting. In order to contrast this risk, the teacher introduces the term “figure” (item 10) which has the effect of keeping students’ attention still on the geometrical objects. In addition the teacher fuels the discussion echoing students and, thus, emphasizing the reference to the dynamical aspects (item 12), which nurtures the construction of the semiotic chains on variation and co-variation.

And, whose intervention is stimulated by the teacher, echoes the use of the sign “figure” and makes explicit exactly the co-variation between the geometrical objects in focus. She also introduces the sign “variable” so establishing a connection between the artefact sign “mobile point” and the sign “variable”.

We are not claiming that the evolution towards the target mathematical signs is completed: a shared and de-contextualized formulation of the different mathematical

notions at stake is not reached yet, as witnessed by Sil's words (item 15), who still makes reference to the use of the artefact in her speech.

14. Teacher A: "Ok, then? Everybody did that, isn't it?"

15. Sil: "We computed the area of the triangle and of the parallelogram, we summed them, and by shifting the mobile point one observed as [the sum of the areas] varied [...]"

The above analysis puts into evidence a number of interventions of the teachers who succeeds in exploiting the semiotic potential of Casyopée, and thus in making the class progress towards the achievement of the designed educational goals.

One can find also episodes in which the teacher's action is not so efficient. The following excerpt is drawn from a discussion held in another class and orchestrated by a different teacher, and it shows an episode in which the teacher does not succeed to exploit the potentialities of the students' interventions. Chi countered the sign "variable" with the sign "variable point" so offering the possibility to dwell on the relationship between not measurable geometrical variables and measurable geometrical variables. The specification of this distinction was considered a key aspect of algebraic modeling, and as such highly pertinent to the designed educational goals. The teacher does not seize the occasion and does not take any action to fuel the discussion on that, she was probably aiming at orienting the discussion along a different direction.

184. Chi: "we put CD as variable, and not by chance CD, in fact we used a fixed point, C, and a variable point on the segment, D"

185. Teacher B: "well, the underpinning idea is to link numbers, and, [...] having observed a link between the position of the point D and [...] the area of the rectangle [...] a link is established between a geometrical world and an algebraic world"

That witnesses the difficulty of mobilizing strategies to foster the evolution of students' signs. One has to constantly keep the finger on the pulse of the discussion and of its possible development. In fact the evolution of students' signs depends on extemporary stimuli asking for a number of decisions on the spot.

CONCLUSIONS

The analysis carried out in the paper confirms the crucial role of the teacher in technology-rich learning environments. In particular, such role may (and should from our perspective) go beyond that of assistant or guide for students' instrumental genesis process. In fact through her interventions the teacher promotes and guides the development of the class discussion, so to foster the production and the evolution of students' signs towards the target mathematical signs, and to facilitate students' consciousness-raising of the mathematical meanings at stake.

Certainly we are aware that the analysis presented is still at a phenomenological level. There is an emerging need for elaborating a more specific model for analysing the teacher's semiotic actions. But there is not only the need of developing tools for finer analysis. We showed an episode witnessing the difficulty of mobilizing strategies to foster the evolution of students' signs. Currently, the Theory of Semiotic Mediation does not equally support analysis and planning. Due to the richness of a class discussion and the number of extemporaneous stimuli which could emerge, one cannot foresee the exact development of the discussion. That makes the teacher's role still more crucial. Nevertheless there is the need of an effort for elaborating more specific theoretical tools for supporting the a-priori design of classroom discussion. All this is also relevant to the more generic issue of teacher's formation.

NOTES

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ESTABLISHING DIDACTICAL PRAXEOLOGIES: TEACHERS USING DIGITAL TOOLS IN UPPER SECONDARY MATHEMATICS CLASSROOMS

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This paper discusses elements of the didactical work of ordinary mathematics teachers using digital tools. The upper secondary school in Norway where the data was collected has run an internal project to integrate the Personal Computer into the mathematics classroom. Using the Instrumental Approach as a framework this paper seeks to describe and interpret elements of teacher practice exploring also the notion of instrumental genesis from a teacher perspective. From the analysis of classroom observations, interviews, meetings, and study of documents three main didactical practices were found to be linked to the introduction of the digital tools: the digital notebook, the digital textbook, and the phenomenon of weaving between tools/instruments in the classroom.

INTRODUCTION

The recent school reform in Norway, Knowledge Promotion 2006, formally acknowledges digital competence as one of the five basic skills students should acquire and develop in their formal schooling¹. This places on schools and individual teachers a responsibility to integrate these tools into classroom practice. This study looks at the practice of two teachers in a comprehensive upper secondary school in Norway who have been using digital tools over a period of five years. In 2007 the school joined the project “Learning Better Mathematics”, hereafter LBM², a developmental project initiated by school authorities through a co-operation with University of Agder. Data used in this paper was collected at the school’s point of entry to the project. The classrooms observed were equipped with a blackboard and a projector with screen and set up as “paperless” environments where all students had their own laptop PC and when observed rarely used paper and pencil in their mathematics lessons: all student work was done on the computer.

THEORETICAL FRAMEWORK

The theoretical approach employed emerged in the mid-nineties in France when researchers became aware that traditional constructivist frameworks were inadequate in the analysis of CAS environments (Artigue, 2002). Artigue claims that this approach is less student centred but provides a wider systemic view also giving the instrumental dimension of teaching and learning more focus (Artigue, 2007). The

¹ Knowledge Promotion (Kunnskapsløftet 2006). These basic skills are given as the ability: to express oneself orally to read, to do arithmetic, to express oneself in writing, to make use of information and communication technology

² The project is supported by the Research Council of Norway

approach uses notions both from the Theory of Instrumentation from the field of Cognitive Ergonomy, and from the Anthropological Theory of Didactics (ATD hereafter) in the field of Mathematics Education (Laborde, 2007).

Cognitive Ergonomy considers all situations where human activity is instrumented by some sort of technology. The theory of instrumentation employs the notion of “instrument” and the notion of “instrumental genesis” (Artigue, 2002). The instrument has a mixed identity, made up of part artefact and part cognitive scheme. It is seen as a mediator between subject and object but also as made up of both psychological structures, called schema which organise the activity, and physical artefact structures such as pencil, paper, or digital tools (Béguin & Rabardel, 2000). For the individual user, the artefact becomes an instrument through a process of instrumental genesis which involves the construction of personal schema or the appropriation of socially pre-existing schemes (Artigue, 2002). This process of instrumental genesis has two elements, *instrumentalisation* the process whereby the user acts on the tool shaping and personalising the tool, and *instrumentation* the process whereby the tool acts on the user shaping the psychological schema (Rabardel, 2003). Instrumental genesis is a process occurring through the user’s activity through participation at the social plane. Guin and Trouche (1999) applied the Theory of Instrumentation in research in mathematics classrooms, studying the process by which the graphic calculator becomes an instrument for the students to learn mathematics. They term the teachers’ role in guiding the students’ instrumental genesis *instrumental orchestration*. This is defined as a plan of action having four components: a set of individuals, a set of objectives, a didactic configuration and a set of exploitation of this configuration (Guin & Trouche, 2002, p. 208).

ATD on the other hand aims at the construction of models of mathematical activity to study phenomena related to the diffusion of mathematics in social institutions, see for example (Barbé, Bosch, Espinoza, & Gascón, 2005). The theory analyses human action including mathematical activity by studying *praxeologies*:

But what I shall call a praxeology is, in some way, the basic unit into which one can analyse human action at large. (Chevallard, 2005, p. 23)

Any human praxeology is constituted of a practical element (praxis) and a theoretical element (logos). The praxis has two components, the task and the technique to solve the task. The logos also has two components, the technology (or discourse) and the theory which provide a justification for the praxis.

Mathematical knowledge in an educational institution can be described in terms of two types of praxeologies: mathematical praxeologies and didactical praxeologies. The object of the didactical praxeologies is the setting up of and construction of the mathematical praxeologies. It is these didactic praxeologies, representing teacher practice, that are of interest to me in my study. Questions arising are: What constitutes or defines the didactical task, technique, discourse and theory? How are the mathematical praxeology and the didactical praxeology entwined? How do the

existing didactical praxeologies change when digital tools are introduced into the mathematics classroom? Laborde's conclusion that, "A tool is not transparent. It affects the way a user solves a task and thinks" (Laborde, 2007, p. 142) should apply equally to both teacher and student.

Research indicates that the interventions of the teacher are critical in relation to student learning of mathematical knowledge when digital tools are introduced (Guin & Trouche, 1999). The teacher's instrumental orchestration is part of the didactical praxeology. As new tools are introduced, the teacher must develop new didactical praxeologies to support the students' instrumental genesis for the particular tool (Trouche, 2004, p. 296). The teacher must also incorporate the new tool into an existing repertoire of tools and didactical techniques. Practically in the classroom, this involves for the teacher: (1) Organisation of space and time, (2) the choosing of the mathematical tasks and the techniques to solve these tasks, and (3) the steering of the mathematical activity in the classroom by discourse.

Aim and research question

This paper aims to identify features of didactical praxeologies that have been established in relation to the introduction of the digital tool and also to describe the process of introduction of the digital tool and changes to practice from the teacher perspective. The research questions are: What features of the teachers' didactical praxeologies can be identified as pertaining to/originating specifically from the introduction and use of the digital tool? Can these features be seen as evidence of a process of instrumental genesis for the teachers in relation to the digital tool? What factors influence this process?

This short paper allows for in depth discussion of only some of the features indicated above. I have therefore selected features that appear to be of significant importance to the teachers when they describe the changing practice in relation to the tool. The paper also seeks to describe only commonalities in teacher practice.

THE EMPIRICAL STUDY

The teachers, their classes and classrooms

The two teachers in this study very generously opened their classrooms and gave of their time to this researcher. Both were active in initiating the ICT project at the school. The ICT project had been established and operated entirely within the school and was not part of any external research, design or development project. It is therefore claimed that it is the practice of two "ordinary" teachers that is described in this paper. In 2005, the school was the only school in the country to conduct final examinations in mathematics entirely on the portable PC.

This part of the study involved classroom visits to two classes of approximately twenty five students. The students were studying the subject "Theoretical Mathematics 1" (1T), which is allocated three double lessons a week, each of 90 minutes duration. These two classes were two of five classes at the school studying

this subject. Each classroom was equipped with a blackboard and a projector with screen. The screen covered part of the blackboard but it was still possible to use the blackboard. The technical features of the environment functioned without difficulties in the observation period. The classrooms observed presented as “paperless” environments as all students had their own laptop PC, leased from the school, and when observed rarely used paper and pencil in their mathematics lessons though this was permitted. All student work including exercises, notes, rough work was done on the computer. I have chosen to refer to this practice as the “digital notebook”. Standard paper textbooks were no longer in use as the teachers have developed their own digital textbooks, which are made available to the students through a Learning Management System (LMS). This practice I refer to as the “digital textbook”. The classrooms appeared very orderly as there were no books, papers, rulers or other items littering the desks. Each student had a PC and perhaps a bag placed on the floor under the desk. The students started work quickly plugging in and turning on the PC, contrasting sharply with “normal” classrooms where students take some minutes to find notebooks, textbooks, pencils and so on. In the observed lessons only the teachers used the projector. Student work was not displayed using the projector.

Data collection and analysis

Data collection over a period of four months involved: audio recording of an introductory meeting between the school and the university where the two teachers, a school leader, two researchers and a project leader from the university were present; lesson observation with video recording of eight lessons; audio recording of three semi-structured interviews before and after lessons with the teachers; audio-recording of seven structured interviews with students (Billington, 2008); and audio data from LBM project meetings where the teachers were present and took part in discussions. The writer was present at all events, taking field notes. In the classroom observations, researchers were present as observers, taking no active part in the planning or carrying through of the lesson. Shortly after each event a preliminary data reduction using the notes and recordings was made. Passages were also transcribed. Later all data was again reviewed, coded and further transcribed. Each data episode renders different information helping to build a picture of teacher practice identifying didactical praxeologies that would not be there without the digital tool. The meetings and interviews tell of the temporal dimension and of the changing nature of the didactical praxeologies from the teacher perspective and also reveal the institutional influences. Classroom episodes record teacher activity in the classroom revealing techniques of instrumental orchestration. Student interviews tell of the students’ instrumental genesis and the teachers’ orchestration from the students’ perspective.

Analysis of data from meetings and interviews

The teachers were very keen to discuss the introduction of the digital tool and there were clear indications in the data that the teachers saw a process of development in their teaching practice. Examples of such comments were as follows:

Teacher 1: ... and it, it has been, been of course, a long process to come this far, this software ...

Teacher 2: But ...there is, as such, a remarkable difference from when we started, now...

Reviewing the data from the meetings and interviews, reoccurring themes emerged. These were first categorised under three headings, justification, implementation and evaluation. I then attempted to interpret these themes in the light of the theory as presented in the table below. In a didactical praxeology, implementation would pertain to the praxis while justification and evaluation would pertain to the logos.

Justification	Implementation	Evaluation
Teachers explained why “we do what we do and continue to do what we do ”	Teachers explained how they organised and carried out the project	Teachers talked about what they identified as affordances and constraints of the tool
Didactical theory – justification of practice	Didactical tasks and techniques	Didactical technology (discourse) – relating theory to tasks and techniques

Table 1: Interview Themes

The most common reoccurring themes under implementation were: the digital textbook, the digital notebook and teaching techniques in the classroom. There was also some discussion input from a school leader, which is relevant to the discussion on orchestration.

Results and discussion of data from meetings and interviews

As stated above, the teachers referred constantly to the introduction of the digital textbook and the digital notebook. Discussion of these two innovative features of the implementation occupied much of meeting and interview time. The teachers referred to the digital textbook as “Learning Book”³. This digital textbook has replaced the usual paper textbook that students would normally buy. It is made available through the functioning LMS. Commenting on the digital textbook, the teachers explained that as the project progressed they found that the students preferred to read the notes that they had made rather than read the paper textbook. As a new syllabus came into force this year they decided to make their own digital textbook from scratch.

Teacher 1: Yes. Totally from scratch, just from the syllabus. Not from any textbook ...We have taken the syllabus point by point ...

Teacher 2: Now we use the syllabus, and it has been extremely useful to go thoroughly into the plans and now we have to make the right choices ... we feel we have to make a good deal of choices ... that we make for the students ...

³ Here literally translated from the Norwegian “læringsbok”

The teachers have been provoked to return to the mathematical goals in the syllabus and build from these. This development is in line with that described by Monaghan (2004). The students save this textbook to their own PC and can write in memos, and notes. All problems and exercises are also made available through the LMS for the students. According to the teachers giving out solutions on the LMS saves time that can be used to other things, for example, “we can go around and help”. The students also retain these files from year to year whereas previously they sold the textbook at the end of the year. In terms of the theoretical framework of ATD this could be interpreted as a transposition of mathematical knowledge (Balacheff & Kaput, 1996) from the syllabus to a form usable on the PC.

The second innovation, the digital notebook, a notebook kept by each individual student where s/he writes and stores all notes, exercises, and rough work on his/her own PC, was also clearly important to the teachers. In fact one teacher gave this aspect some credit for the increased enrolment of girls in these maths courses.

Teacher 1: ...and they (girls) sat on the fence for a year or so. And then a few girls signalled to the others, see here, and then the girls joined in force,That was when the girls saw that this was not about playing games, but this was a way to make it very nice. They got everything very systematic, got a way to keep all their notes in order, and very, very nice presentation, and this, the girls thought was very ok, and the boys too, now they have all their notes from last year and can build on this.

Choosing supporting materials for the student is a didactical task for the teacher. In this case the production of a digital textbook and the promotion of a digital notebook are clearly identifiable as innovations in relation to normal practice and could be interpreted as an instance of instrumentalisation where the user shapes the tool to his/her purpose. Data from the student interviews confirmed that these two innovations were important in the students’ instrumental genesis (Billington, 2008).

This leads to the reoccurring third theme in the meetings and interviews: reflection over teaching practice in the classroom. The teachers expanded on the teaching philosophy on which they have based the project claiming that they tried to avoid the standard structure of theory, example, exercises, and method.

Teacher 2: We have had a main principle since we started with this. These textbooks are always alike, theory, examples, and then exercises exactly like the example, and then examples that are almost the same. As far as possible we try to avoid this. Our philosophy is fewer exercises and they can rather sit and struggle with the same exercise and if it takes the whole lesson that does not matter.

Interestingly the teachers did not expose on the wonders of the digital tool per se, but rather talked of the teaching possibilities with the tool as illustrated by these quotes.

Teacher 1: I have much more influence on my own teaching before...

Teacher 1: The role of the teacher is a bit ...you have greater possibilities, that is what we have seen ...

Teacher 2: But, I must say, for my sake, that I have opportunities that I would never had had without the PC.

These possibilities can be interpreted as new didactical techniques. One teacher claimed that his teaching had changed since the students have now chosen not to use the standard paper textbooks. They discussed the need to focus on understanding rather than the reproduction of algorithms. They saw the creation of the digital textbook as allowing them more freedom to steer the activities of the classroom in line with their philosophy. These reflections I interpret as discourse justifying the praxis element of the didactical praxeologies.

Choosing for students the mathematical tasks, and the techniques and tools to solve these tasks, is a didactical task for the teacher. These tools include the textbook as well as the digital tools, the software and the hardware. The nature of this didactical task has changed for these teachers in the course of the project. They have explained how previously they just followed the book, a routine, but now because of the new situation they have been forced to make new choices. They now worked together to select mathematical tasks themselves rather than following a set up in a book.

Analysis of data from classroom observation

In looking at the data from classroom observations I attempted to identify didactical praxeologies that were a result of the introduction of the digital tool. In the classroom observation data I looked at the teachers' (1) Organisation of space and time, (2) Choice of mathematical tasks and mathematical techniques and physical tools, and (3) Steering of activity through discourse, considering these to be three practical moments of the didactical praxeologies.

In the lessons observed, neither the organisation of space or time nor the choice of mathematical tasks seemed to be dependent on or unique for a classroom where the digital tool of the PC has been introduced. For example, analysis of the time disposition in lessons showed a script with recapping, homework correction, new theory, and then exercises with approximately 50 – 60 % of the lesson time spent with students working alone or in pairs on exercises. Some time however was given to the explanation of the technical aspects of performing the mathematical techniques with the digital tool. This time allocation varied from lesson to lesson.

Deviation from a standard classroom environment without digital tools was observed in the type of tools used by the students and by the teacher and also in the public discourse of the teacher. Choosing the tools for use in the lesson, for the teacher and for the students to carry out mathematical tasks is a didactical task. This is an ongoing task as choices are made in the planning but also in the conduct of the lesson. Two aspects that stood out in the observations were the manner in which the teacher used both the digital tool and the blackboard to support his/her public discourse and the manner, which the teacher referred to and talked about using the digital tool when describing the mathematical techniques to solve the mathematical

tasks. This second aspect involves a too broad discussion to take in this paper but will be discussed in the thesis of which this work forms a part.

Results and discussion of classroom observation

In the classroom observations the teachers used both the blackboard and the screen, which was connected to the PC to support their public discourse. One feature that emerged frequently in each observed lesson, I term “weaving”. Weaving describes the manner in which the teacher moved between the available tools. Three physical tools were noted to be in use when the teacher was holding public discourse: the blackboard, the PC+screen, and gestures with own body such as tracing out a curve in the air. Each of these tools is used in conjunction with the voice and schemas (cognitive apparatus). It appeared that in prepared sequences of the lessons the digital tool was used but in spontaneous situations, for example when pressed for further explanation, the teacher turned to the blackboard or to gestures.

Discussing this weaving with the teachers, one teacher explained, that “we use what is appropriate in the situation”. Teachers seemed to identify affordances and constraints of each tool. It appeared that an affordance of the blackboard was that it allowed more personal and spontaneous expression by the teacher. It may also be the case that such unplanned use of the digital tool requires a high level of skill and familiarity with the tool and as such this is a constraint of the tool. In a later instance one teacher began to draw a circle on the blackboard freehand but suddenly stopped saying; “I have an excellent tool to do this”, and then drew the circle using the dynamic geometry software on PC screen instead. Also the mathematical tasks in use were standard tasks, which could be solved without the digital tool. Had these tasks been more complex or tasks that required the use of digital tools perhaps the response of the teacher would have been different.

CONCLUSIONS AND FURTHER DISCUSSION

Returning to the research questions, three features of the didactical praxeologies as specifically pertaining to and “provoked” by the introduction of the digital tool have been identified and discussed: the digital notebook, the digital textbook, and the phenomena of weaving between tools/instruments in the classroom. The two features that are seen as particularly important by the teachers are the digital textbook and the digital notebook. These could be interpreted as examples of instrumentalisation whereby the teacher as user has adapted the tool to his/her usage. In the classroom, the observation of patterns of inter-dependent mediation between physical tools that have been adapted by the teachers, where they weave between blackboard and the digital tool in response to the situation, could be interpreted as observations of schema or expression of instrumentation as in these cases the tool which is thought to be the most appropriate is used.

Can the project implementation described above be modelled as a process of instrumental genesis for the teachers and is such a modelling helpful in gaining an

understanding of the situation? Further examination of teacher discourse will provide more information about this possible instrumental genesis process though tentative findings in this report seem to lead in this direction. Some issues to be discussed in relation to such a process are for example: the temporal dimension; if instrumental genesis is a process how is it possible to identify the different stages of this process for the teacher; and also as to which observations would indicate the formation of schema. The notion of instrumental orchestration has been discussed earlier. Is the process of instrumental genesis for the teacher also influenced by some constraining factors? Comments by the teachers indicated that, for the teachers, the process is steered in part on an organisational level by the schooling authorities at school, region, and national levels. Financial and policy support from schooling authorities is necessary for the survival of the ICT project. In the meetings, the school leader was highly supportive of the project and expressed the opinion that when students think it (mathematics) is fun, then they use more time on mathematics and so become better at it. Enrolment in mathematics has also increased dramatically. However, more important to the teachers seemed to be the response of the students. In the categories of justification and evaluation the majority of comments by the teachers concerned student learning and engagement as illustrated by the comment below.

Teacher 1: Need to give students a challenge. Students are not educated to work in this way. Now they think it is fun. Looking for methods ...

For the teachers in this study, the students' response to the new situation appears to influence the teachers' use and adaptation of the digital tool. Such comments as above also indicate that the teachers are aware of their role in orchestrating their teaching to support the instrumental genesis of the student.

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DYNAMIC GEOMETRY SOFTWARE: THE TEACHER'S ROLE IN FACILITATING INSTRUMENTAL GENESIS

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In the UK, use of dynamic geometry software (DGS) in classrooms has remained limited. Whilst the importance of the teacher's role is often stated in dynamic geometry research, it has been seldom elaborated. This study aims to address the apparent deficiency in research. By analysing teacher/pupil interactions in a DGS context, the intention is to identify situations and dialogue that are helpful in promoting mathematical thinking. The analysis draws on an instrumental approach to describe such interactions. Elements of instrumental genesis are distinguished in pupils' dialogue and written work which suggest techniques that teachers can employ to facilitate this process.

Keywords: *teacher's role, dynamic geometry, instrumental genesis*

INTRODUCTION

This study aims to elicit teaching techniques that teachers might employ in their classrooms to help pupils engage constructively with dynamic geometry software. Currently DGS has made little impact on classrooms in the UK. Research has tended to focus on elaborating situations of innovative use and student/machine interaction. This study hopes to re-focus on “the teacher dimension” (Lagrange et al, 2003). The author carried out this study in the role of a practitioner-researcher with a high ability year 8 class. Whilst the class cannot be deemed to be representative, nevertheless it is an ‘ordinary classroom’ and therefore this study can claim to respond to the need for research into how dynamic geometry software is integrated into ‘the *regular* classroom’ (Gawlick, 2002).

DGS – A CLASSROOM FAILURE?

Dynamic geometry software (DGS) appears to be following the cycle of oversell and high expectations, ending in limited classroom use identified by Cuban (2001) as a general pattern for technological innovation in education. Research in mathematics education generally presents DGS as a potentially important and effective tool in the teaching and learning of geometry (see for example Holzl, 1996; Marrades and Gutierrez, 2000; Mariotti, 2000). In their survey of geometry curricula, Hoyles et al (2001) state that although most countries seek to integrate ICT into teaching geometry, there is little explicit influence of ICT in classrooms. In the UK, despite recommendations in the Key Stage 3 Mathematics Framework (DfEE, 2001) for using DGS to develop geometrical reasoning, classroom use has remained limited (Ofsted, 2004). Syntheses of research findings generally conclude by favouring the

strong potential of ICT but give few explanations for the contrasting poor reality of classroom use (Lagrange et al, 2003).

THE ABSENT TEACHER

A criticism of educational policy and discourse on ICT is that the predominant focus has been on technology rather than education (Selwyn, 1999). The picture painted by Lagrange et al (2003) of research on ICT within mathematics education is of a field dominated by “publications about innovative use or new tools and applications” where issues of the integration of technology into ordinary classrooms have been largely neglected. In particular, the voice and role of the teacher has been notably absent. DGS is no exception: in his review of research on dynamic geometry software, Jones (2002) suggests that future research could usefully focus on teacher input and its impact, amongst other issues. Although research has begun to examine the role of the teacher in DGS integration, the practices of ordinary teachers in ordinary classrooms remains an area requiring further investigation (Lagrange 2008).

This study was designed with these issues in mind. The instrumental approach, described in the next section, was used to analyse teacher/pupil interactions in order to elicit teaching techniques which might facilitate pupils’ instrumental genesis.

THEORETICAL BACKGROUND

Instrumental genesis is described as the process by which an artefact is transformed into an instrument by the subject or user (Guin and Trouche, 1999). An artefact is a material or abstract object, given to a subject. An instrument is a psychological construct built from the artefact by the subject internalising its constraints, resources and procedures (Guin and Trouche, 1999). Once the user has achieved instrumentalisation, he is able to reinterpret or reflect on the activity he is engaged in. Drijvers and Gravemeijer (2005) describe instrumental genesis as the “emergence and evolution of utilisation schemes”. A utilisation scheme is a “stable mental organisation” including both technical skills and supporting concepts as a method of using the artefact for a given class of tasks (Drijvers and Gravemeijer, 2005). The interrelation between machine techniques and concepts seems important since Drijvers and Gravemeijer (2005) found that the apparent technical difficulties that students had often had a conceptual background.

The instrumental approach has been mainly developed and applied within the context of computer algebra software (Drijvers and Gravemeijer, 2005) and there remains a question over how general its applicability is. Drijvers and Gravemeijer (2005) cite two examples where the instrumental approach has been applied to DGS. Thus it seems instrumental genesis may be an appropriate tool to analyse observations of student behaviour within a dynamic geometry environment.

RESEARCH CONTEXT AND METHODOLOGY

This study was conducted as part of a Best Practice Research Scholarship-funded project on using DGS as a resource for teaching geometrical proof. Much of the previous research on DGS has focused on pupils in upper secondary school (Jones, 2000). It has been suggested that more research is needed on the impact of dynamic geometry software on students in lower secondary school (Marrades and Gutiérrez, 2000). The decision to conduct the research with the researcher's year 8 class was partly influenced by this consideration. Since the pupils were in year 8, there was an added advantage that they were not subject to public examinations, the curriculum is less pressurised and therefore ethical considerations about deviating from schemes of work were somewhat reduced. The school in which the research was conducted is a private day school for girls. The research was conducted with the highest attaining set in year 8, containing 23 pupils, with girls expected to achieve levels 7 or 8 at Key Stage 3 [1]. In common with several other research studies, this was seen as an advantage since students judged to be above average in mathematical ability are most likely to be able to engage with proving processes and therefore allow meaningful data collection to take place (Jones, 2000; Marrades and Gutiérrez, 2000).

In this paper, I consider data drawn from a sequence of 5 lessons in which pupils were engaged in investigating a series of construction problems in pairs using Cabri Geometre. The tasks were based upon the Phase 1 and 2 tasks developed by Jones (2000) and were intended to progress in difficulty. Each task consisted of a figure which the pupils were to construct in Cabri so that it remained constant under drag. The methods for constructing a figure were linked and developed from previous problems to encourage the pupils to examine how additional constraints might affect the resultant shape. They were prompted to say what the resultant shape was and, importantly, how did they know? The point of the teaching sequence was to encourage pupils to justify or prove these assertions.

The pupils were asked to choose a construction of their choice and produce a Power-point presentation on why their construction had worked which was presented to the class. Printouts of the pupils' Power-point presentations and audiotape recordings of their presentations to the class form one part of the data collected. During the lessons, the researcher carried an audiotape so that any teacher/pupil interactions would be recorded: these recordings form another part of the data collected. After the lessons, brief field-notes were made on the major events in the lesson.

The initial stage of data analysis concerned the transcription of tape-recordings made during lessons. Using field notes, the tapes were broken down into major events or "episodes" (Bliss et al, 1996). In the sense described by Bliss et al (1996) these episodes had "an internal coherence"; they were complete conversations which allowed the researcher to "interrupt momentarily, for the purpose of analysis, the 'relentless flow of the lesson'". A second stage of analysis involved going through the transcripts and pupils' work making notes, identifying critical incidents that build

towards detailed accounts of practices. The final analysis was based on a grounded approach using narrative techniques (Kvale, 1996) which moved back and forth between the theoretical viewpoint developed in the review of literature and the pupils' work and transcribed episodes. Each step in this process eased the transition from emotionally involved participant towards objective observer. Using the concept of instrumental genesis to achieve a "rich and vivid description of events" (Hitchcock and Hughes, 1995), this study hopes to tease out the threads of a tapestry of complex social interactions to see if techniques for promoting mathematical thinking can be discerned in the weave.

ANALYSIS

From the analysis of data, three teaching techniques emerged for facilitating pupils' instrumental genesis in Cabri. Using excerpts from teacher/pupil dialogue, these techniques are described below, where T represents the teacher throughout.

Unravelling functional dependency in DGS

In common with other students, Pupils H and C experienced difficulty with specifying where they wanted objects to intersect when attempting to construct two circles sharing the same radius. They constructed the first circle successfully and correctly placed the centre of the second circle on its edge. The difficulty arose when they tried to adjust the size of the second circle so that its edge would pass through the centre of the first circle, thus ensuring that they would share a radius. The problem was that they made it *look* like the edge of the second circle passed through the centre of the first circle rather than specifying to Cabri that the circle should go "By this point" – as the Cabri pop-up phrase suggests if you hover over the required centre point. Although their Cabri drawing looked successful, when it was subjected to a drag-test, the circles changed size in relation to each other instead of maintaining their pattern:

T: Yeahhh. That's it because you see this computer program will only do exactly what you tell it so if you just make it look like it... sort of, yeah. I'm going to be able to change the shape of your circle so if you tell it, look....

crackle: teacher using the computer to show how the circle can still be messed up. Then creates a new one "by this point" method to show the difference

T: Ok now try and mess it up, you try and mess it up now
mess up one of the other circles yeah... ok so...

There follows some unintelligible comments and crackling then...

H: You think a computer's smart but it's not, you can't just sit there and watch it do it for you, you have to know what to do and you have to tell it to do it so it's like a something.... like it's like a lightswitch.

The difficulties that students have in coming to terms with the concept of functional dependency in geometry exemplifies Drijvers and Gravemeijer's (2005) conception of utilisation schemes in which the technical and conceptual elements co-evolve. Pupil H articulates this point very clearly: "you have to know what to do and you have to tell it to do it". Mathematical knowledge is knowing "what to do" and technical knowledge is required in order to tell the computer to do it. The *gap* in H and C's knowledge was an appreciation of the functional dependencies inherent in Cabri: on the one hand, a conceptual gap of the *necessity* of specifying the required geometrical relationship and, on the other hand, a gap in the technical knowledge of *how* to specify the relationship using Cabri. The teacher explains the *need* to specify the geometrical relationship: the "computer program will only do exactly what you tell it". The teacher goes on to illustrate the technical knowledge of *how* to specify the relationship by contrasting the construction 'by eye', which could still be messed-up, to the "by this point" version in which the geometrical relationships remained intact.

Pupil K had similar difficulties to H and C: although she seemed to be clear about how the circle should be positioned, she appeared unaware of the necessity to specify to Cabri that the circle should go "By this point". Again the teacher makes the technical elements explicit:

Ok. Keep your hand ...[K: uhuh] yeah? So if you actually put it on the point and say I want it "by this point" that's how the comp... that's the only bit of IT you're using. [K: But that's...] That's the only knowledge...IT knowledge you've used. And really then you've had to tell it to do that haven't you?

In this case, the teacher is more direct in making the functional dependencies explicit, by guiding the pupil's construction and referring to the software language "by this point". The teacher even describes this technical knowledge of how to specify the relationship as "IT knowledge", unravelling it from the mathematical knowledge of the geometrical relationship. The teacher again refers to the conceptual necessity of specifying the relationship: "you've had to tell it to do that". Drijvers and Gravemeijer (2005) describe instrumental genesis as the "emergence and evolution of utilisation schemes, in which technical and conceptual elements co-evolve". The role of the teacher in supporting instrumental genesis is partly in making the technical and conceptual elements explicit. In the case of dynamic geometry software such as Cabri, the role of the teacher is to unravel the notion of functional dependency by highlighting the *necessity* of specifying the required geometrical relationship and the technical knowledge of *how* to specify the relationship.

Exploiting dynamic variation to highlight geometric invariance

All the figures presented to the pupils for construction were based on the initial construction of a line which was apparently horizontal. Of course, there is no geometrical reason for the line to be horizontal, the figures had been presented in this

way purely for neatness and it had not been given a second thought, until the teacher noticed that all students appeared to be constructing *intentionally* horizontal lines. The pupils had discovered that by pressing the “shift” key whilst constructing a line, the line would snap to the horizontal. According to the pupils, a similar feature of “snapping to a grid” occurs in a piece of completely unrelated software, which was how the discovery was made. Pupil K was insistent that the line should be horizontal:

T: Why do you always insist on that being horizontal? Does it matter if it....

The teacher draws attention to the pupil’s misconception and, by dragging, attempts to convey that the horizontal constraint is artificial, that it can be broken without disturbing the figure under construction. As the pupils were presenting their work to the class, it became clear that all groups had produced figures with horizontal lines. The teacher again attempted to question this feature of their constructions but this time in a whole class context. Pupil MC was asked to reconstruct her solution to Problem 2 (a perpendicular bisector) without starting from a horizontal line. She did this successfully on an interactive whiteboard so that the whole class could see. She then dragged the figure, directed by the teacher, changing its orientation to show its invariance, including the situation with the initial line being horizontal. The teacher exploits dynamic variation to highlight the geometric invariance of the construction in order to help pupils differentiate between geometrical relationships which were or were not crucial.

A similar situation occurred when a pair of pupils, MC and ML, successfully completed the construction leading to a square (Problem 4). They both excitedly told the teacher that the shape they had produced was a diamond. The teacher dragged their construction so that the base of the shape was horizontal, at which point they both concurred that the shape was a square. Upon dragging it back to the original position, ML in particular returned to her previous statement that it was a diamond. Repeated dragging, more and more slowly to emphasise the continuous ‘transformation’ of the shape, convinced both students that the shape was, in fact, always a square. Again the teacher’s strategy is to demonstrate the potential of the software, by exploiting dynamic variation to demonstrate the invariance of the constructed shape. Recognising the potential of the software and making its affordances explicit to pupils is a key element in supporting instrumental genesis.

Making connections between DGS and pencil-and-paper

Pupil N had constructed a rhombus but, as in the examples in the previous paragraph, had difficulty identifying the shape due to its unfamiliar orientation. The teacher employs dynamic variation to convince pupil N that the shape is indeed a rhombus but then continues the explanation on paper:

N: Is this a rhombus? But a rhombus supposed to be like tilted so...?

Teacher manipulating the diagram on screen

N: Oh so it can be, it can be any way up and it [T: Oh!] would still be a rhombus.

T: Well yeah... [N to another pupil: Well it is a rhombus.] it's like, look, this is a well no that's not. This a rectangle isn't it? Ok, it's still a rectangle. It's still a rectangle. However much I turn it, it's still a rectangle. Yeah, ok?

Diagram of rectangle drawn on paper and then the paper twisted and turned as a demonstration that orientation doesn't alter the shape.

Guin and Trouche (1999) suggest that teachers should highlight the constraints and limitations of the software to students in order to support their instrumental genesis. In these cases, the teacher is in fact using the dynamic nature of the Cabri software to highlight the constraints and limitations of the paper-and-pencil environment, exposing a misconception and thereby supporting the pupils' instrumental genesis in the more traditional medium. In the case of the tilted rhombus, the teacher sketched a rectangle on paper in order to further illustrate the concept that orientation does not affect the nature of the shape. This sketch was done on paper at the time mainly because it was quicker than constructing the shape on Cabri. The teacher's return to the paper-and-pencil environment is important because it makes a connection between the two environments: although dynamic variation makes it easier to appreciate that orientation does not affect the shape, the concept still holds in a paper-and-pencil environment. The return to paper-and-pencil is thus an attempt by the teacher to "build connections with the official mathematics outside the microworld", a responsibility which Guin and Trouche (1999) identify as being a crucial part of the teacher's role.

DISCUSSION

From the sequence of lessons, three teaching techniques have been distilled that serve to facilitate pupils' instrumental genesis in a DGS context. These techniques are clearly not exhaustive: exploiting anomalies of measurement in Cabri such as rounding errors might be another way to promote mathematical thinking, for example. These techniques are specific to DGS in general and Cabri Geometre. They are also analogous to teaching techniques used in other contexts. Guin and Trouche (1999) suggest that teachers should highlight the constraints and limitations of the software to students: in the case of Derive, the discrete and finite nature of the software. Similarly, a dynamic geometry environment such as Cabri is only a discrete model of Euclidean geometry, despite its continuous appearance. All tools and resources have constraints and limitations. In the case of paper and pencil, a limitation is the static nature of the environment. Thus techniques such as those identified in this paper may apply to any teaching resource. In a sense, the teaching techniques mentioned here essentially highlight general principles of mathematics teaching applied to a specific context, in this case DGS. The resource provides a context for learning but cannot teach. The focus of research needs to shift away from the context, towards teachers and the teaching techniques they may employ in order

to aid pupils' instrumental genesis. In this way research on ICT may avoid the criticism that the predominant focus has been on technology rather than education.

NOTES

1. Key Stage 3 covers the first three years of secondary schooling in England: Year 7 (age 11-12), Year 8 (age 12-13) and Year 9 (age 13-14). Average attainment at the end of KS3 is at level 5/6. Level 8 is the highest level possible in maths at KS3.

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INSTRUMENTAL ORCHESTRATION: THEORY AND PRACTICE

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The paper concerns the way teachers use technological tools in their mathematics lessons. The aim is to investigate the explanatory power of the theory of instrumental orchestration through its confrontation with a teaching episode. An instrumental orchestration is defined through a didactical configuration, an exploitation mode and a didactical performance. This model is applied to a teaching episode on the concept of function, using an applet embedded in an electronic learning environment. The results suggest that the instrumental orchestration model is fruitful for analysing teacher behaviour, particularly in combination with additional theoretical perspectives.

INTRODUCTION

The integration of technological tools into mathematics education is a non-trivial issue. More and more, teachers, educators and researchers are aware of the complexity of the use of ICT, which affects all aspects of education, including the didactical contract, the working formats, the paper-and-pencil skills and the individual and whole-class conceptual development.

A theoretical framework that acknowledges this complexity is the instrumental approach (Artigue, 2002). According to this perspective, the use of a technological tool involves a process of *instrumental genesis*, during which the object or artefact is turned into an instrument. The instrument, then, is the psychological construct of the artefact together with the mental schemes the user develops for specific types of tasks. In such schemes, technical knowledge and domain-specific knowledge (in our case mathematical knowledge) are intertwined. Instrumental genesis, in short, involves the co-emergence of mental schemes and techniques for using the artefact, in which mathematical meanings and understandings are embedded.

Many studies focus on the students' instrumental genesis and its possible benefits for learning (e.g., see Kieran & Drijvers, 2006). However, it was acknowledged that instrumental genesis needs to be monitored by the teacher through the orchestration of mathematical situations. In order to describe the management by the teacher of the individual instruments in the collective learning process, Trouche (2004) introduced the metaphorical theory of instrumental orchestration.

Until today, however, the number of elaborated examples of instrumental orchestrations is limited. Therefore, the aim of this paper is to investigate the explanatory power of the theory of instrumental orchestration through its

confrontation with a teaching episode. As such, this contribution can be situated in the intersection of themes 2 and 3 of Cerme6 WG7: it concerns the interaction between resources or artefacts and teachers' professional practice, in which students use tools in their mathematical activity.

In the following, we first define instrumental orchestration. Then a description of a classroom teaching episode in which a technological tool plays an important role is provided. The episode is analysed in terms of the theory. This is followed by a reflection on the application and the conclusions which we have drawn.

INSTRUMENTAL ORCHESTRATION: A THEORETICAL MODEL

The theory of instrumental orchestration is meant to answer the question of how the teacher can fine-tune the students' instruments and compose coherent sets of instruments, thus enhancing both individual and collective instrumental genesis.

An *instrumental orchestration* is defined as the intentional and systematic organisation and use of the various artefacts available in an – in our case computerised – learning environment by the teacher in a given mathematical task situation, in order to guide students' instrumental genesis. An instrumental orchestration in our view consists of three elements: a didactic configuration, an exploitation mode and a didactical performance.

1. A *didactical configuration* is an arrangement of artefacts in the environment, or, in other words, a configuration of the teaching setting and the artefacts involved in it. These artefacts can be technological tools, but the tasks students work can be seen as artefacts as well.

In the musical metaphor of orchestration, setting up the didactical configuration can be compared with choosing musical instruments to be included in the orchestra, and arranging them in space so that the different sounds result in the most beautiful harmony.

2. An *exploitation mode* of a didactical configuration is the way the teacher decides to exploit it for the benefit of his didactical intentions. This includes decisions on the way a task is introduced and is worked on, on the possible roles of the artefacts to be played, and on the schemes and techniques to be developed and established by the students.

In the musical metaphor of orchestration, setting up the exploitation mode can be compared with determining the partition for each of the musical instruments involved, bearing in mind the anticipated harmonies to emerge.

3. A *didactical performance* involves the ad hoc decisions taken while teaching on how to actually perform the enacted teaching in the chosen didactic configuration and exploitation mode: what question to raise now, how to do justice to (or to set aside) any particular student input, how to deal with an unexpected aspect of the mathematical task or the technological tool?

In the musical metaphor of orchestration, the didactical performance can be compared with a musical performance, in which the actual inspiration and the interplay between conductor and musicians reveal the feasibility of the intentions and the success of their realization.

The model for instrumental orchestration initially was developed by Trouche (Trouche 2004) and included the first and the second points above, i.e. the didactical configuration and the exploitation mode. As an instrumental orchestration is partially prepared beforehand and partially created ‘on the spot’ while teaching, we felt the need for a third component reflecting the actual performance. Establishing the didactical configuration has a strong preparatory aspect: often, didactical configurations need to be thought of before the lesson and cannot easily be changed during the teaching. Exploitation modes may be more flexible, whereas didactical performance has a strong ad hoc aspect. Our threefold model thus has a time dimension.

The model also has a structural dimension: an instrumental orchestration on the one hand has a structural, global component in that it is part of the teacher’s repertoire of teaching techniques (in the sense of Sensevy *et al.* 2005) and can be reflected in operational invariants of teacher behaviour. On the other hand, an instrumental orchestration has an incidental, local actualisation appropriate for the specific didactical context and adapted to the target group and the didactical intentions.

The instrumental orchestration model brings about a double-layered view on instrumental genesis. At the first level, instrumental orchestration aims at enhancing the students’ instrumental genesis. At the second level, the orchestration is instrumented by artefacts for the teachers, which may not necessarily be the same artefacts as the students use. As such, the teacher himself is also involved in a process of instrumental genesis for accomplishing his teaching tasks (Bueno-Ravel & Guedet, 2007).

In literature, the number of elaborated examples of instrumental orchestrations is limited. Trouche (2004) and Drijvers & Trouche (2008) describe a so-called Sherpa orchestration. Kieran & Drijvers (2006), without mentioning this orchestration explicitly, describe an instrumental orchestration of short cycles of individual work with the artefact and whole-class discussion of results.

THE CASE OF TWO VERTICALLY ALIGNED POINTS

The case we describe here stems from a research project on an innovative technology-rich learning arrangement for the concept of function¹. In this project, a learning arrangement for students in grade 8 was developed, aiming at the development of a rich function concept. This includes viewing functions as input-output assignments, as dynamic processes of co-variation and as mathematical

¹ For further information on the project see Drijvers, Doorman, Boon, Van Gisbergen & Gravemeijer (2007) and the project website www.fi.uu.nl/tooluse/en/.

objects with different representations. The main technological artefact is an applet called AlgebraArrows embedded in an electronic learning environment (ELO). The applet allows for the construction and use of chains of operations, and options for creating tables, graphs and formulae and for scrolling and tracing. A hypothetical learning trajectory, in which the expected instrumental genesis is sketched, guided the design of the student materials. An accompanying teacher guide contained suggestions for orchestrations.

After group work on diverse problem situations involving dependency and co-variation, the notion of arrow chains is introduced to the students. In the third and fourth lessons, students work with arrow chains in the ELO. One of the tasks of the fourth lesson, which some of the students did at home, is task 8, shown in Figure 1.

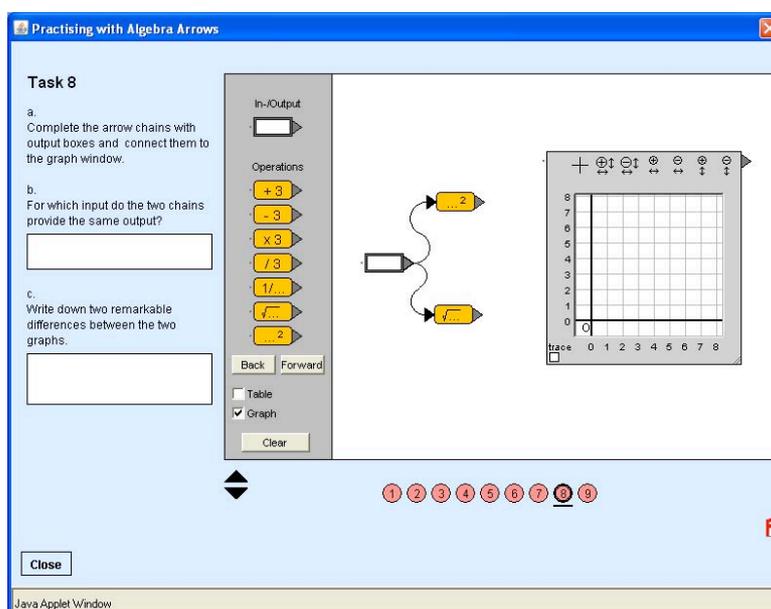


Figure 1 Computer task 8

At the right of Figure 1 is the applet window, which in this task contains the start of the square and the square root chain, and an empty graph window. At the left you see the tasks and two boxes in which the students type their answers. The numbered circles at the bottom allow for navigation through the tasks.

The following verbatim extract describes the way the teacher discusses this task during the fifth lesson.

Using a data projector, the ELO with the list of student pairs is projected on the wall above the blackboard. The teacher T navigates within this list to Tim and Kay's solution for task 8.

T: It says here [*referring to question c*]: what do you notice? Oh yes, I actually wanted to see quite a different one, because they had ...

T navigates to Florence and her classmate's work. The Table option is checked. That leads to 'point graphs' on the screen. The students' answer to question c reads:

"For the square they are all whole numbers, and for the square root they are whole numbers and fractions. And the square of a number is always right above the root. ?"

T: Look here, what this says. [*indicates the students' answer of question c on the screen with the mouse*] For the square they are all whole numbers, okay, and for the square root they aren't whole numbers, we agree with that too, and the square of a number is always right above the square root.

F(lourence): Was that right?

T: I'm not saying.

St1²: Yes, I had that too.

T: What they say, then, is that every time there is...if I've got something *here*, there is something above it, and if I've got something *there*, there is also something above it. [*points vertically in the graph with the mouse*] Why is that, that these things are right above each other?

F: Well, because it...the square root is just...no the square is just, um, twice the root, or something.

St2: No.

T: Kay?

Kay: That's because the line underneath, that's got a number on it, which you take the square root of and square, so on the same line anyway.

T: What are those numbers called that are on the horizontal line then?

St3: The input numbers.

T: The input numbers.

T: Ehm, Florence, did you follow what Kay said?

F: No, but I [...]. It was about numbers and about square roots and about...

Sts: [*laughs*]

St: It was about numbers!

T: Kay said: these are the input numbers, here on the horizontal line. [*indicates the points on the horizontal axis with the mouse*] And for an input number you get an output number. And that is right above it. So if you take the same input number for two functions... [*indicates the two arrow chains with the mouse*]

F: Oh yes.

² St1, St2, .. stands for one of the students

T: ... then you also get...then you get points above it. So that's got nothing at all to do with the functions. It's just got to do with from which number you are going to calculate the output value. Now, if for both of them you calculate what the output value is for 10, they both get a point above the 10 [indicates on the screen with the mouse]. Do you understand that?

F: Oh yes, I didn't know that.

T navigates back to the list of student pairs.

Figure 2 shows the work of Florence and her classmate on this task in Dutch at the end of the teaching sequence. They changed their answer to question c into: “for the square they are always whole numbers, and for the square root they are whole numbers and fractions. The squares get higher with much bigger steps.”

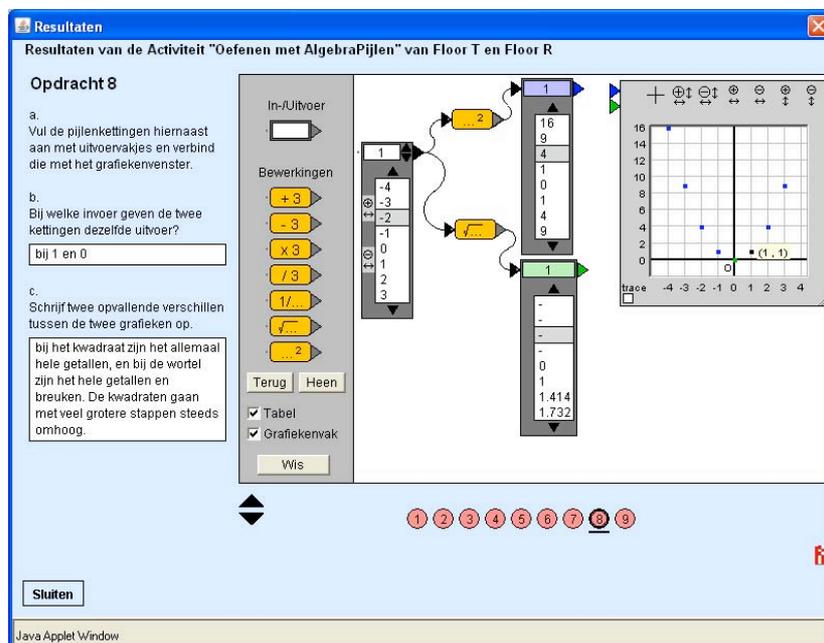


Figure 2 Revision of the answer after whole class discussion

APPLYING THEORY TO PRACTICE

In this section we apply the theory of instrumental orchestration to the above teaching episode, which essentially reflects the teacher's way to treat a misconception of (at least) one of the students, whose use of the Table-Graph technique leads to thinking it is 'special' that two points reflecting function values for the same input value are vertically aligned.

Let us call the instrumental orchestration the teacher puts into action the 'spot and show orchestration'. By 'spot' we mean that the teacher, while preparing the lesson, spotted the students' work in the ELO and thus came across Florence's misconception. The 'show' refers to the teacher's decision to display Florence's results as a starting point for the whole-class discussion of item 8c. The teacher's

phrase “Oh yes, I actually wanted to see quite a different one” and her straight navigation to Florence’s work reveal her deliberate intention to act the way she does.

The *didactical configuration* for the preparatory phase consists of the ELO’s option for teachers to look at the students’ work at any time. As a result, the teacher notices the misconception and decides to deal with it in her lesson. This preparation is instrumented by ELO-facilities that are not available for students. In this sense, the teacher’s artefact is different from the students’ artefact. For the classroom teaching, the configuration includes a regular classroom with a PC with ELO access, connected to a data projector. Apparently, the teacher finds the computer lab not appropriate for whole-class teaching. The configuration includes putting the computer with the data projector in the centre of the classroom. This choice is driven by the constraints of one of the artefacts: if the projector was at the front, the projection would get too small for the students to read. The screen is projected on the wall above the blackboard, thus enabling the teacher to write on the blackboard, which she regularly does – though not in the episode presented here. Both the way of preparing the lessons and the setting in the classroom are observed more often in this teacher’s lessons.

The *exploitation mode* of this configuration includes that teacher’s choice to operate the PC herself. These two aspects of the exploitation mode result in the teacher standing in the centre of the classroom, with the students closely around her, all focused on the screen on the wall. From these and other observations, we conjecture that this exploitation mode enhances classroom discussion and student involvement. Observations of another teacher using the same orchestration in a less convenient setting support this conjecture.

The *didactical performance* starts with the teacher reading the student’s answer with some minor comments (“Look here, ...”). Then she reformulates the answer and asks Florence for an explanation (“What they say...”). When the explanation turns out to be inappropriate, she makes Kay give his explanation, and checks whether Florence understands it. When this is not the case, the teacher rephrases Kay’s explanation and once more checks it with Florence, who now says she understands. Of course, this didactical performance might be different a next time. For example, Florence could be asked to explain her understanding in her own words.

Now how about the link between instrumental orchestration and instrumental genesis? As the episode does not show students using the artefact, we do not see direct traces of the students’ instrumental genesis. We do claim, however, that Florence’s idea of two vertically aligned points being special is part of her scheme of using the TableGraph technique to produce point graphs. Even though this is a misconception, the episode shows that the teacher can exploit the students’ experiences, and those of Florence and Kay in particular, for the purpose of attaching mathematical meaning to the technique they used, which leads to a convergence in a

shared function conception in class. We see the development of mathematical meanings of techniques as an important aspect of instrumental genesis.

This ‘spot&show’ orchestration was one of the options suggested in the teacher guide accompanying the teaching sequence. This teacher used it quite often, whereas she felt free to neglect other suggestions made in the teacher guide. In the post-experiment interview, she indicated to really appreciate the possibility to get an overview of students’ results while preparing the lesson: “The ELO is practical to see what students do, you can adapt your lesson to that.” She seemed to see this ‘spot&show’ orchestration as a means to enhance student involvement and discussion, which she believed to be relevant and seem to be part of her operational invariants . We do not have data, however, that confirm such operational invariants across other teaching settings.

Finally, an interesting aspect of the teacher’s own instrumental genesis is worth discussing. The teacher points with her mouse on the screen, but does not really make changes in the students’ work. Other observations suggest that she doesn’t do so because she is afraid that such changes will be saved and thus affect the students’ work. When she learns that this is not the case as long she uses her teacher login, she benefits from the freedom to demonstrate other options and to investigate the consequences of changes. This behaviour is instrumented by the facilities of the artefact that she initially was not aware of.

REFLECTION ON THE THEORY AND THE CASE

Let us briefly reflect on the application of the theory of instrumental orchestration to the data presented above. A first remark is that the three elements of the model – didactical configuration, exploitation mode and didactical performance – allow for a distinction and an analysis of the relevant issues within the orchestration, and their interplay. As such, the model offers a useful framework for describing the orchestration by the teacher.

As a second remark, however, we notice that it is not always easy to decide in which category something that is considered relevant should be placed. For example, does the fact that the teacher operates the computer herself belong to the didactical configuration or to the exploitation mode? This probably is a matter of granularity: if we study the ‘spot & show’ orchestration, this is part of the exploitation mode. If the focus of the analysis is on students’ activity, it might be identified as a didactical configuration issue.

A third reflection is that the model has the advantage of fitting with the instrumental approach of students learning while using tools. This has proved to be a powerful approach (Artigue, 2002; Kieran & Drijvers, 2006), and it is therefore of great value having a framework for analysing teaching practices that is consistent with it. As such, instrumentation and orchestration form a coherent pair. In terms of instrumentation, we notice that the teachers’ tasks, artefacts and techniques are not

the same as those of the students; still, we can use a similar framework for analysis and interpretation.

The time dimension in the model – ranging from the didactical configuration having a strong preparatory character to the didactical performance with its strong ad hoc character – comes out clearly in the model. For the structural dimension, this is not as straightforward. As a fourth remark, therefore, we notice that operational invariants of the teacher are not limited to the preparatory phases, but also emerge in the performance. For example, the wish to have students explain their reasoning to each other appears as an operational invariant for this teacher, which is more explicit in the performance than in the configuration or in the exploitation mode. As an aside, we are aware that the data presented here do not allow for full identification of the teacher's operational invariants. More observations over time need to be included.

CONCLUSION

As far as this is possible from the one single, specific exemplary case study presented in this paper, we conclude that the model of instrumental orchestration can be a fruitful framework for analysing teachers' practices when teaching mathematics with technological tools. As it is important for teachers to develop a repertoire of instrumental orchestrations, more elaborated examples are needed. Such examples could not only help us to better understand teaching practices, but could also enhance teachers' professional development.

In addition to the need to find and elaborate exemplary orchestrations, a second challenge is to link the theory of instrumental orchestration with complementary approaches. Lagrange (2008), for example, uses additional models provided by Saxe (1991) and Ruthven and Hennessey (2002) to identify and understand teaching techniques. Another interesting perspective concerns the alternation of teacher guidance and student construction, as described by Sherin (2002). In short, the instrumental orchestration approach is promising, but needs elaboration and integration with other perspectives. For the moment, its descriptive power seems to be more important than its explanatory power.

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TEACHING RESOURCES AND TEACHERS' PROFESSIONAL DEVELOPMENT: TOWARDS A DOCUMENTATIONAL APPROACH OF DIDACTICS

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In this paper we propose a theoretical approach of teachers' professional development, focusing on teachers' interactions with resources, digital resources in particular. Documents, entailing resources and schemes of utilization of these resources, are developed through documentational geneses occurring along teachers' documentation work (selecting resources, adapting, combining, refining them). The study of teachers' documentation systems permits to seize the changes brought by digital resources; it also constitutes a way to capture teachers' professional change.

INTRODUCTION

We present in this paper the first elements of a theoretical approach elaborated for the study of teachers' development, and in particular teachers ICT integration.

Technology integration, and the way teachers work in technology-rich environments, have been extensively researched, and discussed at previous CERME conferences (Drijvers *et al.*, 2005, Kynigos *et al.*, 2007). Ruthven's presentation at CERME 5 drew attention on the structuring context of the classroom practice, and on its five key features: *working environment, resource system, activity format, curriculum script, time economy* (Ruthven, 2007). This leads in particular to consider ICT as part of a wider range of available teaching resources. This view also fits technological evolutions: most of paper material is now at some point in digital format; teachers exchange digital files by e-mail, use digital textbooks, draw on resources found on websites etc. Considering ICT amongst other resources raises the question of connections between research on ICT and resources-oriented research.

Many research works address the use of *curriculum material* (Ball & Cohen, 1996; Remillard, 2005). They observe the influence of such material on the enacted curriculum, but also highlight the way teachers shape the material they draw on, introducing a vision of "curriculum use as participation with the text" (Remillard, 2005, p.121). Other authors consider more general resources involved in teaching: material and human, but also mathematical, cultural and social resources (Adler, 2000). They analyze the way teachers interpret and use the available

resources, and the consequences of these processes on teachers' professional evolution.

Such statements sound familiar for researchers interested in ICT, who “consider not only the ways in which digital technologies shape mathematical learning through novel infrastructures, but also how it is shaped by its incorporation into mathematical learning and teaching contexts” (Hoyles & Noss, 2008, p. 89). Conceptualization of these processes is provided by the *instrumental approach* (Guin *et al.*, 2005) and by the work of Rabardel (1995) grounding it; this theoretical frame has contributed to set many insightful results about the way students learn mathematics with ICT. Further refinements of this theory have led to take into account the role of the teacher and her intervention on students instrumental geneses, introducing the notion of *orchestration* (Trouche, 2004). Considering instrumental geneses for teachers has been proposed in the context of spreadsheets (Haspekian, 2008) and e-exercises bases (Bueno-Ravel & Gueudet, 2007). These refinements can be considered as first steps towards the introduction of concepts coming from the instrumental approach and illuminating the interactions between teachers and ICT.

Thus connections between studies about the use of teaching resources, and studies about the way in which teachers work in a technology-rich environment exist; however, elaborating a theoretical frame encompassing both perspectives requires a specific care. We present here an approach designed for this purpose, and aiming at studying teachers' documentation work: looking for resources, selecting, designing mathematical tasks, planning their order, carrying them out in class, managing the available artefacts, etc. We take into account teachers' work in class, but also their (too often neglected) work out of class.

We draw on the theoretical elements evoked above, but also on field data. Some of these data come from previous research in which we were engaged: particularly about use of e-exercises bases (Bueno-Ravel & Gueudet, 2007), and about an in-service training design, the SFoDEM (Guin & Trouche, 2005). Other data were specifically collected: we have set up a series of interviews with nine secondary school teachers. We chose teachers with different collective involvements, different institutional contexts and responsibilities, and different ICT integration degrees (Assude, 2007). We met them at their homes (where, in France and for secondary teachers, most of their documentation work takes place), and asked them about their uses of resources, and the evolution of these ways of use. We observed the organization of their workplaces at home, of their files (both paper and digital), and collected materials they designed or used. The analyses of these data contributed to shape the concepts; in this paper we only use them to display illustrations of the theory. All the interviews took place in France; thus the national context certainly influences the results we display. We hypothesize nevertheless that the concepts exposed are likely to illuminate documentation work in diverse situations.

We present here the elementary concepts of this theory, introducing in particular a distinction between *resources* and *documents*, and the notion of *documentational genesis*. We also expose the specific view of professional evolutions it entails.

RESOURCES, DOCUMENTS, DOCUMENTATIONAL GENESES

The instrumental approach (Rabardel, 1995, Guin *et al.*, 2005) proposes a distinction between *artefact* and *instrument*. An artefact is a cultural and social means provided by human activity, offered to mediate another human activity. An instrument comes from a process, named *instrumental genesis*, along which the subject builds a *scheme* of utilization of the artefact, for a given class of situations. A scheme, as Vergnaud (1998) defined it from Piaget, is an *invariant organization of activity* for a given class of situations, comprising in particular rules of action, and structured by *operational invariants*, which consist of implicit knowledge built through various contexts of utilization of the artefact. Instrumental geneses have a dual nature. On the one hand, the subject guides the way the artefact is used and, in a sense, *shapes* the artefact: this process is called *instrumentalization*. On the other hand, the affordances and constraints of the artefact influence the subject's activity: this process is called *instrumentation*. We propose here a theoretical approach of teaching resources, inspired by this instrumental approach, with distinctive features that we detail hereafter, and a specific vocabulary.

We use the term *resources* to emphasize the variety of the artefacts we consider: a textbook, software, a student's sheet, a discussion with a colleague etc. A resource is never isolated: it belongs to a set of resources. The subjects we study are teachers. A teacher draws on resources sets for her documentation work. A genesis process takes place, bearing what we call *a document*. The teacher builds schemes of utilization of a set of resources, for the same class of situations, across a variety of contexts. The formula we retain here is:

$$\textit{Document} = \textit{Resources} + \textit{Scheme of Utilization}.$$

A document entails, in particular, operational invariants, which consist of implicit knowledge built through various contexts of utilization of the artefact, and can be inferred from the observation of invariant behaviors of the teacher for the same class of situations across different contexts.

Figure 1 represents a *documentational genesis*. The instrumentalization process conceptualizes teacher appropriating and reshaping resources, and the instrumentation process captures the influence, on the teacher's activity, of the resources she draws on.

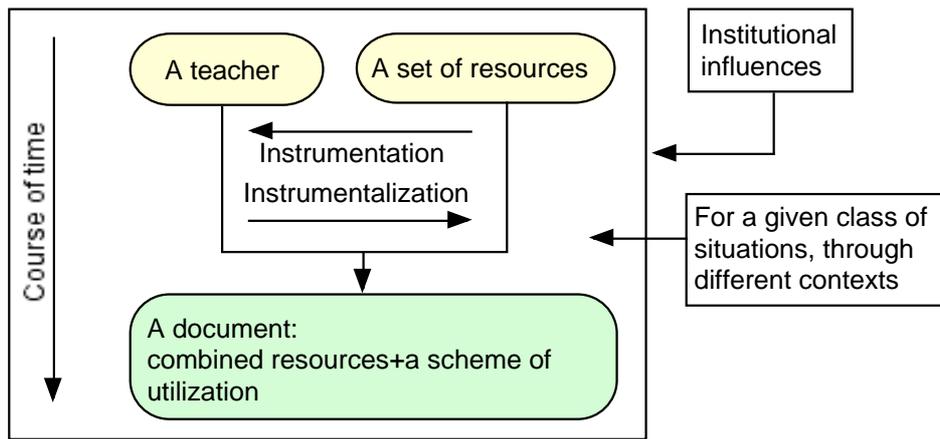


Figure 1. Schematic representation of a documental genesis

DOCUMENTATIONAL GENESES: TWO ILLUSTRATIVE EXAMPLES

We use a first case study (figure 2) coming from our interviews to illustrate the distinction between a set of resources and a document.

Marie-Pierre (aged 40, involved in collective work within an IREM¹ group; no institutional responsibilities, strong degree of ICT integration) is teaching at secondary school for 14 years, from grade 6 to 9. She uses dynamic geometry systems, spreadsheets, and many online resources (e-exercises and mathematics history websites in particular). She has a digital version of the class textbook. Marie-Pierre has an interactive whiteboard in her classroom for three years, and uses it in each of her courses. For the introduction of the circle's area in grade 7, she starts in class by using a website comprising historical references (Archimedes using circular sections to link the perimeter and the area of a circle) and displaying an animation of the circle unfolding and transforming into a triangle (roughly, but that point is not discussed). Then she presents her own course, based on an extract of the class digital textbook. She complements as usual the files displayed on the whiteboard by writing additional comments and explanations, highlighting important expressions etc.

Figure 2. Marie-Pierre, example of a lesson introducing the circle's area

For the class of situations: “design and implement the introduction of the circle's area in grade 7” (figure 2), Marie-Pierre draws on a set of resources comprising the

¹ Institute for Research on Mathematics Teaching.

interactive whiteboard, a website², a digital textbook, and a hard copy of it. The official curriculum texts, about the circle area, only state that “an inquiry-based approach permits to check the area formula”, with no more details. The digital textbook proposes an introductory activity with a digital geometry software: drawing circles, and displaying their areas. Several radius are tested, the radius square and the corresponding area are noted by the students in a table, and they are asked to observe that they obtain an (approximate) ratio table. But Marie-Pierre prefers to draw on the website animated picture (both choices correspond more to an observation activity for the students than to an inquiry-based approach, but we will not discuss this aspect here). So, we claim that she has developed a scheme of utilization of this set of resources, structured by several operational invariants. These invariants are professional beliefs that we infer from our data:

-“A new area formula must be justified by an animation showing a cutting and recombining of the pieces to form a figure whose area is known”. This operational invariant concerns all the areas introduced, it also intervenes in the document corresponding to the introduction of the triangle’s area for example;

-“The circle’s area must be linked with a previously known area: the triangle”; “The circle’s area must be linked with the circle’s perimeter”. These operational invariants are related with the precise mathematical content of the lesson, they were built along the years, with different grade 7 classes (Marie-Pierre uses this website’s animation for three years, with two grade 7 classes each year).

We do not assert that these operational invariants were not present among Marie-Pierre’s professional knowledge before her integration of the interactive whiteboard. But the possibility to display an animation on a website, to complement it by writing additional explanations, to go back to a previous state of the board to link the “official” formula with what has been observed, yielded a document integrating these operational invariants. And we claim that the development of this document is likely to reinforce, in particular, the above presented beliefs. The operational invariants are both driving forces and outcomes of the teacher’s activity.

Documentational geneses are ongoing processes; we use a second case study (figure 3) to emphasize this important aspect. Rabardel & Bourmaud (2005) claim that the design *continues in usage*. We consider here accordingly that a document developed from a set of resources provides new resources, which can be involved in a new set of resources, which will lead to a new document etc. Because of this process, we speak of a *dialectical* relationship between resources and documents.

² http://pagesperso-orange.fr/therese.eveilleau/pages/hist_mat/textes/mirliton.htm

Marie-Françoise (aged 55; involved in collective work within an IREM group, institutional responsibilities as in-service teacher trainer, strong degree of ICT integration) works with students from grade 10 to 12. She organizes for them 'research narratives': problem solving sessions, where students work in groups on a problem and write down their own 'research narratives' (both solutions and research processes). Thus one class of situations, for Marie-Françoise is 'elaborating open problems for research narratives sessions'. For this class of situations, she draws on a set of resources comprising various websites, but also personal existing resources, colleagues' ideas, etc. Students' ideas constitute a major resource for Marie-Françoise, as she told us: "There is the problem and the way you enact it, because students are free to invent things, and afterwards we benefit from the richness of all these ideas, and you can build on it". The design clearly goes on in class. Moreover, the class sessions provide new resources: the students' research narrative, that Marie-Françoise collects, and saves in a new binder, aiming to enrich the next document built on the same open problem.

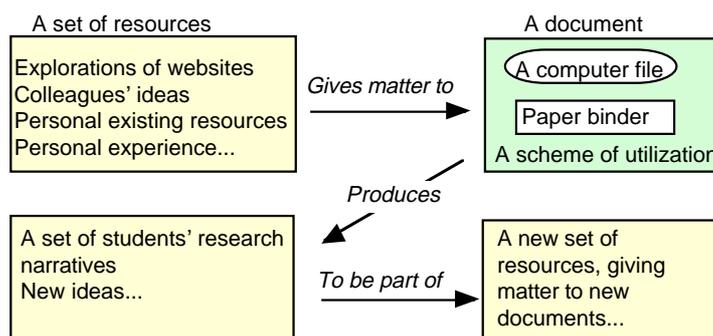


Figure 3. An illustration of the resources/document dialectical relationship

The resources evolve, are modified, combined; documents develop along geneses and bear new resources (figure 3) etc. We consider that these processes are part of teachers' professional evolutions, and play a crucial role in them.

DOCUMENTATION SYSTEMS AND PROFESSIONAL DEVELOPMENT

According to Rabardel (2005), professional activity has a double dimension. Obviously a productive dimension: the outcome of the work done. But the activity also entails a modification of the subject's professional practice and beliefs, within a constructive dimension. Naturally, this modification influences further production processes: the productive/constructive relationship has a dialectical nature.

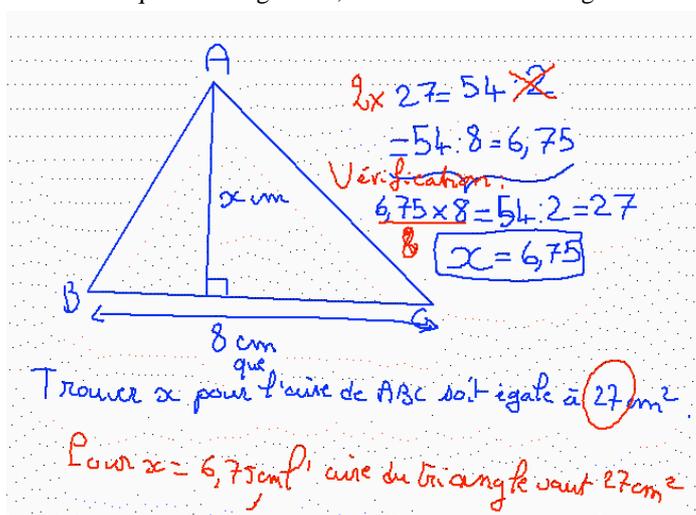
Teachers' documentation work is the driving force behind documentational geneses, thus it yields productive and constructive professional changes. Literature about teachers' professional change raises the question of the articulation between change of practice and change of knowledge and beliefs. We consider that both are strongly intertwined (e.g., Cooney, 2001). The documentational geneses provide a specific view of this relationship. Working with resources, for the same class of situations across different contexts, leads to the development of a scheme, and in particular of rules of action (professional practice features) and of operational invariants (professional implicit knowledge or beliefs). And naturally these schemes influence the subsequent documentation work. All kinds of professional knowledge are concerned by these processes, the evolutions they generate are not curtailed to

curricular knowledge (Schulman, 1986). Thus, studying teachers' documents can be considered as a specific way to study teachers' professional development.

According to Rabardel and Bourmaud (2005), the instruments developed by a subject in his/her professional activity constitute a system, whose structure corresponds to the structure of the subject's professional activity. We hypothesize here similarly that a given teacher develops a structured documentation system.

Let us go back to the example of Marie-Pierre evoked above.

Marie-Pierre keeps all her "paperboards" (digital files with images corresponding to the successive states of the board). She uses these paperboards at the beginning of a new session, to recall what has been written, by herself or by her students, during the preceding session. On her laptop, Marie-Pierre has one folder for each class level. Each of these folders contains one file with the whole year's schedule, and lessons folders for each mathematical theme. The paperboards are inside the lessons folders. The interactive whiteboard screen below corresponds to the introduction of equations in grade 7, in the context of triangles areas.



(Translation: Find x such that ABC area equals 27 cm^2 . For $x = 6.75 \text{ cm}$, the triangle's area is 27 cm^2).

Figure 4. A view on Marie-Pierre's documents

Marie-Pierre's files organization on her computer (figure 4), and her statements during the interviews, clearly indicate articulations between her documents. The document whose material component is the year schedule naturally influenced her lesson preparations; but on the opposite, the documents she developed for lessons preparations during previous years certainly intervened in the schedule design. Documents corresponding to connected mathematical themes are also connected. For a given lesson, the students' interventions can contribute to generate operational invariants that will intervene in preparations about other related topics.

A teacher's documents constitute a system, whose organization matches the organization of her professional activity. The evolutions of this documentation system correspond to professional evolutions. Integration of new materials is a visible of the professional practice, and of the documentation system (in the approach we propose, this integration means that a new material is inserted in a set of resources

involved in the development of a document). When Marie-Pierre integrates the interactive whiteboard in her courses, it entails a productive dimension: she now teaches with this whiteboard. But it also yields other changes of her practice: she makes more links with previous sessions, in particular recalling students productions is now present in her orchestration choices. And it even generates changes in her professional beliefs, for example about the possible participation of students to her teaching. She seems to have developed an operational invariant like: “a good way to launch a lesson is to recall students’ interventions done during the preceding lesson”.

The integration of new material is always connected with professional practice and professional beliefs evolutions. But professional evolutions do not always correspond to integration of new material, and the same is true for documentation systems evolutions. For example, Arnaud (47 years old, no collective involvement, institutional responsibilities as in-service teacher trainer, low degree of ICT integration) presented during his interview “help sheets”, that he designed years ago for students encountering specific difficulties. He now uses the same sheets as exercises for the whole class; thus while no changes can be observed in the material, the action rules associated evolved.

Integration of new material remains an important issue, especially when the focus is on ICT. The study of a given teacher’s documentation system also provides insights in the reasons for the integration or non-integration of a given material. The integration depends indeed on the possibility for this material to be involved in the development of a document, that will be articulated with others within the documentation system. For many years Marie-Pierre prepares her courses as digital files, she uses dynamic geometry software and online resources; the interactive whiteboard articulates with this material. Moreover, Marie-Pierre is convinced of the necessity of fostering students’ interventions, and even of including these in the written courses, and the interactive whiteboard matches this conviction. Possible material articulations are important; but other types of articulations must be taken into account, and the integration of new material also strongly depends on operational invariants, thus on teachers’ professional knowledge and beliefs.

CONCLUSION

This paper is related with the second theme of WG7: *Interaction between resources and teachers’ professional practice*. It introduces a conceptualization of teachers’ interactions with resources and of the associated professional development. Here we just presented the first concepts of a theory whose elaboration is still in progress. Studying teachers’ documentation work requires to set specific methodologies, permitting to capture their work in and out of class, to precise their professional beliefs, and to follow long-term processes: it is the main goal of our research. We did not discuss here the very important issue of collective documentation work, which causes particular processes: its study raises the question of collective documentational genesis and documentation systems, and raises new theoretical

needs. The documentational approach we propose also needs to be confronted with other teaching contexts: primary school, tertiary level; diverse countries; and also outside the field of mathematics. Further research is clearly needed; the present evolutions of digital resources make it a major challenge for the studies of teachers' professional evolutions.

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AN INVESTIGATIVE LESSON WITH DYNAMIC GEOMETRY: A CASE STUDY OF KEY STRUCTURING FEATURES OF TECHNOLOGY INTEGRATION IN CLASSROOM PRACTICE

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The research literatures concerning the classroom practice of mathematics teaching and technology integration in school mathematics point to key structuring features – working environment, resource system, activity format, curriculum script, time economy – that shape patterns of technology integration into classroom practice and require teachers to develop their craft knowledge accordingly. This conceptual framework is applied to an investigative lesson incorporating dynamic geometry use, employing evidence from classroom observation and teacher interview. This illuminates the many aspects of professional adaptation and development on which successful technology integration into classroom practice depends.

INTRODUCTION TO THE STUDY

From synthesis of relevant research literatures, I argued at CERME-5 that successful integration of computer-based tools and resources into school mathematics depends on coordinating working environment, resource system, activity format and curriculum script to underpin classroom practice which is viable within the time economy (Ruthven, 2007). This paper will illustrate –and test– that conceptual framework by using it to analyse the practitioner thinking and professional learning surrounding a lesson incorporating the use of dynamic geometry.

The lesson was one of four cases investigated in a study of classroom practice incorporating dynamic geometry use (Ruthven, Hennessy & Deaney, 2008). In the original study, this specific case was followed up because the teacher concerned talked lucidly about his experience of teaching such a lesson for the first time, and displayed particular awareness of the potential of dynamic geometry for developing visuo-spatial and linguistic aspects of students' geometrical thinking.

This case has been chosen for further analysis because the teacher was unusually expansive in all his interviews, illuminating a range of aspects of practitioner thinking and professional learning. While an exhaustive case analysis in terms of the conceptual framework would require data to be collected with its use specifically in mind, the richness of the evidence from this case provides a convenient interim means of exploring its application to a concrete example.

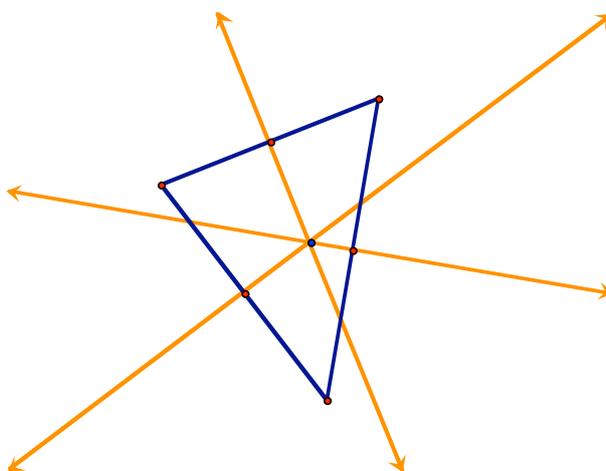
ORIENTATION TO THE LESSON

As the teacher explained when nominating the lesson, it had recently been developed in response to improved technology provision in the mathematics department prompting him to “*to explore some geometry*”:

So we'd done some very rough work on constructions with compasses and bisecting triangles and then I extended that to Geometer's Sketchpad... on the interactive whiteboard using it in front of the class.

He reported that the lesson (with a class in the early stages of secondary education) had started with him constructing a triangle, and then the perpendicular bisectors of its edges. The focus of the investigation which ensued had been on the idea that this construction might identify the 'centre' of a triangle:

And we drew a triangle and bisected the sides of a triangle and they noted that they all met at a point. And then I said: "Well let's have a look, is that the centre of a triangle?" And we moved it around and it wasn't the centre of the triangle, sometimes it was inside the triangle and sometimes outside.



According to the teacher, one particularly successful aspect of the lesson had been the extent to which students actively participated in the investigation:

And they were all exploring; sometimes they were coming up and actually sort of playing with the board themselves... I was really pleased because lots of people were taking part and people wanted to come and have a go at the constructions.

Indeed, because of the interest and engagement shown by students, the teacher had decided to extend the lesson into a second session, held in a computer room to allow students to work individually at a computer:

And it was clear they all wanted to have a go so we went into the computer room for the next lesson so they could just continue it individually on a computer... I was expecting them all to arrive in the computer room and say: "How do you do this? What do I have to do again?"... But virtually everyone... could get just straight down and do it. I was really surprised. And the constructions, remembering all the constructions as well.

For the teacher, then, this recall by students of ideas from the earlier session was another aspect of the lesson's success. In terms of the specific contribution of dynamic geometry to this success, the teacher noted how the software supported

exploration of different cases, and overcome the practical difficulties which students encountered in using classical tools to attempt such an investigation by hand:

You can move it around and see that it's always the case and not just that one off example. But I also think they get bogged down with the technicalities of drawing the things and getting their compasses right, and [with] their pencils broken.

But the teacher saw the contribution of the software as going beyond ease and accuracy; using it required properties to be formulated precisely in geometrical terms:

And it's the precision of realising that the compass construction... is about the definition of what the perpendicular bisector is... And Geometer's Sketchpad forces you to use the geometry and know the actual properties that you can explore.

These, then, were the terms in which the original lesson was nominated as an example of successful practice. This nomination was followed up by studying a later lesson along similar lines through classroom observations and teacher interviews. The observed lesson was conducted over two 45-minute sessions on consecutive days with a Year 7 class of students (aged 11-12) in their first year of secondary education.

WORKING ENVIRONMENT

The use of ICT in teaching often involves changes in the working environment of lessons: change of room location and physical layout, change in class organisation and classroom procedures.

Each session of the observed lesson started in the normal classroom and then moved to a nearby computer suite, a modification of the pattern originally reported. This movement between rooms allowed the teacher to follow a particular activity cycle common to each session, shifting working environment to match changing activity format. The classroom was equipped with a single computer linked to a ceiling-mounted projector directed towards a whiteboard at the front: this supported use of computer-based resources within whole-class activity formats. However, only in the computer suite was it possible for students to work individually at a machine.

Even though the suite was also equipped with a projectable computer, starting sessions in the teacher's own classroom was expedient for several reasons. Doing so avoided disruption to the established routines underpinning the smooth launch of lessons. Moreover, the classroom provided an environment more conducive to sustaining effective communication during whole-class activity and to maintaining the attention of students. Whereas in the computer suite each student was seated behind a sizeable monitor perched on a desktop computer unit, so blocking lines of sight and placing diversion at students' fingertips, in the classroom the teacher could introduce the lesson "*without the distraction of computers in front of each of them*".

It was only recently that the classroom had been refurbished and equipped, and a neighbouring computer suite established for the exclusive use of the mathematics department. The teacher contrasted this new arrangement favourably in terms of the

easier and more regular access to technology that it afforded, and the consequent increase in the fluency of students' use:

Before... you'd book a computer suite, you'd go in and then... you'[d] just not get anywhere, because the whole lesson's been sorting out logging on, sorting out how to use [the software]... And [now] having the access to it so easily and readily just makes a huge difference.

New routines were being introduced to students for opening a workstation, including logging on to the school network, using shortcuts to access resources, and maximising the document window. Likewise, routines were being developed for closing sessions in the computer suite. Towards the end of each session, the teacher prompted students to plan to save their files and print out their work, advising them that he'd “*rather have a small amount that you understand well than loads and loads of pages printed out that you haven't even read*”. He asked students to avoid rushing to print their work at the end of the lesson, and explained how they could adjust their output to try to fit it onto a single page; he reminded them to give their file a name that indicated its contents, and to put their name on their document to make it easy to identify amongst all the output from the single shared printer.

RESOURCE SYSTEM

New technologies have broadened the types of resource available to support school mathematics. Nevertheless, there is a great difference between a collection of resources and a coherent system.

The department maintained its own schemes of work under continuous development, with teachers encouraged to explore new possibilities and report to colleagues. This meant that they were accustomed to integrating material from different sources into a common scheme. However, so wide was the range of computer-based resources currently being trialled that our informant (who was head of department) expressed concern about incorporating them effectively into departmental schemes:

At the moment we're just dabbling in [a variety of technologies and resources] when people feel like it, but we're moving towards integrating [them] into schemes of work now... I'm slightly worried that we've got so much... It's getting everybody familiar with it all.

In terms of coordinating use of old and new technologies, work with dynamic geometry was seen as complementing established work on construction by hand, by strengthening attention to the related geometric properties:

I thought of Geometer's Sketchpad [because] I wanted to balance the being able to actually draw [a figure] with pencil and compasses and straight edges, with also seeing the geometrical facts about it as well. And sometimes [students] don't draw it accurately enough to get things like that all the [perpendicular bisectors] meet at the orthocentre¹ of the circle.

The accuracy, speed and manipulative ease of dynamic geometry facilitated geometrical investigations which were difficult to undertake by hand:

[It] takes hours and hours if you try and do that by pencil and paper... So just that power of Geometer's Sketchpad to move the triangle around and try different triangles within seconds was fantastic. Ideal for this sort of exploration.

Nevertheless, the teacher felt that old and new tools lacked congruence, because certain manual techniques appeared to lack computer counterparts. Accordingly, old and new were seen as involving different methods and having distinct functions:

When you do compasses, you use circles and arcs, and you keep your compasses the same. And I say to them: "Never move your compasses once you've started drawing."... Well Geometer's Sketchpad doesn't use that notion at all... So it's a different method.... I don't think there's a great deal of connection. I don't think it's a way of teaching constructions, it's a way of exploring the geometry.

Equally, some features of computer tools were not wholly welcome: students could be deflected from the mathematical focus of a task by overconcern with presentation. During this lesson the teacher had tried out a new technique for managing this, by briefly projecting a prepared example to show students the kind of document that they were expected to produce, and illustrating appropriate use of colour coding:

They spend about three quarters of the lesson making the font look nice and making it all look pretty [but] getting away from the maths.... I've never tried it before, but that showing at the end roughly what I wanted them to have would help. Because it showed that I did want them to think about the presentation, I did want them to slightly adjust the font and change the colours a little bit, to emphasise the maths, not to make it just look pretty.

Here we see the development of sociomathematical norms for using new technologies, and classroom strategies for establishing and maintaining these norms. Likewise, the way in which dynamic geometry required clear instructions to be given in precise mathematical terms was conveyed as being its key characteristic:

I always introduce Geometer's Sketchpad by saying: "It's very specific, you've got to tell it. It's not just drawing, it's drawing using mathematical rules."... They're quite happy with that notion of... the computer only following certain clear instructions.

ACTIVITY FORMAT

Classroom activity is organised around formats for action and interaction which frame the contributions of teacher and students to particular lesson segments (Burns & Anderson, 1987). The crafting of lessons around familiar activity formats and their supporting classroom routines helps to make them flow smoothly in a focused, predictable and fluid way (Leinhardt, Weidman & Hammond, 1987). This leads to the creation of prototypical activity structures or cycles for particular styles of lesson.

Each session of the observed lesson followed a similar activity cycle, starting with teacher-led activity in the normal classroom, followed by student activity at individual computers in the nearby computer suite, and with change of rooms during sessions serving to match working environment to activity format. Indeed, when the

teacher had first nominated this lesson, he had remarked on how it combined a range of classroom activity formats to create a promising lesson structure:

There was a bit of whole class, a bit of individual work and some exploration, so it's a model that I'd like to pursue because it was the first time I'd done something that involved quite all those different aspects.

In discussing the observed lesson, however, the teacher highlighted one aspect of the model which had not functioned as well as he would have liked: the fostering of discussion during individual student work. He identified a need for further consideration of the balance between opportunities for individual exploration and productive discussion, through exploring having students work in pairs:

There was not as much discussion as I would have liked. I'm not sure really how combine working with computers with discussing. You can put two or three [students] on a computer, which is what you might have done in the days when we didn't have enough computers, but that takes away the opportunity for everybody to explore things for themselves. Perhaps in other lessons... as I develop the use of the computer room I might decide... [to] work in pairs. That's something I'll have to explore.

At the same time, the teacher noted a number of ways in which the computer environment helped to support his own interactions with students within an activity format of individual working. Such opportunities arose from helping students to identify and resolve bugs in their dynamic geometry constructions:

[Named student] had a mid point of one line selected and the line of another, so he had a perpendicular line to another, and he didn't actually notice which is worrying... And that's what I was trying to do when I was going round to individuals. They were saying: "Oh, something's wrong." So I was: "Which line is perpendicular to that one?"

Equally, the teacher was developing ideas about the pedagogical affordances of text-boxes, realising that they created conditions under which students might be more willing to consider revising their written comments:

And also the fact that they had a text box... and they could change it and edit it. They could actually then think about what they were writing, how they describe, I could have those discussions. With handwritten, if someone writes a whole sentence next to a neat diagram, and you say: "Well actually, what about that word? Can you add this in?" You've just ruined their work. But with technology you can just change it, highlight it and add on an extra bit, and they don't mind.

This was helping him to achieve his goal of developing students' capacity to express themselves clearly in geometrical terms:

I was focusing on getting them to write a rule clearly. I mean there were a lot writing "They all meet" or even, someone said "They all have a centre."... So we were trying to discuss what "all" meant, and a girl at the back had "The perpendicular bisectors meet", but I think she'd heard me say that to someone else, and changed it herself. "Meet at a point": having that sort of sentence there.

CURRICULUM SCRIPT

In planning and conducting lessons on a topic, teachers draw on a loosely ordered model of relevant goals and actions that guides their teaching. This forms what has been termed a ‘curriculum script’ – where ‘script’ is used in the psychological sense of a form of event-structured cognitive organisation, which includes variant expectancies of a situation and alternative courses of action (Leinhardt, Putnam, Stein & Baxter, 1991). This script includes tasks to be undertaken, representations to be employed, activity formats to be used, and student difficulties to be anticipated.

The observed lesson followed on from earlier ones in which the class had undertaken simple constructions with classical tools: in particular, using compasses to construct the perpendicular bisector of a line segment. Further evidence that the teacher’s script for this topic originated prior to the availability of dynamic geometry was his reference to the practical difficulties which students encountered in working by hand to accurately construct the perpendicular bisectors of a triangle. His evolving script now included knowledge of how software operation might likewise derail students’ attempts to construct perpendicular bisectors, and of how such difficulties might be turned to advantage in reinforcing the mathematical focus of the task:

Understanding the idea of perpendicular bisector... you select the line and the [mid]point... There’s a few people that missed that and drew random lines... And I think they just misunderstood, because one of the awkward things about it is the selection tool. If you select on something and then you select another thing, it adds to the selection, which is quite unusual for any Windows package... So you have to click away and de-select things, and that caused a bit of confusion, even though I had told them a lot. But... quite a few discussions I had with them emphasised which line is perpendicular to that edge... So sometimes the mistakes actually helped.

Equally, the teacher’s curriculum script anticipated that students might not appreciate the geometrical significance of the concurrence of perpendicular bisectors, and incorporated strategies for addressing this:

They didn’t spot that [the perpendicular bisectors] all met at a point as easily... I don’t think anybody got that without some sort of prompting. It’s not that they didn’t notice it, but they didn’t see it as a significant thing to look for... even though there were a few hints in the worksheet that that’s what they were supposed to be looking at, because I thought that they might not spot it. So I was quite surprised... that they didn’t seem to think that three lines all meeting at a point was particularly exceptional circumstances. I tried to get them to see that... three random lines, what was the chance of them all meeting at a point.

The line of argument alluded to here was one already applicable in a pencil and paper environment. Later in the interview, however, the teacher made reference to another strategy which brought the distinctive affordances of dragging the dynamic figure to bear on this issue:

When I talked about meeting at a point, they were able to move it around, and I think there's more potential to do that on the screen.

Likewise, his extended curriculum script depended on exploiting the distinctive affordance of the dynamic tool to explore how dragging the triangle affected the position of the 'centre'.

This suggests that the teacher's curriculum script was evolving through experience of teaching the lesson with dynamic geometry, incorporating new mathematical knowledge specifically linked to mediation by the software. Indeed, he drew attention to a striking example of this which had arisen from his question to the class about the position of the 'centre' when the triangle was dragged to become right angled:

Teacher: What's happening to the [centre] point as I drag towards 90 degrees? What do you think is going to happen to the point when it's at 90?

Student: The centre's going to be on the same point as the midpoint of the line.

Teacher [with surprise]: Does it always have to be at the midpoint?

[Dragging the figure] Yes, it is! Look at that! It's always going to be on the midpoint of that side.... Brilliant!

Reviewing the lesson, the teacher commented on this episode, linking it to distinctive features of the mediation of the task by the dynamic figure:

I don't know why it hadn't occurred to me, but it wasn't something I'd focused on in terms of the learning idea, but the point would actually be on the mid point.... As soon as I'd said it I thought "Of course!" But you know, in maths there's things that you just don't really notice because you're not focusing on them. And... I was just expecting them to say it was on the line. Because when you've got a compass point, you don't actually see the point, it's just a little hole in the paper... But because the point is actually there and quite clear, a big red blob, then I saw it was exactly on that centre point, and that was good when they came up with that.

In effect, his available curriculum script did not attune the teacher to this property. One can reasonably hazard that this changed as a direct result of this episode.

TIME ECONOMY

Assude (2005) examines how teachers seek to improve the 'rate' at which the physical time available for classroom activity is converted into a didactic time measured in terms of advance of knowledge. The adaptation and coordination of working environment, resource system, activity format and curriculum script are very important in improving this didactic 'return' on time 'investment'.

In respect of this time economy, a basic consideration of physical time for the teacher in this study was the proximity of the new computer suite to his normal classroom:

I'm particularly lucky being next door... If I was upstairs or something like that, it would be much harder; it would take five minutes to move down.

However, a more fundamental feature of this case was the degree to which the teacher measured didactic time in terms of progression towards securing student learning rather than pace in covering a curriculum. At the end of the first session, he linked his management of time to what he considered to be key learning processes:

It's really important that we do have that discussion next lesson. Because they've seen it. Whether they've learned it yet, I don't know... They're probably vaguely aware of different properties and they've explored it, so it now needs to be brought out through a discussion, and then they can go and focus on writing things for themselves. So the process of exploring something, then discussing it in a quite focused way, as a group, and then writing it up... They've got to actually write down what they think they've learned. Because at the moment, I suspect... they've got vague notions of what they've learned but nothing concrete in their heads.

A further crucial consideration within the time economy is instrumental investment. The larger study from which this case has been derived showed that the ways in which teachers incorporated dynamic geometry into classroom activity were influenced by their assessments of costs and benefits. Essentially, teachers were willing to invest time in developing students' instrumental knowledge of dynamic geometry to the extent that they saw this as promoting students' mathematical learning. As already noted, this teacher saw working with the software as engaging students in disciplined interaction with a geometric system. Consequently, he was willing to spend time to make them aware of the construction process underlying the dynamic figures used in lessons:

I very rarely use Geometer's Sketchpad from anything other than a blank page. Even when I'm doing something in demonstration... I always like to start with a blank page and actually put it together in front of the students so they can see where it's coming from.

Equally, this perspective underpinned his willingness to invest time in familiarising students with the software, capitalising on earlier investment in using classical tools:

That getting them used to the program beforehand, giving a lesson where the aim wasn't to do that particular maths, but just for them to get familiar with it... was very helpful. And also they're doing the constructions by hand first, to see, getting all the words, the key words, out of the way.

As this recognition of a productive interaction between learning to use old and new technologies indicates, this teacher also took an integrative perspective on the 'double instrumentation' entailed. Indeed, this was demonstrated earlier in his concern with the complementarity of old and new as components of a coherent resource system.

CONCLUSION

This analysis of a lesson incorporating dynamic geometry illuminates the influence of the key structuring features of working environment, resource system, activity format, curriculum script and time economy on technology use. Although only employing a dataset conveniently available from earlier research, it starts to show the complex character of the professional adaptation on which technology integration

into the classroom practice of school mathematics depends. This points to the value of conducting further studies in which data collection (as well as analysis) is guided by the conceptual framework developed in this paper and its predecessor. Such studies might profitably focus not just on the teacher/classroom level, but on the school/departmental level, and the systemic/institutional level.

NOTES

¹ The point at which the perpendicular bisectors of the sides of a triangle meet is the ‘circumcentre’. However, in the course of the interview, the teacher referred to this centre as the ‘orthocentre’. Note that it is now many years since reference to these (and other) terms – which distinguish the different ‘centres’ of a triangle – was removed from the school mathematics curriculum in England.

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METHODS AND TOOLS TO FACE RESEARCH FRAGMENTATION IN TECHNOLOGY ENHANCED MATHEMATICS EDUCATION

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This paper, addresses the issue of how to successfully bring into school practices the results in technology enhanced mathematics learning obtained at research level. The distance among different research teams and between researchers and teachers is addressed in terms of fragmentation of the research field. A methodology is presented to reduce such fragmentation illustrating a pathway followed at the European level in the EC co-funded projects TELMA and ReMath.

INTRODUCTION

In the CERME 5 conference two plenary sessions (Ruthven, 2007; Artigue, 2007), drawing from the discussions developed in different working groups, highlighted key issues concerning Technology Enhanced Learning (TEL) in mathematics.

According to Ruthven (ibid pp. 52), despite of a generalized advocacy for new technologies in education, these have had a limited success in school. As a matter of fact, he observes that, even if technologies had some positive impact on the instruction of teachers, they remain marginal in classroom practice. This is true, in particular, for mathematics, even if, from the beginning, a wide number of researchers have been concerned with the study of the opportunities brought about by new technologies to the teaching and learning of this discipline (Lagrange, Artigue, Laborde, & Trouche, 2003). As a matter of fact, despite the positive results produced in a number of experimental settings and the budget invested by many governments for equipping schools, actual use of ICT tools in real school environments is still having a limited impact. Recent studies witness difficulties encountered by teachers in implementing teaching and learning activities mediated by technologies due to variables such as working environment, resource system, activity format, curriculum script and time economy (Cuban, 2001; Sutherland, 2004). The coordination of such variables is necessary in order to develop a coherent use of technological tools and to form an effective system. According to Ruthven (ibid pp. 64), this challenge “involves moving from idealised aspiration to effective realisation through the development of practical theories and craft knowledge”. Drawing from our own experience, we identify as a crucial issue the necessity to establish effective interactions among the different actors involved in the process, that is researchers, teachers, policy makers, curriculum developers, software designers, etc.

Such a view is coherent with what is reported in (Pratt, Winters, Cerulli & Leemkuil, in press) from the perspective of educational technology designers. Authors, making reference to the specific field of games for mathematics education, speak of the

necessity of a multi-disciplinary approach to design and deployment of technologies as opposed to the frequently experienced design fragmentation. Such fragmentation is often due to the fact that the different communities involved are not fully cognisant of the structuring forces that impinge on each other's activities. From one hand, discontinuities between design and deployment of technological tools impede the effective use of such tools in school practice and, on the other hand, the development of isolated projects that often do not go beyond experimental settings, do not contribute to cumulative knowledge about the design process that could inform future work. Pratt et al. advocate the need to integrate key stakeholders in the creation of technology enhanced learning tools, as "the problem of design fragmentation remains a real impediment to widespread innovation in the field". They thus state the opportunity of creating multidisciplinary teams focusing on the design and deployment of educational technology that bring together the perspective of different stakeholders: designers, educators, researchers, etc.

Fragmentation, however, is not only a problem experienced among different communities of stakeholders, but it is a problem often experienced also within each community. In particular, as highlighted by Artigue (2007) during her plenary speech at CERME 6, this is one of the key issues of concern within the community of the researchers in mathematics education, and, in particular, within the community of researchers focusing on technology enhanced learning in mathematics. Such a fragmentation is rooted at theoretical level, as witnessed also by the work of the working group 11 of ERME that has been established to discuss such specific issue (Prediger, Arzarello, Bosch & Lenfant, 2008). As a matter of fact the theoretical background of a research team has an important bearing on the epistemological assumptions, the research methodologies, the way in which tools, and, in particular, technology enhanced tools, are perceived and used.

At the European level, where a great variety of different approaches and background is present, there is a specific sensibility to the problem of fragmentation and to the necessity to find feasible ways to overcome it, since, as observed in (Arzarello, Bosch, Gascón & Sabena, 2008) a too wide variety of poorly connected conceptual and methodological tools does not encourage consideration of the results obtained as convincing and valuable. Moreover, in the specific area of TEL, there is the need of designing and implementing tools and methodologies that have a wide scope of application and that are not restricted to a particular community or context. For these reasons, following the impulse given by projects funded by the European Community, efforts have been made to try to overcome such fragmentation.

Our Institute has been involved in European research projects concerned with Information Society Technologies (IST) for several years and, in particular in Networks of Excellence (NoE) and Specific Targeted Research Projects (STREPs). These are two instruments of the European Community 6th and 7th Research Framework Programmes that aim at promoting research integration and collaboration in several fields including technology enhanced learning.

This paper presents some methods and tools, developed within the context of such European projects, which have been developed and tested to address the fragmentation issues discussed above.

Firstly we report on the work performed within the TELMA (Technology Enhanced Learning in Mathematics) initiative that explored the conditions for sharing experience and knowledge among different research teams interested in analysing mathematics learning environments integrating technologies, in spite of the differences in the theoretical frameworks and in the methodological approaches adopted. For this purpose, the notions of “*didactical functionality*” (Cerulli, Pedemonte & Robotti, 2006) and of “*key concerns*” - issues functionally important (Artigue, Haspékian, Cazes, Bottino, Cerulli, Kynigos, Lagrange & Mariotti, 2006) - together with a methodology based on the idea of a “*cross experiments*” (Bottino, Artigue & Noss, in press) were defined and conceptualized as concrete methods to address the problem of fragmentation.

Secondly, we give account of some of the outcomes of the ReMath project that, building on the results of the TELMA project, has addressed the fragmentation problem from the perspective of the design, implementation, and in-depth experimentation of ICT-based interactive learning environments for mathematics, thus involving not only researchers but teachers and technology designers as well. In particular, within the ReMath project, the problem of how to effectively support collaboration in pedagogical planning has been faced. Efforts have been made to provide a solid basis for accommodating the different perspectives adopted, for analysing the factors at play, and also for understanding the initial assumptions and theoretical frameworks embraced. A web-based system, the Pedagogical Plan Manager (PPM), was developed to support researchers, tool designers and teachers to jointly design and/or deploy mathematics pedagogical plans involving the use of technological tools (Bottino, Earp, Olimpo, Ott, Pozzi & Tavella, 2008).

Summing up, in the following sections, we delineate the process that has brought us to afford the problem of the fragmentation of approaches and frameworks, in the field of mathematics teaching and learning mediated by technologies, from different but complementary perspectives.

A COLLABORATIVE METHODOLOGY FOR NETWORKING RESEARCH TEAMS IN TECHNOLOGY ENHANCED LEARNING IN MATHEMATICS

NoEs have been established by the European Commission within the last Framework Research Programmes as instruments to promote integration and collaborative work of key European research teams and stakeholders in given fields. In particular, the network of Excellence Kaleidoscope was established and funded with the aim of shaping the scientific evolution of technology enhanced learning (<http://www.noe-kaleidoscope.org>, accessed March 2009). Since each knowledge domain raises specific issues either for learning or for the design of learning environments, within Kaleidoscope a number of different joint research initiatives, covering a wide range

of domains, have been carried out. Among these, TELMA was specifically focused on Technology Enhanced Learning in Mathematics. It involved six European teams¹ and had as its main aim that of building a shared view of key research topics in the area of digital technologies and mathematics education, proposing related research activities, and developing common research methodologies.

In TELMA, each team brought to the project particular focuses and theoretical frameworks, adopted and developed over a period of time. Most of these teams have also designed, implemented and experimented, in different classroom settings, computer-based systems for supporting teaching and learning processes in mathematics. It was clear from the beginning that, to connect the work of groups that have different traditions and frameworks it was necessary to develop a better mutual understanding and to find some common perspectives from which to look at the different approaches adopted. It was also necessary to develop a common language since the same words were sometime used with different meanings by each team, causing misunderstanding and hindering productive collaboration. Moreover, it became clear that the theoretical assumptions made by each team, were often implicit and thus not accessible to the others.

The notion of didactical functionalities

In order to overcome these difficulties it was decided to focus the work of TELMA on the theoretical frameworks within which the different research teams face research in mathematics education with technology. A first level of integration has been then pursued through the definition of the notion of *didactical functionality* for interpreting and comparing different research studies (Cerulli et al., 2006). Such notion has been used as a way to develop a common perspective among teams linking theoretical reflections to the real tasks that one has to face when designing or analysing effective uses of digital technologies in given contexts. The notion of didactical functionality is structured by three inter-related components:

- a set of features/characteristics of the considered ICT-tool;
- an educational aim;
- the modalities of employing the ICT-tool in a teaching/learning process to achieve the chosen educational aim.

The different didactical functionalities designed and experimented by each team have been compared trying to delineate how different theoretical backgrounds can influence the design of an ICT-based tool, the definition of the educational goals to be pursued, and the modalities of use of the tool to achieve such goals. At the beginning, this analysis was conducted on the basis of a selection of papers published by each team. This approach, even if useful, was considered not sufficient to enter the less explicit aspects of the research work of each team. Thus, TELMA researchers decided to move toward a strategy that could allow them to gain more intimate insights into their respective research and design practices. This strategy relies on the idea of ‘cross-experiments’ and on the development of a methodological tool for systematic exploration of the role played by theoretical frames.

The cross-experiments methodology

The idea of cross-experiments was developed in order to provide a systematic way of gaining insight into theoretical and methodological similarities and differences in the work of the various TELMA teams. This is a new approach to collaboration that seeks to facilitate common understanding across teams with diverse practices and cultures and to elaborate integrated views that transcend individual team cultures. There are two principal characteristics of the cross-experiments project implemented within TELMA that distinguish it from other forms of collaborative research:

- the design and implementation by each research team of a teaching experiment making use of a ICT-based tool developed by one of the other team involved;
- the joint construction of a common set of questions to be answered by each team in order to frame the process of cross-team communication.

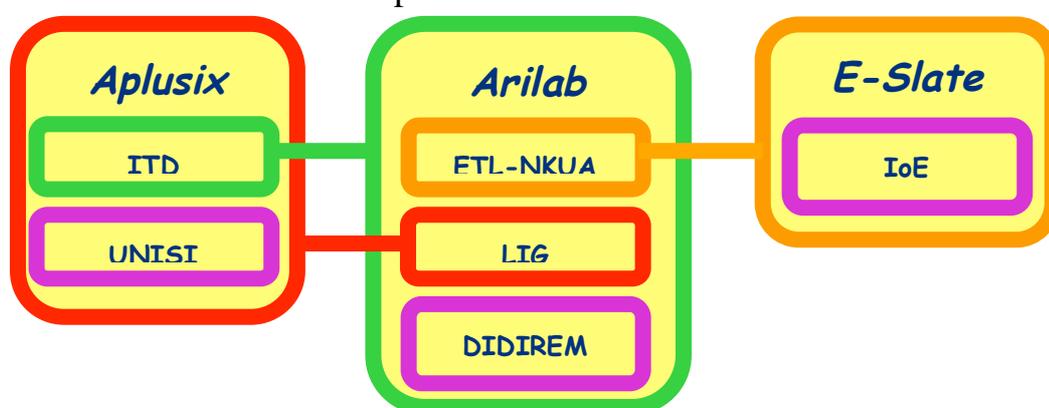


Figure 1: *Aplusix*, developed by Metah, was experimented by ITD and UNISI. *Arilab*, developed by ITD, was experimented by LIG, ETL-NKUA and DIDIREM. *E-Slate*, developed by ETL-NKUA was experimented by IoE.

Each team was asked to select an ICT-tool among those developed by the other TELMA teams (Figure 1). This decision was expected to induce deep exchanges between the teams and to make visible the influence of theoretical frames through comparison of the didactical functionalities developed by the designers of given tools and those implemented by the teams experimenting the tools. Moreover, in order to facilitate the comparison between the different experimental settings, it was also agreed to address common knowledge domains (fractions and introduction to algebra), to carry out the teaching experiments with students between the 5th to 8th grade, and to perform them for about the same amount of time (one month).

Guidelines (Cerulli, Pedemonte & Robotti, 2007) were collectively built for monitoring the whole process: from the design and the a priori analysis of the experiments to their implementation, the collection of data and the a posteriori analysis. Beyond that, reflective interviews (using the technique of "interview for explicitation" (Vermesch & Maurel, 1997)) were a-posteriori organized in order to make clear the exact role that theoretical frames and contextual characteristics had played in the different phases of the experimental work, either explicitly or in a more naturalized and implicit way.

It was hypothesized that, for each team, the use of a non familiar (alien) tool would have made problematic, thus visible, design decisions and practices that generally remain implicit when one uses tools developed within his/her research and educational culture, and that this visibility would have been increased by the guidelines' request of making explicit the choices performed.

The cross experiments provided interesting insights on the complexities involved in designing and implementing mathematics learning environments integrating technology and allowed to make some reflections (Bottino et al., in press; Cerulli, Trgalova, Maracci, Psycharis & Georget, 2008).

The first reflection was on the conditions that can facilitate the sharing of experience and knowledge among researchers in spite of the differences in the theoretical frameworks adopted. Theoretical frameworks, while influencing design and analysis of a teaching experiment, were far from playing the role they are usually given in the literature. As a matter of fact, in the design of the cross-experiments, theoretical frameworks acted mainly as implicit and naturalized frames, and more in terms of general principles than of operational constructs. Even if some variations could be noticed, all the teams experienced a gap between the support offered by theoretical frames and the decisions to be taken in the design process. The acknowledgment of such a gap can be a starting point for establishing a better communication channel not only among researchers but also with teachers. As a matter of fact, a marked emphasis on theoretical assumptions is often too far from the practical needs of teachers. For this reason it is important to establish the exact role that theoretical frameworks play in the planning of an effective teaching experiment. In particular, it was found that researchers tend to overestimate such role, thus making the distance with teachers' needs even bigger. A methodology for making explicit, and justifying, the choices made, proved a useful tool for reducing communication disparities.

A second observation concerns the understanding of what it means to adapt an ICT based tool to a context different from the one it was designed for. In our work this was accomplished by experimenting in each country tools developed in other countries by different teams. Thanks to the adopted methodology and to the request of making explicit assumptions, choices and decisions taken, it was possible to individuate some variables that strongly affect the development of teaching experiments involving the use of technologies. For instance, the attention paid to different *research priorities* (e.g. the detailed organization of the milieu; the social construction of knowledge; the teacher's role) and to *local constrains* (e.g. curricular; institutional; cultural) appeared to be crucial. Such variables are to be deeply considered and made explicit in the communication with teachers to effectively support them to adapt research experiments to their teaching contexts. In other words, researchers should find ways to make explicit all the key assumptions at the basis of their experiments. Of course, this is not enough, since, as suggested in (Pratt et al., in press), it is also necessary to promote a more strict collaboration between researchers,

tool designers and teachers also at the level of the design and the implementation of ICT based tools, and in the planning of the experiments.

Taking into account these needs, and on the basis of the results obtained in TELMA, a new European project was thus developed, involving the same research teams: the ReMath project (IST - 4 – 26751 - STP). In this project the issue of collaboration between different stakeholders was addressed by developing a specific tool to be used to design teaching experiments involving ICT based tools.

A TOOL TO SUPPORT THE COMMUNICATION OF DIFFERENT STAKEHOLDERS IN THE PLANNING OF LEARNING ACTIVITIES INVOLVING TECHNOLOGY

The TELMA project provided a strategy for reducing the difficulties of communication among researchers; this strategy proved to be quite effective, thus it was decided to adapt it to the needs of the ReMath project where communication in a wider community, including software designers, researchers and teachers, has been addressed. The Remath project has two main goals: the development of ICT-based tools for mathematics education at secondary school level and the design and experimentation, in different contexts, of learning activities for classroom practice involving the use of such tools (see: http://remath.cti.gr/default_remath.asp; accessed March 2009). In order to pursue this last goal, a cross-experiment methodology, widening the one developed by TELMA, was adopted. A tool, the Pedagogical Plan Manager (PPM), was, thus, developed to support communication between researchers and teachers when planning learning activities involving ICT tools. The idea was originated by the analysis of some the difficulties, pointed out by researchers in the wide field of learning design (Koper & Olivier, 2004), concerning dialogue and transfer between teachers, researchers and designers. To overcome such difficulties the PPM was realized, relying on the concept of *pedagogical plan*, as a specific system for supporting the process of pedagogical design, namely the description of learning activities to be enacted during cross-experiments (thus also enabling and fostering their reusability).

Pedagogical plans are conceived as descriptions of pedagogical activities to be carried out in real contexts (e.g. a class, a laboratory, etc.) where a number of different indicators could be made explicit, at different level of details (Bottino et al., 2008): educational target (What learning outcomes? What learning contexts? Who are the target learners?); pedagogical rationale (Why those learning outcomes? Why applying a certain strategy? Why using a give tool?); specifications (Which activities are to be carried out? Which roles are to be assumed by the different actors? Which resources and tools are to be used? etc.).

The PPM is a web environment, organized as a flexible structure allowing a three-alike representation of *pedagogical plans* as hierarchical entities which can be built and red at different levels of detail. This structure supports both “authors” of pedagogical plans, providing them with the possibility to work with a top-down

structure, and “readers”, who in top-down organization have a facilitating factor for navigating from the general to the particular and vice versa.

In other words the PPM presents a flexible structure that tries to respond to the different needs of both researchers and teachers; the first, in fact, were mainly interested in sharing ideas about aspects such as the theoretical frameworks and the pedagogical rationale behind each educational intervention, while teachers were mainly interested in retrieving suitable information about the most suitable ways to carry out educational activities in their classes (Earp & Pozzi, 2006).

For space reason, we cannot provide here a detailed description of the model adopted and of the prototype implemented (more details can be found in Bottino et al., 2008). Outputs of its use are currently under examination and will be further analysed at the end of the ReMath project (May 2009).

CONCLUSIONS

Software designers, researchers and teachers may have different needs, different constraints, and different perspectives. This can be an obstacle for the effectiveness of technology enhanced learning in mathematics, also in terms of impact in school practice. The projects briefly presented tried to develop a coherent methodology for reducing the distance between the different stakeholders. In TELMA it was addressed the problem of networking research teams with different backgrounds and approaches by means of a specific collaborative methodology. In ReMath such methodology was extended, also through the development of a specific web-based tool, to involve all the stakeholders in the design, development and deployment of teaching and learning activities involving the use of technologies.

The outlined pathway includes researcher’s explicitation of the actual role played by theoretical frameworks in the effective use of ICT tools and the individuation of the gap between theory and practice. This can help reducing the distance with teachers. The tool for pedagogical planning developed in the ReMath project is aimed at the same goal by involving teachers, from the beginning, also in the design of teaching activities with ICT-based tools. Such activities are seen as integral part in the design process of a technology. In this way we believe it can be possible to develop communities of practice that bring together teachers and researchers so that teaching practice and research could nurture one from each other favouring a better impact of technology enhanced learning in school practice.

NOTES

1. TELMA teams (whose acronyms are indicated in brackets) belong to the following Institutions: Consiglio Nazionale delle Ricerche, Istituto Tecnologie Didattiche, Italy (ITD); Università di Siena, Dipartimento di Scienze Matematiche ed Informatiche, Italy (UNISI); University of Paris 7 Denis Diderot, France (DIDIREM); Grenoble University and CNRS, Leibniz Laboratory, Metah, France (LIG); University of London, Institute of Education, United Kingdom (IOE); National Kapodistrian University of Athens, Educational Technology Laboratory, Greece (ETL-NKUA).

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THE DESIGN OF NEW DIGITAL ARTEFACTS AS KEY FACTOR TO INNOVATE THE TEACHING AND LEARNING OF ALGEBRA: THE CASE OF ALNUSET

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The integration of CAS systems into school practices of algebra is marginal. To integrate effectively digital technology in the teaching and learning of algebra, it is necessary to go beyond the experience of CAS and of their instrumented techniques and to face the design of new artefacts. In this paper we discuss design problems faced in the development of a new digital artefact for teaching and learning of algebra, the Alnuset system. We present the key ideas that have oriented its design and the choices we have worked out to instrument its incorporated algebraic techniques. We compare the quantitative, symbolic and functional instrumented techniques of Alnuset with those of CAS highlighting crucial differences in the teaching and learning of algebra.

Keywords: *Alnuset, Instrumented technique, CAS, Algebraic learning*

INTRODUCTION

In the last 15 years a scientific debate on the role of technology in supporting teaching and learning processes in the domain of algebra has been going on. This debate originates from research studies carried out in different countries with the purpose of studying the use of Computer Algebra Systems (CAS) in school contexts. In particular, near benefits (Heid, 1988, Kaput, 1996, Thomas, Monaghan and Pierce, 2004) obstacles and difficulties have been identified in using this technology by students and teachers (Mayes, 1997, Drijvers, 2000, Drijvers, 2002, Guin & Trouche, 1999). Results of these research works (Artigue, 2005) highlight that the integration of CAS systems into the school practice of algebra remains marginal due to different reasons. CAS expands the range of possible task-solving actions. As a matter of fact, techniques involved in a CAS (instrumented techniques) are in general different from those of the paper and pencil environment. Managing the complexity of CAS instrumented techniques and highlighting the potential offered by the machine to the student is hard work. As shown by some experiments (Artigue, 2005), CAS use may cause an explosion of techniques which remain in a relatively simply-crafted state. Moreover, any technique that goes beyond a simple, mechanically learnt gesture, should be accompanied by a theoretical discourse. For the paper and pencil techniques this discourse is known and can be found in textbooks. For instrumented techniques it has to be built and its elaboration raises new, specific difficulties. Even if the use of CAS seems fully legitimate in the class, in general, instrumented techniques cannot be institutionalised in the same way as paper and pencil ones (Artigue, 2005).

THE RATIONALE

To frame the results carried out by these research studies and the complexity of the processes involved in the educational use of CAS, some French researchers (Lagrange 2000, Artigue, 2002, Lagrange, Artigue, Guin and Trouche, 2003) have elaborated a theoretical framework, named 'instrumental approach', integrating both the ergonomic theory (Rabardel, 1995) and the anthropological theory (Chevallard, 1992). The 'instrumental approach' provides a frame for analyzing the processes of instrumental genesis both in their personal and institutional dimensions, and the effect of instrumentation issues on the integration of CAS in the educational practice. Using this framework, Artigue observes that CAS are extremely effective from a pragmatic standpoint and for this reason professionals (mathematicians, engineers..) are willing to spend time to master them (Artigue, 2002). At pragmatic level the effectiveness often comes with the difficulty to justify, at a theoretical level, the instrumented techniques used. In particular, this is true for users who do not fully master mathematical knowledge and techniques involved in the solution of the task. As a consequence, the epistemic value of the instrumented technique can remain hidden. This can constitute a problem for the educational context where technology should help not only to yield results but also to support and promote mathematical learning and understanding. In educational practice, techniques should have an epistemic value contributing to the understanding of objects involved. *"Making technology legitimate and mathematically useful from an educational point of view, whatever be the technology at stake, requires modes of integration that provide a reasonable balance between the pragmatic and the epistemic values of instrumented techniques"* (Artigue, 2007, p. 73). These results might account for the marginalization of CAS integration into the school algebraic practices. For some researchers, to integrate digital technology effectively in the domain of algebra, it is necessary to go beyond the experience of CAS and of their instrumented techniques and to face the design of new artefacts. As underlined by Monaghan (2007) up to now CAS-in-education workers have paid little attention to design issues, preferring, in general, to work with the design supplied by CAS designers (Monaghan, 2007). Moreover, it should be noted that no comparison between the design of CAS and of technological tools for education has been developed so far. This article aims at pointing out design issues that can effectively support teaching and learning processes in algebra. This goal will be pursued considering the design of ALNUSET (ALgebra on the NUmerical SETs), a system developed to improve teaching and learning of crucial topics involved in the mathematical curricula such as algebra, functions and properties of numerical sets. In particular, in this article we compare design aspects of Alnuset and of CAS and we highlight the relevance of differences in their instrumented techniques for the teaching and learning of algebra.

PROBLEMS OF DESIGN IN DEVELOPING NEW DIGITAL ARTEFACTS

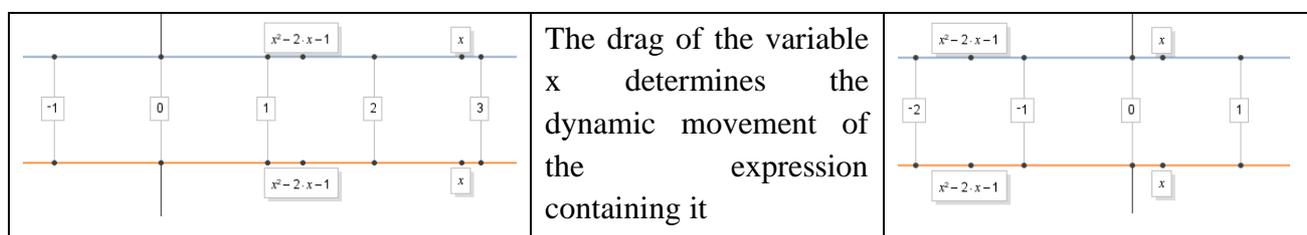
Going beyond the design of CAS requires new creative ideas to instrument techniques for mathematical activity different from those of CAS. The advent of both the dynamic geometrical artefacts and of spreadsheets has evidenced that even a single creative idea can determine a new typology of innovative artefacts. This can occur when new creative ideas allow to instrument mathematical techniques characterizing them with new operative and representative dimensions such as the drag of the variable point of a geometrical construction, as in the case of dynamic geometrical software, or the automatic re-computation of formulas of the table, as in the case of spreadsheet. Moreover, when a technique must be instrumented on the basis of an idea, various types of design problems emerge. They regard the way tasks and responsibilities have to be distributed between user and computer and the management of the interactivity, namely the operative modalities of the input by the user, the representation of the result by the computer (output), the visualisation of specific feedback to support the user action or to accompany the presentation of the result. Moreover, problems of design regard also the way in which the instrumented techniques have to be connected between each other. The way these problems are solved affects the accessibility of techniques, their usefulness for the task to be solved, the meaning that the instrumented technique evidences in the interaction, the discourse that can be developed about it. Hence, the way these problems are solved affects the balance between pragmatic and epistemic values of instrumented techniques within the didactical practice and this can affect mathematics teaching and learning. The anthropological framework is the theoretical tool used to analyse the way in which techniques are implemented and their effectiveness on the educational level. Ideas are evaluated on the base of this framework. We discuss these general assumptions in the domain of algebra referring to Alnuset System.

ALNUSET: IDEAS AND CHOICE OF DESIGN

ALNUSET is a system designed, implemented and experimented within the ReMath (IST - 4 - 26751) EC project that can be used to improve the teaching and learning of algebra at lower and upper secondary school level. The design of ALNUSET is based on some ideas that have oriented the realisation of the three, strictly integrated components: the Algebraic Line component, the Algebraic Manipulator component, and the Function component. These three components make available respectively techniques of quantitative, symbolic and functional nature to support teachers and students in developing algebraic objects, processes and relations involved in the algebraic activity. In the following we present the main ideas that have oriented the realisation of the three components of Alnuset and illustrate the choices and decisions taken to instrument algebraic techniques so that an appropriate balance between their epistemic and pragmatic values can emerge when used in the educational practice.

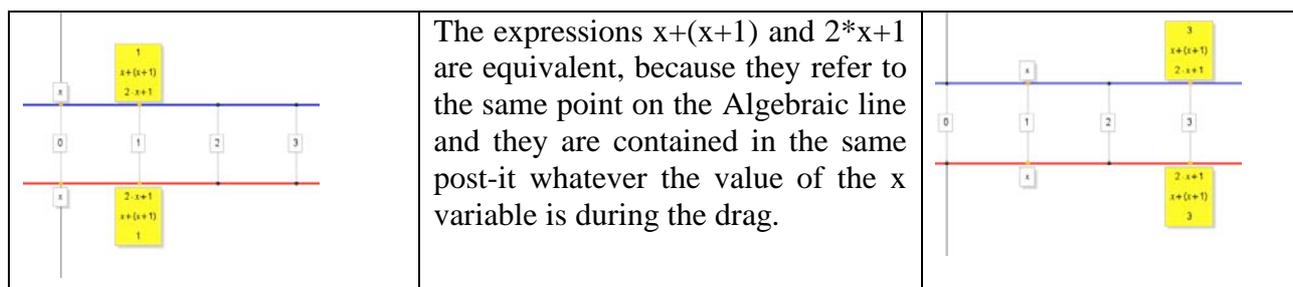
Algebraic line component

The main idea in the design of the Algebraic line component is the representation of algebraic variables on the number line through mobile points associated to letters, namely points that can be dragged on the line with the mouse. In this component the user can edit expressions to operate with. The computer automatically computes the value of the expression on the basis of the value of the variable on the line and it places a point associated to the expression on the algebraic line. When the user drags the mobile point of a variable, the computer refreshes the positions of the points corresponding to the expressions containing such a variable in an automatic and dynamic manner. This is possible only thanks to the digital technology that allows to transform the traditional number line into an algebraic line. The following two figures report the representation of a variable and of an algebraic expression on the lines of this component. Note that the presence of two lines is motivated by operative necessities regarding the use of the algebraic editor based on geometrical models that is available in this component. This editor is not considered in this report.

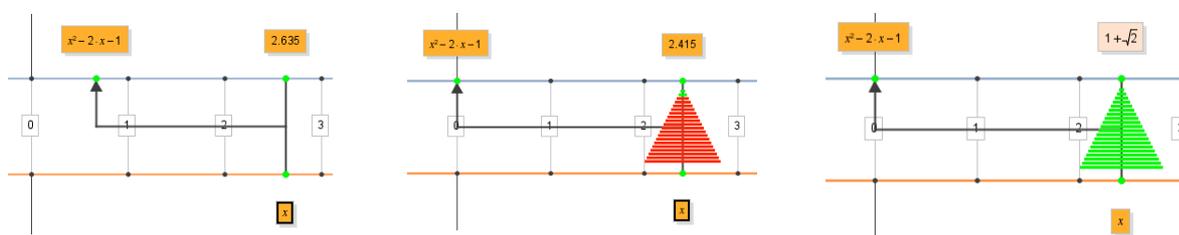


Through its visual feedback, this technique can be used either to explore what an expression indicates in an indeterminate way or to compare expressions. The design of this component is associated to every point represented on the line by a post-it. The computer automatically manages the relation among expressions, their associated points and post-it. The post-it of a point contains all the expressions constructed by the user that denote that point. By dragging a variable on the line, dynamic representative events can occur in a post-it. They might be very important for the development of a discourse concerning the notions of equality and equivalence between expressions. As a matter of the fact, the presence of two expressions in a post-it may mean:

- A relationship of equality, if taking place at least for one value of the variable during its drag along the line
- A relationship of equivalence, if taking place for all the values assumed by the variable when it is dragged along the line.
- A relationship of equivalence with restrictions, if taking place for every value of the variable when it is dragged along the line, but for one or more values, for which one of the two expressions disappears from the post-it and from the line.



Moreover, the algebraic line component has been designed to provide two very important instrumented techniques for the algebraic activity, i.e. for finding the roots of polynomial with integer coefficients and for identifying and validating the truth set of algebraic propositions. The root of a polynomial can be found dragging the variable on the algebraic line in order to approximate the value of the polynomial to 0. When this happens, the exact root of the polynomial is determined by a specific algorithm of the program and it is represented as a point on the line.

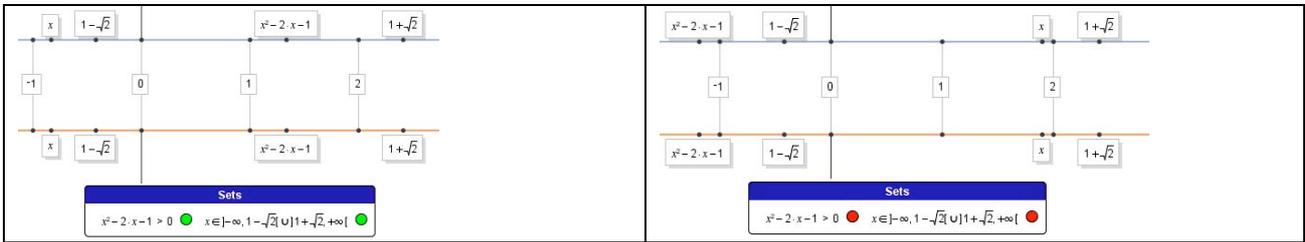


This technique, that can be controlled by the user through his visual and spatial experience, is effective not only at a pragmatic level but also at an epistemic level, because it can concretely support the development of a discourse on the notion of root of a polynomial, as value of the variable that makes the polynomial equal to 0. The truth set of a proposition can be found through the use of a specific graphical editor. Let us consider the inequation $x^2-2x-1>0$, that once edited, is visualised in a specific window of this component named “Sets”. Once the root of the polynomial associated to the inequation has been represented on the line, a graphic editor can be used to construct its truth set (see the figure).

Two open intervals on the line, respectively on the right and on the left side of the roots of the polynomial x^2-2x-1 , have been selected with the mouse. The system has translated the performed selection into the formal language.

Once the truth set of a proposition has been edited, it can be validated using a specific feedback of the system. In the set window propositions and numerical sets are associated to coloured (green/red) markers that are under the control of the system. The green/(red) colour for the proposition means that it is true/(false) while the green/(red) colour for the numerical set means that the actual variable value on the line is/(is not) an element of the set. Through the drag of the variable on the line, colour accordance between proposition marker and set marker allows the user to

validate the defined numerical set as truth set of the proposition (see figure below). The validation process is supported by the accordance of colour between the two markers and by the quantitative feedback provided by the position of variable and of the polynomial on the algebraic line during the drag.



This feedback offered by the system during the drag of the variable is important to introduce the notions of truth value and of truth set of an algebraic proposition and to develop a discourse on their relationships. All the described instrumented techniques that are specific of the Algebraic line component make a quantitative and dynamic algebra possible.

Algebraic manipulator component

The interface of this component has been divided into two distinct spaces: a space where symbolic manipulation rules are reported and a space where symbolic transformation is realised.

<table border="1"> <thead> <tr> <th>Addition</th> <th>Multiplication</th> </tr> </thead> <tbody> <tr> <td>$A+B \Leftrightarrow B+A$</td> <td>$A \cdot B \Leftrightarrow B \cdot A$</td> </tr> <tr> <td>$A+(B+C) \Leftrightarrow (A+B)+C$</td> <td>$A \cdot (B \cdot C) \Leftrightarrow (A \cdot B) \cdot C$</td> </tr> <tr> <td>$A \Leftrightarrow A+0$</td> <td>$A \Leftrightarrow A \cdot 1$</td> </tr> <tr> <td>$A+(-A) \Leftrightarrow 0$</td> <td>$A \cdot 0 \Leftrightarrow 0$</td> </tr> <tr> <td>$A-B \Leftrightarrow A+(-B)$</td> <td>$-A \Leftrightarrow -1 \cdot A$</td> </tr> <tr> <td>$a_1+a_2+\dots \Leftrightarrow x$</td> <td>$1 \Leftrightarrow -1 \cdot -1$</td> </tr> <tr> <td>$n \Leftrightarrow a+b$</td> <td>$1 \Leftrightarrow -1 \cdot -1$</td> </tr> <tr> <th>Powers</th> <td>$A \cdot \frac{1}{A} \Leftrightarrow 1$</td> </tr> <tr> <td>$A^n \Leftrightarrow A \cdot A \dots$</td> <td>$\frac{A}{B} \Leftrightarrow A \cdot \frac{1}{B}$</td> </tr> <tr> <td>$A^{n_1+n_2+\dots} \Leftrightarrow A^{n_1} \cdot A^{n_2} \dots$</td> <td>$\frac{1}{A_1 \cdot A_2 \dots} \Leftrightarrow \frac{1}{A_1} \cdot \frac{1}{A_2} \dots$</td> </tr> <tr> <td>$(A_1 \cdot A_2 \dots)^n \Leftrightarrow A_1^n \cdot A_2^n \dots$</td> <td>$a_1 \cdot a_2 \dots \Leftrightarrow x$</td> </tr> <tr> <td>$(A^n)^m \Leftrightarrow A^{n \cdot m}$</td> <td>$n \Leftrightarrow p_1 \cdot p_2 \dots$</td> </tr> <tr> <td>$A^{-n} \Leftrightarrow \frac{1}{A^n}$</td> <td></td> </tr> <tr> <td>$A^{\frac{1}{n}} \Leftrightarrow \sqrt[n]{A}$</td> <td></td> </tr> <tr> <th>Computation</th> <th>Distribute / Factor</th> </tr> <tr> <td>$A \Leftrightarrow (A)$</td> <td>$A \cdot (B_1+B_2+\dots) \Leftrightarrow A \cdot B_1+A \cdot B_2+\dots$</td> </tr> <tr> <td>Remove extra 0</td> <td></td> </tr> <tr> <td>Simplify numerical expression</td> <td></td> </tr> <tr> <td>Expand</td> <th>Solving</th> </tr> <tr> <td>Collect</td> <td>$A \leq B \Leftrightarrow B \leq A$</td> </tr> <tr> <td>Eliminate variable</td> <td>$A \leq B \Leftrightarrow A-B \leq 0$</td> </tr> <tr> <td></td> <td>$A \leq B+T \Leftrightarrow A-T \leq B$</td> </tr> <tr> <td></td> <td>$A+T \leq B \Leftrightarrow A \leq B-T$</td> </tr> <tr> <td></td> <td>$T \cdot A \leq B \Leftrightarrow A \leq \frac{B}{T}$</td> </tr> <tr> <td></td> <td>$\frac{A}{T} \leq B \Leftrightarrow A \leq B \cdot T$</td> </tr> </tbody> </table>	Addition	Multiplication	$A+B \Leftrightarrow B+A$	$A \cdot B \Leftrightarrow B \cdot A$	$A+(B+C) \Leftrightarrow (A+B)+C$	$A \cdot (B \cdot C) \Leftrightarrow (A \cdot B) \cdot C$	$A \Leftrightarrow A+0$	$A \Leftrightarrow A \cdot 1$	$A+(-A) \Leftrightarrow 0$	$A \cdot 0 \Leftrightarrow 0$	$A-B \Leftrightarrow A+(-B)$	$-A \Leftrightarrow -1 \cdot A$	$a_1+a_2+\dots \Leftrightarrow x$	$1 \Leftrightarrow -1 \cdot -1$	$n \Leftrightarrow a+b$	$1 \Leftrightarrow -1 \cdot -1$	Powers	$A \cdot \frac{1}{A} \Leftrightarrow 1$	$A^n \Leftrightarrow A \cdot A \dots$	$\frac{A}{B} \Leftrightarrow A \cdot \frac{1}{B}$	$A^{n_1+n_2+\dots} \Leftrightarrow A^{n_1} \cdot A^{n_2} \dots$	$\frac{1}{A_1 \cdot A_2 \dots} \Leftrightarrow \frac{1}{A_1} \cdot \frac{1}{A_2} \dots$	$(A_1 \cdot A_2 \dots)^n \Leftrightarrow A_1^n \cdot A_2^n \dots$	$a_1 \cdot a_2 \dots \Leftrightarrow x$	$(A^n)^m \Leftrightarrow A^{n \cdot m}$	$n \Leftrightarrow p_1 \cdot p_2 \dots$	$A^{-n} \Leftrightarrow \frac{1}{A^n}$		$A^{\frac{1}{n}} \Leftrightarrow \sqrt[n]{A}$		Computation	Distribute / Factor	$A \Leftrightarrow (A)$	$A \cdot (B_1+B_2+\dots) \Leftrightarrow A \cdot B_1+A \cdot B_2+\dots$	Remove extra 0		Simplify numerical expression		Expand	Solving	Collect	$A \leq B \Leftrightarrow B \leq A$	Eliminate variable	$A \leq B \Leftrightarrow A-B \leq 0$		$A \leq B+T \Leftrightarrow A-T \leq B$		$A+T \leq B \Leftrightarrow A \leq B-T$		$T \cdot A \leq B \Leftrightarrow A \leq \frac{B}{T}$		$\frac{A}{T} \leq B \Leftrightarrow A \leq B \cdot T$	$a^2 - b^2$ $a^2 + 0 - b^2$ $a^2 + a \cdot b + (-1 \cdot (a \cdot b) - b^2$ $a^2 + a \cdot b + (-1 \cdot (a \cdot b) - b^2$ $a^2 + a \cdot b + (-1 \cdot a \cdot b - b^2$ $a^2 + a \cdot b + (-1 \cdot a \cdot b + (-b \cdot b)$ $a^2 + a \cdot b + (-1 \cdot a \cdot b + (-1 \cdot (b \cdot b)$ $a^2 + a \cdot b + (-1 \cdot a \cdot b + (-1 \cdot b \cdot b$ $a \cdot a + a \cdot b + (-1 \cdot a \cdot b + (-1 \cdot b \cdot b$ $a \cdot (a+b) + (-1 \cdot a \cdot b + (-1 \cdot b \cdot b$ $a \cdot (a+b) + (-1 \cdot (a \cdot b + b \cdot b)$ $a \cdot (a+b) + (-1 \cdot (a+b) \cdot b$ $a \cdot (a+b) + (-1 \cdot b \cdot (a+b)$ $(a+(-1 \cdot b)) \cdot (a+b)$ $(a-1 \cdot b) \cdot (a+b)$ $(a-b) \cdot (a+b)$	<p>This figure shows a part of the commands available with the interface and an example of algebraic transformation.</p> <p>The figure shows a characteristic of the interactivity of this manipulator: the selection of a part of an expression determines the activation of the commands of the interface that can be applied on it. This characteristic can help students to explore the systems of rule for the algebraic transformation and the effects they produce</p>
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The main idea characterizing the design of the Algebraic Manipulator component is the possibility to exploit pattern matching procedures of computer science to transform algebraic expressions and propositions through a structured set of basic rules that are deeply different from those of the CAS. In CAS pattern matching procedures are exploited according to a pragmatic perspective oriented to produce a result of symbolic transformation that could be also very complex, as in the case of

command like factor or solve. As a consequence, the techniques of transformation can be obscure for a not expert user. In the Algebraic Manipulator component of Alnuset pattern matching procedures have been exploited according to three specific pedagogical necessities. The first necessity is to highlight the epistemic value of algebraic transformation as formal proof of the equivalence among algebraic forms. To this aim we have designed this manipulator with a set of basic rules that correspond to the basic properties of addition, multiplication and power operations, to the equality and inequality properties between algebraic expressions, to basic logic operations among propositions and among sets. Every rule produces the simple result of transformation that is reported on the icon of its corresponding command on the interface, and this makes the control of the rule and the result easy to control. Moreover a fundamental function of this component allows the student to create a new transformation rule (user rule) once this rule has been proved using the rules of transformation available on the interface. For example, once the rule of the remarkable product $a^2 - b^2 = (a+b) \cdot (a-b)$ has been proved, it can be added as new user rule in the interface $a^2 - b^2 \leftrightarrow (a+b) \cdot (a-b)$ and it can successively be used to transform other expressions or part of them whose form match with it. Moreover, a specific command allows to represent every transformed expression on the algebraic line automatically. Through this command it is possible to verify quantitatively the preservation of the equivalence through the transformation, observing that all the transformed expressions belong to the same post-it when their variables are dragged along the line. These characteristics of the algebraic manipulator of Alnuset can have a great epistemic importance because they can be effectively exploited to support the comprehension of the algebraic manipulation in terms of formal proof of the equivalence between two algebraic forms. The second necessity is to support the integration of practice of quantitative and manipulative nature. In this manipulator three rules allow the user to import the root of a polynomial, the truth set of a proposition and the value assumed by a variable on the algebraic line from the Algebraic line component to be used in the algebraic transformation. For example the rule "Factorize" uses the root of polynomial found in the Algebraic Line to factorize it. The way in which this rule works, makes the factorization technique of Alnuset different from that of CAS. In CAS this technique is totally under the control of the system, and the result can appear rather obscure for not expert users. In Alnuset, the factorization can be applied on the polynomial at hand only if its roots have been previously determined on the algebraic line. In Alnuset the distribution of tasks between user and computer and the way they interact, can contribute to understand the link between the factorization of a polynomial and its roots. The third necessity is to offer more powerful rules of transformation when needed for the activity and when specific meaning of algebraic manipulation have been already constructed. Two specific rules, also present in the CAS are available in this manipulator. They determine the result of a numerical expression and the result of a computation with polynomials respectively. These rules of transformation contribute to increase the

pragmatic value of the instrumented technique of algebraic transformation in Alnuset and they can be used to introduce to the use of CAS

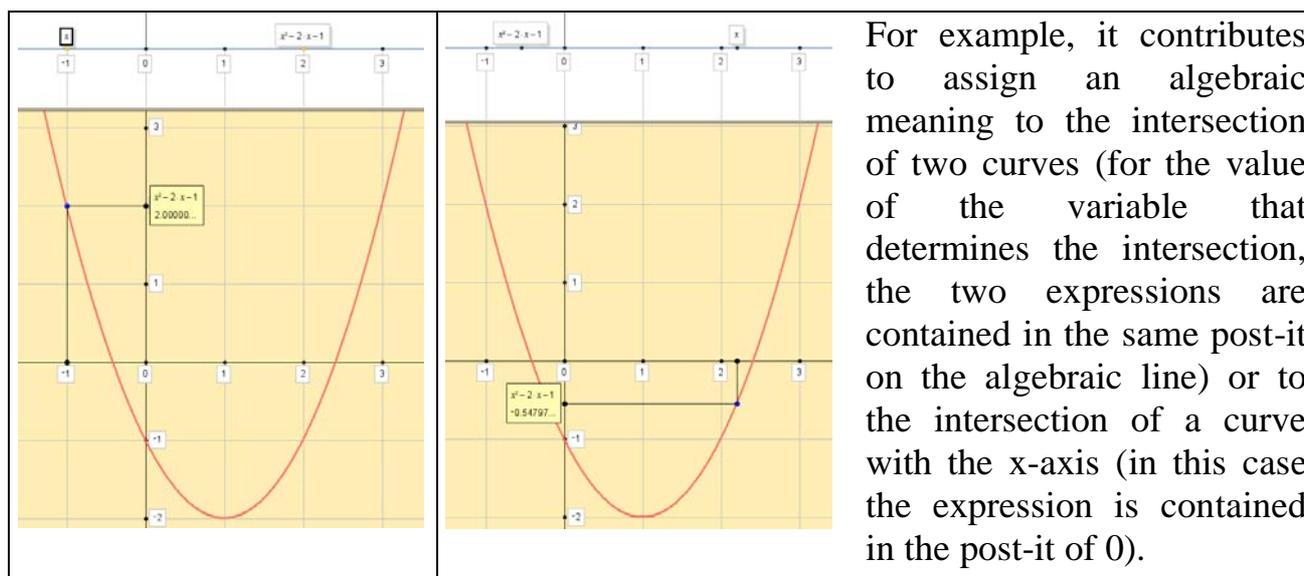
Moreover, the technique of algebraic transformation has been instrumented in this manipulator to provide non expert users with cognitive supports in the development of specific manipulative skills. A first support is the possibility to explore, through the mouse, the hierarchical structure that characterises the expression or the proposition to be manipulated. By dragging the mouse pointer over the elements of the expression or proposition at hand (operators, number, letters, brackets...), as feedback the system dynamically displays the meaningful part of the selected expression or proposition. In this way it is possible to explore all meaningful parts of an expression in the different levels of its hierarchical structure. Another feedback occurs when a part of expression has been selected. Through a pattern matching technique, the system, as feedback, activates only the rule of the interface that can be applied on the selected part of expression. This is a cognitive support that can be used to explore the connection among the transformational rules of the interface, the form on which it can be applied, and the effects provided by their applications.

Functions component

The main idea characterizing the design of the Functions component is the possibility to connect a dynamic functional relationship between variable and expression on the algebraic line with the graphical representation of the function in the Cartesian plane. As a consequence, the interface of this component has been equipped with the Algebraic line and a Cartesian plane. This idea makes this component deeply different from other environment for the representation of function in the Cartesian plane. Through a specific command and the successive selection of the independent variable of the function, an expression represented on the Algebraic line is automatically represented as graphic in the Cartesian plane. Dragging the point corresponding to the variable on the algebraic line, two representative events occur:

- on the algebraic line, the expression containing the variable moves accordingly
- on the Cartesian plane, the point defined by the pair of values of the variable and of the expression moves on the graphic as shown in the following figure.

This instrumented technique supports the integrated development of a dynamic idea of function with a static idea of such a notion (Sfard 1991). The functional relationship between variable and expression is visualized dynamically on the algebraic line through drag of the variable point, and statically in the Cartesian plane through the curve. The movement of the point along the curve during the drag of the variable on the algebraic line supports the integration of these two ideas, showing that the curve reifies the infinite couples of values corresponding to the variable and to the expression on the line. This instrumented technique can be very useful to orient the interpretation of the graphics on the Cartesian plane and to develop important concepts of algebraic nature.



For example, it contributes to assign an algebraic meaning to the intersection of two curves (for the value of the variable that determines the intersection, the two expressions are contained in the same post-it on the algebraic line) or to the intersection of a curve with the x-axis (in this case the expression is contained in the post-it of 0).

Other examples are related to the construction of meaning for the sign of a function (position of the corresponding expression on the line with respect to 0), or to order among functions (positions of the expressions on the algebraic line)

CONCLUSIONS

In this paper we have presented the main ideas that oriented the realisation of Alnuset and the choices we made to instrument specific functions of algebraic activity that can be useful for the teaching and learning of algebra. We have shown that the quantitative, symbolic and functional techniques available in the three environments of Alnuset to operate with algebraic expressions and propositions have characteristics that are deeply different from the instrumented technique of CAS. The technique of Alnuset was designed having in mind two types of users, different from the target user considered by CAS designers. The former type of user is the student who is not an expert of the knowledge domain of algebra and uses the instrumented techniques of Alnuset to learn it carrying out the algebraic activity proposed by the teacher. The latter type of user is the teacher who has difficulties to develop algebraic competencies and knowledge in students and who uses the instrumented technique of Alnuset to acquaint them with objects, procedures, relations and phenomena of school algebra. The technique of Alnuset was designed to be easily controlled during the solution of algebraic tasks, to produce results that can be easily interpreted and to mediate the interaction and the discussion on the algebraic meaning involved in the activity. The techniques of Alnuset structure a new phenomenological space where algebraic objects, relations and phenomena are reified by means of representative events that fall under the visual, spatial and motor perception of students and teachers. This contributes to provide an appropriate balance between the pragmatic and epistemic values of the techniques made available by Alnuset. In the phenomenological space determined by the use of the instrumented technique of Alnuset algebra can become a matter of investigation as evidenced by Trgalova et al. (WG4) and Pedemonte (WG2) of CERME6.

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CASYOPÉE IN THE CLASSROOM: TWO DIFFERENT THEORY-DRIVEN PEDAGOGICAL APPROACHES

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The ReMath project is a European project that addresses the task of integrating theoretical frames on mathematical learning with digital technologies at the European level. A specific set of six dynamic digital artefacts (DDA) has been currently developed, reflecting the diversity of representations provided by ICT tools. Here we consider the DDA Casyopée which was experimented in two different countries: Italy (Unisi team) and France (Didirem team). The paper focuses on the influence of the theoretical frames in the design of these Teaching Experiments.

PROBLEMATIC OF THE REMATH PROJECT

The project focuses on the primary and secondary school level giving a balanced attention to both teachers and students and incorporating a range of innovative and technologically enhanced traditional representations. Specific attention is given to cultural diversity: seven teams from four countries are involved in this project. The work is based on evidence from experience involving a cyclical process of

- a) developing six state-of-the-art dynamic digital artefacts for representing mathematics involving the domains of Algebra, Geometry and applied mathematics,
- b) developing scenarios in a common format for the use of these artefacts for educational added value,
- c) carrying out empirical research involving *cross-experimentation* (Cerulli et al. 2008) in realistic educational contexts, aiming at enhancing our understanding of meaning-making through representing with digital media, in particular by providing new insight into means of using technologies to support learning, and into learning processes in relation to the use of technologies.

Many recent studies highlight the existence of a multiplicity of theoretical frameworks for addressing those themes, and there is a shared increasing need of overcoming the resulting fragmentation (Artigue, 2007). This need is also felt within ReMath project, in which a variety of educational paradigms is present. The issue is addressed through the development of specific methodological tools, some of which are drawn and re-elaborated from the experience of TELMA project (Cerulli et al., 2008).

In this paper we present two different Teaching Experiments designed and carried out within ReMath project, respectively by Didirem team of the University Paris 7 (France), and by Unisi team of the University of Siena (Italy). Both the TEs were

designed around the use of the software Casyopée (partly developed within the project). After describing the main features of Casyopée (exploited in the Teaching experiments) we will focus on the design of the Teaching Experiments, and we will compare them relying on the construct of Didactical Functionality (Cerulli, Pedemonte and Robotti, 2006). Though it would be interesting, a discussion on the actual implementation of the plans in classroom is out of the goals and of the possibilities of the present paper.

THE CONSTRUCT OF DIDACTICAL FUNCTIONALITY

The construct of Didactical Functionality is meant to provide a minimal common perspective, hopefully independent from specific theoretical frameworks, to frame diverse approaches (possibly depending on theoretical references) to the use of ICT tools in mathematics education, as well as the theoretical reflections regarding the actual use of ICT tools in given contexts.

By Didactical Functionality of an ICT tool, one means the system constituted by three interrelated poles: a set of features of the tool, a set of educational goals, and the modalities of employing the specified features of the tool for achieving the envisaged educational goals.

Trivially, through the construct of Didactical Functionality one intends to acknowledge that an ICT tool (or part of it) can be used in different ways for achieving different educational goals, that is one can design or identify different Didactical Functionalities of a given tool. In particular different theoretical perspectives can lead to designing different Didactical Functionalities of a given tool.

THE DDA CASYOPEE

The DDA Casyopée (Lagrange and Chiappini, 2007) is built as an open problem-solving environment with the aim of giving students a means to work with algebraic representation, progressively acquiring control of the sense of algebraic expressions and of their transformations. Functions are the basic objects in Casyopée. Using this tool, students can explore and prove properties of functions. Casyopée takes into account the potentialities that Computer Algebra Systems offer to teaching and learning: going beyond mere numerical experimentation and accessing the algebraic notation; focusing on the purpose of algebraic transformations rather than on manipulation and connecting the algebraic activities. It is expected that students will make sense of algebraic representations by linking these with representations in these domains. See below a screen copy on the algebraic representations provided by Casyopée, it splits into two windows: a symbolic one and a graphical one.

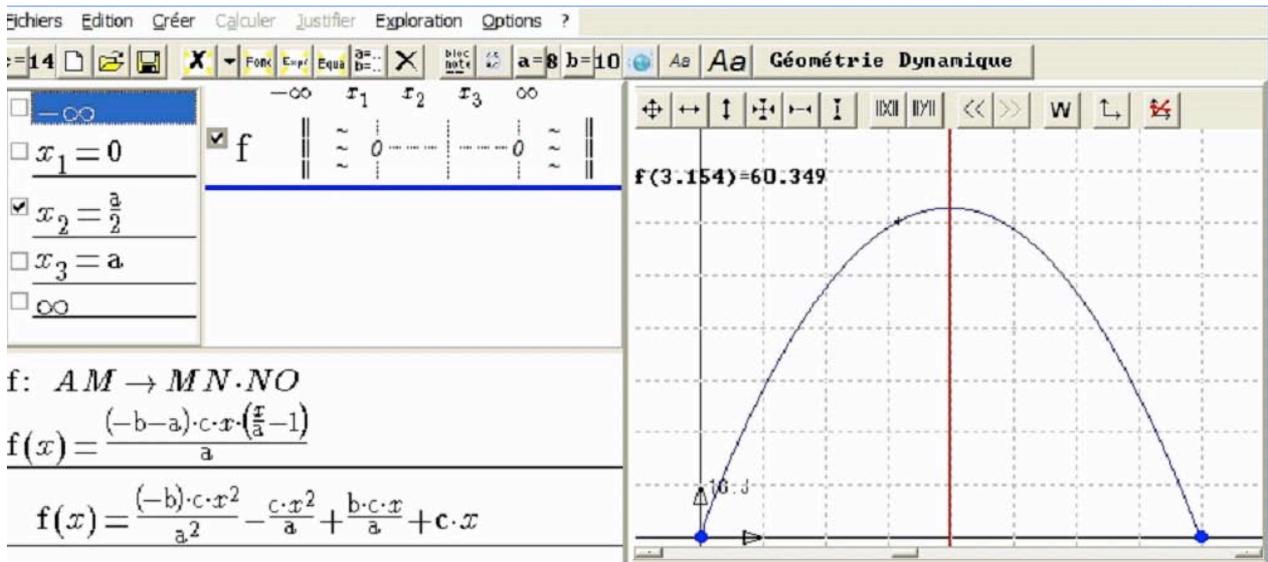


Figure 1: the algebraic setting in Casyopée

In the Remath project, Casyopée has been extended with a geometrical module. The aim is to explore what can be an interesting cooperation between a geometrical problem and its analytic treatment. The goal is not to develop a whole geometric dynamic environment but rather to see how geometric and analytic environments can articulate each other. For instance, a geometrical figure can be a domain to experiment with geometrical calculations. In the screenshot below, students can ask for the measure of the area of the rectangle MNO ^P. Then an algebraic model can be built choosing one of the measures as an independent variable and the other as a dependant variable. Properties of the dependency can be conjectured and proved: they take sense both in the algebraic and in the geometrical settings.

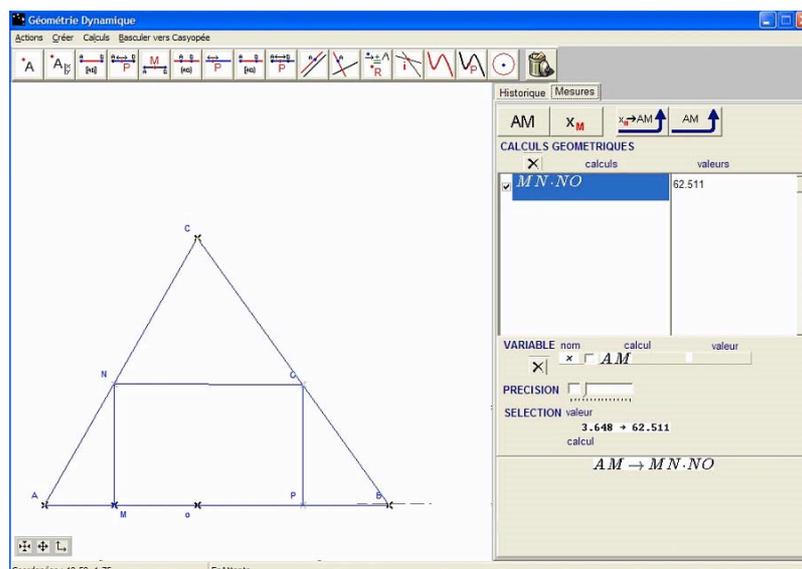


Figure 2: the geometrical window in Casyopée

The main specificity of Casyopée among other dynamic geometrical artefacts is to connect geometric and algebraic approaches. More precisely, the geometrical frame

allows one to consider a geometric calculation and to export it in the algebraic environment. This transfer is allowed by choosing an adequate variable for the geometrical situation. At this point, Casyopée gives a feedback on the choice of this independent variable.

The representations offered by Casyopée have been thought to be close to institutional ones. Casyopée allows students to work with the usual operations on functions such as algebraic operations, analytic calculations and graphical representations. The geometric environment offers commands usually available in other dynamic geometry environments such as creating fixed and free geometrical objects (points, lines, circles, curves)

UNISI AND DIDIREM PEDAGOGICAL PLANS

In the introduction we recalled that different specific methodological tools have been developed within ReMath for fostering the comparability of studies dealing with the use of ICT tools in mathematics education. A new conceptual model of the pedagogical scenario, called Pedagogical Plan (Bottino et al. 2007), is one of those methodological tools. A Pedagogical Plan has a recursive hierarchical structure: each pedagogical plan is conceived as a tree whose nodes and leaves are pedagogical plans themselves. Several components are attached to each pedagogical plan: including the articulation of the educational goals, of the class activities, the specification of the features of the ICT tool used and how they are used, and of the rationale underpinning the whole pedagogical plan and of the theoretical frames inspiring it. A web-based tool (Pedagogical Plan Manager, PPM) has been also developed for supporting teams in designing their pedagogical plans.

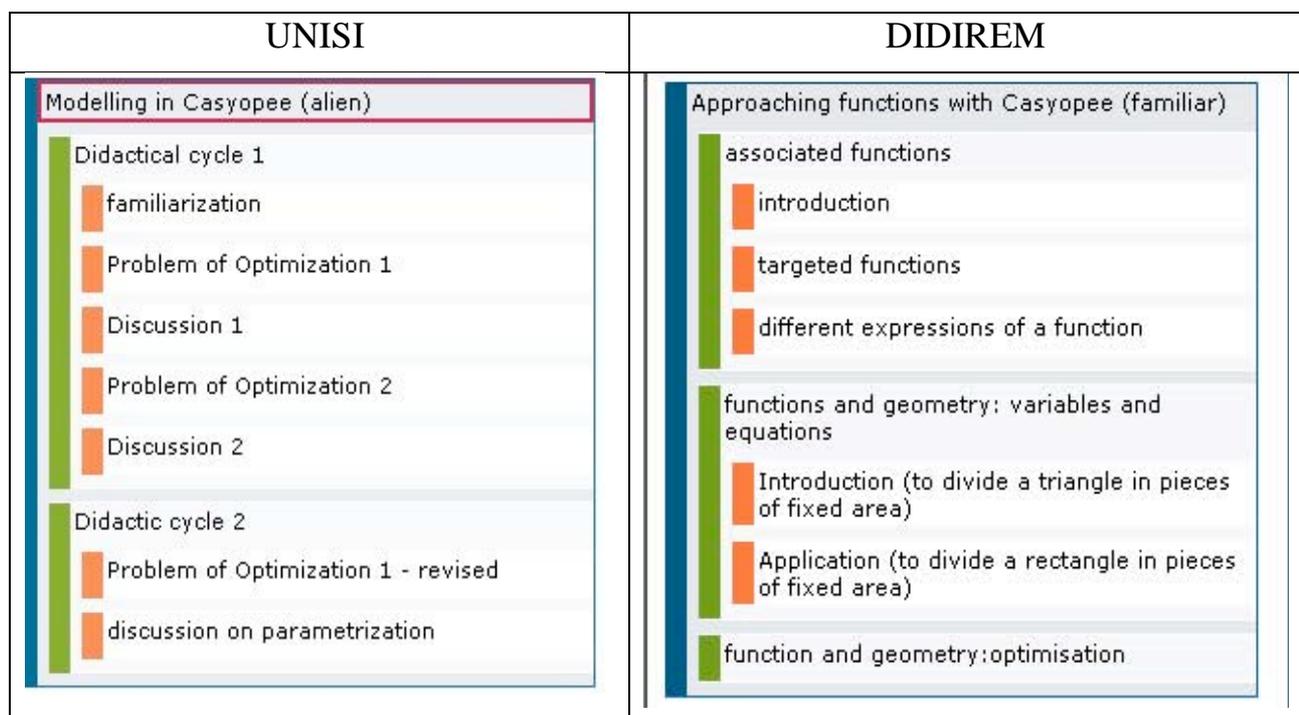


Figure 3: synthetic view of Unisi and Didirem pedagogical plans in the PPM

Figure 3 displays a screenshot from the PPM, and it is meant to provide an overview of the structures of the pedagogical plans designed by the Unisi and Didirem teams.

Details of the Unisi pedagogical plan

The Unisi pedagogical plan is inspired by the Theory of Semiotic Mediation (Bartolini Bussi and Mariotti, 2008) drawn from a Vygotskian perspective. This theory guided both the specification of the educational goals (starting from an analysis of Casyopée) and the overall structure of the planned activities.

The designed educational goals are (a) to foster the evolution of students' personal meanings towards the mathematical meanings of function as co-variation. That regards also the notions of variable, domain of a variable... and (b) to foster the evolution of students' personal meanings towards mathematical meanings related to the algebraic modelling of geometrical situations.

Students are expected to have already received some formal teaching on the notions of variable, function and graph of a function, and on its graphical representation in a Cartesian plane. Moreover, a common experience of researchers and teachers is that meanings related to those notions are rarely elaborated in depth. The aim is to mediate and weave those meanings in the more global frame of modelling.

Hence, the pedagogical plan is not meant to help students become able to use Casyopée for accomplishing given tasks, but instead to foster the students' consciousness-raising of the mathematical meanings at stake.

The whole pedagogical plan is structured in cycles entailing: students' pair or small group activity with Casyopée for accomplishing a task, students' personal rethinking of the class activity (through the request to students of producing individual reports on that activity), class discussion orchestrated by the teacher.

The familiarization session is designed as a set of tasks aims at providing students with an overview of Casyopée features and guiding students to observe and reflect upon the "effects" of their interaction with the tool itself, e.g.:

Could you choose a variable acceptable by Casyopée and click on the "validate" button? Describe how did the window "Geometric Calculation" change after clicking on the button. Which new button appeared?

Besides familiarization, the designed activities with Casyopée consist of coping with "complex" optimization problems formulated in a geometrical setting and posed in generic term, e.g.:

Given a triangle, what is the maximum value of the area of a rectangle inscribed in the triangle? Find a rectangle whose area has the maximum possible value.

The aim is to elaborate on those problems so to reveal and unravel the complexity and put into evidence step by step the specific mathematical meanings at stake.

According to the designed pedagogical plan, the teacher plays the delicate role of guiding students to unravel such complexity and to make the targeted mathematical meanings emerge. The main tool for the teacher to achieve this objective, is the orchestration of the class discussions. The development of a class discussion cannot be completely foreseen *a priori*, it should be designed starting from the analysis of students' actual activity with Casyopée and of the reports they produce, and it would still depend on extemporary stimuli. Nevertheless in the design Unisi team tried to anticipate possible development of the pedagogical plan and to plan some kind of possible canvas for the teachers for managing class discussions.

The pedagogical plan is intended for scientific high schools or technical institutes, grade 12 or 13, and can be implemented over approximately 11 school hours.

Details of the Didirem pedagogical plan

The Didirem pedagogical plan aims to help students construct or enrich knowledge in the following areas: meaning of functions as algebraic objects and meaning of functions as means to model a co variation in geometric and algebraic settings. It is intended for scientific high schools grade 11 or 12 and has been implemented in ordinary classes during approximately 10 school hours. It is inspired both by the Instrumental Approach (Artigue, 2002), the Theory of Situation (Brousseau, 1997) and the Theory of Anthropologic Didactic (Chevallard, 1999).

Specific importance is given to the construction of tasks with an adidactical potential, where students can choose different variables for exploring functional dependencies, and to the connection between algebra and geometry. This connection is supported in Casyopée by geometric expressions that allow expressing magnitudes in a symbolic language mixing geometry and algebra.

The pedagogical plan is built around three main types of tasks:

- First session: finding target quadratic functions by animating parameters (five different tasks according to the semiotic forms used for these functions):

Lesson 1: Introducing associated functions (a function g is associated to a function f if it is defined by a formula like $g(x)=af(x)+b$ or $f(ax+b)$ or similar)

Lesson 2: Target Functions (functions that can be graphed but whose expression is not known; each student have to guess the function graphed by his/her partner)

Lesson 3: Different expressions of quadratic functions

So students should consolidate: the meaning of variable, the distinction between variable and parameter, the meaning of function of one variable with several registers of semiotic representation and the fact that a same function may have several algebraic expressions. The new notion of associated function is worked-out during this session.

- Second session: creating a geometrical calculus as a model of a geometrical situation to solve a problem of relationships between areas, manipulation to experiment co variation between two geometrical variables:

Lesson 4: To divide a triangle in pieces of fixed area

Lesson 5: Application; dividing a rectangle into figures of fixed area

This way students can enhance their knowledge on co variation and develop the ability to experiment and anticipate in a dynamic geometrical situation, and the ability to model a geometric situation through geometric calculations.

- Third session: creating a function as a model of a geometrical situation to solve an optimization problem.

Lesson 6: solving a problem of optimisation in geometric settings by way of algebraic modelling.

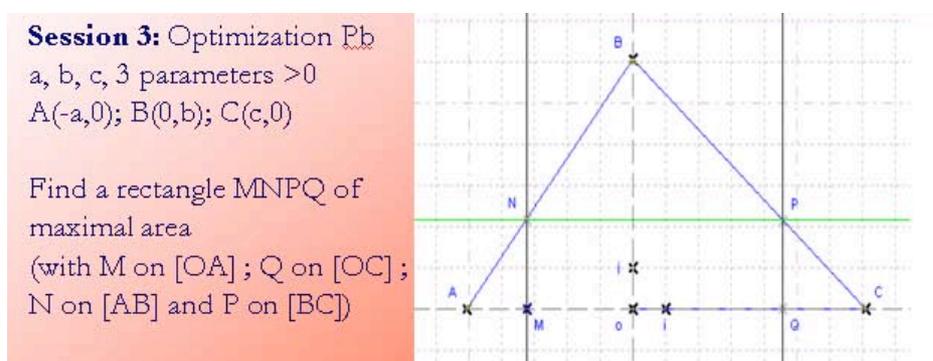


Figure 4: statement of the session 3 in Didirem pedagogical plan

This problem allows both to reinvest abilities to use the DDA, previous knowledge on associated functions and to introduce the notion of optimum in a geometrical situation.

COMPARISON OF THE UNISI AND DIDIREM APPROACHES USING THE CONSTRUCT OF DIDACTICAL FUNCTIONALITY

The two pedagogical plans, described in the previous sections, evidently share some characteristics but also have apparent deep differences. In this section we use the frame provided by the construct of Didactical Functionality to develop a more systematic comparison between the two pedagogical plans.

Tool Features

The two pedagogical plans are not generally centred on the use of the same DDA, but more specifically on the use of the same DDA features. In fact both exploit especially

- (a) features of the dynamic geometry environment: the commands for creating fixed, free or constrained points, for dragging free or bonded points, for creating points with parametric coordinates, and the corresponding feedbacks of the DDA;

- (b) features of the geometric calculation environment: the commands for creating “geometric calculation” associating numbers to geometrical objects, for choosing (independent) variables, for creating function between the selected variable and calculation, and the corresponding feedbacks;
- (c) features of the algebraic environment, including the commands for displaying and exploring graphs of functions, for creating and manipulating parameters, for manipulating the algebraic expressions of functions, and the corresponding feedbacks.

Educational Goals

Different educational goals are associated to the use of those features. More precisely, one can recognize that both pedagogical plans share a common focus on some mathematical notions: function (in particular, conceived as co-variation), variables (independent and dependent) and parameters. Moreover the two pedagogical plans present, among other tasks, two optimization problems sharing the same mathematical core (see sections...). But, besides those surface similarities, there are profound differences.

Other Unisi educational goals are to mediate and weave meanings, related to the notions of function, variable and parameter. With that respect the Unisi team assumes, on the one hand, that those notions are familiar for students, and, on the other hand, that those notions are not elaborated in depth. Hence the Unisi pedagogical plan aims at helping students gain a deeper consciousness of the mathematical meanings at stake and re-appropriate them in the more global frame of modelling. In addition the Unisi objective includes the shared and decontextualized formulation of the different mathematical notions in focus.

The Didirem objectives are mainly to use potentialities of representations offered by Casyopée to introduce some new mathematical knowledge. This knowledge has been chosen for two main reasons: its affordance to the French curriculum and the importance to be studied in several frames of representations.

Modalities of employment

In accordance with the different objectives and the different pedagogical culture, the modalities of use are different as well.

The Unisi pedagogical plan has an iterative structure. Students' activity with Casyopée alternates with class discussions, after each session students are required to produce individual reports on the performed activities. This structure is meant to foster students' generation of personal meanings linked to the use of the DDA and their evolution towards the targeted mathematical meanings together with the students' consciousness-raising of the mathematical meanings at stake. That process is constantly fuelled by the teacher, whose role is crucial. Accordingly the teacher's role is explicitly taken into account in the design of the pedagogical plan, which provides with hints for the possible actions. The tasks used are optimization problems

set in a geometrical frame. Their solution and the reflection on these solutions are fundamental steps towards the achievement of the designed educational goals. Also the familiarization with the DDA has to be considered within that perspective: as already mentioned, it aims at making students observe and reflect upon the "effects" of their interaction with the DDA itself. *Ad hoc* tasks are designed for that purpose.

Instead, the Didirem team pays specific attention to a progressive use of the DDA combining artefact and mathematical knowledge. Indeed, students work only in the algebraic window during section 1, then only in the geometrical windows in section 2; finally section 3 gives an opportunity to reinvests the knowledge in the two environments. Moreover, all the tasks proposed are mathematical ones and are elaborated in order to allow students make progress alone working on the problem and to construct their new knowledge thanks the feedbacks.

CONCLUSION

Those differences can be strongly related with the different theoretical perspectives adopted by the two teams.

The Unisi team has mainly structured its pedagogical plan according to the Theory of Semiotic Mediation which inspired both the specification of the educational goals and the organization of the activities in iterative cycles. In particular the Theory of Semiotic Mediation led the Unisi team to devote attention towards the design of the teacher's action in the pedagogical plan. In fact, the teacher plays a crucial role throughout the whole pedagogical plan, especially for fostering the evolution of students' personal meanings towards the targeted mathematical meanings and facilitating the students' consciousness-raising of those mathematical meanings.

Instead, the Didirem team splits its theoretical approach into several theoretical frames which shape their pedagogical plan: the Instrumental Approach (Artigue, 2002), the theory of Situation (Brousseau, 1997) and at last the theory of anthropologic didactic (Chevallard, 1999). The first frame aims to go further than a simple familiarization with the DDA and to help the students constructing a mathematical instrument. This process goes hand in hand with the learning process. The last optimization problem is used to evaluate the progress of this process. The process is accurately designed through a careful choice of mathematical tasks, with an adidactical potential, whereas the definition of the teacher's actions and role escapes the design of the PP. Finally, the TAD is called upon to manage instrumental distance between institutional and instrumental knowledge.

No doubt that these approaches are complementary. Each team might benefit from this collective work to improve its pedagogical plan in the future. For instance, the Didirem team plans to pay more attention to the teacher's role during the pedagogical plan conception. Nevertheless, the objective is not to elaborate a wide common consensual theoretical frame, but rather to go in depth in the clarification of didactical functionalities, in a shared language.

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NAVIGATION IN GEOGRAPHICAL SPACE

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This study is part of the ReMath project (Remath' – Representing Mathematics with Digital Media FP6, IST-4, STREP 026751 (2005 – 2008), <http://remath.cti.gr>. Twenty four 10th Grade students participated in a constructivist teaching experiment, the aim of which was to investigate children's constructions of mathematical meanings concerning the concept of function while navigating within 3d large scale spaces. The results showed that the utilization of the new representations provided by the dynamic digital media such as Cruislet could reform the way that mathematical concepts are presented in the curricula and possibly approach these mathematical notions through meaningful situations. The new representations provide the opportunity to introduce and study mathematical notions not as isolated entities but rather as interconnected functionalities of meaningful real – life situations.

Functions are a central feature of mathematics curricula, both past and present. Many research studies indicate students' difficulty in understanding the concept of functions. This difficulty comes from a) the static media used to represent the concept, b) the introduction of function mainly as a mapping between sets in conventional curricula, c) the use of formalisation and function graphs as the only representations. With digital media, students can dynamically manipulate informal representations of function defined as co-variation and rate of change, which is an interesting and powerful mathematical concept. Tall(1996) points out a fundamental fault-line in "calculus" courses which attempt to build on formal definitions and theorems from the beginning. Moreover, he suggests that enactive sensations of moving objects may give a sense that "continuous" change implies the existence of a "rate of change", in the sense of relating the theoretically different formal definitions of continuity and differentiability. The enactive experiences provide an intuitive basis for elementary calculus built with numeric, symbolic and visual representations.

The 'Cruislet' environment is a state-of-the-art dynamic digital artefact that has been designed and developed within the Eu ReMath project. It is designed for mathematically driven navigations in virtual 3d geographical spaces and is comprised of two interdependent representational systems for defining a displacement in 3d space, a spherical coordinate and a geographical coordinate system. We consider that the new representations enabled by digital media such as Cruislet can place mathematical concepts in a central role for both controlling and measuring the behaviours of objects and entities in virtual 3d environments. The notion of *navigational mathematics* is used to describe the mathematical concepts that are embedded and the mathematical abilities the development of which is supported within the Cruislet microworld. In this study we focus on how students using

spherical and geographical systems of reference in Cruislet construct meanings about the concept of function.

THEORETICAL FRAMEWORK

A number of research studies suggest that students of all grades, even undergraduate students, have difficulties modelling functional relationships of situations involving the rate of change of one variable as it continuously varies in a dependent relationship with another variable (Carlson et al., 2002; Carlson, 1998, Monk & Nemirovsky, 1994). This ability is essential for interpreting models of dynamic events and foundational for understanding major concepts of calculus and differential equations. On the other hand, the VisualMath curriculum (Yerushalmy & Shternberg, 2001) is an example of a function based curriculum that involves the moving across multiple views of symbols, graphs, and functions. VisualMath uses specially designed software environments such as simulations' software, or other modelling tools that include dynamic forms of representations of computational processes. Yerushalmy (2004) suggests that such emphasis on modeling offers students means and tools to reason about differences and variations (rate of change). Moreover, Kaput and Roschelle (1998) using computer simulations study aspects of calculus at an earlier stage. These simulations (MBL tools), permit the study of change and the ways it relates to the qualities of the situation. In their study Nemirovsky, Kaput and Roschelle (1998) show that young children can use the rate of change as a way to explore functional understanding. In studying the process of the understanding of dynamic functional relationships, Thompson (1994) has suggested that the concept of rate is foundational.

Confrey and Smith (1994) choose the concept of rate of change as an entry to thinking about functions. They introduce two general approaches to creating and conceptualizing functional relationships, a correspondence and a covariation approach. They suggest that “a covariational approach to functions makes the rate of change concept more visible and at the same time, more critical (p. 138). They explicate a notion of covariation that entails moving between successive values of one variable and coordinating this with moving between corresponding successive values of another variable.

Moreover, Carlson, Larsen and Jacobs (2001) stress the importance of covariational reasoning as an important ability for interpreting, describing and representing the behavior of dynamic function event. They consider covariational reasoning to be the cognitive ability involved in coordinating images of two varying quantities and attending to the ways in which they change in relation to each other. On the same line, Saldanha and Thompson (1998) introduced a theory of developmental images of covariation. In particular, they considered possible imagistic foundations for someone's ability to see covariation. Carlson et al (2001) in their study exploring the role of covariational reasoning in the development of the concepts of limit and

accumulation, suggest a framework including five categories of mental actions of covariational reasoning:

1. An image of two variables changing simultaneously
2. A loosely coordinated image of how the variables are changing with respect to each other
3. An image of an amount of change of one variable while considering changes in discrete amounts of the other variable
4. An image of the average rate-of-change of the function with uniform increments of change in the input variable
5. An image of the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function

The proposed covariation framework contains five distinct developmental levels of mental actions. Using this particular framework we will try to classify students' covariational reasoning while studying navigation within the context of Cruislet microworld. We consider navigation as a dynamic function event. The function's independent variable is the geographical coordinates of the position of the first aeroplane, which students are asked to navigate, while the dependent variable is the geographical coordinates of the position of the second aeroplane.

Our approach to learning promotes investigation through the design of activities that offer a research framework to investigate purposeful ways that allow children to appreciate the utility of mathematical ideas (Ainley & Pratt, 2002). In this context, our approach is to design tasks for either exclusively mathematical activities or multi-domain projects containing a mathematical element within the theme which can be considered as marginalized or obscure within the official mathematics curriculum (Kynigos & Yiannoutsou, 2002, Yiannoutsou & Kynigos, 2004).

TASKS

In the tasks that are included in this teaching experiment, students actually engage with the study of the existence of a rate of change of the displacements of the airplanes which are defined in the geographical coordinate system. In particular the displacements of two airplanes are relative according to a linear function. This function will be hidden and the students will have to guess it in the first phase of the activity based on repeated moves of aeroplane A and observations of the relative positions and moves of planes A and B. The second phase, the students will be able to change the function of relative motion and play games with objectives they may define for themselves such as move plane A from Athens to Thessaloniki and plane B from Athens to Rhodes and then to Thessaloniki in the same time period.

This scenario is based on the idea of half – baked games, an idea taken from microworld design (Kynigos, 2007). These are games that incorporate an interesting

game idea, but they are incomplete *by design* in order to encourage students to change their rules. Students play *and* change them and thus adopt the roles of both player and designer of the game (Kafai, 2006).

Initially, students are asked to study the relation between the two aeroplanes, the rate of change of their displacements and consequently find the linear function (decode the rule of the game). In order to decode "the rule of the game", they should give various values to coordinates (Lat, Long, Height) that define the position of the first plane. They will be encouraged to communicate their observations about the position of the second plane to each other and form conjectures about the relationship between the positions of the two aeroplanes.

In the second phase students are encouraged to build their own rules of the game by changing the function of the relative displacements of the two aeroplanes.

METHODOLOGY

The research methodology is a constructivist teaching experiment along the same lines as described by Cobb, Yackel and Wood (1992). The researcher acts as a teacher interacting with the children aiming to investigate their thinking. The researcher, reflecting on these interactions, tries to interpret children's actions and finally forms models-assumptions concerning their conceptions. These assumptions are evaluated and consequently either verified or revised.

Twenty four (24) students of the 1st grade of upper high school, (aged 15-16 years old) participated in this experiment. Students worked in pairs in the PC lab. Each pair of students worked on the tasks using Cruislet software.

The data consists of audio and screen recordings as well as students' activity sheets and notes. The data was analyzed verbatim in relation to students' interaction with the environment. We have focused particularly on the process by which implicit mathematical knowledge is constructed during shared student activity. As a result, in our analysis we use students' verbal transcriptions as well as their interaction with the provided representations displayed on the computer screen.

ANALYSIS

While students were interacting with the Cruislet environment according to the tasks, several meanings emerged regarding the concept of function. We categorise these meanings in clusters that rely upon the concept of function. In particular, there are two major categories:

Domain of numbers

Students navigating an aeroplane in the 3d map of Greece realized that the domain of the geographical coordinates is actually a closed group. The 3d map of Greece is a geographical coordinate system with specific borders. The investigation of the range

of the geographical borders as the domain of the function became the subject of study and exploration through the use of the Cruislet functionalities. In particular, students exploited the two different systems of reference and, experimenting with the values of the geographical coordinates, they define the range of the latitude – longitude values. This specific range of values has been considered as the domain of the functions according to which the displacements of the aeroplanes are relative. Although students didn't refer to the values as the domain of the function, we interpret their involvement in finding them, as a mathematical activity regarding the domain of the function.

Students experimented by giving several values to the geographical coordinates of the airplane's position defining at the same time the range of the coordinates' values. In the following episode students are trying to find out the reason for not placing the airplane in a given position.

S1: Why?? It doesn't accept any value. (they gave values in procedure fly1 and the airplane couldn't go).

R: Do you remember what values the lat long coordinates have?

Isn't lat equals 58 isn't correct? (she also speaks to the next team)

S1: It doesn't accept 32 20 100 either.

S2: Greece hasn't got value 20 (student from another team speak ironically to him)

S1: Why? Was the 58 you used correct?

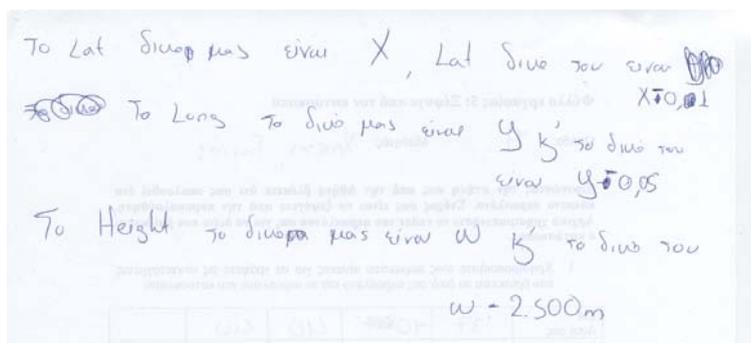
An interesting issue related to the domain of the function, is that the provided representations, i.e. the result of the aeroplane's displacement displayed on the screen, helped students realize that the domain of numbers of the two aeroplanes displaced in relative positions, are strongly dependent. For instance when the first moved to a given position, the second one couldn't go anywhere, but the domain of values was restricted by the first position. In the following episode students realized that the 2nd aeroplane didn't follow them when they flew at a low height. The episode is interesting as it indicates the way students realize the domain of geographical coordinate values that the first aeroplane can take in relation to the other one.

S1: There are some times that it (meaning the other aeroplane) can't follow us.

R: Where? When?

S1: When I'm getting into the sea.

We could say that the characteristics of Cruislet software, such as the visualization of the results of the objects' displacements on the map, acted as a mediator in students' engagement with the domain of function. We have to mention that although the modalities of use of Cruislet software and the communication within the groups didn't reveal that students realized or mentioned anything regarding the concept of



Translation

Our Lat is x , his Lat is $x - 0.1$

Our Long is y and his is $y - 0.05$

Our Height is ω and his is $\omega - 2500\text{m}$.

It is interesting to mention that students separated latitude and longitude coordinates on the one hand and that of height on the other as they were trying to decode the hidden functional relationship between the airplanes' height coordinates. In particular, they didn't encounter difficulties in decoding latitude and longitude relationship in contrast to their attempts to find the height dependency. Although all three functions regarding coordinates were linear, students conceived the functional relationship between height mainly as proportional, in contrast to latitude and longitude that were comprehended as linear, from the beginning. In the following episode, students endeavor to apply the rate of change of the function to decode the height relationship. As they thought the height coordinates had a proportional relationship, they suggested carrying out a division to find it.

S2: When we go up 1000, he goes up 1000.

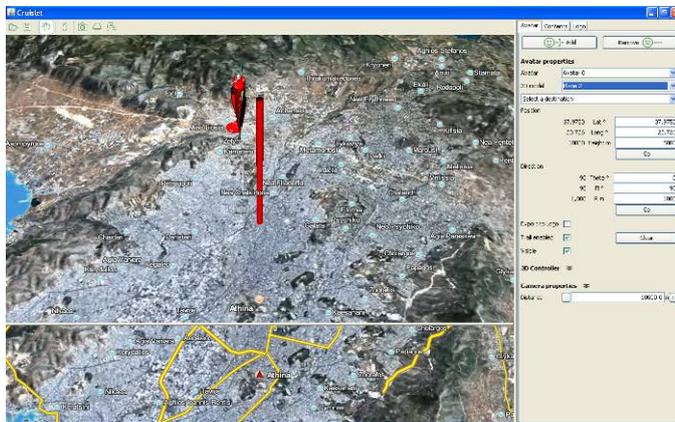
R: Do you mean that if we go from 7000 to 8000 he goes from... let's say 2500 to 3500.

S2: He is at... 3000. No. Give me a moment. At 8000 he was at 5500. At 7000 he was at 4500. At 5000 he is as 2500. And then....

S1: We could do the division to see the rate.

An interesting example was the cases of the variation of the height of the aeroplane every time they pushed the button 'go' in spherical coordinates, when they wanted to make a vertical displacement. In particular, by defining the vector of a vertical upward displacement, students observed that height was the only element that changed in the position of the displacement. Through a number of identical displacements students identified and expressed verbally, symbolically and graphically the interdependency between direction functionality and the height of the aeroplane. Students' reasoning: "the more times we push the button GO the higher the aeroplane goes", suggests that students developed a covariational reasoning ability similar to the second level proposed by Carlson et al (2001) of how the

variables change with respect to each other. Moreover, the retrospective symbolic type developed by students ($h_2 = h_1 + 1000$) indicates that they realized that the rate of change of the height is constant. In the following figures we can see the result displayed on the screen (figure 1) as well as students' writings on the activity sheet (figure 2).



Students' actions:

1. Define the spherical coordinates ($\theta = 0$ $\phi = 90$ $R = 1000$).
2. Push the "Go" button in "Avatar properties" tab resulting in the vertical displacement of the aeroplane.
3. Watch the displacement of the aeroplane on the GUI.
4. Focus on the changes of the height coordinate.

Figure 1

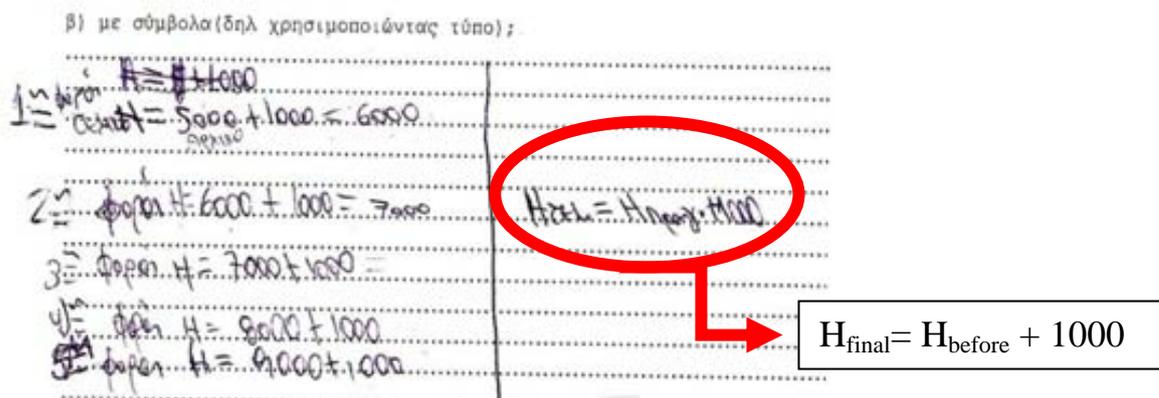


Figure 2

The provided representations of Cruislet software became a vehicle to engage students with concepts related to the concept of function and their expression in a mathematical way. The result of airplanes' displacements on the screen, gave them the chance to realize the dependent relation in 'visual terms' and then express it in mathematical terms. We believe that the results are mainly based on the way that these characteristics were used in the task activity. In particular, the activity was based on the idea of the 'Guess my function' game and the dependent relationship, (built in Logo programming language), was hidden at first. Due to this choice, students focused primarily on the observation of the relative displacements and not on the Logo code underneath it. At the same time perceiving the activity as a game encourages the engagement of students with the activity.

CONCLUSIONS

The study indicated that students exploiting Cruislet functionalities can construct meanings concerning the concept of functions. The provided linked representations (spherical and geographical coordinates), as well as the functionalities of navigating in real 3d large scale spaces actually enable students to explore and build mathematical meanings of the concept of function within a meaningful context. They explore the domain of numbers of a function within a real world situation distanced from the “traditional” formal definitions. On the other hand, they built the concept of function as covariation exploring the variation of the spherical and geographical coordinates. The provided context gave students the opportunity to cope with and explore mathematical concepts at different levels. They navigate within 3d large scale spaces controlling the displacement of an avatar and develop their visualization abilities building mathematical meanings of the concept of function while at the same time they explore the mathematical concepts of spherical and geographical coordinates.

The functionalities of the new digital media such as Cruislet provide a challenging learning context where the different mathematical concepts and mathematical abilities are embedded and interconnected. The role of the teacher becomes crucial in designing mathematical tasks where students’ enactive explorations will reveal these mathematical notions and put them under negotiation. In the case of Cruislet, navigational mathematics becomes the core of the mathematical concepts that involves the geographical and spherical coordinate system interconnected with the concept of function and the visualization ability.

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MAKING SENSE OF STRUCTURAL ASPECTS OF EQUATIONS BY USING ALGEBRAIC-LIKE FORMALISM

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This paper reports on a design experiment conducted to explore the construction of meanings by 17-year-old students, emerging from their interpretations and uses of algebraic-like formalism. The students worked collaboratively in groups of two or three, using MoPiX, a constructionist computational environment with which they could create concrete entities in the form of Newtonian models by using equations and animate them to link the equations' formalism to its visual representation. Some illustrative examples of two groups of students' work indicate the potential of the activities and tools for expressing and reflecting on the mathematical nature of the available formalism. We particularly focused on the students' engagement in reification processes, i.e. making sense of structural aspects of equations, involved in conceptualising them as objects that underlie the behaviour of the respective models.

INTRODUCTION

In this paper we report on a classroom research [1] aiming to explore 17-year-old students' construction of meanings, emerging from the use of algebraic-like formalism in equations used as means to create and animate concrete entities in the form of Newtonian models. The students worked collaboratively in groups of two or three using a constructionist computational environment called "MoPiX" [2], developed at the London Knowledge Lab (<http://www.lkl.ac.uk/mopix/>) (Winters et al., 2006). MoPiX allows students to construct virtual models consisting of objects whose properties and behaviours are defined and controlled by the equations assigned to them. We primarily focused on how students interpreted and used the available formalism while engaged in *reification* processes (Sfard, 1991), i.e. making sense of structural aspects of equations, involved in conceptualising them as objects that underlie the behaviour of the respective models.

THEORETICAL BACKGROUND

Recognising the meaning of symbols in equations, the ways in which they are related to generalisations integrated within specific equations and also the ways in which a particular arrangement of symbols in an equation expresses a particular meaning, are all fundamental elements to the mathematical and scientific thinking. Research has been showing rather conclusively that the use of symbolic formalisms constitutes an obstacle for many students beginning to study more advanced mathematics (Dubinsky, 2000). Traditional approaches to teaching equations as part of the mathematics of motion or mechanics seem to fail to challenge the students' intuitions since they usually encompass static representations such as tables and graphs which

are subsequently converted into equations. Lacking any chance of interacting with the respective representations, students fail to identify meaningful links between the components and relationships in such systems and the extensive use of mathematical expressions (diSessa, 1993). Indeed, students tend to use and manipulate physics equations in a rote manner, without understanding the concepts they convey (Larkin et al., 1980). Sherin (2001) argued that, in order to overcome this obstacle, students need to acquire knowledge elements that he termed *symbolic forms*. The acquisition of *symbolic forms* would help students make connections between an algebraic expression's conceptual content and its structure, which is considered to be crucial for the understanding, meaningful use and construction of physics equations.

In the mathematics education field, the relevant research is mainly based on the distinction between the two major stances that students adopt towards equations: the process stance and the object stance (Kieran, 1992; Sfard, 1991). The process stance is mainly related with a surface "reading" of an equation, concentrated into the performance of computational actions following a sequence of operations (i.e. computing values). In contrast, according to the object stance, an equation can be treated as an object on its own right, which is crucial to the students' development of the so-called *algebraic structure sense* (Hoch and Dreyfus, 2004), i.e. the act of being able to see an algebraic expression as an entity, recognise structures, sub-structures and connections between them, as well as to recognise possible manipulations and choose which of them are useful to perform. This development, linking procedural and structural aspects of equations, has been termed *reification* (Sfard, 1991) and has been considered to underlie the learning of algebra in general.

Recently, students' uses and interpretations of symbolic formalism in understanding mathematical and scientific ideas have been studied in relation to the representational infrastructure of new computational environments designed to make the symbolic aspect of equations more accessible and meaningful to children, especially through the use of multiple linked representations (Kaput and Rochelle, 1997). Adopting a broadly constructionist framework (Harel and Papert, 1991), we used a computer environment that is designed to enhance the link between formalism and concrete models, allowing us to study the ways in which the use of formalism, when put in the role of an expression of an action or a construct (a model), can operate as a mathematical representation for constructionist meaning-making. Our central research aim was to study students' construction of meanings emerging from the use of mathematical formalism when engaged in reification processes. We mainly focused on the development of their understanding on the structure of an equation based primarily on the conception of it as a system of connections and relationships between its component parts.

THE COMPUTATIONAL ENVIRONMENT

MoPiX (Winters et al. 2006) constitutes a programmable environment that provides the user the opportunity to construct and animate in a 2d space, models representing

phenomena such as collisions and motions. In order to attribute behaviours and properties to the objects taking part in the animations generated, the user assigns to the objects equations that may already exist in the computational environment's Equations Library or equations that she constructs by herself.

Figure 1 shows a red ball performing in the MoPiX environment a combined motion both in the vertical and horizontal axis, leaving a green trace behind. As one may observe, the equations attributed to the object incorporate formal notation symbols (V_x , x , t) as well as programming–natural language utterances (ME, appearance, Circle). However, their main characteristic is that they constitute functions of time, as it is stated by the second argument on the parentheses on their left side. For example, the horizontal motion equations attributed to the ball define the object's: horizontal position at the 0 time instance (1), horizontal position at any time instance (2), the horizontal velocity at the 0 time instance (3), the horizontal velocity at any time instance (4) and the horizontal acceleration at any time instance (5). The MoPiX environment constantly computes the attributes given to the objects in the form of equations and updates the display, generating on the screen the visual effect of an animation.

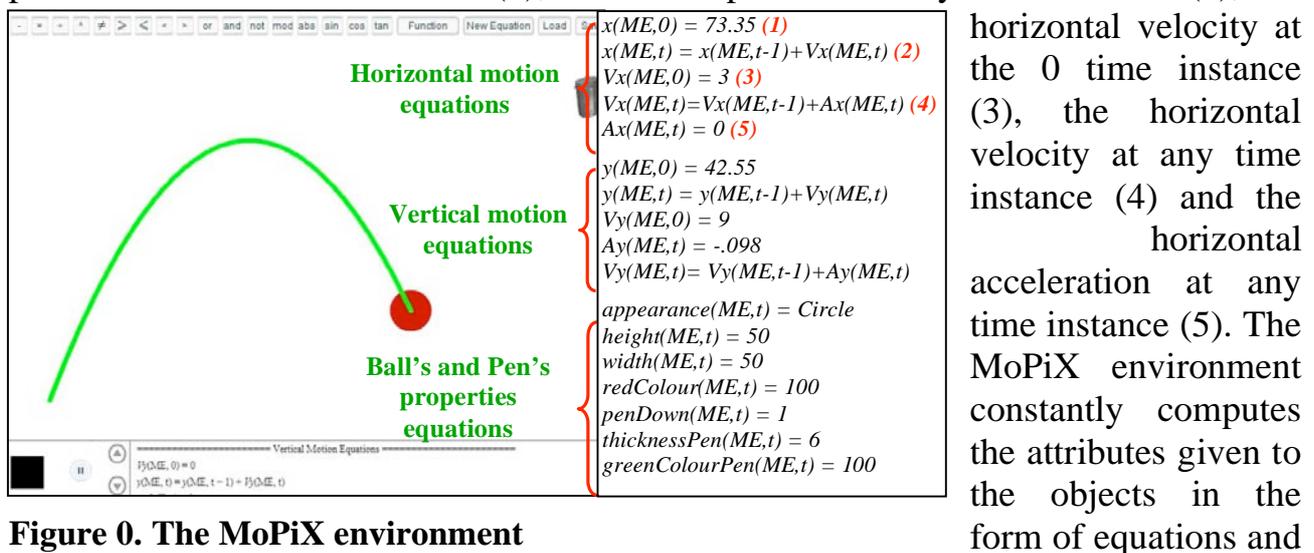


Figure 0. The MoPiX environment

updates the display, generating on the screen the visual effect of an animation.

Some specific features of MoPiX, underlying the novel character of the representations provided, may offer students opportunities to further appreciate utilities of the algebraic activity around the use of equations. The first of these features is that MoPiX offers a strong visual image of equations as containers into which numbers, variables and relations can be placed. The meaningful use of the environment may allow students to easily make connections between the structure of an equation and the quantities represented in it. The second feature of MoPiX is that it allows the user to have deep structure access (diSessa, 2000) to the models animated. The equations attributed to the objects and underpin the models' behaviour do not constitute "black boxes", unavailable for inspection or modifications by the user (for a discussion on black and white box approaches see Kynigos 2004). The third feature of MoPiX is that the manipulations performed to a model's symbolic facet (e.g. changing a value or removing an equation from the model) produce a visual result on the Stage, from which students can get meaningful feedback. "Debugging" a flawed animation demands students' engagement in a back and forth process of constructing a model predicting its behaviour, observing the animation generated, identifying the equations that are responsible for the "buggy" behaviour

and specifying which and how particular parts need to be fixed.

TASKS

For the first phase of the activities we developed, using exclusively “Library” equations, the “One Red Ball” microworld which consisted of a single red ball performing a combined motion in the vertical and the horizontal axis. The students were asked to execute the model, observe the animation generated, discuss with their teammates and other workgroups the behaviours animated and write down their remarks and observations on a worksheet. In order to provoke discussions regarding the equations’ role and stimulate students to start using the equations themselves, we asked them to try to reproduce the red ball’s motion. In this process, we encouraged them to interpret and use equations from the “Library”, add and remove equations from their objects so as to observe any changes of behaviour and link the equations they used to the behaviours they had previously identified. As we deliberately made the original red ball move rather slowly, near the end of this phase, we expected students to start expressing their personal ideas about their own object’s motion (e.g. make it move faster) and thus start editing the model’s equations, using the “Equations Editor”, so as to describe the new behaviours they might have in mind.

For the second phase of the activities we designed a half–baked microworld (Kynigos 2007), i.e. a microworld that incorporates an interesting idea but it is incomplete by design so as to invite students to deconstruct it, build on its parts, customize and change it. In this case we built a game–like microworld –called “Juggler” (Kynigos 2007)– consisting of three interrelated objects: a red ball and two rackets with which the ball interacted. The ball’s behaviour was partially the same as the “One Red Ball’s”. However, certain equations underpinning its behaviour, did not derive from the environment’s “Library” but were created by us. Using the mouse the rackets could be moved around and make the ball bounce on them, forcing it to move away in specific ways.

We asked the students to execute the Juggler’s model, observe the animation generated and identify the conditions under which each object interacted with each other. The students were encouraged to discuss with their teammates on how they would change the “Juggler” microworld and embed in it their own ideas regarding its behaviour. In the process of changing the half–baked microworld, students were expected to deconstruct the existing model so as to link the behaviours generated on the screen to its equations’ formalism and reconstruct the microworld, employing strategies that would depict their ideas about the new model’s animated behaviours.

METHOD

The experiment took place in a Secondary Vocational Education school in Athens with one class of eight 12th grade students (17 years old) studying mechanical engineering and two researchers -the one acting also as a teacher- for 25 school hours. Students were divided in groups of two or three. The groups had at their

disposal a PC connected to the Internet, the MoPiX manual, translations in Greek of selected equations' symbols and a notebook for expressing their ideas. The adopted methodological approach was based on participant observation of human activities, taking place in real time. The researchers circulated among the teams posing questions, encouraging students to explain their ideas and strategies, asking for refinements and revisions when appropriate and challenging them to express and implement their own ideas. A screen capture software was used so as to record the students' voices and at the same time capture their interactions with the MoPiX environment. Apart from the audio/video recordings, the data corpus involved also the students' MoPiX models as well as the researchers' field notes. For the analysis we transcribed verbatim the audio recordings of two groups of students for which we had collected detailed data throughout the teaching sequence and also several significant learning incidents from other workgroups. The unit of analysis was the episode, defined as an extract of actions and interactions performed in a continuous period of time around a particular issue. The episodes which are the main means of presenting and discussing the data were selected (a) to involve interactions with the available tool during which the MoPiX equations were used to construct mathematical meaning and (b) to represent clearly aspects of the reification processes emerging from this use.

ANALYSIS AND INTERPRETATIONS

Interpreting existing equations' symbols

In the first phase of the experimentation, the students in their attempt to reproduce the red ball' motion, started interpreting and using equations that already existed in the environment's "Equation Library". The natural language aspect incorporated in the MoPiX formalism was the element that guided their actions. The equations that they chose to assign first to their object were those whose symbols (at least some of them) were close to everyday language utterances and provided them some indication on the kind of the behaviour they described (e.g. the "amIHittingtheGround" symbol). Equations that contained symbols that didn't satisfy the "natural language" criterion (e.g. the "Ax") were simply disregarded.

As they continued their experimentations with MoPiX, the students seemed to gradually abandon the "natural language" criterion and shifted their attention into identifying the meaning of the symbols. The students of Group B for instance came across two "Library" equations that seemed to describe the velocity in the x axis, the " $V_x(ME,0)=3$ " and the " $V_x(ME,t)=V_x(ME,t-1) + A_x(ME,t)$ ". Their decision to attribute the second one to their object, so as to define its velocity at any time instance, came as a result of a comparison between the two equations' left parts. Yet again, the students seemed to interpret specific symbols of the equations and completely disregard others (e.g. the "Ax" on the second equation's right part).

In a number of subsequent episodes, the same students seem to articulate their understanding not just about particular symbols but also about the whole string of the

equation's symbols and the relations among them. In the following excerpt the students of Group B talk about the " $x(ME,t)=x(ME,t-1)+Vx(ME,t)$ " equation.

- S1 It [i.e. $x(ME,t)$] is the object [i.e. "ME"] in function with time [i.e. "t"].
- R2 What does this mean?
- S1 [goes on disregarding the question and points at the $x(ME,t-1)$] It's your object [i.e. "ME"] in function with time minus 1 [i.e. "t-1"].
- R2 What does "in function with time" mean? Can you explain it to me?
- S1 How much... In every second, for example, how much it moves.
- R2 Meaning?
- S2 Wait a minute! [Showing both parts of the equation] The equation is this one. All of this. It's not just these two [i.e. the $x(ME,t)$ and the $x(ME,t-1)$].
- S1 Minus 1, which means that in every second of your time it subtracts always 1, resulting to something less than the current time. Plus your velocity.

Drawing on his previous experience with the MoPiX equations, S1 starts to independently interpret the equation's symbols moving from left to right. Having interpreted the first two of them, he attempts to also interpret the relationship between them and defines it as the distance that the object has covered in a second of time. S2, who understands the kind of correlation S1 has made, intervenes and stresses the fact that he hasn't taken into account all the symbols in the equation. S1, who up to that point disregarded the " $Vx(ME,t)$ " on the right part, takes an overall view of the equation and interprets it not by merely referring to the comprising symbols but by also referring to the connection between them. It is noticeable that at this point the students' actions demonstrate an emerging awareness of the equation's structure as a system of connections and relationships between the component parts.

Variables and numerical values to control motion animations

As students gained familiarity with the MoPiX formalism, they started expressing their own personal ideas about the ways their objects should move. In order to put into effect those ideas, the students initially modified the existing equations' symbols and left the structure intact. One of the main elements that they often altered was the equations' arithmetic values. The students of Group B, for instance, attributed to their object the " $Vy(ME,0)=0$ " equation which prescribed the object's y axis initial velocity to be 0. The observation of the animation triggered the implementation of a series of changes to the equation's arithmetic values starting with the conversion of the "0" on the right part into "3". The successive changes of the arithmetic value on the equation's right part didn't cause the object to constantly move since the equation referred just to the initial velocity. To make the velocity for "all the next time instances to come" to be "3", the students replaced the "0" on the left part (i.e. an arithmetic value) with "t" (i.e. a variable).

- S2 Do we need a symbol for this?

- R2 Do we need a symbol? It's a good question. How do you plan to express it?
- S2 With symbols we usually express something that we can't describe accurately.
- S1 Plus... t. [*He writes down $V_y(ME,t)=3$*]. [*Showing the "t"*] So, when I look at this symbol
- S2 I'll know it represents the infinity.

We suggest that the students relocated their focus from just attributing arithmetic values, which indicates a process stance to equations, into forming functional relationships. The fact that they were involved in a process of recognizing which manipulations were possible and at the same time useful to perform so as to express their idea, indicates a implicit focus on the structure of the equations. Furthermore, the statements concerning the use of symbols to express "something that we can't describe accurately" seems to constitute an indication of a progressive acquisition of algebraic structure sense through "mixed cues" (Arcavi, 1994) (i.e. interpreting symbols as invitations for some kind of action while working with them).

Relating different objects' behaviours by constructing new equations

The next episode describes how the Group A students, in the course of changing the "Juggler" microworld, didn't just use or edit existing equations but constructed from scratch two new ones. The idea they wanted to bring into effect was to "make a ball on the Stage change its colour according to an ellipse's position". Knowing that there was no such equation in the "Library", they started talking about how they would correlate those two objects using the Y coordinates.

- S1 When it [*i.e. the ball*] is situated in a Y below the Y of this one [*i.e. the ellipse*] for example.
- R1 I'm thinking... Will the ball know when it is below or above the ellipse?
- S2 That's what we will define. We will define the Ys.
- S1 This. The: "I am below now". How will we write this?
- S2 Using the Ys. Using the Ys. The Ys. That is: when its Y is 401, it is red. When the Y is something less than 400, it's green!

Having conceptualized the effect they would like their new equation to have, the students in the above excerpt decide about two distinct elements regarding the equation under construction: its content (i.e. the symbols) and its structure (i.e. the signs between the symbols). Subsequently, encountering the fact that there was no in-built MoPiX symbol to express the idea of an object becoming green under certain conditions, the students came to invent one. The "gineprasino" (i.e. "become green" in Greek) symbol was decided to represent a varying quantity taking two distinct values (1 and 0, according to if the ball was below the ellipse or not). To represent the ball's position they chose to use its Y coordinate in terms of a quantity varying over time (i.e. "y(ME,t)") while for the ellipse's position they chose to use its Y coordinate

in terms of the constant arithmetic value corresponding to the object's position on the Stage at that time (i.e. "274"). Adding a "less than" sign in between, the equation eventually developed was the "gineprasino(ME,t)=y(ME,t)≤274".

Unexpectedly, this equation didn't cause the ball to become green since it described solely the event to which the ball would respond (being below the ellipse) and not the ball's exact behaviour after the event would have occurred (change its colour). To overcome this obstacle, the students decided to construct another equation in which they tried to find out ways to integrate the "gineprasino" variable. A "Library" equation which explains what happens to a ball's velocity when it hits on one of the Stage's sides and the way in which a variable similar to the "gineprasino" was incorporated in it, led students to duplicate this equation's structure, eliminate any content and use it as a template to designate what happens to the ball's colour when it is below the ellipse. The second equation encompassed in-built MoPiX symbols (the "greenColour"), the "gineprasino" variable in two different forms (not(gineprasino) and gineprasino) and numerical values (0 and 100) to express the percentage of the green colour the ball would contain in each case (i.e. the ball being above and below the ellipse). Thus, the second equation developed was the: "greenColour(ME,t) = not(gineprasino(ME,t))×0 + gineprasino(ME,t)×100".

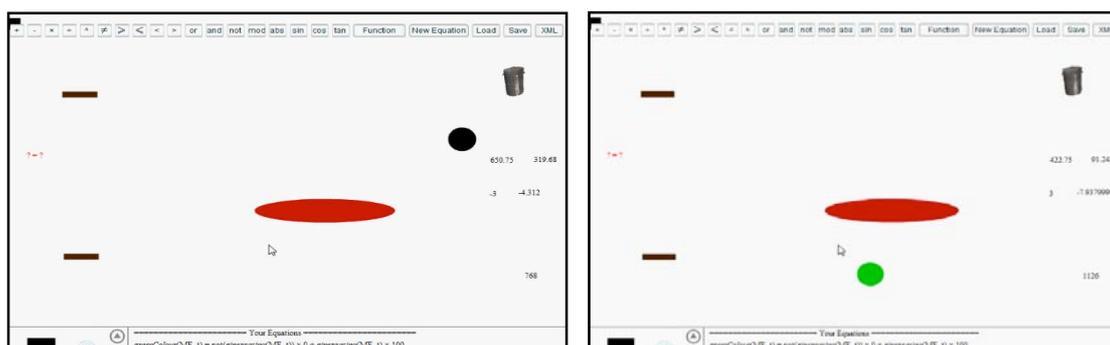


Figure 2: The ball's different percentage of green colour according to its Y position

The above episode contains many interesting events that indicate the existence of a qualitative transformation of the students' mathematical experience in reifying equations that emerged through their interaction with the available tools.

While building the first equation the students got engaged in processes such as inventing and naming variables, relating symbols with mathematical systems (i.e. the XY coordinate system) and manipulating inequality symbols to relate arithmetic values to variables. However, in building the second equation, the meaning generation evolved to include the students' view of equations as objects. The students extracted mathematical meaning from an equation that seemed to describe a behaviour similar to the one they intended to attribute to their ball. Conceptualizing a mapping between the ideas behind the two equations, the students duplicated the similar equation's structure and inserted new terms so as to define a completely novel behaviour for their object. This is a clear indication that they recognised the existence of structures external to the symbols themselves and used them as landmarks to

navigate the second equation's construction process.

The manipulation of the second equation's new terms reveals further their developing structural approach to equations. By inserting in the second equation the the "gineprasino" variable which was introduced in the first one and providing it new forms (i.e. not(gineprasino)), the students seem to have conceptualised the first equation as a mathematical object which it could be used means to encode structure and meaning in the second equation. We think that this reflects a kind of mathematical thinking that has a great deal to do with developing a good algebraic structural sense accompanied with the acquisition of a functional outlook to equations as objects which is a warranty of relational understanding.

CONCLUDING REMARKS

Our purpose in this paper was to illustrate a particular approach to studying the student's construction of meanings for structural aspects of equations, emerging from the use of novel algebraic-like formalism. In the first part of the results, an initial icon-driven conceptualisation of the MoPiX equations seemed to have been leading students towards the development of criteria for an isolated interpretation of the MoPiX equations' symbols. As soon as the students became familiar with testing their models and observing the animations generated on the "Stage", their interactions with the computer environment became strongly associated with the editing of the existing equations' content. As expressed in the second part of the results, the editing of equations revealed a subtle shift from a process-oriented view to equations into an object-oriented one as well as a progressive development of algebraic structure sense. In the last part of the results, students' previous experience with the MoPiX tools seemed to become part of their repertoire, allowing them to construct new equations following specific structural rules, invent variables and specify their values, and use the equations as objects to represent variables in other equations. Concluding, we suggest that in the present study reifying an equation was not a one-way process of understanding hierarchically-structured mathematical concepts but a dynamic process of meaning-making, webbed by the available representational infrastructure (Noss and Hoyles, 1996) and the ways by which students drew upon and reconstructed it to make mathematical sense.

NOTES

1. The research took place in the frame of the project "ReMath" (Representing Mathematics with Digital Media), European Community, 6th Framework Programme, Information Society Technologies, IST-4-26751-STP, 2005-2008 (<http://remath.cti.gr>)
2. "MoPiX" was developed at London Knowledge Lab (LKL) by K. Kahn, N. Winters, D. Nikolic, C. Morgan and J. Alshwaikh.

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RELATIONSHIP BETWEEN DESIGN AND USAGE OF EDUCATIONAL SOFTWARE: THE CASE OF APLUSIX

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In this contribution, we are interested in the design process of Aplusix, a microworld for the learning of algebra and in the impact of usages on this process. In the first part, we present general principles that seem to be guiding the overall design process of the system and the development of tree representation of algebraic expressions, which has been added recently. The second part is devoted to a design and implementation of a learning scenario involving Aplusix. Examples of impact of this empirical study on the software design choices are discussed.

Key words: *Aplusix, algebra, tree representation, pedagogical scenario*

INTRODUCTION

The research reported in this paper is carried out in the framework of the ReMath project (<http://remath.cti.gr>) addressing the issue of using technologies in mathematics classes “*taking a ‘learning through representing’ approach and focusing on the didactical functionality of digital media*”. The digital media at the core of this research is Aplusix, software designed to help students learn algebra. The work has been developed in three phases:

(1) *Design and implementation of a new representation of algebraic expressions.* During this phase, fundamental choices for a representation of expressions in a form of a tree were made collaboratively through interactions between computer scientists and didacticians of mathematics: on the one hand, computer scientists make sure that the new developments comply with general principles of the software, on the other hand, didacticians ensure that these choices are based on didactical and epistemological hypotheses. The choice of theoretical frameworks in both domains has an impact on functionalities of the tree representation. This design phase is presented in the following section.

(2) *Design of a pedagogical scenario.* Based on the choices made in the design phase, didacticians designed a pedagogical scenario to explore possible contributions of this new representation to the learning of algebra. The scenario has to take account of institutional constraints in order to implement it in ordinary classes. The design of scenario may lead to reconsidering certain choices concerning the new representation, or suggesting other. Such cases will be presented further in the paper.

(3) *Experimentation.* The scenario has been experimented in three different classes, which allowed validating underlying didactical hypotheses, as well as assessing the way students manipulate this new representation. This phase is discussed in the last part of the paper.

DESIGN AND DEVELOPMENT OF APLUSIX

When developing computer-based learning environments, designers need to make choices at the interface level and thus at the level of the internal universe of the environment. Thus pieces of knowledge implemented in such an environment will live not only under constraints of the didactical transposition (Chevallard 1985), but also under other constraints proper to the environment resulting from what Balacheff (1994) calls *computational transposition*. Thus, designers of computer-based learning environments have to respond to at least two types of requirements. First, they need to respect basic principles that are characteristic of the environment. The second type is related to the practice of the piece of knowledge in the institution in which it will be used.

Principles governing a design of software are not always made explicit and choices made are rarely explicitly linked to these principles. In what follows, we present a study carried out in an attempt to make explicit principles and choices that were guiding designers of Aplusix (aplustix.imag.fr), software for learning algebra, when they were developing tree representation of algebraic expressions.

General design principles of Aplusix

Aplusix software (Nicaud et al., 2003, 2004) has been developed since 1980s. A new mode of representation of algebraic expressions, a tree representation, is being added to this software. As was already mentioned above, the new developments must not affect the coherence of the whole software and thus have to comply with fundamental principles that guide the design and development of Aplusix. Three main design principles have been identified:

(1) *The student is free to write algebraic expressions.* This principle, influenced by research in the domain of interactive learning environments, considering mainly microworlds, resulted in the development of an editor of algebraic expressions and in the necessity to consider and deal with students' errors.

However, freedom in manipulating algebraic expressions is limited by constraining the selection of sub-expressions, based on the syntactic and semantic dimensions of expressions, which seems to be another important design principle and that can be formulated as follows:

(2) *In manipulating algebraic expressions, their syntactic and semantic dimensions are taken into account.* For example, given the expression $2+3x$, it is not possible to select $2+3$ as a sub-expression. This principle brings the idea of scaffolding since this choice aims at helping understand algebraic expressions and make their manipulation easier.

As regards the interaction between a student and a system, there are two modes of interaction: (1) a test mode in which the student does not get any feedback from the system, and (2) a training mode, in which a feedback is provided both in terms of

equivalence of a student's expression and the given one, and in terms of the correct end of the exercise. Thus the third principle is:

(3) *In a training mode, scaffolding should be provided by the system.* Scaffolding in the training mode requires taking decisions about validation of student's answers. It is important to clarify at this point that Aplusix recognizes 4 basic types of exercises: calculate, expand and simplify, factor and solve (equation, inequality or system of equations or inequalities). For these types of exercises, these decisions have been implemented. For example, for the "solve equation" exercise, it has been decided that the expression $x = 2/4$ will not be accepted as it is written in a non-simplified form, but will not be rejected either as it is not incorrect. Therefore a feedback message is sent to the student saying that the equation is almost solved.

Design and development of tree representation in Aplusix

The decision to implement a new representation system into the existing Aplusix software was taken in relation with the ReMath project focusing on representations of mathematical concepts in educational software. Two possibilities were considered: tree and graphical representations. The reasons for choosing the development of tree representation system are numerous (Bouhineau et al. 2007): (1) from an epistemological point of view, trees are natural representations of algebraic expressions; (2) from a didactical point of view, the introduction of a new register of representation would allow creating activities requiring an interplay between registers, which would enhance learning of algebraic expressions (Duval 1993); (3) from a point of view of computer science, trees are fundamental objects used to define data structures. Indeed, internal objects used in Aplusix to represent algebraic expressions and their visual properties are trees; (4) graphical representation of algebraic expressions is available in a few educational systems, while tree representation is scarcer.

Let us note first that the fundamental choices related to the tree representation were discussed during several meetings among developers (computer scientists and engineers) and didacticians.

Different modes of tree representation

The first idea was to develop the tree representation in a way that the student can see the articulation between the usual representation of an expression and a tree representing it: given an expression in a usual representation, a tree representation is provided progressively by the system, according to the student's command. A "mixed representation" mode has thus been designed where each leaf of a tree is a usual representation of an expression that can be expanded in a tree by clicking at the "+" button that appears when the mouse cursor is near a node; a tree, or a part of a tree, can be collapsed into a usual representation by clicking at the "-" button that appears when the mouse cursor is near a node. The developers considered this idea interesting from the learning point of view. However, it was in contradiction with the principle 1, according to which it was necessary to let the student edit freely a tree. The

development of a “free tree representation” mode, where the student can freely built trees, brought new difficulties the developers had to face: notion of erroneous operator, representation of parentheses, difficulties related to the “minus” sign, to the square root... These difficulties and the ways the developers have coped with them are described elsewhere (Trgalova and Chaachoua 2008).

Based on the principle 3, the developers wished to implement an editing mode providing scaffolding to the student. Design and implementation of scaffolding requires to define new kinds of exercises that would be recognized by the system and the means of validation of these exercises. We will discuss some of these choices below. It led also to the implementation of a “controlled tree representation” mode with constraints and scaffolding when a tree is edited: internal nodes must be operators and leaves must be numbers or variables. The arity of operators must be correct. In the current prototype of Aplusix, 3 modes of editing trees are thus available: free, controlled and mixed representations.

Choices of criteria for validating a student’s answer

According to the principle 3, when the student builds a tree in the free tree representation mode, the system should provide her/him with a feedback. Decisions about the conditions for a tree to be accepted as correct had to be taken and implemented. The student’s tree is compared with the expected one: (1) when, after normalisation of the minus signs (transformation of all minus signs in opposite), the trees are identical, then the student tree is accepted; (2) when the two trees differ only by commutation, the student’s tree is not accepted, but a specific message indicates that there is a problem with order; (3) when there is neither identity between the trees (case 1) nor commutation (case 2) but the two trees represent equivalent expressions, a message is generated indicating that the student’s tree is equivalent but not the expected one; (4) when there is no equivalence between expressions represented by the trees, another message is generated indicating that the answer is not correct.

These choices were made by one of the developers based on *fundamental issues* present in Aplusix such as *the notion of equivalence, the notion of commutation and of associativity*. They are considered as *a first stage choices* that can be discussed and analysed from the didactical point of view, both in terms of messages to be generated and of considering different cases of behaviour.

PEDAGOGICAL SCENARIO

Before presenting a pedagogical scenario we designed in order to validate design choices for the tree representation of expressions in Aplusix, we discuss some theoretical considerations that underpin the scenario.

According to Sfard (1991), mathematical notions can be conceived in two different ways: structurally as objects, and operationally as processes. An object conception of a notion focuses on its form while a process conception focuses on the dynamics of the notion. Algebraic expression, when conceived operationally, refers to a computational process. For example, the expression $5x-2$ denotes a computational

process “multiply a number by 5, and then subtract 2”, which can be applied to numerical values. When an expression is conceived structurally, it refers to a set of objects on which operations can be performed. For example, $5x-2$ denotes the result of the computational process applied to a number x . It also denotes a function that assigns the value $5x-2$ to a variable x . Yet, in the French high school, the operational conception of algebraic expressions prevails in the teaching of algebra. Specific activities are needed to favour the distinction between these two conceptions of an algebraic expression. Examples of such activities are describing the expression in natural language, which requires considering the structure of the expression, or using tree representation of an expression, which highlights its form.

Semiotic representation is of major importance in any mathematical activity since mathematical concepts are accessible only by means of their representations. Duval (1995) calls “*register of representation*” any semiotic system allowing to perform three cognitive activities inherent to any representation: formation, treatment and conversion. These activities correspond to different cognitive processes and cause numerous difficulties in learning mathematics. Duval (2006) claims that while treatment tasks are more important from the mathematical point of view, conversion tasks are critical for the learning. Consequently, conceptualisation of mathematical notions requires manipulating of several registers for the same notion allowing to distinguish between a notion and its representations. As Duval (1993) says, the conceptualisation relies upon the articulation of at least two registers of representation, and this articulation manifests itself by rapidity and spontaneity of the cognitive activity of conversion between registers. Yet, school mathematics gives priority to teaching rules concerning both formation of semiotic representations and their treatment. The amount of activities of conversion between registers is negligible, although they represent cognitive activities that are the most difficult to grasp by students.

Motivated by these considerations, in the design of our pedagogical scenario, we decided to take into account three semiotic registers of representation of algebraic expressions: natural language register (NLR), usual register (UR) and tree register (TR) and to design activities of formation, treatment and conversion between these registers. The pedagogical scenario thus aims at helping the students grasp the structure of algebraic expressions by means of introducing TR and articulating it with UR and NLR. The following hypothesis underpins the scenario: the introduction of TR and its articulation with NLR and UR will have a positive impact on students’ mastering of the usual register of representation of algebraic expressions, which is the one taught in school algebra. The scenario is composed from 4 units: *pre-test*, *learning*, *assessing*, and *post-test* (cf. Table 1). The *pre-test* aimed at diagnosing students’ difficulties in algebra, especially those related to the structural aspect of expressions. On the other hand, the results of the pre-test compared to those of the post-test should provide us with evidence about the efficiency of the pedagogical scenario. Two kinds of activities are proposed in the pre-test: (1) classical school

algebra exercises (calculate, expand and simplify, factor), which are, in Duval's terms, treatment tasks in the register of usual representation, and (2) communication games between students proposing, in Duval's terms, activities of conversion between UR and NLR. The aim of the *learning* unit is to introduce the students to TR, a new register of representation of expressions, as well as to articulate it with the already familiar registers, namely NLR and UR. Then, conversion activities between TR and NLR and UR respectively are proposed. Most of the activities are to be done in a computer lab with Aplusix in the training mode. Eventually, simple tasks of treatment in TR are proposed to assess the mastery of the new register of representation by students. The unit called *assessing* aims at evaluating to what extent TR and conversion tasks between the registers are mastered by the students after having done activities of the *learning* unit. The evaluation is organized in the form of communication games between students similar to those from the pre-test, but this time, TR is involved in the tasks. In the *post-test*, tasks similar to those from the pre-test are proposed in order to enable a comparison of results. Confronting results obtained at the two tests should provide us with evidence confirming or not the underlying hypothesis.

	<i>Activities</i>	<i>Description</i>	<i>Environment</i>	<i>Duration</i>
Pre-test	Treatment in UR	Calculate, Factor Expand and simplify	Aplusix	50 min
	Conversion NLR \leftrightarrow UR	Communication games	Paper & pencil	30 min
Learning	Introduction to TR	Scenario TR introduction	Aplusix in video projection	55 min
	Conversion NLR \leftrightarrow TR	Conversion NLR \rightarrow TR Conversion TR \rightarrow NLR	Aplusix: controlled then free mode Paper & pencil	90 min
	Conversion UR \leftrightarrow TR	Conversion UR \rightarrow TR Conversion TR \rightarrow UR	Aplusix: controlled then free mode	80 min
	Treatment in TR	Calculate in TR Simplify in TR	Aplusix with second view	20 min
Ass.	Formation TR Conversion TR \leftrightarrow NLR (UR)	Communication games	Aplusix: free mode Paper & pencil	55 min
Post-test	Treatment in RU	Calculate, Factor Expand and simplify	Aplusix	30 min
	Conversion NLR \leftrightarrow UR	Communication games	Paper & pencil	20 min

Table 1. Structure of the pedagogical scenario.

EXPERIMENTATION

The scenario was proposed to 3 teachers with a possibility to adapt it to the constraints of their class. In this section, we present one of the experiments that took place in a Grade 10 class (15 years old students) in November 2007.

The pre-test revealed expected errors in treatment tasks within UR, in particular errors showing difficulties to take account of the structure of algebraic expressions, e.g., transforming $2+3x$ in $5x$, and errors with handling powers and minus sign, e.g., transforming $3(-5)^2$ in -3×5^2 or in $\pm 3^2 \times 5^2$. On the other hand, we were surprised by the results obtained in communication games. Algebraic expressions given in UR were described in NLR by the students, but with characteristics of an oral register, i.e., the students described actions allowing to obtain the initial expression (cf. Table 2). This register is based on language structure used to “read” an expression in UR. It presents two specificities: left-to-right reading and presence of implicit elements.

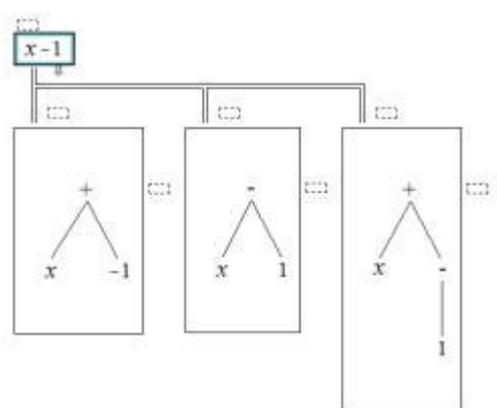
<i>Expression given in UR</i>	<i>Student emitting a message</i>		<i>Student receiving a message</i>	
	<i>Register</i>	<i>Examples of messages</i>	<i>Correct in UR</i>	<i>Wrong in RU</i>
$2x - y$	Oral (left-to-right)	“2 x minus y”	14	0
$2x - y^2$	Oral with ambiguity	“2 x minus y squared”	22	4
$(3x + 2)(3x - 1)$ $a - (x + 2)$	Oral with brackets explicitly stated	“open a bracket, 3 x plus 2, close the bracket, open a bracket, 3 x minus 1, close the bracket, all this over a minus, open a bracket, x plus 2, close the bracket”	7	1
	Oral with brackets explicitly stated and with ambiguity	“open a bracket, 3 x plus 2, close the bracket, open a bracket, 3 x minus 1, close the bracket, over a minus, open a bracket, x plus 2, close the bracket”	19	1
Total			62	6

Table 2. Conversion from UR into NLR.

All messages result from the oral register and they accentuate operational aspect of the expressions rather than structural one. Moreover, more than 66% of messages are ambiguous. Despite of the ambiguities, most of pairs succeeded the game thanks to implicit codes of the oral register the students share and understand and which result from didactical contract (Brousseau 1997). Thus, the goal we assigned to the communication games, namely to lead students to become aware of the limits of the oral register they use in algebra, which does not take into account the structural aspect of expressions, was not achieved.

The learning unit started by an introductory session aiming at introducing tree representation to the students. The teacher asked one of the designers of the

pedagogical scenario to manage this session since he did not feel comfortable enough with the new representation implemented in the software although he uses Aplusix on a regular basis with his students. This introductory session allowed discussing with the students specificities of the tree representation of expressions and introducing vocabulary related to this new register (branch, leave, operator, argument...). Particular attention was paid to reading the expressions. Thus for example, the expression $x+2y$ was read as “the sum of x and of the product of 2 by y ”, which accentuates the structure of the expression, instead of “ x plus 2 y ” highlighting its operational aspect. A particularity of the tree register residing in the fact that several different trees can represent a same algebraic expression was also discussed with the students based on the following example showing different meanings of “minus” sign (Fig. 1):



In the expression $x-1$, the minus sign can be conceived in three different ways leading to three different trees (this difference is hardly visible in UR):

- Sign of a negative number (tree on the left);
- Binary operator “difference” (tree in the middle);
- Unary operator “opposite number” (tree on the right).

Figure 1. Three different meanings of minus sign.

The rest of the scenario was shortened in order for the teacher to be in line with the global pedagogical program shared by all Grade 10 classes in the school. The teacher decided to individualize the implementation of the scenario according to the students in the following way: conversion $NLR \rightarrow TR$ and $UR \rightarrow TR$ in controlled mode only (only one group, denoted G1); conversion $TR \rightarrow NLR$ assigned as homework (whole class); treatment in TR optional (a few students with severe difficulties in algebra).

The G1 group was formed from rather low attaining students. The results obtained in the conversion tasks $TR \rightarrow NLR$ showed a significant difference between the two groups (cf. Table 3). These results can be considered as evidence proving efficiency of the work on conversion tasks $NLR/UR \rightarrow TR$.

	<i>Answer in NLR with structural aspect</i>	<i>Answer in NLR with operational aspect</i>
G1 15 students having worked on conversion tasks with Aplusix in controlled mode	10	5
G2 15 students who have not benefited from the work on conversion tasks	3	12

Table 3. Students' answers to the conversion tasks TR→NLR.

As we mentioned above, the scenario, and thus the new prototype of Aplusix, had been tested in three classes. Feedbacks from students and teachers led the developers to re-examine some choices, which allowed some adaptations and improvements at the interface of Aplusix. Let us take the example of the “second view” functionality that enables visualizing a given algebraic expression represented in two registers at the same time. Initially, the second view displayed only a current step of the transformation. Observing the students using this functionality, we realized that when a student performs the next transformation step, the representation in the second view is updated and the student cannot observe the effects of the transformation in the second register. For this reason, the developers were asked to redesign this functionality in a way for the student to be able to observe the transformation s/he has performed in both registers. At present, the second view displays both current and previous steps.

CONCLUSION

The example of the design and implementation of tree representation of algebraic expressions presented in this contribution shows that the decision to introduce a new register of representation has been motivated by the didactical considerations about the necessity of being able to represent mathematical notions in at least two different registers. Considerations of different nature had an impact on the development of the new register: (1) taking account of a didactical dimension led to make choices allowing the implementation of tasks of conversion between registers, which seem to be essential for conceptual understanding of mathematical notions (Duval 1993); (2) taking account of users' feedback allowed to make some improvements at the interface level. An example was presented in the previous section; (3) respecting the general principles of the development of Aplusix guarantees the coherence of the system after the introduction of the new register of representation of algebraic expressions. As regards the choices made in the design of the Aplusix tree module, it seems that most of them were made internally, i.e., by the developers themselves, and sometimes even individually, i.e., by one of the developers. Decisions are driven by the fundamental design principles in a way that a coherence of the whole system is preserved. Although it seems that the decisions are taken regardless the school

context, both teachers and students are taken into account in the system design. The principles 1 and 3 concern especially students and their interactions with the system. Moreover, the developers are respectful towards the students' ways of editing expressions, which is shown by the decision to make it possible to recover an expression in exactly the same way as the student has edited it, even if the implementation of such a decision was difficult (Trgalova and Chaachoua 2008).

The example of the development of Aplusix illustrates a way the synergy between computer scientists, researchers in math education and users can serve a project of development of educational software.

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CERME 6 – WORKING GROUP 8

QUESTIONS AND THOUGHTS FOR RESEARCHING CULTURAL DIVERSITY AND MATHEMATICS EDUCATION

TABLE OF CONTENTS

Introduction.....	1440
<i>Guida de Abreu, Sarah Crafter, Núria Gorgorió</i>	
A survey of research on the mathematics teaching and learning of immigrant students.....	1443
<i>Marta Civil</i>	
Parental resources for understanding mathematical achievement in multiethnic settings.....	1453
<i>Sarah Crafter</i>	
Discussing a case study of family training in terms of communities of practices and adult education.....	1462
<i>Javier Díez-Palomar, Montserrat Prat Moratonas</i>	
Understanding Ethnomathematics from its criticisms and contradictions.....	1473
<i>Maria do Carmo Domite, Alexandre Santos Pais</i>	
Using mathematics as a tool in Rwandan workplace settings: the case of taxi drivers	1484
<i>Marcel Gahamanyi, Ingrid Andersson, Christer Bergsten</i>	
Parents' experiences as mediators of their children's learning: the impact of being a parent-teacher	1494
<i>Rachael McMullen, Guida de Abreu</i>	
Batiks: another way of learning mathematics	1506
<i>Lucília Teles, Margarida César</i>	
The role of Ethnomathematics within mathematics education	1517
<i>Karen François</i>	

INTRODUCTION

QUESTIONS AND THOUGHTS FOR RESEARCHING CULTURAL DIVERSITY AND MATHEMATICS EDUCATION

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CERME 6, in Lyon 2009, was the 4th meeting of the working group “Cultural diversity and mathematics education” (in previous meetings it was WG10 and it had slightly different titles). The group is particularly interested in understanding learning and teaching mathematics in culturally diverse schools, classrooms and other educational settings. It also acknowledges the relevance of studies on culture and cognition in outside school settings linked with mathematics and, in particular, with ethno mathematics. We constitute a multi-disciplinary group that includes researchers from a variety of disciplines, such as mathematics, education, socio-cultural and developmental psychology, philosophy, anthropology, linguistics, sociology, political sciences, etc. We are in ourselves a multinational community that in Lyon included contributors from Belgium, Brazil, Canada, Cyprus, Denmark, Italy, Portugal, Rwanda, Spain, Sweden United Kingdom and USA.

QUESTIONS RAISED DURING WG8 MEETINGS

The areas covered by the presentations during our meetings were different theoretical and methodological approaches as well as different research domains. Teaching, the relationship between home-family and school, out-of-school practices, particular cultural and linguistic groups were some of the domains discussed. The perspectives that all of us brought to the discussion led, in particular, to interrogating how culture links to diversity, practices and institutions.

Conceptual clarification

The discussion of several papers claimed for clarification of different notions, such as ‘culture’, ‘diversity’ and ‘cultural diversity’. This was considered important both in relation to theoretical papers and to empirical papers. Broad conceptualisations meant that there were issues at stake for data collection. There was agreement that culture is something dynamic but it is also something which is re-interpreted for meaning. In other words, there was interest in the socio-cultural as co-constituted in the psychological. Furthermore, whilst new concepts are introduced into theoretical research others continue to be discussed over time.

Culture in practice

Whilst discussions on the conceptualisation of culture were useful to the group, many felt they needed to make sense of how this shapes and is shaped by practices in the

classroom. Questions were raised such as – how can we teach mathematics whilst respecting cultural diversity? How do teachers/parents of other cultural backgrounds explain mathematical problems? Can culture help us understand identities development in mathematical practices within and outside school?

Culture and institutions

The tensions between the school as a normalising institution and the diversity of students in society were raised. It was questioned what the dangers of bringing culture to a normalising institution may be? When one thinks of school as an institution whose goal it is to transmit culture, one has to think “whose” culture is being referred to. In other words, in which ways do educational institutions reproduce inequalities? It was suggested that this ‘tension’ or ‘gap’ between cultural diversity and the institution is as symbolic as the notion of ‘normal’. The normalised institution, an idea developed and reproduced by school, is also symbolic and can be perceived as exotic and outside the lives of most pupils. Furthermore, institutions are culturally composed by people and these people may influence the institution.

SHARED INTERESTS WITH OTHER GROUPS

During reporting sessions, it was made apparent that there are different overlaps between WG8 and papers presented in other working groups. This was mainly expressed through an interest in a socio-cultural perspective when applied to a specific domain which was covered by another group. This perspective is felt to be more relevant since, nowadays, our schools are recognized to be more and more culturally diverse, and inequity in education has become under socio-political scrutiny.

For some groups, the intersection is wide and obvious. This would be the case with the working group dealing with mathematics and language, since culture is inextricably linked to language. It seems also clear to us that there is an intersection with the group working on Early Years Mathematics, since nowadays it is becoming clearer, especially for this age group, that learning is situated on its context.

For some other groups, one has to go deeper to see the overlapping. However, one of the participants in the Applications and Modelling group explicitly contributed to the reporting session by affirming that “*modelling in mathematics can also benefit if the cultural backgrounds of learners is taken into account while modelling learning situations*”. It did not surprise us either that people that had attended the Algebraic Thinking or Geometrical Thinking groups told that the curricular issues that they have addressed could benefit from a socio-cultural perspective.

AFTERTHOUGHTS

To finish this introduction, we would like to share with the readers how we explain the overlapping with other research groups and the dilemmas that it poses to us as coordinators of the group.

The engagement of participants in WG8, *Cultural Diversity and Mathematics Education*, comes from our shared interest in and commitment to a particular empirical domain, that of multicultural settings. Other CERME working groups are organized either around the study of theoretical perspectives or the content domain of the research –language issues, teacher education, theoretical perspectives, algebraic thinking or modelling, just to name some of them. It is clear that any of the above mentioned focuses could be researched in a multicultural setting. And it is this last point where both our strengths and our weaknesses come from.

Our interest in addressing non-prototypical situations requires that we try to broaden both our theoretical perspectives and our methodological approaches. Both theories and methodologies could be of use to other researchers in mathematics education.

However, each of us as participants to WG8, has once asked him/herself questions such as: Do I want the focus of my presentation to be the fact that I am dealing with a culturally diverse situation? Do I want to stress that I am using a theoretical perspective that is new to mathematics educators? Or do I want to suggest a discussion on curricular issues or content matters? This is where our dilemmas arise. If we keep within our group, the research done in culturally diverse situations becomes closed, making it difficult for others to come to know about our developments. However, if we go to other groups, then we risk losing our primary focus and then a new question arises: who is going to foster research in culturally diverse situations and other neglected empirical domains? What we as a group, and the larger community, will lose or gain if we move from a title of WG8 that has to do with our empirical domain into a title that has to do with a theoretical perspective? How things would change if next meeting WG8 was renamed “Socio-cultural perspectives on mathematics education”?

A SURVEY OF RESEARCH ON THE MATHEMATICS TEACHING AND LEARNING OF IMMIGRANT STUDENTS ¹

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This paper presents key themes that emerged from a review of the literature and from solicited contributions from researchers around the world on the teaching and learning of mathematics of immigrant students. Researchers strongly suggest the need for schools to look at the different kinds of mathematics that immigrant students bring with them and to use this knowledge as a resource for learning. There is a clear need for teachers to gain a better understanding of their immigrant students' and their families' knowledge and experiences. The emphasis on language as "the problem" promotes approaches that segregate immigrant students and raise issues of equity in the mathematics education they are receiving. Little research documents experiences that center on diversity and multiculturalism as a resource for learning.

This paper presents the key themes that emerged from a review of the literature on the topic of the mathematics teaching and learning of immigrant students. This topic was one of the four areas that ICME 11 Survey Team 5 addressed as part of our task to examine the research topic of mathematics education in multicultural and multilingual environments since ICME 10 in 2004. One of my main sources of information for my part of the survey team was the work of researchers actively involved in CERME's working group on Cultural Diversity and Mathematics Education.

The purpose of this paper is to highlight the main findings, advances, challenges, and indicate topics for further research in the area of mathematics teaching and learning of immigrant students. Much of this work is actually centered on research in Europe, hence the role of CERME papers. I also draw on the different contributions received from researchers across the world in response to our survey team's call for contributions. Finally, I also looked at aspects of research in the mathematics education of Latino/a students in the U.S. These three sections (proceedings, contributions, and research with Latino/a students) are discussed at length elsewhere (Civil, 2008b). For reasons of space, in this paper I am only highlighting some of the main ideas with special attention to those that relate to CERME research, as a way to encourage further discussion of this topic, the teaching and learning of mathematics of immigrant students, during the working group sessions.

Different forms of mathematics

Several studies address issues related to everyday mathematics, critical mathematics, community mathematics, school mathematics, and so on. Researchers in Greece have been looking at Gypsy / Romany students' use of mathematics in everyday contexts, in particular computation grounded on children's experiences with their involvement in their families' business (Chronaki, 2005; Stathopoulou & Kalabasis, 2007). These

researchers note that schools and teachers seem to show little interest in what knowledge minority students (in this case Gypsy) bring with them and thus, in how to build on this knowledge for classroom teaching. It may be little interest on the part of the teachers, or it may be due to an unawareness on how to build on this knowledge. Elsewhere I have argued for the complexity of the pedagogical transformation of community knowledge into modules for the classroom setting (Civil, 2007).

In Civil (1996) I raised two questions that still seem relevant today: “Can we develop learning experiences that tap on students’ areas of expertise and at the same time help them advance in their learning of mathematics?” and “What are the implications of critical pedagogy for the mathematics education of ‘minority’ and poor students?” More recently Powell and Brantlinger (2008) discuss some of the tensions around their own work with Critical Mathematics (CM) and write, “CM educators should not be satisfied with engaging historically marginalized students in politicized investigations of injustices (e.g., wage distributions) if they do not have access to academic mathematics” (p. 432). As we consider different forms of mathematics and whose mathematics to bring to the foreground, issues of power and valorization of knowledge become prominent. Abreu has written extensively on the concept of valorization of knowledge (Abreu & Cline, 2007).

Teacher education

Much of the research I reviewed for this topic addressed teachers’ attitudes and knowledge of immigrant students. This body of research presents a rather grim picture and thus opens the door to several possibilities for further research. Reports on an European project that is looking at the teaching of mathematics in multicultural contexts in three countries, Italy, Portugal and Spain, point out that teachers feel unprepared to work with immigrant students. César and Favilli (2005) report that teachers in this study underscore the issue of language as being a problem and do not seem to recognize the potential for richer learning grounded in different problem solving approaches and experiences that immigrant students may bring with them. They also note that teachers seem to have different perceptions on immigrant students based on their country of origin. Overall, these reports point to a deficit view by teachers of their immigrant students.

Abreu (2005) reports that most teachers in the studies she examined tended to “play down cultural differences” arguing for general notions of ability and equity, as in treating everybody the same. Gorgorió (personal communication, April 28, 2008) writes, “teachers tended to make invisible the cultural conflict that would arise in their classrooms as a result of the discontinuities between different school cultures and different classroom cultures.” Abreu points out the need for teacher preparation programs to pay more attention to the cultural nature of learning.

Gorgorió and Planas (2005) discuss the role of social representations in teachers’ images and expectations towards different students. In particular, they write, “unfortunately, too often, ‘students’ individual possibilities’ do not refer to a

cognitive reality but to a social construction. Teachers construct each student's possibilities on the basis of certain social representations established by the macro-context" (p. 1180). Researchers are critical of the public discourse that frames immigration as being a source of problems rather than a resource for learning since this discourse is counter-productive to the education of immigrant children (Alrø, Skovsmose, & Valero, 2005). Unfortunately, as Gorgorió and Planas (2005) point out, some teachers use this public perception as their orientation to assess immigrant students in their classrooms, rather than a direct knowledge and understanding of their individual students and families.

There is a clear need for teachers to understand other ways of doing and representing mathematics (Abreu & Gorgorió, 2007; Moreira, 2007). As Abreu and Gorgorió (2007) write in relation to a teacher's reaction to differences between representations of division in Ecuador and in Spain, "the relevant question is not whether there are any differences in the representation of the algorithm of the division, but how teachers react to the differences" (p. 1564). Related to the need for teachers to know about others' ways of doing mathematics, is a need for an expanded view of what mathematics is. Teachers tend to view mathematics knowledge as culture-free and universal (Abreu & Gorgorió, 2007; César & Favilli, 2005). This relates directly to the previous section on different forms of mathematics. Teacher education programs should address this view of mathematics as being culture-free. Moreira (2007) brings up the need for teacher education programs to prepare teachers to research this locality of mathematics (e.g. everyday uses of mathematics).

Issues related to educational policy

Researchers from different countries are critical of educational policies that push towards assimilation of immigrant students. These policies convey a deficit view on immigrants' language and culture, instead of promoting diversity as a resource for learning (Alrø, Skovsmose, & Valero, 2007). Anastasiadou (2008) writes,

The de facto multiculturalism (...) which now describes the Greek society, ... [which] continues to function with the logic of assimilation (...). In the field of education the adoption of the policy of assimilation means that it continues to have a monolingual and monocultural approach in order that every pupil is helped to acquire competence in the dominant language and the dominant culture. (p. 2)

The work of Alrø et al. (2005) is particularly relevant here as these authors take a socio-political approach to the discussion of the teaching and learning of mathematics with immigrant students. They write about the influence of public discourse and in particular of the view of immigration as a problem rather than a resource:

In Denmark, the sameness discourse has spread into a variety of discourses, which highlight that diversity causes problems – it is not seen as a resource for learning. And this idea brings about a well-defined strategy: Diversity has to be eliminated. (p. 1147)

Then, as researchers in other parts of the world have noted, these authors point to the

emphasis in educational policy on students' acquisition of the Danish language as the priority. The idea that mathematics education is political is particularly true when studying the mathematics education of immigrant students.

Language, mathematics, and immigrant students

Many of the contributions I received from across the world were on this theme. Here I can only give snippets of some of those. Most of them point to a clear concern among researchers for restrictive language policies that limit the use of home languages in the teaching of mathematics. For example, Clarkson (personal communication, May 25, 2008) writes,

Mathematics teaching, like all the teaching that occurs in a school, normally is mandated to be carried out in the dominant language of the society. The use of other languages is normally proscribed. For immigrant children this may be an important matter. If they are from homes that speak a language different to the dominant societal language, then much of their formative early learning undertaken before schooling has begun will be encoded in their home language. Hence for schools to take no or little notice of these extra hurdles that such students have to leap is to simply not be realistic.

Staats (personal communication, June 8, 2008) brings another language-related issue emerging from her work with Somali immigrant students in the U.S. She wonders what happens when students do not really know their home language. She writes,

With the educational history of Somalis they do not know their math vocabulary. It is a point of sadness, in fact, for many young people that they feel they do not know any language well, they might know parts of Somali, Swahili, Arabic, Italian, or English but feel insecure speaking any of these.

Elbers provided thought-provoking comments on the situation of mathematics education in the Netherlands. His comments relate to both the prior section on issues related to educational policy and this section on language:

Realistic Mathematics was also criticized as being not real math (also by leading mathematicians in the Netherlands), and being based more on semantics and interpretation of assignments than on math knowledge and skills. They claim that the Dutch good achievement in math in the PISA studies is because the PISA studies do not test real math. Many plead for a return to transmission of knowledge in classrooms. The bad results of minority children in schools, in the recent debate, was partly explained with a reference to educational methods such as students learning by collaboration and investigation. These methods, the argument runs, depend on students' skills in Dutch and therefore these students, because of their language gap, can never be successful in math. (E. Elbers, personal communication, May 14, 2008)

As we can see, once again, language is singled out as the obstacle to immigrants' learning of mathematics. Elbers' comment is even more pointed as it is focusing on a critique of discussion-rich approaches to teaching mathematics that could be problematic for students for whom Dutch is not their first language. Moschkovich

(2007) addresses this topic in her research with English Language Learners in the U.S. She writes,

The increased emphasis on mathematical communication in reform classrooms could result in several scenarios. On the one hand, this emphasis could create additional obstacles for bilingual learners. On the other hand, it might provide additional opportunities for bilingual learners to flourish (p. 90).

As we have seen, in the eyes of education policy-makers and many teachers, not knowing the language of instruction is seen as a major (and in most cases the main) obstacle to the teaching and learning of mathematics of immigrant students. Hence, the push is for these students to learn the language(s) of instruction as quickly as possible. As Alrø et al. (2005) point out, the emphasis on learning the language of the receiving country may occur at the expense of these students' learning of mathematics. Gorgorió and Planas (2001) have documented a similar situation in Catalonia. In my local context there is long history of changes in language policy for education, with some states now having banned or severely limited bilingual education. In Civil (2008c) I present the case of one student who was Spanish-dominant and had a good command of mathematics (she had already learned much of what she was being currently taught in Mexico), but was in a context in which English was the language of instruction. I raise questions about equity and the opportunities for participation and further learning of mathematics for this student.

What about immigrant parents' views on issues of language policy and mathematics education? This is a less researched topic, but one that is quite prominent in our Center CEMELA (Center for the Mathematics Education of Latinos/as)². For example, in Acosta-Irqui, Civil, Díez-Palomar, Marshall, & Quintos-Alonso (2008), we look at two CEMELA sites (Arizona and New Mexico) that have different language policies (in Arizona, bilingual education is extremely restricted, while in New Mexico it is endorsed in their state constitution). This allows us to contrast the effect of such different language policies on parents' participation in their children's mathematics education. An interesting theme emerging from our research with immigrant parents is that for many of them the language also seems to be the main obstacle to their children's learning of mathematics (this parallels what teachers think as we have illustrated earlier). This is the case in our research with mostly Mexican parents in the U.S. (Civil, 2008a) but is also the case with immigrant parents in Barcelona (Civil, Planas, & Quintos, 2005). As immigrant parents focus on the language as being the main obstacle, I wonder whether they are aware of the actual mathematics education that their children are receiving. In particular, I am referring to issues of placement: are the students placed in the appropriate mathematics classroom (based on their knowledge and understanding of the subject) or are schools basing their placement on their level of proficiency in the language of instruction? I wonder about the thinking behind these placement policies. Not only are parents not aware of the implications of this policy on their children's learning (or not) of mathematics, but also teachers often are not either (Anhalt, Ondrus, & Horak, 2007).

Research with immigrant parents

Most of the research I found on immigrant parents and their views of mathematics education was done by Abreu and her colleagues in the U.K. (Abreu & Cline, 2005; O'Toole & Abreu, 2005) and by Civil and her colleagues in the U.S. (Civil & Bernier, 2006; Quintos, Bratton, & Civil, 2005). Civil, Planas, and Quintos (2005) look at immigrant parents' perceptions about the teaching and learning of mathematics in two different geographic contexts, Barcelona, Spain, and Tucson, U.S. Besides these studies in U.K., U.S., and the one study with immigrant parents in Barcelona and in Tucson, I found one study with immigrant parents in Germany by Hawighorst (2005).

There are three related themes that emerged and that cut across all immigrant parents in these studies. Overall, immigrant parents in the four geographic contexts shared a concern for a lack of emphasis on the "basics" (e.g., learning of the multiplication facts) in the receiving country, a perception that the level of mathematics teaching was higher in their country of origin, and a feeling that schools are less strict in their "new" country. Abreu and colleagues as well as Civil and colleagues have looked at these themes in some depth, thus providing an analysis related to issues of differences in approaches, issues of valorization of knowledge, and potential conflict as children are caught between their parents' way and the school's way.

The research with immigrant parents on their perceptions of the teaching and learning of mathematics underscores the need for schools to establish deeper and more meaningful communication with parents. Parents tend to bring with them different ways to do mathematics that are often not acknowledged by the schools, and vice versa, parents do not always see the point in some of the school approaches to teaching mathematics. Although this may be the case with all parents (e.g., in the case of reform vs. traditional mathematics), the situation seems more complex when those involved are immigrant parents and their children. As the research of Civil and colleagues shows (Civil, 2008a; Civil, Díez-Palomar, Menéndez-Gómez, Acosta-Irqui, 2008) differences in schooling (different approaches to doing mathematics) and in language influence parents' perceptions of and reaction to practices related to their children's mathematics education.

Implications for further research

My hope is that this paper will serve as a starting point to hear from other researchers who are working in mathematics education and with immigrant students. There are several implications that this review points to and that I want to briefly mention here. Abreu, César, Gorgorió, and Valero (2005) raise two important questions that should frame, I think, further research in this field. They write, "Why research on teaching and learning in multiethnic classrooms is not a bigger priority? Why issues of teaching in multicultural settings are not central in teacher training?" (p. 1128)

Based on the research reviewed, there seems to be a clear need for action-research projects with teachers of immigrant students engaging as researchers of their own

practice to counteract what appears to be a well-engrained deficit view of these students and their families. Through a deeper understanding of their students' communities and families (e.g., their funds of knowledge), maybe teachers can work towards using different forms of doing mathematics as resources for learning instead of the current trend that seems to view diversity as an obstacle to learning (there are of course exceptions to this view and I address those in Civil, 2008b). Related to this idea of understanding immigrant students' communities, there is very little research looking at the sending communities. That is, what do we know about the teaching and learning of mathematics in the countries / communities that these immigrant students come from? We have recently started one such project in CEMELA, in which we look at the mathematical experiences of the students who are recent immigrants to the U.S. by studying the teaching and learning of mathematics in some sending communities. Specifically, we are looking at mathematics instruction at one school in Mexico across the border from Arizona to gain a better understanding of Mexican teachers' conceptions about the teaching and learning of mathematics. I argue that there is a need for more research along these lines to gain a better understanding of the background experiences of immigrant students.

There is also a need to analyze the learning conditions in schools with large numbers of immigrant students. What Nasir, Hand, and Taylor (2008) write in reference to African American and Latino and poor students is likely to be the case with immigrant students in many countries:

African American and Latino students and poor students, consistently have less access to a wide range of resources for learning mathematics, including qualified teachers, advanced courses, safe and functional schools, textbooks and materials, and a curriculum that reflects their experiences and communities. (p. 205)

Issues of valorization of knowledge and different forms of mathematics need to continue to be explored, as there are still many open questions. Related to this is the idea of non-immigrant students' views of immigrant students. This topic has received little attention (a notable exception is Planas, 2007), yet it seems important to understand how all the students see and understand the experience of being in a multicultural classroom (Alrø et al. (2007) address this topic to a certain extent).

Another area that needs further research is that of immigrant parents' perceptions about the teaching and learning of mathematics. Furthermore, an important and under-researched area is that of interactions between immigrant parents and teachers and perceptions of each other's in terms of the children's mathematics education. Civil and Bernier (2006) address this to a certain extent, but much more work is needed in this area.

Language is a prominent theme in the research with immigrant students and mathematics education. More research is needed that focuses on multiple languages as resources for the teaching and learning of mathematics, once again to counteract the deficit perspective, particularly in the public discourse that sees the presence of

other languages and not knowing the language of instruction as obstacles to the mathematics education of immigrant children. Issues of placement based on language proficiency and the impact that these decisions have on students' learning of mathematics also need to be studied further.

Finally, a clear implication from the research reviewed on this topic is the need for interdisciplinary teams with expertise in different areas including mathematics education, immigration policy, linguistics, socio-cultural theories, anthropology, just to name a few. There is a need for this interdisciplinary expertise, as well as for the development (or refinement) of theoretical and methodological approaches. I find Valero's (2008) comment on this (in the context of mathematics education in situations of poverty and conflict, which are often the norm in immigrant contexts) very insightful:

The theories that have been used to study mathematics learning build on a fundamental assumption of continuity and of progression in the flow of interactions and thinking leading to learning. (...) When [these theories] are simply applied without further examination the result has often been the creation of deficit discourses on the learners or the teachers. (...) The question then becomes how can (mathematics) "learning" be redefined as to provide a better language to grasp the conditions and characteristics of thinking in situations where continuity and progression cannot be assumed. (p. 161)

I leave the reader with the challenge Valero raises in the last sentence.

Notes

[1] This paper is adapted from a longer paper (Civil, 2008b) prepared for ICME Survey Team 5: Mathematics Education in Multicultural and Multilingual Environments, Monterrey, Mexico, July 2008.

[2] CEMELA is a Center for Learning and Teaching (CLT) funded by the National Science Foundation under grant ESI-0424983. The views expressed here are those of the author and do not necessarily reflect the views of the funding agency.

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PARENTAL RESOURCES FOR UNDERSTANDING MATHEMATICAL ACHIEVEMENT IN MULTIETHNIC SETTINGS

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This paper examines parental understandings about their child's mathematical achievement and the resources they use to go about gaining information in culturally diverse learning settings. This examination takes place within a critical-developmental framework and draws on the notion of cultural models to explicate how resources are used. Three parental resources of mathematics achievement are scrutinised: (i) the teacher, (ii) exam test results and (iii) constructions of child development. The interviews with twenty-two parents revealed that some resources were concrete, such as examination results. Other resources were symbolic, like the representation of child development, and were less likely to be shared with the school community. Either way, these resources were open to parental interpretations and influenced by parents' own experiences and cultural representations.

Key words: parents, resources, cultural models, achievement, ethnic minority

INTRODUCTION

Within the English school system, like many Western/English-speaking countries, there is a strong emphasis on testing and measurable outcomes for success at school. The introduction of the National Numeracy Strategy and nationwide testing in the primary sector led to a greater pressure for parents' involvement in their children's school education (Bryans, 1989). While many could see problems with using parents as teachers in the home, the problems of engaging parents specifically from culturally diverse backgrounds remained largely uncontested.

The education of ethnic minority children has been given some attention, although less seems to be said about mathematics learning in particular in the UK context. The pitting of one ethnic group over another has tended to overshadow the sociocultural composites of school practices or the "gaps" in cultural understandings of what counts as mathematics learning. The current UK government position is to play down cultural influences on home learning even though the precise form in which home learning is delivered depends on the parents' understanding of the individual child and their development (Goodnow, 1988) as well as judgements of value and cultural practices, often filtered by community experience and past experience (O'Toole & Abreu, 2005).

This paper examines parental understandings about their child's mathematical achievement and the resources they use to go about gaining information in culturally diverse learning settings. Resource is a concept which refers to the way in which the

individual is simultaneously a seeker and provider of information which is open to resistance, interpretation and multiple representations. This examination takes place within a framework which suggests that institutional systems like school reflects a dominating and particular way of looking at children's learning where singular pathways to development, often age-related, are considered "appropriate" or "correct" (Burman, 1994). These conceptualisations influence what we think children should learn and what achievement outcomes are necessary by certain stages of development. As such, expectations for children's achievement are "normed" against particular developmental milestones (Fleer, 2006). The "colonization" of the home by school practices does not attempt to reflect or value family practice but marginalises practices which are not represented by White, middle-class groups (Edwards & Warin, 1999). Equally, parents are privy to limited amounts of information about their child's school life, including their child's mathematics learning and therefore seek other avenues for constructing meaning from an environment from which they are largely excluded.

It is also suggested that when parents utilise and incorporate the resources available to them they do so within the boundaries of particular cultural models (Gallimore & Goldenberg, 2001). Cultural models can be understood in terms of a shared understanding of how the individual perceives the way the world works, or should work. A cultural model is described as:

Encoded shared environmental and event interpretations, what is valued and ideal, what settings should be enacted and avoided, who should participate, the rules of interaction, and the purpose of the interactions (p.47).

Cultural models are often hidden and unrecognisable to the individual and quite often assumed to be shared by others around them. As such, mathematical learning also comes with a knowledge structure which is a reflection of the family or community practices (Abreu, 2008). Parents draw on their own understandings of mathematics learning to make sense of how their child is achieving. The resources they use to do so may have concrete or tangible aspects to them such as discussions with the class teacher or examination results. Others err more towards a cultural model that is representational or symbolic. Both are susceptible to miscommunication and interpretation.

A STUDY OF PARENTAL RESOURCES FOR UNDERSTANDING THEIR CHILD'S MATHEMATICS ACHIEVEMENT IN SCHOOL

The twenty-two parents participating in this study had children in primary schools (ages 5-11 years) situated in a town in the South East of England. Eleven of the twenty-two parents were from ethnic minority backgrounds and the remaining participants were White and British born. The children are characterised as being either high or low achievers in mathematics and were placed as such by their teachers. Data collection took place in three multiethnic schools that are known as

school A (mainly White), school B (ethnically mixed) and school C (mainly South Asian). Data from parents was collected using the episodic interview (Flick, 2000), a method which assumes a shared common knowledge on behalf of the participants about the subject under study. It specifically facilitates the exploration of meanings, representations and experiences. The procedure for analysing the interviews was borrowed from Flick (2000) and based upon the analysis of themes.

Although the study was specifically about mathematics, parents within the sample used this opportunity to talk about their child's education as a whole and therefore the data is highly inclusive of other educational issues. For parents, constructing meaning in relation to their children's mathematics education is like fixing together the pieces of a puzzle and this is managed in a holistic way. In their accounts, parents utilised a varied number of resources to help them construct an understanding of their child's "achievement." The three dominating resources were: (a) the teacher, (b) exam test results and (c) constructions of child development.

Using the teacher as a resource for understanding the child's achievement

The teacher was cited most often as the resource of information about mathematics achievement for the parents in the current study. Of interesting note, is that parents of high achieving children mentioned using the teacher as a resource more than parents of low achieving children (19, 11¹). Furthermore, White British parents mentioned using the teacher for this role more than the ethnic minority parents (17, 13). There are a number of potential explanations for why this might be the case. The parents of high achieving children may not have to worry so much about what will be discussed during consultations, therefore there is less at stake in discussing their child's progress with the teacher. Parents of high achieving, and indeed White British parents are more likely to share cultural models of education, teaching and learning with the school. The discrepancies and conflicts in value positions between home and school for those who do not share cultural models with the school have been well documented by Hedegaard (2005).

On the whole parents' communication with teachers tended to centre around the parent-teacher consultation evening on a twice-yearly basis. Communication between parents and teachers surrounding achievement is complex, and teachers couch many of their descriptions of the child to parents using "teacher talk" whereby descriptions could connote two different meanings. For example, if a child is described as having "leadership qualities" this can also be interpreted as "the child is bossy." "Teacher talk" can produce a discrepancy between the teacher's discussion of the child's mathematics achievement and the parent's understanding of that achievement. For instance, Rajesh's mother asked the teacher in the parents' consultation, "how's he getting on, will he be alright?" and Rajesh's mother recalled that the teacher said:

¹ The figures used in this paper are based on the number of times a resource is mentioned, therefore there are times when one parent mentioned a resource more than once.

Rajesh's mother: "he'll be fine, no point to worry or anything...if he just carries on the way he's doing he's fine" (Indian mother: yr 2, LA)

However, the teacher described Rajesh to me as a low achieving child and his family were categorised as having a low level parental involvement. However, this parent has taken at face-value the message. There are opposing cultural models of Rajesh's learning held by home and school here. Rajesh still struggled to undertake calculations with number below ten, whilst curriculum guidelines stipulate that children of his age should be capable of working with numbers up to 20. This parent has assumed that the teacher would offer the most concrete information around her son's mathematical achievement. Another parent, Fazain's mother, reported a similar conversation she had with a teacher at her son's school:

Fazain's mother: Mr. Headworth, he was saying that he is really good in maths because he comes home and you know, because I improve my maths, you know, a lot. So I teach him, and he's coming really good, he's top in his class (Pakistani mother: yr 6, LA)

Age-related views of mathematics learning are representative of generalised and dominant forms of knowledge which places children outside of these brackets of being an achiever. Fazain was by no means top of his class and was described to me by his classroom teacher a low achieving child. Fazain's mother has attempted to align her own models of mathematics with the schools by improving her own learning, but using the teacher as a resource of information still creates discrepancies.

This next quote from Michael's mother shows what can happen if the interaction with the teacher creates a dissonant cultural model of achievement from the one held by the parent. Michael's mother describes a negative parent-teacher consultation she had experienced. In his first two years schooling, Michael's parents had always been told that he was achieving well. At the most recent parent-teacher consultation, Michael's parents were surprised to be told that he was not doing as well as the others. This change in the representation of her son's achievement by the mother, as a consequence of the teacher consultation, prompted her to question the teacher's judgement:

Michael's mother: As I say, this consultation with Mrs. Edwards didn't even sound like Michael...I thought, she doesn't know this child at all, doesn't even sound like him...and I remember being so cross...and I said to [the head teacher] "what does this child have to do to get any praise?" because I thought it was so unfair. Because he was working hard and yet there wasn't a single thing said that was positive. (White British: yr 2, HA)

Although the teacher was an important resource of information for all the parents as a means of understanding their child's achievement, parents may challenge their opinion if it runs counter to well established models of understanding.

On the whole, parents placed a great deal of emphasis and importance on the teacher's judgement of their child's achievement without always realising that teachers' discourse can be framed to connote multiple meanings. One might speculate

that these discrepancies are even more problematic for the more marginalised parent (such as ethnic minority parents, working class parents, or parents of low achievers), like the mothers' of Rajesh and Fazain, who may have been socialised to understand a more literal educational discourse. For example, these parents took at face-value the “no need to worry” teacher talk. This is unsurprising when models of success are more desirable and the teacher is considered the key authority. Using the teacher as a resource means that conversations take place in a setting which is rigidly framed by a White middle-class institutional structure (Rogoff, 2003) and as such, teachers are in a powerful position. Michael's mother has fewer qualms about challenging achievement representations of the teacher. As such she has the resources to challenge the institutional perspective. It was suggested earlier that using the teacher as a resources of information was tangible or concrete and yet “teacher talk” creates models of achievement which are not necessarily congruent with normative age-graded levels, or parents constructions of their child's achievement.

Using examination assessment results as a means of understanding achievement

Examination results from the Standard Assessment Tests (SATs) conducted in year 2 and year 6 were also resources used by some of the parents. Parents of high achieving children were most likely to speak of examination results in relation to achievement (13, 9), although there was no difference between the White British and ethnic minority parents. In principle, parents should be able to use examinations as a concrete means of understanding achievement. Yet how parents come to understand or use these tests for assessing their child's achievement and construct subsequent cultural models is open to considerable interpretation.

For a start, many of the parents failed to understand how the tests are scored (tests are scored using levels rather than A-G classifications which parents are familiar with). Once again though, parents in this sample of high achieving children had a clearer idea of the scoring system used for the SATs tests. Why this should be the case is uncertain, since the scoring is new for all parents of children currently in the school system. It is likely that these parents are confident in accessing resources like the teacher, websites and shop-bought information books.

The majority of the parents who knew that the SATs examinations were taking place had negative feelings about the tests. Some thought the children were too young and therefore ran counter to their cultural models of appropriate child development practices. Others felt that the SATs examinations were for the schools benefit, and not for the children since results are published publicly and are used to measure the school's success. Rajesh's mother was unique in her opinion about testing and its usefulness in understanding achievement. This may have been because she may have been naïve about how the schools use the test results:

Rajesh's mother: I reckon tests are good because it will show him what he needs to go further on and what he needs to learn...I think he's going to have tests his whole life so

he might as well start now...they're not going to judge the kid, if he's bad or anything it just means he needs more help which is good in a way (Indian mother: yr 2, LA)

Rajesh's mother also held the belief that there would be some kind of positive feedback from the tests, which would help her son realise his mistakes and improve. However, once the final examinations had been finished, none of the schools in this sample revisited the papers and other parents had a stronger insight into institutional motives for testing mathematical achievement. Dale's father shared this low opinion on the value of examinations as a resource for understanding his son's mathematical achievement:

Dale's father: I find going into school reinforces my idea that they put you in a pigeonhole at the earliest opportunity; that's the line, you're this side of the line, you'll always be the worst. Well, all right, he's a couple of digits down on a maths test, it's not the end of the world but to listen to them talk sometimes; is that because of the concern for Dale or is it because they're concerned the school is going to get a bad report because the Stats [*sic*] are down...and I sometimes wonder exactly what it's for, this sort of test thing (White British: yr 6, LA)

Parents described how, in their view, SATs examinations have little value as a tool for helping the child, but are instead used as a form of classification. As such institutional practices are at odds with parental cultural models of what counts as a useful learning experience. Also, the parents look at the SATs exercise with justifiable scepticism. Perhaps these parents know better than Rajesh's mother, that the papers will not be re-visited or used as a learning tool.

With two exceptions the parents of low achieving children had more negative feelings towards the examinations than parents of high achieving children. Parents here were concerned about seeing their children fail, something that is more likely to happen to the low achieving children. Parents' difficulties in interpreting the SATs mathematics examination results revealed that even as a concrete resource of information about the child's achievement, examination results can have their own interpretive problems.

Resources of child development for understanding achievement

One other piece of the educational puzzle, perhaps built upon the most symbolic of all the resources for understanding achievement, was the use of models of child development. Juxtapositioned against the need to understand mathematics achievement was the belief that the children were very much in the early stages of their own development. Parents maintained a cultural model of their children as still being very young which are not necessarily shared by teachers or school as an institution. As a consequence of these dissonant models of child development, tensions were created between home and school. The next quote from Rajesh's mother reveals the conflict between her own model of child development and her desire for her child to be successful early in life:

Rajesh's mother: But then I'm thinking like, his education is important at the moment but it's still a bit of a laugh for him so I don't really want to burden, like I don't want to be like a fussy parent saying I'm pushing him or something...but at the moment you think he's only seven, you don't really want to push him too much, cos you're stuck in the middle. Then you think if he has a good start now then he'll have a good start, you know. I don't know, it's a bit difficult (Indian: yr 2, LA)

Her conflicting model of appropriate parenting and educational expectations for achievement are both tied in with her identity as a good parent. Contained within the quote are three messages which are no doubt conflicting but lead back to her model of child development as the resources of understanding. She does value education and considers it important, but for a boy of 7 years old it should be fun. She is also worried about being perceived as "pushy" if she broke away from her own cultural model of child development. However, Rajesh's mother is unaware that it is her own cultural model of child development which is marginalised by against expectations of the school.

Even when parents have a keen awareness of the cultural models held by the school, these may still be challenged by parents own models of child development. Simon's mother drew on her own experiences as a school child to understand the anomalies between her own cultural models of child development to what her son was experiencing:

Simon's mother: I just think that he's seven, he's in the infants and if I related to when I was in the infants, we never brought homework home until; I think we just had reading. And part of me thinks they're just children, let them be children, you know, if they're happy they'll be learning and I don't want too much pressure on him really (White British: yr 2, HA)

Past educational experiences are embedded in cultural models and linked to the settings where practices take place (O'Toole & Abreu, 2005). Based on these past experiences, Simon's mother has a strong model that school is for learning and home for playing/recreation. Once again, she draws on child development as a resource of knowledge for her cultural model.

A recurrent idea running through parents' models of child development was that of learning as a progressive activity. Learning was viewed by many of the parents as a building block, which develops with the child. The stage-theory representation of child development established through developmental psychology is widespread in these parents' accounts. Learning is described as progressive and based primarily in the childhood years. The crux of the problem is that parents' stage-related views on child development are more varied than one might expect. The variations in parents' models of learning and development are strongly influenced by their own values and experiences, which were culturally situated. However, school as an institution in England relies heavily on constructs established by stage-related theories. Moreover,

they are not necessarily congruent with the models held by the teacher. One of the teachers, Richard, in School B told me:

I still think some parents haven't quite caught onto the idea that they're seven so we should be expecting quite a lot of them. Their expectations of what a child can do isn't as high as our expectations...(yr 2, mixed achievement class)

CONCLUSIONS

When parents talk about their children's mathematics learning they draw on much more than just isolated accounts of mathematics as a subject. Parents try to make sense of their child's mathematics experience by using both concrete and symbolic resources. While some resources, like the teacher and examination results might be considered fairly concrete forms of information for parents, they carry their own problems of interpretation and expectation. For example, whilst "teacher talk" may be a kindness to the parents and child, not all parents have the resources to reinterpret the double meaning. In culturally diverse situations there remains the possibility for discrepancy between the cultural models of learning and achievement between home and school through literal educational discourse. It is noteworthy that the two resources most used, the teacher and examination results, come from the most powerful setting where the knowledge is unidirectional; from home to school. Parents with strong cultural models about their child's achievement can challenge the school. Marginalised parents, or those that sit outside White middle-class institutional confines, tend not to have the resources to either challenge the school or recognise incongruent pieces of information. The least tangible cultural model, child development, resides mostly in the home and is born out of values, expectations, practices and past experiences. This is a resource which is least likely to be shared with the school but is still a pervasive influence in the home.

Furthermore, cultural models and knowledge about achievement have a reciprocal influence on each. A question was raised about whether the cultural model is established before the representation of achievement or whether images of achievement precede the model. The use of cultural models and representations of achievement are seen as constituted from each other, in that they have the power to be transformed, reconstructed and rejected based on the resources that are utilised. In other words, new information about achievement (perhaps resourced from test examination results) may change a cultural model. On the other hand, a steadfast cultural model (perhaps resourced from representations of child development) might be resisted or rejected in light of discussions with the teacher about what a child should be able to achieve by seven years of age.

Whilst institutional practices continue to be dominated by universal/western notions of development which are characterised by White, middle-class value-positions then some homes and their cultural practices will be marginalised. Furthermore, these homes and their families will be positioned as incompetent or lacking knowledge.

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DISCUSSING A CASE STUDY OF FAMILY TRAINING IN TERMS OF COMMUNITIES OF PRACTICE AND ADULT EDUCATION

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SUMMARY

This paper focuses on adult mathematics learners working on their children's algebra problems in high school. These "adult learners" have their own characteristics and dynamics as a group. Therefore we define them as a socio-cultural group. In addition we assume that to reach an identity as a member of a group is something good in terms of learning. For different reasons we have chosen Wenger's idea of "community of practice" to look at this socio-cultural group. However we are not looking at this group of parents as a community of practice, but the process of how this group of people becomes it. To understand how a group of people becomes a community of practice may provide some hints to improve our teaching and learning strategies.

KEY-WORDS: Adult Learners, Family Training, Mathematics Education and Communities of Practice.

INTRODUCTION

People who work in the field of education know that classrooms work better, and students achieve better scores, when they identify as members of a community. Many teachers look for strategies to build these complicities at the beginning of the school year, thus students could become a group[1] of people working together to learn. Much research draws on this image by providing supporting evidence to demonstrate that grouping is better in terms of learning strategies (Lou, et. al. 1996). Drawing on the prior research, some relevant questions implicit in the process of building a group of people may include issues such as how the group works, what type of elements provides unity to the group, what are the main characteristics of the "culture" of the group, and so forth. The processes of support, as well as the solidarity between students, stresses the uniqueness of a milieu that encourages inclusion and learning for all the members of the group. The positive interactions held between the different members of the group promotes a working environment that positively strengthens each member. The result in terms of learning is usually better than the one obtained when this group identity is not present (or when it is a group of people with no cohesion).

The idea of "community of practice" is present in a number of articles and books on Mathematics Education (Cobb & Hodge, 2002, Lerman, 2001, Jaworski, 2006). Usually the "community of practice" is related to good practices, because as Renshaw

(2003) claims there is “kindness” in the word “community;” and this “kindness” makes this concept attractive. However, the concept proposed by Lave and Wenger (1991) and developed by Wenger (1998) is a notion precisely used by Wenger in a particular context (the business). It was not created as a tool to be used in the context of educational research. All the research reviewed in this paper use this notion in a finalistic meaning, presenting the group studied as a “community of practice” already established.

Data used in this paper come from a research project titled “Teacher training towards a Mathematics Education of parents in multicultural contexts” (ARIE/2007 program, number of reference 00026), funded by the General Office of Research and Universities (AGAUR) from Catalonia. In this exploratory case study the focus is on families and Mathematics Education. Our main aim is to use the concept of “community of practice” as tool of analysis, in order to understand if people involved in our study are (or not) a community of practice. We consider that the process of how a group of people become a community of practice is an interesting topic to be analyzed. On one hand this transition step is something that has not been studied in the scientific literature, on the other we think that this process may present key elements to understand how this ideal situation of “community of practice” appears, and what aspects play an important role on it. We are not looking at a “community of practice” already built but discuss a process. Data collected suggests that there is some kind of correspondence between the examples found in our study and what Wenger calls a “community of practice” (1998). We look at these situations because previous research suggests that groups working as communities of practice achieve better results than groups where there is not a sense of cohesion. Our research work was held in a classroom with adult people, and as such is a set of people different from other educative targets.

ADULT EDUCATION: TOWARDS AN UNDERSTANDING OF SOME KEY ELEMENTS AROUND ADULTS’ LEARNING AS A CULTURAL GROUP

In this paper we use Woods’s (1990) and Geertz’s (1973) notions of culture to define the adult learners as subjects of our study. The notion of “culture” has been used broadly with many different meanings. The aim of this paper is not to explore the scope of this idea and its definition but we do want to highlight how we use the term “culture” in our research.

Geertz (1973) define culture as a notion that:

“Denotes an historically transmitted pattern of meanings embodied in symbols, a system of inherited conceptions expressed in symbolic forms by means of which men communicate, perpetuate, and develop their knowledge about and attitudes toward life” (p. 89).

According to his definition “culture” is defined in this paper as a characteristic of individuals related not only to the ethnicity, language, country of origin, or social background, but also all the small groups to which these individuals belong to. In this

sense, we define “adult people” as a particular cultural group, with their own characteristics and dynamics, impacting and determining how the educational process works inside the classroom of mathematics. As Woods (1990) claimed, every single group of people has their own culture, thus we need to analyze it in order to understand the practices carried out by the members of this group.

Drawing on phenomenology, Rogers (1969) showed that all persons exist in a world of experience, which is always changing. This “world of experience” becomes the filter through which we perceive all of what is around us. Talking about how adults learn, Rogers (as well as Piaget) argued that there is a cognitive process of adjustment: when somebody finds that some kind of information coming from the outside (the real world) does not accord to his/her previous [cognitive] schemes. This person then assimilates the new information by accommodating it into his/her mental schemes. From this point of view, to learn is a “learner centred” process where the individual tries to solve the incongruence between what s/he perceives and what would represent (according to his/her previous schemas). This argument may explain why many adults have a common set of values and schemes (because their common background), which distinguish them from other social groups.

Other researchers offered key contributions to the learning theory in Adult Education, such as Knowles (1984) and Mezirow’s (1997) who both differentiate adult individuals as a particular cultural group in terms of their own learning. Knowles (1984) claims that adults are individuals who learn by drawing on their own experience and their “self-concept” (that is: the capacity to move from one being a dependent personality toward one of being a self-directed individual). Mezirow (1997) adds that this learning process is grounded in a dialogue. Before Mezirow (1997) was working on these ideas, Freire (1977) discovered the importance of dialogic action. The Brazilian professor had already demonstrated the power of the word (“la palabra”) as a tool to read the world critically. Drawing on this idea, Freire proposed what he called “Dialogical Method of Teaching.”

Drawing on the ideas of Freire and Habermas, among others, Flecha (2000) proposed what he calls “Dialogic Learning Theory.” The most important concept embedded in this learning theory is the egalitarian dialogue: learning is the result of an intersubjective process of interaction that occurs when learners use the egalitarian dialogue in order to share their prior knowledge with others. Thus the learning process is not unidirectional between teacher and students, but the result of a dialogue. Arguments always are discussed grounded on validity claims, not power claims. Flecha (2000) explains this approach using seven principles (egalitarian dialogue, cultural intelligence, solidarity, transformation, creation of meaning, instrumental learning, and equality of differences), which are the central axe of the “Dialogic Learning Theory.” Learning is a powerful experience for adult people; it really transforms their lives. In addition, learning is reached when it makes sense for them. This is a particular difference with children since adult people already have experiences to build upon new knowledge. Drawing on these principles we can affirm

that adult learners are a particular group, with their own ways of thinking and functioning.

THE NOTION OF COMMUNITY OF PRACTICE AS A METHODOLOGICAL TOOL

Wenger (1998) introduces a learning theory grounded on the notion of Community of Practice in his book *Communities of Practice: Learning, Meaning, and Identity*. This concept has the “three dimensions of the relation by which practice is the source of coherence of a community” (Wenger, 1998 p.72) as a key idea. These three dimensions are: mutual engagement, joint enterprise and shared repertoire.

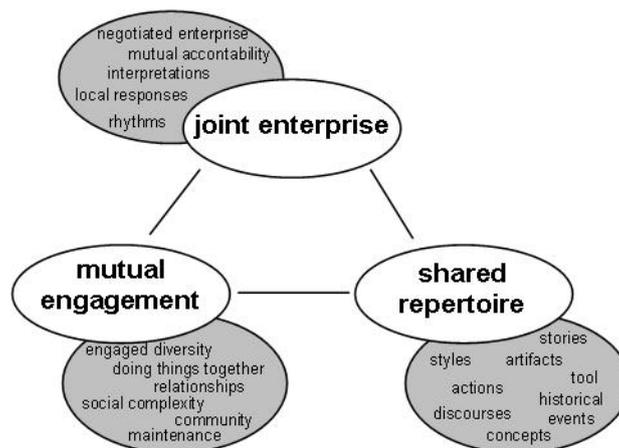


Figure 1. “Dimensions of practice as the property of a community” (Wenger, 1998 p.73)

The concept “community of practice” was created to define a group that acts as an “alive-curriculum” for the learner. For this reason the “community of practice” is a type of community present everywhere, and this is not linked necessarily to a formal system of learning.

The notion of community of practice is more than a group of people with similar (or common) interests, involved in a regular activity. This is not a synonym of group, team, or network. This does not mean (only) to be affiliate to some kind of organization, or to connect with other people (close in terms of geography or social class). This is a dynamic concept, including all members of the community of practice (not just the own participants in the practice which is studied).

Wenger’s (1998) concept of community of practice has been used as tool of analysis more than the theory embedded in it. However this “operationalization” of the theoretical concept cannot be made without taking into account several considerations to avoid doing an incorrect use from the methodological standpoint. [2]

In this paper we use the concept of “community of practice” as tool of analysis, in order to analyze if parents involved in the study became a community of practice (or not). At the same time, we also analyze how this process impacts on teaching and learning practices. Thus the research question is: what type of (social and cultural) processes happen while a group of people became (or not) a Community of Practice?

In order to answer this question, our start points are the 14 “indicators that a community of practice has formed” (Wenger, 1998, p 125). These 14 indicators are specific descriptors of the 3 dimensions quoted before (mutual engagement, joint enterprise and shared repertoire).

These 14 indicators are:

- “1) Sustained mutual relationships – harmonious or conflictual
- 2) Shared ways of engaging in doing things together
- 3) The rapid flow of information and propagation of innovation
- 4) Absence of introductory preambles, as if conversations and interactions were merely the continuation of an ongoing process
- 5) Very quick setup of a problem to be discussed
- 6) Substantial overlap in participants’ descriptions of who belongs
- 7) Knowing what others know, what they can do, and how they can contribute to an enterprise
- 8) Mutually defining identities
- 9) The ability to assess the appropriateness of actions and products
- 10) Specific tools, representations, and other artefacts
- 11) Local lore, shared stories, inside jokes, knowing laughter
- 12) Jargon and shortcuts to communication as well as the ease of producing new ones
- 13) Certain styles recognized as displaying membership
- 14) A shared discourse reflecting a certain perspective on the world” (Wenger, 1998 pp.125-126).

In this paper a series of classroom sessions of mathematics are discussed. Data was collected using videotape. The dynamics generated by the parents involved in the study are analyzed according to Wenger’s 14 indicators. A father and 19 mothers were part of the group. Almost everybody was from Catalonia, although at the beginning of the school year there were also two Latina women. Their children were freshmen in the high school (12-13 years old).

ANALYSING AN ADULT LEARNING GROUP

The group of adult learners took place in a high school classroom in Barcelona city. The learners were a group of parents come together to work on algebra problems. It is a group of people that have deliberately joined together in order to learn mathematics, although some of them knew each other before because they usually came to the high school in order to collaborate in other activities organized by the centre. The group was open to everybody (immigrant and native people, parents of low and high achieving pupils, etc.). Wenger’s (1998) community of practice concept asserts that we can neither build this type of groups as a result of a mandate, nor establish them from the outside. We cannot generate or design these communities either. According

to this viewpoint, “communities of practice are groups of people who share a concern or a passion for something they do and learn how to do it better as they interact regularly” (Wenger, 2007). That means that a group of people may become a community of practice over the time (if they follow the 14 criteria pointed out by Wenger).

Data discussed in this paper comes from the fourth session of the workshop. People involved in this group had been working together for four successive weeks doing mathematics in this classroom. Videotapes show how they were becoming (functional) as a “group” over these four sessions. The identity of every single person of the group became more defined little by little. Analyzing our videotapes in terms of Wenger’s (1998) notion of community, several clips suggest that some of the 14 indicators are achieved (or they are in the way to be achieved), such as indicators 1, 2 and 8 (“sustained mutual relationships – harmonious or conflictual,” “shared ways of engaging in doing things together,” and “mutually defining identities”). A longitudinal analysis of the videotapes indicates that people define their identity collectively (indicator 8). This process produces a number of sustained mutual relationships (indicator 1), and at the same time shared ways of engaging in doing things together appears (indicator 2). The first quote is an example of this type of dynamics. The adult learners are in a classroom placed in a high school and are taking part in an activity of translation: from natural to algebraic language. They are working with first grade equations with one unknown. The facilitator had asked how they solved the problem. Pere is the only man of a group of 20 people (all of them are involved voluntarily in the group). Some of them participate actively in the class. Pere intervenes:

Pere: Me too. Two times x , and then plus two times x .

Facilitator: You wrote two times x , and then?

Pere: One, plus two times x (a noise from the chalk when writing on the chalk board is heard, when the facilitator write on the chalkboard what Pere is saying).

It is interesting to highlight that Pere (who usually is not the protagonist, in the sense that he is not the person who has the highest index of interventions) usually intervenes before the mothers to answer the questions proposed by the facilitator (almost always). This practice always occurs when some kind of explanation or validation is required from the learners. According to this interpretation the role played by Pere is “a person who already has a prior knowledge in mathematics, and who is able to make connections between his ideas and what the facilitator explains, as well as to consolidate this knowledge in the group.”

Another aspect emerging from the data analysis is the definition of learners’ identities as members of the group (indicator 8) in opposition to their children’s identity.

People from the group identify themselves as such because all of them are parents (indicator 6). The variable “generation” becomes a common characteristic of their identity as a group, because it is also connected to their motivation to participate in this workshop of mathematics (and consequently, to consolidate themselves as a group and, perhaps, as a community of practice in the future). This aspect of their identity also helps us to understand the conflict emerging between these people and their children, in terms of teaching and learning mathematics. All these parents have children in the high school, and all the children have difficulties with mathematics. This situation produces a plethora of common experiences shared by all the members of the group. They, as parents, have a different “way to see the world” than their children. This fact, and especially how they have faced this situation as “people who engage in a process of collective learning in a shared domain of human endeavour” (Wenger, 1998), suggests that this group has some characteristics similar to what Wenger defines as a community of practice (1998).

We have observed several clips suggesting that the “parents’ group” and the “children’ group” (implicit in parents’ discourse) have characteristics that may be defined as a culturally different, in terms of Woods (1990). The values shared by parents, as well as the cognitive referents linked to mathematics (ways to act and solve problems), are really different from those used by their children. This difference may explain the “generational” conflict between parents and children, because the culture of each group is not the same. In this next quote the adult learners are once again in a classroom in the high school. The parents are working with a first grade topic “how to solve an equation.” The facilitator solves the problem using one method, and one mother claim that her daughter uses another way to do it. At this point the facilitator explains the method used by the daughter. She has divided the chalkboard into two columns: on the left there is the method used by the facilitator – which is the one known by the mother; on the right the facilitator wrote the daughter’s method – which is the one used by teachers and children in the school):

Facilitator: How it is going? Good?

Mothers: yes... very good (the mom who asked the question is the one who speaks louder).

Mother: We didn’t understand it at home.

Facilitator: eh?

Mother: I didn’t understand it like this at home; this that you have explained to us my daughter used to say “mom, we wrote this here,” and I say “where do you put this?” because I know it in the other w... in the old way (a noise in the background is heard, like admitting she is right) and I was not able to understand it because there is no explanation on the text book.

Facilitator: But, now did you get it?

Mother: (Some mothers agreeing on the background are heard) Kind of, but what happens is that here is so easy... but to me... (She starts to laugh and makes gestures with her hands to say that sometimes the activities are difficult).

Facilitator: ... well... this is the same... but you have to go to....

Mother: (At the same time) now you're getting it, because, because...

Facilitator: (At the same time) to everybody.

Mother: she explains that she does it that way, but I don't know how to explain it....

$2x + 5 = 40 - 3x$ $2x = 40 - 3x - 5$ $2x + 3x = 40 - 5$ $5x = 35$ $x = \frac{35}{5}$ $x = 7$	<p style="text-align: center;"><u>Profe</u></p> $2x + 5 = 40 - 3x$ $2x + 5 - 5 = 40 - 3x - 5$ $2x = 35 - 3x$ $2x + 3x = 35 - 3x + 3x$ $5x = 35$ $x = \frac{35}{5}$ $x = 7$
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Figure 2. Detail of the chalkboard grounded on the field notes.

The problem described in the above quote is common for many families as they experience difficulties in helping their children to solve home mathematics. Those difficulties are sometimes related to mathematics itself and how much mathematics the parents understand themselves. However, other times the problem is the difference between the methods used by parents and the ones used by children (and teachers). One possible reason may be the reforms in mathematics that have changed the procedures used in the classroom to teach mathematics. Figure 2 illustrates the difference between the way used by the mother to solve the equation, and the procedure used by the teacher (of her daughter) to do the same thing. In this figure we can see that while the mother puts all the unknowns[4] together in one side of the equation, and the numbers in the other side of the equal sign, what the teacher does is simplify the expression eliminating the same numbers in both sides of the equation. Both results are the same, but the procedure reasoning implicit is different.

The lack of more opportunities (such as the workshops of mathematics for parents) to connect school and family results in parents having less opportunities to learn what teachers explain in the classroom. Consequently there is no possibility to create a unique discourse about how to teach mathematics. Parents solve the mathematical problems using different strategies grounded on their own methods. But they do not know the methods used by their children (or they just have forgotten them). Then the conflict between them and their children (and more broadly the school) arises. This conflict makes it more difficult for them to get involved in their children' education.

SOME CONCLUSIONS

As a concluding remark, this preliminary data provides evidences that the process of becoming a Community of Practice are not an easy process, neither lineal. It involves definition of roles, interactions, identities, etc. Some indicators appear at different moments, and not according to a prefixed order. In this process some conflicts between actors arise as well. Data shows that there is some kind of generational gap between parents and children (working from a parent involvement approach to the learning of mathematics).

FURTHER RESEARCH

The analysis suggests that when a group is new, every member plays a particular role that becomes part of his/her identity. One question arising from this situation is what is the impact of the role-identity definition process in terms of individual confidence to do and solve mathematical problems? Prior research highlights that self-image (in terms of ability to do/solve mathematics) has a key impact on the self-confidence that everyone has as a mathematics solver/doer. Taking this into account, it is important to analyze the effect that may have the construction of the identity in the process of building a group (being or not a community of practice). Could somebody who is not confident about him/herself feel able to learn mathematics? What is the role of gender in this process? Can the guarantee that everyone has an opportunity to participate ensure that everyone would learn mathematics?

On other hand, in the analysis we have also observed that families and their conflicts with their children doing mathematics may open further analysis to find the elements that affect the relationship between parents and children. The community of practice offers us methodological tools (indicators) to analyze how aspects that define one group could be different for other groups, thus conflicts may be explained because of these differences (contradictions). Consequently a strategy to improve mathematics performances should take into account all the elements that may be defined as “culture” of a particular group (such as prior experience, mathematical knowledge, procedures, etc.) in order to find ways to solve the contradictions (Woods, 1990). In this sense learning approaches such as Dialogic Learning Theory (Flecha, 2000) may be a way forward for further analysis and exploration. However, before that, more in-depth analysis of culture (defined in terms of everyday life) may be needed in order to find hints to bridge the functioning of the different groups. Finally, one more question to be further analyzed is our assumption regarding the impact of “generation” conflict.

NOTES

1. We use the term “group” referring to the people involved in the study because the aim of this study is to elucidate if this “group of people” are (or not) a Community of Practice. For this reason we only use the term “community” when referring to the theoretical concept / definition.

2. “However, it is not clear how to make these learning theories operational from a methodological point of view.” (Gómez, p. 283).
3. All names are pseudonyms.
4. Unknown is “an unknown quantity of variable” (Pearsall, J. (Ed.). (1999). *Concise Oxford Dictionary* (10th Edition). New York: Oxford University Press.

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UNDERSTANDING ETHNOMATHEMATICS FROM ITS CRITICISMS AND CONTRADICTIONS²

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We considered articles from six researchers on the field of mathematics education, in which we identified two categories of criticisms to ethnomathematics: epistemological, related with the way ethnomathematics positioned itself in terms of mathematical knowledge; and pedagogical, related to the way ethnomathematical ideas are implicated in formal education. From this analysis we conclude firstly that it is not easy to criticize a research field so diverse and internationalized as ethnomathematics. Those difficulties are related with the different contexts on which ethnomathematics is pedagogically implicated. Secondly ethnomathematics itself as a research field rejects any dogmatic position, and is aware of contradictions implicated in their pedagogical aims.

Key-words: ethnomathematics, criticisms, contradictions, school, education

THE RADICALITY OF ETHNOMATHEMATICS

To associate the prefix ‘ethno’ to something so well defined, exact and consensual as mathematics can cause strangeness. The idea of a science that is human-proof, as mathematics is in a platonist perspective, is splintered when we associate it with the prefix ‘ethno’. ‘Ethno’ shifts mathematics from the places where it has been erected and glorified (university and schools), and spread it to the world of people, in their diverse cultures and everyday activities. Ethnomathematics as an approach sullies mathematics with the human factor. Not an abstract human, but a human situated in a space and a time that implies different knowledge and different practices to live. Ethnomathematics as a research program is less a complement to mathematics, than a critique to the knowledge that is valorised as being mathematical knowledge.

Ethnomathematics does not restrict its research to the mathematical knowledge of culturally distinct people, or people in their daily activities. The focus could be academic mathematics, through a social, historical, political and economical analysis of how mathematics has become what it is today. As mentioned by Greer (2006), it is

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part of ethnomathematical research to understand the historical development of mathematics as a scientific discipline, the understanding of that development as the intersection between knowledge from different cultures, and the way the validation of what is considered to be true mathematical knowledge is less related with issues of rationality, than with the social and political contexts.

According to D'Ambrosio (2002)³ academic mathematics is the basis of our modern world, upon which rests our faith in science and enlightenment ideas. So, if ethnomathematics aspired to be more than just the study of different mathematical ideas, but also the critical study of the social, political and anthropological aspects of academic mathematics, it assumes itself a critical stance on how mathematics is involved in the maintenance of our modern world. Ethnomathematics wishes to be an epistemological and educational alternative but, above all and this is not always given, a social and political alternative to our modern world.⁴

Given the radicalism of the ethnomathematical program (at least as it is put by D'Ambrosio (2002)), it is not surprising that its emergence has been the target of strong criticism. In our days research on ethnomathematics is numerous and scattered around the world.⁵ It's difficult to have an international perspective on how ethnomathematical research is being done. Hence, to criticize something with so different practices and discourses as ethnomathematical research could result in an unreal chimera, if we don't take into consideration the different contexts on which research is made. A way to surpass those difficulties requires criticizing ethnomathematics as a well defined research program, and by analysing the work of the most important ethnomathematical researchers. That was the path chosen by Rowlands and Carson (2002) and Horsthemke and Schäfer (2006), in the epistemological and educational critique made on ethnomathematics. This critique, we argue, although apparently pedagogical, is an epistemological critique that pretends to highlight academic mathematics as one of the biggest achievements of mankind. In what concerns the pedagogical critique made by the latest researchers, and also by Skovsmose and Vithal (1997), we will articulate the contradictions raised by ethnomathematical researchers. Even among these researchers there are contradictions in how they understand the pedagogical implications of ethnomathematics.

EPISTEMOLOGICAL CRITICISMS

In 2002 Rowlands and Carson wrote an article published in *Educational Studies in Mathematics*, where they make a critical review of ethnomathematics, by comparing the ethnomathematical program to the curriculum of school mathematics. This article was subsequently answered by Adam, Alangui and Barton (2003), which Rowlands

³ But also to the philosopher Heidegger (1977) considered the most important of 20th century by Slavoj Žižek (2006).

⁴ At least, as D'Ambrosio (2002, 2003) put it.

⁵ All those references are present in the bigger version of the paper.

and Carson (2004) later responded to in turn. As raised above, this paper also draws on arguments by Horsthemke and Schäfer who wrote two articles presented at the International Congress on Ethnomathematics in 2006, where they follow most of the arguments presented by Rowland and Carson. Those two sources of criticism present themselves as an educational critique on ethnomathematics but, in the way we analysed the texts, they are above all an epistemological critique, especially the articles from Horsthemke and Schäfer.

Against a nominalist posture assumed by ethnomathematics, Rowlands & Carson (2002, 2004) and Horsthemke & Schäfer (2006) advocate an essentialist position, based on the idea that although knowledge is constructed by humans, remains beyond. This is to say, there is some kind of invariant (an essence) that is repeated in all mathematical knowledge, despite this knowledge being developed in a Mongolian tribe or in a European university, the mathematics involved is the same:

Mathematics is universal because, although aspects of culture do influence mathematics, nevertheless these cultural aspects do not determine the truth content of mathematics (Rowlands & Carson, 2002, p. 98).

The authors positioned themselves against the politicization of science: “mathematics is a science, and its laws, principles, functions and axioms have little to do with issues of social justice” (Horsthemke & Schäfer, 2006, p. 9). Or, as mentioned by Rowlands and Carson (2002) “rationality may be the preserve of an oppressive cultural system but that does not necessarily mean that rationality is in itself oppressive” (p. 82). Represented very strongly in this sentence is the idea that rationality exists *per se*, that is, as something disconnected from the social and political environment. In that sense, mathematics is taken by the authors as a piece of truth and neutral knowledge that could be used to the good and the evil, although mathematics itself is free from judgement: “the odious use of something does not make that something odious” (p. 98).

These authors embraced academic mathematics as a universal human good, shared by all people and considered to be one of the biggest achievements of mankind. This universal knowledge is presented as being the climax of a human evolution, and clearly more precious than others:

The reason we are attempting to ‘privilege’ modern, abstract, formalized mathematics is precisely because it is an unusual, stunning advance over the mathematical systems characteristic of any of our ancient traditional cultures. (Rowlands & Carson, 2004, p. 331)

Finally, the authors adopted an epistemological position in which the genesis and consolidation of knowledge must be understood by analysing the internal logic of that

knowledge and its pragmatic value, suggesting that social and political aspects have no influence in that genesis.⁶

modern conventions of mainstream mathematics have become ‘privileged’ (i.e. accepted by the world’s mathematical community and numerous secular societies) for reasons that have little if anything to do with the politics of nations or ethnic groups, but have much to do with their pragmatic value. (Rowlands & Carson, 2004, p. 339)

EDUCATIONAL CRITICISMS

The tone for the educational critique developed by Horsthemke and Schäfer is the way the application of ethnomathematical ideas into South African schools contributed not to the inclusion, but to the exclusion of children. Ten years before, Skovsmose and Vithal (1997) had developed the same critique, although in a more constructive way. They called our attention to the way ethnomathematical ideas are implicated in schools of countries suffering from ethnic and racial tensions. In the case of South Africa, we can see how those ideas contributed to the creation of a lighter mathematical curriculum (based on students’ backgrounds) to those students considered being ‘ethno’⁷. As a consequence of that politics, those students were systematically excluded from access to academic mathematics then aimed at the white student: “in South Africa bringing students’ background into the classroom could come to mean reproducing those inequalities on the classroom” (p. 146).

This critique on the way ethnomathematical ideas in school could overshadow the access to academic mathematics is also made by Rowlands and Carson. These authors emphasise the dangers involved in not considering formal mathematics as an important part of all students’ education. According to the authors, it is formal mathematics that gives access to a privileged world, and that all students should know how to appreciate that knowledge:

There is every danger that mathematics as an academic discipline will become accessible only to the most privileged in society and the rest learn multicultural arithmetic within problem solving as a life skill or merely venture into geometric aesthetics. (2002, p. 99)

In this sense, the authors defend a clear distinction between the local culture of a student, and the scientific and school culture:

To preserve American Indian cultures, African tribal cultures, traditional cultures of Asia and elsewhere, their uniqueness must be recognised, not collapsed into a dreary and illusory sameness with scientific culture. (2002, p. 91)

Rowlands and Carson are against the use of ethnomathematical knowledge in the classroom, arguing that there may be incommensurable ways of understanding and

⁶ As was done in mathematics during the so called crisis on the foundations of mathematics, where mathematicians like Frege, Hilbert, Russell tried without success to epistemologically understand mathematics by using mathematics. The Gödel results showed what a chimera such enterprise is.

⁷ Black students in the context of apartheid regime.

perceiving mathematics. It is that incommensurability that could make an artificial endeavour in trying to articulate ethnomathematical knowledge with school knowledge. They argue that people can master more than one culture, and school should be the place where people have contact with the more universalized culture, this is, the occidental culture.

Finally, Rowlands and Carson consider mathematics to be a foreign language to all students before they go to school. Contrary to the ethnomathematical stance which argues that students already have non-formalized mathematical knowledge before they start school, these authors argue that *protomathematical* knowledge is not important for learning school mathematics, because all students are equally positioned to learn a new knowledge:

We go to great lengths to point out that children of traditional cultural backgrounds are probably not at any significant disadvantage when it comes to learning mathematics, since it is a 'foreign language' to all novices, regardless their cultural background. (2004, p. 335)

Skovsmose & Vithal (1997) acknowledge the importance of ethnomathematical ideas on a critical mathematics education. They identified four trends in the ethnomathematical research, and stressed that it is in the confrontation with school mathematical curriculum that ethnomathematics finds its greatest challenge, and also the possibility of critique. Firstly, the authors stressed the fact that research in ethnomathematics does not usually specify much about the relation between culture and power. Secondly, they identified a problem with the definition of 'ethnomathematics', and make the question: how can someone educated in formal mathematics identify other mathematics? According to them, ethnomathematics only makes sense through the perspective of academic mathematics. Thirdly, the authors argue that ethnomathematics lacks a critique on how mathematics formatted reality (Skovsmose, 1994). Finally, as mentioned before, Skovsmose & Vithal (1997) think it necessary to *problematize* the idea of students' background, and think not just in terms of the actual culture of students, but also in the aspirations and desires that students have of emancipation, what they called the students' foreground:

Foreground may be described as the set of opportunities that the learner's social context makes accessible to the learner to perceive as his or her possibilities for the future. (p. 147)

According to Skovsmose (1994) all the importance given to students' background could inhibit them from emancipation, and more attention should be paid to the opportunities that the social, cultural and political context could bring to students. By emancipation Skovsmose means the access and participation in a world where mathematical knowledge is central.

SOME COMMENTS ON EPISTEMOLOGICAL CRITICISMS

Before entering into a discussion on the epistemological criticisms made to ethnomathematics, we take the position that the interpretation of ethnomathematics carried out by Rowlands, Carson, Horsthemke and Schäfer is misleading. These authors understand ethnomathematics as an ethnic or indigenous mathematics. In fact, there is a vast diversity of studies in ethnomathematics, and part of them assume that ethnomathematics research consists of understanding, with the tools of academic mathematics, the mathematical ideas of culturally distinct people⁸. In that sense, ethnomathematics is indeed the study of an ‘ethnic’ mathematics:

the prefix ethno refers to ethnicity, this is, to a group of people belonging to a same culture, sharing the same language and rituals, in other words, cultural well delimited characteristics so we can characterize it as a specific group. (Ferreira, 2006, p. 70)

In this sense, the educational implications of ethnomathematics are focused on “how to bring ethnic knowledge to the classroom to allow for a meaningful education? How to establish the bridge between ethnic and institutional knowledge?” (Ferreira, 2006, p. 75). But there are other ways of addressing ethnomathematics. For instance, D’Ambrosio (2004) clearly says that “my view of ethnomathematics try to avoid the confusing with ethnic mathematics, as understood by many” (p. 286). That’s why D’Ambrosio prefers to talk about “ethnomathematics program”, as something more than the study of the ideas and uses of non-academic mathematics. We understand this program as a radical one, in the sense that it endeavours is to criticize, not just mathematics and mathematics education, but social orders and ideologies that feed our current world. As mentioned by D’Ambrosio (2004), “the ethnomathematical program focuses on the adventure of human species” (p. 286). Others like Knijnik (2006) and Powell & Frankenstein (1997) also criticize the idea of ethnomathematics as an ethnic mathematics and have developed investigations where the thematics of power and politics is taken seriously.

The epistemological discussion carried out by Rowlands, Carson, Horsthemke and Schäfer is an echo of a bigger philosophical discussion about the nature of knowledge that was intensively debated in the last decades under the label of “science wars”. As with any philosophical question, there are different ways of analysing it, and everyone has the right to choose the one that better fits its interests. We will not enter in such a discussion here. We just want to call attention to two points. First, in a philosophical line where we can include Nietzsche, Marx, Foucault, Durkheim, Weber, Wittgenstein, Freud, Lacan, Kuhn, Lakatos, Bloor, Restivo, Deleuze, Althusser, Zizek among others, knowledge is perceived from a nominalist perspective, that is, as something which creation, maintenance, valorisation or disqualification has nothing to do with its intrinsic or essentialist value, but with the way knowledge is exercised, whether it is in a language game (Wittgenstein, 2002),

⁸ See for instance the work of Sebastiani Ferreira, Paulus Gerdes and Marcia Ascher.

in the webs of discursive modalities involving power relations (Foucault, 2004), as an ideological discourse (Althusser, 1970), and so on. The meaning and the knowledge we have of something is always contingent, full of historicity, and involved on power relations. As mentioned by Amâncio (2006) the idea of knowledge as something universal, with an existence *per se*, is itself a very ideologically loaded position. Hence, the important aspect of this epistemological discussion is less a discussion on whether knowledge is itself universal or situated, but, as mentioned by Foucault (2004), what intentions, what politics, are behind the claiming that some knowledge (like academic mathematics) is universal?

Secondly, unlike Rowlands, Carson, Horsthemke and Schäfer, we don't think there is a lack of theoretical and philosophical basis for ethnomathematics. Although there is a very diverse and disperse field of research, and also a recent one, there are several studies where the focus is not the ethnomathematical knowledge of groups of people, but philosophy, sociology and political science. Most of those studies use the work of the philosophers mentioned above.⁹

The authors of the essentialist perspective positioned themselves as the guardians of academic mathematics that fuelled this modern world, seen as being superior to any existing society, “the beliefs and practices of other societies are epistemic and vertically inferior to our own” (Horsthemke & Schäfer, 2006, p. 12). From their perspective, we are living the climax of a human evolution, in which academic mathematics is the substrate of a society based on humanistic ideals. This universal society is however problematic. Part of the research on ethnomathematics has been concerned to understand how these universal images of society generate through history¹⁰. As mentioned by Fernández (2006), the idea of such a universal society was possible through “the development of a set of formalisms characteristic of a peculiar way that has a certain tribe, of European origin, to understand the world” (p. 126). That is, the universal society (capitalist society) based on universal knowledge (mathematics and science) suggested by Rowlands, Carson, Horsthemke and Schäfer is a very particular way of understanding time and space, of classifying and ordering the world, of understanding economical and social relations. In short, of conceiving what is possible and impossible to think and do.

CRITICISMS AND CONTRADICTIONS ON THE EDUCATIONAL IMPLICATIONS OF ETHNOMATHEMATICS

Ethnomathematics carries with it a critique on school.¹¹ D'Ambrosio (2003), for instance, compares current school with a factory, where people are components of big machinery that aims uniformity. In school, as mentioned by Rowlands and Carson

⁹ All those references are present in the bigger version of the paper.

¹⁰ See for instance the book edited by Powell & Frankenstein (1997), which collects a set of articles where these ideas are deconstructed.

¹¹ See for instance the work of Ubiratan D'Ambrosio, Gelsa Knijnik and Alexandrina Monteiro.

(2002, 2004), we are introduced to a certain society. And if we are delighted with our current society, as apparently is the case of Rowlands, Carson, Horsthemke and Schäfer, then we must prepare students the best we can to be full members of that society. But part of the studies in ethnomathematics does not share this optimistic view on current society.¹²

Society should be problematized, and not taken for granted, especially when we are aware of the economical politics based on market priorities, and all the ideologies that fuel our way of living (like the liberal view on mankind). What does it mean to educate people to be participative, active authors in a more and more merchandized society? Do we all want “schooling to serve the needs of industry and commerce?” (Rowlands & Carson, 2002, p. 85). Hence, a problematization of society, and the role of school in society is, in our opinion, a priority in a research program like ethnomathematics. But that is far from happening.

For instance, and to speak to one of the criticisms made by Rowlands, Carson, Horsthemke and Schäfer regarding the use of ethnomathematical knowledge in regular schools, we can identify a contradiction on how ethnomathematicians understand this pedagogical implications. On the one hand, as mentioned before, some researchers defend the idea of using students’ ethnomathematical knowledge to construct a bridge for the learning of formal mathematics. But, on the other hand, researchers like Knijnik (2006) clearly said that:

it’s not a matter of establish connections between school mathematics and mathematics as it is used by social groups, with the purpose of achieving a better learning of school mathematics. (p. 228)

Behind these two postures, is the way researchers understand the role of mathematics and school in our society. The problem with the first one, characterized by the “bridge metaphor”, is the reinforcement of the hegemony of school mathematics because the ‘other’ is valorised only as a way to achieve the true knowledge. Thus, it contradicts the critique that ethnomathematics makes to the hegemony of academic mathematics. The same problem identified by the critics regarding the valorisation of background instead of the foreground, is also raised by Knijnik (2006), Monteiro (2006) and Duarte (2006). These authors raise questions about the usually folkloric way ethnomathematical ideas appear in the curriculum. According to them, the use of local knowledge as a curiosity to start the learning of school mathematics could be the cause of social inequalities, as is mentioned by the critics.

But to truly contemplate ethnomathematical ideas in the curriculum is no less problematic. If we focus on a regular school, and take into account its role preparing students to a market orientated society, with all the pressure to learn the mathematics

¹² In Powell & Frankenstein (1997) we can find a set of articles that articulate a critique on mathematics with a critique on society. See also the most recent writings of Ubiratan D’Ambrosio where he developed a social critique, based on the idea of peace.

of the standard curriculum that will be essential to students' approval in the high stakes tests, we can ask ourselves if there is a place for ethnomathematical knowledge (or other local, non scholar knowledge)? Our opinion, according to our review on ethnomathematical research in Brazil, is that those educational implications of ethnomathematics (in a regular school) ended up being phagocytised by a school that, as Rowlands, Carson, Horsthemke and Schäfer would agree, is worried with the uniformization of knowledge. In that sense, we agree with them and also with Skovsmose and Vithal when they say that focussing the learning of mathematics in students' local knowledge could be a factor for social exclusion. But the problem is not just in ethnomathematics, but in school itself. Monteiro (2006), a very well renowned ethnomathematicians makes the definitive question: "Is it possible to developing ethnomathematical work in the current school model?" (p. 437).

Hence, it is not just the valorisation of students' background that should be dealt with care, but also the valorisation of students' foreground. Although we realise the importance of students having the opportunity for emancipation, and for full participation in a technological world (that is also a capitalist world based on a liberal idea of economy that stress the individual above the social), we should criticize naïve and ideologically loaded ideas about society. Preparing students to become participants in a society is also preparing them to assume critical points of view about society, different ways of thinking, acting and doing mathematics. Using the words of D'Ambrosio, we need to emancipate students by learning academic mathematics, but also by reinforcing its roots. If we analyse the role of school in modern societies, this is obviously a paradox.

Critical mathematics education and ethnomathematics, as mentioned by Skovsmose & Vithal (1997), have common concerns. Both developed a critique of the way mathematics is usually understood as one of the biggest achievements of mankind, and the intrinsic resonance (seen as something inherently good) that feeds its education. But in the struggle for a better mathematics education, they should take care when suggesting pedagogical proposals to be implemented in a problematic school. Taking school for granted is the best way to trivializing critical and ethnomathematical ideas.

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USING MATHEMATICS AS A TOOL IN RWANDAN WORKPLACE SETTINGS: THE CASE OF TAXI DRIVERS

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The present study is part of an ongoing study of which the aims are twofold; to provide knowledge about why and how mathematics is involved in specific workplace settings, and to provide student teachers with culturally relevant examples to contextualise school mathematics for secondary school students. Observations and semi-structured interviews were conducted in the workplaces of two taxi drivers, one house constructor and one restaurant manager. The focus here is on taxi-drivers. The analyses draw on ideas from socio-cultural theory and the anthropological theory of didactics. A common main concern was economic profit and risk of loss; level of justification, mathematical problems to solve and techniques used differed. Among the taxi drivers, silent and taken-for-granted cultural knowledge were used.

INTRODUCTION

After the 1994-genocide, the Rwandan society was destroyed and disorganised in all sectors. In order to cater for capacity building, the Government of Rwanda has undertaken several measures in all economic sectors through its Vision 2020 for developing Rwanda into a middle-income country (Republic of Rwanda: Ministry of Finance and Economic Planning, 2000). For instance, in the educational sector, the Ministry of Education (MINEDUC) has embarked on prioritising the teaching and learning of science and technology (including mathematics) to provide human resources useful for socio-economic development through the education system. MINEDUC recommends that learning should be context-bound. This means that in order to serve the local society, teachers and researchers are encouraged to bring material to the students that are taken from national contexts. For instance, exploring mathematics via tasks from workplaces may support students to learn in ways that are personally meaningful (Taylor, 1998). Contextualising mathematics allows students both to understand the role of mathematics in solving different workplace problems and see ways in which mathematics is used out of academic institutions. They can also realize that such activities can be translated into mathematical language that is taught in different institutions.

However, before we embed mathematics in workplace settings, we should have a clear picture of the use of mathematics in such contexts. This is of crucial importance especially in Rwanda where this kind of research is relatively new and where mathematics is mostly seen as an abstract and hidden science which does not provide visible applications in workplaces (Niss, 1994; Williams & Wake, 2007).

In this study the use of mathematics as a mediating tool (Vygotsky, 1978) supports workers to solve problems related to the earning of their income, using culturally

relevant concepts and experiences (Cole, 1996; Abreu, 1999) when seeking survival means is investigated. Therefore, the current study will provide knowledge about why and how mathematics is involved in three workplace settings: daily taxi driving, house construction, and restaurant management. Although the workplace settings are quite different and subject to change over time, the choice was made with the intention to understand mathematics in use in workplace settings where the actors perform differently but aim to achieve the same goal – to earn a good living. Within this study, the present paper will focus on the taxi driving context.

STUDIES ON SITUATED MATHEMATICS

Over the last thirty years, researchers have investigated how mathematics in everyday practices differs from what was taught at school and in academic institutions. In this endeavour Lave (1988) found that mathematics practice in everyday settings is structured in relation to ongoing activities. Based for example on the use of shoppers' "best-buy" strategies, she points out that mathematical practices in work places do not require any imposed regulation. Rather, adults use any available resources and strategies which could potentially help to solve a problem. Also, in a collection of studies related to informal and formal mathematics, Nunes, Schliemann and Carraher (1993) found that there was a discrepancy between street mathematics and school mathematics. This is demonstrated through a mathematical test which was given to the same children who performed better out of school than in a school setting. This discrepancy is due to the fact that at school children tried to use formal algorithms whereas in real situation they did arithmetic based on quantities. It should be noted though that the requested arithmetic procedures were quite simple. In results from a study related to college mathematics and workplace practice, Williams, Wake and Boreham (2001) found that the conventions of school and workplace graphs might be different. Indeed, in a chemical industry, school graph knowledge was not enough to allow a college student to interpret a graph of chemical experiments. However, the college student was able to interpret it with the help of an experienced employee. In a recent study Naresh and Presmeg (2008) followed a bus conductor in India in his daily practice, where they observed that though he performed significant mental mathematical calculations the bus driver's attention was fully concentrated on the demands of his job, making his mathematical work more or less invisible to him.

From the results of the above studies, we conclude that when it comes to solve a particular problem, the way mathematics is used at work is different, however logically organized (Abreu, 2008), compared to how it is used in academic institutions. At a workplace the problem solvers keep the meaning of the problem in mind while solving it in the real situation. In contrast, in the academic institution, the meaning of the problem is often dropped because of the imposed curriculum regulation where the problem solver is expected to employ certain mathematical symbols and conventions.

Researchers have also studied mathematical concepts and processes that are used in different workplace settings. In a study on mathematical ideas of a group of

carpenters, Millroy (1992) found that not only are many conventional mathematical concepts embedded in the everyday practices of the carpenters, but their problem solving is enhanced by their stepwise logical reasoning similarly used in mathematical proofs. Abreu (1999) also found that Brazilian sugar cane farmers used indigenous mathematics to control their income. However, over time, technological innovations in measuring quality requested change to more school-like problem-solving strategies which made farmers prone to abandon traditional units of analysis and value their children's success at school mathematics. A study by Massingila (1994) revealed that mathematical concepts and processes are crucial in carpet laying practices such as estimation and installation activities. Furthermore, she found that measuring and problem solving are two major processes in the carpet laying practice. In their exploratory study related to how mathematics is used and described in workplaces in the context of employees in an investment bank, paediatric nurses, and commercial pilots, Noss, Hoyles and Pozzi (2000) found that practitioners use mathematics in unpredictable ways. Hence, their "strategies depend on whether or not the activity is routine and on the material resources at hand" (p. 17).

A common point to all these studies is that mathematical strategies that are used at workplaces differ to those taught at academic institutions. A mathematical strategy for solving a problem refers to a 'roadmap' that consists of identifying the problem to be solved and the appropriate technique(s) that allow solving that kind of task. However, in the above mentioned studies mathematical strategies are described as applied by workers without details about how they are or may be underpinned by mathematical justifications. Mathematics is seen as a tool to mediate human activity through the lens of workers' goal achievement. None of them looked at mathematics through the lens of its knowledge organisation, including types of problems worked on, as well as methods used to solve them and their justification (cf. Bosch & Gascon, 2006). To fill this gap the current study emphasises mathematical practices and its justifications embedded in mathematical activities found in specific Rwandan workplaces and their relation to academic mathematics.

MATHEMATICS AS TOOL TO MEDIATE WORKPLACE ACTIVITIES

Human activity is always goal-oriented and characterised by two major parallel actions: thinking and acting. The action is shaped by thinking and inversely through available socio-cultural tools for goal-oriented activity. Human mind and activity are always unified and inseparable. This means that the "human mind comes to exist, develops, and can only be understood within the context of meaningful, goal-oriented, and socially determined interaction between human beings and their material environment" (Bannon, 1997, p. 1). In activity theory, social factors and interaction between agents and their environment allow us to understand why tool mediation plays a central role. Tools shape the ways human beings interact with reality and reflect the experiences of other people who have tried similar problems at an earlier time (Bannon, 1997). Tools are chosen and transformed during the development of the activity and carry with them a particular culture. In short, the use

of tools is a means for the accumulation and transmission of social knowledge. At the same time, they influence the nature of external behaviour and the mental functioning of individuals.

Engeström's (1993) model of basic human activity systems comprises six main elements: subject, object, tools, rules, community, and division of labour. He also suggests that such systems always contain "subsystems of production, distribution, exchange, and consumption" (ibid., p. 67). The present study is located in the subsystem of production which is mainly characterised by interactions between subject, tools and object. Within the production activity, subjects chose and transform useful tools that match a prior defined object to achieve a desired outcome.

However, our study will not elaborate on the production process as such. It will rather focus on the sub-production related to the selection and transformation of useful mathematics that facilitates the concerned subjects to achieve their goal on their respective workplaces. In other words, the study will investigate how the selected mathematics is organised so that the workers may interpret it in terms of the outcome of their activities. At that stage, it was imperative to add a complementary theory which explains deeply about the organisation of mathematical knowledge.

We will thus use a theoretical model from the anthropological theory of didactics (ATD), viewing teaching and learning as an activity situated in an institutional setting (Chevallard, 1999; Bosch & Gascon, 2006). By engaging in this activity, the participants elaborate a target piece of knowledge for which the activity was designed. This perspective sets a focus to the knowledge itself as an organisation system (a *praxeology*), including a *practical block* of types of tasks and techniques to work on these tasks, and a *theoretical block* explaining, structuring and giving validity to work in the practical block (Barbé, Bosch, Espinoza, & Gascon, 2005). This praxeological organisation of knowledge can be used to describe very systematic and structured fields of knowledge (such as mathematics or any experimental or human science) and its related activities, with explicit theories, a fine delimitation of the kind of problems that can be approached and the techniques to do so. Considering the mathematics teaching and learning process, we can find two different (intimately related) kinds of praxeologies: mathematical ones, corresponding to the subject knowledge taught, and didactical ones, corresponding to the pedagogical knowledge used by teachers to perform their practice. For the purpose of the present paper we will look into the mathematical praxeologies (or *mathematical organisations*) observed at the different workplaces.

Aims and research questions

The study reported in this paper is from the first part of an ongoing research project aimed at finding ways to contextualise school mathematics within cultural mathematical practices in Rwanda. In this project, the researcher documents the rationale and characteristics of mathematical practices in local workplace settings, to serve as a source to design contextualised mathematical activities for student teachers

in a teacher education programme. From the experiences of working on such problems, the student teachers will design tasks contextualised in the local culture for secondary school students, whose work on these tasks will then be analysed. In this three-stage process, the didactical transposition (see Bosch & Gascon, 2006) of the workplace mathematical practice, via the mathematical tasks designed for and solved by student teachers, to the school students' contextualised mathematical work will be analysed.

The general question about why and how mathematics is involved in specific Rwandan workplace settings was split into specific research questions. First it was important to clarify what motivates the workers to involve mathematics in their daily activities (the why-question). In this regard, the interest was on what problems workers solve at their workplaces. Next there was a need to look at how those mathematical problems were solved. The answer to these questions raised the issue of justification of mathematical techniques used (the level of logos in the mathematical organisation observed). Using the ATD framework the following research questions were thus set up: What types of mathematical problems do workers solve at their workplaces? What techniques do they use to solve their mathematical problems? How are the techniques used justified?

THE EMPIRICAL STUDY

Method

In this interview study the data-collection was performed by the first author who is familiar to the field. Four workers from the three workplace settings volunteered to participate in the study, a female restaurant owner, a male constructor and two male taxi drivers. Three visits were conducted to each workplace. The purpose of the first visit was to inform the participants why and how he wanted them to be involved in the research. On this occasion, they agreed that he was permitted to observe and interview them about the use of mathematics in their daily activities. On the second occasion, after three weeks, the purpose was to observe and conduct the first semi-structured interview in order to understand how mathematics helps the workers to achieve their goals in their respective work sites. Three months later, a third visit was conducted to strengthen the understanding of the mathematical organisations. On that occasion, supplementary semi-structured interviews and observations were conducted. The interviews were performed in Kinyarwanda, a common language to all involved parties. Field notes were taken and interviews were tape recorded and transcribed at all visits. In the analysis we have used ideas from activity theory in which we draw on the object of activity to elucidate mathematics as one among the involved mediating tools in the activity. The analysis does not encompass the whole activity system; rather it focuses on the subsystem of production. The reason is that the purpose of the study is specifically to shed light on mathematics as a tool to help the participants to achieve their outcome. This part of the analysis illuminates the mathematical problems that are embedded in the workers' activity. Regarding how mathematics is used by workers on workplaces, the analysis draws on ideas of ATD,

especially on its notion of mathematical organisation (MO). To perform this analysis we will build on a reference MO (Bosch & Gascon, 2006, p. 57), based on our own knowledge of academic and applied mathematics, in order to be able to analyse the observed MO in the workplace settings and on the interview data.

Findings

Due to space limitations detailed data on the observed mathematical organisations will be reported only from the taxi driving workplace. We will provide knowledge about the mathematical basis they use to determine the estimated transport fee charged to the customer. The taxi driving profession in Rwanda is mostly exercised by citizens with limited school background. The majority of taxi drivers consider the driving license as their core means of generating income. Some of them drive their own cars whereas others are employed. Taxi driving is mostly done in towns where you find financially potential people able to use taxi as a means of transport. Rwanda has not yet any explicit policy or norms and regulations that taxi drivers should follow to charge their customers. Because of lack of taximeters in the cars, the cost is negotiated between the taxi driver and the costumer.

From the transcripts of the interviews conducted with two taxi drivers, an employed (A) and a car owner (B), their main concern seems to be a non fixed level of profit and to avoid the risk of loss. Due to the difficulty of determining the number of customers every day, the estimation of costs depends mainly of considering control of factors such as road condition (good/bad), trip distance (in kilometres), quantity of petrol that the car consumes for a given trip (measured by money spent), waiting time (if necessary), and the time of the day (different day and night tariffs). Following an agreement between driver A and the employer, A was not responsible for expenses such as taxes, insurance, spare parts and so on. Also, A and his employer had agreed that A must deposit 5000 Frw every day to B and A's monthly salary was 30000 Frw. When the drivers were asked about their mathematical reasoning process while estimating costs, they always referred to authentic examples like pre-fixed estimations and rounded numbers without detailed calculations. In the interview, A gives an example of how he calculated the costs for a trip Kigali – Butare on a high quality tarmac road.

Interviewer: Ok.. let's take an example. Has it happened to you that you have taken a client from here [Kigali] to Butare?

Driver A: Yes, many times.

Interviewer: Could you explain to me how you have estimated the price?

Driver A: A one way of that trip is about 120 kilometres. The estimated cost for that trip was 30000 Frw. It means that I considered the cost of the petrol about 12000 Frw and I remained with 18000 Frw ...

But sometimes it happens that while I am on my way of returning back, I meet customers and depending on how we negotiate the cost I charge him

3000 or 5000, it depends ... But when estimating the price with the customer before the departure, I ignore this case because there is no guarantee to have this chance

This extract shows that the estimation of cost was made with respect to the cost of petrol and the driver's profit only. Road conditions were probably not mentioned as both interviewer and interviewee were assumed to be familiar with it. Transports between Kigali and Butare are frequent as contacts between the National University in Butare and official administrators or foreign aid agencies and others in Kigali take place on a daily basis. The next example is taken from a less frequented distance.

Interviewer: OK. Ok let's take the case of a Kigali – Bugesera trip. Although the road is now becoming macadamized it was always used as a non macadamized road. How much do you estimate for instance when you bring somebody there?

Driver A: ...distance is almost 50 kilometres...then the return trip is 100 kilometres. But because of the poor road conditions, the cost is estimated at 15000 Frw. In that case I assume that the car is going to consume petrol for 5000 and I remain with 10000.

In the above extract, the estimation of the trip cost was made according to road condition, cost of petrol and the driver's profit. A seems to assume that more petrol is needed if the road is of bad standard but looking at Example 1 the same unit (10 km for 500Frw) is used. However, in Example 2 the driver does not seem to expect to be able to pick up a new passenger for the return trip.

In the second interview with B, the owner of the taxi, he explains how he estimates costs in relation to distance, price of petrol and time.

Interviewer: Let me ask you one explanation... for example when you charge a customer a cost of 1500Frw ... what is your basis for that price?

Driver B: Do you remember I told you that with the petrol of 1000 Frw, I usually go 20 kilometres? Now when the customer tells me the destination I start to think of the number of kilometres to reach there. Then you say this time one litre of petrol costs for example 550 Frw... Approximately my car consumes 50 Frw to go one kilometre. This means that to go a distance which is not more than 10 kilometres for a return trip my car uses 500 Frw. So if I transport the customer to that destination without any waiting time I should have 1000 Frw for a work time less than 20 minutes... Do you get my point?

Like driver A, B calculates with rounded thirds, one third for petrol, one third for time spent and one third as a profit. As he is the car owner he could also have calculated with taxes and other costs involved with keeping a car.

Analysis of the observed MO

To characterise the MO observed in this taxi driving workplace setting, the type of problems involved could be described as varying versions of calculating the value of a function symbolically written as $W = F(x, y, z, t) + P$, where W is the estimated cost that the driver suggests to the customer. This cost consists of a non-fixed profit P and a cost F for the driver, estimated from all or a few of the four variables road condition (x), covered distance (y), petrol consumption (z) and time (t). Referring to the examples shown above, in the case of waiting for the customer the problem simplifies to $W = F(t) + P$, while the case with a short distance on a bad road will increase both the time and petrol needed: $W = F(z(t(x))) + P$. When the road is good but the distance longer it is the distance which is the deciding variable, $W = F(z(t(y))) + P$, which in the case of also a bad road changes to $W = F(z(t(x, y))) + P$. The techniques used by the drivers to solve these different types of problems are based on rounded estimations of basic costs, without providing a rationale of the amounts mentioned, and when needed elementary arithmetic operations are performed on these rounded numbers. For example, for the Kigali-Butare trip the model $W = F(z(t(y))) + P$ was used, with $y = 2 \times 120$ km and $W = 30000$ Frw with $z = 12000$ Frw and $P = 18000$ Frw. In the case of the Kigali-Bugesera trip the road was not macadamized and thus in a bad condition and the model $W = F(z(t(x))) + P$ was applied, where $W = 15000$ Frw and $P = 10000$ Frw with $y = 2 \times 50$ km. Technologies included number facts of addition and subtraction of natural numbers, and simple multiplication facts such as doubling. All numbers used were contextualised with units of distance and currency and no justification of the mathematical techniques used was referred to. Rather, it could be described as silent knowledge, adopted by experience and exchange with colleagues.

CONCLUSIONS

In Rwandan society as well as elsewhere in the world, the utility of mathematics is recognized through several activities. Those activities are seen on the one hand in academic institutions such as in schools and universities, where mathematics is used and learned for the purpose of developing knowledge about the subject per se; and on the other hand at different workplaces, where mathematics is used as a mediating tool to facilitate production within the workplace. The present study is partly an answer to policy departments' demands for a more contextualized mathematics education with a move away from using pseudo-problems to more culturally adapted problems. However, one aim is also to meet a theoretical challenge that attempts to combine sociocultural theories with Chevallard's anthropological theory of didactics. The latter makes possible an analysis of the observed knowledge organisation of workplace mathematics (in this case of taxi driving in Rwanda) that deepens the understanding of the purpose and function for the worker of using mathematics.

In the current study our focus was on taxi driving. A pre-determined common object for the drivers was to avoid any risk of loss while generating their income. The taxi drivers chose an appropriate mathematical organisation (MO) among other tools to

mediate their activities, as described above. The observed techniques used by the subjects build on basic arithmetic related to addition and subtraction. Taken-for-granted cultural knowledge is seen in the example when the drivers request a higher profit for the distance Kigali – Butare as most local people travel this distance by frequently running minibuses. Taxis are for those who can pay. For community members the return fee to Kigali is subject to negotiation.

The way in which elementary arithmetic is applied should be understood in the context of continuous control of changing situational and cultural factors which make up a fundamental basis for the drivers' success. The observed MO is characterised by techniques which are functional to the problems at hand, the cultural constraints and the educational background of the drivers. As long as they are pragmatic for the goals of the activity, no further justification of the techniques is needed, resulting in a MO with undeveloped logos. This is reflected in the evident fact the drivers' goal is not to develop knowledge in the discipline of mathematics. What is functional at workplaces may in some cases be less functional in an educational context, where levels of justification often play an important role. However, these sets of constraints will form a background to the series of didactic transpositions that will occur before workplace mathematics can be used to contextualise school mathematics. This is a challenge for continuing research in this field. Moreover, the documentation of constraints and possibilities with which taxi drivers operate contribute to the ecology of mathematical and didactical praxeologies.

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PARENTS' EXPERIENCES AS MEDIATORS OF THEIR CHILDREN'S LEARNING: THE IMPACT OF BEING A PARENT-TEACHER

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This article discusses the way parents' past experiences influence the construction of their mathematical identities, their representations and their valorizations of current school mathematics, and how these factors mediate involvement with their children's mathematical learning. Two different groups of parents, with and without teaching experience, were interviewed. Participants within the groups showed similarities in the ways they constructed their own mathematical identities, and differences in how they constructed representations and valorizations of current school mathematics. Whilst those with teaching experience generally held more positive representations of current practices, the way they valued these practices changed according to their perceptions of their child's needs, and the various roles they adopted.

Key words: *parents; home-school; identities; representations; valorizations*

INTRODUCTION

The William's Report argues that parental involvement in schooling is a powerful force, and that 'parents are a child's first and most enduring educator, and their influence cannot be overestimated' (Department for Children, Schools and Families, 2008, p.67). However, research indicates that parental involvement in their children's education is complex. In a study reported by the Department for Children, Schools and Families (2007), it was found that whilst 73% of parents feel it is extremely important to help with homework, confidence amongst parents to become involved has decreased in recent years. Barriers to successful interaction may be particularly evident when parents and children work together over mathematics homework (Abreu & Cline, 2005; O'Toole & Abreu, 2005). Societal and cultural changes (e.g. National Numeracy Strategy, UK, 1999; immigration) are among the factors which have resulted in very different experiences of mathematics learning by both parents and children (O'Toole & Abreu, 2005). Abreu and Cline (2005) found that many parents were confronted with differences between their own ways of tackling mathematics and methods their children learned at school. Parents developed sophisticated representations of these differences, the most common concerning teaching methods and tools (e.g. calculators) available in the classroom. They also found that even when parents share knowledge of different methods to approach calculation, they may have a different understanding of how these methods are valued, and it is the position they adopt towards these shared representations that may affect how they organize mathematical practices for their children (Abreu & Cline, 2003). Abreu (2002, 2008) proposes that it is *participation* in particular practices

which enables individuals to master cultural tools, and to understand how these are socially valued. For parents whose experience of learning mathematics was algorithmic rather than conceptually based, new methods of learning may remain inaccessible and they may be expected to support their children's learning in ways that don't make sense to them (Remillard & Jackson, 2006).

THE RESEARCH QUESTIONS

Many studies have examined the response to perceived differences in numeracy practices in minority cultural groups (Abreu, 2008; Abreu & Cline, 2005; O'Toole & Abreu, 2005; Quintos, Bratton & Civil, 2005; Civil & Andrade, 2002). In Abreu's previous studies, it was apparent that both parents' own experience of mathematical learning in a different cultural setting, and their lack of direct exposure to current school mathematics, impact on their understanding of their children's mathematical learning. This study seeks to understand further parental participation in their children's learning within the majority (White-British) cultural group, in terms of how this group experiences their children's mathematical learning in the context of historical changes between their school education and the education of their children. In addition, the study seeks to explore further the impact of parents' personal histories on their involvement with their children's learning, in terms of their experience of direct participation in current methods of learning. In this way, the study can shed light on issues that are specific to curriculum changes over time within a society, and issues that are more related to minority cultural groups. The study explores the experiences of two different groups of parents, those with teaching experience (direct participation in current teaching practices) and those without, with a view to determining similarities and differences in the way the participants in each group interpret their past experiences, construct current representations, and use these representations to mediate interaction with their child. The research questions investigated were: (1) What are the similarities and differences between the parents of these two groups in the way they construct their mathematical identities, and how does different adult experience affects these identities? (2) How do the parents from the two groups construct representations of current school mathematics, and how do they value perceived differences between current school mathematics and their own? (3) How do the parents from the two groups use their representations and valorizations of school mathematics to mediate interaction with their children's learning?

METHODOLOGY

Two groups of six White-British parents were interviewed. All participants had attended schools in the UK during the late 1960's - early 1970's, were university-educated, and all had children currently attending Primary schools. One group ('parent group') had no teaching experience, and were recruited through a Primary

school in Oxford. The other group ('parent-teacher' group) had varying teaching experience. Four of this group had teaching experience prior to the National Numeracy Strategy, had taken a career break, and were selected from a Return to Teaching course organised by the Teacher Development Agency. These parents had undertaken recent placements in Primary schools which involved teaching numeracy, and could therefore compare their experiences of teaching numeracy both before and after the educational reform. The remaining two parent-teachers had recently trained as Primary teachers, and were able to draw on their experience of helping their children with their homework prior to their training.

Procedure and tools for data collection: An episodic interview (Flick, 2000) format was used as this method of questioning encourages participants to give their opinions about the subject matter, and to give concrete examples of situations in their past. The interview covered basic information, and explored the interviewee's biography in relation to their mathematics learning, current uses of mathematics, and their experiences of helping their children with school homework. For parent-teachers, their teaching experience was also explored. All participants were interviewed in their own homes for approximately 45 minutes, and interviews were audio-recorded.

Data analysis: The interviews were fully transcribed and analysed using thematic analysis (Braun & Clarke, 2006), taking into account the research questions, key concepts from the literature, and new information emerging from the data. The coding was supported by NVivo qualitative analysis software. Initial thematic maps grouped sub-themes together into super ordinate themes as described in Table 1. The data was then examined for similarities and variability between the two groups of participants.

Table 1. Superordinate themes and sub-themes.

Superordinate themes	Sub-themes
1. Parent's mathematical identities	1. Memories of mathematics learning - emotions 2. Perceptions of own ability 3. Social value of mathematics in family/peer group 4. Effect of parent's identity on child's identity
2. The effect of adult experience on identity	1. Effect of work experience on identity 2. Effect of teaching experience on identity
3. Parents' representations of school mathematics	1. Knowledge/understanding of current methods 2. Perception of own school mathematics as same/different 3. Effect of teaching experience on representations
4. Parents' valorizations of	1. Equivalence of/confidence in different methods

different practices	
5. How different representations and valorizations influence interaction	<ol style="list-style-type: none"> 1. Effect of representations and valorizations on interaction 2. Valorization of methods by parent and child 3. Effect of teaching experience on interaction 4. Emotional aspect: frustration/fear of confusing child

FINDINGS AND DISCUSSION

Parents' mathematical identities

Three main themes were revealed in participants' perceptions of themselves as mathematics learners: their perception of their ability, memories of the emotive nature of their mathematics learning experiences, and their status as a learner amongst family and peer group. Participants in both groups were similar in that their assessment of their cognitive competence in the cultural tools of mathematics formed a significant part of the way in which they constructed their mathematics identity. The data also indicates that participants' view of their mathematics ability did not solely rely on their perception of their competence, but was strongly influenced by their feelings about their experiences. For example, Table 2 shows that there were parents from both groups for whom learning mathematics was remembered as a struggle and was associated with fear and panic. Tilda talks about 'feeling lost for ever, for ever after'.

Table 2. How emotions mediate mathematics identity.

Parent group	Parent-teacher group
<p><i>P</i>: I can remember saying, "I don't understand," and him trying to explain it, and I was none the wiser. I can actually remember saying, "Help!" I mean he tried but it was no good, and then I can just remember being lost for ever, for ever after ... I think I was always quite good at just basic maths, but with algebra or anything like that, I'd always be frightened. [I felt] a sort of terror, fear. <i>Tilda, parent</i></p>	<p><i>P</i>: I think it got to that point where sometimes you'd go, "Oh, I can't do that!", and your brain freezes, and your brain would stop working and decide that it can't do this. <i>Rebecca, parent-teacher</i></p>

For both parents and parent-teachers, their mathematical identity relied strongly on how they were identified by significant others, for example, parents and teachers, and their perceptions of their ability in comparison to siblings and peers. Parents in both groups hoped that their child would construct a positive mathematics identity, and for many, it was more important that their child have a confident relation with

mathematics, than be expert in the subject. The consequences of parents identifying themselves, or their children, as less competent, resulted in participants from both groups positioning themselves, or their child, as an ‘arts’ person rather than a mathematician. In positioning themselves in this way, they devalued mathematics as something not necessary to succeed. Consequently, this may have limited their capabilities in mathematics, or their expectations for their child. Many showed awareness of how their own parents’ mathematics identity had influenced the way they perceived themselves as mathematicians, and how this could, in turn, influence their children’s identity. As illustrated in Table 3, Tilda felt it was extremely important not to let her daughter know that she wasn’t a confident mathematician, whilst Clare understood that her own identity was interlinked with her father’s.

Table 3. How parents’ mathematical identity can affect their children’s.

Parent group	Parent-teacher group
<p><i>P:</i> I’ve got a friend that says, “I was crap at maths, so my kids are crap at maths”, that’s what she says. And she has a daughter who isn’t doing so well in maths, but she’s taking it as an absolute given that that is how it will be and I suppose I don’t ... I’ve never said to [Lily] I wasn’t any good at maths because that would be a dirty little secret I would keep to myself! <i>Tilda, parent</i></p>	<p><i>P:</i> My dad was a maths teacher for a while, and he used to get really frustrated with me, helping me with maths, because he’s sort of mathematically-gifted, he sort of finds it easy. So there was this conflict in my relationship with my dad ... and I didn’t see myself as a natural mathematician. <i>Clare, parent-teacher</i></p>

The effect of adult experience on identity

The research revealed that parents in both groups felt that they had developed a more positive relation with mathematics due to experience during adulthood (see Table 4).

Table 4. The effect of adult experience on mathematical identities.

Parent group	Parent-teacher group
<p><i>P:</i> I think it’s practical maths ... because once you actually leave school and you start working, you have to use maths on a day to day basis, and suddenly it all starts to make sense, and depending on the kind of work you do ... I’ve always learnt by rote, managed to get through, and then latterly actually as</p>	<p><i>P:</i> It’s interesting actually as I think my own feelings about mathematics really changed when I did my teacher training ... Suddenly I saw the beauty of numbers, it all fell into place and I could see how all the different parts of mathematics relate to each other ... revisiting it I had this sudden enthusiasm for maths that I’d never had before ... I’m not suddenly a better mathematician because I’m doing more</p>

you get older, you realize why that goes with that, and it's a late discovery. Suddenly it's like, "Oh! Oh yes!" <i>Lisa, parent</i>	advanced level maths, I'm a better mathematician because I understand the basics in a different way. <i>Clare, parent-teacher</i>
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Often, those parents who described a change in their mathematical identity, experienced a transformation of their understanding of activities through participation in different contexts for mathematics practice. A number of parent-teachers experienced transition from being an anxious mathematics learner, to a confident teacher of mathematics, through participation in different contexts for mathematics *learning*. Clare reveals that the experience of 'revisiting' mathematics during teacher training allowed her to acquire an understanding of the concepts of mathematics she felt she lacked as a child. Many parent-teachers attributed this greater understanding to current conceptually-based methods, in comparison to the algorithmic approach they had experienced themselves.

Whilst participants in both groups had experienced changes in their relation with mathematics during adulthood, there was variability between the groups in how the participants constructed their relation to mathematics due to the differing nature of these experiences. Those in the parent group tended to associate the change in their mathematics identity with maturity, or to using mathematics in daily life. Those in the parent-teacher group, however, were more likely to associate change with the opportunity to revisit mathematics, and participate in practices which differed from those they were familiar with.

Parents' representations of their children's school mathematics

Whilst having clear memories of certain aspects of their own learning, many participants, particularly in the parent group, had unclear ideas of how their children were currently learning mathematics. As Table 5 shows, this lack of knowledge sometimes produced a strong emotional response. Lisa, for example, talked of feeling 'closed' to the new methods because they didn't make sense to her, whilst Karen experienced frustration and could not view the school's methods in a positive light.

Table 5. The effect of parents' lack of knowledge of current methods.

Parent group
<p><i>P</i>: I know I'm not open, I feel that I'm quite closed to these new methods because I look at them and they don't make sense to me. I get the impression that they're trying to make maths meaningful and I just think it isn't meaningful, it only becomes meaningful if you start to use it in life. And if you're one of those people that it's not obvious to, the way they're doing it, it's not making it more obvious, it's actually making it more obscure. <i>Lisa, parent</i></p>
<p><i>I</i>: Can you show me any ways that you think they're doing it?</p>

P: Oh, God, I can't. I mean, no, I can't. I must be really honest here, I don't actually understand how the mathematics is taught or why the mathematics is taught in the way it is ... And the point is, I don't actually know whether there are advantages to the way they do it, I just don't know, because I don't understand it, and I don't know how they're teaching it. Karen, parent

Although many felt unclear about the new methods, all participants remembered their learning as very different to the ways their children learn now, and these differences were explained as historical changes within Primary education. The representations of these differences were similar in the two groups of parents in terms of teaching methods used, and different mathematical strategies for calculation. Current methods were viewed by participants in both groups as having a greater emphasis on underlying meanings and relationships, whereas a significant feature of their learning had been the repeated practice of 'rules' or 'formulae' for calculation. The groups differed, however, in their conception of whether current or old methods placed a greater focus on mental strategies. Indeed, it became clear that what was meant by 'mental strategies' was quite different to the groups. Those in the parent group tended to equate mental strategies with basic mental arithmetic, and felt strongly that there was less emphasis on this in current teaching. The participants in the parent group valued the repeated practice which had allowed their mental skills to become 'second nature'. The participants in the parent-teacher group, however, viewed current methods as having a *greater* emphasis on mental strategies, but saw this in terms of children having more opportunities to discuss concepts, and have a greater range of mental strategies to tackle calculation. Although parents from both groups talked about valuing mental mathematics, how they constructed their representations and valorizations of mental mathematics was quite different.

Parents' valorizations of different mathematical practices

Whilst participants in both groups shared the view that current school mathematics was different to their own school mathematics, the way the groups valued different practices was quite diverse. The parent-teacher group participants had a clearer idea of the purpose of the new methods, saw the changes as predominantly enhancing children's global abilities in mathematics, and as providing them with a more solid platform for later mathematical study. They spoke positively of children talking about mathematics, developing a greater ability to reason, and a greater understanding of the concepts of mathematics. They were more likely to value conceptually-based learning, and less likely to value an algorithmic approach. Their view of current mathematics was often in comparison with how they remembered their own experiences of learning which whilst enabling them to perform calculation procedures well, had also meant they adopted an 'automatic approach' without understanding how numbers worked together. Their accounts of the way in which

they learned may have been mediated by their greater knowledge of the aims of current methods and their current perceptual frameworks.

Most of the parent group participants, on the other hand, saw the changes predominantly in terms of confusion and complexity. They described the new methods as too numerous and more complicated, and were anxious that the focus on understanding the concepts of mathematics was at the expense of rigorous training in the acquisition of basic mental skills. They viewed that this would result in a gap in their children's cognitive skills, particularly if they perceived their children to have a less confident relation to mathematics. Amongst the participants in this group, differences in methods were not described in neutral terms, and were not treated as equal alternatives. Parents used language such as 'simple', 'straightforward' and 'logical' to describe their own form of mathematics, and 'long-winded', 'complicated', and 'obscure' when describing new ways. Parents in this group were more likely to value an algorithmic approach, and less likely to value an emphasis on conceptual understanding. As they possessed less knowledge of the new methods, they were more likely to feel new methods inadequate or confusing, and to feel closed towards them.

How different representations and valorizations influence interaction

The data revealed that many of the parent group participants experienced difficulties in understanding practices in which they did not have direct participation, and were often dependant on children's explanations about how they use particular procedures. That children themselves were often unable to explain clearly often resulted in a breakdown in communication between parent and child. Table 6 shows that Karen felt frustration that her incomplete knowledge prevented her from helping in anything more than a checking role, whilst Susie described how lack of information made her feel there was nothing she could do, and compromised the amount of effort she was prepared to invest. Not only did those parents who lacked knowledge of current methods feel excluded from helping their children, they couldn't judge their child's competence in comparison with their own ability at a similar stage, and felt they did not know what could be expected of their child.

Table 6. The effects of parents' lack of information on interaction.

Parent group
<p><i>I</i>: ...Do they think they're good at maths?</p> <p><i>P</i>: Yes, I think so. The problem is it's difficult for me to know whether they're good ... obviously they seem to get their maths homework right ...but I don't know what that means, are they good beyond that? Are they capable of more than that? ... I sort of feel like, and this is my lack really, I feel I should be more sort of involved with their mathematics ... I feel I'm not involved enough, because I basically just sit and look at it and any that are wrong I'll check them, but only from a distance really ... So I do find it difficult to support them as much as I could. I don't feel I can get as</p>

involved as I would if he was learning in the same way as I did. *Karen, parent*

P: Well, a lot of the time if I don't understand what method is to be used, I just throw up my hands. There's nothing I can do. I don't feel ... I don't feel anything really, it's a waste of energy really. There's nothing I can do, but I sometimes feel sorry for Molly, because she gets really upset and there's nothing I can do. *Susie, parent*

As parents talked about the way in which they interacted with their children, it became clear that many children valued school's methods more highly than methods their parents showed them. This was not necessarily because the school's methods were better or clearer, but that children perceived them to be the 'right way of doing it'. Parents in both groups talked of how their children 'revered' school more than their parents, and of their child's resistance to being shown other ways. This often resulted in discordance between parents and child, and led to homework as a source of conflict. The data also revealed that responses to mathematical practices differed according to which practices parents valued more highly. Whilst not wanting to undermine school methods, many in the parent group displayed frustration that their own tried and tested methods were being devalued, whilst they perceived other methods as resulting in confusion for their children. Those in the parent-teacher group, on the other hand, generally had more favourable representations of current school mathematics, and were more willing to support methods which they viewed as enabling their children to achieve a positive relation with mathematics. They reported that their teaching experience had enabled them to develop a greater understanding of current school mathematics, and this allowed them to be more confident in assessing their child's ability, and in participating in mathematics homework. However, the data also revealed that although most of the parent-teachers understood and appreciated the use of multiple methods, they adopted different positions towards these approaches if they perceived their own child was confused and this, in turn, affected how they organized mathematical practices for their children (see Table 7).

Table 7. Parent-teacher's valorizations of their own methods.

Parent-teacher group

P: Milly, I know, knows one method, and if something else is being taught, then I'm afraid I'm saying to her, ignore it, because I'm worried that she will mix it as well. I'm saying forget what Mrs Woods tells you, I keep telling her, which is very naughty, but stick to what you know, because you can do it that way. *Jane, parent-teacher*

P: I think we took the right decisions for Luke at the time, but I think potentially it could have been even more confusing to him, because I could explain to Luke, yes, you can do it these different ways at school, but you know if Dad's shown you this way and you're happiest with that way, then you do it that way. *Cathy, parent-*

<i>teacher</i>

Although some parent-teachers were unwilling to devalue the school's methods, others felt they were right to encourage their children to use only one method, if their child continued to be confused. Jane talks about actively encouraging her own daughter to ignore the school's insistence on multiple methods because of her fear that she will become confused. The effect of teaching experience, then, was generally positive in terms of parents' representations and valorizations of current school mathematics. However, although, many parent-teachers recognized that multiple methods may enhance understanding by providing 'the bigger picture', they constructed different representations of new methods as too numerous and too complex if they perceived their own child to be confused by them. Even with a good knowledge and understanding of new methods, and sympathy towards the aims of the National Numeracy Strategy, their position in relation to the numeracy practices changed according to the particular role, as professional or parent, they had to adopt at any given time. It was the position parents adopted towards these representations which affected how they interacted with their children's mathematical learning.

CONCLUSIONS

The research set out to explore how parents' past experiences influence the way in which they construct their mathematical identities and their representations of different mathematical practices, and how these factors influence the ways they interact with their children's learning. The findings illustrated that both those with, and those without teaching experience, construct their mathematical identity in similar ways and this identity was shown to evolve through participation in different contexts of mathematical practice and learning. Participants in both groups were similarly aware that their own mathematical identity could affect the way in which their children approached mathematics.

The study revealed that both those with and without teaching experiences perceived current school numeracy practices to be very different to those they had experienced when learning. Varying levels of knowledge, and different levels of participation in current methods resulted in the participants from both groups valuing different mathematical practices in different ways; those with teaching experience tending to attribute a higher value to current methods than those without teaching experience. However, the study indicated that although in many areas, those with teaching experience were able to bridge the gap between differing mathematical practices more easily, when confronted with their child's continuing confusion about mathematics, parents may revert to the methods they formerly depended on, despite holding positive representations of current methods. Parents' perception of their child's ability in relation to certain mathematical practices was, therefore, a more significant resource for parents, and contributed more significantly to the way in

which they interact with their children, than their overall representations of current methods.

This research indicated that it is the opportunity for participation in different mathematical approaches which allows parents to construct more positive representations of varying practices, and in turn, to understand how they are socially valued. This has implications for how schools communicate the way they approach mathematics, and the opportunities they offer to parents for understanding these practices and for raising confidence amongst parents to become involved. The study also explored the transitions parents experience between their roles as parent and teacher, and how the subjective knowledge they developed during these transitions is adapted for each role. Further study of teachers' representations and practice in the classroom, in the light of the interaction they experience with their own children, would contribute to research on the ways in which valorization of numeracy practices affect support both within the home and at school.

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BATIKS: ANOTHER WAY OF LEARNING MATHEMATICS

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This paper examines a micro-project that was developed in an 8th grade class. Students elaborated batiks and then they discussed mathematical tasks based in their batiks' elaboration process. This research is based in two research projects: Interaction and Knowledge (IK) and IDMAMIM. We assume an interpretative approach and a case study design. Results illuminate the potentialities of these classroom practices, illustrated through the analysis of some video taped peer interactions. The focus of analysis is in the didactic contract, based in collaborative work, and in the nature of the tasks that were part of this micro-project.

THEORETICAL BACKGROUND

Portuguese schools are multicultural settings (César, 2007; César & Oliveira, 2005). Considering Nieto's definition (2002), culture is "(...) the ever-changing values, traditions, social and political relationships, and worldview created and shared by a group of people bound together by a combination of factors (...), and how these are transformed by those who share them" (p. 53). According to this definition, in school we find a great diversity of cultures. Not only origin cultures but also many others, including the school's culture, or some teenagers' group culture.

Sometimes the school culture is so far away from students' cultures that they focus their energies on other directions (Säljö, 2004). School needs to facilitate the emergence of "thinking spaces", a construct coined by Perret-Clermont (2004) that stresses the role played by securing spaces in which students may discuss doubts, conjectures, solving strategies, learning difficulties, developing their critical sense, learning autonomy, but also their "sense of identity" (Zittoun, 2004), of belonging to that particular learning community. As César (2007) claimed, becoming a legitimate participant in a learning community, namely in formal educational settings, facilitates students' engagement in academic tasks but also their construction of identities and the management of the dialogical I-positioning (Hermans, 2001) that are often conflictive when the student's culture is much different from the school's culture.

Schools also need to be more inclusive (Ainscow, 1999; César, 2003, 2007, 2009; César & Santos, 2006) and to promote interactions among community members and cultures. Intercultural (mathematics) education facilitates the emergence of dialogical interactions, namely among students from different cultures (D'Ambrósio, 2002;

Favilli, César, & Oliveras, 2004; Peres, 2000; Powell & Frankenstein, 1997; Teles & César, 2007). Ouellet (1991) has already stressed that this education is for everyone, based on the comprehension, communication, and promotion of interactions. Collaborative work among students (and with the teachers) was studied by many authors. It acts as a facilitator and mediator for student's knowledge appropriation when it is part of a negotiated and coherent didactic contract (César, 2007; César & Santos, 2006; Schubauer-Leoni & Perret-Clermont, 1997; Teles & César, 2005), and it also facilitates transitions (Abreu, Bishop, & Presmeg, 2002; César, 2007, 2009).

The development of intercultural (and interdisciplinary) microprojects related to handicraft activities promotes students' performances and academic achievement (Favilli et al., 2004). They underline the cultural dimension these activities give to the learning processes, also contributing to the mobilisation/development of competencies. Their social marking of the tasks, i.e., the possibility of connecting them to students' daily experiences and social life, plays an important role on students' engagement and mathematical performances (Doise & Mugny, 1981; Teles, 2005). It also plays an important role when teachers aim at changing students' social representations about mathematics. Social representations are often stated as being an important contribution for students' performances and school achievement (Abreu & Gorgorió, 2007; César, 2009).

METHOD

We assume an interpretative approach, inspired in ethnographic methods. This study is based in two research projects: *Interaction and Knowledge (IK)* and *IDMAMIM*. The first one was developed during 12 years (1994/95-2005/06) and its main goal was to study and implement social interactions in formal educational scenarios (for more details see César, 2007, 2009). The didactic contract that was negotiated in this class was clearly shaped by this project's features. Teachers' practices, based in collaborative work, were also shaped by this project's pedagogical ideals. *IDMAMIM* project was developed in some towns of Spain (Granada), Italy (Pisa) and Portugal (Lisbon). Its two main goals were: (1) to identify didactic needs in order to develop an intercultural mathematics education; and (2) to elaborate intercultural didactic materials, like the ones based in the batiks elaboration, and its later exploration in mathematics classes (Favilli et al., 2004). The mathematical tasks used in this class were part of this project.

This case is part of a broader study including 4 case studies. In all these case studies students developed an intercultural microproject, based on the elaboration of batiks. Batiks are a handicraft from Java, that was then developed in other parts of the world, namely in Cape Verde, where we collected information about how to elaborate them. Batiks assume different ways of being produced in different parts of the world, according to the native cultures of each country, and also to their economic conditions. In Cape Verde, as it is a very poor country, they use flour, water and lime,

instead of wax in order to make the production process cheaper. Thus, even discussing the different ways of production of batiks, that students discover in the internet before elaborating them, it is a way to explore a critical mathematics approach. This is complemented by the discussion of the video we made in Cape Verde in which batiks are being produced. This way of approaching the microprojects also allows them to be explored in a multidisciplinary way, as teachers from different subjects may participate and, for instance, explore the texts from the internet in English language subject, the production process in Chemistry, the evolution of batiks around the world in History, the elaboration of the templates in Arts. In this paper we focus in the one of the mathematical tasks that was solved after elaborating the batiks. Thus, the research question that we analyse in this paper is: What are the contributions of intercultural and collaborative microprojects to students' mathematical knowledge appropriation?

The participants were the students from a 8th grade class (13/14 years old), their mathematics teacher, external observers and evaluators. This class had 21 students, one of them categorized as presenting special educational needs (SEN). There were 12 girls and 9 boys. These students were from different cultures and some of them were born, or had families that were born, in other countries. But even Portuguese students belonged to different cultures and socio-economical backgrounds. The mathematics teacher described this class as “(...) a working, engaged, interested and challenging class” (Teacher's final report, p. 7), as some of these students experienced underachievement in previous school grades in mathematics. Thus, many of them presented a negative social representation about mathematics in the beginning of the school year (September), according to the data of the *IK* project (students' protocols – for more details about the first week procedure, see César, 2009 or Teles, 2005). Some of these students usually did not participate in mathematics activities during classes, in previous school grades. They did not disturb the class work. They simply did not do anything and just waited for the end of the class to go to the break. Thus, many of these students never went to the blackboard after solving mathematical tasks, or participated in the general discussion. For these reasons, one of the main teacher's practices aims during the first month of classes was to promote students' participation in mathematical activities, and to avoid having only three or four of them – always the same ones – participating. The dyad whose peer interaction we chose to discuss is a paradigmatic one: J. was one of the students who experienced underachievement in mathematics in previous school grades while her peer loved participating in mathematics classes. Thus, the teacher tried to promote J.'s participation and, in this episode, we can see that she is no longer silent, or just trying to be unnoticed. She is already able to go to the blackboard, during the general discussion, after dyad work, and to explain to the whole class her dyad's solving strategy. Thus, this dyad illuminates some of the processes that could be observed in many other excerpts from the videos, and that were shaped by the collaborative work these students developed during the whole school year in mathematics classes.

Data was collected through observations, questionnaires (*IDMAMIM*), interviews (*IDMAMIM*), the teacher's and external evaluators' reports and students' protocols. In this paper we focus in the analysis of some video excerpts, the teacher's report and in students' protocols.

In this episode, students were solving a mathematical task in dyads, after elaborating their batiks. A batik is a pure cotton wrap tainted with colours where a drawing is contrasted. This elaboration process uses mathematical knowledge that can be explored further in later mathematics classes (for more details, see Favilli et al., 2004; Teles, 2005). They were discussing about the following situation:

Ms. Bela made a batik. It was in a square piece of cotton whose side measured 60 cm. Mr. Evaristo is interested in buying a batik. But he wants one with the double of the size.

- Ms. Bela, how much is a batik like that with the double of this size? – asked Mr. Evaristo.

- Look, Mr. Evaristo, this batik costs 18€ And I can sell you the other batik at the same price each m^2 .

- Then, I offer you 36€ Do you accept my offer?

1.1. What do you think: Should Ms. Bela accept Mr. Evaristo's offer? Explain your reasons.

1.2. Complete the table below, considering the correspondence f that associates a square batiks' side (x) to its area (y).

Length of the side of the batik,		20		6
Area of batik, cm^2 (y)	0		1600	



Figure 1: Batik

RESULTS

This episode is an excerpt of an interaction between two students: J (a girl – 13 years old) and N (a boy – 12 years old). They are both Portuguese, but their family cultures are differentiated: N. comes from a highly literate family, whose parents have an university graduation; J. comes from a family whose parents have jobs related to commerce and services. From the economical point of view their families are from a class that is not very high or very low. They could be characterised as paradigmatic teenagers, with the hobbies, dressing code, language, and friendships of most of the teenagers in Portugal. J and N are on 8th grade for the first time but they have different previous experiences with mathematics. J does not like mathematics. In the first term she still experienced some underachievement (she got Level 2, a mark that is negative, in the Portuguese educational system, in which students' marks vary from Level 1 – the lowest - up to Level 5 – the highest). But during the next two terms she was engaged in mathematics classes and she was able to achieve. N is a student with

a calm and pleasant relationship with mathematics. He always succeeded in this subject. He shows a high self-esteem, namely an academic one, while J was less confident about her abilities and competencies, in particular in mathematics and during the first months of the school year. It was precisely their differentiated characteristics as mathematics students, and when they addressed the mathematics tasks – in the beginning of the school year J tended to give up very easily or even not try at all to solve them – that were the criteria for choosing them to be discussed and analysed in this paper, as they both represent many other similar students we had in this class, and even in the other three cases from the *IDMAMIM* project.

In this episode, they are solving the question 1.2. It is N who starts the interaction writing on his notebook his reasoning in order to explain it to J.

b) Preencham a tabela:

Comprimento do lado do batique, em cm (x)	0	20	40	60
Área do batique, em cm^2 (y)	0	800 400	1600	2400 3600

Figure 2: J and N resolution (Question 1.2.b))

1 N: It is: 20, 40, 60. It is half of 1600 [He understood that 20 is half of 40; then the table should be completed with half of 1600, i.e., 800]. It is 800. It is the double of this [He points]. Then, here it is 40 is 1600; then 20 is 800.

2 J: A little confusing!

3 N: What is the part you don't understand?

4 J: This part [she points to the sum]. Why is this plus this?

5 N: Because... This plus this equals 1600. Teacher!?

[The teacher approaches them]

6 N: Could you see if my reasoning is correct?...

7 J: So, what do you [turning to J] think about his reasoning?

This piece illuminates the role of the didactic contract of this class (César, 2003; César & Santos, 2006; Schubauer-Leoni & Perret-Clermont, 1997; Teles & César, 2005): students can start their resolution of the task by individual work but they need to explain their reasoning to his/her colleague from the same dyad. They need to discuss the solving strategies they used in order to find a consensus. But they also need to understand each other's solving strategy because one of them may be asked to represent their dyad in the general discussion and to explain to their colleagues their solving strategies. As they are both engaged in this type of didactic contract, they know that just having an answer produced by one of them is not enough. Thus, J is trying hard to understand her peer's solving strategy and this is exactly what her

teacher aimed: to improve her participation in the mathematical activities, during mathematics classes. Their teacher was trying to create what Perret-Clermont (2004) designates as thinking spaces, facilitating students' reflection upon their solving strategies and some mathematical concepts.

They also know that discussing their solving strategies is a way of learning for both of them. For the one who used this solving strategy as s/he has to clarify its steps in order to explain them and to answer to his/her peer's doubts and questions; and to the one who is, at that moment, acting as the less competent peer (Vygotsky, 1932/1978), as it helps him/her progressing in his/her mathematical performances and in knowledge appropriation. These features of collaborative work, that we can also see in other parts of this episode presented below, also help students develop their positive self-esteem – particularly clear in the way of acting of J, in this episode, namely when she goes to the blackboard during the general discussion and is able to explain her dyad's solving strategy without taking any sheet with their resolution in her hands (according to the video record, she acted like this due to her teacher's suggestion). Thus, it helps them to begin acting as legitimate participants and not as peripheral ones (César, 2007). This changing form of participation is illustrated by the ways J acts, during the different parts of this episode, as well as by the external observers reports, during the school year, and by the analysis of other episodes that were also video recorded.

In Turns 5 and 6 N asks for their teacher's help and assumes this dyad's leadership. He is assuming the role of the more competent peer (Vygotsky, 1932/1978). This happened in this dyad during the first month they worked together, as J considered N "much better than me" (questionnaire, January) and it took some time before she was able to express her opinions, solving strategies and arguments before listening to N. It must be added that while analysing many other pieces of videotapes from this class it was clear the teacher's effort in order to promote the positive self-esteem of J and to make her feel more confident. Her aim, according to the features of collaborative work, inclusive education and this particular didactic contract, was to be able to have the role of more competent peer assumed by each one of them, in different mathematical tasks, or even in different moments/steps of their solving strategies. But when one of the students usually performed much better than the other in previous school years, achieving this point takes time and needs a lot of knowledge about how to act from the teacher's point of view.

J considers N's resolution "A little confusing!" (Turn 2). Thus, N tries to realise what J did not understand. Then, he tries to explain J what she did not understand (Turn 5). But he is not very clear in his explanation. He realises that J is still confused and thus he asks for their teacher's help, trying to legitimate his reasoning (Turn 6). According to the didactic contract, their teacher does not answer him. Instead, she asks J's opinion about N's reasoning (Turn 7) and tries to promote a dialogical interaction between these students. The teacher assumes the role of a mediator of learning (Vygotsky, 1932/1978). She is more concerned with students understanding and with

the interaction between them than just with the validation of students' answers. Their teacher's reaction illuminates how the expert other can facilitate students – in this case, J's – change from a peripheral to a legitimate participation (César, 2007, 2009; Lave & Wenger, 1991). As we stressed in other cases we analysed in other papers, this is an essential move in order to promote more inclusive formal educational settings, and an intercultural education (for more details see César, 2007, 2009; César & Santos, 2006; Teles, 2005).

8 N: It is 20...

9 T: But, I don't want that answer! [Points to Question 1] Well... explain! I said that we'll correct Question 1. So, I want you to explain me why you wrote this and...

10 N: 36€ 36€ is the double of batik that cost 18€ Ms. Bela's batik cost 18€

11 T: It measures 60cm in this side.

12 N: It is 60cm of side but we want the double of this batik...

13 T: You want a batik with the double of these dimensions [she points at each side of the batik].

14 N: Yes. Yes.

15 J: So, it is the double of this one.

1.1. O que vos parece: a D. Bela deve aceitar a proposta do Sr. Evaristo?
Expliquem o vosso raciocínio.

Não, porque a D. Bela vai ficar a perder com a oferta do senhor Evaristo, porque para ter um batik quadrado com o dobro das dimensões vai ter que pagar 4 vezes o batik de 18 euros.

$A_1 \text{ batik} = 60 \times 60 = 3600 \text{ cm}^2 = 0,36 \text{ m}^2$	$18 - 0,36$	$x = \frac{18 \times 4 = 50€/\text{m}^2}{0,36}$
$A_2 \text{ batik} = 120 \times 120 = 14400 \text{ cm}^2 = 1,44 \text{ m}^2$	$x - 1$	
Custo do batik = $1,44 \times 50 = 72€$		

No, because Mrs. Bela would loose money with Mr. Evaristo's offer. Because in order to have a square batik with the double of the dimensions of the first one, he has to pay 4 times more, i.e., four times 18€

Figure 3: J and N resolution (Question 1.1.) and students' answer translation

N starts the interaction with their teacher again (Turn 8), and explains the solving strategy they used to answer to Question 1. He answers the teacher's questions, but J also participates in this dialogue and concludes N's argumentation (Turn 15). But another interesting feature appears in Turn 9: these students, although engaged in solving the task, were not answering to the part their teacher had asked to be solved. This illuminates the importance of the teacher's role during classes, even when students are working in an autonomous way, it is only by observing closely what is going on that the teacher can help students to learn how to self-regulate their work in a more adequate way. In the excerpt, we understand that both students know the solving strategy they used and they can explain it because they co-constructed it together, according to the rules of the didactic contract (César, 2007, 2009; César & Santos, 2006; Teles, 2005). But in order to understand their different solving strategies students also need to establish an intersubjectivity that allows them to understand each other's arguments and solving strategies (Valsiner, 1997; Wertsch, 1991), as illuminated in the following piece:

16 T: Is it?

17 N: It is the same as we have another batik here, together.

18 T: Is it? I didn't think like this! Put two batiks together and confirm if it is a batik with 120cm of side.

19 J: We did 18×2 .

20 T: I understood! But, I'm asking you if this is correct!?

21 N: Maybe!

22 T: Maybe? So, imagine that this is a batik. And you have another batik here ...

23 J: It has 120cm of side.

24 T: Here [she points in their sheet of answers].

25 J: Yes.

26 T: And here? [she points again]

27 J: It doesn't. It is 60.

28 T: Ah... I want a square batik! 120 per 120. But, if you put two batiks together it has 120 per 60. Ah! Why? I said that I want the double of dimensions. The first one had 60 per 60 and this one has to have 120 per 120. Right?

An interesting point here is their teacher's care to avoid any evaluative comments on their work. She asks challenging questions as she seeks to encourage the students to realise their mistake (Turns 18, 20, 22, 24, 26, and 28). Their teacher wants these students to question themselves about what they did. Thus, she chooses to ask them

questions and to pretend she does not understand what they did and why they did it this way (Turns 18 and 20). But her tone of voice is a kind one, she smiles from time to time, the interaction has an easy-going mood, and students, although paying attention, also have a smiling face.

As we can observe, J participates actively in this discussion, in spite of her usual introverted mood and her lack of confidence in her competencies (Turns 19, 23, 25 and 27). She believes on what she did with N.

29 J: Right! It is impossible!

30 T: Impossible!?

31 N: The teacher wants the double of this one. So, we have to add... we have to divide batik for all sides!?

32 J: What!?

33 T: To divide batik for all sides!? I don't understand.

34 J: I don't understand it either.

35 N: I don't understand it too.

J does not understand what their teacher told them, and thus she considers this problem impossible (Turn 29). Her attitude illuminates her lack of confidence and persistence in the activity, when she fails. This situation makes their teacher look for other alternative ways to promote students' interest and increase their positive academic self-esteem.

36 T: Let's think a little bit more. You are saying that ... I think that you already understood that if you put another batik here... the other is the double, isn't it?...

37 J: If we put here (down side), it is not enough. It isn't 120.

38 T: [We can't understand]

39 J: But, here (down side) is not enough. It is 60.

40 T: And? You are about to have a square.

41 N: It is a square.

42 T: In the question they say that it is a square after we cut the batik. Think a little bit more.

Facing students' doubts and this impasse, their teacher decides to change the direction of the resolution because she wants them to go on trying to solve this problem. But, she starts from what she believes the students already understood (Turn 36). J's interest seems to increase during this interaction. She participates actively in the discussion. But, even more important, she goes on trying to solve this task when

the teacher goes away again. Thus, although this episode ends without a resolution, students' discussion around that question continued. During the general discussion (whole group discussion) J went to the blackboard and was able to explain to their colleagues their solving strategy. She did it in a convincing way, explaining their solving strategy clearly and she was even able to answer to two colleagues doubts. Thus, J showed different I-positioning as mathematics student during this resolution. Basically, she passed from a non-confident I-positioning, typical of a low achieving student, to a confident I-positioning, that let her be considered a competent peer in the resolution of this task.

FINAL REMARKS

To get students' engagement a teacher needs some effort and creativity. Students' access to the rules of the didactic contract can help them understanding their role in that particular classroom and at school. It also facilitates facing the academic tasks in a confident and responsible way. As we could observe both N and J knew the rules of the didactic contract. They discussed their reasoning to find a consensus and they asked for their teacher's help only when they couldn't solve an impasse.

The teacher's role is another important feature. In this episode we could observe a teacher that assumes a mediating role. She did not tell students the right answer. She helped them to realise their mistake and she gave them assistance in order to facilitate their progress in their solving strategy. This teacher believed in the students' competencies and she aimed at facilitating the mobilisation and development of other students' competencies.

The nature of the task is another relevant feature to achieving students' engagement. In this episode the task was about batiks, which students elaborated in previous classes. The social marking of the task helped students' understanding of the task. As they elaborated batiks, they knew the process of elaboration and they were able to give a meaning to this mathematical task. Thus, the social marking of the task facilitated students' learning processes and also their knowledge transition from one situation (elaborating batiks) to another (mathematics class, solving problems).

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THE ROLE OF ETHNOMATHEMATICS WITHIN MATHEMATICS EDUCATION

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Abstract

This paper considers the field of enquiry called ethnomathematics and its role within mathematics education. We elaborate on the shifted meaning of ‘ethnomathematics’. This “enriched meaning” impacts on the philosophy of math education. Currently, the concept is no longer reserved for ‘nonliterate’ people, but also includes diverse mathematical practices within western classrooms. Consequently, maths teachers are challenged to handle people’s cultural diversity occurring within every classroom setting. Ethnomathematics has clearly gained a prominent role, within Western curricula, becoming meaningful in the exploration of various aspects of mathematical literacy. We discuss this enriched meaning of ethnomathematics as an alternative, implicit philosophy of school mathematical practices.

Key-words: Ethnomathematics, Diversity, Politics, Philosophy, Values.

INTRODUCTION

Until the early 1980s, the notion ‘ethnomathematics’ was reserved for the mathematical practices of ‘nonliterate’ – formerly labeled as ‘primitive’ – peoples (Ascher & Ascher, 1997). What was needed was a detailed analysis of the sophisticated mathematical ideas within ethnomathematics, which it was claimed were related to and as complex as those of modern, ‘Western’ mathematics. D’Ambrosio (1997), who became the “intellectual father” of the ethnomathematics program proposed “a broader concept of ‘ethno’, to include all culturally identifiable groups with their jargons, codes, symbols, myths, and even specific ways of reasoning and inferring”. Currently, as a result of this change within the ethnomathematics discipline, scientists collect empirical data about the mathematical practices of culturally differentiated groups, literate or not. The label ‘ethno’ should thus no longer be understood as referring to the exotic or as being connected with race. This changed and enriched meaning of the concept ‘ethnomathematics’ has had its impact on the philosophy of mathematics education. From now on, ethnomathematics became meaningful in every classroom since multicultural classroom settings are generalized all over the world. Every classroom nowadays is characterized by (ethnic, linguistic, gender, social, cultural ...) diversity. Teachers in general but also math teachers have to deal with the existing cultural diversity since mathematics is defined as human and cultural knowledge as any other field of

knowledge (Bishop 2002). The shifted meaning of ethnomathematics into a broader concept of cultural diversity became meaningful within the community of researchers working on the topic of ethnomathematics, multicultural education and cultural diversity. Where the topic was absent at the first two conferences of the Conference of European Research in Mathematics Education (CERME 1, 1998; CERME 2, 2001), the topic appeared at CERME 3 (2003) as *Teaching and learning mathematics in multicultural classrooms*. At CERME 4 (2005) and CERME 5 (2007) the working group was called *Mathematics education in multicultural settings*. At CERME 6 (2009) the working group was called *Cultural diversity and mathematics education*. From now on, there is an explicit consideration to the notion of cultural diversity.

DEALING WITH CULTURAL DIVERSITY IN THE CLASSROOM

Ethnomathematics applied in education had a Brazilian origin, but it eventually became common practice all over the world. It has been extended from an exotic interpretation to a way of intercultural learning that is applicable within any learning context. Dealing with cultural diversity in the classroom is the universal context within which each specific context has its place.

The meaning of the *ethno* concept has been extended throughout its evolution. It has been viewed as an ethnical group, a national group, a racial group, a professional group, a group with a philosophical or ideological basis, a socio-cultural group and a group that is based on gender or sexual identity (Powell 2002, p.19). This list could still be completed but since lists will always be deficient, all the more because some distinctions are relevant only in a specific context, we use the all-embracing concept of *cultural diversity*. With respect to the field of mathematics, and in line with Bishop's (2002) consideration on mathematics as human and cultural knowledge, there appears to be a change in the meaning of ethnomathematics as diversity within mathematics and within mathematical practices. This view enables us to see the comparative culture studies regarding mathematics that describe the different mathematical practices, not only as revealing the diversity of mathematical practices but also to emphasize the complexity of each system. In addition there is interest in the way that these mathematical practices arise and how they are used in the everyday life of people who live and survive within a well-defined socio-cultural and historical context. Consequently there has to be a translation of this study to mathematics education where the teacher is challenged to introduce the cultural diversity of pupil's mathematical practices in the curriculum since pupils also use mathematical practices in their everyday life.

This application exceeds the mere introduction in class of the study of new cultures or – to put it dynamically – new culture fields (Pinxten 1994, p.14). These are the first 'ethno mathematical' moves that were made, even before dealing with cultural diversity arose. Diversity within mathematical practices was considered as a practise of the 'other', the 'exotic'. It was not considered relevant to mathematics pupils from a westernised culture. That is why the examples regarding mathematics (and adjacent sciences) are an enquiry of all kinds of exotic traditions such as sand drawings from

Africa, music from Brazil, games such as Patience the way it is played in Madagascar, the arithmetic system of the Incas or the Egyptians, the weaving of baskets or carpets, the Mayan calendar, the production of dyes out of natural substances, drinking tea and keeping tea warm in China, water collection in the Kalahari desert, the construction of Indian arrows, terrace cultivation in China, the baking of clay bricks in Africa, the construction of African houses. The examples are endless (Bazin & Tamez 2002). Notwithstanding the good intentions of these and similar projects, referring to Powell & Frankenstein (1997) we would like to emphasize that these initiatives may well turn into some kind of folklore while originally intending to offer intercultural education.

However, we also stress that we are not advocating the curricular use of other people's ethnomathematical knowledge in a simplistic way, as a kind of "folkloristic" five-minute introduction to the "real" mathematics lessons. (Powell & Frankenstein 1997, p.254)

In line with the empirical research by Pinxten & François (2007) on mathematical practices in classroom settings, one can prove many appropriate examples that pupils' mathematical practices may be used in class, not as some kind of exoticism but as the utilization of a mathematical concept. Starting from pupils' mathematical knowledge and their everyday mathematical practices is a basic principal of the new orientation towards realistic mathematics education and the development of innovative classroom practices (Prediger 2007). The question remains how one can move from a teacher centered learning process towards a pupil centered learning process where pupils' mathematical practices can enter the classroom. Cohen & Lotan (1997) describe how interactive working can be structured and they also explain the benefits of interactive learning in groups to *deal with diversity*. For that purpose the *Complex Instruction* theory was developed which they implemented in education. Meanwhile this didactic has had an international breakthrough in Europe, Israel and the United States and it has been elaborated to the didactic of *Cooperative Learning in Multicultural Groups (CLIM)* (Cohen 1997: vii). This teaching method has been tested in a number of settings, in distinct age groups and with regard to different curricula (Cohen 1997, Neves 1997, Ben-Ari 1997). Besides the acquisition of mathematical contents was part of this. *Complex Instruction* is a teaching method with equality of all pupils as its main objective. This teaching method tries to reach all children and tries to involve them in the learning process, irrespective of their diversity (François & Bracke 2006). In order not to peg cultural diversity down to a specific kind of diversity Cohen (1997) in this context speaks of working in heterogeneous groups. Heterogeneity can be found in every group structure. Even a classroom is characterized by a diverse group of pupils where every pupil has in some way his or her everyday mathematical practices. If pupil centered learning is taken seriously, teachers are challenged to deal with this present mathematical practices while teaching mathematics. In this way, ethnomathematics became a way of teaching mathematics where cultural diversity of pupils' everyday mathematical practices are taken into account (François 2007).

ETHNOMATHEMATICS IN EVERY CLASSROOM

The extended notion ethnomathematics as dealing with pupils' everyday mathematical practices has equality of all pupils as its main objective. Ethnomathematics becomes a philosophy of mathematics education where mathematical literacy is a basic right of all pupils. The teaching process tries to reach all pupils and tries to involve them in the learning process of mathematics, irrespective of their cultural diversity. All pupils are equal. This notion of mathematics for everyone fits in with the ethical concept of *pedagogic optimism* that is connected with the theory of *egalitarianism*. This ethical-theoretical foundation on which the project of equality within education is based, assumes that the equality is measured at the end of the line. As reported by the justice theories of John Rawls (1999) and Amartya Sen (1992) pupils' starting positions can be dissimilar in such a way that a strictly equal deal will prove insufficient to achieve equality. A *meritocratic* position – which measures the equality at the start of the process – thus cannot fully guarantee equal chances (Hirtt, Nicaise & De Zutter 2007). An *egalitarian* position starts from a pedagogic optimism and it needs to take into account the diversity of those learning in order to give equality maximum chances at the end of the line.

By extending the notion ethnomathematics to cultural diversity and mathematics education, the distinction between mathematics and ethnomathematics seems to disappear. Hence the critical question can be raised whether the achievements of ethnomathematics will not become lost then. On the contrary the distinction between ethnomathematics and mathematics can only disappear by acknowledging and implementing the ethnomathematics' achievements in the mathematics education. The issue on the distinction between ethnomathematics and mathematics has been raised before within the theory development of ethnomathematics (Setati 2002). Being critical on the dominant Western mathematics was the basis out of which ethnomathematics has developed and now the time is right to raise the critical questions also internally, within the field of ethnomathematics itself. What exactly distinguishes ethnomathematics from mathematics? Setati raises this question in a critical review on the developments within the ethnomathematics as a theoretical discipline that dissociates and distinguishes from mathematics (Setati 2002). Setati sees mathematics as a mathematical practice, performed by a cultural group that identifies itself based on a philosophical and ideological perspective (Setati 2002). Every maths teacher is supposed to use a series of standards that are connected with the profession and with obtaining the qualification. The standards are philosophical (about the way of being), ideological (about the way of perceiving) and argumentative (about the way of expressing). Both mathematics and ethnomathematics are embedded in a normative framework. So the question can be raised as to whether the values of mathematics and ethnomathematics indeed are that distinctive.

It cannot be denied that ethnomathematics was based on an emancipatory and critical attitude that promotes the emancipation and equality of discriminated groups (Powell & Frankenstein 1997). This general idea of emancipation can also be found in the UNESCO's view on education. Moreover we see in its mission a tight connection with the socio-economic development, with working on an enduring and peaceful world, while respecting diversity and maintaining human rights. Education here is obviously connected with the political factor.

UNESCO believes that education is key to social and economic development. We work for a sustainable world with just societies that value knowledge, promote a culture of peace, celebrate diversity and defend human rights, achieved by providing education for all. The mission of the UNESCO Education Sector is to provide international leadership for creating learning societies with educational opportunities for all populations; provide expertise and foster partnerships to strengthen national educational leadership and the capacity of countries to offer quality education for all. (UNESCO 1948)

Taking into account these general stipulations we have to conclude that the explicit values of the general education objective connect to the values of equal chances for all pupils which are central within ethnomathematics. Consequently the expansion of ethnomathematics as a way of teaching mathematics which takes the diversity of pupils' mathematical practices into account can be justified. There is a kind of inequality in every group and the real art is to learn to detect the skins of inequality and the skins of cultural diversity. Instead of a depreciation of the concept 'ethnomathematics' this extended notion could mean a surplus value in situations where heterogeneity and cultural diversity are less conspicuous.

Within ethnomathematics education two aspects are highlighted. First there is the curriculum's *content*. Often this is the first step when implementing ethnomathematics. Besides the mathematics that can be found in the traditional curriculum, there will now be additional space to be introduced to more exotic or traditional mathematics practices. Powell & Frankenstein (1997) also emphasize this aspect in their definition of the enrichment of a curriculum through ethnomathematics. Stressing other mathematical practices offers the opportunity to gain a better perception in the own mathematical practice and its role and place in society (D'Ambrosio 2007a). It also offers the opportunity to philosophize and critically reflect on the own mathematical practice. In language teaching it goes without saying that it is better to learn more than one language. It broadens the outlook on the world and offers a better adaptation to dealing with other people in this globalized world. Knowledge of several languages is undoubtedly an advantage and besides it broadens the knowledge of the mother tongue. This comparison could even be extended to the mathematics education where knowledge of mathematical practices of several cultural contexts and throughout time proves to be advantageous. A second aspect within ethnomathematics is the *didactics*, the way that the learning process is set up. Here an interactive approach is crucial (Cohen 1997, César 2009). The two aspects obviously have mutual grounds. An interactive approach results in

contents being defined also by the learning with an active participation in the learning process. This aspect is strongly emphasized by researchers who investigate the integration of so-called traditional groups within the academic context. This is expressed as one of Graham's key questions in his enquiry into mathematics education for aboriginal children: *what do the children bring to school?* (Graham 1988, p.121). With the extended notion ethnomathematics as cultural diversity and mathematics education and with the emphasis on dealing with pupils' everyday mathematical practices, ethnomathematical practice is now closer to the social environment of the pupil and unlinked it from its original (exotic) cradle. Both the theory and practice of ethnomathematics have opened up the eyes and broadened the minds. It immediately answers the question as to what exactly could be of benefit to the highly-educated countries – with their outstanding results in international comparative investigations – regarding ethnomathematics as it originally developed, as a critical and emancipation theory and as a movement that aimed to give all pupils equal chances. In a final section about ethnomathematics we would like to link up mathematics education and politics.

ETHNOMATHEMATICS AS HUMAN RIGHT

D'Ambrosio, who is the mathematician and educationalist of the mathematics on which ethnomathematics is based, situates mathematics education within a social, cultural and historical context. He can also be considered the first to explicitly link mathematics education and politics. Mathematics education is a lever for the development of the individual, national and global well-being (D'Ambrosio 2007a, 2007b). In other words the teaching and learning of mathematics is a mathematical practice with obviously a political grounding. D'Ambrosio advances the political proposition that mathematics education should be accessible to all pupils and not only to the privileged few. This proposition has been registered in the OECD/PISA report, which is the basis for the PISA-2003 continuation enquiry.

Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen. (OECD, 2004, p.37)

This specification of mathematical literacy clearly implies that this form of literacy is a basic right for every child, such that it gets a chance to participate to the world in a full, constructive, relevant and thoughtful way. We will see this proposition recurs later in the essays of Alan J. Bishop (2006) where he demonstrates the link between mathematics, ethnomathematics, values and politics.

Mentioning mathematics education and education of values in one and the same breath does not sound unambiguous because mathematics is undeniably being perceived as non-normative.

It is a widespread misunderstanding that mathematics is the most value-free of all school subjects, not just among teachers but also among parents, university mathematicians and

employers. In reality, mathematics is just as much human and cultural knowledge as any other field of knowledge, teachers inevitably teach values [...]. (Bishop 2002, p.228)

It is predominantly within D'Ambrosio's' ethnomathematics research program that the link of mathematics and mathematics education with values is extended to the political domain, not in the least with the intellectual father of ethnomathematics. According to D'Ambrosio still too many people are convinced that mathematics education and politics have nothing in common (D'Ambrosio 2007a). He will take the edge of this cliché. In his recent work D'Ambrosio (2007a, 2007b) departs from the Universal Declaration of Human Rights where articles 26 and 27 highlight the right to education and to share in scientific advancements and their benefits.¹ This declaration concerning education is further developed and confirmed within the UNESCO's activities by means of the *World Declaration on Education for All* in 1990 and ratified by 155 countries. Finally the declaration has been applied in mathematical literacy in the OECD/PISA declaration of 2003. D'Ambrosio regrets that these declarations are not well-known by maths teachers since they play a key role in the emancipation process. In line with the World Declaration, 'mathematics education for all' implies a critical reflective way of teaching mathematics. According to D'Ambrosio, this way of teaching does not receive sufficient opportunities. Following Bishop (1997) he criticizes the technically-oriented curriculum with its emphasis on technique and drill and where history, philosophy and critical reflection are not given a chance. D'Ambrosio develops three concepts to focus on in a new curriculum regarding the usage of the international (UNESCO) emancipatory objectives - *literacy*, *matheracy* and *technoracy*.

Literacy has to do with communicative values and it is an opportunity to contain and use information. Here both spoken and written language is concerned but so are symbols and meanings, codes and numbers. Mathematical literacy is undoubtedly a part of it. *Matheracy* is a tool that offers the chance to deduce, to develop hypotheses and to draw conclusions from data. These are the base points for an analytical and scientific attitude. Finally there is *Technoracy* which offers the opportunity to become familiar with technology. This does not imply that every pupil should or even could get an understanding of the technological developments. This elementary form of education needs to guarantee that every user of a technology should get to know at least the basic principles, the possibilities and the risks in order to deal with this technology in a sensible way or deal not at all with it.

With these three forms of elementary education, which can be developed throughout the ethnomathematics research program, D'Ambrosio wants to meet the Universal Declaration of the Human Rights that relate to the right to education and the right to the benefits of the scientific developments.

CONCLUSION

This paper considered the shifted meaning of ethnomathematics and its role within mathematics education. Ethnomathematics is not longer reserved for so-called

nonliterate people; it now refers to the cultural diversity in mathematics education. Math teachers are therefore challenged to handle pupils' diverse everyday mathematical practices. In line with the UNESCO declaration (1948) on education and the OECD declaration (2004) on mathematical literacy, ethnomathematics clearly gained a more prominent role. Within Western curricula, ethnomathematics became meaningful to explore as an alternative, implicit philosophy of school mathematical practices. The extended notion of ethnomathematics as dealing with pupils' cultural diversity and with their everyday mathematical practices brings mathematics closer to the social environment of the pupil. Ethnomathematics is an implicitly value-driven program and practice on mathematics and mathematics education. It is based on an emancipatory and critical attitude that promotes emancipation and equality (Powell & Frankenstein 1997). Where the so-called academic Western mathematics still is locked in the debate on whether it is impartial or value-driven, the ethnomathematics' purposes stand out clearly right from the start. The historian of mathematics Dirk Struik postulated the importance of ethnomathematics. He validates ethnomathematics as both an academic and political program. There mathematics is connected to its cultural origin as education is with social justice (Powell & Frankenstein (1999). D'Ambrosio even puts it more sharply: *Yes, ethnomathematics is political correctness* (D'Ambrosio 2007a, p.32).

The implication for research is threefold. First, research has to reveal the (explicit and implicit) values within mathematics, mathematical practices and mathematics education. Second, research has to investigate thoroughly the use and integration of pupils' mathematical practices in the curriculum. Third, pupils' daily mathematical practices have to be studied.

NOTES

1. Article 26. (1) Everyone has the right to education. Education shall be free, at least in the elementary and fundamental stages. Elementary education shall be compulsory. Technical and professional education shall be made generally available and higher education shall be equally accessible to all on the basis of merit. (2) Education shall be directed to the full development of the human personality and to the strengthening of respect for human rights and fundamental freedoms. It shall promote understanding, tolerance and friendship among all nations, racial or religious groups, and shall further the activities of the United Nations for the maintenance of peace. (3) Parents have a prior right to choose the kind of education that shall be given to their children. Article 27. (1) Everyone has the right freely to participate in the cultural life of the community, to enjoy the arts and to share in scientific advancement and its benefits. (2) Everyone has the right to the protection of the moral and material interests resulting from any scientific, literary or artistic production of which he is the author. (United Nations Educational, Scientific and Cultural Organization. 1948)

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CERME 6 – WORKING GROUP 9

DIFFERENT THEORETICAL PERSPECTIVES AND APPROACHES IN MATHEMATICS EDUCATION RESEARCH

TABLE OF CONTENTS

Introduction.....	1529
<i>Susanne Prediger, Marianna Bosch, Ivy Kidron, John Monaghan, Gérard Sensevy</i>	
Research problems emerging from a teaching episode: a dialogue between TDS and ATD	1535
<i>Michèle Artigue, Marianna Bosch, Joseph Gascón, Agnès Lenfant</i>	
Complementary networking: enriching understanding.....	1545
<i>Ferdinando Arzarello, Angelika Bikner-Ahsbabs, Cristina Sabena</i>	
Interpreting students' reasoning through the lens of two different languages of description: integration or juxtaposition?	1555
<i>Christer Bergsten, Eva Jablonka</i>	
Coordinating multimodal social semiotics and institutional perspective in studying assessment actions in mathematics classrooms.....	1565
<i>Lisa Björklund-Boistrup, Staffan Selander</i>	
Integrating different perspectives to see the front and the back: The case of explicitness	1575
<i>Uwe Gellert</i>	
The practice of (university) mathematics teaching: mediational inquiry in a community of practice or an activity system.....	1585
<i>Barbara Jaworski</i>	
An interplay of theories in the context of computer-based mathematics teaching: how it works and why	1595
<i>Helga Jungwirth</i>	
On the adoption of a model to interpret teachers' use of technology in mathematics lessons	1605
<i>Jean-Baptiste Lagrange, John Monaghan</i>	
The joint action theory in didactics: why do we need it in the case of teaching and learning mathematics?.....	1615
<i>Florence Ligozat, Maria-Luisa Schubauer-Leoni</i>	
Teacher's didactical variability and its role in mathematics education	1625
<i>Jarmila Novotná, Bernard Sarrazy</i>	

The potential to act for low achieving students as an example of combining use of different theories	1635
<i>Ingolf Schäfer</i>	
Outline of a joint action theory in didactics	1645
<i>Gérard Sensevy</i>	
The transition between mathematics studies at secondary and tertiary levels; individual and social perspectives.....	1655
<i>Erika Stadler</i>	
Combining and Coordinating theoretical perspectives in mathematics education research.....	1665
<i>Tine Wedege</i>	
Comparing theoretical frameworks in didactics of mathematics: the GOA-model.....	1675
<i>Carl Winslow</i>	

INTRODUCTION

DIFFERENT THEORETICAL PERSPECTIVES AND APPROACHES IN MATHEMATICS EDUCATION RESEARCH - STRATEGIES AND DIFFICULTIES WHEN CONNECTING THEORIES

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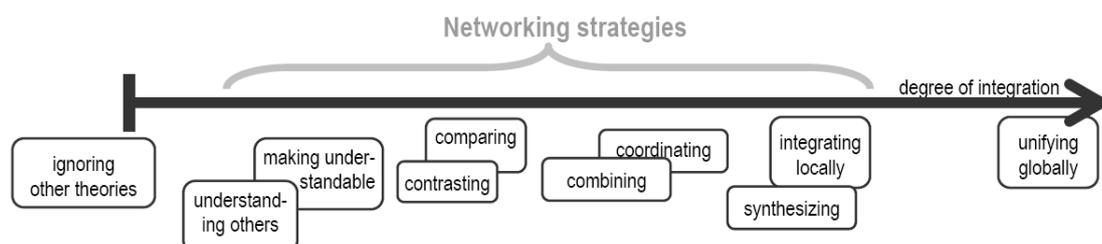
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A large diversity of different theoretical perspectives and research paradigms characterize the European mathematics education research community. Since CERME 4, the ‘Theory Working Group’ has explored differences between these theories, their expression in different research practices and possible ways to deal with this diversity (see Artigue et al. 2006, Bosch et al. 2008 and Prediger et al. 2008).

Exploiting diversity as a rich resource for grasping complex realities (Bikner-Ahsbabs & Prediger 2006) requires developing strategies for connecting theories or research results obtained using different theoretical approaches. In 2007, the Theory Working Group continued its efforts in this direction and reflected on opportunities and difficulties of what we call ‘networking theories’. We noted different intentions behind researchers efforts to network theories. In some cases, the goal is to investigate the complementary insights that are offered when we analyze given data with different theories (Kidron, 2008). In other cases, the intention is to explore the insights offered by each theory to the other theories and, at the same time, to highlight the limits of such an endeavour (Kidron et al., 2008; Radford, 2008).

The call for papers for the Theory Working Group at CERME 6 was guided by the idea of avoiding an overly abstract discussion without a concrete basis. That is why we called for papers with *concrete case studies* in which two or more theoretical approaches were connected. After an intensive peer review process, 15 substantial papers were chosen for discussion in the working group and for publication in these proceedings. The most important issues arising in the discussion of these case studies can be sketched under some key words structured according to the landscape of networking strategies as proposed by Prediger, Bikner-Ahsbabs & Arzarello, (2008).



Main issues arising in comparing and contrasting: Dimension of comparison

Comparing theories requires categories for comparison. A variety of categories have been suggested by Prediger, Arzarello & Bikner-Ahsbals (2008). The discussion this year was influenced by the following:

- the delimitation of empirical data and the kind of questions that arise, as well as the concrete formulation of results (see Ligozat & Schubauer-Leoni in this volume);
- the distinction between theoretical approaches and perspectives (discussed by Wedege in this volume);
- an ontological characterization of theories such as that proposed by Winsløw (in this volume) called the *GOA*-Model, which distinguishes theories according to nature of their objects of research, namely groups (*G*) structured by certain relationships, the organisation (*O*) of knowledge and practice, and artefacts (*A*) used to access and communicate in and about *O*.
- an epistemological characterization of theories such as that proposed by Radford (2008), distinguishing between their basic principles, their methodology, and the paradigmatic questions that are approached.

Main issues arising in combining and coordinating: Compatibility

In order to combine or coordinate different theories, it appears to us that the theories must, in some sense, be compatible; but what exactly does this mean? In working group discussions of the case studies presented in the papers, different levels were posited as possible locations for potential incompatibilities:

- the level of general principles, e.g. epistemological principles about how to interpret mathematical knowledge;
- the level of basic ‘paradigms’, the potential danger of hastily combining stability-oriented with transformation-oriented perspectives;
- the level of central constructs: although the sense or denotation of constructs may not be identical over different theories, they should not be contradictory (Gellert in this volume shows an interesting example of networking around the construct “rules”);
- the level of practical consequences: if coordinating theories in empirical work leads to contradictory practical consequences with regard to learning, then there is a need to continue reflection (see Bergsten & Jablonka in this volume);
- the level of ontology: this does not seem to present as many difficulties as some of the above since different grain sizes of analyses and focuses might help in combining theories (see, for example, Jungwirth in this volume).

In the working group discussion it was suggested that when paradigmatic research questions and/or objects diverge in different perspectives, the combination of these

perspectives in the course of analysing an empirical phenomenon might produce incommensurable, but not contradictory, results, as shown by the paper of Bergsten & Jablonka (in this volume). This raises the question of whether it is acceptable that different results can, without contradiction, lead to radically opposed interpretations.

On the other hand, we found some aspects that facilitate the connection of theories. Theories might be linked more easily when they are not too strong with respect to their grammar or their methodologies (i.e. when they are at an early level of elaboration) or when they are complementary with respect to their hypothetical scope or empirical load (see Jungwirth in this volume).

Main issue arising in integrating and synthesizing: Substrategies

The working group discussion regarding strategies for integrating and synthesizing theories led to the tentative proposal to identify substrategies which included: ‘bricolaging’ (that is adapting non-conflicting principles, notions or local analysis methods of different grand theories); ‘subordinating’ (see Gellert); ‘zooming in and out’ (see Jungwirth); and ‘metaphorical structuring’, the use of single concepts based on metaphors from one theory that converge into another (see Gellert with regard to rules).

As Radford (2008) stated, although connections between theories are possible, there is a *limit* to what can be connected and this limit is determined by the *goal* of the connection and the *specificities* of the theories that are being connected. In the following, we differentiate between different goals in the networking process.

Networking with different aims

In order to link theories beyond comparing and contrasting, we discussed the aims of the papers.:

- Some of the papers propose networking strategies with the aim of understanding an empirical phenomenon that seems difficult to entirely grasp within one single theory. These can be described as having an initial combining strategy that ends up with the construction of local coherence between the notions or principles used. In this sense, Arzarello, Bikner and Sabena (in this volume) combine theories for analysing data about a failed teaching strategy and integrating them (very) locally for the purpose of making sense of the situation described. The paper of Schäfer (in this volume) combines theories for constructing a local theory that improved his potential to approach a ‘practical’ question about low achieving students. Wedege (in this volume) presents a study in which some aspects of two theoretical perspectives are coordinated. Stadler (in this volume) coordinates different perspectives within one empirical study, describing how a research interest in the transition between mathematics studies at secondary and tertiary levels generates the need for different theoretical approaches.
- A different goal presented by some papers is to network with the aim of dealing with new problems. For example Ligozat & Schubauer-Leoni’s and Sensevy’s papers are hybrids which borrow constructs from distinct theories for local integra-

tion with conversions in order to address specific research problem, the issue of joint action of the teacher and the students.

- Networking is also an important tool to elaborate existing theories with the aim of increasing their scope by questioning them from the outside. Artigue, Bosch, Gascón & Lenfant (in this volume) show how a theory can evolve locally when an effort is made to approach a question formulated by another theory. The strategy here is to work within one theoretical framework and develop it in interaction with others, for instance by enlarging the set of paradigmatic research questions or its empirical unit of analysis. The work of Jungwirth (in this volume) presents a method of synthesizing local theories for ‘zooming in and out’ of the data.
- Other papers consider networking with the aim of satisfying the need for an enlarged framework in relation to some new domain of research, assuming the existing frames are insufficient. For instance, Lagrange & Monaghan (in this volume) incorporated Saxe’s four parameters model in order to understand the situation of teachers using technology. To these authors, the existing frameworks they considered for viewing teachers’ activities in technology-based lessons are insufficient because they focus on teachers’ established routines but technology interferes with these routines.

Different kinds of dialogues

Within these aims we may distinguish different kinds of *dialogues* between theories. We use the word ‘dialogue’ not only to describe that which enables mutual understanding in the way we communicate our theories but also to emphasize differences in the use of language. Different kinds of dialogues were offered in the papers by Ligozat & Schubauer-Leoni, by Sensevy and by Artigue et al. One important characterization is that the dialogues in these papers are between neighbouring approaches - theoretical approaches which were born in the same educational and didactic culture, which may be considered as belonging to the same ‘paradigm’. Even so, when we explore the dialogues in depth important differences between the theories can be seen and some interesting questions arise:

- Do these “neighbouring approaches” use the same words with the same meanings? For instance, is the word *milieu* in the Anthropological Theory of the Didactic (ATD) equivalent to the *a didactic milieu* in the Theory of Didactic Situations (TDS)? The same question could be asked in relation to other terms, e.g. *institution* or *contract*. The question could arise also for theories which are not necessarily *neighbouring approaches*.
- Do the different theories deal with different ways of addressing similar issues? For instance, comparing the Joint Action Theory in Didactics (JATD), as described in both Ligozat & Schubauer-Leoni and Sensevy’s papers, with ATD and TDS, we may ask what is the difference between *ATD media milieu dialectic*, *TDS a didactic and didactic situations*, and *JATD dialectic between contract and milieu*. Sensevy states that in order to situate JATD in relation to TDS and ATD it can be ar-

gued that whereas these two theories initially focus, from a logical point of view, on the nature of *knowledge* (what is the knowledge which is taught?), JATD initially focuses on the *diffusion process* (what is going on when a specific piece of knowledge is taught?). The aim of the networking is to construct a new theory JATD which makes use of existing theories, ATD and TDS. Therefore we may ask what supplementary insights and/or what new questions/problems are offered to ATD and TDS by JATD's analysis of the *diffusion process*? For example, the JATD may raise the following question: within the contract-milieu dialectic how may the teacher link the topogenesis and the chronogenesis processes with respect to the piece of knowledge at stake, and how might these processes lead the teacher, in specific cases, to enact a new learning game? In this question there are some notions from ATD and TDS which are reconceptualized in that they are used in a new way, and there is a new notion (learning game). From an abstract viewpoint, this kind of question is not impossible in ATD and TDS, and it is clearly understandable in these two theories. But the probability that this question is raised in these two theories is not high because their fundamental concerns are not focused on the problems of didactic joint action even though they are interested in didactic action.

In Artigue et al. (in this volume) the notion of 'minimal unit of analysis' appears as a basic aspect of the modelling of educational phenomena proposed by each theory. Starting from the way each perspective reformulates a given research question, we could specify what units of analysis are considered in each case and how they can be connected. The authors add that this could be a good way to improve our capacity for describing and comparing not only the concrete research or practical problem formulated by each theory but also the types of problems that can be proposed, the kind of empirical data needed and the set of 'acceptable answers' that can be provided. When we choose a *specific unit of analysis*, we make decisions not only about the empirical data we consider but also about our different priorities with regard to the focus of the analysis (Bosch & Gascón, 2005).

Final remarks

The discussions that took place in our working group about affordances and constraints of different networking strategies made us aware that the theoretical frameworks used in our research are 'living entities' that evolve through our studies. Some have been around and have developed for many decades, others are less mature. They are our working tools, providing us with new ways of looking at reality, new descriptions of empirical phenomena, new methods of analysis and new possible answers to the difficulties of teaching and learning mathematics. They are imbedded in researchers' social, cultural and institutional inheritances and their development is also impregnated with the personal interactions between researchers and the cooperative work done in our community. When we embody 'theories' into research practices that, at the same time, use theories and produce them, it becomes clear that our reflec-

tions about ‘networking theories’ are methodological reflections, referring to the kind of tools we can or cannot use, the basis and the aim of our research, as well as the kind of rules we follow.

Considering the networking of theories as the *networking of research practices* may lead us further not only in our capacity to collaborate between different groups of researchers (and thus accumulate efforts and results) but also to gain insight about the very nature – and the rationale – of our own research in mathematics education.

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RESEARCH PROBLEMS EMERGING FROM A TEACHING EPISODE: A DIALOGUE BETWEEN TDS AND ATD

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When approaching an empirical teaching episode or data related to it, theoretical approaches always select and highlight some aspects in detriment of others, globally interpreting the episode using their own conceptual categories and methodological tools. Therefore, different theoretical approaches often construct different research problems, often making their comparison difficult or even impossible. The fact that the Theory of Didactic Situations and the Anthropological Theory of the Didactic share their main assumptions and their ‘research programme’ (in Lakatos’ terms) makes it easier to contrast them in the way each one reinterprets and reformulates the problems raised by the other. Starting with ‘neighbouring approaches’ thus appears as a sensitive way to approach the complexity of networking theories.

According to Rodríguez et al. (2008), we assume that any strategy to compare, contrast or network theories has to take into account the way theories question reality and formulate problems about it. This assumption leads us to consider as a *networking methodology* the comparison between the reformulations proposed by different theories of a research question raised by one of them. In this case, the question emerges from an empirical episode and a given set of data. We start this ‘exercise’ with the case of two theories close to each other, the Theory of Didactic Situations (TDS) and the Anthropological Theory of the Didactic (ATD). We first present the context where this study takes place, and then analyse the exchanges between the co-authors of this contribution around a particular research question, before entering a more general discussion about the potential of this methodology.

1. THE CONTEXT FOR THIS STUDY

This study is part of the work on the comparison of theoretical frames of a collective that emerged at CERME4, and whose first outcomes have been presented at CERME5 (Arzarello et al. 2008, Kidron et al. 2008, Prediguer 2008). Since CERME5, the group has orientated its work towards the development of networking methodologies. Different strategies are used for that purpose. One of these, which presents some similarity with the strategy used in the ReMath European project (Artigue 2007, Mariotti 2008), is the comparison between the formulations proposed by different theories when confronted to a given set of data and a research question raised by one of them. In our case, the research question emerged from the analysis of a video, which, from the very beginning, played a crucial role in the work of the group. It corresponds to a classroom session at grade 10 in Italy on the exploration of the properties of exponential functions in the Cabri-géomètre environment, and more

precisely to the observation of a group of two students. In a first phase of the work, the different teams involved in the group analysed the video from their respective theoretical perspectives, what made clear that all of them, except the Italian colleagues, could not find what they needed for completing the analysis they aimed at in the information initially provided: the video and some documents about the classroom session. Each team was thus asked to make clear the kind of information it needed, and the demands of the different teams were discussed at a post-CERME5 meeting. One of the results of this discussion was a questionnaire to be answered by the teacher in charge of the class observed. When the extra information agreed upon, including the teacher's answers to the questionnaire, was disseminated, each team tried to complete its analysis, and the results were presented during a joint meeting in Barcelona. In their respective presentations, several teams referred to a particular answer made by the teacher, pointing out that, from their perspective, such an answer raised important and non trivial issues and deserved further discussion. The question and the answer were the following:

“During a lesson of this type, under what circumstances do you decide to get involved with a pair of students, and what kinds of things do you do?”

“I try to work in a zone of proximal development. The analysis of video and the attention we paid to gestures bring me to become aware of the so called ‘semiotic game’ that consists in using the same gestures as students but accompanying them with a more specific and precise language in relation to the language used by students. A semiotic game, if it is used with awareness, may be a very good tool to introduce students to institutional knowledge.”

This episode of our collaborative work and the potential we soon suspected it could have if analysed in depth, was the source of the networking methodology we then developed. This methodology obeyed the following organization: the team working in TDS formulates a research problem using its own terminology; each team converts the problem according to its theoretical perspective; the team working in TDS comments on the new formulations, looking at the generic and specific issues; each team works on its specific question and reflects on the process followed.

In what follows we describe the exchanges that this methodology generated between the TDS and ATD perspectives, and analyse their networking potential.

2. EXCHANGES ON “SEMIOTIC GAMES”

2.1. A first perspective inspired by TDS

As mentioned above, a series of comments regarding the teacher's answer and the articulation of some precise questions was first elaborated within the TDS perspective by two of the co-authors of this contribution (MA & AL). We summarize the main lines of their argumentation below, the teacher being denoted by T.

First, MA & AL observe that the answer expresses the confidence that T has in the so called semiotic games to face a major didactic problem: the connection between, on the one hand, the mathematics produced by students in an adidactic situation, through the interaction with the adidactic *milieu* of this situation,¹ and on the other hand, the institutional mathematical knowledge aimed at. They add that this connection generally requires at least changes in the ways the mathematics at stake are expressed in order to progressively tune them with more conventional forms of expression; and that T obviously considers that he has a specific mediating role to play for making this connection possible and uses semiotic games as a tool for that purpose. In other terms, semiotic games can be considered as components of the praxeology (or more certainly of the different praxeologies) that T has developed in order to solve this didactic task. It is interesting to point out that this last sentence uses terms coming from ATD and not TDS, organizing a first bridge between them.

MA & AL then point out that this answer raises two interesting didactic issues:

The first one is that the situations proposed to students for building new mathematical knowledge do not necessarily have the adidactic potential that is necessary to enable the students to produce the mathematics to be produced under the constrained conditions of the classroom. What is achievable and achieved through an adidactic interaction with the *milieu* is often far from allowing the teacher to easily establish a meaningful connection with the mathematical knowledge aimed at. The discrepancy leads to different phenomena that have been discussed in TDS research (for instance Jourdain effects or “*dédouplements de situation*”), all the more as the teacher feels obliged to maintain the fiction that the mathematics knowledge he or she is expecting has to be produced by the students.

The second one is that the situations proposed to students for building new mathematical knowledge are very often what the TDS calls situations of action. They can lead to a linguistic activity but language issues are not their main concern. The characteristics of the *milieu*, the feedback available, do not make the productivity of the interaction with the *milieu* strongly dependent on the language used by the students. This is a fundamental difference with situations of communication often associated with the dialectics of formulation in the TDS.

Referring to their analysis of the video, MA &AL claim that the associated situations have a rich adidactic potential but also that this potential is a priori not sufficient to ensure the production of all the mathematical knowledge aimed at according to T's answers to the whole questionnaire. They also add that, even if the students have to produce narratives, the three situations which can be identified in the observed ses-

¹ The notion of *milieu* was introduced by Guy Brousseau as a main element of the Theory of Didactic Situations (Brousseau 1997). It refers to a system without any didactic intention that constitutes a key element of any adidactic situation. The reader unfamiliar with the TDS can find a very accessible introduction in Warfield (2006).

sion are closer to situations of action than to situations of formulation. They thus conclude that these situations constitute a priori good material to examine in context the potential and limits of semiotic games.

They also point out the specific status of T who is an expert teacher, but much more than that, due to his research engagement. According to them, this means that the confidence he expresses in the potential of semiotic games certainly has a solid experiential basis both in his personal practice and also in the practices of the research community he is involved in. Nevertheless, MA & AL's personal experience leads them to look at these semiotic games carefully, all the more when they are said to provide techniques for solving what are considered difficult didactic problems, and to try to understand under what conditions and why they can become efficient didactic techniques helping teachers face the difficulties described above.

The research question resulting from this analysis is the following:

How to identify characteristics of the semiotic game technique that would help us to understand its potential for:

- Compensating the possible limits of the interaction with the didactic milieu to achieve the expected mathematical goals?
- Fostering the linguistic evolution linked to the needs of institutionalization processes?
- Identifying conditions required to activate this potential?

How to identify possible difficulties in the management of such semiotic games and possible effects of their possible malfunctioning?

2.2. Conversion of the research question within the ATD perspective

The answer to this question analysed below comes from the other two co-authors of this contribution (MB & JG) who work in the ATD perspective.

2.2.1. Some preliminary considerations

The ATD describes human practices (including doing mathematics and its teaching and learning) in terms of *praxeologies* composed by two complementary folds: a *praxis* or practical block (the “know-how”) made of *types of tasks* and *techniques* to carry out these tasks; a *logos* or theoretical block (the “knowledge” in its narrow sense) that appears as an assemblage of discourses to describe, explain and justify the *praxis*.² The question formulated by MA & AL starts from a rather vague notion of ‘semiotic game’ that, in the ATD, can be considered as a *didactic technique* that T describes as follows: “the teacher starts using ‘the same gestures as students but accompanying them with a more specific and precise language in relation to the language used by students’”. T's comments on the episode also reveal some *theoretical components* explaining and justifying the use of this technique, formulated in terms

² For the reader unfamiliar with the ATD, see Bosch & Gascón (2006).

of ‘working in a zone of proximal development’. At the same time, the comments refer to a type of *teaching task* that is supposed to be performed with this technique: ‘introducing students to institutional knowledge’. We are thus considering a *didactic praxeology* as it is evocated by the teacher.

Any ‘didactic problem’ (that is, a problem related to the teaching, learning, studying or diffusion of knowledge) can be generally identified both with a ‘teaching problem’ (that is, a question or difficulty that appears in the teacher’s practice and that requires an appropriate *didactic praxeology*) and with a ‘research problem’ (that is, an open question for research in mathematics education). In both cases the problem is formulated in relation to a *teaching and learning process* and connected to a given mathematical content (which is a *mathematical praxeology* or a set of mathematical praxeologies). In this sense, the ‘expected mathematical goal’ that appears in the formulation of the question, as well as the ‘proposed institutional knowledge’ are mathematical praxeologies that can have different ‘size’: point, local, regional or even global.³ According to Bosch & Gascón (2005), the ATD postulates that the minimal unit of analysis of didactic processes has to contain at least a *local* mathematical praxeology. Furthermore, this local level is considered as privileged or basic because, in order to be studied in an operative way, any didactic problem formulated beyond this level of analysis needs to be ‘projected’ into its local components. For MB & JG, in the ATD perspective, the initial research question can thus be situated in the very general problem of the study of *the conditions that make the building of local mathematical praxeologies in a given institution possible and the restrictions that hinder it*.

2.2.2. *The dialectic media/milieu*

At the beginning, the process of building local mathematical praxeologies can start from questions that arise within a point praxeology or in a small set of them. In any case, the driving force of the didactic process, what provokes the need to study or build a local praxeology integrating and completing the point praxeologies, is the emergence of questions that cannot be answered within the point praxeologies. How these questions arise in a given didactic process? What conditions are needed for a study community to ‘take them seriously’? What ‘*media*’ can help the study community to generate provisional answers and what ‘*milieu*’ is available to test and modify these answers? These are still open questions and an in-depth analysis of what is called the ‘dialectic media/milieu’ seems essential to answer them.

³ A *point* praxeology is generated by a unique type of tasks and is often characterized by a unique technique to deal with them; a *local* praxeology is generated by the integration of several point praxeologies within the same technology; a *regional* praxeology is obtained by coordinating, integrating or linking several local praxeologies through a common mathematical theory and a *global* praxeology is a connection of some regional praxeologies (Rodríguez *et al.* 2008).

According to Chevallard (2004), the elaboration of an answer to a real question supposes ‘resources’ or ‘milieus’. In close connection with the TSD terminology, a ‘milieu’ is a system without any didactic intention in the interaction we can have with it during the study process. In this sense a milieu behaves as a fragment of ‘nature’. Besides the notion of ‘milieu’, the ATD introduces the notion of ‘media’ as any system the main goal or intention of which is to supply information about a given issue. In any knowledge construction process a dialectics between a media providing new knowledge or information and a milieu able to give evidence of the validity of this information takes place. An extreme situation is when one takes the message coming from the *media* as it appears, without any need for testing it. The opposite side is the construction of knowledge from scratch, through only the confrontation with a *milieu*. The existence of a vigorous (and rigorous) dialectics between *media* and *milieus* appears to be a crucial condition for a study process not to be reduced to a simple copy of previously elaborated answers spread over different social institutions.

2.2.3. Formulation of the question in the ATD frame

(a) ‘Semiotic games’ and the limitations of the didactic milieu

The general didactic problem we are considering is the study of the didactic tools, devices or praxeologies that are necessary for the teacher to lead and for the students to carry out the process of building local praxeologies. With respect to the problem of the ‘limits of the interaction with the didactic milieu’, it is important to notice that, from the perspective of the ATD, the dialectic media/milieu supposes that any milieu has limitations in the didactic process consisting in building a local praxeology as the progressive answer to a problematic initial question. Even if a given milieu can help contrast a partial answer to the initial question, it will always provoke the need of new *media* introducing new information having to be tested with new milieus, and so on. In this context, T’s ‘semiotic game’ considered as a didactic technique, may be interpreted as a resource used by the teacher – acting as a ‘media’ – to supply students with praxeological components of the praxeology that is to be built.

(b) Institutional didactic praxeologies underlying the ‘semiotic games’

Beyond the didactic techniques a given teacher can ‘create’, research in the ATD frame is interested in the *didactic techniques a given institution makes available* to the teacher and the students to manage the construction of mathematical praxeologies and, more particularly, to manage the media/milieu dialectics.

This institutional dimension is essential because it strongly determines the ecological conditions required by these didactic techniques to normally evolve in the considered institutions. More particularly, the existing institutional conditions influence the kind of *technical gestures* that can usually be made in the institution, as for instance the ‘semiotic games’. Like any other didactic technique, ‘semiotic games’ need an insti-

tutionalised didactic *technology* to describe, justify, interpret and control their role in the didactic process. Beyond the technological level, it is also interesting to study what *theoretical* foundation supports this teaching technique and technology.

2.3. Back to the TSD and the formulation of new research questions

After the re-formulation of the research question raised by TDS, a second exchange took place between the teams. We extract from it what concerns the TDS and ATD perspectives. In their comments, MA & AL first point out that, considered as didactic techniques both in TDS and ATD, semiotic games are given two different functionalities according to the theoretical perspective chosen. They also suggest that from this situation can emerge interesting insights regarding the relationships between ATD and TDS:

“According to ATD, a condition for a study process not to be reduced to a simple copy of previously elaborated answers is the existence of a strong dialectic between appropriate media and milieus. Such a theoretical position presupposes that any milieu has limitations in the didactic process consisting in building a local praxeology, a process which is seen as the progressive answer to an initial question. Within this approach, T’s semiotic games find their place as a didactic technique used for the management of the media/milieu dialectic. We think that it will be interesting from this point of view to compare the vision that will be proposed concerning these semiotic games on the one hand by the ATD analysis projecting them in the media/milieu dialectic and on the other hand by TDS projecting them at the interface between adidactic and didactic processes. Having its origin in a theoretical context both distinct from ATD and TDS, it may provide a good opportunity for understanding better the similarities and differences between these two theoretical approaches regarding these crucial aspects.”

Another element stressed by MA & AL is that the conversion of the initial questions within an ATD perspective makes a new dimension move from the periphery to the centre: the institutional dimension. ATD indeed obliges the researchers to consider that the study of any kind of didactic technique has to be situated within an institutional perspective. It cannot exist and develop without any institutional legitimation, any institutionalised didactic technology used to describe, justify, interpret and control its role in the didactic process. Within this perspective, what is of interest for research is clearly not the study of semiotic games as practices of individual teachers but the study of their institutional status and ecology, of their relationships with other institutional techniques available to teachers for managing the dialectics between media and milieus. MA & AL add that, in this particular case, the experimental status of the course to which the observed session belongs means that at least two institutions are involved and should be considered: the research institution and the high school institution.

Finally, the exchanges also make MA et AL reflect more globally on the first phase of the work, and the limitation of the perspective underlying it. The first phase con-

sisted of using the TDS and ATD theoretical constructs to reflect about semiotic games, their didactic potential and limit, but the converse movement is also possible, leading to investigate what can be offered to TDS and ATD by having the ideas of semiotic game and the ‘zone of proximal development’ as functioning in T’s ‘practical theory’ (Ruthven, 2006) entering the scene. This converse movement can also be insightful regarding relationships between TDS and ATD, and the possibilities of networking between them.

2.4. Main features of a didactic research problem

At this point of the networking between TDS and ATD, and in order to pursue the network with other theoretical frameworks, it seems necessary to locate the dialogue in a new position, more general and relatively neutral from an epistemological point of view. Three main features seem important to distinguish.

2.4.1. Institutional dimension of the didactic problems

In the ATD perspective, the expression ‘semiotic game’ appears as an element of the *teacher’s didactic theoretical discourse*: it helps him interpret what happens in the classroom, take decisions, etc. In this sense, we are dealing with a component of the *spontaneous didactic praxeology of a concrete teacher*. A first difficulty appears concerning the personal or institutional dimension of this didactic praxeology.

Institutional praxeologies (and their ecology) are the ATD’s primary object of study. To study them, we take as an empirical basis the personal manifestations of these praxeologies as well as their more collective or institutional manifestations: regular practices, discourses, texts, official documents, etc. The dialectic between *persons* and *institutions* can be made more explicit in the following terms. The institutions where praxeologies take place are composed of persons. Reciprocally, persons are always *subjects* of a complex of institutions and, as such, have a personal relation to praxeologies that can be explained to a great extent by the analysis of the institutional praxeologies they have encountered.

2.4.2. Mathematics as a core component of didactic problems

Taking into account the educational institutions’ vision of teaching and learning processes is a basic methodological principle of the ATD. Otherwise, we run the risk of taking for granted the description of phenomena proposed by each institution – which can furthermore differ from one institution to another. More particularly, research on *didactic transposition processes* (Bosch & Gascón 2006) has shown the necessity for research to construct its own models of mathematical knowledge (or mathematical activities) in order to avoid taking for granted the models imposed by the dominant institutions. These models of mathematical knowledge should include the description of its construction, development and diffusion (and, thus, the mathematics teaching and learning processes).

2.4.3. *The importance of the unit of analysis*

Any essay to contrast or compare theories has to face a dilemma. On the one hand, to contrast theories, we need a ‘common’ empirical universe and, thus, we have to remain close to the educational institutions. On the other hand, each theoretical perspective constructs its own vision of this empirical universe, moving away from the educational institutions (to ‘escape’ from their dominant vision). This detachment is necessary in order to approach problems related to the teaching and learning of mathematics in a more operative way. However, it has always to maintain an accurate distance to the reality one wishes to study – and modify!

The notion of ‘minimal unit of analysis’ (section 2.2.1) appears as a basic aspect of the modelling of educational phenomena proposed by each theory. Starting from the way each perspective formulates MA & AL’s question, we could make explicit what units of analysis are considered in each case and how they can be connected. This could be a good way to improve our capacity of describing and comparing not only the concrete research or practical problem formulated by each frame but also the type of problems that can be proposed and the kind of empirical data needed.

3. CONCLUSION

This contribution illustrates a methodology of ‘networking theories’ based on the study of a question, considering how the different research frameworks engaged in the networking can formulate and approach it, through a sequence of exchanges and progressive refinements. We have taken the interaction between TDS and ATD as a study case, considering two close frameworks that share the same scientific project. This proximity makes the networking easier because the discussion on the fundamental background of the theories can be avoided. It is important to recall that ATD emerged within the TDS, thus integrating the original research programme, its basic assumptions, the nature of the considered problems and phenomena and, more particularly, the need to question and model mathematical knowledge (that is, to take it as a specific object of study). Making this methodology productive with more distant approaches raises the necessity to make the basic assumptions of each one explicit and to contrast them. Another positive consequence of this methodology stems from the fact that the theories involved are questioned from an external construction, which in our case has given rise to two main contributions. The first one is the institutional dimension assigned (by the ATD) to the ‘semiotic games’ and the way it can be taken into account by the TDS. This issue has long been explored and largely discussed by research in both the TDS and ATD perspectives (Sensevy et al 2005). The second contribution is the comparison between the projections ‘didactic – adidactic’ and the ‘media and milieu dialectics’. They emphasize an obvious difference in the way both theories take into account the milieu’s insufficiencies and the changes in our relation to knowledge led by the technological evolution. It is important to note finally that, till now, little advantage has been taken from the inverse networking movement: considering the contributions made by the external perspectives to the development of

our own one. For instance, by formulating problems which are not of first priority in our research programmes but the study of which can open unexpected lines of development. We finally postulate that making explicit the position adopted by research perspectives to the features considered in section 2.4 constitutes an essential step for the networking. This position is important because it delimitates what is considered a 'didactic research problem' and, consequently, contributes to characterise the object of study of our discipline.

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COMPLEMENTARY NETWORKING: ENRICHING UNDERSTANDING

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Our analysis of data about one learning situation from two theoretical perspectives yields results that on the surface seem to be in conflict. Through networking of two theories we produce a fresh combined analysis tool, which deepens our understanding of the data in an integrated way. We elaborate this example to make explicit our two theoretical approaches and our networking strategies and methods.

INTRODUCTION

The goal of the paper is to show how networking different theories can help researchers in entering more deeply into their research questions. More precisely, we will illustrate the limits of two theoretical approaches when used alone to analyse a classroom teaching situation, and the benefits of networking. As a result, data analysis and learning processes understanding is strongly enriched.

The main question faced in our research concerns how mathematical knowledge about the growth of the exponential function is achieved in a specific socially supported learning processes. This requires properly defining the objects of our research, the method and the tools for observation (Prediger et al., 2008). As to the objects, we distinguish two deeply linked components: the social interaction among the subjects, and the epistemic issues in such learning processes.

Our networking strategy is worked out through analyses of empirical data. The same teacher-student-interaction is analysed from two theoretical perspectives that on the surface seem to be in conflict: the *interest-dense situation* and the *semiotic bundle* analysis. Using the former, it appears that the thought process of a student is disturbed by the social interaction with the teacher. However, no disturbances appear using the latter. We will show that through adding an epistemological perspective this conflict can be cleared away since the results can be integrated into a common view deepening our insight from both theoretical perspectives. This experience will be a starting point for a case of local integration of the two theoretical perspectives and some methodological reflection concerning networking strategies and methods.

ADOPTING TWO DIFFERENT PERSPECTIVES

Interest-dense situations and its epistemic process

So called interest-dense situations (Bikner-Ahsbahs, 2003) are those in which a maths class shows interest in the mathematical topic or activity, they occasionally occur within discursive processes in everyday maths lessons. In these situations the students become deeply involved in the mathematical activity, deepen their mathematical insight constructing further reaching mathematical meanings and begin to appreciate

the mathematics they learn. To achieve some mathematical knowledge the students activate epistemic actions (actions that are executed in order to come to know more). Through social interactions the class collectively coordinates the epistemic process. In this way collective epistemic actions are constituted by social interaction. In contrast to non interest-dense situations, all interest-dense situations lead to the epistemic action of structure seeing (perceiving a mathematical pattern or rule referring to an unlimited number of examples).

The genesis of interest-dense situations is supported by a special kind of social interactions: The students are driven by their own way of thinking. They follow their own questions and ideas about the mathematical object that they want to know more about. In this case the students' actions are independent of the teacher's expectations. In interest-dense situations the teacher's expectations do not control the situation. Rather the teacher focuses on supporting the students' thinking. If the teacher's behaviour is controlled by his own expectations the emergence of an interest-dense situation is interrupted, and the learning process is disturbed (Bikner-Ahsbabs, 2003).

The ways in which the teacher and students socially interact can be analysed on the three levels (Davis, 1980; Beck & Meyer, 1994). Speaking, a person expresses something on three different levels. On the locutionary level he/she says something, on the illocutionary level, he/she tells something through the way of saying something. The perlocutionary level is concerned with effects: "a speaker saying something produces an effect on feelings, thoughts, or actions of the audience, other persons, or himself" (Davis (referring to Austin and Searle), 1980, p. 38). In our example, G locutionarily says: "for a very big variable a , when the exponential function ($f(x) = a^x$) and this straight line (which he assumes), meet each other, it (meaning the straight line) approximates the function very well because..." being interrupted by the teacher's request: "what straight line, sorry?". By using broken language, G tells the teacher that (illocutionarily) he is working out his train of thought while speaking. Starting the sentence with "because", he indicates on the illocutionary level that his train of thought is not yet finished. On the perlocutionary level we observe an effect; the teacher's request. In order to comprehend how the epistemic process in a discursive learning situation is socially supported or hindered; the analysis of social interactions is done on these different levels and is complemented by an analysis of the epistemic process. The term "non-locutionary level" will embrace the illocutionary and perlocutionary level.

The semiotic bundle perspective

The semiotic bundle perspective lies on two basic assumptions:

- the teaching-learning process inherently involves *resources of different kinds*, in a deep integrated way: words (orally or in written form); extra-linguistic modes of expression (gestures, glances, ...); different types of inscriptions (drawings, sketches, graphs, ...); different instruments (from the pencil to the most sophisticated ICT devices), and so on (for some examples see Arzarello, 2006);

- such resources may play the role of signs (according to Peirce's definition¹) and therefore can be considered as *semiotic resources*.

Differently from other semiotic approaches, the semiotic bundle construct allows us to theoretically frame gestures and more generally all the bodily means of expression, as semiotic resources in learning processes, and to look at their relationship with the traditionally studied semiotic systems (e.g. written mathematical symbolism):

"A semiotic bundle is a *system of signs* — with Peirce's comprehensive notion of sign — that is produced by one or more interacting subjects and that evolves in time. Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question. Possibly the teacher too participates to this production and so the semiotic bundle may include also the signs produced by the teacher" (Arzarello et al., in print).

In teaching-learning contexts the different semiotic resources are used with great flexibility: the same subject can exploit simultaneously many of them, and sometimes they are shared by the students and by the teacher. All such resources, with the actions and productions they support, are important for grasping mathematical ideas, because they help to bridge the gap between the worldly experience and the time-less and context-less sentences of mathematics. An interesting phenomenon that has been identified within such an approach is the so called *semiotic game* (Arzarello, 2006; Arzarello et al., in print). A semiotic game happens in the teacher-students interaction when the teacher tunes with the students' semiotic resources and uses them to guide the evolution of mathematical meanings. We have analysed various examples in which the teacher repeats a student's gesture, and correlates it with a new term or with the correct explication given using natural language and mathematical symbolism (ibid.). Semiotic games constitute therefore an important strategy in the process of appropriation of the culturally shared meaning of signs.

An example analysed from the two perspectives

In this example, students (grade 10 of a scientific oriented high school) are working in pair on an exploratory activity on the exponential function. They are using a dynamic geometry software to explore the graphs of $y = a^x$ and of its tangent line² (a is a parameter whose value can be changed in a sliding bar). At a certain point the teacher has asked the students the following question: what happens to the exponential function for very big x ? We propose a short excerpt from the interaction between the teacher and one pair of students (G and C) about this question.

¹ As *sign* or *semiotic resource*, we consider anything that "stands to somebody for something in some respect or capacity" (Peirce, 1931-1958, vol. 2, paragraph 228).

² The line is actually a secant line; the secant points are so near that the line appears on the screen as tangent to the graph. This issue has been discussed in the classroom in a previous lesson.

1 [00:00] G: but always for *a* very big this straight line (pointing at the screen), when they meet each others, there it is again...that is it approximates the, the function very well, because...

2 T: what straight line, sorry?

3 G: this ...(pointing at the screen) this, for *x* very, very (00:14) big



00:14 G: the hand goes upwards

4 T (00:16): will they meet each other (00:17)? [suggestive connotation in the sense of "do you really think so?"]



00:16 T: pointing two forefingers



00:17 T: crossing the two pointed forefingers



00:19 G: two forefingers touching each other

5 G: that is [cioè]³, yes, yes they meet each other (00:19)

6 T: but after their meeting, what happens?

7 G: eh..eh, eh no, it make so (00:24)

00:24 G crosses the left hand over the right one; T is keeping the previous gesture



8 T: ah, ok, this then continues (00:27), this, the vertical straight line (00:28), has a well fixed *x*, hasn't it? The exponential function later goes on increasing the *x*, doesn't it (00:31)? Do you agree? Or not?



00:27 T moving rightwards his left hand



00:28 T: right hand vertically raised



00:31 T: moving rightwards his right hand

³ The expression "cioè" in Italian means literally "that is". Over-used by teenagers, it introduces a reformulation of what just said. As it is likely in this case, it can have the connotation of "I am sorry but".

9 G: yes [...]

10 T (addressing C): He [G] was saying that this vertical straight line (pointing at the screen) approximates very well (00:43) the exponential function

11 G: that is, but for very big x (00:46)

12 T: and for how big x ? 100 billions? (00:51) $x = 100$ billions?



00:43 T raises both hands



00:46 G moved his left hand high wards



[00:51 T: raised his hand and keeps it still]

13 G: that is, at a certain point...that is if the function (00:57) increases more and more, more and more (00:59) then it also becomes almost a vertical straight line (1:03)



00:57 G raises his left hand



00:59 G moves his hand upwards



01:03 G's hand is vertical

14 T: eh, this is what seems to you by looking at; but you have here $x = 100$ billions (01:08), is this barrier overcome sooner or later, or not?



01:08 T: keeps his right hand in the vertical position

15 G: yes

16 T: in the moment it is overcome (01:12), this x 100 billions (01:13), how many x do you have at disposal, after 100 billions? (01:14)



01:12 T crosses left forefinger over right hand



01:13 T raises his right hand



01:14 T moves right hand rightwards, repeatedly

17 G: infinite

18 T: infinite... and how much can you go ahead after 100 billion?

19 G: infinite points

20 T: then the exponential function goes ahead for his own business, doesn't it? [01:26]

The analysis from the perspective of interest-dense situations

How is the emergence of an interest-dense situation supported or hindered? In line 1 G begins to construct mathematical meanings about the growth of the exponential function in broken language as described above. In this moment the teacher interrupts him: Apologising, the teacher illocutionarily indicates that he normally would not interrupt the student, but in this case an interruption is necessary. The teacher perlocutionarily might want G to feel accepted, however, saying sorry indicates also that there is something wrong with the “straight line”. Locutionarily the teacher says: ‘tell me what straight line you mean’. However, G does not react on the locutionary level; he describes the condition for his explanation in line 1: “for very big x ”; just as he was asked to do in the task. The teacher’s question “They will meet each other?” is (illocutionarily) posed in a suggestive way. Perlocutionarily, the teacher wants to get the answer: ‘no, they don’t meet’. However, G withstands the teacher’s demand and answers *that they meet* (5). This is supported through adopting the teacher’s finger crossing gesture (6, 7). On the locutionary level, we would see only the question and the answer. On the non-locutionary levels there is negotiation underneath. Looking only at the lines 1 to 5, an interest-dense situation is about to emerge. From the theory of interest dense-situation we could predict how the teacher could support or hinder the emergence of interest-density. Focussing on the student’s ideas he would support it, acting according to his own thinking process or his expectations he would interrupt the emergence of it.

In the sentence that follows, the teacher starts to build up an argumentation as a proof of contradiction following his own train of thought and not that of the student. In line 8, he constitutes his base of argumentation. In order to include G into the process, his rhetorical questions “do you agree? Or not?” demands G’s agreement. Summarising G’s statement from line 1 grammatically more precise (10), the teacher establishes the statement that he wants to prove being false. G’s modification “but for very big x ” locutionarily looks like a complementary argument, but illocutionarily he corrects the teacher. G only partially agrees, because his description was based on ‘very big x ’ (11). Again, G indicates that his train of thought is a bit different. Perlocutionarily G succeeds at this moment because the teacher changes his focus; locutionarily taking up the student’s idea in the question: “for how big x ?” (12). G seems to feel encouraged to explain: “that is, normally does not arrive at a certain point, the function increases more and always more, then still it becomes almost a vertical straight line ...”. Again, an interest-dense situation is about to begin. Then, on the non-locutionary level, the teacher expresses understanding G’s view (14). However, through saying that, he also says that the student’s way of arguing is false. He proves this by a proof of contradiction which he closes by the rhetorical question: “or not?” After the proof,

G gives up to follow his own train of thought. The emergence of interest-density dries up.

Semiotic-bundle analysis

We see both student and teacher enacting a semiotic bundle composed by words, gestures, and inscriptions on the screen of the laptop. The basic point of discussion regards the behaviour of the exponential function for big base a and big values x . G thinks that in this case, the function can be approximated by a vertical line (#1-3). Such a conjecture is fostered by the image from the dynamic geometry software the students are using (see Figure 1): the tangent line appears in fact as almost vertical,

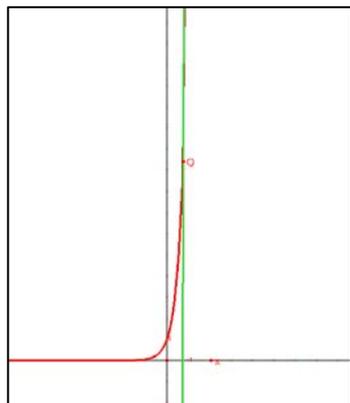


Figure 1

and the exponential function comes to be perceptually confused in it. The teacher wants to clarify whether the student is thinking to a vertical asymptote (#4-6). Asking about an hypothetic meeting of the function with the straight line, he is representing the graphs by means of his iconic gesture (00:17): his right forefinger stands for a vertical line, and his left forefinger is inclined to represent the exponential function graph. G (#5-7, 00:19 and 00:24) is tuning with the teacher's semiotic resources, both speech and gesture. With his hand, he represents the graph of the exponential crossing the vertical line (00:24): he is answering the teacher's question by means of the gesture. The teacher (#8) accepts such

an answer and endeavours in making explicit the idea that the domain of the exponential function is not limited, and therefore its graph intersects any vertical line. To do so, he uses both speech and gestures (see #8-20, and the related pictures). Let us enter into the dynamics of the semiotic bundle. In order to include C in the discussion, the teacher reports G's observation. By repeating G's words (#10) he is tuning with the student's semiotic resource (speech). But through gestures (00:43, 01:12, 01:13), he is making explicit the behaviour of the exponential function, i.e. the fact that it crosses any vertical line. The teacher is showing what we call a semiotic game, in that he is tuning with the student's semiotic resource, and is using another resource to make meanings evolve towards mathematical ones. The gesture appears a powerful resource, since it allows him to refer to what cannot be seen in the representation on the screen, and that is still difficult for the students to be conveyed in speech. In particular, gesture seems a suitable means to refer to very big values and to evoke their infinite quantity (01:14). If we now turn to G, we see that he does not appear to have profited from the teacher's semiotic game. Let us focus on lines 11-13 and related pictures. In his words we can see that he is still insisting on the idea that the function will become "almost a vertical straight line", but above all his gestures appear very different from the teacher's ones. In fact, whereas the teacher's gestures link big values of x with the right location in space (hand moving rightwards: 00:31, 00:51 and 01:14), the student's ones link big values of x to top location in space (hand moving upwards (00:46, 00:57, 00:59 and 01:03). From a cognitive point of view, they are

adopting different metaphorical references and only the teacher's one is consistent with mathematical signs (i.e. the Cartesian plane).

AN EMPIRICALLY BASED INTEGRATION

Based on the theoretical account and the empirical analysis, we can consider the two theories as *complementary*: they shed light on different aspects of the teacher-students interaction. However, by using the two theoretical lenses separately it appears that there is something important missing in each case. The strength of the interest-dense situations perspective is the possibility to predict their emergence according to the type of social interactions that hinder or foster it. In fact it includes the analysis of the locutionary and non-locutionary levels of speech and shows negotiations underneath the content. This approach is able to describe how the epistemic process proceeds and provides deeper insights into the social interaction process that foster or hinder the emergence of interest-dense situations, including structure seeing. However, the student and the teacher are not able to merge their argumentations although there is a lot of negotiation about whose train of thought will be followed. Neither the teacher nor the student is able to engage with the other's perspective. The analysis shows a gap that cannot be overcome, but is unable to give the tool to find out why this is so. By looking at a wide range of signs (in Peirce's sense), the semiotic bundle analysis identifies the semiotic game between teacher and student, and allows the game to be properly described. However the theory is not able to fully explain the reason why the student does not gain much from such semiotic game. In most other cases we had observed that the students succeeded to learn through semiotic games (e.g. see Arzarello et al., in print). One difference that can be identified within the theoretical frame is that this time the semiotic game applies the gesture-speech resources in reverse way with respect to semiotic games analysed as "successful". In this case, in fact, the teacher tunes with students' speech and uses gesture to foster meaning development; in other cases (see Arzarello et al., in print) it was the other way round: tuning with gestures and fostering meanings through words. We could conjecture that the characteristics of gestures as semiotic resource are not apt to this kind of didactical support, and indeed this can be a research problem to investigate. But within the semiotic bundle theory we are not able to say why such semiotic game did not work. The discussion so far leads us to argue that the simple juxtaposition of the two perspectives is not enough to deeply understand what's going wrong in the analysed episode. To go a step further, we start from the example to combine and locally integrate the two theories. The *combination* provides a tool to investigate how each sign of the semiotic bundle may contribute to the locutionary or non-locutionary aspects of the interaction. For instance, a gesture can support locutionary as well as non-locutionary features that play important roles in the interaction (see Figure 2). In the episode, gestures illustrated in pictures 00:19 and 00:24 at the locutionary level show the behaviour of the graph in iconic way, and at the non-locutionary they show that the student is trying to agree with the teacher's perspective. The hands in fact are used in the same configuration as the teacher (observe the teacher in the same pic-

tures); in the entire episode this is the only case in which it happens. In all the other cases, G's gestures have very different configurations. Concerning the words, a similar situation is constituted; at the locutionary level G's words affiliate to the teacher's perspective. But at the non-locutionary levels the teacher and G do not fully agree with each other using words.

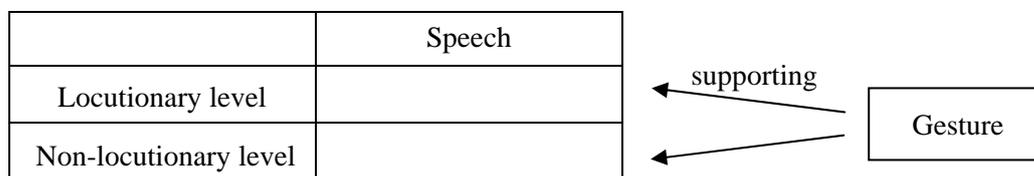


Figure 2: Two-level-analysis of semiotic resources

With the aim to answer the question what exactly did not work in the student-teacher interaction of the episode, we propose an integration of the two combined theories adding an *epistemological* dimension to the analysis above; that means to carefully consider the epistemological points of view of the teacher and of the students. By *epistemological points of view* we mean the background of the piece of knowledge that a subject thinks can give sense to a specific situation. The epistemological point of view is not always explicit: it appears not only from the locutionary dimension of the semiotic resources used by a subject but also from the non-locutionary ones. Moreover, it can be partially revealed by the epistemic actions produced by the subject. Of course the epistemological point of view with respect to a situation can vary with the subjects. For example, that of a student can be different from that of the teacher or of another student. But this difference might not be apparent although the dynamics of a didactic situation in the classroom might be deeply influenced by it, especially when the teacher is not aware of it or does not take into account the epistemological points of view of his students. This is exactly what happened in the episode analysed above. We observe a semiotic game articulated in a tuning in words and a dissonance in gestures: the teacher is repeating G's words (#11-12), but he is performing completely different gestures (see, that in 00:46 G's hand is moving upwards, to indicate big values, whereas in 00:51 the teacher's hand is moving rightwards). The dissonance in gesture is a signal that the teacher and the student are showing different points of view: the teacher relies on a formal theory (Weierstrass definition of limit) using potential infinite; the student relies on his perception imagining what happens "for very big x " (#11). It is not so clear what the student means: possibly he has been influenced by perceptive facts (see the discussion above) and perhaps he is thinking within an "actual infinite" perspective, even if this point is not so explicit here. The analysis of the semiotic game including the epistemological dimension allows us therefore to say that there is an *epistemological gap* between teacher and student, and to hypothesise that this gap prevents the teacher from suitably coaching the student's knowledge evolution and the student from profiting by the interaction with the teacher. Therefore the emergence of an interest-dense situation was not successful.

CONCLUSIONS

Presenting an empirical case of networking of theories, we showed that through a local integration two theoretical approaches can be enriched (Prediger et al., 2008). This was possible because the theories provided two complementary observation tools: one at the level of discourse analysis describes social interactions and their epistemic processes; the other at the level of gesture analysis describes learning from a semiotic perspective. The starting point of the theoretical integration was based on the empirical data analysis whose meaning was not clarified by any of the two theories. This stall was overcome by suitably combining the two approaches: adding an epistemological dimension made possible to locally integrate the two theories, so uncovering blind spots in both.

The results of our analysis could have important didactical consequences: in fact from them it seems possible to design a fresh role for the teacher in supporting students' learning processes. According to the combined analysis of the semiotic and linguistic features, integrated with the epistemological dimension, the teacher could develop suitable interventions, taking care both of the social interaction and of the epistemological issues with the help of semiotic resources.

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INTERPRETING STUDENTS' REASONING THROUGH THE LENS OF TWO DIFFERENT LANGUAGES OF DESCRIPTION: INTEGRATION OR JUXTAPOSITION?

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This contribution exemplifies the interpretation of a common set of data by using two languages of description originating from different theoretical perspectives. One account uses categories from a psychological and the other from a sociological perspective. The interpretations result in different explanations for the students' struggles with sense making. However, the results cannot be integrated into a combined insight, but only be juxtaposed.

INTRODUCTION

The role of theory in mathematics education research has many facets so that comparisons of outcomes of research carried out within different perspectives remain a challenging and complex task (Silver & Herbst, 2007; Radford, 2008). The observed diversity of theories, paradigms, and frameworks in the field has called for serious efforts of understanding, comparing, contrasting, coordinating, combining, synthesising, or integrating different perspectives (Prediger, Bikner-Ahsbals, & Arzarello, 2008). In line with this work, this paper, by way of an example, sets out the task to construct two *accounts* of a transcript from a video taped problem solving session for the purpose of comparing and contrasting different *accounts* for it (Mason, 2002), based on two languages of description stemming from two different theoretical traditions. In the session pairs of students were working on tasks on limits of functions, a topic where most of the research about students' sense making has been done from a cognitive psychology approach (Artigue, Batanero, & Kent, 2007). For an alternative account, we have chosen a sociological approach, which is rather uncommon but has the potential of overcoming deficit orientated interpretations of students' struggles.

Much of the research that aims at accounting for the problems students have, focuses on a distinction between "intuitive" and "formal conceptions" of limits (e.g. Harel and Trgalova, 1996, pp. 682-686). The notion of limits of functions is conceived as one where intuitive conceptions of infinity may prove insufficient or even contradictory to a formal mathematical treatment (Núñez et al, 1999). As an exemplary of approaches that account for students' problems with limits of functions in terms of the individual's cognition, we produce an account of the data that draws on the work of Alcock and Simpson (2004, 2005). Their conceptualisation describes an interplay between modes of representations and beliefs about oneself and the role of algebra in reasoning about limits.

Starting from a sociological perspective, in a second attempt, we outline an account of the students' productions in terms of the dilemma they face when participating in

different types of discourses. This interpretation draws on a language of description developed in the context of studies of recontextualisation that represent a structuralist tradition (Bernstein, 1996). In drawing on Bernsteins's theory, a successful student can be described as being able to realize in which context she participates and produces what is expected in this context, that is, the student must have access to "recognition rules" and "realisation rules" in order to produce "legitimate text". The ultimate agenda of such an approach is to explain how the students' access to these rules is distributed unevenly with respect to their different backgrounds. For our account of the empirical text from the problem solving sessions, we use categories of expression and content of mathematical problems from the perspective of recontextualisation of different types of discourses about limits of functions.

THE INTERVIEW SITUATION

Six beginning engineering students from a first semester calculus course volunteered to participate in the video study, where they were working in pairs to solve problems on limits of functions. Each session lasted for about 45 minutes. After an introductory question about the concept of a limit and its definition, the students were asked to investigate the limits of functions. The type of problems chosen were similar to the ones they encountered in the course: to find the limits as $x \rightarrow \infty$ and as $x \rightarrow 0$ for the three functions $f(x) = \frac{2x}{x^2 + \sin x}$, $g(x) = \frac{1}{x} - \frac{1}{x^2}$, and $h(x) = \frac{\ln(1+x^2)}{x}$.

For our accounts presented below, we used the transcribed protocol from the work of two pairs (A and B) of students on the function $h(x)$ and on the introductory question.

At the time of the interview the lectures had covered the definitions and basic properties of limits and continuity, and introduced and proved theorems about standard limits such as $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$, as well as worked examples. The textbook provided an exposition of an introductory calculus course based on the standard $\varepsilon - \delta$ definition of limits and continuity. In particular, standard limits were proved within this theory and used as theoretical tools to investigate the limits of functions given in algebraic form. Other techniques taught include removing dominating factors, extension by the conjugate expression, and change of variable. The approach was algebraic and non-numerical. Occasionally, diagrams were used. The teacher of the course sets out his agenda as follows (see Bergsten, 2007, p. 63):

I want to present, to make things seem true, the most important I think is that students believe they understand better what a concept means. To exemplify what you can handle practically, to illustrate the standard way of doing things.

In the lecture the teacher made some efforts to integrate formal algebraic treatment with non-formal ideas about limits and behaviour of elementary functions (ibid.).

ACCOUNT 1: INDIVIDUALS' BELIEFS AND PREFERENCES

The style of work of Anne and Adam is dominated by algebraic manipulations across all tasks, where the observed notations are used mainly as keys for performing procedures that hopefully will lead to a possibility to apply a standard limit. This is done immediately when starting a new task, without prior discussion about how to attack the problem or what can be “seen” by considering properties of the functions involved. In the transcript, when discussing the case where x tends to infinity for the function $h(x)$, Anne immediately suggests making a change of variables:

- Anne: Change of variables.
- Adam: ...yes ... I think you get ... the logarithm can be rewritten, the function inside.
- Anne: No, we can't touch the function inside [writes, Adam looks at her seemingly puzzled] there is no expression for $\ln X$ plus $\ln Y$ equal to $\ln X$ plus Y .
- Adam: Yes yes but you can write it as \ln one plus X ... that part [points] one plus X square can be written as ... one plus ... one minus X .
- Anne: Yes, equal to \ln [inaudible, Anne writes].
- Adam: It does not help much in this case.
- Anne: No [erases what she wrote].

While solving this task no diagram is drawn or point made on properties of the functions involved that could lead the process forward. Standard limits and comparison tables are recalled as incitements and as clues to continued algebraic manipulation. Uncertainty in recalling these facts correctly does not prevent them from proceeding the algebraic explorations, possibly thinking it will eventually lead to a result:

- Anne: I must elaborate further on that one and see if it works.

The work goes on along the same lines in all tasks, trying to remember what one can do and trying out different algebraic methods, sometimes ending up in what could be called an algebraic mess, using expressions like “this is just impossible”. In the following excerpt the students substitute $1+x^2$ by t .

- Adam: If we in the original expression extend with ... the square root of minus one ... T minus one in the denominator, $\ln T$ the square root of T minus one ... that one was not much better [looks at Anne].
- Anne: [writing] This is also unnecessary because we can't do this, it is the same shit ... doesn't matter ... than we have that this one moves this one moves and then this one moves.
- Adam: Yes all tend to infinity.
- Anne: To be honest, I think that infinity is the answer, as ... when I changed variables.

The last sentence indicates a weak “internal authority”, as she cannot find a method that works, and on another occasion (on problem f) this is directly expressed:

- Anne: The question is if it is correct. Now I just want to know the right answer.
 Interviewer: You don't feel confident with the result?
 Adam: I can't say it should be another result, but this is a kind of task where I feel I could easily make a mistake.
 Anne: Yes, me too.
 Adam: By some change of variable it can be possible to make it tend to zero. /.../
 Anne: I think it is zero in both cases. What was the answer?

This predominantly algebraic way of working seems to be in contrast to the response to the opening question on the meaning of a limit, where they initially describe it verbally as a dynamic process using words like “approaching” but then prefer to make a drawing and add gestures when talking about it. However, as these images do not seem to have a link to their subsequent work on the problems they may lack a sufficient generality to justify their reasoning (cf. Alcock & Simpson, 2004).

Also the students in pair B describe the mathematical notion of limit as a dynamic process of ‘approaching’ but seem to accept both a potential and actual infinity, as when they discuss the arrows commonly used to denote limits:

- Bob: Yes I would maybe miss a little arrow ...
 Ben: Yes.
 Bob: ... in front of A [i.e. the limit], tends to A, but I don't know if ...
 Ben: it gets so very close, yes goes to A.
 Bob: Yes, you usually don't have those arrows like that. But the function attains the value A when X is infinitely large, is a very very large number, don't know if I need to add more.
 Interviewer: Do you agree?
 Ben: Yes.

They also state that it is more easy to explain when using a diagram. However, their diagram is more elaborated and seems to support their thinking during the work with the problems. For pair B this work proceeds in quite a different manner from pair A, dominated by more informal reasoning about the size of the quantities of the different parts of the given functions. They frequently use the expressions “a very small number” and “a very large number”. In ‘simple’ cases this way of reasoning is functional but in the case $\lim_{x \rightarrow \infty} h(x)$, this kind of intuitive method proves insufficient to find the limit even after 15 minutes of work:

- Bob: Zero times infinity is ok, almost zero times infinity is more tricky, it is not really zero but only tends to it. So it can be almost anything. Do we get anywhere? [looking at Ben]
 Ben: No [Bob laughing].
 Bob: Yes, but which one goes more, does that one go more to zero than that one to infinity? No it goes more to infinity than to zero, I think. [silence]

It seems as if algebraic methods, shown in the lectures, here are tried only when the

conceptual approach does not produce an answer. However, when it does these students do not feel any need to verify the solution formally by the use of proven theorems on standard limits. They rely on “internal authority”.

Internal authority is also evident by the use of the words “I think we are done” in the case $\lim_{x \rightarrow 0} h(x)$, after identifying a standard limit and applying it after expanding the term by x . But again no algebraic manipulations are performed on $\lim_{x \rightarrow 0} g(x)$, where they reason about approaching zero from the right or from the left. They conclude, after testing a numerical value, drawing a diagram and comparing infinities, that $g(x)$ tends to negative infinity. However, Bob is not fully satisfied:

- Ben: So this [i.e. when approaching zero from the right] must also be negative infinity, don't you think so?
- Bob: Yes, but it is kind of delicate when you take infinity minus infinity, it is kind of vague. But if we accept this way of reasoning with infinities of different size, then we have found that, if it is correct.

Thus, relying on internal authority might have prompted questioning the bases of their arguments and imply an uncertainty about the correctness of the result.

ACCOUNT 2: WEAKLY / STRONGLY INSTITUTIONALISED DISCOURSE

For the purpose of analyzing the recontextualisation of domestic practices in school mathematics texts, Dowling (2007) introduces a “relational space” of domains of action that differentiates between content and expression of a text, both being weakly or strongly institutionalised (see Table 1). Esoteric domain text refers to the conventional institutionalised mathematical language and its strongly classified specific meanings. In descriptive domain text, the expression is conventional mathematical language though its object of reference is not institutionalised mathematics. In expressive domain text, a mathematical concept or procedure etc. is expressed via signifiers that are not or weakly institutionalised (in an extreme case via non-mathematical signifiers). Public domain text is text with both weakly institutionalised forms of expressions and content.

The following interpretation employs these notions. As the context is a university lecture in calculus, public domain text cannot be expected to be found. The oral discourse in the lecture analysed in Bergsten (2007) included metaphorical language

Expression (signifiers)	Content (signifieds)	
	strong institutionalisation	weak institutionalisation
strong institutionalisation	<i>esoteric domain</i>	<i>descriptive domain</i>
weak institutionalisation	<i>expressive domain</i>	<i>public domain</i>

Table 1: Domains of Action (Dowling, 2007, p. 5; layout adjusted)

and gestures describing graphs of functions in terms of motion and direction as well as hints about what to do when applying standard procedures.

The written discourse focused on algebraic representations. The topics were presented with very detailed formalisations, very much in line with the textbook (co-authored by the lecturer), that is, as esoteric domain text drawing on strongly classified and institutionalised language and meanings. So it is the oral discourse that is situated in another domain, a domain of visuo-spatial and movement metaphors that are used for describing the Cartesian graphs, “the behaviour”, of functions and their limits (in terms of shape, growth, getting bigger and smaller and approaching). The meanings in this discourse are weakly classified, as are the modes of expressions. In the course of establishing the esoteric discourse, this discourse is re-contextualised from the perspective of an algebra of functions and their limits, and in doing so the first is subordinated to the latter. The students attempts to solve the tasks in the interview situation can be interpreted as a struggle to produce a legitimate text, that is an esoteric text. However, if they discussed with their peers and approached the solutions in terms of the weakly classified oral discourse, they were faced with a problem of recontextualisation. However, in the introductory question of the interview, they were asked to explain the concept of limit, which is a quite different challenge. The interviewer shows to the students a piece of technical language from the course: “ $\lim_{x \rightarrow \infty} f(x) = A$ ” and asks:

Interviewer: Imagine you have a friend who just started such a course in calculus and has never seen this. How would you explain to him what this means?

The students are faced with the problem to recognize what a legitimate text in this interview situation would be. Into which domain has the expression to be translated for this imaginary friend?

Anne and Adam interpret this question as a task to produce expressive domain text. They first have to establish this new domain and start negotiating the translation and eventually agree that this new domain includes drawings of examples of functions. The technical language comprises “x”, “function”, “A” (which remains untranslated), “LN-function”. The expression “ $\lim_{x \rightarrow \infty} f(x)$ ” is translated into “the limit”.

Anne: This is an expression for the limit. One looks at how a function behaves when X tends to infinity...and when X tends to infinity and the function approaches a constant which is called capital A, so it is convergent, as one calls it. This means that one can say the function then approaches a value if it does not go on ...

Adam: It approaches a finite value then, so it is bounded, a bounded function.

Anne: This is a little hard without drawing it.

Adam: Yes, this is hard to explain, it is more easily explained with a figure, I think.

After two comments of Adam who talks about the value going “closer and closer”, the interviewer interferes by asking them whether they would want “to draw a figure for that friend”:

- Anne: I think the friend should get a clearer picture in any case [Adam draws quietly, Anne watches] ... yes [approves the figure and holds up the paper to the friend and smiles].
- Adam: This is a function that approaches but never really reaches [illustrates with a gesture].
- Anne: A bit like LN one can say
- Adam: Yes, LN-function.
- Anne: This looks like an LN [both laugh].

In their conversation while solving the tasks $\lim_{x \rightarrow 0} h(x)$ and $\lim_{x \rightarrow \infty} h(x)$, they focus on associating it with a standard limit they have encountered in the lecture. They eventually solve the version for x approaching zero by expanding the expression by x and substituting $x^2 = t$, that is, by producing esoteric domain text. However, they do not explicitly refer to the “multiplication rule” for limits from the lecture to justify their conclusion. They are not successful in their attempt to solve the second part of the task. As they adhere to a strategy to formalize their informal approaches and employ some methods suggested in the lecture, this can be seen as a production of descriptive domain text, which in parts, switches into the esoteric domain when they are trying out different algebraic transformations. Anne several times refers to “writing” it down properly, which indicates that she recognizes what type of text they are usually supposed to produce. The episode, in which the pair tries to solve the second part of the question, ends with a remark about the criteria for producing legitimate text:

- Anne: Now you have made a writing mistake. You have to write X, or T, T goes towards infinity... They will like that at the mathematics department ... also when we are very detailed.

The second pair also takes the interviewer’s question as a prompt to produce expressive domain text by describing the meaning in terms of the weakly institutionalised oral discourse. Bob refers to the limit as “the value A when X is infinitely large, is a very very large number, don’t know if I need to add more” and talks about “the little arrow” (see the transcript from the first account). After another prompt of the interviewer, they expand their explanation:

- Ben: Yeah, the function value A as X tends to infinity, or? [silence ...] Then we have drawn [moving his hand as if he is drawing], have we not? [glancing at Bob]
- Bob: Yes, it gets like that, x tends to infinity, it is very simple if you make a sketch [raising his hand with the pencil but does not draw, making drawing gestures while talking]. If we have A at a certain part of the y-axis we can say, we get such a horizontal line. The function starts at zero maybe and then goes up, kind of approaching A all the time, getting thinner, the bigger the x-value the closer you get ... and ... I don’t know if I should bring that in too, you can always get closer than you already are, that is this thing with limits. That is the whole point, as in this case it will finally be as close as ... you can’t say as close as you can because you can always get closer but ...

They solve the task $\lim_{x \rightarrow 0} h(x)$ by reducing it to a standard limit, talking about substituting x^2 and decide about a solution. However, Ben seems unsure about the status of the solution produced by Bob (who does not refer to “multiplication rule”):

Bob: And this her goes towards zero, that X goes towards zero. One times zero.

Ben: One times a very small number next to zero.

Bob: This is what I also would like to say, indeed one times zero becomes zero.

Ben: I think we are clear with this one.

The last remark indicates that they do not adhere to the criteria for legitimate text established in the lecture. In the course of the solution of the second task, they remain in oral discourse and use visuo-spatial and movement metaphors for describing the shapes of standard functions and the “limit” as “approaching and coming closer”. However, they are not successful in re-contextualising this discourse from the perspective the formal algebraic discourse. However, as the other pair, they seem to know the criteria for legitimate text, as Bob says at one occasion: “You can’t do it like this mathematically /.../ It can be done, there is a method”.

None of the pairs interpreted the first question of the interviewer as an invitation to establish the meaning for a novice by introducing her into the technical language and its institutionalized meanings, that is, to come up with a definition. Both pairs seem to realize that the legitimate text for successful participation in the course is located in the esoteric domain.

DISCUSSION

One goal of this exercise has been to see whether both interpretations can in combination produce useful insights about the students’ reasoning about limits in the context of a university calculus course.

The first interpretation pictures those students showing an external sense of authority as the ones who tend to use the mathematical notations as keys to apply algebraic procedures. A conclusion could be that they lack an “intuitive feeling” for the mathematical objects involved, which should form the basis for using algebraic techniques. The second pair is pictured as showing an internal sense of authority and a preference for an “intuitive” approach. They often “know” by informal reasoning what the limit is and occasionally express a need to use algebraic representations. A conclusion could be that they lack an ability to use algebraic representations to formalise their reasoning. As the first approach focuses on the individuals’ cognition it does not include the relation of their preferences to the context, in which these arose, as a specific research question.

The second interpretation shows that both pairs were, for different reasons, not able to produce solutions that would satisfy the criteria for legitimate text established in the lectures. The first pair did not have full access to the technical language and its institutionalized meanings, which they tried to employ, the second did not recon-

textualise their own productions from a formal algebraic perspective. This account draws attention to the structural complexities that relate to the ways in which the re-contextualisation by means of formal algebra of the oral discourse about functions and limits employed in the lecture operates. It includes the establishment of a link of the students' productions to the discourse, in which they participate, as a *paradigmatic research question* (Radford, 2008) by conceptualising it in terms of their possession of recognition and realisation rules for producing legitimate text.

The two approaches also differ in terms of the methodology. While within the first framework the interview situation is a method for gaining insights into the students' beliefs and preferences, the second interpretation takes into account that the conversation during the problem solving sessions can also be conceived as a situation, in which the students are faced with the challenge of producing legitimate text. However, the students can neither have recognition nor reproduction rules for such a situation because it is the first time they participate in a study like this. They seem to have interpreted the interview situation differently, as more (Anne and Adam) or less (Bob and Ben) identical with the context of the course they were attending and thus more or less identifying the researcher with the official side of the university course. This interpretation would account for the fact that the second pair did not spend so much effort to translate their versions into a formal algebra as the first one and that they were mostly convinced that their solutions are reasonable, perhaps because of recognizing the context as informal. The first pair, in contrast to their following productions, engaged in weakly institutionalized discourse only as a response to the introductory question, perhaps recognizing the story about the imaginary friend as not belonging to the esoteric domain. In contrast, the first interpretation takes the students' explanations that follow the introductory question as an indication of their understanding of the concept of limit, or alternatively as an indicator of whether they know a definition in formal algebraic terms.

From the second perspective, “understanding” can be framed as having access to both of the discourses identified, as well as to the principle by which the oral discourse can be recontextualised from the perspective of the written one. The “intuitive” approach is only represented in the oral discourse. Both interpretations suggest a tension between these discourses that cannot easily be resolved.

It remains a highly questionable undertaking to look for combined insights stemming from interpretations that use languages of description, which stem from different theoretical traditions, particularly if issues of validity are at stake (cf. Gellert, 2008). The two interpretations presented here illustrate their points by selecting different episodes from the transcript. Considering that the research situation is re-interpreted in the second account (and thus taking the interviewer's questions as a piece of data), one could say that the two accounts are not interpretations of the same “data”. In addition, different background information about the course has been used.

The outcomes of this interpretational exercise do not result in conflicting readings of the data. However, the results cannot be integrated into a combined insight, but only be juxtaposed. This is because the basic principles of the theories from which the approaches originate have established two different “universes of discourse” (Radford, 2008) in which the paradigmatic research questions are formulated.

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COORDINATING MULTIMODAL SOCIAL SEMIOTICS AND AN INSTITUTIONAL PERSPECTIVE IN STUDYING ASSESSMENT ACTIONS IN MATHEMATICS CLASSROOMS

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What can a multimodal social semiotic perspective in coordination with an institutional perspective make visible? In this paper we describe how we coordinate these two perspectives in order to look at the same empirical material with different focuses. The research interest is assessment actions in mathematics classrooms, an interest that also affects research objectives and possible results. When coordinating the different perspectives, we have chosen, for the analytical frame-work, to develop the social semiotic meta-functions by adding a new, fourth, meta-function: the institutional. For the detailed analysis, we connect to these four meta-functions other compatible concepts to create an analytical framework.

BACKGROUND

The focus of this paper is to describe how we coordinate two theoretical perspectives, multimodal social semiotics and an institutional perspective, in order to create a structured and nurturing analytical framework for the analysis of assessments during lessons in mathematics. We will start out by describing some of our central notions of assessment.

Assessment – a broad concept

Both in cases where some people realise that they actually are “capable” in mathematics, and in other cases where people think that they will never come to terms with it, we can notice “hidden” stories about assessment. Obviously, assessment explicitly takes place when students are given their mathematics test results. But often enough, assessment is implicit during teacher-student interaction in learning sequences. One example is the following: a student asks the teacher about a certain mathematical “rule” and wonders where it comes from. The teacher’s answer, by way of different communicational modes, shows that this particular student does not have to bother about such a question. S/he is just asked to follow the rule. But when another student asks the same question, the teacher engages in a discussion about the historical development of this particular rule. The first student in this example learns, through this implicit assessment, that the teacher does not consider her/him capable enough to understand this kind of question. Our assumption is that both the explicit assessments and the implicit assessments in mathematics classrooms play a key role for students’ learning. The empirical examples we use in this paper focus on implicit assessment actions.

COORDINATING TWO THEORETICAL PERSPECTIVES

As stated above, we hold that we are coordinating two different theories. Prediger et.al. (2008) make a distinction between *coordinating* and *combining* theories. They define “coordinating” as a term for bringing theories together that contain assumptions that are compatible, whereas “combining” is when the theories are only juxtaposed.

A multimodal social semiotic perspective

In a multimodal approach, all modes of communication are recognised (Kress et.al. 2001). Communication in a multimodal perspective is not understood in the same way as communication in a narrow linguistic perspective, focussing on verbal interaction only. Rather, all kinds of modes have to be taken into consideration, such as gestures, and gazes, pictorial elements and moving images, sound and the like. Relevant modes in (most) mathematics education are, for example, speech, writing, gestures and gazes as well as graphs, diagrams, physical objects, symbols, pictures and virtual animations. Modes are socially and culturally designed in different processes of meaning-making, so their meaning changes over time. It is also the case that one “content” in one kind of configuration (for example as speech), will not necessarily be the “same” content in another configuration (for example as illustration). Different representations of the world are not the “same” in terms of content. Rather, different aspects are foregrounded. In verbal texts we read linearly, within a time frame, whilst a drawing will be read within a space frame. And a graph does not represent a knowledge domain in the same way as numbers does. The modes that are “chosen” in a specific situation reflect the interest of the sign maker, and they are therefore not arbitrary. We argue for the importance of understanding multimodal communication to be able to fully understand a phenomenon as assessment. Language, in a broad sense, “may serve as a crucial window for researchers on to the process of teaching, learning and doing mathematics” (Morgan 2006, p 219).

We also argue that the assessment of learning (in a deeper sense) is about understanding *signs of learning*, as shown by different communicative modes (see Kress 2009, Pettersson 2007, Selander 2008b). This perspective is based on an understanding of learning as an increased engagement in the world, and as an increased capacity to use signs, modes and artefacts for meaningful communication and actions (Selander 2008a).

Institutional perspective

Within social semiotics, there are acknowledgements of institutional aspects, even though they are not always as clearly outlined as in the following:

Detailed studies of the use of a given semiotic resource are interesting in their own right, but they also demonstrate a theoretical point. They show how the semiotic potential of framing is inflected on the basis of the interests and needs of a historical period, a given

type of social institution, or a specific kind of participant in a social institution (van Leeuwen 2005, p 23, see also Morgan 2006)

Institutions are often taken for granted by the researcher who “knows” the situation. But without some idea of the communicative situation, it is very difficult to draw conclusions from, for example, a conversation. Here, we will go one step further in addressing “the institution” in its historical context. We understand that the interactions between teacher and student are situated in a context characterized by dominant mathematics education discourses, the use of artefacts developed over time, framings in terms of specific resources for learning, division of labour and time, established routines, classroom structure and authority.

Douglas (1986) argues that institutions (rituals, norms and classifications, what counts as centre or periphery etc.) affect the decisions made by individuals, for example the way they classify “phenomena” and “things”. Existing classification systems are often taken for granted. In this paper, we take the stance that classifications are products of social and cultural negotiations (Bowker & Star 1999). Wertsch and Toma (1995) emphasise that powerful institutional parameters constrain classroom discourse (see also Bartolini Bussi 1998, Lerman 1996). Our understanding of the term institution is also to be seen as being in line with a dynamic view:

Importantly, however, the thinking and meaning-making of individuals is not simply set within a social context but actually arises through social involvement in exchanging meanings (Morgan 2006, p 221).

Institutional framings have both direct and indirect effects. Decisions may be made on different “levels” in the school system, which have a direct impact on the classroom work. However, in this paper we will try to outline the indirect aspects, such as classificatory systems, norms and traditions developed over time. We will also use the institutional aspect already in the creation of analytical categories, not only as an overall umbrella-tool for reflecting over the results (see Björklund Boistrup 2007).

AN INSTITUTIONAL PERSPECTIVE IN RELATION TO META-FUNCTIONS

Inspired by Halliday (2004), social semioticians usually talk about three communicative meta-functions: the ideational, the inter-personal and the textual. In Morgan (2006), these functions are used with a focus on the construction of the nature of school mathematics activity. In this paper, we start out with the meta-functions as used by Kress et.al. (2001), focussing on assessment in mathematics.

As we see it, the three meta-functions are strong concepts for discussing situated communication and learning. However, two different kinds of restraints need to be noted. The first concerns the fact that not *all* possible communicative aspects can be captured by the three concepts. For example, expressive modes are not well captured (van Leeuwen 2005). Secondly, to be able to fully address institutional discourses in the situated communication and learning (as in this study), a wider notion of institu-

tional framing (norms, institutional practices, classifications of good or bad performance etc.) seems to be needed. Communication in a classroom has different characteristics than communication in court or in a medical consultation. We add a fourth, *institutional* meta-function (proposed by Selander 2008c).

META-FUNCTIONS AND RESEARCH OBJECTIVES

In this paragraph, we describe the four meta-functions and relate them to the research objectives of an ongoing research project on assessment actions in mathematics classrooms in grade 4 (10-year-olds). Even if all four meta-functions are present in all cases, in each and everyone of them, one function is in the foreground and the others are in the background. Thus, the division into four meta-functions related to four research objectives is meant to be seen as an *analytical* framework.

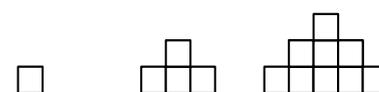
The ideational meta-function – aspects of mathematical competence

The *ideational* meta-function is related to human experience and representations of the world (Halliday 2004). When using this meta-function and aligning it with the *research interest* of assessment, the aim for the research project is to investigate what aspects of mathematical competence that are represented and communicated in the assessment actions.

In order to find a structure which can serve as part of the analytical framework for the more fine-grained analysis, we draw on a structure presented by Skovsmose (1990). He discusses mathematics education and the possibilities for mathematics to serve as a tool of democratisation in both school and society. He presents a structure of three aspects of mathematical competence:

- Mathematical knowledge itself
- Practical knowledge. Knowledge about how to use mathematical knowledge.
- Reflective knowledge. A meta-knowledge for discussing the nature of mathematical constructions, applications and evaluations.

In the following sequence, the students in the class are working in pairs on patterns. A boy (B) and a girl (G) are working together. Before the teacher approaches, these two students are discussing whether they need to count the squares one by one in order to find how many they are, or if they can use the pattern from an earlier task (1, 4, 9...). The excerpt shows what takes place when the teacher approaches the group. In the first line of the transcript, the students' speech (SS) and the teacher's speech (TS) are noted. In the next line, we find the students' and teacher's gestures (SG and TG), and in the bottom line the students' and teacher's body movements and gazes (SB and TB). The actions that occur simultaneously are written above each other. The teacher starts by asking how things are going.



First three figures in pattern.

SS: G: 25

Yes, it's going great!

B: *This was strangely difficult.*

TS: *Are things going well?*

Why is it strange?

SG: G is writing.

B stops writing.

TG:

SB: G is looking at her work and at T.

B looks at T and at his work.

B looks at T.

TB: Approaches. Looks at G's paper. Moves close to G's desk.

Looks at B's work. Moves closer to B's desk. Leans forward.

Time →

We suggest that, during this lesson, the students get to show “Mathematical knowledge itself” related to patterns. The girl’s comment that things are going great might be a sign that she feels that she has been able to handle the patterns well so far. The boy seems to have a different opinion. The teacher asks him and it becomes clear that this comment is mainly related to the aspect of mathematical competence focused on structuring one’s notes. He has run into problems when drawing the figures:

SS: B: *You add this, but then it does not show that this one is this and that this one is this.*

TS:

No they are close now, but you can still see it I think. You'll have to leave more space between them.

SG: B points at the figures on his paper.

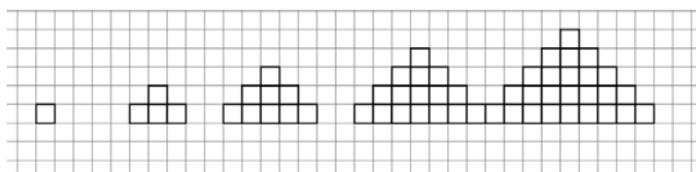
TG:

SB: B looks at his work.

B looks at T and down.

TB: Looks at B's work.

What he explains and shows by pointing is that two of his figures are drawn too close together on his paper, like this:



The teacher’s comment is related to this “note-structuring” since she suggests that he should try to leave more space between the figures.

The interpersonal meta-function – feed-back, feed-up and feed-forward

The *interpersonal* meta-function is about how language (used in a broad sense in this paper) enacts “our personal and social relationships with the other people around us” (Halliday 2004, p 29). Morgan (2006) connects interpersonal aspects with assessment in an analysis of a classroom sequence. This is compatible with the way we use the

interpersonal meta-function in this paper. Our *research interest* in relation to this is to find out what kind of assessment in the form of feedback and self-assessment is taking place in the interaction between teacher and student.

The structure for the detailed analysis is inspired by Hattie (2007). He suggests three kinds of feedback:

- feed-back – what aspects of competence has the student shown?
- feed-up – how can the aspects shown be related to stated goals?
- feed-forward – what aspects of competence might it be best to focus on in the future teaching and learning?

Using the same example as earlier, we find that the signs of assessment are shown both through the students' self-assessment and through the teacher's responses. Both the girl's and the boy's comments are within the category feed-back. The teacher's responses are connected both to feed-back and to feed-forward. We consider them as feed-back when the teacher communicates to the boy that his way of drawing the figures is acceptable; "No, they are close now, but you can still see it, I think". At the same time, she addresses a way of handling the very same issue during his continuing work, which we regard as feed-forward: "You will have to leave more space between them".

The textual metafunction – different communicative modes

The *textual* meta-function is related to the construction of a "text", and this refers to the formation of whole entities which are communicatively meaningful (Halliday 2004), in this case to other kinds of existing assessment systems and procedures. Teacher and students communicate in mathematics education with speech, gestures, gaze, pictures, symbols, writing and so on. According to this meta-function and our *research interest*, the objective is to investigate how different communicative modes (Kress et.al. 2001) are used and accepted by the teacher and the students. The boy shows his self-assessment on "note-structuring" by way of speech, gestures and drawings. The teacher listens and looks at the boy's work. Both the student and the teacher seem to accept different modes.

The institutional meta-function – tradition versus active participation

When it comes to institutional aspects of Swedish mathematics education, a dichotomous picture is often noticed (e.g. Palmer 2005, Persson 2006). On the one hand, the discourse of mathematics education is seen as "traditional", whereby students are expected to spend a good deal of time solely on solving all the problems in a textbook. On the other hand, the "wanted" discourse of mathematics education which emphasises a joint exploration in which, for example, students are invited to be active participants in problem-solving. These two discourses of assessment are similar to the discourses described in the literature on assessment in general (see Gipps 1994, Lindström & Lindberg 2005). The two discourses of assessment in mathematics can be summarised in the following way:

“Traditional” discourse	“Active participant” discourse
Focus on the correct answer	Focus also on processes
Focus on teacher’s guidance	Focus on the teacher promoting thinking
Focus on the number of finished tasks in the textbook in mathematics	Focus on the quality of the mathematical accomplishments
Focus only on the aspects of mathematical competence the student shows on her/his own	Focus also on the aspects of mathematical competence the student shows when working with peers
Focus only on written tests in mathematics	Focus also on documentation of the learning in mathematics
The teacher is the only one who assesses	The student is also part of the assessment

With inspiration from Lindström & Lindberg (2005).

In the following example, we keep to these dichotomous discourses. However, during the full analysis we will broaden the scope of discourses in relation to the findings. We will now go further on in the sequence from the classroom. We start out with the girl asking the teacher if it is possible for her to read what she has written and drawn on her paper. The teacher asks if the student understands it herself. The girl answers yes and the teacher says that she also understand the notes. Then the girl makes this comment:

SS: G: <i>Just so that you don't mark it wrong "here you are wrong"</i>	
TS:	"laughs" <i>Is that what I usually do?</i>
SG: B & G are writing.	
TG:	
SB: B & G look at own work. G smiles.	
TB: Looks at G's work.	Looks at B's work.

As we see it, the girl’s comment refers to the traditional discourse of assessment in mathematics, since she proposes that the teacher might regard her notes as either wrong or right. The teacher engages in the discussion and asks if that is what the girl assumes that she as a teacher normally does. The girl answers no to this question and suggests that the teacher sometimes asks about notes that she does not understand. The teacher acknowledges this and the girl continues:

SS: G: <i>It is actually quite good to ask if you don't know what the children have done</i>	<i>Yes..</i>
TS:	<i>Well, that is the only way to get to know. Mm</i>
SG: G & B are drawing.	G stops drawing.
TG:	

SB: G & B look at their work.

G looks at T

TB: Looks at B's work.

Looks at G.

Nods.

Here, the other discourse is present, and by (finally) looking at each other, they seem to agree on this. To be able to assess the students' notes, the teacher might have to ask for clarification. The implicit assessment in this described activity is not just a matter of what is right or wrong. It is a matter of active participation by the student as well.

REFLECTIONS ON THE COORDINATION OF THEORETICAL FRAMEWORKS

We argue that the three meta-functions need to be understood in the light of institutional framings (also see Morgan 2006). The fourth meta-function is a way to both *understand* and *describe* institutional discourses as situated in history, and to address what it is that is at stake in conflicts and negotiations of assessment procedures and standards.

We find the theoretical perspectives described in this paper fruitful with regard to several aspects of the research process. We understand assessment as an act of meaning-making through a multimodal use of language. When defining the research objectives, the four meta-functions provide means to focus on different aspects of assessment actions.

In the short examples in this paper, we have shown how the aspects of mathematical competence that are present (the ideational meta-function) at first seem to be in patterns. But through the boy's speech, gestures and drawings, our understanding shifts to the structuring of notes. When it comes to the interpersonal meta-function, we find that both teacher and students show signs of feedback, and in the end the teacher also gives feed-forward. The textual meta-function gives us clues as to how the teacher and students use, and show acceptance of, different modes of assessments. Finally, the institutional meta-function makes it possible to describe the discourse as related to a strong tradition in mathematics education, but also in the ways in which new ideas can be ideationally, interpersonally and textually meaningful. In relation to this issue, we have described a situation in which the girl positions the teacher in a "traditional" discourse of assessment in mathematics (right-wrong). When analyzing what the teacher's gaze is focused on, we can notice that she initially looks at the boy's work when she is talking to the girl. But finally, when she turns towards the girl, they look at each other and the gazes reveal an "active participant" discourse.

This coordination of perspectives, including an analytical framework, seems to be a fruitful (and sufficient) basis for the full analysis of the empirical material in the project, in order to be able to describe, understand and discuss assessment in the mathematical classroom in a way that has not earlier been done (in Sweden).

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INTEGRATING DIFFERENT PERSPECTIVES TO SEE THE FRONT AND THE BACK: THE CASE OF EXPLICITNESS

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The paper contributes to the ongoing discussion on ways to connect theoretical perspectives. It draws explicitly on the introductory article and the concluding article of the Theory Working Group publication ZDM – The International Journal on Mathematics Education 40(2), particularly on the strategy of local theory integration. In the first part of the paper, a classroom scene is presented to provide some footing in empirical data. This data is used to illustrate the theoretical propositions, made from two theoretical perspectives, on the topos of explicitness in mathematics teaching and learning. In the second part, the two theoretical accounts are locally integrated resulting in a deepened and more balanced understanding of the role of explicitness. In the last part, this example is used to differentiate three modes of local theory integration: bricolage, recontextualisation and metaphorical structuring.

PRELIMINARY REMARKS

According to Lakoff and Johnson (1980), the attribution of a front and a backside to something is metaphorical in nature and depending on the experience and interest of the attributor. A front-back orientation, they cogently argue, is not an inherent property of objects but a property that we project onto them relative to our cultural functioning. The front is what we see. If we want to see the back of it, we need to walk around it or to turn it round. This is quite clear for concrete objects like, say, mountains and fruits. Attributing a front-back orientation to the abstract concept of explicitness is different because there is no cultural agreement about what the front and the back of it may be. By projecting categories that emerge from direct physical experience onto non-physical constructs, a metaphorical structuring occurs which transmits the connotations of the former to the latter. It is thus no value-neutral endeavour to discuss the concept of explicitness in terms of its front and its back. In many cultures the front of something is regarded as being more important than its back, but otherwise the front may be taken as just a surface and you need to look at the back of it to see the ‘real thing’. I will come back to some consequences of this issue, in terms of Radford’s (2008) conceptions of theories, at the end of the paper.

In the paper, I present empirical data from a 5th grade mathematics classroom for looking at the degree of explicitness in a case of mathematics teaching. I draw on the consequences of this teaching practice for the students’ learning of mathematics from two theoretical perspectives, a semiotic (“the front”) and a structuralist (“the back”) one. While arguing that both perspectives connect fruitfully, I use this example for taking on the ongoing discussion of the challenges and possibilities of connecting theories in mathematics education (Prediger, Arzarello, Bosch & Lenfant, 2008).

THE EMPIRICAL DATA

In most federal states in Germany, primary school ends after 4th grade. From 5th grade on, the students are grouped according to achievement and assumed capacity. Those students, who achieved best in primary school, attend the *Gymnasium* (about 40% in urban settings). The data I am drawing on in this paper is the videotape of the first lesson of a new *Gymnasium* class, which consists of 5th graders from different primary schools. The teacher and the students do not know each other. It is the very first lesson after the summer holidays. The teacher starts the lesson by immediately introducing a strategic game known as “the race to 20” (Brousseau, 1975, p. 3). [1]

Teacher: Well, you are the infamous class 5b, I have heard a lot about you and, now, want to test you a little bit, that’s what I always do, whether you really can count till 20. [*Students’ laughter.*] Thus it is a basic condition to be able to count till 20, so I want to ask, who has the heart to count till 20? [*Students’ laughter.*] Okay, you are?

Nicole: Nicole.

Teacher: Nicole, okay. So you think you can count till 20. Then I would like to hear that.

>[2] Nicole: Okay, one two thr ...

>Teacher: Two, oh sorry, I have forgotten to say that we alternate, okay?

Nicole: Okay.

Teacher: Yes? Do we start again?

Nicole: Yes. One.

Teacher: Two.

Nicole: Three.

Teacher: Five, oops, I’ve also forgotten another thing. [*Students’ laughter.*] You are allowed to skip one number. If you say three, then I can skip four and directly say five.

Nicole: Okay.

Teacher: Uhm, do we start again?

Nicole: Yeah, one.

Teacher: Two.

Both continue ‘counting’ according to the teacher’s rules. In the end, the teacher states “20” and says that Nicole was not able to count till 20. Then he asks if there were other students who really can count till 20. During the next 7 min. of the lesson, eight other students try and lose against the teacher whilst an atmosphere of students-against-the-teacher competition is developing. While ‘counting’ against the teacher, the tenth student (Hannes) draws on notes that he has written in a kind of notebook –

and he is winning against the teacher. After Hannes has stated “20”, the following conversation emerges:

Teacher: Yeah, well done. [*Students applaud.*] Did you just write this up or did you bring it to the lesson? Did you know that today ...

Hannes: I have observed the numbers you always take.

Teacher: Uhm. You have recorded it, yeah. Did you [*directing his voice to the class*] notice, or, what was his trick now?

Torsten: Yes, your trick.

Teacher: But what is exactly the trick?

During the next 5:30 minutes the teacher guides the mathematical analysis of the race to 20. In form of a teacher-student dialogue, he calls 17, 14, 11, 8, 5 and 2 the “most important numbers” and writes these numbers on the blackboard. He makes no attempt of checking whether the students understand the strategy for winning the race. Instead, he introduces a variation of the race: you are allowed to skip one number and you are also allowed to skip two numbers. The students are asked to find the winning strategy by working in pairs. After 10 minutes, the teacher stops the activity and prompts for volunteers to ‘count’ against the teacher. The first six students lose, but the seventh student (Lena) succeeds. After Lena has stated “20”, the following conversation emerges:

Teacher: Okay, good. [*Students applaud.*] Well, don’t let us keep the others in suspense, Lena, please tell us how you’ve figured out what matters in this game?

Lena: Well, we’ve figured it out as a pair.

Teacher: Yes.

Lena: We have found out the four most important numbers and, in addition, the other must start if you want to win.

Teacher: Do you want to start from the behind?

Lena: From behind? No.

Teacher: No? Okay, then go on.

Lena: Okay, well if the other starts then he must say one, two or three. Then you can always say four. [*Teacher writes 4 on the blackboard.*] When the other says five, six or seven, then you can say eight. [*Teacher writes 8 on the blackboard.*] And when the other says nine, ten or eleven, then you can say twelve. [*Teacher writes 12 on the blackboard.*] And when the other says thirteen, fourteen or fifteen, then you can say sixteen. [*Teacher writes 16 on the blackboard.*] And then the other can say seventeen, eighteen or nineteen and then I can say twenty.

Teacher: Yeah, great. What I appreciate particularly is that you have not only told us the important numbers, but also have explained it perfectly and automatically. Yes, this is really great. Often, students just say the result, they haven't the heart, but you have explained it voluntarily. That's how I want you to answer.

In the next two paragraphs the focus is on the theoretical issue of explicitness. First, it is argued from a semiotic perspective that implicitness is a precondition for learning and that an exaggerated explicitness counteracts mathematical learning in school. Second, the structuralist argument that students benefit differently from invisible pedagogies is explored. The data is used to illustrate the theoretical propositions. [3]

THE FRONT: IMPLICITNESS AS A PRECONDITION OF LEARNING

From a theory of semiotic systems, Ernest (2006, 2008) explores the social uses and functions of mathematical texts in the context of schooling, where the term 'text' may refer to any written, spoken and multi-modally presented mathematical text. He defines a semiotic system in terms of three components (Ernest, 2008, p. 68):

1. A set of signs;
2. A set of rules for sign use and production;
3. An underlying meaning structure, incorporating a set of relationships between these signs.

According to this perspective, the learning of mathematics in school presupposes the induction of the students into a particular discursive practice, which involves the signs and rules of school mathematics. Whereas signs are commonly introduced explicitly, the rules for sign use and production are often brought in through worked examples and particular instances of rule application. The working of the tasks, the reception of corrective feedback, and the internalisation gradually enrich the students' personal meaning structures. It is only at the end when the underlying mathematical meaning structure is made explicit.

By referring to Ernest's semiotic system, we can make sense of the 5th grade teacher's actions: First, he is explicitly stating that counting the normal way till 20 is well-known for all students and he is playfully introducing a (growing) set of rules for sign use. Second, the strategies for winning the different races to 20 remain on an exemplary level and are not transformed into a general rule. Third, he leaves any exploration of the underlying meaning structure completely to the students.

Regarded from the adopted semiotic perspective, the teacher is inviting the students to a very open and not much routed search for regularities and more general relationships between signs. This way of teaching avoids what Ernest calls the "General-Specific paradox" (Ernest, 2008, p. 70):

If a teacher presents a rule explicitly as a general statement, often what is learned is precisely this specific statement, such as a definition or descriptive sentence, rather than

what it is meant to embody: the ability to apply the rule to a range of signs. Thus teaching the general leads to learning the specific, and in this form it does not lead to increased generality and functional power. Whereas if the rule is embodied in specific and exemplified terms, such as in a sequence of relatively concrete examples, the learner can construct and observe the pattern and incorporate it as a rule, possibly implicit, as part of their own appropriate meaning structure.

Apparently the teacher is introducing his mathematics class as a kind of heuristic problem solving. He is giving no hints for finding a route through the mathematical problem of the race to 20. When Hannes has succeeded in the race, the teacher is explicitly framing the solution as a “trick” that is useful in the particular task under study. He then continues by modifying the rules. This may allow the students to come closer to a general heuristic insight: It may be an appropriate strategy to work the solution back from 20. However, the teacher is not insisting upon Lena explaining backwards. The ‘official’ underlying (heuristic) meaning structure of the race to 20 is not made explicit during the lesson, though the students are gradually inducted into the generals of heuristic mathematical problem solving.

THE BACK: EXPLICITNESS AS A PRECONDITION OF ACCESS FOR ALL

From a structuralist position, Bernstein (1990, 1996) polarises two basic principles of pedagogic practice: visible and invisible. A pedagogic practice is called visible “when the hierarchical relations between teacher and pupils, the rules of organization (sequence, pace) and the criteria were explicit” (Bernstein, 1996, p. 112). In the case of implicit hierarchical and organisational rules and criteria, the practice is called invisible. He argues that in invisible pedagogic practice access to the vertical discourses, on which the development of subject knowledge concepts ultimately depends, is not given to all children. Instead, evaluation criteria remain covert thus producing learners at different levels of competence and achievement.

In terms of Bernstein’s differentiation of pedagogic practices, invisible practice dominates the 5th class’ first mathematics lesson. When comparing the teacher’s talk with Hannes and with Lena, it can be seen that the teacher keeps the students in the dark about some essential aspects of the mathematical teaching that is going on. Although students, who read between the lines of the teacher’s talk, may well identify some characteristics and criteria of the pedagogic practice they are participating in, the teacher transmits these characteristics and criteria only implicitly. All those students who do not notice these implicit hints, or cannot decode them, remain in uncertainty about:

... if the race to 20 is meant as a social activity of getting to know each other (It is the very first lesson!) or as a mathematical problem disguised as a students-teacher competition,

... if thus students should fish for “the trick” or heuristically develop a mathematical strategy and

... if thus successful participation in this classroom activity is granted when the race has been won or when a strategy has been established by mathematical substantiation.

Only at the end of Lena's explanation, the teacher makes the criteria for successful participation in 'his' mathematics class explicit. As a consequence, students' successful learning has been contingent on their abilities to guess the teacher's didactic intentions. Recording the numbers the teacher always takes (Hannes) without transcending the number pattern for a mathematical rule, is only legitimate to a certain extent. As long as the hierarchical and organisational rules and the criteria (which Bernstein (1996, p. 42) calls respectively the "distributive rules", the "recontextualizing rules" and the "evaluative rules") remain implicit, students are intentionally kept unconscious about the very practice they are participating in. Only visible pedagogic practices facilitate that students collectively access, and participate in, academically valued social practices and the discourses by which these practices are constituted (cf. Bourne, 2004; Gellert & Jablonka, in press).

CONNECTION: INTEGRATING THE TWO PERSPECTIVES

The contrasting perspectives on explicitness reveal that the rules and criteria of mathematics education practice remain – in part as a matter of principle – implicit. On the one hand, the need for implicitness is due to the very character of the learning process: whoever strives for whatever insight cannot say *ex ante* what this insight exactly will be. Ernest's "General-Specific paradox" is an interpretation of this issue. On the other hand, the principles that structure the practice of mathematics education remain implicit to the participants of this practice, without any imperative to do so for facilitating successful learning processes.

However, for that the general can be fully acquired, the students indeed need to understand that the specific examples and applications have to be interpreted as the teacher's means to organise the learning of the general. Successful learning in school requires the capacity to decode some of the implicit principles of the teacher's practice. The structuralist perspective supports the argument that the students actually benefit more from teaching-the-general-by-teaching-the-specific if they are conscious about the organising principle that is behind this teaching practice. By making the organisational and hierarchical rules and the criteria of the teaching and learning practice explicit, the teacher provides the basis for that all students can participate successfully in the learning process.

It is quite clear from the empirical data presented above that the teacher is partly aware of this relation: In the end of the passage, he explicitly explains to the students the characteristics of legitimate participation in 'his' classroom. However, as this explanation is given retrospectively and in a relatively late moment of the lesson it seems that some of the pitfalls of the implicit-explicit relation have not been avoided:

(1) It is neither obvious from their behaviour nor does the teacher check whether this very important statement is captured by all students. Particularly those students, who

did lose interest in the mathematical activity because they do not know where it can lead to, might not pay attention. (The fact that some students do not listen to the teacher's statement can be observed in the videotape.)

(2) By giving the explanation retrospectively, the teacher has already executed a hierarchical ordering of the students. Although no criteria for legitimate participation in the mathematical activity of the race to 20 has explicitly been given in advance of the activity, the teacher favours Lena's over Hannes' participation: Hannes is offering a "trick" (which might be more appropriate for playing outside school) while Lena is giving a mathematically substantiated explanation of her strategy. Apparently, Lena demonstrates more capacity of decoding the teacher's actions than Hannes does.

(3) It might be difficult for many students to transfer the teacher's statement to their mathematical behaviour during the next classroom activity. Indeed, the teacher is giving another specific statement, which the students gradually need to include in their meaning structure. This is another case of teaching-the-general-by-teaching-the-explicit: a general expectation ("students explain voluntarily") is transmitted by focussing on a specific example (Lena's explanation). Again, and on a different level, the students need to decode the teacher's teaching strategy: the teacher's statement is not only about legitimate participation in the race to 20, but also about participation in 'his' mathematics class in general.

Particularly the point (3) shows how the local integration of two theories may lead to a deepened and more balanced understanding of the issue of explicitness and its role within the teaching and learning of mathematics.

REFLECTIONS ON THE 'GENERAL': CONNECTING THEORIES

The connection of the two perspectives has structurally woven the front ("learning requires implicitness") into the back ("making hierarchical and organisational principles of classroom practice explicit"). A structuring of theoretical perspectives has thus taken place. But what is the nature of the new structure, and what are the characteristics of the process that has taken place?

Radford (2008) develops a conceptual language for talking about connectivity of theories in mathematics education. He takes theories as triples $\tau = (P, M, Q)$ of principles, methodologies and paradigmatic research questions. For questions about connectivity of theories, he argues that the principles seem to play a crucial role as "divergences between theories are accounted for not by their methodologies or research questions but by their principles" (Radford, 2008, p. 325). Indeed, at first glance, Ernest's semiotic perspective and Bernstein's structuralist perspective share an attention to the explicitness and implicitness of rules. The divergence of the two perspectives becomes apparent when the mode of these rules and their status is considered. Whereas from the semiotic perspective rules are rules for sign use and sign production and thus closely linked to the individual student's capacity of using and producing mathematical signs (P_I), the structuralist perspective takes rules as the

constitutive elements of classroom practice (P_2). Ernest's semiotics is concerned with text-based activities where the texts are mathematical texts and the semiotic system is school knowledge. Bernstein's set of rules is the mechanism that provides an intrinsic grammar of pedagogic discourse. Although this looks like a fairly different understanding of rules and their respective theoretical status, the principles P_1 and P_2 of the two theories seem to be "close enough" to each other" (Radford, 2008, p. 325) to allow for integrative connections.

Prediger, Bikner-Ahsbals and Arzarello (2008, p. 173) describe "local integration" as one of the strategies for connecting theories. Acknowledging that the development of theories is often not symmetric, the strategy of local integration aims at an integrated theoretical account of a local theoretical question (e.g., Should rules be made explicit?). As a matter of fact, the principles P_i and P_j of two theories τ_i and τ_j deserve closer attention: How get P_i and P_j connected, what modes of mediating their divergence exist?

Bricolage. The mode of integration of theories Prediger et al. refer to is Cobb's notion of "theorizing as bricolage" (Cobb, 2007, p. 28). Cobb describes a process of adaptation of conceptual tools from the grand theories of cognitive psychology, sociocultural theory and distributed cognition. His goal is to "craft a tool that would enable us to make sense of what is happening in mathematics classrooms" (Cobb, 2007, p. 31). Here, the mode of mediation between theoretical principles is essentially pragmatic: Non-conflicting principles $P_{g1}, P_{g2}, P_{g3}, \dots$ of the grand theories $\tau_{g1}, \tau_{g2}, \tau_{g3}, \dots$ are adapted for fit into the bricolage theory τ_b . As the goal of the integration is the development of a tool, τ_b is essentially an externally oriented language of description of empirical phenomena. Cobb's theorizing as bricolage is reminiscent of Prediger et al.'s (2008, p. 172) "coordinating" strategy. As the bricolage theory τ_b is a theory *en construction*, it is problematic to make the criteria for the selection of non-conflicting principles explicit.

Recontextualisation. Another mode of integration of theories is recontextualisation, "the subordination of the practices of one activity to the principles of another" (Dowling, in press, ch. 4). This is the case when the principles P_i of the theory τ_i dominate the principles P_j of the theory τ_j . An example of theory recontextualisation can be found in Gellert (2008) where an interactionist methodology M_i is subordinated to structuralist conceptual principles P_s . This process results in an asymmetrical role played by the methodologies M_i and M_s as a consequence of a hierarchical ordering of the principles of the corresponding theories (P_s over P_i ; cf. Radford, 2008, p. 322f.). Hierarchical organisation of theories in the mode of recontextualisation is a device for avoiding theoretical inconsistencies.

Metaphorical structuring. A third mode of integration of theories is mutual metaphorical structuring. As Lakoff and Johnson (1980, p. 18f.) remark, "so-called purely intellectual concepts [...] are often – perhaps always – based on metaphors". Since metaphors aim at "understanding and experiencing one kind of thing in terms of

another” (Lakoff & Johnson, 1980, p. 5), this is again a case of subordination: metaphorical structuring. If we talk about the learning of mathematics in terms of rules, then the learning of mathematics is partially structured and understood in these terms, and other meanings of mathematics learning are suppressed. Similar things occur when concepts from one theory are infused into another theory. For an example see the infusion of the General-Specific paradox into the principles of a visible pedagogy. The argument that the advantage of a visible pedagogy relies on the explicitness of its criteria becomes differently structured when understood in terms of the General-Specific paradox: How can criteria be made explicit without producing blind rule-following and a formal meeting of expectations only? Infusing the term decoding capacity into the components of the semiotic system has produced a mutual effect: The teacher’s strategy of teaching-the-general-by-teaching-the-specific is effective only if the students are able to decode the respective activities.

CONCLUSION

Bricolage, recontextualisation and mutual metaphorical structuring show different effects on the theoretical components that become locally integrated. This is still a complex issue and it might be very useful to further develop a meta-language for the connection of theoretical perspectives. I am convinced that a systematic description of the organising principles of local theory integration is an essential part of this developing language.

NOTES

1. The transcript presented, here, is my translation from the German original. Students’ names are pseudonyms.
2. The sign > indicates overlapping of speech.
3. For a detailed analysis of what these passages can tell us about the exigencies that students face in mathematics classes, see Gellert and Hümmer (2008).

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THE PRACTICE OF (UNIVERSITY) MATHEMATICS TEACHING: MEDIATIONAL INQUIRY IN A COMMUNITY OF PRACTICE OR AN ACTIVITY SYSTEM

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Theoretical perspectives of ‘community of practice’ and ‘activity theory’ are used along with constructs of ‘inquiry’ and ‘critical alignment’ to theorise developing mathematics teaching at university level. The paper introduces and explains the theories and relates theory to issues in the ongoing development of a mathematics course for engineering students. It focuses on developmental research which seeks both to chart developmental progress and lead to more informed teaching relating to the goal-directed activity of those involved, the systems of which they are a part and the tensions/issues within which development occurs.

INTRODUCTION

In recent writing (e.g. Jaworski, 2007, 2008a) I have focused on *communities of inquiry* in developing mathematics teaching and learning. I have drawn particularly on Wenger’s (1998) concept of *identity* based in modes of *belonging* to a *community of practice*. This has been in the context of *developmental research* – that is research that seeks to develop practice while charting that development (see also, Goodchild, 2008). Here, I want to look more closely at how theoretical and methodological perspectives not only complement each other but are intertwined in the complex process of improving practice in teaching and learning mathematics.

I distinguish two areas of theory here. The first is Wenger’s theory of *belonging to a community of practice*. The second is theory of *inquiry*, based in Vygotskian ideas of *activity, mediation* and *tools*. The complex notion of *identity* and its relation to *community* is a central unifying force. I have used these theoretical ideas previously to address analysis of data in a longitudinal study of developing mathematics teaching and learning in schools through collaboration between teachers and didacticians in Norway. Many sources document this research (e.g., Jaworski, 2007; 2008a; Jaworski, Fuglestad, Bjuland, Breiteig, Goodchild and Grevholm, 2007; <http://fag.hia.no/lcm/papers.htm>). In this paper, I focus on the beginnings of research into developing mathematics teaching in a university mathematics department, focusing on my own practice as a (novice) mathematics teacher in this context.

The structure of this paper is as follows. First I give accounts, separately, of the two areas of theory, relating them explicitly to practices in mathematics teaching and learning. Then I turn to research into my own practice as a university mathematics teacher – a rather different form of practice from that of teaching mathematics in schools which has been my main focus in previous papers. I will expose some of the differences and related dilemmas and ways in which the two areas of theory cohere to support a theorising of practice and analysis of data. In doing this, I will address the

nature of developmental research, its importance in contributing to development in mathematics teaching and learning, and issues in its operationalization

BELONGING TO A COMMUNITY OF PRACTICE

The term ‘community’ designates a group of people identifiable by who they are in terms of how they relate to each other, their common activities and ways of thinking, beliefs and values. Wenger (1998, p. 5) describes *community* as “a way of talking about the social configurations in which our enterprises are defined as worth pursuing and our participation is recognisable as competence”.

Within a university school of mathematics I recognize mathematicians, mathematics educators and our students at various levels as part of a community. In this community we engage with mathematics in various ways: learning mathematics, teaching mathematics and doing research into mathematics or into learning or teaching mathematics. Mathematics itself and what it means to do mathematics is central to this community. We can recognize both individuals and groups: that is to ascribe *identity* to both. Holland, Lachicotte, Skinner and Cain (1998, p. 5) write, “Identity is a concept that figuratively combines the intimate or personal world with the collective space of cultural forms and social relations”. Identity refers to *ways of being* (Holland, et al. 1998) and I talk here about ways of being in the university mathematical community. For example, people who teach mathematics have identity with relation to what it means to teach mathematics within a university environment, and within one particularly.

Within this community we all engage in some forms of *practice*: Wenger writes of practice: “The concept of practice connotes doing, but not just doing in and of itself. It is doing in a historical and social context that gives structure and meaning to what we do”. (1998, p.47). So *doing* within the school of mathematics means *engaging in the practice of university mathematics*. This *includes* doing mathematics, whether this is on the part of undergraduate learners or of research mathematicians; it *includes* students and academics researching aspects of the learning and teaching of mathematics, and associated contexts such as use of technology in teaching and learning and mathematics support for learners at all levels.

Wenger talks about identity in communities of practice as being about *belonging* to a community of practice. He suggests three modes of belonging: engagement, imagination and alignment. We *engage* in practice with others: our participation requires us to *do*, not just to observe the practices of which we are a part. Students have to engage with learning, teachers with teaching. All engage with mathematics. Engagement is the fundamental activity in doing. In order to engage we have to make sense of what we do; *imagination* allows us to interpret its various aspects and conceive of ways to achieve what we see as the goals of practice. We are not alone in our enterprise: the community of practice has developed over time and has norms and expectations of what will be done and how. We need to *align* with the norms of practice –

alignment provides the sociohistorical dimension within practice by which the practice is recognisable, sustainable and continuing.

Seeing university mathematics as a social practice is becoming a familiar basis for research in mathematics education related to learning and teaching mathematics in a university (e.g., Burton, 2004; Hemmi, 2006; Nardi, Jaworski & Hegedus, 2006) which has a long history and tradition, both in universities generally and in any one in particular. Recognisable aspects are university terms or semesters, lectures and tutorials, courses organised across several years of study in calculus, analysis, algebra and so on, and forms of assessment. Mathematics itself has an even longer history, with traditions in philosophical groundings, how topics are grouped and how learning and understanding mathematics are perceived. As mathematicians engage, whether in teaching or research, they bring imagination to interpret courses or research topics and they align with accepted practices, perpetuating a status quo and ensuring ongoing traditions. Students coming in fresh to the practices learn quickly acceptable forms of engagement and, imaginatively, how to make the system work for them according to their own, more familiar, communities of practice. They align with norms of practice developed over centuries and experience insights and obstacles familiar to cohorts of their forebears.

However, perpetuation of tradition is not always helpful in ensuring effective learning outcomes, especially if cohorts of learners no longer fit traditional moulds. Difficulties at the transition between school and university have been extensively reported (Hawkes & Savage, 2000). Existing research describes the mismatch between university lecturers' expectations of mathematics undergraduates and student competencies (London Mathematics Society, 1995). Brown, Wiliam, Barnard, Rodd & Macrae, (2002) reported how mathematics undergraduates' attitudes change and many become disillusioned with the style of teaching mathematics in university. In a study of teaching in university mathematics tutorials, Nardi, Jaworski and Hegedus (2005) suggested a variability of pedagogic awareness, in the teaching of university mathematicians, shifting from the naïve and dismissive to the confident and articulate. Hemmi (2006) studying mathematicians' and university students' attitudes to *proof* found distinct differences in the ways students and their teachers perceived mathematics learning and teaching at university level, and categorization of mathematicians interview responses showed significantly varying views on the nature of teaching. Burton's (2004) interview study of 70 mathematicians revealed both common traditions in mathematics teaching and research and particular viewpoints and idiosyncrasies. Such sources have highlighted both significant issues related to traditional practices and new concerns relating to changing traditions in which more research is urgently needed.

ACTIVITY, MEDIATION AND TOOLS: THE ROLE OF *INQUIRY*

Doing mathematics, for students at any level, requires engagement with abstract concepts which are not readily visible in the world around us. Although we can see particularities of mathematics in our familiar social worlds (examples of numbers or shapes, use of ideas of probability or statistical tools), expression of mathematical generality, necessarily, is abstract and requires abstract means of expression and justification.

Schmittau (2003), drawing on Davidov, speaks of mathematics as involving *scientific concepts* which require “pedagogical mediation for their appropriation” (p. 226). Scientific concepts are concepts which cannot be learned spontaneously in engagement with everyday life (Vygotsky, 1986). Some form of mediation (going between) is needed for students to meet mathematical concepts and engage with them in meaningful ways. Particularly, Vygotsky talks about tools and signs which mediate the process of learning – *mediating artefacts* (see Figure 1). Such artefacts include both physical and intellectual tools; for example books and writing on paper, and language in which ideas and concepts are expressed. Technological tools can be helpful mediators for learning mathematics and teachers can orchestrate the use of technology to promote learning. *Pedagogical* mediation refers to the role of a teacher in creating opportunity for students to learn. The simple mediational triangle (Figure 1) deriving from Vygotsky and Leont’ev (e.g. Leont’ev, 1979) has been extended by Engeström (e.g., 1998) to include mediation *in social worlds* captured by the terms “rules”, “community” and “division of labour” to which he refers jointly as “the hidden curriculum” (1998, p. 76). (See Figure 2). It is “hidden” because the factors involved are often not considered or questioned overtly as mediating factors in the education enterprise.

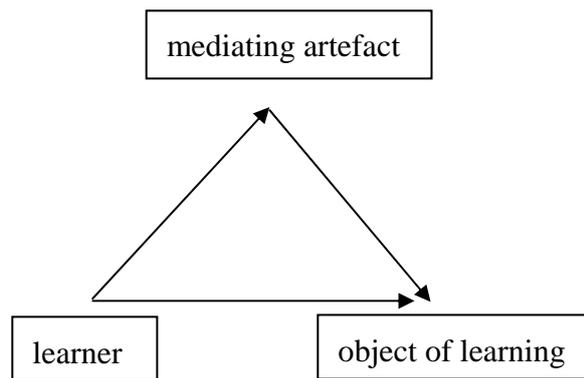


Figure 1: A simple mediational triangle

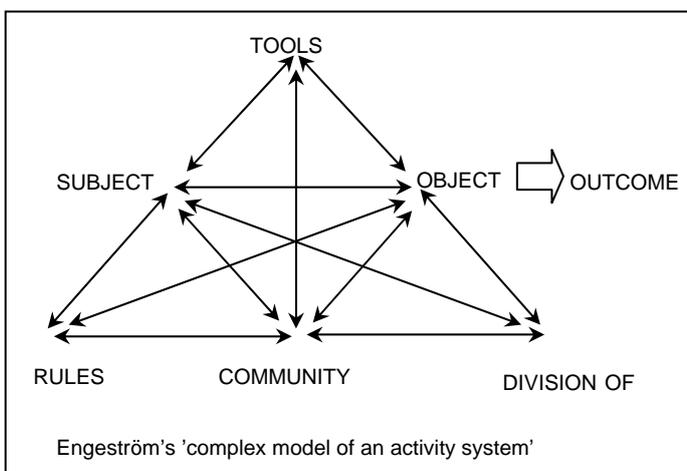


Figure 2: An expanded mediational triangle

involved are often not considered or questioned overtly as mediating factors in the education enterprise.

In university mathematics education, the *rules* include courses to be taken, measures of success in a course or programme, expectations of participation; *community* encompasses those who engage in processes of mathematics learning and teaching with the purpose of advancing mathematical knowledge and understanding, primarily students and

teachers; *division of labour* encompasses the differing roles and responsibilities of those within the community, for example teachers to teach and students to learn. Thus, for a learner (the *subject* of the learning process) with an *object* of learning mathematics, the activity of engaging in mathematics in a mathematical community is mediated by all of these factors as well as the artefacts commonly used to support learning.

Engeström refers to the system defined by the relationships illustrated in Figure 2, as an *activity system*, following a theory of *activity* deriving from Vygotsky and Leont'ev. Briefly, all activity is motivated, and comprises actions which are explicitly goal directed. Thus, in any such system, participants act according to goals and their actions are mediated by the various elements of the system (Leont'ev, 1979; Jaworski & Goodchild, 2006). An issue that arises in the learning and teaching of mathematics in a university is that of potentially conflicting communities where the goals of activity are concerned. So within a broad activity system of university mathematics (including students, teachers, researchers, learning, teaching and so on) we see subsystems which relate to the activity of certain groups. For example, teachers working within the established university system and its mathematical community have expectations of how students will act in relation to the norms and expectations of learning mathematics in a university. They have goals for students' learning and their actions are a consequence of their goals.

For students however, the system looks different. They come from different traditions in school systems and wider society. They are used to the kinds of relationships with teachers and peers that are afforded by pre-university education. They are highly influenced by popular culture and their peers. Stepping into the university system requires a re-alignment in their engagement; imagination, relating to the various communities of which they are a part, inspires their re-alignment. Lave and Wenger (1991) have offered a theory of legitimate peripheral participation to account for the transition for a novice into a community of practice. Here, I draw rather on Wenger's tri-partite characterisation of *belonging* and to activity theory to account for the dichotomies that emerge from collision of communities. Engeström's (1998) use of the expanded mediational triangle shows recognition of tensions in and between activity systems which can help address dichotomies. I say more on this below.

The place of *inquiry* in these theories and systems is central to my arguments in the paper. I see inquiry first of all as a *tool* mediating mathematics learning, teaching and development and then as *a way of being* in practice (Jaworski, 2006). When we start to inquire, we can be seen to use inquiry as a tool. Through sustained use the tool becomes a part of our identity as well, possibly, as of the identity of our community. Concepts relating to inquiry in practice, and its relation to these two established areas of theory, have emerged from 5 years of research in Norway (Jaworski et al., 2007). Seeing inquiry first as a tool emphasises its mediational characteristics within an activity system. Teachers and students, inquiring into the processes of learning and

teaching, achieve “metaknowing” (Wells, 1999, p. 65ff) through inquiry practice. Inquiry in mathematics involves asking questions and working on problems which engage participants and lead to new awareness and ultimately knowledge – we see this both in the activity of research mathematicians (e.g., Burton, 2004) and, where an inquiry pedagogy is in place, in classroom mathematics. Inquiry in teaching mathematics involves teachers in asking questions and working on problems in didactics and pedagogy; inquiring into ways in which opportunity can be created fruitfully for mathematical learning. Inquiry is also central to a developmental research process in which research into aspects of learning and teaching mathematics leads to enhanced knowledge in the academy and, importantly, to more *informed* practice (Goodchild, 2008; Jaworski 2008a).

Seeing inquiry as *a way of being* shifts inquiry from its status as a tool, to a more fundamental constituent of an activity system in which it becomes part of the “hidden curriculum”, having a consequence of *making the hidden curriculum less hidden*. To manifest inquiry as a way of being requires inquiry to become part of the fabric of learning and teaching, what is taught and how it is approached, to such an extent that it permeates the rules, community and division of labour. It therefore offers a response to tensions and dichotomies that leads to metaknowing and possibilities for more knowledgeable practice. In order to explain this, I have introduced the concept, of *critical alignment*. Before discussing this in theory, I turn now to the context of university teaching and learning, and my own practice as a (novice) university teacher.

TEACHING MATHEMATICS TO FIRST YEAR ENGINEERING STUDENTS

At my university, the engineering faculty entrusts the mathematics teaching of its students to the *Mathematics Education Centre* which is the smaller of two parts of the *School of Mathematics*¹. As I write this, I am currently in my second year of teaching a cohort of students in materials engineering some of whom have relatively low mathematical qualifications². In the first year, I taught the weakest of these students (16 of them) separately from the rest and was able to develop good individual relationships. This year, all the students are together (around 70) and the approach to teaching is influenced strongly by this larger number. I want all students to be able to engage with mathematical concepts, to develop both conceptual understanding and procedural fluency and to be able to apply these to their engineering tasks. So, one area of inquiry is how I teach: what I do, how I do it, and what it achieves; included within this is encouraging students to inquire as part of their learning of mathematics. I bring an *inquiry way of being* as a result many years of experience, but nevertheless

¹ The other part is the Department of Mathematical Sciences. Members of both departments teach mathematics. Largely, those in the DMS do research in mathematics; those in the MEC do research in mathematics education.

² Some have not done mathematics beyond GCSE (the national examination at 16+). Others have very low grades in A level mathematics (the national examination at 18).

in this new arena I need to use inquiry overtly *as a tool*, both for myself and for my students. Methodologically, I engage in research and development cycles (Goodchild, 2008), planning, observing and analysing teaching and learning as it progresses; collecting data through teaching plans, reflective memos, student work, assessment tests, a student survey and student interviews.

Due to limitations of space here, I focus on just one aspect of teaching, for both year-groups of students. In the first year, to extend a more direct focus on curriculum topics, I offered a weekly investigative problem for students' exploration, requiring mathematical concepts with which students needed to develop strength and confidence³. It was introduced in a class session (we had two 50-minute sessions per week for 30 weeks); students were asked to continue to work on it in their own time, singly or in groups, and each one to give me some of their working and findings from the problem. Attendance at class sessions was very variable, but most of those who came handed in some work on which I wrote comments and returned to them. I learned about each student's mathematical skills and understanding from this activity. Observation over these weeks showed a willingness to engage with mathematics in non-routine ways on the part of more than half the students, and a classroom atmosphere in which questions could be asked and addressed and students mainly contributed actively (speaking up, asking questions, coming to the board) in class.

It became clear that some students had very weak mathematical skills, especially relating to algebra. When we came to the topic of exponential and logarithmic functions, I anticipated the difficulties that this topic would present. It seemed necessary to put all time and energy into the topic, and this halted the weekly problems. While maintaining an active questioning approach, I moved into a more direct approach to the topic: involving the class in sketching graphs, noting functional characteristics and relationships, expressing meanings aloud and addressing fundamental questions, and a strong emphasis on the rules of exponents and logs and their use in solving equations. Two outcomes were (a) in the related class test, several students achieved more highly than in two previous tests; (b) in a questionnaire in which I asked students to comment on their participation in the course, the level at which they rated their understanding of this material seemed more realistic and accurate than in relation to earlier topics. In my own reflections, while I was regretful of the demise of the weekly problem (it was not reinstated), I recognised that the teaching approach to *exp* and *log* had also achieved significant outcomes. I then had to rethink the objectives of my approach overall and their practical interpretation within constraints of time, curriculum and so on (Jaworski, 2008b). This has had implications for the current teaching. With a cohort of 70 the investigative problems with quick feedback would not be possible. The more direct approach has been maintained to a strong degree, and liai-

³ For example, the *painted cube* problem which affords experience with algebraic formulation and manipulation – a wooden cube is painted on the outside and then sliced into smaller cubes all the same size; how many cubes have paint on one face, two faces, three faces?

son with the engineering department has started to produce problems relevant to the study of the particular students. An investigative element has been included using a *GeoGebra* medium.

The activity outlined above incorporated an *inquiry cycle* (plan → act and observe → reflect and analyse → feedback to planning) which led to growth and recognition of knowledge which should feed back into planning for teaching both locally and globally. Issues addressed included problems of variable attendance, a wide range of mathematical experience within the class, the time factor in focusing on a problem of the week, the demands of concepts that students found difficult and so on. *Aligning* within the university system was and is a necessity, but the element of inquiry has allowed a questioning of what is possible, experimentation and critical review of outcomes, and modification according to observation and analysis. This shows *critical alignment* in practice with related growth of knowledge and understanding.

An activity theory analysis shows some conflicts/tensions in these issues. For example, the problem of the week afforded development of confident mathematical participation and opportunity to work algebraically. The more direct addressing of mathematical concepts and associated skills afforded a greater achievement in curriculum-related summative assessment. Time and other factors militated against inclusion of both of these approaches. These issues can be seen as breaks in the mediating links in Engeström's triangle and highlight areas where the system is in conflict. Such conflict fosters the meta-knowledge that is needed to move forwards productively (e.g., Engeström 1998, p. 101; Jaworski & Goodchild, 2006).

I contrast here the two ways of theorising teaching development. Seeing *critical alignment in practice* emphasises *the inquiry process* in belonging to the community of practice which allows modification and change within engagement, imagination and alignment. The practitioner here brings an overtly critical eye to the practice and finds ways of adjusting her alignment. An activity theory analysis allows juxtapositioning of key elements of the activity system and examination of their relationships. Tools (e.g., the investigative problems), rules (e.g., lecture timetables), community norms (e.g., students who do not attend lectures) and division of labour (e.g., the expected roles of students and lecturers) can be seen to be in tension. Thus the analyst finds here a valuable tool in revealing the issues, their nature and relationship. This is both explanatory and predictive: it offers ways of seeing the status quo and reveals possibilities for consequent activity.

I see these two theoretical frames to have rather different functions. The first is closely related to action in practice: recognising where alignment is required and where it can be adjusted. It offers a practical interpretation in the use of inquiry as a tool, and aids development of an analytical awareness of how the inquiry cycle can both raise and address issues. The second allows a more holistic vision of the various factors and issues with a framework, a set of constructs, with which to characterise and link, and through which to see where the tensions lie. This allows further activity

to be planned from the outside. Seen in these ways, the two frames offer complementary insights to the developmental process and the hidden curriculum.

THEORETICAL FRAMES AND ONGOING PRACTICE/ACTIVITY

One reviewer of this paper asked why students' goals had not been taken into account. This is an important question. With the first cohort of students, a questionnaire was completed asking about their course participation, understanding and achievement and some interviews were conducted (Jaworski, 2008b). Both cohorts completed the standard university evaluation of the course. In another research project into university teaching we have tried to organise focus groups with students to discern their perspectives. A discussion of analysis of these sources is beyond the scope of this paper. However, a future study would valuably bring students' goals to centre stage, particularly in an activity theory analysis in juxtaposition with teachers' goals. For example, in the use of GeoGebra as an exploratory tool, indications are that students do not so far see what the teacher perceives as value in its use. An activity theory analysis suggests that we have here tensions between the teacher's goals for creating conceptual understanding and students' goals for instrumental success. This could be shown by juxtapositioning of two activity systems, one for the students and one for the teacher. However, stronger data is needed before this would make sense. Critical inquiry into how GeoGebra can be used by students to achieve conceptual understanding is proposed as action.

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AN INTERPLAY OF THEORIES IN THE CONTEXT OF COMPUTER-BASED MATHEMATICS TEACHING: HOW IT WORKS AND WHY

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Abstract: *I analyze the interplay of theories within a study on computer-based mathematics teaching. I will address divergences in their conceptualization of the empirical realities, influences on the interpretation of data, characteristics of my linking strategies, and issues of compatibility.*

Keywords: impact of theories on data analysis, theory development, compatibility of theories, micro-sociology, linguistic activity theory

INTRODUCTION

Amongst many others (Lester, 2005; Mason & Waywood, 1996; to name two only), “interpretative” research in the German speaking community of mathematics education has highlighted the crucial role of theory in research (Bikner-Ahsbahr, 2003; Jungwirth & Krummheuer, 2008; Maier & Beck, 2001). Accordingly, on the one hand, this research invests much in the development of theoretical frameworks, on the other hand, it aims at a development of locally limited, grounded theories. The outcome of research is thought of as a reconstruction of phenomena that is always theoretical in the sense that it transcends data and thus is an ideal type of reality (Bikner-Ahsbahr, 2003; Jungwirth, 2003). The Austrian research project “Gender – Computers – Maths&Science Teaching” by H. Jungwirth & H. Stadler was based on the above position. The aim was to reconstruct participants’ “relationships” to mathematics, physics and computers in computer-based classrooms, and the role gender plays within their interactive development (Jungwirth, 2008b; for the mathematics-related part). Apart from theorizing those relationships, a theoretical approach to classroom processes being appropriate for a comparison of both subjects had to be developed. It had to provide a notion of teaching as an ongoing process (in order to scaffold the investigation of the establishment of relationships) and as a whole (in order to be able to specify the contextual conditions of both subjects). My previous research suggested a use of micro-sociological theories and of a supplementary theory that was located in the context of activity theory. In this paper I want to deal with these theories and their networking restricted to mathematics teaching (Jungwirth, 2008a; for the related findings). As my aim is not to present the study itself I just mention briefly that the data consisted of 21 common Austrian, mostly CAS-based mathematics lessons, that all were videotaped and transcribed, and analyzed according to the standards of that “interpretative” research which means that interpretation follows hermeneutics and text theory in order to go beyond participants’ (i.e. teachers’) subjective

understandings, and beyond everyday life readings of the analyzed events. The overall procedure to elaborate the final set of hypotheses is borrowed from grounded theory (Glaser & Strauss, 1967).

MICRO-SOCIOLOGICAL THEORIES

A micro-sociological perspective on mathematics teaching and learning has already proven fruitful in a variety of studies. To be precise, the attribute does not denote a single perspective but refers to different theories that share a basic understanding of social reality. Its structures are assumed to be established by the members' of society mutually related acting. Those theories that figure in the project are symbolic interactionism (Blumer, 1969), and ethnomethodology (Garfinkel, 1967).

According to symbolic interactionism, interaction is the key concept to grasp social reality. Within interaction objects (anything that can be pointed, or referred to) get their meanings, and meanings are crucial for people's acting towards objects and, in that, for establishing reality. Interaction is thought of as an emergent process evolving between the participants in the course of their interpretation-based, mutually related enactment. Thus, social roles, content issues, or participants' motives as well are not seen as decisive factors; rather, they are also objects that undergo a development of their meaning. Consequently, neither the course of an interaction nor its outcome is predetermined. The term "interaction" is not restricted to events having outstanding qualities in respect to number of participants, topics, kinds of exchanges a.s.o. This means that classroom processes do not need to meet special demands in order to be a proper research object. From the perspective of symbolic interactionism, attention will always focus on the meanings objects get in local interaction, and on the very development of that interaction. As all participants matter from the standpoint of that theory, students are considered to be equally important as the teacher.

Ethnomethodology, too assumes that social reality is made into reality in the course of action but addresses the issue that despite of its formation social reality is taken as a given reality. This is due to the reflexive character of everyday activities. By accomplishing their affairs the members of society provide explanations for their doing and thus make it the normal way of doing. Ethnomethodology tries to reconstruct those methods. Accordingly, it helps in taking into account the methods by which teachers and students make computer-based mathematics teaching a matter of course whatever it will be about. Because of the shared stance towards reality the micro-sociological theories are treated here as "one" approach.

However, both theories are not sufficient. First, they address even large joint actions under the aspect of formation by separate acts of the participants; that is, they do not foreground the idea of a whole that has its specifics and thus can be spoken of as an entity. Hence it is difficult to think of teaching as a business that has an overall orientation. Secondly, both theories may induce a bias towards verbal events. There is a tendency to focus on verbal processes because of the prominent role of participants'

indications to each other which are indeed often verbal. Yet in an analysis of computer-based mathematics and, even more, experimental physics teaching all kinds of doing have to be covered.

LINGUISTIC ACTIVITY THEORY

The added theory (Fiehler, 1980) is a linguistic branch of activity theory (Leont'ev, 1978) that is not specialized on teaching and learning issues. Its basic concepts are activity, and activity complex. Activities are not merely actions but lines of conduct aimed at outcomes, or consequences. An activity complex can be thought of as a network of, not necessarily immediately, linked activities of some people that is oriented towards a material, or a mental outcome; that is, the concept always indicates a purposive stance. Linguistic activity theory in particular elaborates on the idea that there are three types of activities: practical activities (being accomplished by manipulations of material objects, or by bodily movements), mental activities, and communicative activities (in the sense of verbal activities). It foregrounds the interplay of these types of activities; actually between practical and verbal ones as the involvement of mental activities is a matter of inference. Two kinds of activity complexes – verbally, and practically dominated ones – are postulated in which the orientation towards verbal, or practical outcomes shapes the interplay in specific ways. As for my concern, linguistic activity theory helps me think of computer-based mathematics classrooms as entities having their own character. In particular, attention is turned to their global objectives. This is a relevant issue since in computer-based mathematics teaching IT plays an important role and could become a matter of teaching of its own right. Thus, there might be a further objective. The micro-sociological point of view is open to this option. But linguistic activity theory is in particular conducive to an identification of such cases as it helps in recognizing modes of activities and their interplay.

STRATEGIES FOR NETWORKING

As for the strategies of networking (Prediger, Bikner-Ahsbabs & Arzarello, 2008), “contrasting” theories has taken place so far and revealed that they play rather complementary roles. In particular, this holds for the micro-sociological approach on the one side, and for linguistic activity theory, on the other side. Each of them provides perspectives that are not covered by the other one but are needed to form a better whole: on situational adjustment and formation, on the one hand, and on certain aspects of structure and overall sense, on the other hand.

This two-sided approach has been used for a certain conceptualization of computer-based (mathematics) teaching: Its overall appearance depends in particular on predominating activities and objectives that are put into effect. These features give evidence of certain activity complexes that are the outcome of a multitude of similar negotiations among participants. Different types of computer-based mathematics teaching can be assumed to be established, ranging from a highly verbal teaching emphasizing mathematical aspects to a teaching that is totally devoted to carrying out ma-

nipulations at a computer. That conceptualization can be seen as a nucleus of a theory of computer-based mathematics teaching.

Thus, because of combining theories for the sake of the development of a local theory, synthesizing is a networking strategy in my research. The micro-sociological theories contribute by a “close-up”: the step-by-step formation of an activity complex becomes visible. Linguistic activity theory provides a “long shot”: a multitude of interactions can be spoken of and treated as an entity.

However, in order to elaborate that nucleus of a grounded theory it has to be applied to the data. Empirical phenomena are interpreted in its light. This means that the basic theories are also co-ordinated. Networking also serves the purpose to reconstruct concrete computer-based mathematics teaching. But as the research aims at a local, grounded theory, co-ordinating turns out to be synthesizing.

NETWORKING OF THEORIES: AN ILLUSTRATIVE EXAMPLE

The transcript is taken from an 11th grade classroom. During the lesson the class was given an introduction into maximum-minimum problems in which Derive should be used. The initiating task was: “A farmer has 20 metres of a fence to stake off a rectangular piece of land. Will the area depend on the shape of the rectangle?” A table should help to systematize the findings. In a first step, the students developed a conjecture based upon examples being subject of the first part (lines 01-26). In the following section of teaching (which is disregarded here) Derive was used to note the examples and to build the table. At the beginning of the second part (lines 134 ff) that table, containing columns for length (x), width (y), and area within the range of the examples, is visible to the students by a data-projector showing the solution of Erna who had to provide the official solution in Derive in interaction with the teacher.

- | | |
|-------------|---|
| 01 Teacher: | Our question is. All these rectangles with circumference 20. Do |
| 02 Sarah: | [inarticulate utterance] |
| 03 Teacher: | they have the same area. For example which ones can we take. |
| 04 Boy1: | No. |
| 05 Boy2: | No. |
| 06 Teacher: | Which range can you give an example length width |
| 07 Boy: | Six and four? |
| 08 Teacher: | Six times four is |
| 09 Boy: | 24 |
| 10 Teacher: | Another example |
| 11 Eric: | Five times five this is the square |
| 12 Teacher: | Five times five would be a square having which area |
| 13 Eric: | 25 |
| 14 Teacher: | Or a smaller one. Is there a smaller area as well |
| 15 Carl: | For instance three times seven |
| 16 Teacher: | Three times seven is 21. Or another one. |

- 17 Carl: One times two sorry one times ten
- 18 Teacher: One times ten is ten or if we make it still smaller half a meter
- 19 Girl: [inarticulate]
- 20 Teacher: No. One times ten does not work one times nine would be OK. If the length will be ten what will happen.
- 21 Boy: I see
- 22 Teacher: Length ten what will we get if we take ten for the length
- 23 Arthur: It is a line, a line [smiles], an elongated fence
- 24 Boy: Not at all [continues inarticulately]
- 25 Teacher: A double fence without an area thus the area can range from zero to. What was the largest so far
- 26 Eric: 25
- <...>
- 134 Teacher: OK. This is OK. [to Erna] We can see if x is zero the width
- 135 Boy: Ten
- 136 Teacher: The area
- 137 Student: Ten?
- 138 Teacher: Yes. But now I like to have names for the columns x y z sorry x y the area. This we can do in the following way. We did it never before. Through a text object. Insert a text object [to Erna] this is not the proper place [it is above the table] but it does not matter no delete it. [she does] We want it below the table please click into the table and a text object above. Yes. And now you have to try. Use the cursor to place x y and area x in order that it is exactly above yes x y and the area. [she has finished] I do not know another way. I have figured out just this one. OK. We can see now the area change from zero 9 16 21 24 25 24. Hence the areas differ.

The episode 01-26 is about a response to a question. An analysis following symbolic interactionism can work out what participants' taken-to-be-shared consensus concerning that response actually is. Participants deal with the question in the way that they first present a concluding answer (04, 05, maybe 02, too) and then demonstrate its correctness by giving several examples. Thus the response becomes a moot point again, and participants establish an everyday argument of the kind "statements about parts of a whole hold for the whole as well" (Ottmers, 1996) that confirms the initial response. As for the development of the interaction, specifying length, width, and area serves as a format for giving examples but the binding character of the format does not come about at once. For instance, the second student foregrounds his own point and brings into play the shape as well (11). The teacher is always just one party in an interaction. Also his dealing with the wrong combination of length one and width ten (20) is a reaction to the events.

Ethnomethodology enables me to reconstruct the ways in which the whole process of responding becomes a matter of course. For instance, students keep to presenting length and width as factors (11, 15, 17); or, in the case of disturbance (11), the

teacher's ineffective acknowledgement of the square, consisting of a confirmation and an immediate question about the area (12), proves appropriate for stabilizing the format. In the end, it is quite normal that responding is about making sure that the areas differ and about finding out their range. The reference to the square (although not irrelevant at all) turns out to be already beyond the established scope.

Both theories do not provide a more global understanding of the event. In particular, the question may arise what this episode is good for in the light of the research it belongs to. Linguistic activity theory helps to recognize a general purpose of the first part of the episode. It can be taken as a part of an activity complex: of an introduction to maximum-minimum problems. Accordingly, in the presented part a mathematical matter is made plausible that constitutes a problem that, in a generalized version, will have to be solved by means of calculus involving Derive. Besides, linguistic activity theory makes the solely verbal accomplishment of the response task a more remarkable fact; it springs to mind that, for instance, the table is not drawn on the blackboard. Conversely, however, this theory does not provide insight into the specific way of arguing that turns out to be the solution of this task in the end.

In a nutshell, in a co-ordinated theoretical perspective a mathematical event is established that has the role of a preparatory step in a computer-supported task solving. The subject matter-related potential of the interaction is realized as far as it answers this purpose of preparation though, in the light of that role, the pseudo-reasoning about the difference of the areas appears somewhat artificial. Participants produce that event through a fine, inconspicuous verbal adjustment of their acting.

At the beginning of the second part of the episode (134-137) participants demonstrate how the table has to be read. The values in the first line are used to explain what the output means. In a smooth-running process the teacher and two students establish a shared understanding of the table. After the reading has been clarified the table could be used (and this actually happens afterwards) to check the maximum area conjecture by further examples that are not confined to integer-sized rectangles (to be precise: an adapted version has to be used that provides numerical values in between). However, beforehand headings for the columns in the given table are produced. A second meaning of the table emerges. The table that was designed as a means for the solution of a mathematical task turns into a mere scheme being subject to completeness. The switch is initiated by the teacher, and shared by the students (for example, Erna's immediate adjustment to the new task; 138). All the time manipulations are carried out, and the utterances refer to them. That makes a difference to the first part of the episode. There is much talking again but the accomplishment of the practical activities shapes the verbal process. The completion of the table in Derive becomes the subject of the episode. The situation offers an occasion for such a change; apart from that options of a program will always have to be introduced in some task context. However, as the table was already interpreted well and should help to systematize the findings, the switch is rather a surprise. But: If teaching in that introduction to maxi-

mum-minimum problems aimed at accurate products at a computer this turn towards the completion of the table would not be an extraordinary event. It just had to have priority then. This interpretation hypothesis grounding on linguistic activity theory would neither reject the possibility that those products at a computer could be conducive to mathematical ambitions nor exclude that there could be entirely mathematics-related negotiations. Thus, in its light the first episode need not be an exceptional event; it can even get an important role: it gives the computer-oriented business a mathematical air.

In a modified version, this hypothesis is the overall résumé of my research: Computer-based mathematics teaching of the observed type is a technologically shaped practice. The connection of the theories has also given insight into the particular features of that practice (Jungwirth, 2008a).

To combine theories of different grain sizes seems to be rather a successful strategy for co-ordinated data analysis and theory development (Prediger, Bikner-Ahsbahr & Arzarello, 2008; for some examples). In the following sections I want to address aspects of the theories featuring in my research that may further explain the fruitfulness of networking of theories in my case, and even beyond.

EMPIRICAL LOAD OF THEORIES

The first aspect is the “empirical load” of a theory (Kelle & Kluge, 1999). Accordingly, theories can be classed by the risk of empirical failure: whether or not they comprise concepts and statements from which categories and hypotheses can be deduced that can be examined, and thus refuted through data. In the first case a theory has empirical substance, in the second one a theory has no empirical substance. These are the poles of a spectrum of states.

Symbolic interactionism is at the second pole. It is a stance towards the world that can be held, or rejected. It is not possible, for instance, to formulate refutable hypotheses for the position that objects get their meanings in the course of interaction, or to deduce categories for those meanings from the theory. Ethnomethodology too is a theory that lacks empirical substance. There is no empirical decision-making whether or not people’s methods to settle their everyday affairs make these commonplace affairs, and to fix in advance those methods.

Empirically empty theories have the role of “sensitizing concepts” (Blumer, 1954), that is, of mere perspectives from which data can be looked at. The outcome in the given case has to be worked out in the data analysis. Data can never make such a theory plausible; rather, conversely, interpretations of the data can be plausible in the light of the theory. Qualitative research often draws upon sensitizing concepts because they favour its approach to reality that tries to take into account participants’ own interpretations of that reality (Schwandt, 2000).

Linguistic activity theory has some empirical substance. Observable hypotheses can be built and examined through data. The category of activity and its properties “verbal” and “practical” can be used for this. For instance, it is possible to decide whether or not practical activities dominate and replace talking at all in certain manipulation contexts.

A use of empirically rich theories is characteristic, or even necessary, for quantitative research as the hypotheses to be formulated need a ground they can be deduced from. Within qualitative research referring to such theories may go beyond expectations concerning the rules for that kind of research. Accordingly, literature on methodology (Kelle & Kluge, 1999) points to the risk that properties of categories and hypotheses formulated in advance could dominate and interfere with the intended reconstruction of reality. However, it is not necessary to use empirically rich theories as it is done in quantitative research (Hempel, 1965); a researcher is not obliged to restrict her/himself to examinations of fixed properties and hypotheses.

My study gives evidence that empirically empty and empirically rich theories are compatible, and, moreover, that combining them is a practicable mixture. It seems that this does not hold in my case only. Such a constellation can make connecting theories on a level involving empirical analysis particularly effective. Certainly, applying solely theories without an empirical substance has proven fruitful in qualitative research (in mathematics didactics as well); however, it may be harder to elaborate typologies. Besides, empirically rich theories enhance the development of grounded theories as they help to carry out the check of interpretation hypotheses being strictly demanded in Strauss' version of grounded theory (Strauss 1987).

CONCORDANCE OF BASIC ASSUMPTIONS (PARADIGMS)

The second notable aspect is the compatibility of basic assumptions theories make for the subject under investigation. To put this concern more clearly I present it in well-established terms: it is about theories' belonging to paradigms. The concept of paradigm has quite a lot of meanings; I will adopt here the broad view of Ulich (1976) in which a paradigm is thought of as a socially established bundle of decisions concerning the basic understanding of the section of reality a theory wants to cover.

According to him, the duality of stability and changeability of social phenomena is a crucial aspect for theories that deal with social processes and settings. Consequently, he has made it a starting-point for a typology of paradigms. “Stability-oriented” paradigms regard regularities as manifestations of stable, underlying structures. Theories in that tradition try to grasp invariabilities. “Transformation-oriented” paradigms ascribe regularities to conditions that are changeable because they are seen as having been established by the members of society. Thus, theories try to reconstruct the constitution of regularities and to find out conditions for change.

The theories I refer to differ in their origins and their concerns. Yet despite of all differences they share the idea that regularities are established regularities; that is, that

they are outcomes of practice that can change if inner conditions change. This is obvious for the micro-sociological theories but it holds for linguistic activity theory as well. According to activity theory in general, society is a man-made society; order and stability of societal phenomena reflect the cultural-historical development of human labour and living conditions (although there is an inner logic in that development). Thus, all theories belong to the transformation-oriented paradigms. Symbolic interactionism and ethnomethodology are usually considered to be representative of the “interpretative” paradigm (Wilson, 1970) but that is, in the given typology, simply the micro-sociological version of the transformation-oriented ones.

This common ground justifies an approach to activity complexes under the aspect of local development and, as a consequence, the above conceptualization of computer-based mathematics teaching. If linguistic activity theory thought of human practice as an invariable, “given” entity, networking would not be honest at least. Actually, the idea that an interaction is determined by the roles of the participants, and the idea that an interaction is a negotiation process from which (also) roles emerge could not be combined to an integrated view on interaction serving as a base for analysis.

The general issue arising from the discussion above is which elements of their respective grounds theories have to share in order that networking on the level of some synthesis of theories, or of an integrated analysis, can take place.

To summarize: The last sections should shed some light on the compatibility of theories. It seems that it depends on, or at least benefits from the aspects addressed.

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ON THE ADOPTION OF A MODEL TO INTERPRET TEACHERS' USE OF TECHNOLOGY IN MATHEMATICS LESSONS

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This paper examines why researchers adopt a theoretical model in reporting the results of their research. It describes the development of two researchers investigating teachers' use of digital technology in their lessons. The two researchers were dissatisfied in their attempts to understand the difficulties that the teachers they were researching experienced and they got round this dissatisfaction by augmenting their theoretical positions by the adoption of Saxe's four parameter model. The paper introduces Saxe's model, provides accounts of the researchers' development and ends with a discussion of issues raised.

INTRODUCTION

There has been considerable recent work on theories in mathematics education, reflecting researchers' efforts to be explicit about their theoretical assumptions and the links between different theories. CERME has been a focal point for many of these reflections. But why do researchers adopt a (particular) theoretical model in reporting the results of their research? There are many possible answers including: researchers are expected to adopt a theoretical model; a particular model may be 'in vogue'; the researchers work in a culture where a particular model is the accepted model; the model addresses central questions that the researchers seek to understand. We are two researchers, with different national backgrounds, who used Saxe's (1991) cultural framework and especially the four-parameter model to understand teachers' activities in using technology in their classrooms. We look at this model with regard to central issues we sought to understand. The paper addresses CERME Working Group 9's call for papers questions: *What divergences appear in the way different perspectives conceptualize empirical realities, tackle practitioners' problems? What is the influence of the different frameworks used on the research process? What is their influence on the interpretation of data?* The paper is a report of what Prediger (2008, p.285) calls 'problem solving "in the wild" of ordinary classroom practices' and considers the dual nature of this theoretical problem solving (theory and researcher). The paper first sets out Saxe's model, then describes why and how Saxe's model was used and ends by discussing issues arising.

SAXE'S MODEL

Saxe's model centres on emergent goals under the influence of four parameters: activity structures; social interactions; prior understandings; and conventions and artefacts (see Figure 1). Emergent goals are not necessarily conscious goals but are goals that arise from a problem in an activity and once the problem is solved the emergent goal usually vanishes. Saxe's model was conceived to explain mathematical practices

in cultural transition (the Oksapmin tribe dealing with decimal money transactions) and is cultural-historical in its conception of artefact and interpersonal mediation in social practice. It has been applied in studies of street-sellers' practices (Saxe, 1991) and technicians' volume calculations (Magajna & Monaghan, 2003). It is, in our view, quite general in its application and particularly suited to the interpretation of innovative technology-based activity, such as teachers using digital technology due to unexpected goals emerging in this activity and the influence of cultural views regarding technology. The four parameter model is the first component of a three component theory: analysis of practice-linked goals; form-function shifts in cognitive development; the interplay of learning across contexts, i.e. Saxe's model is a construct and is part of Saxe's broader theoretical framework.

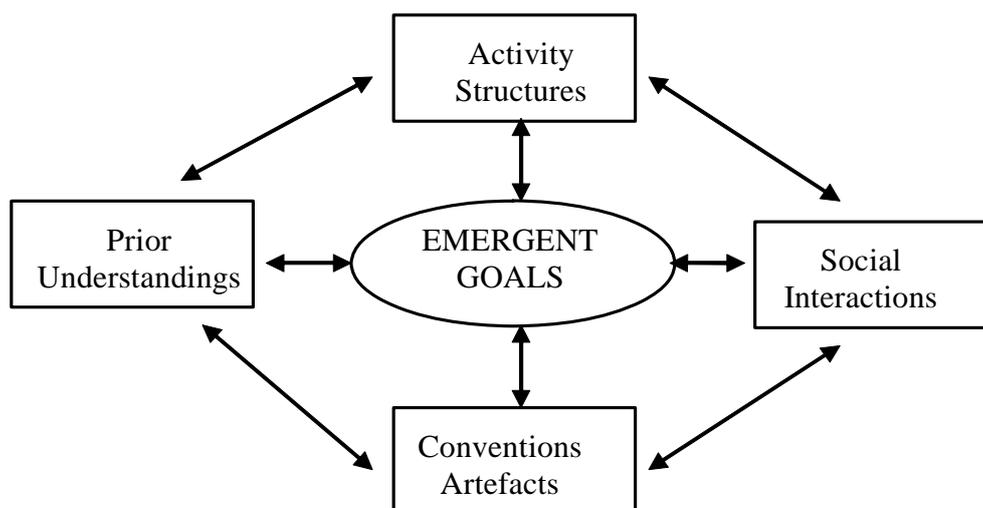


Figure 1 Saxe's four parameter model

We provide examples from Monaghan (2004) to illustrate the parameters, in the case of teachers using ICT, their interrelatedness and their impact on emergent goals.

The *activity structures* parameter “consists of the general tasks that must be accomplished in the practice- and task-linked motives” (Saxe 1991, p.17). In mathematics lessons this parameter concerns tasks that the teacher sets and the lesson structure. The tasks students engaged with in non-technology lessons were textbook exercises and the lesson structure was teacher exposition and examples followed by students doing textbook exercises. The tasks and cycles of the technology-based lessons varied considerably over the teachers and over time for each teacher.

The *social interactions* parameter concerns relationships between participants, teachers and students, in lessons and how these relationships influence participants' goals. It is very difficult to summarise differences between technology and non-technology lessons with regard to social interactions so we provide one example. Teachers spent much more time speaking to two or more students (as opposed to speaking to an individual) in technology lessons. Further to this the computer tools not only performed mathematical actions but also recorded the product of these actions and this provided a common basis for a group of students to collaborate.

The *conventions and artefacts* parameter, consists of “the cultural forms that have emerged over the course of social history” (ibid p.18). Cultural forms in mathematics lessons include techniques linked to traditional, not computer-based, tasks and tools and these can clash with new practices using new tools. A teacher using a spreadsheet planned a lesson focusing on ratio but the students’ and her emergent goals in the lesson were on getting the spreadsheet cells right, not only the correct equation but a suitable cell format. She commented after the lesson that she was unhappy with this focus on ‘cell-arithmetic’ and questioned “is this maths?”

The *prior understandings* parameter, includes teachers’ content, pedagogical and institutional knowledge, “the prior understandings that individuals bring to bear on cultural practices both constrain and enable the goals they construct in practices” (ibid p.18). The term ‘individuals’ is important because the different levels of experience participants in practice “bring to bear different (arithmetical) understandings on practice-linked problems and consequently their goals differ” (ibid., p.18). One teacher commented that with technology it was “back to being like a student teacher” because you are not prepared for any eventuality.

These parameters interact and impinge on practice-linked emergent goals. With regard to *conventions and artefacts* and *prior understandings* and the teacher who questioned whether cell arithmetic was mathematics, for example, this question was legitimate for her because her prior understanding of mathematics was formed in a public understanding of what (school) mathematics is. Further to this she voluntarily planned the task and wrote a worksheet which resulted in a focus on cell arithmetic and this discomfort only emerged in practice because her emergent goals in the lesson were shaped by the need to get the spreadsheet cells right.

HOW AND WHY WE CAME TO EMPLOY SAXE’S MODEL

We, in turn, state why we adopted Saxe’s model in our search for answers to central questions in our research.

Monaghan’s case

I have a long history of using digital technology in my own teaching and in working with other teachers who endeavoured to use it (some found it easy, others found it very difficult). In the late 1990s I ran a research project where I deliberately set out to work with teachers who had not used digital technology in their classrooms but who wished to do so. I worked closely with 13 secondary school teachers over a full school year, leading training sessions and conducting many interviews and observations. Teachers chose the technologies they would use which included computer algebra and dynamic geometry systems, graphic calculators and computer graphic packages and spreadsheets. Each teacher was video-recorded several times over the year (51 recordings in total) including one recording of a lesson at the beginning of the year where they did not use digital technology. Video-recordings were analysed using systematic classroom analysis notation (SCAN; Beeby et al., 1979). SCAN

analysis involves viewing lessons as a series of activities, e.g. teacher exposition, students working, teacher-student dialogue. Each activity is viewed as a series of episodes, e.g. coaching, explaining. Events sub-divide the episodes into social and linguistic categories, e.g. managerial, confirmation. Coding consisted of categorising 30-second blocks with regard to the teacher, the students and the episode. I wrote and co-wrote a number of papers on this work but I still felt ‘unsatisfied’ – there were difficulties that the teachers had experienced in their practices that I could not explain in a satisfactory manner. In one paper (Monaghan, 2001), for example, based on SCAN analysis, I produced fairly strong empirical evidence that teachers using technology did not change from being ‘didacticists’ to ‘collaborators-with-students’ (as some constructivists would have it). I showed, for example, that many teachers became what I called ‘techno trouble shooters’ and I described the material basis for this (the set up and use of classrooms and computer-rooms) but this was not the deep understanding I was looking for.

Of the many intellectual influences on me at that time (\approx 2000), one that fitted with my thinking was Olson’s (1992) work on teachers’ routines. Olson views the study of teachers’ routines as a means to interpret teachers’ actions.

Through classroom routines teachers express themselves. To understand what is being said in classrooms it is important to know what the routines are because such routines are rituals – performances involving significant symbols. These symbols belong to the tacit dimension of practice – what is said in the classroom that is not spoken directly.

As a teacher-educator who is familiar with teachers’ routines these words ring true to me but as a researcher in this project with teachers using digital technology I had a problem with a focus on routine – my project teachers, who were using digital technologies in the classroom for the first time, did not have routines – they were experimenting and doing lots of different things (according to the material conditions of their classrooms). I needed another means to interpret the difficulties my project teachers experienced and the diversity of in-class practices they exhibited. I had, with Zlatan Magajna, used Saxe’s model in his work on technicians’ mathematical practices and I considered analysing my project teachers’ practices via Saxe’s model. Initial considerations looked promising. I feel it is worthy to note, for discussion at CERME WG9, that this analysis via Saxe’s model was quite different to my SCAN analysis. The SCAN analysis was “local” in as much as it concerned categorising actions in specific (30 second) time intervals; further to this it was procedural and, as far as is possible in qualitative analysis, objective. The analysis via Saxe’s model was “holistic” in that whole lessons and often sequences of lessons informed categorisations and took the form of confirming or not the influence of parameters in teachers’ practices.

Lagrange’s case

My approach is to consider theories to address an overarching question: considering the potentialities of technology and the strong emphasis that society puts on its educa-

tional uses, why are these uses so rare, and why, when they exist, are they often deceiving? In this approach, I was brought to focus on the teacher using technology and especially on his(her) classroom activity, and to search for theoretical frames that could help in that endeavour. This approach is reflected in the contributions I wrote for CERME 2, 3 and 4 and in a recent paper (Lagrange, Ozdemir-Erdogan, to appear).

In CERME2 (Lagrange, 2002) I reflected on a meta-study conducted by a group of French researchers of a comprehensive corpus of international publications about research and innovation on the integration of technology into mathematics. The study built a framework of several dimensions in order to account for trends in the corpus. A statistical analysis provided evidence that dimensions considering the impact of technology upon the learner and mathematical knowledge were addressed by a wealth of studies and theories giving account of successes of the use of digital technologies mostly in ‘laboratory conditions’. The other dimensions related to the ‘ecology’ of technology in educational settings were poorly addressed in term of research studies as well as in terms of theoretical frameworks that could give account of successes but also of failures in ‘real school conditions’. We considered a ‘teacher dimension’ but found very few studies addressing this dimension.

In CERME3 (Lagrange, 2004) I focused on problematising teachers using technology. Returning to the overarching question of a discrepancy between the potentialities of technology and the actual uses, my interpretation was that innovators and researchers made an implicit assumption: new technologies and the associated didactical knowledge could easily be transferred to teachers by way of professional development and training. I thought that this assumption had to be questioned because, in a country like France, uses of technologies are deceptive although efforts have been made to train teachers. In my hypothesis the existing corpus of didactical knowledge and frameworks about digital technologies use was not sufficient to really help teachers integrate technology. Thus research had to study the teacher and try to look at his(her) action in the light of new frameworks.

Analysing research (especially Kendal & Stacey, 2001 and Monaghan, 2004) about the teacher and digital technologies strengthened the idea of a difficult integration, contrasting with research centred on epistemological or cognitive aspects. Kendal and Stacey brought evidence that, even in a research project, teachers’ use of technology can be very different to what was intended because of the influence of teachers’ beliefs and habits on the way they use technology in the classroom. Monaghan did a thorough analysis of teachers’ classroom activity showing that innovators’ expectations for a more open classroom management and for more emphasis on mathematics in teacher-students interactions were not fulfilled.

These studies were a first entry into the complexity of teachers’ relationship with technology use. To give account of this complexity and to think of new strategies for a better integration, I considered that an activity theory framework was needed. The

reason is that, while teacher's activity in the classroom is problematic, it has its own logic and consistency. I believed that an activity theory framework would help to elucidate the difficulties encountered by teachers using technology in the classroom, while giving insight on how their activity and professional knowledge evolve during these uses.

In CERME4 (Lagrange, Dedeoglu & Erdogan, 2006) I tried out models of teachers' practices when using technology. Working with two doctoral students, observing and analysing teacher practices in two fields – teachers at lower secondary level using dynamic geometry and teachers at upper secondary level non-scientific stream using a spreadsheet, we (Lagrange, Dedeoglu & Erdogan) noted that classroom use of technology reinforces the complexity of teacher practices by introducing a number of new factors. Our aim was to understand the impact of these factors on systems of teachers' practices, and the conditions for classroom use of technology. We considered Robert and Rogalski's (2005) "dual approach" and we tried to complement this approach by using models dedicated to teacher use of technology: Ruthven and Hennessy's (2002) model addressed teachers' views of successful use, whereas Monaghan (2004) developed a model of teacher classroom activity inspired by Saxe (1991), as outlined above.

We noted in the conclusion that, combined with classroom observations, this model can help to make sense of phenomena in the classrooms that we observed. For instance, it is a general observation that teachers teaching in a computer room devote much time to technical scaffolding when they expected that technology would help their students to work alone and that they could act as a catalyst for mathematical thinking. Ruthven and Hennessy's model helped us to understand how a teacher can connect potentialities of a technology to her pedagogical needs, overlooking mathematically meaningful capabilities. The observation of two teachers using dynamic geometry showed what happens when the connection does not work: the teacher tries to re-establish the connection by becoming a technical assistant.

Saxe's model was chosen to appreciate teachers' specific positions using the parameters and to make sense of their classroom activity in similar lessons. We considered two teachers, one positively disposed towards classroom use of technology, and the other not, both of them experienced and in a context in which spreadsheet use was compulsory: a new curriculum in France for upper secondary non-scientific classes. We contrasted the two teachers through the viewpoint of Saxe's parameters and analysed their activity. In the classroom observations, we noted that teachers had to face repeatedly episodes marked by improvisation and uncertainty. The notion of emergent goals was central to analyse this flow of unexpected circumstances and questions challenging teachers' professional knowledge and parameters helped to understand how teachers react differently with regard to this flow. We also used other didactical constructs like *instrumented techniques* (Lagrange 2000) and *milieu* (Brousseau, 1997) that helped to highlight weak points in these teachers' activity: teachers seemed

not to be able to open a clear dialogue with the students about why it is better to use spreadsheet techniques than usual paper pencil techniques. They also seemed to not have a clear view of the *milieu* they should establish for their teaching goals. Saxe's approach helped to understand the reasons for these weaknesses, mainly grounded in the different cultural representations between students and teachers (Lagrange & Erdogan to appear).

The analysis clearly separated the two teachers. One teacher was at an impasse. Her tendency to act on an exposition/application activity format and a teacher/student individual interaction scheme had been reinforced by the spreadsheet and consequently application was replaced by narrow spreadsheet tasks. With regard to individual parameters, the other teachers' dispositions towards technology integration were, in our opinion, excellent, but globally they conflicted and this teacher had to make real efforts to get herself out of such conflicts. Saxe's approach helped us to understand why good dispositions are not a guarantee of easy integration.

Using Saxe's model gave us more than what we expected. Because it is a cultural approach, it drew our attention to how cultural representations of the spreadsheet can differ, making it difficult for teachers to anticipate and understand what students do with the spreadsheet.

DISCUSSION

We consider issues raised above under two headings: the need for an augmented framework; how to evaluate the productivity of a theory.

The need for an augmented framework

Although we have developed as researchers in different countries we have, for many years, corresponded on matters concerned with the use of technology in the classroom. The constructs available to us, however, and in our opinions, for viewing teachers' activities in technology-based lessons were insufficient because they focused on teachers' established routines and technology messes up teachers' routines. Saxe's model, with its central emergent goals, provided us with a construct to view teachers' activities in technology-based lessons precisely because emergent goals arise from unexpected things that happen in such lessons.

A second reason for augmenting a theoretical framework lies in the gap between data analysis and data interpretation one can trust. Very often researchers conduct research with a framework that integrates methodology and theoretical approach, where data analysis leads the researcher to data interpretation. This appears very sensible unless one finds that the data analysis does not answer 'why' questions. This happened with Monaghan. SCAN analysis revealed large differences between teacher time spent (in technology and non-technology-based lessons) in teacher-whole class exposition, eliciting ideas from students, etc. (see Monaghan, 2001 for further details) but did

not contribute to a deep understanding of why this was happening. Saxe's model, in Monaghan's opinion, provided a means to a deep understanding of these phenomena.

In augmenting a framework one should ensure that the augmentation is consistent with the underlying assumptions of the broader framework. In the case of Saxe and us there is a shared value of the importance of activity and mediation through artefacts and people. Further to this Saxe's model as a construct makes few assumptions. We have focused on emergent goals and parameters which interrelate with them. Emergent goals are ubiquitous in every human activity – so much so that we rarely notice them. Saxe's model has what Dawkins (2008), in discussing Darwin's theory, calls a large explanation ratio, 'what it explains, divided by what it needs to assume in order to do the explaining – is large'.

How to evaluate the productivity of a theory?

In our opinion two outcomes impinge on the usefulness of a theory or model, understanding and widening the research focus/questions. First, the theory or model should provide specific understanding with regard to the focus of the research. Comparing the contribution of Saxe's model to other frameworks helps to evaluate this specificity.

In Lagrange's national context two frameworks are dedicated to learning (Theory of Didactical Situations, Anthropological approach) and a framework is dedicated to the teacher (Robert and Rogalski's (2005) 'dual approach'). These frameworks were useful, but the conclusions we drew did not constitute sufficient progress towards understanding the situation of teachers using technology.

As said above, considering how teachers dealt with the "milieu" and the spreadsheet techniques helped to highlight weak points in their activity. But it was not our central question. The question was why it is specifically difficult, even for experienced teachers, to develop a consistent activity when using technology. Then, the question is, why are those teachers not aware of these weaknesses, or, if they are, why do they not change their activity? Saxe's framework provided a means for a deeper understanding of these weaknesses: rather than a poor didactical analysis, they reflect teachers' uncertainty, and differences between students and teachers, with regard to spreadsheet representations and the fact that it was difficult for teachers to anticipate or understand what students do with spreadsheets.

Robert and Rogalski's approach assisted a consideration of the complexity of teachers' activity. We learnt from that that we would have to consider a plurality of factors with complex links between them. We anticipated and observed that, rather than bringing solutions, technology amplifies complexity. This result is, however, too general and did not account for the uncertainty experienced by teachers using technology in the classroom. The 'dual approach' postulates that practices are complex *and* stable, that is to say that teachers' practices do not change easily because they are constructed to deal with the complexity. In contrast, teachers' practices in dealing

with the complexity of classroom use of technology are far from stable and Saxe's framework assisted an analysis of this instability as a flow of emergent goals.

A second criterion for the useful contribution of a theory or model is that it helps to widen the research questions. The main reason for choosing Saxe's model was the uncertainty of teachers' activity when using technology and the need for a holistic approach of this activity. We were attracted by the model rather than by the whole framework: goals and parameters seemed adequate to analyse teachers' classroom activity, and they actually were. But after using the model, we reflected why this model was productive. We realized that there should be something in common between our teachers and the New Guinea Oksapmin from which Saxe built the model. This should be that both had to deal with a new artefact involving deep cultural representations. In the Vygotskian perspective, Saxe was interested by the impact of culture upon cognition and he chose the Oksapmin people because in their case there was a conflict of cultures: these people have a traditional way of counting, using parts of the body as representation of numbers; some of them trade in the modern way, but their traditional way does not permit them the calculations that this trade requires. This comparison brought us to consider cultural systems involved in classroom use of technology. Students saw the spreadsheet as a means to neatly display data. It is consistent with the social representations of technological tools. People are generally not aware of the real power of the computer, which is the possibility of doing controlled automatic calculation on a data set, even when they used spreadsheet features based on this capability. In contrast, the teachers saw the spreadsheet as a mathematical tool. They were disconcerted because they were not conscious of the existence of other representations. Clearly, Saxe's approach helped us to widen our reflection about the impact of cultural views associated to computer artefacts upon classroom phenomena.

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THE JOINT ACTION THEORY IN DIDACTICS: WHY DO WE NEED IT IN THE CASE OF TEACHING AND LEARNING MATHEMATICS?

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In this paper, we reflect on the Anthropological Theory of Didactics and the Theory of Didactical Situations in Mathematics as the roots of an emergent framework: the Joint Action Theory in Didactics. Disclosing some of the boundaries of the two major French theories in didactics allows us to sketch an integrative scheme of certain of their principles and concepts within the background of socio-cultural and pragmatist approaches to teaching and learning practices.

This paper aims at contributing to the discussion that has progressively given rise to a "theory networking space" in the previous Working Group sessions. We regard this work as an important step for several reasons. First, it accounts for the paradigmatic partition of the main theories currently used in mathematics education, ranging from the more *cognitive* ones that focus on the understanding processes of individual learners, to the more *cultural* ones, that are oriented by institutional and collective structures in which knowledge is subjected to social transactions. It sheds a new light on certain theories we are familiar with, since they are contrasted with some others on certain aspects like the role of social interaction, the role of learning environments, the role of the teacher...etc. Second, some very interesting mechanisms are disclosed about the ways researchers may attempt to connect these theories, while preserving their specificities. We especially value the tension between *integration* possibilities and *boundaries* to preserve, but also the triplet [principles, methodologies and paradigmatic research questions] that is worked out by Radford (2008).

As we support the development of comparative studies in didactics, these questions are of premium interest for delineating both the generic and the specific (i.e. content knowledge related) principles of the intricate processes of teaching and learning. More particularly, the work in progress in this CERME Working Group is an opportunity for us to reflect on the development of the Joint Action Theory for Didactics (JATD), for the purpose of grasping teaching and learning complexity under ordinary classroom conditions.

PART I : SKETCHING A NETWORKING SPACE FROM ATD AND TDSM

In the first part of this paper, we contrast the two major theories developed by the French didactics of mathematics, *i.e.* the Anthropological Theory of Didactics (ATD; Chevallard, 1985/1991; 1992) and the Theory of Didactical Situations for Mathematics (TDSM; Brousseau, 1997). Since these frameworks have developed over more than 30 years, this has to be drastically reduced to their major orientations, without having here the opportunity to decline the various branches that they inspired further

on. Indeed, what we are most interested in is their epistemological stance rather than outlining these theories *per se*. In line with one of the most important principle underlying both the ATD and the TDSM, we consider that theories, like knowledge, emerge as a collective elaboration to face a set of problems and questions that human groups experience in the development of societies. Thus, a good starting point for inquiring into theories may be to compare the realm of reality they account for, through *their paradigmatic research questions* (Radford, 2008) along with *their epistemological roots* in human sciences.

From an historical standpoint, the theorization of an "experimental epistemology for mathematics" that was worked out by G. Brousseau in the mid 70's is a mean to account for the generation of meaningful mathematical knowledge in classrooms. Then, in the early 80's, Y. Chevallard's anthropological analysis of the conditions of knowledge dissemination within institutions, shed a new light on knowledge taught as reworked from its genuine context of emergence in expert (or academic) communities. Therefore, the knowledge coherence and legitimacy as presented in school, has to be studied in terms of epistemic affordances and constraints. In both cases, the epistemological account of the knowledge content at stake as the third pole of the didactical system opened the era of the didactics of mathematics as a science taking off from the psycho-pedagogical stance on teaching and learning.

Since the early works, the ATD relied upon an assumed structuralist point of view of knowledge development within institutions that can be referred to the background of a Durkheimian sociology and eventually to certain socio-cultural approaches. In line with Douglas (1986), the basics of the ATD are that (1) ways of thinking of individuals are shaped by the collective practices to which they partake and (2) these collective practices are oriented by purposes whose coherence defines the primary goal of an institution as a social organisation bound to achieve a type of task. In the case of educational institutions, the transmission of a socially agreed culture is the core of the activity, relayed by an "intention to teach" and an "intention to learn" at the level of the teacher and the students respectively. Thus, the determination level of what the participants do is to be studied in the institutional patterns of the teaching and learning culture. Early works from Chevallard (1985/1991) have stated that the way mathematical knowledge is ordinarily presented within educational institutions does not match the epistemological way the mathematics are built (*i.e.* the mathematical praxeologies in the ATD). Differences in goals generate differences in tasks to be achieved and so the patterns of school mathematics are somewhat distant from academic mathematics. The *transposition process* as the starting point of the ATD accounts for the specific organisation of knowledge in the purpose of its transmission within educational institutions. In particular, the didactical transposition process is characterised by (1) a *decontextualisation* of mathematical practices from the problems they originally attended, into sequence of topics to fit the curricula constraints and the frames of teaching time; (2) a *recontextualisation* of these topics by the teachers, in order to make the students encounter the knowledge to be taught within

the classroom practices. This process has long been regarded to be consubstantial to the functioning of didactical systems as ruled by institutional practices. In recent works, it has been refined by featuring the didactical praxeologies as a set of practices, combining with each other, in order to describe the possibility of studying the process of mathematics in the classroom. It is structured in terms of moments that are *theoretically inherited* from the praxeological structure of the mathematical knowledge, *i.e.* the two levels of practices that correspond respectively to the techniques for solving a type of problem and the formulation / justification of these techniques. In furthering this, the ATD also attempts to account for the role of words, graphics and gestures as "ostensive objects" that shape the mathematical activity. Ostensives encapsulate the socio-cultural definition and values of the mathematical knowledge and they provide tools for a praxeology to develop. In our view, the ATD's paradigmatic research questions attend to a *top-down systemic approach* of the mathematical studying process. A description of the mathematical tasks and the possible didactical praxeologies are attempted as forms of institutional practices.

The epistemological roots and the research questions of the TDSM are more complex to depict. Brousseau's well-known starting point is that a given mathematical knowledge can be functionalised by a *fundamental situation* gathering the epistemological conditions for the emergence of the considered piece of knowledge in the human culture. This major underlying principle is somewhat compatible with the definition of the mathematical praxeologies in the present works of the ATD. Whereas ATD considers this principle as a mean to *describe the possible structures of human practices in studying mathematics*, at the level of institutions, the TSDM refers to the same principle for *modelling the epistemological conditions* in which the students may develop some meaningful mathematical knowledge, within the classroom.

A major concern in G. Brousseau's work is to identify such fundamental situations in the primary school mathematics and to derive some didactical situations from them. In such situations, students encounter some constraints requiring an adaptation of their prior knowledge towards the learning of a new one. The students have to work out the solution of a problem in which specific knowledge cannot be avoided. Brousseau explicitly refers to the Piagetian theory of learning. The core of the learning process relies upon the students' adaptation to a *milieu* as a set of epistemological constraints. The milieu is designed to orient the students' actions by providing some positive or negative feedbacks to the strategies used. To achieve meaningful learning, the students have to take the responsibilities of their game (devolution) without relying on the teacher's feedbacks. This is what Brousseau defines as an *a-didactical situation*, in which the student is supposed to focus his/her interest on a "game" against the milieu and "forget" the teacher's expectations at least for a while. From the student's point of view, the outcome of the game is a new "connaissance" that is being progressively socialised within the classroom debate. Typically, the student first *acts* to find a local solution to the problem, then *formulates* his/her strategies through a communication game and finally, the strategies may be *validated* within a

controversial debate in the classroom. Moving from the peculiar answer to the problem to a generalised pattern of knowledge is supported by *some changes in the milieu* with which the student interacts. Then, the institutionalisation process managed by the teacher makes sure that the "connaissance" constructed by the students within the didactical situation, is adequate to the definition of knowledge in curricula. Thus, the outer horizon of Brousseau's didactical situations remains coherent with a cultural approach of knowledge. However, the kernel of this theory relies upon *a constructivist epistemology* where the student-milieu relationship primes the learning process, by the mean of the a-didactical situation. Social interactions come into play for anchoring the "connaissances" built by students as individuals, within the pre-existing socio-cultural knowledge. As noticed by Radford (2008), they are "a mere facilitator of individual's development of mental structures"(p320). In our view, the paradigmatic research questions that the TDSM addresses is *the design of epistemic models of knowledge, i.e.* situations that enable an adaptive shift of the student towards the construction of new knowledge, without relying onto the teacher's indications at some points of the didactical contract.

Both these theories attempt a model of teaching and learning mathematics as a three poles system where the "being teaching" (teacher) and the "being taught" (student) are two epistemic instances constrained by the knowledge structure. In the ATD framework, the diffusion of mathematical knowledge is studied merely at the collective level of the social structures whereas the TDSM attempts to link the conventional patterns of knowledge and the connaissances constructed by individuals in a rather functionalist way (the milieu originates in the student's actions /formulations/validations). These structural and / or functional stances on the teaching and learning process were crucial in the development of the French didactics of mathematics. We regard it as a major epistemological break from the merely psychological approaches to students' difficulties in mathematics and the pedagogical positivism more generally. It afforded the premises of a science of the teaching and learning phenomena in mathematics, and it also inspired other subject matter didactics in the French speaking community. However, moving back to the major features of each theory allows to highlighting some irreducible boundaries between them.

The epistemological boundary: The TDSM draws strongly on the student – milieu interactions, as an epistemic model of the adequate conditions for reconstruction of knowledge to occur within didactical conditions. The teacher's role in the devolution and the institutionalisation phases is an add-on. In between, the teacher organises the constraints of the milieu to sustain the optimal interactions. The dualistic relationships between the student and the milieu exclude the vision of the classroom social environment as a "thought collective" (Douglas, 1986) to which each student is subjected *ipso facto* through the use of language and more generally signs that are socially agreed. *The predominance of the milieu, as a pre-structured environment made of material, symbolic and social objects to which students have to adapt themselves, shadows the reflective activity that they may also activate to make meanings from*

collective practices. The adaptive function of the milieu addresses the individual minds as independent structures that become intertwined through the formulation and validation games. The reference to the collective practices is not continuous in the participants' experience as it is supposed to be in the underlying principles of the ATD framework. However, one can also argue that the ATD focuses on the institutional practices mainly but the way individuals may get the ownership of these practices and eventually make them evolve, is not accounted for. Very few elements describe what the participants effectively do within the didactical system, in order to teach and learn. As stated by Arzarello, Bosch, Gascon & Sabena, "the non-ostensive objects exists because of the manipulation of the non-ostensive ones within specific praxeological organizations" (2008, p181). *The interpretative process of the collective meanings by individuals are shadowed by the schemes of institutional practices that (over)structures local purposes and psychological processes*. Although the concept of "mesogenèse" was promisingly introduced (Chevallard, 1992) to account for the dynamics of the relations between individuals and objects in their environment, it did not deepen, for instance, how the semiotic systems handled by students (i.e. ostensives) may generate meanings, i.e. non-ostensives (Schubauer-Leoni & Leutenegger, 2005).

The methodological boundary: Early works from Chevallard stated that, ordinarily, the knowledge presented to students in classrooms does not appear according to the epistemological conditions in which it was born, due the decontextualisation and sequentialisation processes in curricula. From this point of view, the works carried out by Brousseau's team may be regarded as an attempt to *counter* the transposition process by redesigning school mathematics into meaningful situations that are not ordinarily supported by didactical institutions. Indeed, a didactical situation is supposed to restore some of the epistemological conditions for knowledge to be built, by designing specific learning environments. A series of fascinating designs were produced in which cultural knowledge is genuinely functionalised (numbering with integers, measuring capacities, introducing rational and decimal numbers, Euclidean divisions, linear functions...etc.). But the way ordinary school institutions may incorporate these situations is not investigated, leaving some opportunities to misleading interpretations of certain examples of didactical situations in some teaching materials. Furthermore, the design process tends to minimize the teacher's work which is then strongly supported by the research team. One can say that it shunts the "*repersonnalisation*" process of the institutional patterns of knowledge, which is ordinarily carried out by the teachers. The relationships between the milieu to be organised and the interaction arena which is ruled by the reciprocal expectations of the didactical contract is the main concern. But the relationships between the ordinary resources that the teachers use and the effective teaching environments they implement cannot be investigated from Brousseau's paradigmatic research questions because they strongly rely upon research designs.

From these boundaries, we argue that (1) the TDSM cannot be regarded as a direct continuation of the ATD framework in terms of classroom practices and interactions among individuals ; (2) the structuro-functionalist stances that are consubstantial to both these theories does not allow an account of the interpretative motions of the subjects within the didactical system as an social institution. These two points could be said to be out of synch with the purposes of those researchers who actually work with one or another theory. Nevertheless, we argue that if didactics is to be a science of the teaching and learning phenomena about a given content knowledge, then some new research questions have to be addressed.

PART 2 : THE GROWTH OF J.A.T.D. AS AN INTEGRATIVE THEORY

In this part, our purpose is not to feature details and examples of use of the Joint Action Theory in Didactics, since this is presented in Sensevy (this group of papers). We rather would like to present the conditions of emergence of its paradigmatic research questions and how some principles and concepts may be borrowed from the ATD and TDSM, by the mean of a conversion process in the light of some pragmatist theories to match a socio-historical perspective of knowledge development in teaching and learning (Forget & Schubauer-Leoni 2008; Ligozat, 2008).

Many empirical studies have reported that the specific role played by the milieu in TDSM's is a feature that is hardly observed as controlled by the teacher in ordinary classes. Most of time, the set of objects partaking to the situation is not self-sufficient to enable students develop an epistemic relation to the problem or task to be achieved. Or, to reformulate this in the terms of the ATD, consistent bodies of mathematical praxelologies are hardly managed by the teacher. However, in these ordinary conditions, that we consider to be the most common teaching and learning reality for mathematics, we cannot envision that no learning happens at all. It progressively leads us to consider that *didactical situations that would be a priori endowed with some a-didactic affordances may not be an adequate model* to theorize the ordinary teaching and learning practices. In other words, the "obdurate reality" of classrooms as an empirical field has to be investigated. What kinds of meanings are constructed in students' "ordinary" learning experience? How does the teacher support them? What kind of common ground is being built for the whole class and how does it fit with the cultural definition of knowledge? What do we know about the way teachers select, structure, refine and adjust instructional settings? ...etc. Such questions arose from empirical observations of classrooms at primary school mainly and with an increasing demand for professionalizing teacher education. The institutional location of researches in didactics in teacher training institutes (IUFM in France, since the early 90's) and/or in some department of educational sciences (e.g. Geneva) has broadened the scientific scope of the subject matter didactics toward a comprehensive account of the didactical phenomena as an educational matter. The realm of studies of the didactics of mathematics as a science meet the opportunity to grow from a merely epistemological programme to a quest for an account of human practices that are specified by the conveyance of a socio-historically built culture. In this

context, the *paradigmatic research questions of the JATD are new ones compared those featured by the ATD and the TDSM*. The teacher and the students cannot be regarded any longer as epistemic instances merely subjected to the structure of knowledge. The interpretative part of their activity within the educational institutions as a social framework has to be accounted for too. To be clear, we are not arguing that the JATD could replace the fields of investigation that are at the focus of the ATD and/or the TSDM. We would like to point out that it is a complementary framework aiming at giving a status to the subjects' actions and interpretations relatively to the institutional contexts for teaching and learning a given subject matter.

In producing such a framework, we call in some principles that are rooted in both human activity as primarily social and historically built *and* in a pragmatist view of the situations in which the activity develop. Against this background, *the transposition process* sketched by Chevallard and *the didactical contract* theorized by Brousseau, can be viewed as the starting point of *a hybridizing plot*.

First, we postulate that the interpretation of classroom events cannot be performed by focusing solely on either the teacher's actions or the students' ones. We propose to look at *the teacher and students "joint" action* to account for both the historical and the situated interdependence of the classroom actions. Such a joint action may involve separate and distinctive acts that are bound together to make the collective action progressing in some cooperative patterns. The genesis of joint action is based partially on orderly, fixed and repetitious definitions of previous acts through the collective memory that is relayed by the use of signs (graphical, gestual, or vocal). Of course, such joint action is also open to uncertainty and so the transformation of the use of signs to sort new tasks and problems. These statements are general to many actions in human activity (Clark, 1996). A way of specifying them is to consider both the specific purposes of educational institutions and the forms of knowledge to be taught.

i) From TDSM, the didactical contract is probably the most likely principle to address the problem of the individuals' interpretation of contextual practices. We consider that the intention to teach a given topic supported by the teacher generates an expectation to learn "something" from the students. Regularities in the functioning of the classroom as a didactical institution progressively makes the students aware that a teacher usually has "something" in mind beyond the concrete tasks or questions they have to sort. On his/her side, the teacher organises didactical time slots for making the students develop a reflection, an inquiry, the achievement of a task...etc. As soon the student is aware of what is being taught, he/she supposed to know, and the teachers moves on toward another topic. Therefore, teachers and students always remain in an asymmetrical relationship due to the difference in the respective status of their knowledge. We consider *the cultural stance of the didactical contract* as a system of reciprocal expectations merely, according to which the teacher and the students adjust their actions. The asymmetrical status of the teacher and the students relative to their respective relationship to knowledge is consubstantial to the chronogenesis and topo-

genesis processes that were initially sketched by Chevallard (1985/1991) to describe the structure of recontextualisation of knowledge in the classroom.

ii) However, we do not maintain the constructivist stance of the didactical contract, *i.e.* the contract as regulating an antagonist set of objects that would constrain the students' actions. A converting plot is then required to describe the relationships of the participants to the objects partaking to the situation. Following Mead's definition of the social act (Mead, 1934), we consider that individuals indicate the objects to themselves in line with the function these objects have in collective practices. The meaning-making process is supported by actions –gestures and discourses- in communicative situations. Objects have a meaning for one-self only because they have also have a meaning for otherselves in the situation but also in the culture pre-existing to the situation. Such processes, as indications of objects within the background of language games (Wittgenstein) are actually under investigation for describing the articulation of collective practices and meanings made by individuals. The distinction of "which object counts for which participant", or "from whom this kind of relation comes out" and "who grasps it" is important in determining 1) the set of objects that participants indicate to themselves, 2) the meaning that they may ascribe to their own actions with these objects, 3) the control they gain from it and that may be re-allocated in further experiences. This threefold meaning-making process over time is described as *a mesogenesis*.

iii) Then, it follows that the topogenesis and the chronogenesis are strongly related to the teacher's actions because of his/her leadership in the didactical relation. The teacher is the one supposed to orient the student's actions in order to help him/her learn, but also to notice the student's elaborations in order to designate them a new knowledge. Therefore, some *chronogenetic and topogenetic techniques contribute to the building of a common reference* (objects, relations) in the mesogenetic process. Chronogenetic techniques are anything that the teacher may do in order to *orient the students' actions* toward the piece of knowledge to be learnt. The topogenetic techniques are anything that the teacher does to *regulate his/her involvement* in the joint action and to assign a role to the students all together or as individuals. The devolution and institutionalisation categories for the teacher's action primarily exist in Brousseau's didactical situations, but they may be revised as generic to any teaching process.

iv) The specification of the joint action also operates through the epistemic tasks that are to be achieved. The pre-existing culture necessarily comes in when studying how knowledge to be taught is presented in the teaching materials and curriculum texts. But the purposes of the ordinary practices in classrooms may be rooted in some multi-determination levels other than merely mathematical ones. Thus, acknowledging for the individuals' interpretations of the situations they encounter lead us to reconsider the transposition of knowledge within the didactical institutions from a *bottom-up* point of view that is coupled with the *top-down* analyses typically performed by the ATD framework. We conduct an analysis of the epistemic tasks that are em-

bodied in the teaching materials that the teacher uses (Ligozat & Mercier, 2007). For instance, from the worksheet proposed by the teacher to the students, we may inquire 1) what could be learnt in performing it and then 2) what could be taught according to the curriculum of a given grade. At this step, the fundamental mathematical situations or the mathematical praxeologies provide some useful ways of modelling the epistemic knowledge. The possible gaps and contradictions that are issued by the decontextualisation process may be disclosed against the background of the mathematical practices. Then a bottom up process aims at reconstructing the meanings that objects, situations and practices may have for the participants to the classroom joint action. In this second process, the epistemic model of mathematical knowledge is used as reference to understand 1) what is actually taught and learnt in the joint actions; 2) what the distance left toward the cultural knowledge is and 3) what the epistemic necessities that bend the joint action in some specific ways are. This type of analysis may be carried out at various scales of analyses (a classroom episode, a whole lesson, a teaching unit spread over several lessons...etc.) that can be nested together. The coupling of both the transposition and the social transactions analysis with the classroom supports the investigation method in the JATD framework. A full study of the course of joint actions in the classroom against the transposition of measurement at primary school was achieved in Ligozat (2008).

CONCLUSION

The JATD attempts to encompass a huge programme for didactics as a scientific domain studying the human transactions organised about the transmission of a socio-historically built culture. The need for a theory that aims at theorising teaching and learning practices as they occur in ordinary classroom seems unavoidable. However, in its present state, the JATD has to face different kinds of problems: 1) defining its *identity* as a generic theory for the study of the didactical facts but which develops and produces results by accounting for the specificity of knowledge domains; 2) the further clarification of *its epistemological stances* with respect to the principles and concepts that are borrowed from other theories and 3) the definition of some methodological *units* from its very extended realm of reality, that may be worked out independently without taking the risk of generating some misleading interpretations. The very intention of this paper can be regarded as an attempt to contribute to the first and second points with respect to relationships the JATD has with other theories concerning specific domain didactics. However the clarification of the epistemological stances of the action theories that we invoke still remains a major stake for the works in progress.

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TEACHER'S DIDACTICAL VARIABILITY AND ITS ROLE IN MATHEMATICS EDUCATION

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We are looking for the explanation of the differences in learners' flexibility when using the learned knowledge in new contexts. The main aim of our contribution is to combine various theoretical perspectives of investigating teachers' variability and students' flexibility when applying the learned knowledge. We consider the interpersonal differences as an effect of the teacher's didactical variability. Sarrazy (2002) claims that the question of the use of algorithms and taught theorems by students is more an anthropological than psychological problem. The contribution relates to the question B2: Do different frameworks make us look at different aspects of the learning process, that is, at different research questions and different data, or at different interpretations of the same data about the learning process?

1 INTRODUCTION

Learning mathematics is successful only when the learner is able to identify conditions for the use of knowledge in new situations. These conditions, however, are not present in the algorithms itself and cannot be carried over by teachers to their learners. This is one of the didactical contract paradoxes: "The more the teacher gives in to her demands and reveals whatever the student wants, and the more she tells her precisely what she must do, the more she risks losing her chance of obtaining the learning which she is in fact aiming for." (Brousseau, 1997, p. 41).

In (Novotna, Sarrazy, 2005) we presented two studies originally carried out as independent entities both dealing with the same topic: problem solving. One of them belonged more to the psychological perspective while the second one examined the effects of variability in the formulation of problem assignments on students' flexibility when using taught algorithms in new situations; the research was developed in the framework of the theory of didactical situations. These two studies proved themselves to be perfectly complementary. The first one allowed the detection of a set of phenomena, whereas the second gave them precision through an action model of the problem focusing on the variability in word problems. Connecting these two approaches allowed opening interesting perspectives for a better understanding of the role of problem solving in teaching and learning mathematics by giving precision to certain conditions of their use.

Why is it worth to combine the two approaches? Novotná (2003) showed that the analysis of models created by students enables the teacher to help them in case that

their effort to solve the problem correctly is not successful (mainly in determining the type of obstacles the student has faced). The individual differences in the form of graphical models could be explained by the internal student's cognitive processes (Novotná, 1999). However, this approach did not enable us to explain the striking difference “spontaneity versus copying” in the student groups. The psychological perspective did not offer any explanation of the fact observed. It was to be searched for outside the psychological approach. A suitable tool for the explanation was found in the frame of the Theory of didactical situations by Brousseau (1997), namely in the of variability of teachers introduced by Sarrazy (see Part 3).

Sarrazy (2002) presents a model based on the following idea: The more versions of realisations a particular form includes, the more uncertainty is attached to this form. To satisfy the teacher's expectations, the student must ‘examine’ the domain of validity of his/her knowledge much deeper than a student who is exposed to strongly ritualised (repetitive) teaching and therefore considerably reduced variability.

2 INTERPRETATION OF EFFECTS OF VARIABILITY

We are investigating effects of variability of teachers on learners' flexibility in applying algorithms from three perspectives (for more details see Novotná, Sarrazy, to be published):

a1 – *Psychological interpretation*: Variability gives priority to the change of learners' operational register by diversifying their relationship to the object of teaching or to their action (Richelle, 1986; Dré villon, 1980). In fact, the diversity of modes of relationship to the object of teaching, which is typical for didactical environments with strong variability, brings in an alternation between the phases of knowledge integration and differentiation in their usage. Dré villon (1980, p. 336) states that learners would possess a plurality in their access to objects that would be efficient to help “not only to proceed to the operational formal stage but to construct a repertoire of cognitive registers. This repertoire enables, if asked or needed, to examine a problem and solve it at the functional level, i.e., practical and objective, or to extract the operational quintessence and thus to construct a more general activity model”¹.

According to Piaget (1975, 1981), it is also possible to consider variability as one of the sources of perturbations resulting from variations of didactical environments; this variability enables to provoke cognitive adaptations (accommodations) and thus to increase the student's cognitive register in relation to a conceptual field – e.g., additive and multiplicative structures studied by Vergnaud (1979, 1982, 1994).

This first aspect can be précised didactically by changing the frameworks as proposed by Douady (1986) in the theory of “dialectic ‘tool-object’ (outil-objet)”: “A student possesses mathematics knowledge if he/she is able to provoke its functioning as explicit tools in problems he/she must solve [...] if he/she is able to adapt it when the normal conditions of its use are not exactly satisfying for interpreting problems or for posing questions with regards to it”² (Douady, 1986, p. 11).

a2 – *Anthropological interpretation*: When interpreting variability effects in relationship to what could be called “school culture” of the class, variability creates a characteristic of the environment in which learners develop and learn mathematics. In case of weak variability, a repetitive teaching, poorly varying in its forms of organisation and in the content, leads the learners to a hyper-adaptation to proposed situations. In order to adapt themselves to the usual teacher’s demands, the learners develop strategies of *coping* (Woods, 1990) with the criteria usually used. They can easily detect indicators allowing them to adapt their decisions and their behaviour to their teacher’s didactical requests. In that case, learners can very well apply suitable behaviour without exactly understanding the sense of the lesson or of the problem they were assigned. In case of strong variability, the learners cannot rely solely on the “rituals” because they can neither anticipate nor manage the succession of sequences or behaviours expected by the teacher. The learners’ engagement in the situation is much more probable.

It is well known that a particular teacher’s attitudes create educational environment, let us call it climate. Flanders (1966) showed the influence of teachers’ ways of functioning on the class climate. This climate was defined as “common attitudes that learners have, in spite of their individual differences, with respect to the teacher and the class”. In individual cases, this climate can support or block learners’ future successful development of their relation towards learning. Certain works in the domain of didactics of mathematics, e.g., Perrin-Glorian (1993) or Noirfalise (1986) support the previous interpretation.

The authors observe that some teachers focus their teaching more on the content to be taught while others on their learners privileging the relationship with the student. The first mainly look for progress in the subject matter and gaining new knowledge, they appreciate all attitudes with which the learners manifest their interest in what they are taught; the latter prefer production of ideas and communication among students. Achievements obtained by students differ significantly according to the considered domains: focus on the content favours success in algebra while focus on the students leads to better results in geometry and to making mathematics more attractive for the student.

a3 – *Didactical interpretation*: As mentioned in a1, Douady’s results (1986) allow clarifying the processes enabling to report on the effects of variability. This research is done in two frameworks: Theory of conceptual fields by Vergnaud (1990) and Theory of didactical situations by Brousseau (1997). For Douady, teaching a mathematical concept requires a transformation, a completion to see even the rejection of learners’ previous knowledge. The proposed problems must be perceived in such a way that the learners have an opportunity to engage at least one basic solving strategy but this strategy is insufficient: the taught knowledge (*object*) must correspond to the *tool* best adapted to the problem.

Douady distinguishes 6 different phases constituting the process of the “dialectic tool-object”:

Phase a – Mobilisation of “former”: Corresponds to the phase of the problem adaptation by the student.

Phase b – “Research”: Corresponds to the phase of action of the Theory of didactical situations (Brousseau, 1997). During this phase, students encounter difficulties caused by the insufficiency of their previous knowledge and consequently look for new, better adapted instruments.

Phase c – “Local explication and institutionalisation”: The teacher points out the elements that played an important role in the initial phase and formulates them in terms of the object with the condition of their use at the given moment.

Phase d – “Institutionalisation” (in the sense of the Theory of didactical situation by Brousseau, 1997): The teacher gives a cultural (mathematical) status to the new knowledge and he/she requests memorization of current conventions. He/she structures the definitions, theorems, proofs, pointing out what is fundamental and what is secondary.

Phase e – “Familiarisation - reinvestment”: It concerns the maintenance of what was learned and institutionalised in the various exercises.

Phase f – “Complexification of the task or a new problem”: The aim of this last phase is to allow the students to make use of the new knowledge in order to allow new objects to occupy their position in the students’ previous knowledge repertoire.

According to Douady, the aim is to exploit the fact that most mathematical concepts operate in several frameworks – in fact in diverse types of problems. For example, a numerical function can be presented at least in three frameworks: numerical, algebraic, and geometrical. These changes of frameworks (“game of frameworks”) allow varying the significances (supports of significations) for the same concept and allow avoiding that the learners make them function in a partial or in over-contextualised ways. The interactions among diverse frameworks allow, according to Douady, to make the knowledge progress and to keep all the conceptual potential of the taught object.

3 EXAMPLE: SARRAZY’S MODEL OF TEACHERS’ VARIABILITY

For the characterisation of teachers’ modes of didactical activity, typology of modes and examination whether these modes enabled awareness of the differences in the sensitivity to didactical contract in groups of students, Sarrazy (1996) introduced a model that allows describing the modes of teachers’ actions. This model is sensitive in learners’ treating of problem types. It uses the following three dimensions, the six variables being defined in order to measure variability in organisation and management of the teacher’s work during and between lessons:

- i) *Didactical structure of the lesson* (what the teacher really does from the perspective of the knowledge to be taught);
 - v₁. What is the type of didactical dependence? Does the teacher proceed from simple to more complex tasks or the other way round?
 - v₂. Place of institutionalisation: At which moment does the teacher present a solving model? Closer to the beginning or to the end of the lesson? Or only at the beginning or at the end?
 - v₃. Types of validation: How are the students informed about validity of their answers? Does the teacher always use the same type of evaluation and assessment (by the milieu, by direct evaluation, by the Topaze effect³, by peers ...)?
- ii) *Forms of social organisation* (this domain corresponds to the teacher's activities regarding class management)
 - v₄. Interaction modes: teacher-student(s), student(s)-student(s)
 - v₅. Management with regard to the students' groupings: the whole class, small groups, individual work
- iii) *Variability of arithmetical problem assignment*
 - v₆. The variable is related to editing the problem assignment. It is given by an indicator which measures the teacher's "capacity" to consider diverse modalities of the same didactical variable in the assignment.

This model makes it possible to describe the teacher's teaching practices from a triple perspective: presentation of the content (i), desired forms of teaching (ii) and variety of the proposed situations (iii). It is not an isolated variable that affects the students' learning (mainly defined by the notion of sensibility – i.e. their ability to use the taught algorithms in various contexts). On the contrary, it is an effect linked to a set of variables (that may be called a profile of the didactical action); this profile enables a characterization of one way of letting the students do mathematics. This is why we proceeded to a hierarchical classification in order to show similarities by clustering of variables.

Using the above variables, teachers' different profiles were hierarchically classified (Sarrazy, Novotná, 2005, where the experimental disposition, that allowed characterising teacher's variability and thence to show the influence on the way how the students do mathematics, is presented; the crucial role of didactical contract and the sensitivity to it is documented).

Let us recall here the general idea: Submission of students to a teaching style poorly varied (and strongly repetitive in the forms of organisation in the presentation of the content) will decrease the possibilities of opening the didactical contract; vice versa, more variable the teaching is, the more the students will be confronted with new situations and the more flexible their use of the taught algorithms will be. Let us con-

sider a simple (and therefore caricaturing) example which serves as an illustration of the theoretical position:

The mother spent 13 EUR at the market. Now, she has 19 EUR. How much had she had when she went to the market?

This problem, although simple, presents several difficulties to the students. These difficulties are based on the fact that the problem evokes the framework of subtraction but the numerical operation to be executed is addition. Here is an example of the variety: the more the student will be confronted with the situations that involve diverse contexts of the use of additive structures, the higher the probability that his/her answer will be guided by conceptualising the relations in play; vice versa, the less diverse the situations are, the more the students will be led to rely on the apparent characteristics of the tasks when producing their answer (e.g.: every time seeing the verb “spent” they will subtract, “anybody” divide etc.).

Using the above variables we defined three teaching styles of the school culture that are in strong contrast:

“*Devolving*”: This style corresponds to what, in the first approximation, could be called “active pedagogy” in which the students need to be “active”. This style is characterised by strong variability in the organisation and management of situations: the teachers regularly use group work although they by no means restrict only to this form of student work; generally speaking, the problems are complex; classroom work is very interactive (students interact spontaneously, “choral” answers are not rare, ...); in the lesson, institutionalisation is diverse. These are the main features of the first style.

The other extreme is the “*institutionalising*” style. This climate is characterised by a weak introduction and a weak variety of situations presented to students; we could call it ‘classic teaching’ in which the scheme “show–remember–apply” seems to be the rule. These teachers institutionalise one solving model very quickly and then present students with exercises of growing complexity. First, the exercises are corrected locally – the teacher passes through the rows and corrects them individually. Then the teacher gives the complete correction on the blackboard; here he/she gives details of the solution and, depending on the time he/she has, occasionally invites some students to the board either to make sure that they are paying attention, or to recall certain knowledge. Now, the interactive climate is quantitatively as well as qualitatively very different from the interactive climate of the preceding style: Students’ spontaneous interactions or “choral” answers hardly ever occur.

The third style is the “*intermediary*” style. As its name indicates, this style is closer to the institutionalising style, even if the teachers ‘open’ the situations more and more frequently. In any case, here the students have more chances than students of “institutionalising” teachers to encounter research situations, and debate, but markedly less than those exposed to the “devolving” style.

As we expected, we observed strong internal coherence of each of the styles (climates) confirmed by the stability of the results acquired using various methods of data analysis (implicative analysis, dynamic clusters, hierarchical classification, and so on). It seems to provide evidence in favour of the existence of an organising principle for the practices. This organising principle could at the same time be linked with didactical conditions (meant in relation to the knowledge dealt with) and with anthropological conditions (independent of knowledge but linked with teachers' pedagogical or political convictions, with influences of fashionable constructivist, cognitive, and other psychological models).

4 CONCLUDING REMARKS

There are two concluding topics to be discussed: the consequences of the presented results for teacher training and the theoretical positions of the studies about variability.

The presented results are of great interest for improving the teaching of mathematics by focusing on the flexibility in the use of the taught algorithms. But is it possible to foster an increase in the variability of the teachers? It seems to be difficult to directly influence the conditions allowing increasing the variability of teachers. Even if we find it important to present teachers with models of the analysis of problem assignments (e.g. those of Vergnaud concerning additive and multiplicative structures), there are good reasons to believe that mere presentation is not sufficient. In fact, on the one hand these models when only presented to teacher trainees to have a look at them do not affect their variability directly (Sarrazy, 2002); on the other hand, we could observe that variability is the dimension of the teacher's activities that is statistically linked with other dimensions of his/her didactical activities (e.g. the use of group work, the volume of didactical interactions, his/her pedagogical philosophy). Variability should be understood as one of the elements of the teacher's system of didactical activities that interacts with other components.

This last aspect bids for discussion of its theoretical status. We do not pretend to submit here a new theoretical concept of a teacher's didactical activity but more modestly, we situate this approach as an extension of the Theory of Didactical Situations by Brousseau (1997). During the "ordinary" teaching situations that we observed, we found rarely those where the "milieu" contained an a-didactical component, i.e. those where the situation allowed to delegate to students the retroaction to their actions. We believe that a developed variability when the a-didactical "inside" of the situation is absent, would allow the students to establish a quasi a-didactical relation only. As we indicated, it is the consequence of the fact that they cannot go upon the formal aspects of the proposed assignments.

An important question arising from our research is: What kind of training is likely to increase the variability of teachers? Although it is certainly an important question, we find solving it premature as long as the problem of conditions favouring the variabil-

ity has not been clarified. This problem, first opened in anthropo-didactical approach in DAESL about fifteen years ago, needs to be explored in further research in the area where didactics and pedagogy meet.

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¹ Translation from French by J. Novotná. Original text: « non pas seulement à passer au stade opératoire formel mais à construire un clavier de registres cognitifs. Ce clavier permet à la demande, et en cas de besoin, d'examiner un problème et de le résoudre au niveau fonctionnel, c'est-à-dire pratique et objectif, ou d'en extraire la quintessence opératoire et de construire ainsi un modèle plus général de l'activité. »

² Original text: « Un élève a des connaissances en mathématiques s'il est capable d'en provoquer le fonctionnement comme outils explicites dans des problèmes qu'il doit résoudre [...] s'il est capable de les adapter lorsque les conditions habituelles d'emploi ne sont pas exactement satisfaites pour interpréter des problèmes ou poser des questions à leurs propos ».

³ *Topaze effect*. When the teacher wants the pupils to be active (find themselves an answer) and when they can't, then the teacher suggests/disguises the expected answer or performance by different behaviours or attitudes without providing it directly. Example: Teacher: 6×7 ? Pupils: 56. Teacher: Are you sure?

THE POTENTIAL TO ACT FOR LOW ACHIEVING STUDENTS AS AN EXAMPLE OF COMBINING USE OF DIFFERENT THEORIES

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In dealing with low achieving students one needs a fine grained measure for their gain in knowledge. I will show that the concept “potential to act” helps to understand the students’ difficulties and to support their construction of knowledge. The concept connects parts of theories of different scope: a model for abstraction in context, self-determination theory and a psychological theory of action. The relevant parts of the theories will be discussed, and, more specifically, to which extend they are compatible. I shall utilize an example to illustrate the concept of the “potential to act” and to show the interplay of the different theories at work. Further, I will explain how their combining use gives rise to additional insight about the construction of knowledge.

INTRODUCTION

As part of an on-going project at the mathematics education department of the University of Bremen, I am working on a theory of support for low achieving students in Hauptschule[1], aged between 13 and 18. In the project, we want to identify what kind of potential to act in certain situations these students have in order to be able to adapt the supporting lessons better to them, and to understand how they construct and reconstruct mathematical knowledge. For this it is necessary to get finer information about the students’ gain of knowledge than is possible by error analysis of direct tasks or questionnaires.

We are not discussing the phenomenon of low-achieving students in terms of “dyscalculia” or similar notions (cf. (Moser Opitz, 2007) for a recent review). Those studies concentrate mainly on primary school students and on typical problems with arithmetic and numeracy tasks. In contrast, I am interested in the problems of motivation for low-achieving students, which seem to have gained little interest so far. A notable exception is the article of Pendlington (2006), where the author describes the effect of supporting lessons on self-esteem.

I will not use the concept of self-esteem in this paper, but I will make use of self-determination theory for the motivational aspect. Furthermore, I complement this approach with the theory of abstraction in context and a theory of action. By applying these parts of different theories we can accomplish a more complete understanding of the learning process for low-achieving students.

In this paper I present a case of combining three different theories that in their cores may not be fully compatible and this case raises the question what compatibility means in this context.

THEORETICAL BACKGROUND

I will restrict the description of the three theories to their main parts.

Abstraction in context – the RBC model

Hershkowitz, Schwarz & Dreyfus (2001, p. 202) regard abstraction as “an activity of vertically reorganizing previously constructed mathematics into a new mathematical structure”. This means that abstraction is an activity in the sense of Leont'ew's activity theory that comprises actions. Hershkowitz et al. identify three characteristic epistemic actions, namely recognising (R), building-with (B), and constructing (C). Recognising is described as an action in which a student becomes aware of a familiar mathematical structure in the situation, and building-with as “combining structural elements to achieve a given goal” without gaining new complex knowledge about the situation. When this happens constructing takes place.

These epistemic actions are observable in social interaction and provide evidence that a process of abstraction is taking place. The actions are nested, e.g. constructing requires that the subject has already recognised and built with existing structures to construct a new mental structure.

Self-determination theory (SDT)

The self-determination theory (Deci & Ryan, 2000; Ryan & Deci, 2000b) explains how different kinds of motivation emerge. For this the existence of three innate psychological needs is postulated: the need for autonomy, the need for competence and the need for social relatedness. These needs “specify the necessary conditions for psychological health or well-being” (Deci & Ryan, 2000, p. 229) and are indispensable for intrinsic motivation or integration of extrinsic motivation. Following Bikner-Ahsbabs (2005) I specify the innate needs for students in mathematics as follows: *autonomy* as the experience of being able to initiate learning processes and decide about them, *relatedness* as the experience of integration in the social environment and of social support. Bikner-Ahsbabs's definition of *competence* as experience of broadening or deepening one's mathematical abilities seems to be too narrow for our purpose, because low achieving students might get a feeling of competence simply by successful application or reproduction of their mathematical knowledge.

Theory of action

Oerter (1982) discusses the notion of action and the relation of objects and action. He follows the tradition of Leont'ew's activity theory and considers action to be of “primary reality” for each subject, i.e. action is the sole link between an individual and its environment.

“There is no remembering, imagining or thinking as such, other than with respect to the objects of the environment.” (Oerter, 1982, p. 103, transl. by the author)

This implies that any kind of relation to objects or between different objects can only be accomplished by action. There are three layers of object relations[2].

1. *no separate object*, i.e. the object is a fixed part of the situation and cannot be thought of after the current action. It will not even be recognised as an object.
2. *object separated from subject*, i.e. a relation beyond the current action. A subject can recognise the object and name it after the current action but it may still be dependent on the given situational context.
3. *abstract, formal object*, i.e. the common structure of the contextualized objects.

Our experiences with low-achieving students lead to the hypothesis that these students often fail at the transitions from one level to the other. For example, let us take a quarter of a certain cake. At the first level, the student does not realize a separate object at all, i.e. this quarter has no meaning by itself and after it has been eaten there is nothing left to think about. At the next level, the meaning of a quarter of this cake can be transferred to similar situations. So, we might think of a quarter of a piece of chocolate, but all of those quarters are still tied to their context. Finally, at level three a student might have a concept of a quarter of something, meaning one of four equal parts of an entity. Thus, this concept has become abstract and does not depend on the concrete action.

THE POTENTIAL TO ACT

We start with the definition: *The potential to act* consists of all possibilities a subject has to act in a given situation with respect to given objects. This rather abstract definition requires some explanation and we shall discuss it in a more concrete setting:

Imagine that you are working with a student on some mathematical concept, e.g. division of natural numbers. Using a traditional test you have already found out that he fails to solve most division tasks. Furthermore, you have experienced that he cannot make use of most basic ideas associated with division of natural numbers. However, if you ask him to explain how something might be divided in a certain family situation, he can explain some of these basic ideas. In this case his potential to act includes these concepts in the family situation, but not in the written test. So, using the family situation, you might be able to help him enlarge the potentials to act for division tasks.

It is obvious that it is impossible to describe the potential to act of a given student completely. Nevertheless, by looking at the real actions (in contrast to the potential ones) a researcher is able to identify indicators for them and can develop hypotheses about how the student's potential to act might look like in this specific situation and similar ones.

A potential to act can be described by two dimensions: the cognitive dimension and the motivational dimension. The RBC-model and the SDT provide tools to gain indicators in these dimensions. Let us briefly describe what these dimensions mean and how to get indicators for their description.

The motivational dimension is thought of as the degree of intrinsic motivation. Whenever an innate psychological need is satisfied, we interpret this according to SDT as an indicator for an increase in the motivational dimension. If the needs for competence, relatedness or autonomy are not satisfied, we infer that intrinsic motivation will decrease. At this stage of research we use the words *increase* and *decrease* in a qualitative sense without any quantification.

The epistemic actions of the RBC-model may serve as indicators for the cognitive dimension of the potential to act. This dimension inherits the hierarchy of the nested epistemic actions.

Besides the cognitive and motivational dimension, one has to cope with situational aspects of the potential to act including the objects involved. The layers of object relation are used as a tool to structure and categorize the objects in different situations.

Let me briefly comment why those three theories were chosen for the aspects of the potential to act. In order to have a framework for the notion “potential to act”, I chose the theory of action according to Oerter, which has the advantage to offer a description of relations to the objects. The theory of abstraction in context is used, because it allows gaining information about the process of construction of knowledge and fits well with Oerter’s framework of action. Self-determination theory was chosen, because it captures the motivational aspects of the potential and has already been successfully used in describing the motivational problems of low-achieving students in general (Skinner & Wellborn, 1997).

SOME DATA

The data shown below stems from an explorative study conducted at the University of Bremen to explore the potential to act for a group of low achieving students. The students were of age 14 to 18 and took part in weekly supporting lessons, which were done either for groups of three students or individually. The lessons were videotaped and the video was analyzed afterwards to reconstruct the potential to act and to set up the tasks for the next lesson based on this analysis.

The following transcript shows part of supporting lessons that were intended to help the student (S) to understand the concept of equivalence of fractions. This specific task was chosen to help S to develop connections between different representations of extending fractions. S is 14 years old and has been taught by a special school teacher in mathematics for over a year before she came into our project. In her math class fractions had already been introduced the year before and were again the topic of various lessons in class during the weeks before this episode was conducted. After S

has been given a worksheet showing figure 1 the teacher (T) asks her to explain the diagram.

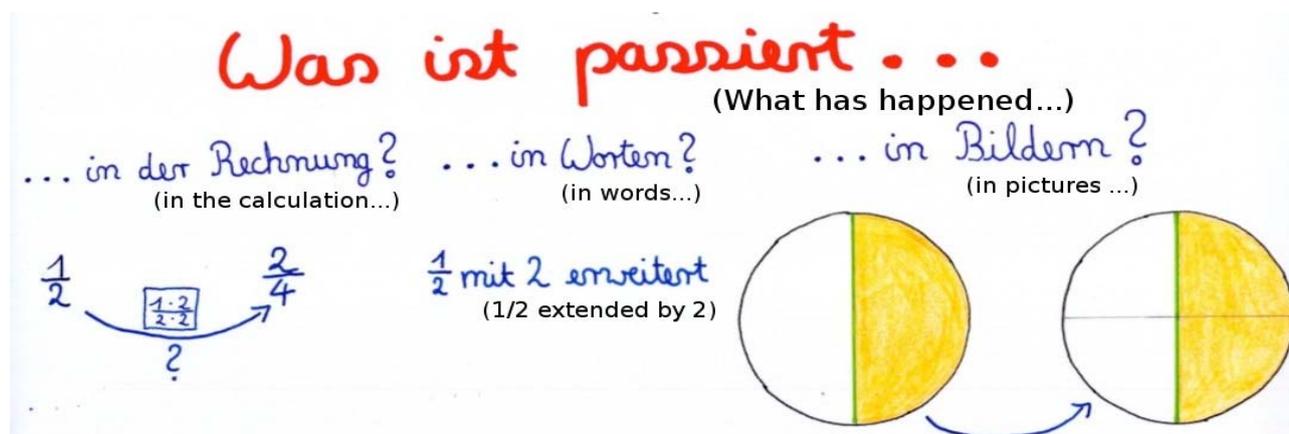


Figure 1: “What has happened ...?” (translation by the author)

- | # | Speaker | |
|----|---------|---|
| 2 | S | Well – erm – they have one half – times – they have calculate one times two – up here, haven’t they? (S points at calculation in the denominator) |
| 3 | T | Hmm. |
| 4 | S | And – erm – what then four – erm – to get four as a result, they have calculated two times two. |
| 5 | T | Hmm, exactly. |
| 6 | S | Well, they have extended by two. |
| 7 | T | - And what is this picture? |
| 8 | S | Erm, that is one half and ... quarter. Two quarters. |
| 9 | T | Hmm. And what exactly has this picture to do with – erm – the calculation? |
| 10 | S | This is one half and this – and these are two halves. (S points at $\frac{1}{2}$ in calculation and left circle, $\frac{2}{4}$ in calculation and right circle in fig. 1) |
| 11 | T | Hmm – exactly, fine, and – er – now in here there is this, this calculation described, isn’t it? You have said this correctly already. Erm, can you find this, what has happened here, this calculation. Can you find it in here again? |
| 12 | S | (S pauses for 17 seconds) one times two is this (S points at left circle in fig. 1) and two times two this (S points at right circle in fig. 1) two and two (S smiles) – |
| 13 | T | (T shrugs, then smiles) Erm, two times two – where does it say that? |
| 14 | S | Down there. |
| 15 | T | Erm. And do you know, what it means, if it is written down there? |
| 16 | S | (S pauses for five seconds) If it says 2 times 2 below, then we must do “times two” above. |

Transcript 1: “What has happened...?” (translation by the author)

Before we analyze the potential to act, let us first summarize the situation. The lines 11 – 13 point at the crucial situation. The student is asked to explain how the process of extension by two is visualized in the picture. She is expected to say that this is done by refining the given fraction. While the teacher explains to S after line 16, what the answer should have been, S is looking out of the window and seems frustrated. S does not engage herself anymore in the rest of the supporting lesson and is very serious.

We reconstruct S's potential to act in three steps. First, let us consider the epistemic actions. There are a number of recognising actions in lines 2, 4 and 6. S recognises the calculation in the numerator and the denominator of the fraction in the blue box in fig. 1. She also recognises the left circle as a half and the right circle as two quarters (8) and is able to relate them to the corresponding fractions in the calculation. In line 11 she is asked where to find the calculation inside the blue box in the picture. After a short pause, she identifies "one times two" as the left circle, and "two times two" as the right circle. This should be considered as a building-with action, because she puts together the things she has already recognised and she has to think about this question. In line 16 she also builds-with, because she states a general rule for the objects. Unfortunately, we do not know why she thinks this rule is valid.

What about the motivational component in this situation? There is no experience of autonomy in this transcript, because the task is very explicit and she has not been given much choice how to deal with it on her own. But we can see some experiences of competence here. She is able to identify the fractions in lines 8 and 10, and the teacher supports her by saying "exactly" and "fine". This experience of competence is deepened by S's answer to the question in line 11. S thinks for 17 seconds and manages to give an answer that makes her smile; she seems content with her own abilities. But the reaction of the teacher (shrug) and the teacher's later explanations reverse this experience of competence into the opposite. S realizes that her answer was wrong and may feel even more incompetent because she did not manage to understand that this answer was wrong. Likewise the need for relatedness might be fulfilled by the support S gets from the teacher and the smiling, a bit later this support might seem hollow and misleading. In summary, none of the three innate needs is satisfied here.

Using Oerter's layers of object relation we may interpret this episode further. For S the calculation is not one object, but likely she thinks of a pair of objects, i.e. two separate multiplications. Therefore she looks for a corresponding pair of objects that are given by the two circles in fig. 1. She uses the name "extend by 2" only once in line 6 and it may just be, because it is written on the sheet. Given she names the process of extension on her own, then her relation to this process as an object is in the second layer. But she does not even seem to be able to identify this process as an object of its own right (Oerter's first layer). Thus, her relation to the object "extension by 2" is somewhere between the first and second layer. Line 16 indicates that she

might actually be closer to the second layer, but we do not know, why S thinks one “must do ‘times two’ above”. We do not know whether she is really able to understand this extension as an object of its own right, i.e. as a process that transforms one fraction into an equivalent one.

In summary, S is involved in the situation up to line 16, recognises and builds-with the corresponding mathematical objects. Her innate psychological needs are satisfied up to here. Since S is not able to identify the calculation in the picture correctly, T starts explaining how to understand the picture after this episode, which leads to the experience of incompetence for S. Using the layers of object relation we argue that S cannot correctly identify the extension process for the circles because she is only partially able to think of the extension by two as an object. Thus, she cannot recognise it or build-with. Moreover, this information in mind future supporting lessons can be planned to foster S in the transferring to the next layer of object relation.

The analysis above demonstrates that the use of only one theoretical perspective is not enough to understand the data in sufficient generality for the given purpose. Using the RBC-model we saw that S built-with the structures she recognised, i.e. she was engaged in the process so far. SDT can explain why her engagement stops and in terms of the layers of object relation we can understand her epistemic problem and why she could not construct or reconstruct the concept of “extension by 2” in the given situation. Leaving out one perspective results in serious loss of information, e.g., if the SDT was left out, we would know the epistemic problem but could not explain the sudden change in S behaviour.

SOME PRELIMINARY FINDINGS

It should be kept in mind that the following results are only some preliminary findings from the explorative study. They should be thought of as hypotheses for a larger study to be tested.

Low achieving students seem to make use of a large repertoire of avoidance strategies in order to cope with given tasks. Especially, if their basic psychological needs were not satisfied the students responded by withdrawal, denial or similar actions, as seen above.

Furthermore, the students’ potential to act seems to be very dependent on the situational context. Frequently, their relations to the objects were found to be at the first or second layer, hence, the students had no abstract understanding of the objects. If the object relation was at the first layer, the students were not able to recognise the objects and thus could not do building-with actions. At the second layer students frequently developed different versions of an object depending on the context, e.g., a student had developed two different and unrelated object relations of a hexahedron having only the name in common.

TOWARDS THE USE OF THE DIFFERENT THEORIES

Prediger, Bikner-Ahsbahs & Arzarello (2008) suggest a landscape of strategies for connecting theories, which can be ordered by the degree of integration of theories. I shall now explain where the position of my approach in this landscape is.

I use the three theories as a way to understand the different dimensions and aspects of one concept. In terms of Prediger et al. I combined the different parts here “in order to get a multi-faceted insight into the empirical phenomenon in view” (Prediger et al., 2008, p. 173). It may even be that I coordinated, i.e. developed “a conceptual framework built by well-fitting elements from different theories” (ibid., p. 172). For this “a careful analysis of the mutual relationship between the different elements” is necessary and it “can only be done by theories with compatible cores” (ibid., p. 172). To decide the question whether I combined or coordinated let us consider the relationship of the theories:

From the broadest perspective, we have two psychological theories (SDT and the theory of action) and a theory originated in mathematics education research (RBC). SDT and RBC focus on the individual, Oerter’s theory on social interaction, but there is no obvious contradiction at this level between these approaches.

The epistemic actions of the theory of abstraction in context have their roots in activity theory (Pontecorvo & Girardet, 1993). Oerter’s concept of action is also motivated by activity theory and as far as foundations and basic assumptions are concerned, both theories are compatible.

How do these theories relate to SDT? SDT is a theory in cognitive psychology and at its core are the three innate psychological needs, which act as inner regulation processes that regulate and determine behaviour:

“SDT describes and predicts the occurrence of distinct processes by which behavior is determined or regulated, some of which are characterized as autonomous and some as controlled or amotivational. We assume not only that these forms of regulation differ experientially, but they also differ in their antecedents, their consequences, and their neuro-psychological underpinnings.” (Ryan & Deci, 2000a, p.330)

It seems impossible to express the above quotation from Oerter’s point of view. His fundamental critique is that action should not be thought of as an intentional but as the primary concept in psychology (Oerter, 1982, p.102). Every other concept has to be developed based on and connected to action. It is not clear to me, whether this implies contradicting basic assumptions, since the notion of “behaviour” by Deci and Ryan is not compatible with Oerter’s actions.

What are the relations between different terms in the theories? The potential to act is a concept defined in the notions of Oerter’s framework. The epistemic actions are expressed in terms of activity theory and can be understood in Oerter’s framework without any change. The three innate psychological needs are defined through experiences of the subject that are the results of certain actions. Autonomy, for example,

was defined as the experience to be able to initiate learning processes and decide about them. This experience is the result of a successful initiation or decision action by the individual itself or by the social group, e.g. the class. In this way the potential to act and all terms used to investigate it can be coherently expressed in terms of the theory of action.

Since the main difference between coordination and combination of theoretical frameworks is whether the theories are compatible, which includes non-contradicting assumptions, I cannot say which one I did, although I have built up a coherent philosophical base above.

SUMMARY AND OUTLOOK

In this paper I presented the definition of the potential to act and applied it to an example using empirical data. It was utilized and helped to gain insight in the process of the construction of knowledge and the motivational aspects of it.

The interplay of the three theoretical parts in the potential to act was described and I tried to position myself into the landscape of connecting theories following Prediger et al. (Prediger et al., 2008).

Bearing in mind the difficulties I had to find the position of my approach, I ask what the meaning of the notions “compatibility of theories” and “non-contradicting cores of theories” is. Does it mean a theory is compatible with another one just because their terms are incommensurable? When do basic assumptions contradict? Cobb (Cobb 2007) remarks that there is no algorithm how to deal with different theoretical perspectives. I suppose that there is also no algorithm to guarantee enough compatibility such that one has not build up “inconsistent theoretical parts without a coherent philosophical base” (Prediger et al., 2008, p. 173), but there might be general strategies which can serve as guide lines for the process of analyzing compatibility.

The “potential to act” is part of my research on low achieving students. The long-term goal is to have a theory of support for low achievers which builds upon the enlargement of the potential to act. A first explorative study has been done on this and my next step is to use the experience gained there in a larger study on support for low achieving students.

NOTES

1. Hauptschule is a secondary school for children, which are supposed to be in the lowest achievement category
2. It should be noted, that these layers are simplified versions of Oerter’s layers adapted for the purpose at hand.

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OUTLINE OF A JOINT ACTION THEORY IN DIDACTICS

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ABSTRACT

In this paper my goal consists of presenting aspects of the Joint Action Theory in Didactics on the principle of a twofold specification (Didactic Game and Learning Game), after integrating it in a more general picture. I first make a general presentation of the epistemological background against which the Joint Action Theory in Didactics could be seen. Then the second part of the paper is devoted to the description of a system of tools which constitutes the core of the JATD. In the third part, I give an example of empirical analysis in order to illustrate the categories presented previously. In the last part of the paper, I make some conclusive remarks in order to contribute to the networking process that this group is elaborating.

INTRODUCTION

In this paper I present some aspects of a collective work (Sensevy & Mercier, 2007; Schubauer-Leoni, Leutenegger, & Forget, 2007; Ligozat 2008), which functions as a *collective thought* from which I take most of the ideas I express in this contribution.

1. THE JOINT ACTION THEORY IN DIDACTICS: AN EPISTEMOLOGICAL BACKGROUND

1.1 The logic of practice, language-game and semiosis

In Social Sciences, the main challenge is probably to understand the meaning-making process in practices, thus understand the logic of practice on which people base their behaviors. In our conception, acting according to the logic of a practice is to be able to master a specific language-game in a particular life-form (Wittgenstein, 1953/1997). In order to master this language game, one has to be able to decipher signs of various kinds in an appropriate way. Acting according to the logic of the practice is therefore to be able to participate in a specific *semiosis* process (see Lorenz, 1994). To do that, people have to draw the same conclusions from a given environment, to give the same meaning to the prominent features of this environment. Inside this frame, I argue that the fundamental meaning-making process is an inference process, by which one can grasp and express the logic of the practice, and, doing that, can demonstrate understanding and agency.

1.2 The inference-reference process: institution and thought style

I assume that meaning is mainly processed in analogical inferences. In order to understand how these analogical inferences are made, one must consider that they are processed in context, the analogies being produced from a context to another. A theoretical point is thus to characterize what is a context, that I consider as an *institutional milieu*. Such an institutional milieu can be viewed as a specific reference, a background against which the agreement on inferences (“joint inferences”) is made. Lan-

guage-game mastering, semiosis process, and inference-reference strategies in an institutional milieu gather in the process of *recognition of forms*, which is the central feature of our conception of cognition and language. A way of conceptualizing the inference-reference process occurring during this ongoing attempt of recognition of appropriate forms is to consider meaning-making as unfolding in institutions (Douglas, 1987, 1996), which produce thought collective and thought styles (Fleck, 1934/1979). A thought style can be viewed as a kind of shared semiosis, by which people infer similar meanings from signs perceived in a same way, in a common recognition of forms. This common recognition of forms can be seen as a *seeing-as* (Wittgstein, 1953/1997), which is a habit of perception, and make possible the *joint-inferences*. The whole teaching-learning process can be viewed under this description (Sensevy, Tiberghien, Santini, Laubé & Griggs, 2008).

1.3 The logic of practice: the grammar of situations

In analyzing the social world, our concern is a *grammatical* one. We do think that every practice is unfolded according a specific logic, which over-determinate a great deal of it. Thus, as researchers we take a *grammatical stance*, which means that we try to understand the specific situational logic, the *peculiar grammar*, of a given practice. This concern logically stems from the conception of cognition and language we outlined below. If meaning-making is a matter of recognition of forms which are given by the collectives we are in, the description of meaning-making process rests on the identification of such forms, that is, a grammatical perspective. We must point out that a general way of understanding the logic of the practice lies in the comprehension of the situations in which this very concrete practice unfolds. The logic of practice is the logic embedded in the situations of practice. This kind of description helps understand why the meaning making process is viewed as mainly analogical. If the logic of practice is determined by the logic of the situations of the practice, meaning is made by relating the actual situation in which we are acting to the previous ones which resemble to the current one.

1.4 Game, situation, institution

In order to describe the grammar of the situations, we use a way of describing the social world in terms of *games*, by developing a “bourdieusian” perspective (Bourdieu, 1992). We consider the human activity as developing in games. By using the notion of game, we may use the following descriptors: the stakes of the game; the investment of the players in the game; the “feel for the game” that the players can or cannot display; the different kind of *capitals* related to the different games, that is, a way to acknowledge power phenomena in the social world. Thus the game is for us a fundamental grammatical structure, as a model of the social world, and also as a mean to relate institution and situation. Learning to act in a specific part of the social world is learning to play a certain game in situations embedded in institutions.

2. THE JOINT ACTION THEORY IN DIDACTICS: SOME TOOLS

2.1 The Didactic Game as a general pattern

We can try to describe the didactic interactions between the teacher and the students as a game of a particular kind, a *didactic* game. What are the prominent features of this game? It involves two players, A and B. B wins if and only if A wins, but B must not give directly the winning strategy to A. B is the teacher (the teaching pole). A is the student (The studying pole). This description allows us to understand that the didactic game is a collaborative game, a *joint* game, within a joint action (Clark, 1996). If we look at a didactic game more carefully, we see that B (the teacher), in order to win, has to lead A (the student) to a certain point, a particular “state of knowledge” which enables the student to play the “right moves” in the game, which can ensure the teacher that the student has built the right knowledge. At the core of this process, there is a fundamental condition: in order to be sure that A (The student) has really won, B (The teacher) must remain tacit on the main knowledge at stake. The teacher has to be *reticent* in order to let the student build proper knowledge, *her* proper knowledge. The teacher has to withhold information, because the student must act *proprio motu*. The teacher’s scaffolding must not allow the student to produce the “good behavior” without mastering the adequate knowledge. This *proprio motu* clause is necessarily related to the *reticence* of the teacher. Indeed, according to us, the didactic game, with the *proprio motu* clause and the teacher’s reticence, provides a general pattern of didactic interactions.

2.2 From the Didactic Game to the Learning Games

The Didactic Game refers to what we consider to be the fundamental grammar of the teaching-learning process. In order to deeply characterize this process, we use a system of concepts that we aim to unify under the notion of Learning Game. Learning Game, as a way of describing the Didactic Game as it occurs *in situ*, requires itself a structure of particular descriptors : the didactic contract/milieu doublet ; the genesis triplet (mesogenesis ; chronogenesis ; topogenesis) ; the game quadruplet (defining, devolving, monitoring and managing the certainty/uncertainty dialectic, institutionalizing). In the following, we will give some rapid descriptions of this system of concepts. First of all, a Learning Game can be identified by describing the didactic contract and the milieu referring to the piece of knowledge at stake.

The didactic contract and the milieu

We consider the didactic contract (Brousseau, 1997) according to a threefold viewpoint. The didactic contract can be viewed as an implicit system of mutual *expectations* (Mauss, 1989) between the teacher and the students, about the knowledge at stake, an implicit system of *joint habits* (Dewey, 1922) about this knowledge, and an implicit system of mutual attribution of *intentions* (Baxandhall, 1985). It is important to point out that this definition emphasizes the permanent features of the contract, and may explain the analogical process of meaning-making. We consider the didactic milieu under a 2 components description. On the one hand the milieu is a cognitive context, as a common ground, which notably provides the expectations and the mutual attributions of intentions on which the didactic contract rely. With this respect,

the milieu is a system of shared meanings which makes possible the joint action. But this kind of description is not efficient enough to provide a good understanding of the teaching-learning process. One has to acknowledge that in order to learn, the students have to encounter an antagonist milieu (that Brousseau called adidactic milieu), a kind of *resistance* to their action, which is also a resistance to the joint action. Thus this notion refers to the part of knowledge that the students cannot directly assimilate, which resists to their habits, and which prevents them to play the *right game*. The way in which the milieu provides such a resistance can be figured out (or not) *a priori* by the teacher, and even modelled by a researcher. It is important noticing that encountering the resistance of the milieu requires a certain grasp of consciousness. Indeed, by experiencing this resistance, the students have to encounter their ignorance, and the need for a specific piece of knowledge which will bridge this “ignorance gap”.

The dialectic between contract and milieu

When students try to play a learning game, some moves are directly given to them by the habits of action related to the knowledge they have recognized as the knowledge at stake. Some of these moves don't enable them to act accurately to meet the didactic situation requirements. In some cases, it is why they encounter a resistance to their action, and they just no longer play the game. It is critical to understand that these encounters and the shared awareness of their reality are a matter of joint action. Among all categories which are used for the description of learning games, the relationship between contract and milieu holds a prominent position. In order to characterize the didactic joint action, one has to identify how the students orient themselves, either by enacting the didactic contract habits or by establishing epistemic relations with the milieu. It means that empirical studies have to reveal what kind of dialectic is built between the “contract-driven students' orientations” and “the milieu-driven students' orientations”, in order to understand the Didactic Joint Action and the way mathematical knowledge is processed.

The game quadruplet

What we call “the game quadruplet” is a set of categories that we use to describe the way the teacher has the students playing the game in the joint action (Sensevy, Mercier, Schubauer-Leoni, Ligozat, & Perrot, 2005). *Defining*. The defining process can be viewed as a way of introducing the definitory rules of the learning game, in order for the students to be able to play this game. *Devolving*. When a game is defined, it has to be accepted by the students. That means that the students have to elaborate an adequate relation to the milieu. *Monitoring, managing the certainty/uncertainty dialectics*. The monitoring process refers to any teacher's behaviors produced to modify the students' behavior in order to enable them to produce the relevant strategies they need to win the game. In doing so, the teacher plays on the level of certainty/uncertainty of the students' action. *Institutionalizing*¹. In the ongoing didactic process, the teacher needs to recognize parts of the targeted knowledge in the students' activity as the relevant one for the learning game at play. In doing so, it

¹ The terms 'devolving' and 'institutionalizing' refer to Brousseau's concepts (1997).

makes the student understand that their activity reached the knowledge at stake, which is not only the “classroom knowledge”, but also the knowledge of a social community, which is larger than the school community.

At another scale and with other purposes, we consider a triple dimension that describes the teacher’s work, relative to starting and maintaining a didactic relationship (Chevallard, 1991, 1992; Sensevy, Mercier, Schubauer-Leoni, Ligozat, & Perrot, 2005) in the playing of the game.

The genesis triplet

Mesogenesis (i.e. the genesis of the milieu) describes the process by which the teacher organizes a milieu, with which the students are intended to interact in order to learn. *Chronogenesis* (i.e. the genesis of the didactic time) describes the evolution of knowledge proposed by the teacher and studied by the students, as it unfolds in the joint action. The teacher has to monitor the knowledge process through a lesson or several lessons, in order to meet his didactic intentions. *Topogenesis* (i.e. the genesis of the positions) describes the process of the division of the activity between the teacher and the students, according to their potentialities. The teacher should define and occupy a position, and enable the students to occupy their positions in the didactic process.

3. AN EMPIRICAL ILLUSTRATION

We focus now on an empirical example. The learning game occurred in an adidactic situation: the puzzle situation (Brousseau, 1997, p. 177) within a very large “didactic engineering” (N & G. Brousseau, 1987). I will make a first analysis of this episode, before trying a more general description of the same episode. The puzzle situation is a first situation for the study of linear mappings. It is put to students as following (Brousseau, 1997): “Here are some puzzles (Example: “tangram”). You are going to make some similar ones, larger than the models, according to the following rule: the segment that measures 4 cm on the model will measure 7 cm on your reproduction. I shall give a puzzle to each group of four or five students, but every student will do at least one piece or a group of two will two. When you have finished, you must be able to reconstruct figures that are exactly the same as the model”. *Development*: after a brief planning phase in each group, the students separate. The teacher has put an enlarged representation of the complete puzzle on the chalkboard.

In the studied episode, as usual in this case, the students have added 3 cm to every dimension. The result, obviously, is that the pieces are not compatible. The teacher comes to a group at this moment. We give the transcription of the dialogue between the teacher and the students.

The puzzle episode

1.	Student	There’s a problem it looks as if one is missing
2.	Teacher	There’s a problem, yes
3.	Student	But already here it’s leaning a lot here and then it’s there
4.	Teacher	Yes and it should be leaning in the same way?
5.	Student	Here we can see that the pike/point it touches the other one here again there is a problem and here it should be there it does like this there it does like this it would have been correct
6.	Teacher	And everywhere here you have added 3 are you sure you’ve added 3
7.	Student	yes
8.	Teacher	1,2,3, 1,2,3, 1,2,3
9.	Student	Well not to this one
10.	Teacher	1,2,3, have you added 3 everywhere?
11.	Student	Well it is correct

12.	Teacher	Then what must be challenged?
13.	Student	Well It's wrong well this piece is a good one
14.	Teacher	Well no it's not because it doesn't make up the good puzzle
15.	Student	Here it doesn't make 3
16.	Teacher	Where 3?
17.	Student	It only makes 2
18.	Teacher	Well 3? It's 3 more where?
19.	Student	On each side
20.	Teacher	If I were you I'd think about the method I used maybe this is what's not good
21.	Students	Yes
22.	Teacher	Maybe it's you're sure you've added 3 you didn't make any mistakes when you cut out the pieces, ok ? Everyone has cut on the lines?
23.	Students	Yes
24.	Teacher	Well so maybe you mustn't add 3 you must do something else
25.	Tony	But from 4 to find 7
26.	Teacher	Ah
27.	Student	There's a problem here too
28.	Teacher	Are you listening Tony
29.	Students	Yes
30.	Teacher	Go on try to look into this problem

3.1 The puzzle episode: a first description

A possible structure of the episode

In ST (Speech Turn) 1, the student acknowledges that “there is a problem”. We can analyze the excerpt by structuring it into four parts : in the first part, from 1 to 11, the teacher wants the children to agree that if there is a mistake, it is not a measurement mistake; the ST 12 (*Then what must be challenged?*) is the teacher's first try to give to the students an incentive to challenge their method, but without effect; in the second part, from 13 to 19, the teacher and the students return to the discussion of the measurement method, notably by arguing about what is a “good piece” (13-14); in the third part, from 20 to 26, the teacher takes a high topogenetic position, in order to focus the students' attention on the “proper signs” of the situation; in the fourth (last) part, one can think that the students are beginning to challenge their methods (25-27), so the teacher leaves the students and goes to another group.

Some teacher's moves in the Joint Action

We can focus on several teacher's moves in this excerpt. 1) In ST 4 (*It should be leaning in the same way?*), the teacher holds a “come-along position”, which means a low position in the topogenesis, at the same level as the students. We can think that a good student's answer could be something like “Yes, because the model and the reproduction must have the same dimensions, the same properties” (this answer would be based on the conservation of proportions), but the students do not really understand the question. 2) In ST12 (*Then what must be challenged?*), the teacher's move is produced in order to make the students understand that they have to change their way of conceiving the problem. It is worth noticing that this calls for a different position from the teacher: not a “come along position”, but an “analysis position”, in which the teacher does not use the same kind of reticence about his knowledge. But this move does not work, for the students go on discussing about their measurement. 3) In ST 20, (*If I were you I'd think about the method I used maybe this is what's not good*) the teacher takes a higher position, in a very interesting utterance: “*If I were you*” functions as a prominent sign in the didactic contract. For the students, that may

mean that the teacher is saying an important thing; by using the word “*method*” the teacher draws the students’ attention to the fundamental meaning in this episode; 4) In ST 22, the teacher makes a summary of the students’ work that one could paraphrase by saying “Are sure that your measurement was right?”. It functions as a kind of frame for an inference which could be: if you are sure that your measurement was right, then you have to challenge the method. 5) In ST 24 (*Well so maybe you mustn’t add 3 you must do something else*), the teacher draws herself the inference (if it is not a measurement error, then it is a method error). Tony’s reaction is very informative of his endorsing of the additive strategy; it’s a kind of encounter of ignorance. For the first time in the episode, the additive strategy is questioned, which may function as a sign for the teacher that the learning process is going on.

3.2 The puzzle episode: a re-description

Here the learning game takes place inside an *adidactic* situation (Brousseau, 1997)². First of all, the students have to encounter their ignorance, with the resistance of the milieu. In this learning game, as we have seen, they have to make a clear distinction between what is a *measurement error* and what is a *method (mathematical) error*. In order to move the didactic time forward, the teacher has to be sure that the students are convinced they have not made a measurement error. It is a necessary condition for them to challenge their method (i.e. the additive method). We can re-describe the episode using some theoretical tools of the JATD.

Reticence and proprio motu ; topogenesis and chronogenesis

The topogenetic characterization of this learning game enables us to understand how the teacher is progressively taken more and more responsibility in the didactic transactions. From a low topogenetic position (ST2, *there’s a problem, yes*), he reaches a rather high topogenetic position (ST 24, *Well so maybe you mustn’t add 3 you must do something else*). At the beginning of the episode, the reticence is very important, and the teacher does not unveil his didactic intentions. At the end of the episode, even if the teacher has displayed a part of his intentions, the reticence remains important. Indeed, nothing has been said about the proportional reasoning, which is at the core of this situation. The state of the milieu makes possible such an evolution, for there is a kind of agreement between the teacher and the students that the measurement is right. Thus we can acknowledge the specific interplay between chronogenesis and topogenesis in this rather short episode. The high topogenetic position is possible only because the didactic time - which is the knowledge time - has gone by, as we can see in the comparison of ST 2, 12, and 24. The teacher’s “feel for the game” enables her to accomplish gradually this topogenetic rising while keeping an effective didactic reticence.

² In order to be understood properly, this episode would have to be replaced in a more general structure, investigated at different scale-levels. We are focusing here on the micro-level of the didactic transactions, but a complete inquiry would necessitate a meso-level and a macro-level investigation (on this point, see Ligozat, 2008). This is a fundamental methodological issue for the Joint Action Theory in Didactics, which rests on the necessity to provide enquiry processes with a plurality of description levels, using for this purpose specific tools (in particular *synoptic table* and *didactic plot*).

The contract-milieu dialectic

At the beginning of the episode, the students are caught in the didactic contract enacted by the situation. As a student said, “from 4 to find 7” one *has to make* an addition. This “additive contract” could be considered as a thought style in this episode, which provides a way of perceiving and a way of acting. Another feature of the didactic contract at play could be found in a lack of experimental culture which prevents the students to distinguish the “measurement realm” from the “conceptual realm”, and which brings a kind of “experimental fuzziness”. Thus the present learning game stems from the students’ observation that the puzzle pieces do not fit together. This observation has to be seen as a resistance of the milieu, a relevant feedback for the modification of the students’ strategy. But this resistance is not obvious for the students, and the teacher’s work consists of helping the students “read” the milieu. For the researcher (and for the teacher as well) a fundamental aspect of this episode consists in acknowledging how the contract/milieu dialectic needs to be built in the transactions. The milieu feedback is not at all naturally perceived by the students. In the uncertain didactic transactions, what counts as an evidence for the teacher (the pieces do not fit together), which provides an accurate inference (the additive strategy does not work) is very far from the students’ relationship to the milieu, given that this relationship is shaped by i) the “additive contract” and ii) the “experimental fuzziness”. The students have to build another relationship, and they can’t do that alone. The teacher’s monitoring is fundamental to foster the students’ relevant relationship to the milieu and its events, which will enable them to “resist” to the contract habits and to renew them. In that, for the teacher, enacting the contract-milieu dialectic in the didactic transactions is a way of taming the uncertainty while building a relevant certainty, and enabling the students to accurately recognize the “empirical facts”.

4. NETWORKING MATHEMATICS EDUCATION THEORIES: SOME BRIEF CONCLUSIVE REMARKS

0. The Joint Action Theory in Didactics (JATD) is a didactical Theory. It responds to the fundamental definition of Didactics as a science: the science of conditions and constraints under which *the diffusion of knowledge* is enacted. In order to situate this theory (JATD) in relation with the Theory of Didactic Situations and the Anthropological Theory of the Didactic, we can argue that while these two theories first focus, from a logical point of view, on the nature of *knowledge* (what is knowledge which is taught?), the JATD first logically focus on the *diffusion process* (What is going on when a specific piece of knowledge is taught). This is what we may call *the actional turn* of the JATD. This difference of logic means a difference of problems: the kind of problems the JATD attempts to solve, in a bottom up process, are that of the *didactic action*.

1. Prediger (2008) proposes an interesting way of characterizing theoretical conceptualizations according to three types, as idealized poles: “individual learning”, “class teaching”, “institutional structuring”. In this perspective, it seems to me interesting to notice that a crucial point for the JATD consists in an attempt to understand how the institutions, in Douglas’ meaning (1987, 1996) shape the individuals’ personal life in thought styles (Fleck, 1934/1979). So, one can say that in the JATD the “institutional

concern” is the first one. It does not mean that the JATD is not interested in “individual learning” or in “class teaching”. On the contrary, we believe that the development of mathematics education theories needs a theory of didactic experience, if we call “didactic experience” these life events which enable people (and not only students or teachers) to gain knowledge as power of acting. But an essential feature of the JATD lies in the theoretical principle which assumes that meaning-making is mainly at work in the situations that institutions enact.

2. In the same paper, Prediger (2008) proposes another interesting way of characterizing studies with respect to the “prioritized types of research intentions”. Thus the studies are located on an axis from “improved understanding” to “improved practices”. As the other theoretical endeavors in French didactics, the JATD is rather on the “improved understanding” pole. But I would like to say that this type of reasoning could be dangerous, if researchers do not succeed in building a kind of normativity. This normativity, rationally and empirically grounded, could enable them to identify some principles in order to understand the didactic value of teaching-learning practices.

3. As a conclusion I would refer to Radford’s paper (2008) about the problems of networking theories. In this paper, Radford considers theories as “flexible triples” of “principles, methodologies, and paradigmatic research questions” (Radford, 2008, p. 322). He then argues that “If we dig deep enough, we will find that difficult to connect theories are more likely to have fundamental differences in their system of principles” (Radford, 2008, p. 325). As any theory, the JATD rests on some principles. It seems to me that it could be useful to distinguish epistemological principles, which represent a theory of knowledge for a given theory, from theoretical tools, which are used directly in the enquiry process. In a good deal of published papers, the epistemological principles in the background of the research, which one can see as the roots of the theoretical tools, are not really worked out. It seems to me very important to clarify these epistemological roots if we want to network theories. In this perspective, a primary concern, following Kidron et al (2008), could be to shed more light on the role of social interactions in theoretical approaches, with respect to their epistemological roots. As Kidron et al show, all the researchers agree on the importance of taking into account this type of interactions in their theoretical frameworks, but what is the meaning and the value of such an agreement ?

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THE TRANSITION BETWEEN MATHEMATICS STUDIES AT SECONDARY AND TERTIARY LEVELS; INDIVIDUAL AND SOCIAL PERSPECTIVES

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The aim of this paper is to illustrate how an empirical research interest in the transition between mathematics studies at secondary and tertiary levels generates a need for different theoretical approaches. From interviews with teacher students before and during their university studies in mathematics, three crucial aspects of the transition have been discerned; Mathematical learning objects, Mathematical resources and Students as active learners. Whereas the two former have both individual and social dimensions, the latter can be regarded as relational, constituting a link between the learning environment and the student in his or her intention to learn mathematics.

Keywords: teacher students, transition, individual, social, grounded theory

INTRODUCTION

My ongoing research project examines the transition between mathematics studies at secondary and tertiary levels, from now on termed “the transition”. This research interest stems from novice university students experiences with increased difficulties and changes in the conditions of mathematics studies at university, compared to upper secondary school. When novice university students begin their studies at university, they learn mathematics in a new learning environment. From a student’s perspective, this situation presents new challenges in terms of, or changes in, their knowledge, skills and self-image. Dynamic processes are going on, whereby students and their learning environment are mutually influencing each other. There are no obvious theories or methods at hand for dealing with this complex and extensive research area. Consequently, this study exemplifies the question raised by Arzarello, Bosch, Lenfant and Prediger of “how empirical studies contribute to the development and evolution of theories” (2007, p. 1620). Thus, an important part of the study has been to develop an analytical framework for the transition as seen from a student’s perspective. In this paper, I will give an account of the theoretical considerations this empirical problem brings to the fore.

TRANSITION-RELATED RESEARCH

Learning mathematics at university level is a well examined area. Many studies have focused on students’ learning and understanding of specific topics within university mathematics, for example limits of functions, derivatives, linear algebra and group theory (Dorier, 2000; Juter, 2006; Nardi, 2000). Other studies have considered how students struggle with advanced mathematical thinking, and with changes in the subject itself, including transformations from concrete and intuitive to more abstract,

formal and general forms of mathematics (Tall, 1991) or demands for new ways of approaching the mathematical content (Lithner 2003, Schoenfeld, 1992). A common characteristic of these studies is an approach that focuses primarily on the students as individual learners. From a more situated and cultural perspective, the issue of transition between different contexts of mathematical practices has been carefully examined by de Abreu, Bishop and Presmeg (2002). They define transitions as individuals' experiences of movements between contexts of mathematical practices. The transition as seen from a student's perspective can be captured by studying students' actions and interactions in a learning situation, looking for traces of conflict between different learning cultures, or variations of meaning that students ascribe to phenomena in the learning situation.

Artigue, Batanero and Kent (2007) suggest that research in learning mathematics at the post-secondary level must go beyond notions of for instance advanced mathematical thinking and also involve more comprehensive perspectives on mathematical thinking and learning. In their article, they refer to Praslon, who states that the transition cannot be defined as a shift from school mathematics to formal mathematics, or from an intuitive approach to mathematics to a more rigorous one. Instead, the transition is rather a question of an accumulation of small changes in mathematical culture. It is a shift from studying specific mathematical objects towards an extraction of mathematical objects from more general conditions. It is a change from applying specific algorithms to a category of tasks towards general methods and techniques. According to Praslon this is a consequence of the increment of the mathematical content to be learnt, and the impossibility of learning a specific algorithm for every kind of task in a relatively short period of time.

By gathering many research studies from different areas with different perspectives, it is possible to grasp a more complete picture of the transition. This has been done in a recently published study by Gueudet (2008), who states that the transition involves individual, social and institutional phenomena that call for different theoretical approaches. From my brief overview of transition related research it can be concluded that research concerning the transition has been conducted both from individual (von Glasersfeld, 1995), situated (Wenger, 1998), and cultural perspectives (Säljö, 2000). To examine the transition from a student's perspective, where the transition is defined as learning in a new environment in light of previous experiences is to simultaneously consider individual and social perspective on learning. Thus, the challenge is to combine an individual a social perspective on a local level within one empirical study.

From a more general point of view, this issue refers to the discussion of whether individual and social perspectives on learning can be unified. Cobb and Yackel made an important contribution to this debate with their *Emergent perspective* (1996). Their notions of sociomathematical norms and mathematical beliefs and values coordinate an individual and a social perspective on the collaboration between the teacher and the students in classroom environments. The strong emphasis on interaction in

the classroom can be regarded as a strength of this perspective. However, the transition from a student's perspective is not limited to the classroom. Instead, an essential part of the study must concern individual previous experiences of learning mathematics, requiring one to base findings on interviews. Thus, there is a mismatch between the methodological implications of Cobb and Yackel's Emergent perspective and the requirements of the research design of my study. My study requires a theoretical perspective that considers both an individual and a social perspective on the transition but from a methodological point of view, it requires more variety of data sources. Consequently, I was without a suitable theoretical framework and a pre-defined set of methods to follow to gather data and empirical considerations based on my definition of the transition had to serve as a starting point for the choice of research methods instead.

RE-ARRANGEMENT OF THE METHODOLOGICAL SEQUENCE

From a more general point of view this question also refers to a future challenge, raised during the Cerme 6 conference in Lyon, France, namely the discussion of how to find methodologies for networking theories, where the link between theory, empirical data and research results should be more highlighted. Methodological considerations link theoretical perspectives with appropriate research methods. Often, the formulation and intention of a research question is formulated within a theoretical discourse that results in a specific theoretical perspective. Thus, the research process, frequently used in mathematics education can schematically be described as follows:

Question → Theory → Method → Result

Or alternatively:

Theory → Question → Method → Result

Here, theory may refer to a more comprehensive theoretical perspective, for example a social or situated perspective, but may also refer to a more local theoretical framework as the Emergent perspective. The point is that often decisions about method seem to follow almost automatically once the initial choices of research question and/or theoretical perspective have been described. My research approach has been somewhat different. The starting point for my study has been a real world situation, from which the aim and the definition of the transition were developed. Because the definition of the transition - the students' learning of mathematics in a new setting in the light of their previous experiences requires the study to combine an individual and a social theoretical perspective, there has not been a given choice of methodological approach. Instead, my intention to study the transition from a student's perspective has been used as a methodical starting point, whereby the results contribute to new theoretical approaches and relations.

This approach can be summarised as follows:

Aim → Method → Result → Theory

With this rearrangement of the methodological sequence I want to emphasise how a real world problem implies a research process that ends up with a theoretical description of this phenomena. These descriptions have a local and specific character. However, based on their construction, they contain theoretical elements of both individual and social character. Thus, by studying them, conclusions can be drawn about how different theoretical perspectives come into play on a more general level. In accordance with my definition of the transition three main parts can be discerned, namely the students' previous experiences with mathematics studies, their learning of mathematics at university level, and the university as a new learning environment. To cover these parts empirically, I have collected different kinds of qualitative data from five teacher students during their first mathematics courses at university, i.e. individual interviews, observations from lectures and tutorials and written solutions to exercises and examinations. In this paper, I present some extracts from interviews with two of the students, Cindy and Roy. The pre-interviews were carried out after the students had enrolled at the university but before they had begun take courses in mathematics. The aim was to gain a picture of essential aspects of the students' understanding of mathematics studies in general and in particular of their experiences from upper secondary school. During their first courses in mathematics, the students were frequently interviewed to follow shifts in their thinking about mathematics and the learning of mathematics as they progressed through the courses. The interviews were audio-recorded and transcribed in full. Transcriptions have been analysed using methods inspired by Grounded theory (Charmaz, 2006). The data have been coded and sorted into categories, and axial coding has been used to analyse how the categories relate to each other. The result is a local theoretical description of essential aspects of the transition that could be discerned in the empirical data. However, these descriptions will contain aspects of individual and social theoretical perspectives from a more general point of view. How they interact within these concepts can also spread light of how different theoretical perspectives can be connected, coordinated, combined or networked.

RESULTS FROM INTERVIEWS

During the pre-interview, Cindy tells that she always liked mathematics and describes it as “her subject”. She particularly enjoyed solving equations, which according to her demands accuracy and concentration. In lower secondary school, she was one of the best in her class, but in upper secondary school, she experienced that mathematics became more difficult. In her last courses, she had to “struggle to survive”, and “integrals, strokes and such were not easy”. A mathematics lesson usually started with a 10-15 minute lecture about the type of exercises the pupils were to work with. Next, the students would work individually with exercises from the textbook. During mathematics lessons, Cindy would collaborate with two classmates in a spontaneous group. By working together on the same exercise at the same time, they could explain

to each other how to solve many exercises. To work on her own was meaningless to Cindy, because she would get stuck and could not continue on her own. When Cindy did not manage to understand the mathematical content, she simply tried to learn how to solve different types of exercises. She emphasises that there is a huge difference between knowing what to do and understanding mathematics, but her experience is that she often had to be content with the former. A new experience concerning exercises is that even if one finds the right answer, one cannot know if the solution is correct. For example, Cindy says that if she finds the limit of a function, she does not know if she has based her conclusion on the correct arguments or if she was simply lucky.

Cindy also thinks that another difference between mathematics studies at upper secondary school and university is that “it is harder” at university. She experiences that the mathematical content is more difficult and that everything is always completely new. During a mathematics lecture at the university an extensive amount of mathematics is covered, which results in many new things at the same time. This increases the risk of forgetting the first things that were said during the lecture. Cindy feels that the university teacher is good. When answering individual questions, he gives detailed explanations from the beginning. On the other hand, Cindy remarks that it is hard to get a straight answer or a simple explanation. Cindy feels that the most useful part of the lectures is when the teacher shows examples on the whiteboard, and when all steps in the solutions of the examples are demonstrated.

In the pre-interview, Roy tells that during upper secondary school he studied all available mathematics courses and got the highest grades. According to him, the first mathematics courses at upper secondary school were too easy. The majority of the mathematics consisted of using algorithms in a mechanical way and solving many similar exercises. This felt meaningless and bored him. It was not until later courses that Roy also met some challenges, which he defines as a need to “think for yourself”. He tells that probability was one of his favourite subject areas, because it offered the opportunity to reason logically and to try different solution strategies.

Roy remembered that mathematics lessons usually began with a short demonstration by the teacher. During the remaining part of the lesson, the pupils worked individually or in spontaneous groups, solving exercises from the textbook. Roy’s strategy was to look at the last exercises in the chapter. If he managed to solve them, he concluded that he could also solve the previous ones and that he had understood the content of the lesson. Most of the time, Roy worked on his own. However, if he did get stuck, he preferred discussing with his classmates instead of asking the teacher. He also frequently helped other students in his class and enjoyed explaining things to others. At university, Roy prefers working with peers rather than on his own, because it makes him more disciplined. From a social point of view, it is nice to meet with others and it makes studies more enjoyable. Often, he has solved more exercises than

his peers have, but Roy likes to help the others solve exercises and feels it is a good opportunity to review the mathematical content.

When Roy compares mathematics studies at upper secondary school and university, he says that the main differences at university are longer lectures, a higher tempo, less time to work on exercises during lessons, the importance of “being in phase”, and really understanding. Another difference is that mathematics is no longer only a question of understanding or not understanding; it is also necessary to read about mathematics and learn some things by heart. This results in a need to study mathematics, not only to work on exercises. It is also essential to truly understand what one is doing and not just work on exercises. Roy says that he is very satisfied with the teacher, who works thoroughly on “building up the concepts with understanding” and states that he can “buy his explanations”. He also states that understanding is more important than ever, because if he is going to become a teacher, he needs a deep understanding to be able to explain even to gifted students. He feels very highly motivated.

ANALYSIS OF INTERVIEWS

From the interviews with Cindy and Roy, portraits of two individual students appear with very different experiences and abilities for mathematics studies. In the following, I will give an account of three central aspects that can be discerned from the interviews and that seem crucial to mathematics education in a learning environment, namely mathematical learning objects, mathematical resources and student as an active learner.

There are a number of objects and relationships that play an important role in students’ mathematics education, for example the teacher, peers, the textbook and time. Cindy’s and Roy’s stories illustrate how these come into play in different ways and how they support their learning of mathematics. Thus, empirical data implies that students use both tangible and intangible issues to accomplish what they consider as learning of mathematics. Results also show that to obtain mathematical learning demands making use of different entities in the environment. The *Mathematical learning object* refers to the main target of mathematics studies in a wider sense from the student’s point of view. This concept captures the very essence of what students think that mathematics is and what should be learnt. Though Cindy and Roy study the same mathematics courses, they give very divergent descriptions of the subject. While Cindy feels that mathematics gets harder and harder, Roy characterizes the increasing difficulty as a stimulating challenge. Cindy’s statement about integrals and strokes can almost be considered drivel, which in turn indicates a superficial view and memory of the mathematical learning object. Students use *Mathematical resources* to obtain mathematical learning objects. In the interviews, Cindy and Roy explain how they collaborated with peers during mathematics lessons. However, while peers were an essential resource for Cindy to be able to solve exercises, peers rather had a motivational and self-confirmational function for Roy. Thus, a mathematical resource is

relational rather than absolute and is constituted by students' usage of it. Which mathematical learning objects students focus on and what they experience as understandable and meaningful can also be related to which mathematical resources the students are able to use. Different ways of interpreting mathematical understanding, their assignments and what it means to learn mathematics will also influence their mathematical study methods, which mathematical resources they choose to use, and how they view themselves as learners. One example that is worthwhile to examine further is their view of what a mathematical problem is, and what it means to solve it. Thus, it is plausible that how students perceive the mathematical learning object affects them as active learners, which in turn actualizes diverse mathematical resources and puts them into play in different ways.

There is a mutual relationship between mathematical resources and mathematical learning objects. Students use mathematical resources to obtain mathematical learning objects, but on the other hand, a mathematical learning object requires students' use of different mathematical resources. How they come into play depends on the characteristics of *Students as active learners*, which can also be discerned from the interviews. Students as active learners highlight the activities and actions they undertake to learn mathematics, and the intentions behind them. In the interviews, Cindy and Roy tell how they participated in the mathematics education and their thoughts and feelings about it. From these narratives, central aspects are, for example, the students' self-conception, motivation and identity. Cindy and Roy show clear differences between most of their learning activities, but they also carry out the same activity with different intentions.

As an example of how these three aspects interact, and how they interact in different ways for Cindy and Roy, I will return to an empirical example from the interviews. Even though Cindy wants to study mathematics, she often experiences the mathematical content as difficult. From her perspective, the content can be described as *inaccessible*. As a learner of mathematics, Cindy can be characterized as *dependent* with a view of the mathematical content as sometimes unmanageable and hidden. From her perspective, peers and teacher constitute a basic condition for her mathematical learning by helping her to find solutions to exercises and explaining things. By using them as a mathematical resource, she gains access to her mathematical learning object. For Roy, the mathematical content is *accessible*. To gain access is rather a question of his motivation for, and time spent on, studying. Roy can be described as an *independent* learner with a great portion of self-confidence in relation to the mathematical content. In his interaction with peers, they serve as a source of self-confirmation. Thus, peers as a mathematical resource have a more social and motivational character for Roy. The words *dependent* and *independent* as a description of students as learners and the *accessibility* or *inaccessibility* of mathematical content may be interpreted as inherited properties. However, this is not the way they should be understood. Instead, these characteristics are activated in the dynamic and inter-relational interplay between the individual and the social environment. The concept

of dependent-independent can rather be interpreted as an individual concept used with a social meaning. In the same way, the notion of access is used from an individual perspective. Thus, this example emphasizes and confirms that the transitions merge an individual and social perspective on learning. In relation to previous studies of mathematics studies at university level and the secondary-tertiary transition, it is obvious that the transition cannot be understood by limiting to learning a specific topic, ways of reasoning or advanced mathematical thinking. Instead, the interviews show that it is rather a question of an accumulation of small changes in the mathematical culture (Praslon in Artigue, Bataneri & Kent, 2007). However, these changes occur as a consequence of both changes in the learning environment and students' intentions and abilities to relate to them in a favourable way.

To further elucidate mathematical learning object, mathematical resources and the students as active learners, I will relate my analysis to the theoretical framework of Wenger (1998) regarding communities of practice. According to him, a practice is about meaning as an experience of situated activities. There are two interactively constituted processes involved in the negotiation of meaning within a practice, namely participation and reification. While the former is used in a common sense, the latter needs some clarification. According to Wenger, reification refers to "the process of giving form to our experience by producing objects that congeal this experience into 'thingness'" (Wenger, 1988, p. 58). Thus, reification is tightly connected with the creation of meaning in relation to concrete or invisible objects and entities in the surroundings. From the above description, a parallel between Wenger's concepts of participation and reification on the one hand and my concepts of students as active learners and the mathematical content on the other can be discerned, whereby the mathematical resources constitute an interface between participation and reification or as the bridge between students as active learners and the mathematical learning object. Thus, it is clear that the concept of mathematical resources is more embracing than simply referring to something that gives rise to cognitive conflicts for the individual student from a constructivist point of view (von Glasersfeld, 1995). Neither does a mathematical resource equal a sociocultural artefact (Säljö, 2000). Instead, mathematical resources must be considered relational and dynamic. They come into play in the interaction between a student's intentional actions to learn mathematics in an actual situation, surrounded by a specific learning environment. From Cobb and Yackel's "Emergent perspective" (1996), students as active learners and the mathematical content can be related to both a social and a psychological perspective at all levels, while the mathematical resources appear between the individual and social columns in their model.

CONCLUDING REMARKS

My intentions with this paper is to show how a research interest can give rise to new theoretical concepts that do not fit in more established theoretical frameworks about thinking and learning. The case in question concerns secondary-tertiary transition.

The emergence of mathematical learning objects, mathematical resources and the students as active learners are a result of my initial statement that the transition is best understood from both an individual and a social perspective. For example, a mathematical learning object can be constituted by a specific mathematical concept or entity, but the shape of the learning object and which mathematical resources the student uses are both a matter of individual pre-knowledge, identity and overall aim with his or her studies, as well as the learning situation and availability of potential mathematical resources in the setting. There is a constantly ongoing interplay between these individual and social dimensions of the transition. The dynamical aspects of these categories capture essential aspects of the transition from the students' perspective. The transition may change the students' roles as active learners by contributing to shifts in their intentions with learning mathematics and in their actions in different learning situations. In turn these shifts may change the students' use of mathematical resources and their focus on different mathematical learning objects. This captures the core of the transition from the students' perspective, but also elucidates the interplay between individual and social theoretical aspects, raised from a complex "real world situation" that lacks an obvious choice of theoretical approach. The next step is to analyse observations of students working with mathematics in tutorials and in clinical settings, both when they work alone, under the guidance of the teacher and in collaboration with peers. These analyses are to contribute to a more sophisticated definition of the concepts, which can be used to characterize different learners and their paths through the transition.

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COMBINING AND COORDINATING THEORETICAL PERSPECTIVES IN MATHEMATICS EDUCATION RESEARCH

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The author presents and discusses general issues related to combining and coordinating different theoretical perspectives and approaches in ongoing work on people's affective and social relationships with mathematics. The discussion is based on two concrete examples: Coordination of a sociological perspective (habitus) with an anthropological perspective (situated learning) in combination with a theoretical gender perspective on the analyses of qualitative data. The ambition of the paper is to bring a terminological clarification of differences between "perspective" and "approach" into the work on networking strategies for connecting theories.

INTRODUCTION

For the last 15 years a new international research field has been cultivated in the borderland between mathematics education and adult education. In order to study adults learning mathematics, conceptual frameworks and theoretical approaches has been imported from the two neighbouring fields and restructured (Wedege, 2001). Mathematics education research has welcomed and incorporated this new field where adult numeracy versus mathematical knowledge is continuously debated (FitzSimons et al., 2003). In this context, "diversity is not considered as a problem but as a rich resource for grasping complex realities" — as is stated in the call for papers from Working Group 9, *Different theoretical perspectives and approaches in research*, CERME6. As a consequence "we need strategies for connecting theories or research results obtained in different theoretical approaches", and Prediger, Bikner-Ahsbabs and Arzarello (2008) propose a terminology for dealing with this issue in the article "Networking strategies and methods for connecting theoretical approaches". As they state this is the "first steps towards a conceptual framework", which is based on the work in the Theory Working Group of CERME5:

The terminology of *strategies for connecting theoretical approaches* is presented as pairs of strategies (understanding others / making understandable; contrasting / comparing; combining / coordinating; synthesizing / integrating locally) within a scale of degree of integration from "ignoring other theories" to "unifying globally". The term *coordinating* is used when a conceptual framework is built by well fitting elements from different theories. This can only be done by theories with compatible cores. The term *combining* is used when theoretical approaches are only juxtaposed. This does not require complementarity or compatibility. Even theories based on conflicting principles can be combined. Finally, the term *networking strategies* is used to conceptualize those connecting strategies, which aim at reducing the number of unconnected

theoretical approaches while respecting their specificity (Prediger et al., 2008, pp. 170-173). In this paper, I also follow Radford (2008) when he suggests considering theories in mathematics education as triples $\tau = (P, M, Q)$, where P is a system of basic principles “which includes implicit views and explicit statements that delineate the frontier of what will be the universe of discourse and the adopted research perspective” (p. 320); M is a methodology supported by P; and Q is a set of paradigmatic research questions.

The research project *Adults learning mathematics in school and everyday life* is an example of effort to grasp complex realities by connecting different theoretical approaches and perspectives (see http://www.mah.se/templates/Page_____76536.aspx). Here, the purpose is to develop a comprehensive theory on conditions for adults learning mathematics, i.e. to establish an interdisciplinary theoretical framework to describe, analyse and understand the conditions of adults’ learning processes — including social and affective aspects (Evans & Wedege, 2004; Wedege & Evans, 2006). In the research process, we find the relational interplay between theoretical investigations and empirical studies crucial when developing the theoretical framework, and different connecting strategies are used. Below, strategies of combining and coordinating are presented with two examples from this work. In the article “To know or not to know mathematics – that is a question of context” (Wedege, 1999), two theoretical perspectives (habitus and situated learning) are coordinated in the analysis of the data from a mathematics life history interview. In the paper “A gender perspective on adults’ motivation to learn mathematics” (Wedege, 2008), a theoretical gender perspective was adopted in the analysis of existing qualitative data from a large English research project on adults’ reasons for studying mathematics.

In this paper, I present and discuss theoretical and methodological issues from the work in progress on people’s affective and social relationships with mathematics, drawing on the work of the CERME Working Group. The focus is on the influence of combining and coordinating different theories on the research process. But first, I shall propose a terminological distinction between a theoretical approach and a theoretical perspective.

THEORETICAL APPROACHES VERSUS THEORETICAL PERSPECTIVES

I adapt the understanding of “theory” as proposed by Prediger et al. (2008); i.e. the basic frame – or working definition – for discussion of conditions for connecting theories is “a dynamic concept of *theory* [or *theoretical approach*] whose notion is shaped by its core ideas, concepts and norms on the one hand and the practices of researchers – and mathematics educators in practice – on the other hand” (p. 176; my insertion and italic). According to this dynamic understanding, theories and theoretical approaches are constructions in a state of flux and theoretical approaches guide and are influenced by observation (p. 169). The notion of theory is broad when “theory” is synonymous with “theoretical approach”. A first consequence is that theory is not only a guide for thinking but also for acting – for methodology. In the article

“Theories of mathematics education: Is plurality a problem?”, Lerman (2006) examines the diversity of theories. He does not define “theory” but by looking at the examples and the proposed categorization of social theories within the mathematics education research community (1. Cultural psychology; 2. Ethnomathematics; 3. Sociology; 4. Discourse) it is obvious that Lerman’s understanding of “theory” encompasses methodology and even *problematique* understood as a paradigm for mathematics education research (cf. Wedege, 2001). This conception is in contrast to Niss (2007) who presents a static definition of theory as a stable, coherent and consistent system of concepts and claims with certain properties; for example, the concepts are organized hierarchically and the claims are either basic hypotheses and axioms or statements derived from these axioms.

Another consequence of “theory” and “theoretical approach” being used as synonyms is that “theory” is implicitly distinguished from “theoretical framework”, which does not automatically involve a methodology. The same goes for “theoretical approach” versus “theoretical perspective” and, in what follows, I shall suggest a terminological clarification of the latter pair.

I start by looking at the syntax and semantics of the two English nouns in the context of the debate in the Theory Working Group. According to the dictionary, “approach” is a verbal noun meaning the act of approaching (begin to tackle a task, a problem etc.). “Perspective” means a view on something from a specific point of view (seen through a filter) (Latin: *perspicere* = looking through). In our context, the noun does not have a verbal counterpart. The Danish verb “*perspektivere*” meaning “to put something into perspective” is not suitable here. In order to distinguish the two terms, I propose the following clarification: A *theoretical approach* is based on a system of basic theoretical principles combined with a methodology, as defined by Radford (2008), hence, guiding and directing thinking and action. A *theoretical perspective* is a filter for looking at the world based on theoretical principles, thus with consequences for the construction of the subject and problem field in research; that is the field to be investigated (cf. Wedege, 2001). For example, in the literature reference is often made to socio-cultural perspectives on mathematics education, simply meaning that social and cultural aspects of the educational phenomena are taken into account in research. Within the suggested terminology, it would not make any sense to talk about socio-cultural approaches without a reference to a specific theory, e.g. a socio-cultural approach – or *problematique* – like Engeström’s (2001).

In order to exemplify how different theoretical perspectives which share an emphasis on the social dimension in mathematics teaching and learning lead to different interpretations and understanding of a short transcript of students’ collaborative problem solving, Gellert (2008) compares and combines “two sociological perspectives” on mathematics classroom practice meaning. In order to “emphasise the theoretical grounds” of the two perspectives as he says, Gellert terms them “structuralist” and “interactionist” respectively. In this text, he is using the two terms “perspective” and

“approach” alternatively without any terminological clarification. However, it seems that his choice of terms is deliberate and that his usage matches the distinction proposed above. He is talking about theoretical and methodological “approaches to research in mathematics education” (pp. 216, 220, 222) and “research approaches” (pp. 220, 221), and he concludes:

The methodological approach I am sketching reflects a change of theoretical perspectives: Having identified relevant passages within the data material (from the structuralist point of view), these passages are analysed according to the standards of interactionist interpretation techniques (Gellert, 2008, p. 222).

In his discussion of the general issue of combining two theoretical perspectives, Gellert uses a piece of data – a short transcript of sixth-graders’ collaborative problem solving. He states that “by selecting and focusing on this particular piece of data I have already taken a structuralist theoretical perspective” because, from this perspective, the passage is “a key incident of specification of inequality in the classroom” (p. 223).

COORDINATING AND COMBINING THEORETICAL PERSPECTIVES

A consequence of the terminological distinction between a theoretical approach and a theoretical perspective suggested above is this: In the network strategy of combining, theoretical approaches and theoretical perspectives are juxtaposed and they do not have to be complementary or compatible. But, in the strategy of coordinating, where well fitting elements from different theories are built into a conceptual framework, I consider only theoretical perspectives and they have to be complementary or compatible.

When theories are combined, a subject area is studied with different theoretical approaches. The area is structured into different problem fields to be investigated and different results are produced. When compatible or complementary theoretical perspectives are coordinated, the subject area is studied from an integrated perspective and one result is produced. According to Prediger et al. (2008) the strategies of coordinating and combining theories are mostly used for a networked understanding of an empirical phenomenon or a piece of data. In the following examples the aim of the networking is partly this and partly directed towards developing a theoretical framework.

Coordinating theoretical perspectives

As an example of coordinating theoretical perspectives for networked understanding of a piece of data, I have chosen the analysis of a life history interview (Wedege, 1999). In a narrative interview with a 75 year old woman, Ruth, about mathematics in her life there is a contradiction which is well known in adult education: many adults resist in learning mathematics in formal settings while they are mathematically competent in their everyday life. This particular woman, who had really bad experiences with mathematics in secondary school, went to a Technical School to be a draughts-

man as 50 year old and she got the top grades in mathematics. But her dispositions towards having to do with mathematics did not change, neither did her beliefs about herself and mathematics. While some adults change their attitude to mathematics during a training course, others fail to do so. For some people, this means something for their image of themselves and their life project, for others not. These differences cannot be explained solely within the educational context and the students' current situations and perspectives. In order to expand the context for analysing learning processes and drawing a link to the lives lived by adult students, I have attempted to combine Lave and Wenger's concept of situated learning with Bourdieu's concept of *habitus*, i.e. systems of durable, transposable dispositions as principles of generating and structuring practices and representations (Bourdieu, 1980).

Lave and Wenger (1991) see learning as a social practice and the context of their analysis of learning processes is the current community of practice. The theory of *situated learning* is about learning as a goal-oriented process described as a sequence from legitimate peripheral participation to full participation. Throughout her life Ruth has participated in a number of different communities of practice (family, school, work, etc.). She learned a number of things in her mathematics lessons: that she was stupid at mathematics, that she was not interested in it, and that in any case mathematics had no relevance for her life. She was confirmed in this by never having failed in practical situations due to a lack of mathematics knowledge. When, much later in her life, Ruth got the highest grade in the subject of mathematics while being trained as a draughtsman, this did not change her idea of mathematics, the world around her, or herself. But the theory of situated learning does not present the possibility of explaining why her perception of herself had not changed, and why she never had any appreciation of mathematics.

Ruth's motivation to be a draughtsman made her overcome her blocks, but not her resistance to learning mathematics. Her intentions had changed but not her dispositions towards mathematics, incorporated through her lived life. According to the theory of Bourdieu, the habitus of a girl born 1922 in a provincial town as a saddler's daughter, of a pupil in a school where arithmetic and mathematics were two different subjects, at a time where it was "OK for a girl not to know mathematics", and the habitus of a wife and mother staying home with her two daughters is a basis of actions (and non-actions) and perceptions. Habitus undergoes transformations but durability is the main characteristic.

I have argued that the concept of habitus, developed and belonging in a sociological problematique as a concept of socialisation, can be coordinated¹ with Lave and Wenger's concept of situated learning in a problematique of mathematics education (Wedegé, 1999). In the first place, Bourdieu emphasises that the theory of habitus is not 'a grand theory', but merely a theory of action or practice (Bourdieu, 1994). The

¹ The word I used in (Wedegé, 1999) was "combined" and not "coordinated".

habitus theory has to do with why we act and think as we do. It does not answer the question of how the system of dispositions is created, and how habitus could be changed in a (pedagogical) practice. This means that the concept of habitus can be used in a descriptive analysis of the conditions for adults learning. Lave and Wenger's theory of situated learning is also a partial theory, a theory of learning as an integral part of social practice. They are precisely trying to find an answer to the question of how people's dispositions are created and changed through legitimate peripheral participation (Lave & Wenger, 1991). Bourdieu and Lave/Wenger both aim at challenging the dichotomies of subject-object and actor-structure. Both are critical of phenomenology and structuralism while simultaneously having social relations as the focus of their subject areas. Bourdieu set himself the task of constructing a theory of action as social practice and Lave a theory of learning as an integral part of social practice.

A common core – or basic principle – in both theories is the understanding of learning as social practice. Furthermore, the two theories reject the idea of internalisation of knowledge and attitudes/norms, respectively. They mention instead active incorporation. Thus, the theory of habitus, as a social practice theory, does not encompass the theory of situated learning, but I have argued that the two theories are compatible and that the concept of habitus, which is developed and belongs in a sociological problematique, can be imported into an educational problematique about adults' learning mathematics together with the concept of situated learning.

Combining these with a theoretical gender perspective

In the interview with Ruth, gender was an obvious aspect which might have been in the foreground of the analysis. The theories of habitus and of situated learning do not exclude gender aspects, but are a background dimension. In this section, I present another example of networked understanding of a piece of data – this time by combining the above with a theoretical gender perspective.

Complexity is a characteristic of the problem field in mathematics education, and diversity (gender, ethnicity, social class etc.) calls for multi- and inter-disciplinary studies and for different research methodologies. However, focus and methodology of any study are determined by its purpose, theory and research questions. For example Evans and Tsatsaroni (2008) have argued that research into gender within a social justice agenda requires both quantitative and qualitative methods.

When the research problem is formulated and the method and the sampling strategy are to be decided, the researcher has to choose among a series of factors and dimensions to reduce complexity. Gender is one of the aspects to be decided upon. In some studies, gender is a dimension in the *foreground*: the study is designed to investigate gender and mathematics – and gender is focussed in the purpose and the research question. In other studies, gender is a variable in the *background*: gender is just one independent variable among others.

Gender is in the foreground as an important analytical dimension in our on-going work on people's motivation and resistance to learn mathematics (Wedegé & Evans, 2006). So far we have not designed a new empirical study with gender in the foreground but we have access to rich empirical data from 81 semi-structured interviews with students (2/3 female and 1/3 male) from an English research project on adult students' reasons for learning mathematics, "Making numeracy teaching meaningful to adult learners" (Swain et al., 2005). In this project gender is in the background: none of the research questions are about gender but information about gender is available in the data. In a pilot case study with one of these students, Monica, I have tried to adopt a gender perspective for a small part of this data (Wedegé, 2008). The theoretical framework for this analysis consists of four analytical *gender viewpoints*² (*structural, symbolic, personal, and inter-actional*) (Bjerrum Nielsen, 2003). The analysis shows that the framework of gender viewpoints can be productive in locating gender in the data collected in the English project. The four gender viewpoints – separate or inter-connected – create new meanings to Monica's narrative.

From the structural gender viewpoint, gender constitutes a social structure, and men and women are, for instance, unevenly distributed in terms of education. For Monica, not having a high level of education has been a structural consequence of being a woman. As in many other families, girls were not educated in her family. They were brought up to fulfil traditional women's roles. Today, Monica is a single parent. In England – as in Scandinavia – the situation of being a single parent is closely connected with being a woman. Talking about reasons for attending the numeracy course, the students talked about the new governmental demands that single parents have to go back to work or alternatively go into training.

The core of our ongoing work is understanding motivation as a social phenomenon, which is also the case in the English project. Their theoretical framework is based on the work of, for example, sociologist Bourdieu and anthropologists like Lave (Swain, 2005 p. 31 ff) whom we have also used in our research. This theoretical choice had consequences for the questions asked to the students during the interviews, which in the case of Monica, for example, made it possible for her to talk about her childhood.

In the majority of studies in mathematics education, we find gender in the background. Hence, internationally, we have a large amount of data which has not been investigated from a theoretical gender perspective. In a recent overview of mathematics education research in Denmark and Norway, it was shown that very few studies were designed with gender in the foreground (Wedegé, 2007). However, a series of Nordic researchers intend to bring gender into the foreground and, through the latest 15 years, they have presented papers with a focus on gender. These presentations were based on data from their own previous research (quantitative or qualitative stud-

² The term used by Bjerrum Nielsen (2003) is "perspectives". However, due to terminological constraints from the discussion in this paper, I have changed the term into "viewpoints".

ies) with gender in the background. That is, the researchers returned to their “own” data with questions related to their original problem.

CONCLUSION AND PERSPECTIVE

Diversity of theoretical approaches and perspectives is a challenge in research on adults learning mathematics, as in mathematics education research generally speaking. Inter-disciplinarity is also a significant feature of this field where theoretical frameworks are imported and restructured (Wedegé, 2001). However, the researchers often import concepts from other disciplines, like psychology, sociology and anthropology, without any reflections on the process of import, integration and restructuring of the framework. Hence, there is a need for strategies for connecting theories from disciplines. Another problem is terminology and I see the present work, on developing terminology in parallel with strategies (Prediger et al., 2008), as very important in terms of quality. Hence, I hope that the proposed clarification of differences between the two terms “theoretical approach” and “theoretical perspective” will be adopted in the continuation of this work.

As mentioned above, the purpose – or the overall aim – of the research project “Adults learning mathematics in school and everyday life” is to develop theory, thus research with a *top-down profile* (cf. Arzarello et al., 2007). But if we look at the research process beginning in the 1990s, the aim of networking theories in the studies of adults learning mathematics alternates between top-down development and *bottom-up development* with the aim of understanding a concrete empirical phenomenon. The theoretical investigations and constructions iterate in continual interplay with empirical studies. In Wedegé (1999), the aim of coordinating theories is understanding and explaining a concrete empirical phenomenon combined with intentions of theory development; in Evans & Wedegé (2004) and Wedegé & Evans (2006), the purpose is conceptual clarification and development; and in Wedegé (2008), the intention is to combine with a theoretical gender perspective to revisit empirical data for new purposes. The aim of coordinating theoretical perspectives on habitus and on situated learning was to understand and explain a mathematical life history. But the arguments for compatibility of the two perspectives were general and not restricted to the data. In this and in the other studies, the development is driven by the concrete study combined with a general interest.

Combining and coordinating theories are steps on the road towards networking theoretical approaches in a new theory, but it is too early to say if our final networking strategy will be *synthesizing* between two or more equally stable theories or *integrating locally* some concepts or aspects of one theory into another more elaborated theory.

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COMPARING THEORETICAL FRAMEWORKS IN DIDACTICS OF MATHEMATICS: THE GOA-MODEL

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In this paper we propose a meta-model for comparing different theoretical frameworks in didactics, focusing on three components of the study object of didactics: a set of human beings with relations (e.g. students and teachers in a classroom), an organisation of human practice and knowledge, and a set of artefacts used to mediate and relate the previous two. We argue theoretically and through an example (related to the transition from secondary to tertiary education) that this meta-model helps identifying complementarities, similarities and differences among four leading theories or models of the didactical field, and thereby to facilitate rational justifications for selecting a theoretical framework with respect to a given purpose of research.

1. INTRODUCTION

The comparative study of theoretical frameworks in didactics of mathematics (for short, *didactics*) was the subject of a special issue of ZDM (no. 40, 2008), drawing on papers and discussions from working groups at CERME-4 and CERME-5 (cf. ermeweb.free.fr), as well as on other papers, many in previous issues of ZDM. Prediger et al. (2008, Fig. 1) subsumes the “landscape of strategies for connecting theoretical approaches” as ranging from “ignoring other theories” to “unifying globally”, between which we find intermediate positions for “finding connections as far as possible (but not further)” that the authors call “networking strategies”. Some consensus seems to have emerged to pursue the latter type of strategies, while considering the uses of a small number of theories (mostly 2-4) in concrete “cases” for research, such as studying or developing a classroom design based on a simple task. A general “metalanguage” to compare theoretical frameworks was proposed by Radford (2008, 320): a theory is considered as based on a triple consisting of a set of implicit and explicit principles of the theory, a methodology and a set of paradigmatic research questions. This idea seems to be applicable to theories in any field of research, and focuses essentially on aspects of the *epistemology* afforded by theories.

This paper proposes another, possibly complementary, approach to the issue: namely to compare the characteristic ways in which different theories build models of *the object of study in didactics*. The basic hypothesis is that significant differences among theories of didactics come from *focusing on different phenomena* within the complex reality of mathematics teaching and learning. In short, we propose a meta-model for the *ontology* of the theories, understood as the models they propose of their object.

2. EPISTEMIC SYSTEMS – THE GOA MODEL

Every science is about “something” – the *objects of study*. For an empirical science like didactics, which sets out to study a certain realm of mental, social or physical entities, the objects of study are delimited and to a certain extent constituted by the

development of theoretical *models*. Such models are more or less *systemic* in the sense that they imply relations among the objects; models are not simply lists of independently defined objects.

Without assuming (or saying) much, the “object” of didactics can be loosely described as the teaching and learning of a specific knowledge domain. Teaching and learning implies subjects who teach and learn – that is, teachers and students, or more generally a *structured group of people* (where structure implies that members of the group may have different roles and relations to each other, such as being teachers or students). The knowledge domain itself can be modelled and analysed as a coherent *organisation of knowledge and practice*. Finally, knowledge and “knowers” (be they teachers or learners) cannot be related without *artefacts* of different forms (texts, media, other tools and materials of various sorts). Given these basic observations we suggest that the systems of objects studied in didactics can be described as a triple

$$(G, O, A)$$

where: G is a group of people structured by a certain set of relationships, O is an organisation of knowledge and practice which G enacts, and A is a set of artefacts which G uses to access and communicate in and about O . Notice how relations on $G \cup O \cup A$ are crucial not just to study but also to define the triple. We call such a triple an *epistemic system* (ES) because the system involves use, circulation, development or even production of knowledge. Of course, not all ES are likely to be objects of didactical research, but surprisingly many types could need to be taken into account.

An ES may be considered in synchronic and diachronic ways, corresponding to a snapshot of its state at a given time (or a shorter period where it can be considered as relatively stable), and to its development over a period of time. It is also important to notice that (G, O, A) may be considered as general systems corresponding to an *institution* (e.g. a professional community or workplace) where the artefacts may include such diverse objects as buildings, tools, texts and so on, giving identity and delimitation to the institution. Finally, an ES may be naturally divided into “subsystems” (G_i, O_i, A_i) , such as different divisions within a workplace.

Here are four special cases which are of particular importance in didactics, in themselves and in interaction; they also show how varied phenomena ES include:

2.1. Didactic systems may be described as the case where G consists of one or (rarer) several *teachers* and a class of students, engaged in the teaching and learning of a knowledge organisation O while mobilising, possibly in different and changing ways, a set of artefacts A (including objects within the classroom). The knowledge organisation could be based on one or more problems or questions, mediated and tackled using A , and potentially mobilising or enabling the construction of the “intended knowledge or practice” (also part of O). In fact, these intentions – of the teacher(s) – are an important factor in didactic systems, but it could take many forms.

2.2. *School systems* consist of a certain collection of didactic systems, e.g. with G comprising all students and teachers of a given school, or of all schools within a given region or country; the boundaries of a school system (as regards all three components) are sometimes institutional boundaries in the sense that they are defined quite explicitly, such as by law, or they could be considered pragmatically as (observable) systems of persons common aims, practices, and material surroundings.

2.3. *Teaching systems* are parts of school systems but with G being a group of teachers, who may work alone, or together, to construct or reflect upon one or more didactic systems. The knowledge organisations and artefacts involved in such systems may, of course, also be quite different from those involved in didactical or adidactical systems. For instance, teachers could be involved in developing or sharing teaching plans and other teaching material (artefacts) related to the teachers' knowledge and practice enacted within didactic systems.

2.4. *Noospheric systems* consisting of a group of people G involved in generating, delimiting or defining all or parts of the knowledge and practice organisations O to be worked on in didactic systems, using or producing artefacts to this end; for instance, G may be one or more authors of a textbook (part of A) aimed to support O , or a group responsible for producing standards for school systems (A in this case involves documents setting up requirements or recommendations regarding the practice, target knowledge and artefacts of these). The term *noosphere*, originally coined by Chevalard (1985), ironically refers to the "thinking about" school systems which takes place outside these systems from a peripheric yet superior position.

3. The GOA model as a "meta-model" for comparing theories in didactics

The above model of ES can be thought of as a *meta-model* since its use in practice requires finer models for each of its components and their interrelations. Supplying these details, we recover several "real" models or theoretical frameworks commonly used in didactical research. We now do this for some important ones, familiar to us.

3.1. *The theory of didactical situations* (TDS, cf. Brousseau, 1997) considers, as its primary objects, *didactical situations* evolving around *didactical milieus* and regulated by *didactical contracts*. The situations are themselves modelled as the interplay between students and teachers (forming G) and the milieu, which in turn is a compound of both material elements (forming A) and a particular organisation O_M of practice and knowledge. The system as a whole is analysed in terms of a wider organisation O of intended and prescribed forms of practice and knowledge, which includes also a dialectic between *personal knowledge* of the different members of G , and *shared knowledge* which develops over time. This means that the entire didactic system (G, O, A) is considered diachronically, albeit mostly over shorter periods (corresponding to a lesson or a sequence of lessons). The didactical contract consists of (mainly) implicit rules which govern the whole system, in particular the interactions within G and between G and the milieu. In sum, this theoretical framework models G

as consisting of a teacher and a group of students, with different relations to both O and A , a relation which varies over time and is interpreted as being governed of a rule system (contract) corresponding to expectations and obligations of the members of G . It can be said to be more “naturalistic” as regards G and A as such, and focuses particularly on the evolution of the relation between G and O . Moreover, diachronically, the theory focuses on subsystems existing at times where the teacher does not interact with the students, called *adidactical situations*; this refers to shorter time spans for a didactic system, which at other times involves interaction between teachers and students.

3.2 *The anthropological theory of didactics* (ATD) involves highly intricate models of O (mathematical and didactical organisations, cf. Chevallard, 2002), corresponding to forms of practice and knowledge related to mathematics and the teaching of mathematics, respectively. More precisely, it models both of these as organisations of *praxeologies*, each of which consist by definition in a quadruple (type of task, technique, technology, theory). Praxeologies are organised at various levels according to the techniques, technologies or theories they share. The researcher constructs a *reference model* to observe and analyse these organisations within different systems. Among artefacts explicitly considered in this theory are *ostensives* mediating and embodying the techniques and technologies of O , including also discursive media and tools. In this theory, G is mostly implicit, except for the strong emphasis on *institutions*, viewed as the human ecologies in which praxeological organisations live and between which they are transposed. The theory also contains a structured view of institutions successively determining each other at different levels (Chevallard, 2002), from a didactic system considered synchronically (e.g., Barbé et al., 2005), to the noospheric systems (including the level of societies) considered in diachronic development (e.g. Chevallard, 2002). Finally, a recent development in this theory, to describe the long term developments of didactic systems, is Chevallard’s notion of *research and study programme* (see eg. Barquero et al, 2006), focusing again on O but with a community of learners G being perhaps more explicit in recent empirical studies of how such a programme evolves (ibid.). However, even more than TDS, the ATD focuses primarily on the analysis of the O component.

3.3 *Socio-constructivist theory of mathematics learning* (SCT) exists in many forms and variants; we consider here the approach to didactic systems developed by Cobb and associates (e.g. Cobb et al., 2001). As the name suggests, the model has dual roots (ibid., 119-120): on the one hand, in *constructivist* learning theories going back to pioneers such as Steffe, Skemp, and ultimately Piaget; and in *socio-cultural* theories, with a lineage involving names such as Bauersfeld, Lave and Vygotsky. The idea is to study the learning – in particular mathematics learning – of participants in a classroom situation (students, teachers and even researchers) both as individuals and for the group collectively within a socio-cultural context. In fact,

there is an extremely strong relation between what we have described as the social and psychological perspectives that does not merely mean that the two perspectives are interdependent. Instead, it implies that neither perspective exists without the other in that each perspective constitutes the background against which mathematical activity is interpreted from the other perspective (ibid., 122).

The researchers' interpretation of classroom activity aims to clarify this dynamics of (mainly students') *individual beliefs* and *sociomathematical norms* developed and shared by G collectively. It is based on careful analysis of video recordings of classroom activity (as a primary form of data, among others such as field notes and interviews). This allows for observing not only *discursive* and *embodied* practices related to a mathematical task, and thereby the emerging organisation O of practice and knowledge found in the classroom, but also the role played by artefacts (discursive, semiotic, material...). It is important to note that while G and A are theorised e.g. as *communities of practice* and *semiotic ecologies* (ibid., 153), the individual and shared knowledge organisations (including beliefs and norms) are considered to *emerge* from the interaction within G and between G and A : *we take the local classroom community rather than the discipline as our point of reference* (ibid., 120). In other versions of SCT, such as Ernest (1997), a wider perspective is adopted.

3.4 The cognitive-semiotic theory (CST, cf. Duval, 1995) focuses on the relationships (mental schemes or processes) which exist for the members of G between a collection of signifiers (primary elements of A , organised in semiotic systems) and signifieds (mathematical objects within O). The fact that these relationships may be different for different members of G (and develop over time) is explicated in variants of this model through a *triadic* model of the sign relationship, including also the different interpretations or schemes for the relationship between semiotic artefacts and their "meaning". Particularly important for the objects of mathematics is *multimodal representations*, which occur in two distinctive forms (cf. Duval, 2000): different representations of an object within the same semiotic system (*register*), which are obtained by *treatments*, sometimes based on complicated algorithms; and representations in different *registers* (like a function being represented symbolically and graphically), obtained from each other by *conversion*. Coordination of different representations of a given mathematical object is a key requirement in many common mathematical tasks. Notice that this model may be applied to all kinds of ES, but with a special focus on "semiotic" artefacts and the corresponding schemes, and sometimes relatively implicit models of O (although for the case of mathematics, the mathematical objects and their properties are often considered as *constructed* or even *consisting* in those schemes, cf. Winsløw, 2004).

3.5 Comparison. The above four "snapshots" of theoretical frameworks enables a first comparison of them, as the modelling of certain *parts* of (G , O , A) occupy the foreground within each of them. In TDS, the interaction of teacher and students (G) around the didactical milieu (part of (O , A)), in ATD the praxeological organisations

(O) in their institutional context (G, A), in SCT the community of practice (G) with its evolving shared norms and individual beliefs which contribute to determine O , and in CST the semiosis and associated schemes (part of (G, A)) as a condition for accessing and enacting O . To compare these theoretical frameworks, it is crucial to realise that they model, to some extent, *different parts* of a common reality (such as a didactic system). To a much lesser extent do we find apparent oppositions in their basic constitution, such as the deliberate absence in SCT of reference models for O “outside the classroom”, versus the strong emphasis on such models within ATD.

4. CASE: THE TRANSITION FROM SECONDARY TO TERTIARY

For about a decade, I have been studying the transitions problems which arise for students at the beginning of university programmes in mathematics, along with development projects aiming at enhancing the outcome of students’ work. The difficulties students encounter – and the strategies one may envisage to help them overcome those difficulties – may be approached using any of the theoretical frameworks considered in the previous section (as well as others, of course). In this section, a simple example will be used to show how contributions associated to each framework are different because they model the relevant ES with different foci and notions. Notice that Gueudet (2008) presents an overview of literature explicitly addressing the transition from secondary to tertiary, including more theoretical perspectives than those considered here.

Globally, transition concerns *students* who move from one type of ES, (G, O, A), to another one, (G', O', A'), in which there may be some overlap in all three components, including (by definition) the students themselves within $G \cap G'$. An obvious place to locate the obstacles for students within (G', O', A') is in the set of practices and knowledge components O' which they have to acquire, as opposed to those they have previously known (O). As difficulties appear most strikingly in the setting of concrete tasks which the students experience as difficult or impossible, many studies focus on such tasks and how they relate to the global transition. Here, we shall consider the following task, and expand our analysis of it as presented in (Winsløw, 2007):

- a) Show that $f(t) = t/(1+t)$ defines an increasing function on $[0, \infty)$.
- b) Show that with f as above, $f(s+t) \leq f(s) + f(t)$ for all $s, t \geq 0$.
- c) Show that the formula

$$d(a,b) = \frac{|a-b|}{1+|a-b|}$$

defines a metric on \mathbb{R} .

In fact, c) is the enunciation of a text book task (Carothers, 2000, p. 37) while a) and b) are provided as hints in the book. This is a typical task given to students as they

begin to study the concept of a *metric*, defined axiomatically by three properties: on a space M , a metric is a function on $M \times M$ which satisfies, for all x, y, z in M : $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$. In the case of c), the first two properties are immediate and the last one is verified using a) and b), bearing in mind that $\delta(a, b) = |a - b|$ defines a metric on \mathbb{R} (corresponding to the usual concept of distance on \mathbb{R}).

In actual practice, our observations of numerous exercise sessions show that students take the “hint” to rephrase the exercise as a three step procedure, as formulated above; that almost all solve part a) by computing the derivative and showing it’s positive; that few students solve b), often by round-about methods involving functions of two variables (and sometimes even a computer algebra system); and that very few students were able to make use of a) and b) to solve c), in fact in 6 out of 8 groups of 25-30 students observed, no student had succeeded to do so. Another point is that students having failed with b) did not even try to tackle c).

4.1. TDS approach. The exercise can be considered as a didactic milieu devolved by teachers to students and presenting certain obstacles, the overcoming of which are the price of the experience which the teacher aims for the students to have, of applying the definition of metrics. The first parts seem more familiar to the students and its form activate existing contracts, in the sense that surface parts of the enunciation (“increasing” and “ \leq ”, corresponding to artefacts in the milieu) trigger certain techniques of calculation. In particular, to show that a concrete function is “increasing” one computes the derivative and “see” that it is positive. To “see” an inequality may take some rewriting; the presence of two variables in b) (again, artefacts of the milieu) is responsible for many complicated attempts to use partial differentiation and the like. Finally, no contract has been established for a “right way” of showing that something is a metric; the tripartite nature of the definition is no doubt part of the problem (one has to verify three properties instead of one). In the classroom situation, the teacher can get no further than to make the students recite (or look up) the text book definition. The teacher’s (and the text book author’s) expectation that this will be a simple experience with applying the definition thus fails because the milieu leads the students to identify the problem with contracts at the “micro”-level of the individual steps. A more global contract is visible in the fact that failure with b) kept many from even considering c), amounting to an understanding of exercises as built of from increasingly difficult parts (“if you can’t do b) then you certainly can’t do c)”). The outcome of this analysis, in the setting of TDS, is that the milieu will have to be *re-designed* to better fit the teachers’ intentions (the target knowledge, surely to be further analysed!) as well as the contractual phenomena evidenced by the students’ response to the original tasks. In particular, the properties of metrics – the key element of O' for the above task – may have to be (re)constructed by students first, as a response to a situation with a milieu that relates to their existing experience with (O, A) .

4.2. *ATD approach.* Winsløw (2008) presents a two-step transition in terms of the praxeological organisations present in secondary schools and in universities: first practice blocks are completed to entire praxeologies (with theory blocks), then new practice blocks are built with tasks that take objects from “old” theory blocks. In the task above, it is mainly the second step which is in play: the function f and its (theoretically proved) properties are used to build a new object for a practice block related to metrics (task type: show that a given two variable function is a metric; technique: verify the axioms). Also, the “standard distance” δ (fundamental to the theory of calculus on \mathbb{R}) becomes one among an infinity of objects that this task type takes as an object. The institutional point of view provides a framework for explaining the apparent necessity of this two-step transition $O \rightarrow O'$. In secondary school, epistemic systems are constrained by noospheric systems pursuing aims that go much beyond the school institution itself (a range of continuing study programmes, a central examination, etc.). At universities, two types of ES coexist, those of research and those of teaching (cf. Madsen and Winsløw, to appear); the overlapping group of users consists of “professors” (research mathematicians who also teach). The complete praxeologies O' pursued in undergraduate mathematics programmes in research intensive universities aim to converge towards those pursued in research (O''). In particular, the practice of checking that a given object is a metric, as well as theories based on metric spaces, are indispensable in several branches of research mathematics. Tasks of the kind considered above are thus, at least to some extent, consequences of the *choice* that O' should approach O'' .

4.3. *SCT approach.* Sharing some concerns with the TDS analysis presented above, a SCT analysis focuses more sharply on the beliefs and norms evidenced by the discourses found in the classroom where the exercise is discussed. While we have no space to provide even excerpt of relevant data, these might well turn out to present a cleavage in the group G' , between the teachers and the few students on the one hand who have formed a conception of metrics – and a technical level in algebraic manipulations – that allow for understanding and completing the task; and those students who try, quite desperately, to relate the task to norms and beliefs which they have acquired in secondary school practices. This may or may not proceed towards a progressive inclusion of the latter subgroup into a community of practice with shared norms and beliefs; but in this isolated episode, this seems to be out of reach. The fact that the majority of students did not get to consider the “real” task – because of their inability to follow the “hints” (or complete the preliminary tasks) – leads to a form of (at least) local alienation from the intended meaning-making. Such phenomena are frequently observed in studies of undergraduate mathematics within a socio-constructivist approach. Dreyfus (1999, 106) subsumes the transition which many students fail to make as moving from questions of type ‘What is the result?’ to questions of type ‘Is it true that ...?’, after arguing that even within the community of mathematicians, there are no universal criteria for whether an answer to the latter

type of question is complete and correct (cf. also Ernest, 1998). A SCT approach thus focuses on the process of building at least local consensus in G' about this matter, given that G has mainly been engaged in practices where it does not occur.

4.4. CST approach. Here, the first question could well to be: when addressing the three parts of the task, what are the forms of representation (including the variety of semiotic artefacts) available – actually and potentially – to students (relation of G to A)? For part a), the students can easily graph the function and thus become intuitively convinced of the claim (part of O). For a more formal argument, they can use the algebraic register and either differentiate f to get $(1+t)^{-2} > 0$, or use a treatment like $t/(1+t) = 1-1/(1+t)$. The latter (and simpler) *ad hoc* argument did not occur among students or even teachers, because it is not produced by an algorithm, and the standard procedure is reasonably easy. For b) there is no easy general procedure and the treatment required is equally non-standard; so only few students succeed. In fact, the complexity of required treatments *not given by a standard algorithm* also explains why the almost all students fail with c): here one has to *combine* previous results with the validity of the axioms for δ . Moreover, unlike a) and to some extent b), representations of the involved objects in other registers, such as the graphs or tables of the functions d and δ , are of no help to understand or intuitively support the sought conclusion. By contrast, in secondary level mathematics, the predominant mode of thinking involves *coordination* of several registers (e.g. graphical, symbolic, numeric) of the objects considered. While this is both a challenge and a support at the secondary level, it tends to disappear for more abstract mathematical objects at tertiary level. In the concrete case, a task on metrics on \mathbb{R}^2 (with ample possibilities for illustrating the different metrics) might help to enable a multimodal first encounter with the notion, on the condition that students succeed in coordinating the involved forms of representation.

5. CONCLUDING REMARKS

In this paper we compared four theories in general (see 3.5). While it would be too simplistic to maintain that the considered theoretical frameworks model only one or two of the three components of ES, they do exhibit very different foregrounds in the sense that each provides highly developed notions and principles for analysing certain components or relations between them, while leaving other in the background. One might also talk of *ontological foregrounds* in the sense that different parts of didactical reality are identified or constructed through these models. This is also illustrated by the case study (section 4). Serious integration of theoretical frameworks may eventually become possible and even useful to some extent; but I personally feel that it is more urgent to develop the rationality with which we *choose* frameworks according to *a given purpose of research*. Analysing theoretical frameworks within the GOA model may contribute to this end because research purposes and ontological foregrounds are strongly interdependent.

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CERME 6 – WORKING GROUP 10

MATHEMATICAL CURRICULUM AND PRACTICE

TABLE OF CONTENTS

Introduction.....	1688
<i>Leonor Santos, José Carrillo, Alena Hospesova, Maha Abboud-Blanchard</i>	
Effective ‘blended’ professional development for teachers of mathematics: Design and evaluation of the “UPOLA” Program	1694
<i>Lutz Hellmig</i>	
Teachers’ efficacy beliefs and perceptions regarding the implementation of new primary mathematics curriculum.....	1704
<i>Isil Isler, Erdine Cakiroglu</i>	
Curriculum management in the context of a mathematics subject group.....	1714
<i>Cláudia Canha Nunes, João Pedro da Ponte</i>	
Gestures and styles of communication: are they intertwined?.....	1724
<i>Chiara Andrá</i>	
Teachers’ subject knowledge: the number line representation	1734
<i>Maria Doritou, Eddie Gray</i>	
Communication as social interaction. Primary School Teacher Practices.....	1744
<i>Antonio Guerreiro, Lurdes Serrazina</i>	
Experimental devices in mathematics and physics standards in lower and upper secondary school, and their consequences on teacher’s practices	1751
<i>Fabrice Vandebrouck, Cecile de Hosson, Aline Robert</i>	
Professional development for teachers of mathematics: opportunities and change.....	1761
<i>Marie Joubert, Jenni Back, Els De Geest, Christine Hirst, Rosamund Sutherland</i>	
Teachers’ perception about infinity: a process or an object?.....	1771
<i>Maria Kattou, Michael Thanasia, Katerina Kontoyianni, Constantinos Christou, George Philippou</i>	
Perceptions on teaching the mathematically gifted.....	1781
<i>Katerina Kontoyianni, Maria Kattou, Polina Ioannou, Maria Erodou, Constantinos Christou, Marios Pittalis</i>	

The nature on the numbers in grade 10: A professional problem.....	1791
<i>Mirène Larguier, Alain Bronner</i>	
A European project for professional development of teachers through a research based methodology: The questions arisen at the international level, the Italian contribution, the knot of the teacher-researcher identity	1801
<i>Nicolina A. Malara, Roberto Tortora</i>	
Why is there not enough fuss about affect and meta-affect among mathematics teachers?.....	1811
<i>Manuela Moscucci</i>	
The role of subject knowledge in Primary Student teachers' approaches to teaching the topic of area	1821
<i>Carol Murphy</i>	
Developing of mathematics teachers' community: five groups, five different ways	1831
<i>Regina Reinup</i>	
Foundation knowledge for teaching: contrasting elementary and secondary mathematics.....	1841
<i>Tim Rowland</i>	
Results of a comparative study of future teachers from Australia, Germany and Hong Kong with regard to competences in argumentation and proof	1851
<i>Björn Schwarz, Gabriele Kaiser</i>	
Kate's conceptions of mathematics teaching: Influences in the first three years	1861
<i>Fay Turner</i>	
Pre-service teacher-generated analogies for function concepts	1871
<i>Behiye Ubuz, Ayşegül Eryılmaz, Utkun Aydın, Ibrahim Bayazit</i>	
Technology and mathematics teaching practices: about in-service and pre-service teachers	1880
<i>Maha Abboud-Blanchard</i>	
Teachers and triangles.....	1890
<i>Sylvia Alatorre, Mariana Saíz</i>	
Mathematics teacher education research and practice: researching inside the MICA program	1901
<i>Joyce Mgombelo, Chantal Buteau</i>	
Cognitive transformation in professional development: some case studies	1911
<i>Jorge Soto-Andrade</i>	
What do student teachers attend to?.....	1921
<i>Nad'a Stehlíková</i>	
The mathematical preparation of teachers: A focus on tasks.....	1931
<i>Gabriel J. Stylianides, Andreas J. Stylianides</i>	

Problem posing and development of pedagogical content knowledge in pre-service teacher training	1941
<i>Marie Tichá, Alena Hospesová</i>	
Sustainability of professional development	1951
<i>Stefan Zehetmeier</i>	
A collaborative project as a learning opportunity for mathematics teachers	1961
<i>Maria Helena Martinho, João Pedro da Ponte</i>	
Reflection on Practice: content and depth.....	1971
<i>Christina Martins, Leonor Santos</i>	
Developing mathematics teachers' education through personal reflection and collaborative inquiry: which kinds of tasks?	1981
<i>Angela Pesci</i>	
The learning of mathematics teachers working in a peer group	1991
<i>Martha Witterholt, Martin Goedhart</i>	
Use of focus groups interviews in mathematics educational research.....	2000
<i>Bodil Kleve</i>	
Analyses of interactions in a collaborative context of professional development.....	2010
<i>Maria Cinta Muñoz-Catalán, José Carrillo, Nuria Climent</i>	
Adapting the knowledge quarter in the Cypriot mathematics classroom	2020
<i>Marilena Petrou</i>	
Professional knowledge in an improvisation episode: the importance of a cognitive model	2030
<i>C. Miguel Ribeiro, Rute Monteiro, José Carrillo</i>	

INTRODUCTION

FROM A STUDY OF TEACHING PRACTICES TO ISSUES IN TEACHER EDUCATION

Leonor Santos (Portugal)

José Carrillo (Spain)

Alena Hospesova (Czech Republic)

Maha Abboud-Blanchard (France)

Group 10 is particularly interested in theoretical, methodological, empirical or developmental papers on issues concerning teachers' practices, professional knowledge and teacher education. Several themes are possible to be discussed, such as teachers' beliefs, teachers' activity, the role of the teacher in the classroom, professional knowledge, professional development, strategies for teacher education, and links between theory and practice, research and teaching, and teacher education and collaborative research.

This group received 57 proposals (48 for papers and 7 for posters). Each proposal was reviewed by the leader of the group and two authors, in general including one of the others co-leaders. Some proposals were immediately accepted (8 papers, 3 posters), others were asked some revisions (31 papers, 4 posters) and 9 proposals for papers were recommended to be transformed into posters. Fifty five authors from 19 nationalities participated in the sessions of the working group during the conference, through the presentation of 35 papers and 5 posters, all of them accepted to be included in the proceedings.

All the papers and posters have been grouped in different topics that constituted five panels. Each panel began with short presentations (5 minutes each), where the authors presented their paper contributions to the topic and posed three questions (maximum) to be dealt with in the working groups and the further discussion. This first part ended with a comment related with all the presentations (10 minutes), made by a previous invited participant of the working group. Afterwards, a discussion part took place. In general, this discussion had a first moment in small groups and a second one with the whole group.

The organisation of the sessions was highly valued by the participants, as well as the atmosphere. Nevertheless, due to the high number of presentations, the time for discussion was sometimes less than desirable. The group leader presented a different way to organize the working group for the future (some panels may occur in parallel), if the participation maintains so high, informing in advance the distribution of the papers in the different panels. One participant suggested that each author would be in a different small group permitting that the work in that group focuses on that author's

paper. It has been also proposed a possible change in respect of presentations: the participants would present other participant's paper. We didn't get to any final agreement on this last proposition.

Panels

We present the emerging issues and ideas that rose during the different panels.

Panel I: Mathematical curriculum and practice

- Is it possible a renewal of the curriculum, which implies changes in the teacher's style of work into the class, without any external stimulus (working at school in group, consulting only textbooks, even with the help of some experienced teachers)? If yes, what conditions are necessary at schools, and more widely in the social context?
- How can one develop a new curriculum in a mode that integrates top-down and bottom-up approaches?
- There is a specific role for mathematics educator, but which one and when? And for research?
- How does curriculum management influence students' learning of mathematics?
- Is the study of teachers' efficacy meaningful without taking into account the teachers' views about mathematics?
- What is the incidence and availability of such research, at international level? Can we think about common research on any topic in Europe without taking into account cultural and social differences among the countries?

Panel II: Professional knowledge

There are uses of similar, but different terms, within the notion of professional knowledge: knowledge base for teaching; pedagogical content knowledge; competence: disciplinary, didactic, and relational; subject didactical competence; practical knowledge (beliefs and knowledge)

- How can one present mathematics for the teachers to contribute to the development of their pedagogical content knowledge?
- What tasks can we use to diagnose the (students) teachers' subject matter knowledge (its possible weakness)?
- How can one change teachers' conceptions on mathematical communication (as information transmission) through a collaborative work (eg. centered on teachers' reflection on their own practice)?
- How can one promote lasting classroom culture among teachers, one of its focus being the discussion of students' (right or wrong) strategies?

Panel III: Professional development

As for primary teachers, also for secondary teachers, mathematical content knowledge and pedagogical content knowledge must be interrelated in teacher education (having a mathematics degree isn't enough to understand the mathematics to teach).

- Professional development is about becoming autonomous and critical at designing and conducting classroom teaching. How do teachers develop professionally? In particular, what is the role of:
 - theory (listening to lectures, reading papers, discussing issues, ...)?
 - practice (appropriating ideas from the practice of others, transforming ideas from his/her own practice)?
 - reflection (reflecting on what? how? with what purpose?...)?
- How is it possible that groups of teachers develop towards a real learning (inquiry) community? What kind of impulses do they need?
- Which role could/should researchers/teachers' educators play in such professional development (taking account of their experience in international projects, in research studies, in the use of supporting tools of analysis...)
- How is it possible to promote real changes in the beliefs and the teaching practices of in-service teachers?
 - How can we measure the sustainability of this professional development?
 - What is the impact (if any) of the changes on the mathematical experience and learning of pupils?
- Co-learning is a means to promote professional development. But how to combine the expertise of teachers and that of mathematics educators/researchers in a way that can be useful to the two partners?

Panel IV: Approaching reflection and collaboration in mathematics teachers' professional development

Collaborating is not just sitting or working together and reflecting is not just thinking about or thinking aloud. Content and depth of reflection are determinant. Reflection is a privileged way for professional enhancement. Collaboration is a mean for professional development and for research strategy.

- What strategies, settings and content can we design to promote reflection and collaboration amongst teachers and between teachers and researchers in order to achieve a real professional development?
- How can we categorise data, statements, and phenomena? And why?

- What data should be analysed to measure the improvement of teaching via (joint) reflection?

Panel V: **Models to analyse the practice**

The practice of teachers includes classroom teaching, as well as training and other professional development contexts, ... There are different examples of models to analyse the practice, such as: focusing on teachers' cognitions; focusing on interactions in a collaborative environment (bottom-up); and focusing on teachers' use of curriculum materials, textbook in particular.

- Enquiring into teachers' beliefs about teaching and learning mathematics through *focus* groups:
 - What other uses might the focus group interview have in teacher education/teaching development?
 - What are the special techniques for managing a focus group interview?
- How can we manage to make research results and instruments useful for teachers as means in their professional development, and for educators in training contexts?

PAPERS

Panel I: **Mathematical curriculum and practice**

Hellmig, L. *Effective blended professional development for teacher of mathematics: Design and evaluation of the “UPOLA” Program.*

Isler, I. & Cakiroglu, E. *Teachers’ efficacy beliefs and perceptions regarding the implementation of new primary mathematics curriculum.*

Nunes, C. & Ponte, J. *Curriculum management in the context of a mathematics subject group.*

Panel II: **Professional knowledge**

Andrá, C. *Gestures and styles of communication: are they intertwined?*

Doritou, M. & Gray, E. *Teachers’ subject knowledge: the number line representation.*

Guerreiro, A. & Serrazina, L. *Communication as social interaction. Primary School Teacher Practices.*

Hosson, C.; Vandebrouck, F. & Robert, A. *Experimental devices in mathematics and physics standards in lower and upper secondary school, and their consequences in teacher’s practices.*

Joubert, M. *et al.* *Professional development for teachers of mathematics: opportunities and change.*

Kattou, M. *et al.* *Teachers’ perception about infinity: a process or an object.*

Kontoyianni, K. *et al.* *Perceptions on teaching the mathematically gifted.*

Larguier, M. & Bronner, A. *Nature on the numbers in second: A professional problem.*

Malara, N. & Tortora, R. *A European project for professional development of teachers through a research based methodology: The questions arisen at the international level, the Italian contribution, the knot of the teacher-researcher identity.*

Moscucci, M. *Why is there not enough fuss about effects and meta-effects among mathematics teachers?*

Murphy, C. *The role of subject knowledge in Primary Student teachers’ approaches to teaching the topic of area.*

Reinup, R. *Developing of mathematics teachers’ community: five groups, five different conceptions*

Rowland, T. *Foundation knowledge for teaching: contrasting elementary and secondary mathematics.*

Schwarz, B. & Kaiser, G. *Results of comparative study of future teachers from Australia, Germany and Hong Kong with regard to competences in argumentation and proof.*

Turner, F. *Kate’s conceptions of mathematics teaching: Influences in the first three years.*

Ubuz, B. *et al.* *Pre-service teacher-generated analogies for function concepts.*

Panel III: Professional development

Abboud-Blanchard, M. *Technology and mathematics teaching practices.*

Alatorre, S. & Saíz, M. *Teachers and triangle.*

Mgombelo, J. & Buteau, C. *Mathematics teacher education research and practice: researching inside the MICA program.*

Soto-Andrade, J. *Cognitive transformation in professional development: some case studies.*

Stehlíková, S. *What do student teachers attend to?*

Stylianides, G. & Stylianides, A. *The mathematical preparation of teachers: A focus on tasks.*

Tichá, M & Hospesová, A. *Problem posing and development of pedagogical content knowledge in pre-service teacher training.*

Zehetmeier, S. *Sustainability of professional development.*

Panel IV: Approaching reflection and collaboration in mathematics teachers' professional development

Martinho, M. H. & Ponte, J. *A collaborative project as a learning opportunity for teachers.*

Matins, C. & Santos, L. *Reflection on Practice: content and depth.*

Pesci, A. *Developing mathematics teachers' education through personal reflection and collaborative inquiry: which kinds of tasks?*

Witterholt, M. & Goedhart, M. *The learning of mathematics teachers working in peer group.*

Panel V: Models to analyse the practice

Kleve, B. *Use of focus groups interviews in mathematics educational research.*

Muñoz-Catalán, M. C.; Carrillo, J. & Climent, N. *Analyses of interaction in a collaborative context of professional development.*

Petrou, M. *Adapting the knowledge quarter in the Cypriot mathematics classroom.*

Ribeiro, C., Monteiro, R. & Carrillo, J. *Professional knowledge in an improvisation episode: the importance of a cognitive model.*

EFFECTIVE 'BLENDED' PROFESSIONAL DEVELOPMENT FOR TEACHERS OF MATHEMATICS:

DESIGN AND EVALUATION OF THE "UPOLA"-PROGRAM

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The paper describes the implementation and evaluation of UPOLA, a one-year-long blended learning professional development (PD) program for teachers of mathematics. The use of polyvalent tasks in classes as the main issue of UPOLA proved to be appropriate to support changes in classroom practice. Based on a short overview of the concept of polyvalent tasks, a description of the design of the blended professional program is given by considering multiple dimensions of 'blending'. The evaluation of the program shows a shift in participants' perception over the time from rather environmental variables towards the impact of UPOLA for teachers' acting and students' learning. Furthermore, some findings on the implementation of web-based communication and collaboration are presented.

Keywords: Professional Development, Blended Learning, Co-Operation, Evaluation, Polyvalent Tasks

INTRODUCTION

The current practice of teachers' PD in Germany is predominantly a set of single events of limited time, with little impact on teachers' classroom activity and students' learning. Given the current situation in the field of PD of practicing teachers, a lack of effective, job-embedded PD for teachers can be observed (Sowder, 2007). Limited-time events, rarely longer than a single day, are the current practice of teachers' further education in Germany. The impact of most of these lectures, meetings, or workshops is weak, since they do not affect teachers' behavior and students' learning. A detailed analysis of the present state is given by Jäger and Bodensohn (2007).

According to Loucks-Horsley (2003) and Guskey (2000) PD should be an ongoing, intended and systemic process. However, there is no clarity about attributes of effective PD. A comparative study by Guskey (2003) shows that "[...] most of the identified characteristics [are] inconsistent and often contradictory" (Guskey, 2003, p. 4). Overall, implementing peer-cooperation and collaborative activities are frequently named as key features to ensure changes in classroom practice (i.e. Garet, Porter, Desimone, Birman, & Yoon, 2001; McGraw, Arbaugh, Lynch, & Brown, 2003).

Following Jäger and Bodensohn (2007), a successful PD-program has to consider the specific needs of participating teachers. Inside-differentiation in heterogenic classes is one of the most evident general issues for PD of teachers of mathematics (Jäger & Bodensohn, 2007). In the German province of Mecklenburg-Western Pomerania,

where heterogenic classes in grade 5 and 6 have been established since 2006 in opposite to the common trinomial school system, teachers identify a higher need for differentiation especially in their classes.

UPOLA, which means "Teaching by using Polyvalent Tasks" (*in German: "Unterrichten mit POLyvalenten Aufgaben"*), focuses both on offering an appropriate topic (polyvalent tasks) to meet the needs of teachers and on a holistic blended approach for the design of PD. To adjust the ongoing program and to identify its strengths and weaknesses, evaluation on multiple stages was an essential part of the program.

POLYVALENT TASKS – AN ISSUE OF PROFESSIONAL DEVELOPMENT

According to the idea of "Open-Ended Approach" (Becker & Shimada, 1997), Sill and Hellmig (2008) defined the concept of "polyvalent math tasks". A mathematical task is polyvalent, related to a group of students, if (1) every student is probably able to find a solution, and (2) the task has a set of solutions on different levels according to the use of mathematical skills. These attributes distinguish a relative small set of polyvalent tasks from a broad range of general open tasks. Thus, polyvalent tasks are highly appropriate to meet the needs of differentiation.

Asserting the benefits of these tasks requires an apposite style of teaching, which is different from the general practice in Germany. Hellmig *et al.* (2007) suggested a time-ratio of about 50% to 50% for two phases of implementing polyvalent tasks in classroom: First the students are asked to find answers to the task individually, by cooperating in pairs or in small groups. During the second phase students present their solutions. The teacher encourages less successful students to show their ideas first; further other students are asked to present different solutions with a higher degree of complexity. The aim of this phase is to develop a culture of communication about mathematics in classes. The course material (Hellmig *et al.*, 2007), provided to every participant in the program, described the characteristics of these tasks, their use in classes, and contained a collection of 70 tasks for grade 5 and 6 students.

The use of polyvalent tasks in classroom supports the idea of openness, communication and cooperation. To take the mentioned ideas into teachers' practice, the design of the program itself is dedicated to these characteristics.

DESIGN OF THE PROJECT – A BLENDED APPROACH

General considerations

"All learning is blended learning." (Oliver & Trigwell, 2005, p. 20) Designing PD is always a blend of different goals, contents, and methods. Inspired by Cross (2006) the author sees a complementary interaction on several dimensions of PD with the main dimensions (1) instruction/construction, (2) presence/ distance, (3) individual/collaborative learning, (4) content/experience focus, (5) "traditional" media/e-learning. Regarding these dimensions, the project UPOLA was blended

through

- Leading the course by two moderators; one with theoretical background, the other with more practical background.
- Giving content-related input (during meetings) and constructing knowledge by the participants through activity, reflection and discussion.
- Combining individual learning by teaching and reflecting with collaborative learning. This included discussions on didactical issues and about lessons, which were taught by the participants, as well as joint planning of lessons.
- Using a guideline linked to the curriculum during the school year and self-directed teaching, reporting and discussing.
- Meetings "off the job" and phases of experience and reflection "on the job".
- Using traditional channels and web-based environments to communicate.

A factor for transferring the topics of PD into classrooms is engaging more than one teacher per school. Transfer is influenced by organizational support of principals and acceptance by staff members of a school (Guskey, 2000; Krainer, 2002; Loucks-Horsley, 2003; Gräsel, Fussangel, & Parchmann, 2006). Thus, every teacher in grade 5 of the participating schools has been invited to attend the program. We assumed that a vast amount of fruitful peer communication and co-operation during PD could affect the growth of the local professional communities of the participating schools.

Implementation of UPOLA in 2007/2008

After a pilot study in 2006/2007, "UPOLA" was put into practice in 2007/2008. We grouped 44 teachers of grade 5 classes of Mecklenburg-Western Pomerania and Berlin into five courses. These courses were integrated in "Mathematics Done Differently", an initiative for PD of teachers of mathematics. A key feature of the programs in "Mathematics Done Differently" was the moderation by a tandem of a school- and a university-teacher (Rösken & Törner, 2008).

We combined four meetings "off the job" between August 2007 and May 2008 with three phases of PD "on the job"; each segment lasted 8-12 weeks in duration. This combination of presence and distance learning supports co-operative and collaborative work, associated with social interaction and flexible time management, which is important for preventing high drop outs (Lynch & Dembo, 2004; Nash, 2005; Picciano, 2006). A valuable list of factors for blended PD-programs was given by Wideman, Owston, and Sinitskaya (2007). We used the learning-management-system (LMS) "moodle" for online communication.

Meetings

The meetings mostly took place at the participating schools, the workplace of the attendants. We ensured a suitable atmosphere for the meetings, offered refreshments and agreed on an informal style to communicate with each other, even between

participants and facilitators. Typically, a meeting started with a structured group interview as a review on the recent period of work, which often turned into a spirited discussion. The review ended by writing a collective summary. Second, a facilitator linked selected theoretical topics to the issue of polyvalent tasks and encouraged a discussion. Finally, participants selected a concerted task for the next on-the-job-phase and outlined first thoughts on teaching with the chosen task. Each meeting closed with a short written feedback on two open questions. A substantial amount of time of the first two meetings was spent for introducing the LMS "moodle" and the characteristics of asynchronous communication.

Phases of experience and asynchronous communication

During an "on-the-job-phase", the attendants planned and conducted a lesson about the chosen polyvalent task. They were asked (1) to report and reflect upon their own lesson, (2) to comment on the reports of their peers, and (3) to discuss different teaching approaches with polyvalent tasks by using moodle.

For setting up the LMS we had to consider the skills and the attitudes of the attendants towards information technology. A certain number of teachers felt uneasy and tried to avoid the use of computers; some of the participants had to struggle with technical issues and deficient skills along the entire course. Hence we designed the structure of the moodle-course to be as clear and simple as possible into a general block and three topic-blocks, each for one on-the-job-phase. The main activity of each topic block was a discussion board for reporting everyone's experience in teaching polyvalent tasks and to discuss about didactical issues. Beyond that, we provided additional material such as manuals (i.e. how to write a report) and files of course-related content.

EVALUATION

Success of PD depends both on content and design. Hence, the evaluation followed two main questions: (1) Are polyvalent tasks appropriate to address a broad range of students with different skills and encourage communication about mathematics in class?, (2) How far is this kind of blended learning applicable for teachers' PD and what sort of items can increase the outcome of the program? In this paper, we put our attention to the second question.

Methodology

Guskey (2000) describes a model of evaluating teachers' PD that comprises five stages. We utilized this model, and gathered data for (1) participants' reactions, (2) participants' learning, (3) organizational support and change, (4) participants' use of new knowledge and skills, and (5) student learning outcomes. The author subclassified the second stage into (2a) process, and (2b) results of participants' learning.

Determined by our blended view of professional development, we had to separate

two points of view from each other. On the one hand, we examined five courses in their entirety with certain attributes to find general correlations. On the other hand, we had to regard the participants as individual learners and teachers by case studies.

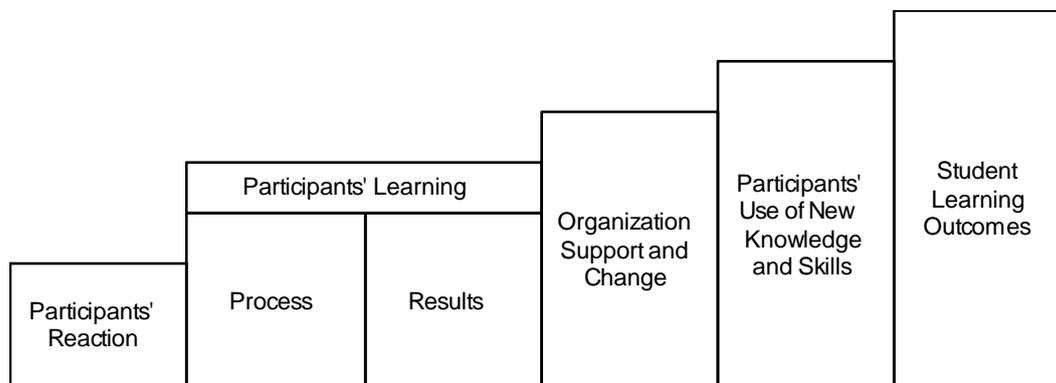


Figure 1: 5 Stages of Evaluation adopted from Guskey (2000)

Use of different means for evaluation was necessary to gain reliable data. The most important means were different questionnaires, interviews with teachers and principals, classroom observations, and monitoring discussion groups by quantitative and qualitative criteria. Finally, a modified method of the Repertory Grid interviewing technique (Collet & Bruder, 2006) was employed to capture the system of participants' personal constructs regarding math tasks before and after the course. Reflective reports and discussions during every face-to-face-session delivered very rich and useful "soft" data to get insights in participants' learning. The variety of tools for evaluation generated two separate sets of data: a set of personalized data, gathered by interviews, online- and face-to-face-discussions, and sampled classroom observations; and a set of anonymous data, collected by surveys and Repertory Grid. On the one hand, it was not possible to avoid getting some personalized data of the participants; on the other hand, protection of privacy is a precondition to get objective and reliable responses by participants. Three examination papers about the influence of polyvalent tasks on grade-5-students with different abilities were written.

Focusing on the use of the LMS, we analysed the number of insights in documents hosted on moodle, and quantitative and qualitative parameters of discussion threads. First, we simply counted the number of postings by every participant, differentiated by opening a thread and giving reactions to a posting. To rate the vitality of the discussion, we defined a scale for grading every thread. Beginning with the lowest degree we distinguished (1) posting by the moderator without a reaction, (2) posting by a participant without a reaction, (3) posting and one answer (one by the moderator) (4) posting and one answer without commitment of the moderator, (5) discussion (at least one posting regarding an answer) between a participant and the moderator, and (6) discussion without participation of the moderator. Furthermore, we viewed the dates of the postings to assess the continuity of participation. An analysis of qualitative variables (i.e. use of new terminology, deepness of reflection) complemented the observation of web-based communication. We compared these

data with additional attributes, such as group-size, schedule of school-year activities and holidays.

Additionally, we could compare online activity of the participants with their contribution to the "off-the-job-meetings", and in some cases by observing classroom-activities concerned with the implementation of the subject.

UPOLA, as a part of "Mathematics Done Differently", was also evaluated externally by the Centre for Educational Research (zepf), University of Koblenz-Landau. Since that external evaluation was designed for one-day-events of PD, the usability of these data and the comparability with our self-evaluated data was limited.

Findings

The description of the findings of the evaluation is grouped according to Guskey's (2000) five stages of evaluation.

On stage 1, participants' reactions, participants appreciated the open and informal atmosphere of the meetings with possibilities to share experience with facilitators and colleagues. They reported about the importance of face-to-face-communication, many felt more comfortable to participate verbally rather than by online-written contributions. Participants attended the meetings regularly; we rated a small drop out (4 of 48) as an indicator of general satisfaction.

On stage 2, participants' learning, we observed that participants shared their individual approach to implement polyvalent tasks in profound discussions. We saw the quality of these discussions as a demonstration of increasing knowledge of participants. Frequently we heard that participants would rather communicate face-to-face than by using a discussion board.

In general, the use of the LMS for asynchronous communication felt short of our expectations. Although we defined a common and clear task for each experience phase, the number of postings by many participants did not match our demands. Most of the discussion-"threads" were only reports without a response by other participants. In some cases, participants received responses, but discussions developed rarely. We can confirm that the group size is an influential factor for the activity and intensity of discussion. Like Caspi, Gorski and Chajut (2003) and Wideman *et al.* (2007) we saw a better performance of courses with ten participants or more. The participants did not contribute postings continuously. First of all, the majority of the postings were written within the last two weeks before the meetings. This is critical regarding to the aim of developing discussions. Furthermore, we placed meetings into the last week before holidays. As a result, stimuli and motivation given during the meetings, faded out immediately due to the holidays.

To keep the attention of participants, daily alerts of ongoing activities had an influence on the activity of participants. Components of the LMS without delivering alerts (downloadable materials as well as some discussion groups) received

measurably less attention or responses from participants. Since reading e-mails was not a daily routine for some participants, facilitators had to contact and motivate some teachers by using additional channels of communication, i.e. by making phone calls.

Participants started to reflect about their lessons just by giving an overview about different approaches of the students to solve polyvalent tasks. By continuing the program many of the attendants included thoughts concerned with planning or reflecting about their lessons.

Evaluating higher levels (stages 3-5 of Guskey's model) of the impact of UPOLA has to regard the conditions of the attendants' workplace in addition to the program. Our research underlines the findings reported by Beaudoin (2002), who reported that a lack of online activity does not implicate a lack of adopting knowledge by participants. Observations of lessons of the UPOLA-project showed that in some cases teachers demonstrated sophisticated skills in teaching with polyvalent tasks, however, they gave no or very few reports to the discussion. Other participants admitted that they did benefit from ideas and experience of others, but hesitated to give themselves a reflection about their own work.

Finding relationships between teachers' PD and students' outcome is crucial, but challenging. Polyvalent tasks are usually not suitable for grading students by giving marks. Effects of polyvalent tasks were anticipated and observed in terms of motivating students, especially of students with lower skills, to think mathematically and to communicate about mathematics. Attendants reported that polyvalent tasks gave them the possibility to observe and assess their students in a broader variety of classroom settings. At this point, evaluation of the design of the program is closely linked to the evaluation of content.

Overall, an obvious change in teachers' perception of the PD program was observable. By classifying the comments of attendants on feedback-sheets (often so-called "happiness-sheets") it has become clear that teachers shifted their attention about the meetings from assessing the atmosphere or appreciating refreshment (after the first meeting) to higher-order categories such as content, quality of cooperation, or transferability. Although we encouraged teachers with the last feedback-sheet to report explicitly on their adapted 'knowledge', they focused more than before on their use of knowledge in classroom. In many cases, a possible impact on students' outcome was considered. Figure 2 shows the development of teachers' thinking towards students' learning over time, and indicates a solid impact of the program, according to Guskey's model of evaluation.

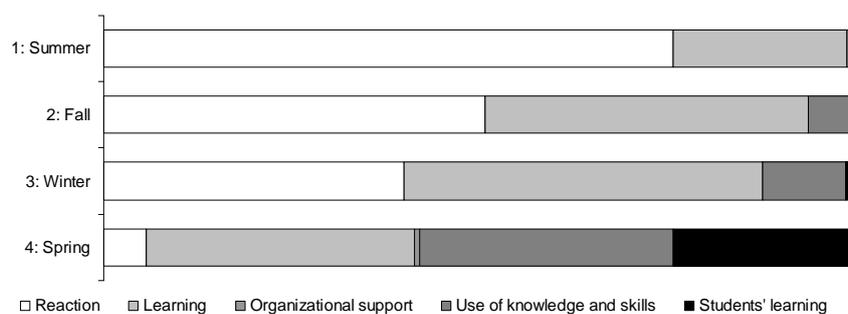


Figure 2: Ratio of participants' responses after each meeting, on Guskey's (2000) five levels of evaluation

In general, data of the internal evaluation was confirmed by the results of the external evaluation by the zepf, Landau.

CONCLUSION

Constructing and developing lasting knowledge, skills and beliefs through teachers' PD must be seen as a process, which needs sufficient time and possibilities to gain experience situated at the workplace and to share ideas and experience in a collaborating group. Using a blended-learning setting – four face-to-face-meetings connected with three experience phases "on the job" – can be one way to meet the needs of participating teachers and to change classroom practice sustainably. We did not merely use a LMS-course to offer instructional and supporting material, but rather the teachers were asked to report and to discuss their lessons using discussion groups in the same moodle-course.

We identified a high acceptance of the topic and of the main structure of UPOLA. Teachers reported the importance of collaboration and discussion among teachers for their situated learning, and their own work. Still, the participants met our expectations about the use of a learning management system only partially.

Different types of weaknesses in terms of remote communication and co-operation have been observed. First of all, teachers were challenged by the faint culture of reflection and discussion about their own work, particularly in a written form. In some cases we identified a lack of motivation for continuous distance learning; teachers had not been aware of the benefits of informal, situated learning and ongoing cooperation. Insufficient technical skills and little experience and confidence, related to asynchronous communication with information technology, hindered the development of a vital and deep discussion. It was indicated that some attributes of the course-design, number of participants per group, dates of face-to-face-meetings, clear tasks for teachers' reports are key for the quality of web-based cooperation. Groups with a certain minimum of participants have to be built to ensure a vital discussion; however, exceeding a maximum of attendants could be a hindrance for developing social relationships.

Further suggestions for planning subsequent projects are to synchronize the course-structure with the schedule of teachers' workload during one school year, to avoid face-to-face-meetings that are immediately followed by holidays, and to design a plain and clear structure of the e-learning-platform, which requires no more than elementary technical skills. In addition, sufficient time and support has to be given to develop technical skills of every participant, including a prior phase for signing in and discovering the LMS through the participants themselves.

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TEACHERS' EFFICACY BELIEFS AND PERCEPTIONS REGARDING THE IMPLEMENTATION OF NEW PRIMARY MATHEMATICS CURRICULUM

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Abstract

The purpose of this study was to investigate primary school and mathematics teachers' efficacy beliefs and perceptions in the context of the new primary mathematics curriculum in Turkey and identify differences, if any, in teachers' efficacy beliefs and perceptions based on their area of certification, gender, and experience. The sample consisted of 805 teachers, 696 of whom were primary and 105 of whom were mathematics teachers working in elementary schools located in 5 cities of Turkey. The questionnaire administered to participants was adapted by the researchers throughout the study. The results of the MANOVA analysis indicated that teachers' area of certification and experience had a significant role on the collective dependent variables, gender did not.

Keywords: Teacher Efficacy Beliefs, Teachers' Perceptions about the Curriculum, Mathematics Curriculum Implementation, Teachers' Practices, Primary and Mathematics Teachers

THEORETICAL FRAMEWORK

Mathematics curriculum change for elementary and middle grades was initiated in 2004 in Turkey. After a period of piloting, a new curriculum was started to be implemented in public and private schools throughout Turkey. Parallel with mathematics education reform movements in many countries, the new elementary and middle grades mathematics curriculum requires a significant shift in the teaching and learning of mathematics within the classroom. Compared to its precursor, the new Turkish curriculum includes a larger emphasis on learner-centered instruction, problem solving, open-ended explorations, modeling real-life situations, and the use of technology as a tool to support mathematics learning (MNE, 2005). Teachers are considered to have a critical role for the actualization of the ideas in the new curriculum. Hence, no matter what the curriculum suggests, it is the teacher who makes the ultimate decisions about what is going on in the classroom. Teachers' potential to learn and adapt to innovations can lead to students' learning and acquaintance with the innovations in classrooms. In that sense, teachers are seen as both the means and ends of curriculum reform movements (Cohen & Hill, 2001). Therefore, any curriculum change should pay attention to what teachers know and believe. The purpose of this study was to investigate teachers' efficacy beliefs about the implementation of the new national mathematics curriculum in Turkey. More

specifically, it was aimed to investigate possible differences in teachers' efficacy beliefs based on their area of certification, gender, and experience.

Teacher efficacy has emerged as an important construct in teacher education over the past 25 years. It has been defined as “teachers’ beliefs in their ability to actualize the desired outcomes” (Wheatley, 2005, p. 748). Teacher efficacy has been linked to teacher effectiveness and appears to influence students in their achievement, attitude and affective growth. Researchers have shown that teacher efficacy has positive effects on teacher effort and persistence in the face of difficulties (Soodak & Podell, 1993), professional commitment (Coladarci, 1992), student motivation (Midgley, Feldlaufer & Eccles, 1989), and openness to new methods in teaching and positive teacher behavior (Ghaith & Yaghi, 1997). In addition, teachers with a high sense of efficacy are more likely to use student-centered teaching strategies, while low-efficacious teachers tend to use teacher-directed strategies, such as didactic lectures and reading from textbooks (Czerniak, 1990). Thus, the importance of teacher efficacy is well established.

Teachers’ sense of efficacy and reforms in curriculum has many common points (Smith, 1996). The changes teachers apply to their practices and adaptation to innovations require that they have a high sense of efficacy. Nevertheless, while both the implementation of reform in mathematics education and teacher efficacy beliefs have been studied in depth over the years, there have been very few research studies completed on the possible connection between the two.

The current study aimed to make a contribution to teacher efficacy research in the context of a major curriculum change initiated in Turkey. Furthermore, teachers’ sense of efficacy has been described as “context and situation specific” (Bandura, 1986). Thus, many scales have been developed to serve different purposes, and some of them have been extensively used in different cultures. Therefore, for the specific purpose of the study, a questionnaire was adapted and utilized throughout the study to assess teachers’ efficacy beliefs and perceptions regarding the implementation of the new curriculum.

METHODOLOGY

In this study, a survey research design was employed. In the sampling method, schools rather than individuals were randomly selected. 57 schools selected for the study were public schools. The participants of this study included 696 primary teachers and 109 mathematics teachers who are teaching at upper primary level. Overall, there were 503 female and 302 male participants.

The data in this study were collected through a survey instrument, one part of which was adapted from another instrument called “Teachers Assessment Efficacy Scale (TAES)” (Wolfe, Viger, Jarvinen, & Linksman, 2007) and the other part constituted of “Teacher’ Sense of Efficacy Scale (TTSES)” (Capa, Cakiroglu, & Sarikaya, 2005) which was originally developed in English by Tschannen-Moran and Hoy (2001).

INSTRUMENTATION

Within the adaptation process, the TAES was translated in respect to the Turkish school culture. A conceptual translation method was employed. This method “uses terms or phrases in the target language that capture the implied associations, or connotative meaning, of the text used in the source language instrument” (Braverman & Slater, 1996, p. 94). Moreover, there were no negatively worded items in the original scale. However, Gable and Wolf (1993) suggest that both positive and negative items should be included in an instrument in order to control the response style. Therefore, some of the items were reworded to include a negative stem by maintaining the corresponded sub-dimension of the item. In addition, the confidence items were rephrased with “can” as Bandura (2006) suggested using “can” to refer to capability while developing efficacy scales because self-efficacy is a perceived capability. After the adaptation process of the instrument, various expert opinions were obtained for the content validation.

The final draft of the instrument consisted of four parts. The first part included 11 items measuring teachers’ demographic characteristics such as gender, experience, educational level and area of certification. The second part included 22 items on a 5-point Likert type agreement scale (1-strongly disagree, 3-undecided, 5-strongly agree) related to the sub-dimensions of (1) efficacy beliefs in terms of the implementation of the new curriculum (e.g. I can prepare assessment tasks in accordance with the new curriculum) (2) beliefs about the impact of the new curriculum on classroom instruction (e.g. When based on the new curriculum, mathematics classes motivate the students to learn), and (3) perceptions about the utility or practicability of the new curriculum (e.g. The new curriculum can help me to identify the knowledge a students must master). The third part included 24 items on a 5 point Likert type frequency scale (1-never, 3-sometimes, and 5-always) about teachers’ perceived utilization of the new curriculum (e.g. I use the new curriculum to plan problem-solving tasks for my students). Twelve items were added to the original sub-scale in order to assess teachers’ utilization of special techniques such as cooperative group work and their use of manipulatives during instruction (e.g. I organize cooperative group work activities for my students). The fourth and the last part included the short form of Turkish teachers’ sense of efficacy scale (TTSES) which consisted of 12 9-point scale items (1- inadequate, 5-moderately adequate to 9-extremely adequate) (e.g. How much can you do to control disruptive behavior in the classroom?).

In this study, common factor analysis was employed in order to discriminate the unique variance of each variable from common variance (Costello & Osborne, 2005). Factor analysis was conducted in two stages: factor extraction and factor rotation. Maximum Likelihood analysis with Direct Oblimin was used for each part of the questionnaire. Kaiser-Meyer-Olkin Measure of Sampling Adequacy (KMO) produced values higher than .9 for all parts of the questionnaire which means the sample size is appropriate for factor analysis (Field, 2005). Moreover, Bartlett’s Test

of Sphericity was significant evaluating the correlation matrix is not an identity matrix (Tabachnick & Fidell, 2007).

Results of exploratory factor analysis suggested six dimensions: Utility and Impact of the Curriculum, Impact of the Curriculum regarding Efficacy Beliefs, Efficacy Beliefs regarding the New Curriculum, Utilization of Curriculum, Utilization of Special Techniques, and Teachers' Sense of Efficacy. The reliability coefficients of the sub-scales produced high levels of reliability coefficients except the Efficacy beliefs regarding the new curriculum subscale.

Reliability of the subscales were satisfactory (Field, 2005) which were given in table 1.

Table 1. Reliability Statistics of the Sub-scales

Sub-scale	Cronbach's Alpha (α)	Number of Items
Utility and Impact of the curriculum	.873	9
Impact of the curriculum regarding Efficacy beliefs	.821	8
Efficacy beliefs regarding the new curriculum	.670	5
Utilization of Curriculum	.910	11
Utilization of Special Techniques	.864	13
Teachers' Sense of Efficacy	.912	12

DATA ANALYSIS

For the inferential results, MANOVA was employed because of its advantage of controlling the risk of Type I error. Furthermore, MANOVA also provides univariate ANOVAs in the output to observe the separate effects of independent variables on each dependent variable (Field, 2005); however the significance of the follow-up tests should be evaluated by using Bonferroni method by dividing the alpha by the number of dependent variables in the analysis. In this study, three independent variables were chosen for investigations which were: teachers' area of certification, gender, and experience. Therefore, the alpha level was adjusted first dividing by three ($0.05 \div 3$) and then by the number of dependent variables ($0.02 \div 6$). The assumption the homogeneity of population covariance matrix for dependent variables of MANOVA was checked by inspecting Box's M Test of Equality of Covariance Matrices and Levene's test.

RESULTS

The results of the MANOVA indicated that teachers' area of certification and experience had a significant role on the collective dependent variables, while gender did not (Table 2).

Table 2. MANOVA Results for Area of Certification, Gender and Experience

Effect	Wilks' Lambda	F	Hypothesis df	Error df	P	Partial η^2	Observed Power
Area of certification	.976	2.800	6.000	697.000	.011	.024	.884
Gender	.966	4.124	6.000	697.000	.000	.034	.977
Experience	.929	4.124	24.000	2401.335	.001	.018	.993

Further follow up analyses revealed that primary teachers ($M = 3.76$, $SD = .538$) had significantly stronger efficacy beliefs about the new curriculum than mathematics teachers ($M = 3.57$, $SD = .545$).

Moreover, teachers with 11 to 15 years and 21 and more years of experience were significantly found to perceive a higher utilization of special techniques than teachers with 10 years or less experience. In a similar sense, teachers with 16-20 years of experience were found to have a significant higher perceived utilization of special techniques than teachers with 5 years or less experience.

Table 3. Utilization of Special Techniques according to Teaching Experience

Teaching Experience	<u>M</u>	<u>SD</u>
5 years or less	3.61 ^a	.485
6-10	3.68 ^a	.484
11-15	3.90 ^a	.473
16-20	3.86 ^a	.458
21 or more years	3.88 ^a	.521

^a The possible highest score is 5; the possible lowest score is 1.

DISCUSSIONS

Results indicated that primary teachers had significantly stronger efficacy beliefs about the new curriculum than mathematics teachers. This result is interesting in the sense that primary teachers who teach all subjects possessed higher efficacy beliefs in the implementation of the curriculum than mathematics subject-matter teachers. One of the reasons may be that primary teachers teach younger students

than mathematics teachers. For example, Ross (1994) noted that declines occur in teacher efficacy when the grade levels taught are increased. Also, Capa (2005) found that elementary school teachers were more efficacious about student engagement than secondary school teachers in their first-year of teaching. Another possible reason for the lower sense of efficacy in the mathematics teachers may be because the new mathematics curriculum has been implemented since 2005 and it was first conducted in primary grades (1-5), then in the upper primary grades (6-8). Therefore, primary school teachers have been implementing the new curriculum for a longer time than mathematics teachers; thus, primary school teachers may be more acquainted with the new curriculum. Furthermore, primary teachers may have more congruent practices with the new curriculum such as developing and using hands-on activities with their students in the primary levels. Therefore, they may have felt more efficacious than mathematics teachers in the implementation of the new curriculum. A study was conducted by Wilson and Cooney (2002) including mathematics and primary teachers. The results showed that while the mathematics teachers focused on content knowledge; elementary teachers focused on different views of instructional strategies that claimed to have more “constructivist-oriented” views (p.143). Another claim for this result may be, in the grades between 6 through 8, middle grades, there are national examinations held at the end of each year for the purpose of placement of students to high schools after the 8th grade. Therefore, mathematics teachers may focus more on the scope of these examinations during their instructions rather than the requirements of the new curriculum, so that they may feel less efficacious about the new curriculum than primary teachers.

Results indicated that teachers with 11 to 15 years and 21 and more years of experience had significantly higher perceived utilization of special techniques than teachers possessing 10 or less years of experience. Moreover, teachers with 16-20 years of experience possessed significantly higher perceived utilization of special techniques than teachers with 5 or less years of experience. The first five years of teaching profession is a period where teachers are in the beginning of experiencing the learning to teach and developing ideas about themselves as a teacher. This may be a reason of why less experienced teachers perceive themselves to utilize the specific techniques suggested in the new curriculum less frequently. Ghaith and Shaaban (1999), founding their measurement on Veenman’s (1984) list of teaching problems pointed out that teachers’ concerns about teaching decrease after 15 years of experience. Therefore, more experienced teachers were expected to integrate special techniques more frequently than their beginning or less experienced counterparts since they may have less concerns about other issues such as maintaining classroom management and discipline. Veenman (1984) also called the first-year experience of teachers as a “reality shock” because of the gap between the theory they learned and the practice they are engaged in.

The study also revealed that, although found to be insignificant, teachers’ efficacy beliefs about the new curriculum increased when teaching experience increased (Table 4).

Table 4. Efficacy Beliefs regarding the New Curriculum according to Teaching Experience

Teaching Experience	<u>M</u>	<u>SD</u>
5 years or less	3.64 ^a	.521
6-10	3.70 ^a	.523
11-15	3.77 ^a	.510
16-20	3.71 ^a	.512
21 or more years	3.75 ^a	.581

^a The possible highest score is 5; the possible lowest score is 1.

The findings of other studies in this issue is somewhat varying. Wenner (2001), for instance, indicated in his study with pre-service and in-service teachers that experience leads to greater perceived efficacy of teachers. De Mesquita and Drake (1994), on the other hand, investigated primary school teachers' attitudes and efficacy beliefs towards a nongraded state mandated educational reform and found that teachers possessed a lower-sense of efficacy when their experience increased. However, in the current study teachers' sense of efficacy beliefs, was found to increase when teaching experience increased although this increase was not statistically significant.

Moreover, gender did not reveal a significant difference in this study. However, descriptive results revealed that the sense of efficacy beliefs of male teachers was higher than females; despite not being statistically significant. On the contrary, Evans and Tribble (1986) found that females had higher teaching efficacy than males and Cheung (2006) found that female teachers had significantly higher general efficacy beliefs than male teachers by employing TSES. However, there have been some studies which indicate no relationship between gender and teacher efficacy (Hoy & Woolfolk, 1993; Ghaith & Shaaban, 1999).

It should be noted that change is a process rather than an event. Therefore, the teachers' adaptation process should not be underestimated. In-service trainings may aim to develop new sources for teachers' efficacy beliefs compatible with the reform efforts especially for mathematics teachers. For the design of the in-service training sessions, collaboration between schools and universities may provide educational opportunity for teachers. Furthermore, the in-service training should be parallel to the approach of what is expected from teachers as conductors of the curriculum, so that the teachers may gain mastery experiences which may provide them more efficacious about the new approaches of the innovation. In order to achieve the intended changes through implementation of the new curriculum, teachers' practices and beliefs in the adaptation process should continue to be analyzed well. Moreover, qualitative studies may be conducted to support teachers' self-report measures such as classroom

observations and interviews in order to gain in-depth data about teachers' efficacy beliefs regarding the new curriculum and their adaptation processes to the new curriculum.

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CURRICULUM MANAGEMENT IN THE CONTEXT OF A MATHEMATICS SUBJECT GROUP

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This paper analyses how Simon, a mathematics teacher, manages the curriculum, uses the textbook to plan his practice and conducts students' assessment. It also seeks to understand the relationship between such curriculum management and the collaborative work undertaken by the mathematics teachers' subject group. This is a qualitative and interpretative case study, with data collection through participant observation, interviews and documents. The results show that the teacher manages the curriculum adjusting the expectations of different educational players (colleagues, students and parents) and his own expectations. They also show that curriculum management supported by the collaborative context generates tensions when a teacher makes decisions that diverge from those assumed collectively.

Key-words: Curriculum management, mathematics, mathematics subject group.

A key aspect of professional practice is the way the teacher manages the official curriculum in order to meet the stated objectives, taking into account the students' characteristics and the conditions and resources of the school. In Portugal, curriculum management is particularly complex, giving the social tensions concerning mathematics teaching, largely fuelled by the performance of the students in mathematics in national (GAVE, 2002) and international assessments (OCDE, 2004). Innovative teaching practices are increasingly challenged in many forums, particularly in the public media. This paper aims to describe and analyze how a teacher manages the official curriculum, including the strategies and resources that he uses and how he assesses his students' learning.

CURRICULUM MANAGEMENT IN MATHEMATICS

Different levels of the curriculum may be distinguished. There is the prescribed (or formal) curriculum of official documents, the available curriculum mediated by school textbooks, the planned (or shaped) curriculum by the teacher, the curriculum in action put in place by the teacher in the classroom, the curriculum learned by the students, and the curriculum evaluated, for example, through national examinations (Gimeno, 1989; Stein, Remillard & Smith, 2007).

Curriculum management refers to the actions of the teacher that contribute to the construction of the curriculum in the classroom (Gimeno, 1989; Ponte, 2005). The focus of the management process is students' learning, and it is according to such learning (at least in theory) that decisions are taken. Curriculum management has to do, essentially, with the way the teacher interprets and shapes the curriculum, on two levels: a *macro* level, concerning the overall planning of teaching for an extended period, and

a *micro* level, corresponding to the teaching process in the classroom. The teacher makes decisions selecting tasks, strategies, and materials appropriate to the objectives and purposes of mathematics teaching, taking into account his/her students and working conditions. The teacher adjusts the curriculum as he/she evaluates and periodically reflects on his/her professional practices.

As a curriculum manager, the teacher faces new challenges. The cultural diversity of the student population requires the implementation and management of a dynamic curriculum that seeks to meet the demands of modern society. At the same time, the role of the teacher is changing from a “deliverer” of knowledge, to that of a facilitator of learning (Brooks & Suydam, 1993; Ponte, 2005). When planning his/her teaching, the teacher selects the tasks to propose to the students. These may be all similar (usually, exercises) or diversified (including, for example, problems, investigations, projects, and modelling tasks, as well as exercises) (Ponte, 2005). Tasks may be framed in mathematical contexts or refer to other contexts. According to current curriculum documents (ME-DGIDC, 2007; NCTM, 2000), the tasks should help the student to develop a comprehensive view of the mathematics activity, increase their understanding of mathematical processes, and help them to develop their mathematical reasoning.

School textbooks are important resources for curriculum management. Their use changes according to different perspectives on their role in different contexts (Ponte, 2005). In Portugal, the *Relatório Matemática 2001* (APM, 1998) indicates that textbooks are the teaching material most used by teachers from grades 5 to 12 (82% of the teachers use them always or in most classes). Textbooks have a large tradition in the field of education and occupy a central role in the classroom, influencing the work of teachers, and helping in delimitating the knowledge students are supposed to learn (APM, 1998). In general, teachers use textbooks to organize their classroom activity and to select tasks to propose to students to do in the classroom or at home. In this way, textbooks are key mediators between the different dimensions of the curriculum, particularly the curriculum taught and prescribed by the central government and the curriculum learned by students (Pires, 2005; Ponte, 2005).

Students’ assessment is closely linked to curriculum management, playing a regulatory role in the teaching and learning process. Santos (2002), for example, suggests that assessment should be diversified and occur in formal and informal situations, with the active participation of students, contributing to their development and to the success of learning. The negotiation and establishment of an appropriate contract for assessment are important issues that can determine the success of students’ learning (Nunes, 2004).

These challenges require teachers to work collaboratively, in order to frame and solve the many problems that arise in developing and adjusting the curriculum. It also requires the ability to reflect on teaching practice and students’ learning, creating dynamics that promote their professional development and the school culture (Hargreaves, 1998; Nunes & Ponte, 2008). For schools to make a significant development

in curriculum management and teaching practices, teachers' active involvement in innovative projects, carried out collaboratively, is an essential condition (GTI, 2008).

METHODOLOGY

This study follows a qualitative approach (Erickson, 1986), with a case study design (Stake, 1994; Yin, 1989). The study involves a group of 14 mathematics teachers of a secondary school with 12-18 years old students. The mathematics subject group has an extensive experience of working collaboratively and in recent years has developed various projects at the school. Most of these projects emerged from the need felt by the teachers to improve their practice and to help students to overcome their difficulties. During the school year 2007/08 the subject group developed the project "Investigations, proof and problem solving tasks in textbooks and in curriculum management", involving all classes from grades 7 to 12. This project aims to diversify tasks in the mathematics classroom, to encourage the students' in learning mathematics.

This study focuses on the group of teachers of the project and within that group, on three teachers: Ana, the coordinator of the subject group, Matilde, a new arrival to school and to the group, and Simon a teacher at the school for 28 years. These cases provide several contrasts that may enable understanding to the relationships between professional knowledge and curriculum management, as well as with collaboration and leadership at the school. In this article, we present the case of Simon and give special attention to his curriculum management, because of his professional experience and role in the group.

Collection of data was done during the school year 2007/08 and includes participant observation (Jorgensen, 1989) of the group working sessions and two classes, with record of field notes in a research journal, two interviews with each of the three teachers selected for case studies, and collection of documents (Adler & Adler, 1994; Patton, 2002; Yin, 1989). According to the research plan, data analysis began simultaneously with data collection, to identify the need for further collection of data. The second level of data analysis involves the development of categories focused on professional knowledge, curriculum management, collaboration and leadership that may provide an interpretation of the data. The third level of analysis seeks to explain the meaning of the data, to provide contributions to the understanding of the phenomenon under study (Merriam, 1988).

SIMON: MANAGING THE CURRICULUM

Simon is a teacher with 28 years of experience teaching mathematics classes from grades 7 to 12. Throughout his career he played several roles in his school such as deputy head teacher, in-service teacher education coordinator, department coordinator, and project coordinator (of mathematics projects and of other school projects). He is an in-service teacher educator in professional development courses and belongs to several working groups in and outside his school. Because of his professional experience and the initiatives he promotes in the group, Simon is recognized by his colleagues as the leader of the group. This academic year he has only grade 12 classes.

Planning. At the beginning of the school year, Simon makes the annual plan together with his colleagues who are teaching the same grades. This planning begins with the group of teachers browsing the school textbook and, together, making changes in the annual planning of the previous year. When questions arise, particularly about the number of lessons to assign to each unit, the group uses the mathematics curriculum and its “roadmap” with the methodological guidelines for planning. Once the overall plan is made, he and his colleagues direct their attention for the planning of the first unit [Group Working Session (GWS), 11/Sept/07]. At this stage, from inside his textbook, Simon hands several sheets, handwritten in pencil. In a table with just two columns he registered an analysis of all the tasks of the textbook, in a uniform way:

This is my “curriculum management.” These sheets are worth gold! I have done this for all the textbooks that I use. (...) The first approach is always the textbook. I solve all exercises (...) This symbol [a ring], here around the number indicates that the task is very important and I note those tasks that are more difficult [marked with an arrow] and those that do not interest, because they are poorly structured or have errors [marked with a cross]. They [the teachers from the group] always ask me for my sheets. [GWS, 11/Sept/07]

They [the students] know that everything I have decided to do I have solved before. I also see other textbooks, especially when I am introducing new units. [Interview, 16/Oct/07]

Simon seeks to be well prepared for his teaching. Therefore, he knows well the textbook that he uses, reading the sections on the subject that he is going to teach and solving all the exercises. His individual working plan is based on his vision for teaching mathematics. For him, the most important thing is that his students enjoy what they are doing and develop capacities that allow them to be autonomous and mathematically competent:

To learn, students have to like what they are doing, then what I like most is that they solve their own problems. First, I would like them to be able to read a problem and not turn their arms down, not discouraging, therefore grasp the problem. (...) Achieving that with my classes is to get weapons to grasp and solve the problems which arise. [Interview, 16/Oct/07]

To achieve these goals, Simon diversifies both the tasks that he proposes to his students and the strategies he uses to solve them. However, he begins by assuming that it is not always possible to manage the mathematics curriculum diversifying the situations proposed to students. The major obstacles are time, or lack of it, coupled with the need to meet the official curriculum, taking into account the external evaluation of students at the end of grade 12:

What I have more in mind, but I do not do always, is diversity, both of tasks and resources. I think it makes the lessons more attractive. Difficult things, easy things, open [tasks], closed [tasks], some [done] in groups, other individual (...) [I use] several resources: calculators, computers, manipulatives... I think some-

times I have to do more! Until grade 12 I do. In grade 12 I do too little, just the calculator with great strength. [Interview, 16/Oct/08]

Tasks. In addition to the tasks suggested in the textbook, Simon selects other tasks to offer his students a variety of experiences to foster the different aspects of their mathematical competency. However, in grade 12, this is not always the case. It is perceptible that, at this grade level, he assigns an important role to problems that require using the calculator and to tasks that promote the development of written communication in mathematics. In such work, he highly values the textbook:

First the textbook, then the other things. (...) We have a grade 12 textbook that has so many proposals that we have difficulty in selecting things. (...) We have to give everything and then we have no time for anything else! (...) Unfortunately, the textbook doesn't have much open tasks, but (...) problem solving, it has a lot. And it also suggests the use of technology, a little bit the computer, the calculator a lot. (...) The worksheets we have done [Law of Laplace, Slope, Lighthouse] were things related to communication, a bit following last year's project [project communication in mathematics]. [Interview, 8/Apr/2008]

Simon believes that the selection of tasks is not an easy job, and through the discussion that he develops with his colleagues who teach grade 12, he attempts to address their difficulties: "The collaborative work between colleagues can be a great help to feel more secure and confident on what we do and we developed in our classes and the materials we propose to our students" [Final reflection, 14/Jul/08].

Curriculum materials. The textbook is the curriculum material most often used by Simon when he is planning the work and assigns it a central role in the classroom. Therefore, he considers vital to choose a good textbook, highlighting as key elements in a textbook the nature and the diversity of the tasks. For other curriculum materials, he likes to diversify its use, but he acknowledges that in grade 12, because of the national examination, he just uses the calculator. However, he states that this is not always so: "I do not use the computer in grade 12 and I always use it in other grades" [Interview, 8/Apr/08].

Classroom work and assessment. Simon argues that the classroom work must be focused on the student. So, he seeks to promote since early the students' autonomy:

Another thing I do is also autonomy, and as the years go they [the students] are increasingly autonomous. (...) I guide them! I say: "Look, I think that you should do this or that!" After, each one follows his/her path! There are some that do everything, others who do very little and I am not concerned to control it. The other day in a classroom, (...) they had questions in some exercises but they were all in different exercises and it could not be a lesson for all at the same time, so they made a request: "Look, do this and this and this," and I did it! [Interview, 8/Apr/08]

When performing tasks constructed by the mathematics group, Simon uses different strategies in the classroom, according to its purpose. Usually, he demands that stu-

dents work in the tasks in pairs or in small groups. In assessment tasks, students work individually and in two phases.

Decisions about assessment provide an interesting episode concerning the relationship of Simon and the group. In fact, the other grade 12 teachers felt that the students should do assessment tasks just in one phase. That is what Ana and Diogo indicate:

Ana – I think that if the task is to assess the students' learning then it has to be done individually. (...) I do not agree to give a second chance, because there are students with private tutoring and already know the task and many of them can provide ready-made answers.

Simon - I think that they perform much better in a second stage. And I do not agree with you [Ana] that the reason is that they have external help and they already know the task.

Diogo - I agree with Ana. In addition, if it counts for assessment, we have to do all in the same way, so that some [students] benefit and others do not. [GWS, 20/Nov/07]

However, Simon decided to use a different strategy. He chose to give a second chance to his students to improve their first response to the task, once corrected and commented. He did so because he strongly believes that this helps students to improve their learning. As he mentions, “students learn from the mistakes they do and a second chance allows them to improve their performance” [Interview, 8/Apr/08].

That decision was discussed in the following working session, as Simon announced his decision and suggested the group to analyze and reflect on the performance of his students in both phases. There were some negative reactions, especially from Ana and Diogo who have disagreed with Simons' decision [GWS, 4/Dec/08]. The issue was taken up later at meetings in which the group built tasks and discussed how to implement them in the classroom [GWS, 15/Jan/08; 19/Feb/08; 8/Apr/08, 6/May/08]. As a result, some other members of the group began to use Simon's strategy. In particular, at the end of the study Diogo admitted that this strategy can help students improve their learning, as he has verified with his own classes [GWS and Final reflection, 14/Jul/08].

The assessment of the students is one of the tasks that Simon acknowledges be the toughest for him. A major problem is the classification of the open tasks and its visibility in the students' final grade. With the collaboration of the subject group, he tried to overcome the difficulties, investing more in the construction and assessment of diversified tasks and testing different criteria for classification, starting from the criteria used in the national examination. Also the review of the assessment criteria established by the department of mathematics and the construction of a self-assessment grid helps to minimize this issue:

I add under the formula, the four tests we had done so far, the three compositions [from open tasks], participation in the classroom in the first and second

school period... In terms of knowledge and attitudes, and I gave a number. [Interview 2, 8/Apr/08]

Simon believes that to make decisions concerning curriculum management and to adjust his practices, the information concerning the work that he develops with his students in class is more useful than the one he collects from the tests:

The assessment that I do all the classes is much more useful. Because everyone thinks they know [what I'm talking about], but when I come to the conclusion that they do not know I have to come back to do it in a different way. [Interview 2, 8/Apr/08]

However, the external assessment has a crucial role in the teaching strategies of Simon. That is visible in how the students do independent work in the classroom, in the tasks that he proposes, the curriculum materials and assessment instruments he most often uses (textbook, calculator, and tests). He is very concerned with the quality of his students' learning and their success, particularly in the mathematics' national exam and access to higher education. He also notes that,

We [the math teachers] are always together, to speak of what happened [in class], and what we are going to do. (...) The assessment instruments are always made [together] and they are always the same. There are no complaints from our group, from anyone: the school community and parents. (...) The school realizes that we [subject group] work very, very in group. [Interview, 16/Oct/07]

Simon's words suggest that he seeks to take into account the expectations of students and parents. In this sense he also builds with his colleagues the assessment tools that he uses in order to harmonize them with the views of the other teachers and to support the decisions about his students' assessment.

Work with the mathematics group. Simon says that the discussions that the group has done in the project working sessions have been very "interesting" for him. In particular, he stresses the construction of open tasks, the definition of criteria to assess and to reflect on the results of students:

The construction of tasks with a group of proofs, problems and explorations and investigations and their implementation in the classroom, the discussions we had in the sessions, has always been very enriching, and the exchange of ideas and clarification of points were a highlight of this project. (...) Discussions on the grading of the students' work on their achievements and to give them feedback were undoubtedly very important aspects for my learning. The contributions of all colleagues made me to reflect on my practice in these aspects, questioning what we did and discovering ideas and suggestions perfectly workable in practice in the future. [Final reflection, 14/Jul/08]

The collaborative work developed in the group played an important role in the individual work of Simon. His activity has also a major influence in the way the mathematics group works, with a culture of collaboration that has been strengthened over

the years with the development of various school projects. This culture of collaboration seems to have been fostered by the way almost all teachers of the group have been involved in the project by joining and participating with enthusiasm. They appear to think that these initiatives are essential to their growth as teachers. These initiatives seem to be the key to the way they work as a group and have contributed to their working culture, where exchange of ideas, experiences and materials are welcome.

DISCUSSION AND CONCLUSION

The curriculum management carried out by Simon at the macro level contains a collective and an individual side. The collective side involves the annual planning and the construction of units and tasks. In this process, we can see that he is an important element, particularly in its preparation, solving all the tasks of the textbook and feeding in this way the discussions of the group. Simon's curriculum management at micro level is markedly individual. He seeks to promote his students autonomy in mathematics learning, encourages them to take responsibility in their own actions and to be independent thinkers. This is much in line with the innovative teaching described by Boaler (1998). That is, mathematics education carried out in line with current curriculum orientations is possible at school level, both in Portugal and England.

His decisions have as a starting point, first, the school textbook. He seeks to understand the proposals presented and selects tasks in order to diversify the learning situations (planned curriculum). In addition to the tasks of the textbook, he offers other tasks to his students constructed together with his colleagues, and uses them for assessment. The information that he gets from his daily practice with his students helps him to regulate the teaching-learning process. The test is the instrument that he uses most. However, the formal assessment of students at the end of each term takes into account the information from students' work in the open tasks and involves the students' active participation. Simon manages the curriculum on the context of the mathematics teachers' group, but there is an individual mark that differs from the group. For example, the classroom strategies that he uses to perform the tasks in two phases differ from those initially supported by his colleagues. Also, we see that he tries to conduct the curriculum management dealing with the tension between different expectations in teaching and assessment of pupils, parents and colleagues and his own personal views. On the one hand, he proposes tasks from the textbook and, on the other hand, he gives his students more open and contextualized tasks which require the use of technology. Simon manages the curriculum taking into account its various dimensions. His practice (curriculum in action) goes beyond teaching from the textbook (mediated curriculum), exploring open tasks that involves students in significant mathematics activity (Boaler, 1998).

Second, his formal assessment practices essentially use the results of the students in tests and open tasks. However, to regulate his teaching practices he uses the information that it collects from his daily work with students (Nunes, 2004; Santos, 2002). Simon accepts the challenge of keeping diversifying his assessing practices, despite

considering this to be one of the most difficult tasks of his work as a teacher. The experience of Simon, the various projects in which he participates and the collaborative work that he develops within the mathematics group of his school are key elements to help him to manage the curriculum in order to promote his students' learning (Hargreaves, 1998). Equally essential, seems to be his ability to address and solve issues of professional practice, reflecting in action, and about action (Schön, 1983).

Finally, the various initiatives of the group, in particular, its projects, are a key to the sustainability of the culture of collaborative work (Nunes & Ponte, 2008). This dynamic and working context seem to motivate the involvement of the teachers in teaching and learning. In particular, such dynamic appears to support the professional development of Simon and his capacity to accept new challenges. There are situations that generate conflicts in the group, especially when most participants favour some decision and some individual practices diverges from that. One important conclusion that we draw from this analysis is that Simon, the natural leader of the group, nurtures his relationship with his colleagues using curriculum management as a focal activity. The professional practice of these teachers, supported by this working environment, shows that current curriculum orientations may be implemented not just at an individual or small group level, but by a whole school mathematics subject group. From this study new issues emerge for future research, namely: How teacher's practices and curriculum management influence students' learning of mathematics? What conditions are necessary at schools, and more widely in the social context, so that this kind of collective curriculum management takes place, very much in line with current curriculum orientations?

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GESTURES AND STYLES OF COMMUNICATION: ARE THEY INTERTWINED?

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The resources used by mathematics teachers include gestures, drawings and extra-linguistic modes of expressions, which can be analysed through a semiotic frame. Teacher's words may go with his gestures, his written signs on the blackboard or slides projection on a screen. Depending on the emphasis given to one among these three possibilities, the styles of communication could be classified into three main trends, where the body of the speaker, the speech and the blackboard play different roles with respect to each tendency. Gestures and styles of communication seem to be intertwined, since giving importance to the body or the written signs leads to different communicative styles; conversely, the style of communication influences the type, the frequency and the role of gestures/written signs accompanying the speech.

Key-words: teacher, gesture, communication, multimodality, semiotic bundle.

INTRODUCTION

This paper focuses on teacher's use of gestures, drawings and extra-linguistic forms of expression when talking about mathematical subjects. It investigates whether it is possible to define a relation between teacher's modes of using gestures and his style of communication. An answer is given through a case study. Moreover, in the same case study possible effects on students' learning process are shown.

Different resources, spreading from words to gestures to ICT instruments, are employed by teachers in the class. Sometimes they become communicative tools, supporting students in their comprehension and learning process. A semiotic approach to teaching-learning processes in mathematics is useful to understand the personal appropriation of signs by persons within their social contexts (Arzarello, Paola, Robutti & Sabena, in print).

At a more or less deep conscious level, any teacher formulates his communication strategy. An analysis of communication strategies chosen by teachers is useful to understand the way mathematical concepts are told to the students. Specifically, it can be interesting to focus on the *objectives* of the message (in the case of mathematical lessons they mainly concern giving information and knowledge), on the *target* to which the lesson is managed; and on the *definition* of messages.

It can be fascinating to combine both semiotic and communication approaches, when examining the acquisition of knowledge by students. In this paper teachers' way of communicating mathematical concepts is considered. How they use gestures, what gestures they make, and which tools support their lesson, is taken into account.

This paper is divided into five main parts, this *Introduction* and a conclusion. *Section 1* focuses on the semiotic bundle, introduced by Arzarello (2006), who adopts a Vygotskian approach and presents an enlarged notion of semiotic system, which reveals particularly helpful for framing all the semiotic resources found in the learning processes in mathematics. *Section 2* is centred on communication strategies (Di Raco, 2000) adopted by teachers. Considering a mathematical lesson, common features and a classification based on styles of communication is presented. *Section 3* presents the methodology used in the case study. In *Section 4* the analysis of some videos is sketched and the main traits of different styles of communication are modelled on both bases of semiotic bundle and of communication strategies. *Section 5* reports some considerations about the relation between teacher's communicative choice and its impact on students' feelings. The *Conclusion* closes the paper.

THE SEMIOTIC CONTEXT OF SIGNS

In a semiotic approach to mathematical teaching, the role of signs and the way they are adopted by individuals within their social context is central (Arzarello, Ferrara, Paola & Robutti, 2005). According to Peirce, a sign is anything that “stands to somebody for something in some respect or capacity” (Peirce, 1931-1958). Within this wide perspective, Arzarello (2006) has introduced the semiotic bundle, which allows studying gestures – and teaching-learning processes – in a multimodal approach. Recent discoveries in neuropsychology (Gallese & Lakoff, 2005) underline the embodied aspects of cognition and show that the brain's sensory-motor system is multimodal rather than modular. Multimodality consists in interactions among the different registers within a unique integrated system, composed by different modalities: gestures, oral and written language, symbols, and so on (Arzarello & Edwards, 2005 and Robutti, 2005).

An important example of semiotic bundle is given by the unity speech-gesture. McNeill claimed that gesture and spoken utterance should be regarded as different sides of a single underlying mental process (McNeill, 1992). Gesture and language constitute a semiotic bundle, made of two deeply intertwined semiotic sets. Researches on gestures have discovered some important relationships between the two, for example match and mismatch has been studied (Goldin-Meadow 2003).

The term “gesture” includes a variety of behaviours that do not form a single category. According to McNeill, the term designates any spontaneous movement of the hands and arms that people perform when talking. Gestures are characterized by the following features (McNeill, 1992): they begin from a position of rest (the *preparatory phase*), move away from this position (the *peak*), and then return to rest (the *recovery phase*).

McNeill (1992) identifies two types of gestures: the *propositional gestures*, which have a main pictorial component, and the *non-propositional gestures*, which are discourse gestures. The propositional gestures could be *iconic gestures*, if they bear a relation of resemblance to the semantic content of discourse; *metaphoric gestures*,

similar to iconic ones, but with the pictorial content presenting an abstract idea that has no physical form; *deictic gestures*, if they indicate objects, events or locations in the concrete world. Among the non-propositional gestures, McNeill distinguishes the *beats* (e.g. the hands move along with the rhythmical pulsation of speech, lending a temporal or emphatic structure to communication), and the *cohesive gestures*, that tie together thematically related but temporally separated parts of the discourse.

Since recent findings in psychology show that gestures can contribute to creating ideas (Goldin-Meadow, 2003), investigating how gestures are used by the teacher can be useful. In fact, it has been shown that – when gestures accompany the discourse – the listener retains more information with respect to a situation in which no gestures are performed (Cutica & Bucciarelli, 2003).

The types, the frequency and the use of gestures vary not only from teacher to teacher, but also depend on the choice of supporting tools like the blackboard or the slide projector, during the lesson (Andrà, in print).

STRATEGIES OF COMMUNICATION

Semiotic activities are classically defined as communicative actions utilizing signs. This involves both sign reception and comprehension via listening and reading, and sign production via speaking and writing. In researches of the Turin group (Robutti, 2006), it has been investigated both the role of gestures and written signs in the mathematical discourses of students, and the role of teachers' gestures with respect to the learning processes of students: how they are shared among students and how they influence their conceptualisation processes (Furinghetti & Paola, 2003).

In order to analyze the phases that a teacher follows to prepare a lecture, the classification used by Di Raco (2000) is adopted. The first phase is the phase of *knowing*, which consists of defining theoretical objectives, choosing communication policy and investigating about expectancies and needs of the target to which he refers; in this phase, the teacher get conscious of the teaching-learning situation in which he is involved.

The phase of *designing* consists in modifying theoretical objectives and adapting them to the target, creating events and communicative situations, selecting communication channels and identifying tools that can help the teacher to talk as more clearly as possible. In this phase the teacher chooses tools that can support him while teaching (the blackboard or the slide projector).

The phase of *planning* consists in defining lengths of time, resources, structure and style of the communicative activity.

The phase of *implementing*: it is the only part that the researcher can analyse when watching videos (as it is the case of this paper), and by this examination it is possible to know something about the previous phases.

METHODOLOGY

The case study focuses on teacher's use of gestures, drawings and extra-linguistic forms of expression when talking about mathematical subjects. Defining a relation between teacher's modes of using gestures and his style of communication is the purpose. Only university lectures have been chosen for the analysis, in order to avoid any noise given by lack of discipline from students.

In a first step, seven videos have been analysed: they concern university lessons on mathematical subjects and each one lasts about 30 minutes. They have been examined from both the semiotic context and the communicative strategies perspectives. Contributions from communication strategy researches supply a background for the semiotic analysis that is the core of this paper. The results of the analysis in the first step are reported in the next section.

In a second step, six new lectures (speakers are labelled respectively F, G, H, I, L, M) had been analysed, following the classification defined in the first step. At the end of each lesson, a questionnaire was given to students, in order to have an immediate feedback on their feelings. The questionnaire was structured in four parts: the first one contains a series of couples of opposite adjectives describing the teacher's attitude (the students and the teacher were asked to agree at a certain level to one between the two adjectives of each couple); in the second part an opinion about the rhythm of the lesson was requested; the third part was focused on students' perception of understanding: how they take notes, whether or not they remember previous lessons and what was the subject of the lecture. In the last part, an opinion about teacher's gestures was asked. A similar questionnaire was given to each speaker, in order to have the possibility of comparing the teacher's intentions with the student's receptions. The number of students involved in answering the questionnaire is 178: 35 students in lecture F, 18 in G, 70 in H, 26 in I, 24 in L and 5 in M.

GESTURES AND COMMUNICATION STYLES

From a semiotic perspective, it is possible to distinguish four phases in each lecture. In fact, the semiotic unity speech-gesture evolves in time. Each phase corresponds to a particular relation between words and use of signs, gestures, drawings and so on.

The "*zero*" phase consists of the first few minutes: the speaker ties with his audience. In this phase, either the speaker does not gesticulate, or his gestures have few relevance. The *introductory phase* is characterized by a great number of gestures: during this phase the teacher introduces the language that becomes shared between him and his audience. The strong relation between speech and gestures is evident. The *main phase* is more extended temporally than the previous one, but is characterized by a decreasing number of signs. In fact, the teacher has already introduced the main concepts he needs and the words he uses evoke themselves the ones – combined with signs – he has utilized in the previous phase. Some signs, utilized in the introductory period, are utilized again. The *concluding phase* varies from teacher to teacher, but a common feature is that an increasing frequency of signs

is observed. A possible explanation could be that in this phase there is the need of fixing the concepts firstly introduced and then explained in the previous phases.

On the side of communication strategies, all videos have in common some main features. In fact, the objectives are mostly cognitive and didactical ones (transmitting knowledge is at the core of the activity); the professor speaks neither to equals nor to a generic public: the target is a group of professionals with a lower level of knowledge; messages he communicates are mathematical contents; and channels of communication consist always in front lessons.

There are some differences, from speaker to speaker, in communication policies and in tools accompanying talks (slides projection, blackboard...). Focusing on the semiotic bundle speech-gesture leads to consider also such supports the teacher may use. The role of such instruments is crucial. The choice of the communication policy influences not only the quantity and the quality of signs but also the preference for certain tools accompanying talk, instead of other ones.

Referring to these choices, in analysed videos it is possible to distinguish three distinct trends. When the communication takes place mainly through *the body* of the speaker, iconic and metaphoric gestures are predominant, because it is the same body of the teacher that talks with the audience. In the speech-gesture unity, the second component has a central role. The use of the blackboard or slide projection is limited or it is absent. Among non-propositional gestures, beats are numerous. In the “zero” phase the teacher does not make signs nor gestures. The introductory phase is characterized by a great number of iconic and metaphoric gestures, and some signs are pictured on the blackboard. The strong relation between words and gestures is clear and it reveals its potential power. Gestures used in this phase are repeated in the subsequent phase. The speaker is introducing the lecture and the concepts he is talking about will return during his speech in the next phase. He will broaden these concepts, and gestures utilized at this time would be repeated, going with words as an inseparable unity. During the main phase the creation of iconic and metaphoric gestures falls off, while the number of beats holds steady. Some iconic and metaphoric gestures of the previous phase are utilized. At times cohesive signs are used, for example to connect what the teacher is telling to what had been written on the blackboard. Signs written on the blackboard are not erased and accompany the whole speech. Written signs enrich the semiotic bundle made of words and gestures. In the last phase gestures utilized during the introductory one get back.

In the second trend observed in those videos, the communication takes place mainly through *the blackboard*, i.e. through written signs that are contemporary of speech. The unity speech-written sign is central in the semiotic bundle, and gestures serve to enrich it. Deictic and cohesive gestures are dominant. In the “zero” phase the blackboard is already at the centre of attention, because the speaker is writing on it or because he just points it (e.g. no sign has already been made, but the speaker indicates, while he is introducing concepts, the point where he will start to write few

minutes later). The introductory phase is characterized by the use of the blackboard. Cohesive and deictic gestures as well as beats are frequent. At the beginning of the central phase the blackboard is erased. It is continuously utilized and it is erased many times. In the final phase the blackboard is employed in a manner that is, in some way, symmetric with respect to the introductory phase.

In the last tendency identified, the communication happens substantially through the projection of slides. In this case the signs produced by the speaker are very limited in number. Iconic and metaphoric gestures are absent. Beats are slightly incisive. It is hard to distinguish the phases shown for the previous trends. The semiotic bundle is made mainly of words and of signs projected on the screen.

The reader is referred to Andrà (in print) for an exhaustive analysis of those seven videos.

IMPACT ON STUDENTS

It has been shown that it is possible to piece together theoretical aspects belonging to the semiotic context and to strategies of communication. The result of this mix is a framework in which one can analyse a didactical activity such as a lecture from a more complex point of view. Four different phases in the teacher's speech have been distinguished. These phases are characterized by aspects referring to both gesture studies and to communication techniques. Different styles of communication involve different uses of signs, in quality and in quantity. And how a speaker uses his body rather than other didactical tools such as the blackboard determines different strategies for the communication of mathematical concepts.

The question of interest is now about the effect of each strategy on students' feeling. Till now, the semiotic analysis of gestures has focused only on the teacher. The teacher, however, communicates to students. Students are listening to him, they are learning the concepts he teaches. Following Vygotsky (1986), how do the choices he has made influence the way students internalize what he has said?

According with the analysis from the six new lectures and the questionnaire, two professors (F and G) followed the first communication strategy: their body plays a central role when they speak. I, L and M followed the second communication strategy: the blackboard was the main tool to teach. Speaker H used slide projections in conducting her lesson. In tables 1, 2 and 3 the main trends in students' answers are reported. When the proportion of students choosing a certain response is lower than $\frac{1}{4}$, it is not reported, since it has revealed as little significant.

In table 1 the six couples of opposite adjectives describing the teacher's attitude are shown. For each couple, the major trend is indicated for each teacher's style (the students' proportion of the main trend is given). Looking at table 1, when in the unity speech-gesture the second component (i.e. the body) prevails, students' perception is mainly in involvement. Students feel them near to the teacher's world. If the blackboard plays a central role, this involvement is a little lost and it is not perceived

when the blackboard is replaced by the slide projections. In this last case, students' perception of conciseness and of a schematic presentation increases with respect to the other two cases.

	F (body)	G (body)	H (slides)	I (blackb.)	L (blackb.)	M (blackb.)
Interesting Boring	80% appealing		60% quite boring	60% appealing		
Involving Detaching	70% involving		60% detaching	50% involving		
Concise Lengthy	> 50% lengthy		60% concise	50% quite lengthy		
Schematic Convolutated	>50% quite convolutated		80% schematic	50% quite convolutated		
Clear Confused	60% sufficiently clear		50% clear	60% in the middle		
Passionate Cool	80% passionate		70% quite cool	50% passionate		

Table 1: Main trends (percentages) in judging teachers' attitude are compared

The opinion on the rhythm of the lesson varies from one strategy to another. How students perceive the speed of the lesson may reveal how quickly they interiorize concepts explained. If the rhythm is suitable or slow for a student, probably he finds little difficulty in understanding what the teacher is saying.

	F (body)	G (body)	H (slides)	I (blackb.)	L (blackb.)	M (blackb.)
Teacher's rhythm	45% suitable 45% quite fast		25% slow 25% suitable 35% fast	30% slow 30% suitable 30% fast		

Table 2: Main trends (percentages) in judging teachers' rhythm are compared

Table 3 reports the main trends in students' perception of understanding. The body-style had lead to a broaden spread of key-concepts perception. In the slide case, on the contrary, the key-concept is definitively perceived by a larger percentage of students. A possible interpretation is that grasping mathematical knowledge seems to be easier when slide projections are employed, rather than when the teacher speaks with no support like this.

	F (body)	G (body)	H (slides)	I (blackb.)	L (blackb.)	M (blackb.)
Notes?	100% often		50% only displayed	60% often		
Previous lessons?	70% remember		30% don't remember 30% remember 30% know where in the program this lecture is	50% remember 20% know at what point in the program this lecture is		
The subject of this lecture?	50% skin-deep concepts		70% quite good understanding	30% quite deep 40% superficial		

Table 3: Taking notes, remembering previous lessons and understanding the analysed lecture are shown by comparing the main trends

Finally, an opinion on teacher's gestures was asked. Students had to indicate whether the teacher had made signs during his lesson and whether these gestures were bothersome. The purpose was of knowing students' perception of gestures and words as a unitary entity: if students did not notice teachers acts, movements or signs, one can hypothesize that gestures are felt as intertwined with the speech.

In the body-centred case, iconic and metaphoric gestures are heavily utilized, but a percentage of 20% of students had never noted them, an analogous percentage said that the teacher wrote on the blackboard mostly and only a half of students realized that the speaker made gestures, and they were not bothersome.

In the blackboard-centred case, only 5% of students said that the teacher wrote mostly on the blackboard, 40% said that he did not make signs or that it had never been noticed and 60% that the speaker gesticulated mainly.

In the slide-centred case, 45% of students said that the teacher gesticulated but it was not bothersome, 40% said that they had never noticed it and 15% that the speaker did not make signs.

It seems that the main tool chosen by the professor in communicating has not been noticed: students' attention is driven on the other supports (on the blackboard in the body-centred lessons, or the body in the blackboard-centred ones). One can suppose that the main tool (the body, the blackboard and the slides respectively) has been perceived by the students as an underlying entity, which forms a semiotic unit with the speech. Conversely, students noted that the teacher has been using different tools, those tools he did not concentrate on.

CONCLUSION

Both semiotic standpoint and researches on communicative strategies can help to frame teacher's way to conduct his lesson. It has been shown that types, frequency

and the use of gestures are closely related to the style of communication chosen by the speaker. The impact of each strategy on students learning process has been analysed from four distinct perspectives: how the teacher's attitude has been perceived by students, how the rhythm of the lesson has been felt, what level of perception of understanding students had and how teacher's gestures had been noticed.

Students seem to be mostly involved in the case the professor used mainly his body when speaking. When the blackboard plays a central role, a little lost of such involvement has been observed and, when the blackboard is replaced by the slide projections, it has not significantly perceived. In the slides case, conciseness and precision have been more perceived, rather than in the other two cases.

When the teacher used his body to communicate, students often take notes and are able to remember the previous lecture. When the slides were utilized, the notes taken are less, because they wrote only fundamental concepts, but a greater percentage of students was able to indicate in which part of the program the lesson was located.

If the blackboard is heavily used, further investigation is needed. It is not clear neither if students remember the subject of the previous lesson, nor how they take notes. Their level of understanding is not evident. A possible interpretation of this fact is that the use of the blackboard assumes all the students be able to capture the concepts at the same speed, namely the speed of the teacher's writing.

As a final consideration, it has to be pointed out that students reversed the rule between the main and the accessory tools chosen by the teacher. For example, they had said that teacher F mainly wrote on the blackboard while he had primarily used his body, but with a regular pacing on the blackboard: in the introductory phase he wrote the concepts he recalled at the end of the lecture, without erasing them. The main tool is perceived as integrated with the speech. The rhythm of the lecture is beaten by the use of this tool (e.g. the body). Students noticed a change in the rhythm (associated to a change in the tool used, for example from gestures to the blackboard), rather than the smooth use of the main tool. Accessory tools became central in their perception, since they corresponded to a change in the rhythm of the lecture.

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TEACHERS' SUBJECT KNOWLEDGE: THE NUMBER LINE REPRESENTATION

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This paper considers the perceptions that trainee and experienced teachers have of the number line. Grounded within the theoretical perspective highlighted by Herbst (1997) the paper examines the interpretations that 'teachers' place on a core classroom representation advocated for teaching the number system in English schools (DfEE, 1999; 2006). The outcome suggests that primary school teachers have conceptions of the number line that do not portray conceptual understanding of its abstract nature as a representation of the number system. Descriptive characteristics of visual models, ambiguity and an emphasis on use overshadow the deeper understanding that would lead to the realisation of the potential as a valuable metaphor.

Key-words: Number Line, Teachers, Conception, Interpretation, Ambiguity.

INTRODUCTION

This paper brings to surface teachers' knowledge about the number line representation. A representation used extensively within English mathematics classrooms and that appears frequently within curriculum documentation – the National Numeracy Strategy (DfEE, 1999; 2006) and the National Curriculum for Mathematics (DES, 1991). Within these two documents, there is no explicit reference to the conceptual knowledge associated with the number line's form and use, despite the fact that this representation is identified as a "key classroom resource". The number line appears not only as an alternative version of the number track, but it also is frequently fragmented to emphasise particular features of the number system such as whole number and fraction. The difference between a number track and a number line lays both in the perceptual and the conceptual sense identified by Skemp:

The number track is physical, though we may represent it by a diagram. The number line is conceptual – it is a mental object, though we often use diagrams to help us think about it. The number track is finite, whereas the number line is infinite. ... On the number track, numbers are initially represented by the number of spaces filled, with one unit object to a space. ... On the number line, numbers are represented by points, not spaces; ... The concept of a unit interval thus replaces that of a unit object.

(Skemp, 1989, pp. 139-141)

Using evidence drawn from the way in which practicing teachers and teacher trainees perceive and talk about the number line, this paper indicates that knowledge based on the perceptual characteristics of the number line together with an ambiguous use of the term number line with that of the number track, express an incomplete and

compartmentalised understanding of the conceptions associated with a representation which is used on a frequent basis in their primary school practice.

THEORETICAL FRAMEWORK

The association between number (real number) and line has been evident since Babylonian times (Wilder, 1968). The Greeks intuitively conceived real numbers as corresponding to linear magnitudes. The Greek idea of “magnitude”, which is substituting magnitude for number, implied that one may think of “numbers as measured off on a line” (Bourbaki, 1984, p. 121). The number line is, therefore, an abstraction of a representation strongly associated with the notion of a measure instrument since continuity underscores it. Starting from the Euclidean line, a “sense of continuity” can be created for and by the individual and the result be used as a number line to represent natural numbers.

Herbst (1997) concurs that the number line is a metaphor of the number system and in order to form a number line:

one marks a point 0 and chooses a segment u as a unit. The segment is translated consecutively from 0. To each point of division one matches sequentially a natural number. (Herbst, 1997, p. 36)

All kinds of numbers can be represented on it. If a series of different number lines each introducing different numbers is built, then the number line could be in one-to-one correspondence between numerical statements and number-line figures. Growing sophistication with its formation supports representation of a number line containing natural numbers, followed by number lines illustrating the positive rationals, the integers, the negative rationals and finally one containing all numbers — the real numbers number line, which would include all numbers. It is these features that would appear to suggest the use of the number line as a pedagogical tool whilst the “dense” quality of the number line enabled Herbst to write about what he calls the “number line metaphor” and the “intuitive completeness” (Herbst, 1997; p.40) of the number line, evolving from plane geometry.

Such features are relevant in the context of teachers’ subject knowledge and awareness of conceptual issues associated with understanding the nature of the number line. Shulman (1986) defines subject matter (content) knowledge as “the amount and organisation of knowledge per se in the mind of the teacher” (p. 9) and distinguishes between the aspects of knowing “that” and knowing “why”. Aubrey (1994) suggests that every teacher has different subject knowledge and personal beliefs about teaching and learning, which are factors affecting their work in classroom; and in order for teaching to be effective, conceptual understanding of knowledge is essential. It is suggested, therefore, that in the context of the number line, teachers would be effective if they conceptualized the representation as a “metaphor” of the number system.

Ball (1990) argues that subject matter knowledge for teaching not only entails ‘*substantive knowledge of mathematics*’ – specific concepts and procedures – but also ‘*knowledge about mathematics*’ – mathematics as a field. Examining what teacher trainees understood about division with fractions as they entered formal teacher education, she focusing on what they have learned as students and what they need to know as teachers. She concluded that the students’ had narrow understanding of division that was compartmentalized and based on rules. This was a view supported by Ball, Hill & Bass (2005) who, as the result of their attempt to measure teachers’ mathematical content knowledge (an amalgam of common content knowledge and specialized knowledge) for teaching, concluded that teachers in general lack strong mathematical understanding and skill.

This paper aims to present one aspect of primary school teachers understanding of the number line as identified by their conceptions of what the number line is. The insight may provide some indication of their potential effectiveness.

METHOD

The results presented in this paper form part of a broader study carried out during 2003 and 2004 (Doritou, 2006) that, given the explicit recommendations regarding the use of the number line within curriculum material, investigated the relationship between teacher’s presentation and children’s understanding of the number line. The study is a case study of an English primary school that follows guidance within the National Numeracy Strategy (DfEE, 1999). The issues addressed the primary school teachers’ perception and understanding of the number line. This paper address one aspect of these issues but it draws its data from two samples that are considered to be related and complementary: (a) teacher trainees and (b) practicing teachers.

As part of an examination of their understanding and perception of the number line the full final year cohort of BA(Ed) students within the Education Department of a large Midland University were invited, through a questionnaire, to “Define a number line”. The response to this question forms part of the focus of this paper. The 69 teacher trainees in the sample, had had the benefit of a four year course associated with the content and pedagogy of primary school (children aged 5-11) mathematics, were fully conversant with the contents of the National Numeracy Strategy, had experience teaching it within school and been provided core lectures associated with the number line. The respondents were followed a mixture of subjects, such as English, art, music and a third of them followed mathematics and science.

The full-time teachers’ sample (also referred to as practicing teachers) contained teachers who taught mathematics within each of the year groups 1 to 6 (median ages 5.5 to 10.5). Through lesson observation and informal interviews on a one-to-one basis the teachers’ perspective of the number line at a personal level and the way they presented it to the children as a pedagogical tool was investigated. Placing the trainees conceptions of the number line within a perspective associated with

practising teachers, it is hypothesised a valuable insight may be gained into what primary school teachers think a number line is.

RESULTS

Teacher Trainees' Conceptions of the Number Line

When the participating Teacher Trainees (TT) were asked as part of a questionnaire to define a number line, only one student provided a definition that implied that the number line was infinite and contained all numbers:

A line that contains all rationals and irrational numbers. It is an infinite line. (TT4)

One other suggested it was:

A continuous line of all of the numbers within our number system. (TT1)

Two others provided definitions that evoked either the notion of infinity but with no further explanation, one indicated that the number line was limited to rational numbers, whilst one other defined a number line with a response that may be interpreted as an association with magnitude:

A sequence of numbers arranged on a line which has an infinite number of divisions. (TT23)

A line of numbers on which any number can be placed. (TT48)

A line where you may place all the rationals at some point on the line. (TT32)

Representation of value according to how far the number is along the line. (TT43)

None of the above students gave any explicit reference to the notion of a repeated unit, which could be partitioned, although partitioning may be implied from the statement of TT4. However, almost one quarter of the students (16/69) did make reference to some form of equal spacing associated with the line, although there was some evidence of little formality about the way they articulated this underlying feature:

A line which is separated equally into different portions. (TT2)

A straight line with equal distances marked. (TT7)

A piece of apparatus with equal divisions marked. (TT10)

13 of these sixteen students associated the notions of equal spacing with numbers although in two instances the students referred to digits:

A line with digits equally spaced along it. (TT47)

A line with numbers attached at equal intervals. (TT66)

A line which numbers are spaced evenly across it in a specified pattern. (TT17)

An equally segmented line, each segment numbered in ascending order. (TT20)

Although it is not certain, TT20's definition suggests that she is thinking about a number line that only has positive numbers. This type of definition was relatively common:

Numbers placed at identical intervals marked on a line in ascending order. (TT15)

and indeed, no student made explicit reference to the notion that a number line could contain negative numbers.

TT17's reference to pattern was, together with notions of order and sequence, a feature of the number line identified by 42% of the respondents:

A string of numbers in a pattern. (TT27)

Numbers in a correct order. (TT9)

A sequence of numbers in a row. (TT22)

A sequence of numbers ordered from left to right. (TT24)

A line in which there is a number sequence reaching from lowest to highest number. (TT11)

An ordered set of numbers in sequence, horizontal. (TT6)

Here again we see no explicit reference to negative numbers. The implications in two of these quotes (TT24 and TT11) suggest that the number line only contains whole numbers, an issue confirmed by the comments of some trainees:

A line with number patterns on it — or from zero to a number. (TT12)

Numbers that have been arranged in some form of sequence mainly from 0 to 10. (TT35)

A horizontal line with a series of digits on it that have a pattern: one to ten; ten to one hundred. (TT42)

The above comments also give the sense that the number line is finite and none of these particular trainees made any reference to the notion of partitioning the intervals. However, one student did provide an indication that partitioning was associated with the line by using the word "divided":

A horizontal line divided into ten equal sections allowing it to be divided into fractions or quantities. (TT64)

Interestingly, in addition to these students who explicitly mentioned order, pattern or sequence, six others introduced the word "chronological" to define the number line:

A chronological line of numbers. (TT37)

A line with marked number intervals in chronological order. (TT56)

A horizontal line where positive numbers ascend in some sort of chronological order. (TT61)

We can see from the definitions provided by the trainees identified through the above examples, that reference to the underlying qualities of Herbst's (1997) definition — the consecutive translation of a segment U as a unit from zero, the partitioning of U in an infinite number of ways — is extremely limited. We note that only three students referred to infinity, but only one of these implied that through partitioning all numbers could be represented. However, though there was no reference to the notion of “consecutive partition”, almost 25% of the teacher trainees indicated that a number line possessed equal divisions but these definitions appear to be founded upon partitioning rather than the continued replication of a defined unit.

Herbst further indicated that a number line could be formed by choosing a unit, repeating it from zero and then attaching to the end of each repeated unit a natural number. Though just over 80% of the teacher trainees associated the notion of the number line with a number or numbers, the majority of the remainder focussed on defining the number line as a tool (see below) but, as TT6 (above) indicated, there was also some evidence that the reference to numbers was not linked to the notion of line.

The overall impression left from the trainees' definitions of the number line was that they did not define it, but instead indicated how it may be seen. The sense was that they were describing a specific number line but often this specificity was limited to the more obvious perceptual characteristics rather than conceptual aspects of the line. In doing this, essential features were often omitted. Only in the first six instances quoted above do we see the trainees' explanations rise above specificity to give more sophisticated responses.

An additional feature of the trainees' definition of the number line was its identification as a tool. Almost 10% of the trainees suggested that the defining feature of the number line was either its use in calculation or in solving mathematical problems:

A continuous line in which numbers can be placed and used to aid calculations. (TT3)

A piece of apparatus with equal divisions which children use to help them count. (TT10)

A line with numbers on representing intervals, aid to solving mathematical problems. (TT34)

or associated it with the notions of counting:

A device to aid learning, involving counting on and counting back. (TT39)

A method used to count on or back horizontally. (TT62)

To aid children when counting up or down. (TT65)

In one instance, the identified process was left open to interpretation:

A way of roughly finding out any numbers between any two given extremes at each end.

(TT52)

Although the above responses emphasise the nature of the number line as a “helping tool” – used as a metaphor to support thinking – and although Herbst (1997) suggested that its dense nature meets such a requirement, there seems little indication from these particular responses that other qualities could be associated with the number line. Additionally, the responses suggest that those students who emphasise use are drawing upon experience, either as learner or as teacher and, it is hypothesised, were drawing upon episodes from within that experience.

Practicing Teachers’ Perception of the Number Line

When the practicing teachers were each informally interviewed about their conceptions of the number line, one issue that was raised was whether or not they thought that the number line was a good representation of the number system.

3 of the 5 teachers identified the number line as a good representation of the number system because it carried the very ideas that 42% of the trainee teachers expressed with their definitions of the line. That is an emphasis on order and sequence:

Yes! I suppose it is because it is natural order in a sequence, isn’t it? (Y2 Teacher)

It’s a good representation for them to be actually able to see it! It has it (numbers) all in order and they can see it! (Y5 Teacher)

The fact that children could ‘see’ the number line was one of the reasons why a Year 4 teacher (teaching children with a median age of 8.5) thought the number line was a good representation of the number system

Because it’s visual and children like visual things, and they can come up and interact with it. (Y4 Teacher)

Having something to see enabled some of the teachers to be quite specific in talking about the number line although there was evidence that this could lead to the sort of confusion identified by Skemp (1989), particularly if we recognise the hundred square as a segmented number track:

I have got the number line, which is really useful, but because it’s so long, it is quite hard... It’s at least two metres (a number line on laminated card under the board). I do refer to it quite a lot, but I do use the number square as well. I do try and encourage the children that it’s the same. (Y2 Teacher)

This similarity between the hundred square and the number line was also volunteered by the Year 3 teacher. He indicated that the number line is a good representation of the number system when used to develop subtraction, but not so easy as the hundred square which is

easier than sometimes using the number line. Really, they're sort of similar things, but this goes zero to one hundred, this goes from one to one hundred, so it's the same really... (Y3 Teacher)

Other evidence associated with seeing and with accessibility came from the Year 1 teacher, who when asked if there was a difference between a number line and a ruler, replied:

I just use the ruler, because it's a good individual tool and easily accessible. So if they want to use the number line it's immediately accessible. (Y1 Teacher)

Within her teaching of the classroom lessons, this teacher and the Year 2 teacher both drew an analogy between the number line and the ruler:

A ruler is a bit like a number line. (Y2 Teacher)

The number line here is like a ruler. Use a ruler¹ as a number line to help you. (Y1 Teacher)

However, the Year 2 teacher preferred to use the hundred square

I do use the hundred square as well in the classroom, coz that's easier to display to be honest. (Y2 Teacher)

One of the teachers explicitly thought the number line was a good representation of the number system, because of the arithmetic that could be done with it:

Yes! Very good! Use it to bridge through multiples of ten. Partition the numbers and then the tens and then the units, if they're doing addition. And if you're working out subtraction. (Y3 Teacher)

The teacher teaching Year 6 among other classes was the only teacher who gave a response that made any reference to the fact that the number line (although finite in her terms) was a representation of the number system:

... I think Year 6 children are quite good to see that the number line represents quarters, halves, numbers up to a thousand or even negative numbers.

This teacher's response to the question "Is the number line a good representation of the number system?" bore remarkable similarities to the trainee teachers' conceptions of the number line. Two of the five teachers referred to pattern, order and sequence. There was reference to the number line as tool, but only one reference to the variety of numbers that could be represented on it. However, whilst all of the teachers could talk about what it may look like or what it may be associated with, none provided a sense of its continuity and density. Those teachers who referred to the hundred square or to the ruler did not make a distinction between the abstract nature of the number

¹ The ruler the teacher referred to and given out to the children was one that represented a number track. It was a wooden 30cm stick divided in squares, with the first coloured yellow, the next green, the next yellow and so on and so forth. Within each box a natural number was written, starting from 1.

line representing continuity and the more concrete nature of the alternatives that represented the discrete nature of number.

DISCUSSION

In their consideration of effective teachers Askew, Brown, Rhodes, Wiliam & Johnson (1997) suggest that effective teachers can be distinguished from less effective teachers in terms of increased fluency in discussing conceptual connections in the context of classroom practice whilst less effective teachers may express a more procedural rather than conceptual personal subject knowledge. The former, generally identified as “connectionist”, appeared to value both pupils’ methods and teaching strategies, in an attempt establish links with mathematical ideas. The latter, those associated with the notion of “transmission”, appeared to prioritise teaching over learning and considered mathematics to be a collection of routines and procedures.

The data presented in this paper would suggest that connectionist values associated with the number line seldom featured in the responses of either trainees or practising teachers. Indeed, most of the English curriculum material presents the number line as a concrete model supporting actions with little if any reference to its strength as an abstract representation of the number system. Such a focus may be more strongly associated with, and possibly even instrumental in, promoting beliefs that are associated with transmission. Though the classroom teachers in this survey applauded the pedagogical benefit of the number line as a tool, neither they nor the trainee teachers provided little explicit or implied indication that this benefit had a formality based upon the repetition of a unit interval and the partition of this interval. Instead we see that perceptual features, frequently implicitly associated with episodes and with a particular “line”, dominated the definitions and additional comments obtained, though, particularly in the case of the teachers, these were frequently tempered by representational ambiguity and supported by counting episodes associated with moving backwards and forwards.

Gray and Doritou (2008) suggest that such conceptions lead to similar conceptions amongst children and though these do not appear to mitigate against the success of younger children in elementary arithmetic they eventually led to confusion amongst the older children. Specific interpretations of the features and use of a number line fail to provide children with a platform from which they may recognise its potential to contribute to the development of a global perspective on the number system. They also fail to contribute towards procedural efficiency as number size increased.

Bright, Behr, Post & Wachsmuth (1988) suggest that the number line is currently an extensively used model in the teaching of mathematics in elementary school, and whilst generally effective is also the source of difficulty both in instruction and its use by children. This paper provides one explanation for this difficulty – a very limited conceptual understanding of the representation by the teachers who use it. It was more general for the number line to be conceptualised as a series of discrete

representations of particular elements of the number system. The notion that it evolved from a unit that could be repeated and partitioned was less important than the notion that actions could be carrying out with it. This emphasis essentially associated with transmission caused ambiguity in the way teachers referred to a number line and, it is hypothesised, a consequent limited understanding of a sophisticated representation by the children who are faced with it.

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COMMUNICATION AS SOCIAL INTERACTION

PRIMARY SCHOOL TEACHER PRACTICES

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Abstract. *This article reports the reflections of a primary school teacher on her communication practices in the classroom and the interaction between the students. It is part of a large research which intends to study the evolution of collaborative work among three teachers and the first co-author of this article, with regard to the knowledge of and development of processes of mathematical communication and interaction in the primary school classroom.*

Key-words: mathematical communication; collaborative work; teaching practices; professional knowledge; teacher education.

Communication as an instrument of the relationship between teacher and students has been the target of widespread dispute in the field of education, given its relevance in the teaching and learning process. The greater value given to the role of dialogue and the sharing of information is opposed to a more traditional form of communication based on a one-way discourse, undertaken by the teacher (Brendefur & Frykholm, 2000).

From this point of view, the transference of information and codes (linguistic and others) is not approached nor studied in itself, but in its use, and communication is characterised as a process of social interaction. It is in this process of interaction that the subjects as well as society itself undergo their construction, through the negotiation of meaning between individuals (Yackel, 2000).

Founded on this desire to understand the role of communicational changes in the teaching and learning of mathematics, the first co-author of this article developed a collaborative research into mathematical communication with three primary school teachers, with the supervision of the second co-author.

This article proposes to explore the way in which the communicative practices of the teacher can raise the value of the communication among the students in the classroom. It results from the work undertaken with one of the teachers who participated in the study – Laura.

COMMUNICATION AS A PROCESS OF SOCIAL INTERACTION

From the point of view of communication as interaction, learning by the subjects arises from interactions between the individual and the culture (Sierpinska, 1998), including the interactions between students and the teacher.

Communication is characterized as a process of social interaction, which permits the subject to identify himself/herself with the other, and at the same time, express and affirm his/her singularity (Belchior, 2003), and has the function of creating and maintaining understanding between individuals.

Thus, teaching is understood as an interactive and reflective process, with a teacher continually engaged in differentiated and updated activities for his/her students. With these activities, meanings are formed in the process of interaction between the subjects, and not only in the transmission of a codified knowledge which is given beforehand (Cruz & Martínón, 1998; Godino & Llinares, 2000; Yackel, 2000).

It is assumed that mathematics teachers' knowledge is a specialized knowledge of and about mathematical (Ball, 2003), practical and personal knowledge (Chapman, 2004; Elbaz, 1983) that teachers develop through the process of reflection. Thus the collaborative work between teachers and researcher is a privileged way for knowing the teachers' professional practices (Boavida & Ponte, 2002).

METHODOLOGICAL OPTIONS

The background investigation for this article fits into a qualitative methodology (Bogdan & Biklen, 1994), which adopts the interpretative paradigm and follow the design of a case study (Stake, 1994; Yin, 1989). Three primary school teachers participated in this study, in a context of collaborative work with the researcher, regarding the reflection about their professional practices concerning mathematical communication.

The study has been conceived in two phases: the *characterization phase* in order to characterize the participants and interpret *the state of the art* (carried out during 2006/2007 academic year) and the *collaboration phase* in order to work together on mathematical communication in the process of teaching and learning (carried out during 2007/2008 academic year).

The data collection consisted of initial and final interviews (audio taped) with the teachers, description of the collaborative meetings (audio taped) between the researcher and the teachers (collectively and individually) and classroom reports (audio and/or video taped). The data were transcribed and reduced in expressive episodes.

In the *characterization phase* an interview was carried out with each teacher. The researcher attended two lessons of each one and carried out two meetings with them. In the *collaboration phase* there were two meetings with each teacher and five meetings of collaborative work. The researcher attended nine lessons of each teacher. The final interviews were carried out at the beginning of 2008/2009 school's year.

The collaborative work implicated theoretical framing discussion, elaboration of mathematical tasks for the classroom and the reflection based on the transcription of teachers' lessons and the video of teachers' and students' communication practices in the classroom.

The data analysis was organized in case studies. Each one with the characterization of the teacher and school context, namely the teacher's mathematical communication conceptions and practices. These were the reflections of the teachers about the facts and situations that gave added value to social interaction between students and mathematical learning.

INTERACTION IN THE CLASSROOM

This section shows how the interaction in Laura's classroom evolved from the beginning of the study and throughout the collaborative work.

The initial reflections (in the characterization phase) of Laura about the interaction among the students in the classroom, in the class group, seem to reflect conceptions associated with the notion of communication as transmission of information:

Normally explaining how they did things, the reasoning, the calculations, but also in relation to the problems. [Interview, December 2006]

This presentation of strategies and reasoning is conducted by the teacher, requiring sometimes the participation of the rest of the class. The students were presenting their productions of rectangular panels constructed with twelve paper squares. (Appendix 1):

Teacher: Which was the first one that you made together?

[The students in the group, up by the blackboard, point to one of their stuck-on designs]

Teacher: That one. How did you make this one here? [Points to the first rectangle]

Student: Four...

Teacher: Four.

Student: Four, four and four...

Teacher: Was it like that?

Another student: Four, three and three...

Teacher: And the second one?

Student: We made it two by two and four by four.

Teacher: Not four.

Student: One, two, three, four...five, six.

Teacher: Ah, and the last one, how was that one? You just said: "We have to make three, three, three...", I said, "no, you already have three, three, three...", "ah, of course there is. So we have to make four, four, four...", "but you already made that here", "Ah, of course that's right. So we have to make two", "but you already have that here". What did you say to me then?

Student: We can make it one by one. [First Year Class, June 2007]

The omnipresence of the teacher in the classroom, allied to the *monologue* of the students, appears to result in an understanding of communication as a way to put forward previously constructed ideas which have been validated by the teacher.

Interaction and Exploration of Error. The avoidance of error in the construction of mathematical knowledge seems to be one of the causes of this constant validation of

the activities of the student by the teacher. As Laura tells us, her main worry in relation to the work of her students was the attempt to avoid error, “*always to get the thing right*” [Collaborative Work, October 2007], given that “*we really love it when they get it right straightaway*” [Idem].

The reflection, in the collaborative group, on the role of off-the-cuff validation and of error, implied that teachers involved in the study made an effort to try to avoid validation of the activities of their students when group work was taking place.

Laura tried to get the students to interact among themselves, in spite of her very much present mediation. As Laura says, despite trying not to interfere so much, the students constantly need her approval, “*Mine look at me and wait for me to say something*”, while they are putting questions to each other [Collaborative Work, November 2007].

In the development of this strategy of communication among the students priority was given to presentation of the incomplete or wrong strategies of the students and consequently to the discussion of the mathematical aspects or other causes for the errors put forward.

In the problem of the River Crossing (Appendix 2), the teacher opted to begin the discussion with a solution that was incongruent with the conditions of the problem. The student Monica presented the solution of her group, writing:

Little Johnny takes the rabbit in the boat. Little Johnny takes the cabbage in his lap and the dog on one side, and they go on their way

While the student was writing on the board, some students were waiting with their hands up, as a sign that they wanted to question their colleague.

Teacher: There are hands up.

The teacher alerted Monica to the questions of her colleague and she ended her presentation and chose one of the other students to ask her a question. After an intervention directed towards the correct solution, one of the students who had identified the incongruency of the resolution with the statement of the problem explained:

Gonçalo: The group wrote “the cabbage in his lap and the dog on one side” but he can only take one animal.

Teacher: Where?

Gonçalo: In the boat.

Teacher: One thing. But three things went.

Gonçalo: Yes, but the cabbage can't go on Little Johnny's lap. There can only go the dog or the cabbage, only one thing. [Second Year Class, March 2008]

The teacher valued the interaction among the students and passed this conclusion on to the group which was at the blackboard, highlighting the impossibility of more than two *passengers* in the boat. Faced with this *rejection*, one of the members of this same group – Tiago – presented a new proposal for the solution, writing:

First goes the dog [the students become agitated because they consider what their colleagues wrote to be wrong]. Second goes the cabbage. And last goes the rabbit.

Gonçalo, observing the solution written by Tiago, says:

Gonçalo: I know what's wrong.

Teacher: So go up there Gonçalo. Go to the blackboard and say what's wrong.

Gonçalo went up to the blackboard and put his reasons to Tiago.

Teacher: Tiago, stay there to defend yourself.

Gonçalo: The dog can't go first, because if Little Johnny took the dog.... If Little Johnny crossed the river with the dog, then the rabbit would eat the cabbage [idem]

The comments of the teacher were intended to promote the interaction between the students – “*There are hands up*” – and to encourage the justification of student's reasons - “*stay there to defend yourself*”. This attitude of this teacher promoted a greater interaction between the students in the classroom.

Interaction and Teaching and Learning. Laura recognizes and values the students change in attitude towards communication by the students, emphasizing that they have also changed their attitude in the other subject areas:

I try to get them communicating among themselves, no matter what the subject is.

[Meeting of the Teacher with the Researcher, April 2008]

This attitude of the students also appears to be related to a significant change of the teacher's attitude in the classroom, in particular with regard to her expectations about students:

I bide my time, I wait, listening more carefully, because at times what they say is important, although sometimes it isn't. [Idem]

This seems to have contributed to a greater autonomy of the students in the learning and construction of knowledge:

[The students] are more at ease, they have a different dynamism. They participate more.

They are more attentive to what they are doing. [Idem]

The development of communication and interaction among the students has changed the way of working in the classroom. As Laura says, “*we are working at a deeper level because there's more discussion*”. [Idem]

SOME FINAL CONSIDERATIONS

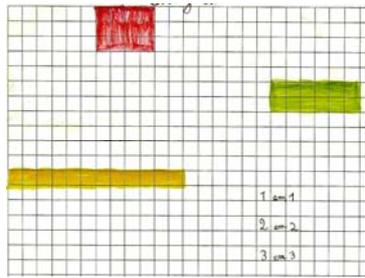
The teacher's practice in relation to the interaction among the students is initially associated with the valorisation of the attitude of exposition of their activities according to the role of the teacher in explaining mathematical concepts.

Teachers were involved in reflecting on their classroom practices in mathematics. With this reflection they began to give more importance to the role of error in mathematics learning, and to allowing students to interact with their peers. This led to increase the interaction among students, either mediated by teachers or not.

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Appendix 1



Appendix 2

River Crossing - The hunting dog, the rabbit and the cabbage

Little Johnny was crossing a dry, unshaded field on the way to his grandfather's house. He was taking with him a hunting dog to go with his grandfather on the hunt, a jack rabbit for his grandmother to put in her rabbit hutch with a pretty female rabbit and a nice cabbage for lunch.

All along the way, the dog wanted to eat the rabbit and the rabbit to eat the cabbage. Little Johnny had to be very careful as he walked along to avoid anything going wrong. After a while Johnny came to a river he had to cross.

In order to cross the river there was a small boat which he could use, but it was so small that he could only take with him one passenger at a time: the dog or the rabbit or the cabbage. He could never leave the dog alone with the rabbit, nor the rabbit alone with the cabbage, so how can he get all of them across without any problem? You are going to have to help to resolve this problem.

EXPERIMENTAL DEVICES IN MATHEMATICS AND PHYSICS STANDARDS IN LOWER AND UPPER SECONDARY SCHOOL, AND THEIR CONSEQUENCES ON TEACHER'S PRACTICES

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The new French Standards for the teaching of science subjects in secondary school advantage the experimental dimension by a revival of words such as "experiment", "experimental" and by the introduction of quite new teaching concepts such as "inquiry-based teaching" and "practical experiment test". Our study deals with the introduction of a new teaching paradigm which includes a strong experimental dimension in both mathematics and physics instructions. The "double approach" frame, including both didactic and ergonomic approaches, constitutes the global frame for the analysis of the teachers' practices we wish to focus on. This allows us to go back over some variables that could be essential to take into account in order to choose appropriate educational devices.

The new French Standards for the teaching of science subjects in lower and upper secondary school advantage the experimental dimension by a revival of words such as "experiment", "experimental" and by the introduction of quite new teaching concepts such as "inquiry-based teaching" and "practical experiment test". This novel approach is common to mathematics, physics, chemistry and biology instruction in lower secondary school. Conversely, in upper secondary school specificities appear depending on each scientific subject. In mathematics, this specific approach leans on more or less implicit references to the use of ICT.

Our study deals with the introduction of a new teaching paradigm which includes a strong experimental dimension in both mathematics and physics instructions. First, we will survey the meaning and the possible place of experiments in the physics and mathematics learning by examining the textbooks and standards. Then, we will focus on the practices of the teachers who intent to implement such experimental elements in their classroom. In that perspective, we use a common frame of analysis ("double approach didactic and ergonomic") in order to raise the predictable complexity of the recommended approach. Some examples are given which analysis leads to the conclusion that either the approach suggested by the teacher is too open and nothing happens or it is too restrictive or reductive, and students have no real access to what is required.

In the "double approach" frame, a didactic point of view and an ergonomic one are interwoven. It constitutes the global frame for the analysis of the teachers' practices we wish to focus on (Robert & Rogalski, 2005, Robert, 2008, Pariès, Robert, Rogalski, 2007). This frame allows us to describe both planed sequence and expected

tasks proposed by teachers (in terms of available knowledge and adaptations), and to confront them to an analysis of the possible children's activity.

To conclude, we will go back over some variables that could be essential to take into account in order to choose appropriate educational devices, that is, concepts or situations that fit with a relevant experimental approach. At the same time, the efficiency of our methodological frame will be thus attested.

THE PLACE OF THE EXPERIMENT IN THE MATHEMATICS AND PHYSICS CURRICULA

In an international context, a lot of researchers looked into experimental activities and enlightened their different objectives. Their results led the authors of curricula to take new directions for science education. It consists in showing a richer image of scientific processes, giving more autonomy to pupils and proposing more open tasks allowing them to develop higher level cognitive activities: the statement of scientific questions, the statement of hypotheses, the design of experimental protocols, the choice and treatment of data and the communication of the results. These different elements have been made explicit in several projects, such as Science for All Americans or in the recent report ordered by the European Commission. More particularly in France, this kind of process in the classroom at low secondary school, is a continuation of a pedagogical practice implemented at primary school since 2000. In France, it appeared in the curriculum in 2005, and was reasserted in 2007 under the name of “*démarche d'investigation*” in French, that has been translated here into “*inquiry-based teaching*” (IBT). This process concerns both mathematics and science teaching.

Despite this common educational text for both mathematics and physics instruction (grade 6 and 7), it seems difficult to implement and to analyze this type of approach in the classroom in the same way in mathematics and physics, insofar as the actual objectives are on both sides different. Indeed, this requires at least to question the very nature of the subject itself (in an epistemological point of view) and the different type of problems involved in a scientific process learning such as modeling the real world, complex operating of tools previously elaborated, etc.

In mathematics, the experimental test in upper secondary school (end of grade 12) includes a consistent and open problem. Students can be asked to model a part of this problem, but this is not systematic (BOEN HS n°7, 2000). From the perspective of potential acquisitions, the experimental test doesn't seek to introduce new knowledge but to make students' knowledge (assumed available) operate. This type of process includes rich, various and possibly new adaptations of this knowledge. Students often face a number of choices: choice of cases to deal with specific software, choice of the software itself, etc. It seems appropriate to *a priori* consider what we want to “win” in terms of students' knowledge (start-up knowledge, knowledge supposedly already there, and also the distance between the two). It is to estimate how students can stage

and work with the "experimental" part itself, given the management developed by the teacher that determines the whole work in the classroom and also number of other constraints such as time, material organization, etc.

In the IBT context, physics teachers are now invited to elaborate problems that are favorable to the development of processes and construction of new knowledge by the pupils themselves (BOEN HS n°6, 2007). At the same time, pupils are given more responsibility and autonomy (the statement of hypothesis or conjectures, the elaboration of an experimental device in order to test these hypotheses). At last, teachers are expected to know pupils conceptions in various subjects and be able to exploit them in the elaboration of sequences that would aim at making these conceptions evolve by using a hypothetico-deductive process. The implementation of the IBT in the classroom requires profound changes in science teachers' practices and experience. A focus on the spontaneous transition between IBT in the curriculum and teachers' practices leads us to draw a picture of the way teachers appropriate the new instructions and allows us to identify the underlying difficulties.

SOME COMMON ELEMENTS OF METHODOLOGY FOR ANALYSING TEACHERS' PRACTICES IN THE CLASSROOM

The « double approach frame » (Robert & Rogalski, 2005) postulates that the analysis of teachers' practices requires for the researcher to draw what tasks are chosen by the teacher for its pupils, and to derive the way its courses are organized. The corresponding analyzes lead to reorganize the activities the pupils could have performed. These analyzes are guided by the choices of the teachers, but they remain inadequate to understand teachers' practices as a whole. Other analyzes, inspired by the ergonomic framework complete the former ones: they include the constraints and the resources associated with the profession of "teacher": institutional constraints (connected with the curricula), social constraints and the constraints connected with the personal resources of the teacher, that is, his beliefs, knowledge and experience.

This theoretical framework is not a model; it is drawn from the Activity Theory (Leont'ev, 1984, Vygotsky, 1997, Vergnaud, 1990). The conversion of fundamental elements of this theory into specific theoretical elements adapted for mathematics or physics and for learning situation allows us to question teachers' practices and to legitimate our research questions whether there are local or global. Thanks to this approach, our questions can be in kipping with a unique framework associated with specific methodologies.

These methodologies involve on the one hand the presentation of a large planed-teaching course that includes the analyzed sequence(s) (either because many sequences are involved or at least to clarify the place of the sequences into the whole course), and on the other hand, the statement of the possible activities of the students. The latter is done trough the confrontation of an *a priori* analysis (including the study

of expositions or instructions and the examination of the data given by tools) with an analysis of the teaching processes.

The *a priori* analysis provides the tasks the students should perform and the corresponding knowledge (Horoks and Robert, 2007). The second analysis (**the analysis of the teaching processes**) refines the *a priori* analysis by taking into account teachers' interventions. This concerns the organization of students' work (including the timing of the different phases) and this also covers their actual work (self-working, part of initiatives, students' involving, teachers' help to the making tasks, aid to overcome the action, reports). Starting from **the recovery of students' activities** we can question and understand the choices done by the teachers and think about alternatives strategies that take into account the standards, different constraints (e.g. time), the habits of the job, and individual characteristics.

CASE-STUDIES

In mathematics

We develop in this communication two examples of grade 12 teaching sequences (12th grade). The two sequences last one hour, with pupils working alone on a computer, and with the teacher helping them individually.

The objective of this session is to discover a property of the slope of the exponential curve, then to prove this property.

EXPERIMENTAL PHASE

To answer this question, you will use the software **Geogebra**

1) Realization of the diagram

(...)

2) Experimentation

Vary the point A on the curve. Observe simultaneously the X-coordinates of A and B

3) Hypothesis

What property seems to be true for all positions of the point A? Try to imagine a method to confirm this hypothesis with experimentation.

RESEARCH OF A PROOF

1) Let a be a real and A the point on the curve $y=\exp(x)$ which X-coordinate is a . Find the equation of the slope T of the curve on A.

2) Can you use this equation to prove your hypothesis?

3) Make the proof of the hypothesis.

Table 1: exposition given in the first example of mathematics teaching sequence

The a priori analysis shows that the experimental activity potentially made by the pupils is banished. Indeed, the ICT tool to be used is given and the objective “discover a property of the slope of the exponential curve” is too hazy to allow an autonomous pupils’ activity. Then, the experimental construction is given by the exposition “realization of the figure” (question 1) and the activity described as experimental (question 2) is reduced to vary a point on the curve and to observe the conjecture as an evidence (question 3): “The X-coordinate of A is always the one of B plus 1”. There is no more one demonstration exercise fairly traditional with no experimental dimension anymore. Even if the introduction of the parameter and the calculation of the equation of T is explicitly asked in the exposition, some intermediary tools have to be introduced by pupils. So this traditional exercise is complex in comparison with the task.

The analysis of **the teaching process** confirms this complexity: the teacher says that “even the best student asks for an indication” and that she finishes the session by showing in a collective way how to do the proof. So, in this first example, there is no experimental activity of students but only several immediate applications of some explicit pieces of knowledge.

The exposition for the second studied sequence is the following:

Let k be a real positive. **We are interested about the number of roots of the equation $\ln(x)=kx^2$ for x positive..**

1. Open the software **Géogébra**.
2. In the entry windows, enter $f(x)=\ln(x)$ then validate. Enter x^2 then validate. Do the same with $0.5x^2$, then $0.1*x^2$ and then $-x^2$. Fill in the table :

Value of k				
Number of roots according the graphical curves				

3. We want now to determine in a more precise way the number of roots. Click on “Fenêtre”, then “Nouvelle Fenêtre” and then let appear the curve of the function \ln in this new frame.
4. Enter $k=1$ in the entry window then validate. This number appears in the algebra window. In the entry window, define now $g(x) = kx^2$.
5. Vary the number k , then click with the right button of the mouse on this number, then click on “Afficher l’objet”. A cursor appears. Click on  to define the mode “déplacer”, and then displace the cursor with the mouse.
6. **Conjecture following the values of k the number of roots of the equation $\ln(x) = kx^2$.**

Call the teacher to validate your answer.

7. If $k > 0$, graphically find a value of k with two right digits after the decimal point for which the equation admits only one solution (you can right click on k and then on “Propriétés”, “Curseur”, to reduce the increment inside the

interval)

Call the teacher to validate your answer.

Demonstrate on your sheet that for any negative value of k , the equation admits a unique solution.

Table 2: exposition given in the second example of mathematics teaching sequence

This second example assumes a level of software's competencies which is lower than the first one but we don't want to enter in this problematic for this communication. However, **the a priori analysis** shows that the experimental construction is again given by the exposition (questions 3, 4, 5, 6). Moreover, the exposition initiates the activity of testing some particular cases of the whole open problem (question 2). The so called experimental part is again isolated from the one more strictly mathematical. This last one is cut in two sub tasks (questions 7 and 8) while a real experimental activity should lead to treat the whole task.

The teaching process shows that lot of pupils don't see the link between the curve they draw during question 2 and some particular cases of the problem. The teacher says that they didn't see how to fill in the table. This reinforces the idea that there is not at all experimental activity during this sequence. Moreover, the question 8, even if it is simplified by the exposition, remains very difficult for pupils.

With these two examples, we understand that the expositions, as in educational texts, are effectively open problems: "to discover a property of the slope of the exponential curve" and "we are interested in the number of roots of the equation $\ln(x) = kx^2$ for x positive". But the field of activity is too large to allow an autonomous activity of students and the tasks are simplified by the expositions: "Realization of the figure", "Fill out the table". In other words, the experimental management is not in charge of the students but it is explained by the detailed expositions. So the hypotheses are evidenced at the end of the explained manipulations. There is no reason to question these hypotheses even if some questions can be awkward in this direction, as in the first example: "Try to imagine a method to confirm this hypothesis with experimentation".

Then, a classical proof ("research of a proof", question 8), isolated from the manipulation phase, is asked. Moreover this proof can be difficult for students because of complex uses of available knowledge and because manipulations don't help for this purpose at all. However, in general, we think that there could be an interaction between the two parts of the session. For instance, in the first example, the manipulation of software Géogébra requires the internalisation of some commands. More precisely, the command "curseur" of the software is deeply associated to the introduction of parameter to prove the hypothesis. So there could exist a thought to help students to introduce parameters in their proofs by training them to associate parameters and "curseur" in ICT environment.

In physics

An analysis of 26 teachers' worksheets available on pedagogical websites and supervised by the educational authorities was conducted a few months ago (Mathé & al. 2008). This analysis revealed important gaps between IBT in the curriculum and teachers' perceptions or appropriation. In particular, it has been shown that few of them make pupils' conceptions explicit in their worksheets and build their sequence in order to destabilize these conceptions. Moreover, while the curriculum comprises a phase of statement of hypotheses, only 11 worksheets ask pupils for stating hypotheses. Furthermore, only 9 protocols are entirely designed by the pupils. In the other worksheets, the teacher plays a more or less important part: whether he designs the protocol himself or he imposes the experimental equipment, or corrects the pupils' propositions (Mathé 2008, Mathé & al. 2008).

The sequence we take as an example concerns combustion processes. The new knowledge aimed by the sequence is exposed as following:

- the combustion of carbon requires oxygen and produces carbon dioxide;
- a fire naturally occurs when air, heat and fuel are combined.

These three elements form the "fire triangle". When one of these elements is missing, the fire stops. The problem to be solved –"How to extinguish a fire"– is connected to an everyday-life starting situation which is supposed to motivate the pupils. They are asked to go outside the classroom, to find all the anti-fire and fire protection devices of the school and to explain the way they operate. Doing so, the teacher expects the children to make hypotheses on combustion process such as "oxygen is necessary for the combustion process" or "combustion produces carbon dioxide". This hypothesis should be tested by appropriate experiments elaborated and performed by the children. The sequence is implemented with grade 7 children and last two hours. It is video-recorded and transcribed. We focus here on specific heading: children's conceptions, the statement of hypotheses, and the hypothetico-deductive process.

The a priori analysis shows that the tasks proposed to the pupils can't destabilize children's conception about fire such as "fire is an object endowed with material properties" widely studied by philosophers and science education researchers (Bachelard 1938, Méheut 1982), and we wonder to what extent it doesn't strengthen it. Indeed, attention to the anti-fire devices operation does not automatically leads to the idea that the air supply is necessary in the combustion process. Consequently, the problem to be solved can't lead to the statement of the expected hypotheses either. Thus, no spontaneous hypothetico-deductive process can be expected.

The analysis of the teaching processes confirms this difficulty. Children are easily involved in the preliminary activity which consists in describing the anti-fire and fire protection devices of the school. A difficulty appears when the teacher asks them to describe the way the devices operate. We observe a misunderstanding between the teacher's expectation which concerns the underlying chemical process and the pupils'

answers that exclusively focus on the description of the way the device is used. This unexpected difficulty leads the teacher to formulate a more precise and guided question: “can you explain why these devices extinguish the fire?”. At that time, a second difficulty occurs which is directly connected to the way that “the fire” is considered in pupils’ mind. As an example, pupils think that fire-resisting doors close in order to prevent the fire to move forward. According to them when the doors are closed the fire “bounces” on them. None of the pupils spontaneously establish a link between the air (specifically the oxygen) and the existence of the fire. This difficulty is widely underestimated by the teacher during the effective sequence. Finally, after one hour of discussion, the expected hypotheses are given by the teacher himself: “oxygen is necessary for a fire to exist” and “a fire produces carbon dioxide”. Pupils are then invited to elaborate experiments in order to test the hypotheses. In this phase, they must isolate the different air contents to prove that only the oxygen plays a part in the combustion process. They also have to elaborate an experiment in order to evidence the carbon dioxide. In the next course, contrary to what was planned, the experiments are imposed and performed by the teacher. This is directly connected to management constraints.

According to the a priori analysis, we observe significant gaps between the teacher’s intentions and what really occurred during the effective sequence. Children’s ideas about the burning process and the fire are not destabilized by the inquiry-based activity itself. The teacher plays a determining part in the knowledge transmission and the starting situation doesn’t allow the implementation of a cognitive-conflict as expected in the IBT. Moreover, the teacher asks the pupils to design an experimental protocol but he finally imposes his own experiment.

CONCLUSION

We assume that no generalities can be asserted as the analyses previously presented remain clinical. Nevertheless, some regularity seems to emerge that form tracks to explore.

What is specific to us is the need for teachers to make a quadruple prior analysis, lighter than the researcher’s one of course, in order to effectively implement this type of process in their classroom:

- an analysis of the aimed knowledge or the knowledge to be used (different from a subject to another);
- an analysis of the available knowledge to permit an autonomous activity of pupils;
- an analysis of the role played by the experimental process in the connection between the aimed knowledge (or knowledge to be used) and the available knowledge considering both content and teaching processes;
- an analysis of the way the teacher can manage this experimental activity.

Moreover, in physics, depending on the nature of the referred content, an inquiry-based teaching can be adapted or not. IBT in the classroom requires choosing relevant scientific content and problem that aim at destabilizing pupils' conceptions and that allow the implementation of a hypothetico-deductive process by the pupils implying more autonomy for the statement of hypotheses and the design of a protocol. However, it may be that students cannot develop hypotheses highlighting their misconceptions. In that perspective, the choice of the scientific subject remains fundamental.

In mathematics, we have seen that there is a problematic amalgam between an experimental approach of mathematical activity and an activity with ICT tools, these tools being able to lead pupils easily to emit correct conjectures for complex problems. The experimental constructions being given by the expositions, the experimental activity can only exist in a one to one correspondence between manipulations (not experimentations) and proofs. This activity, even if it is far from scientific one, can be interesting for using mathematical knowledge (activity with available knowledge or activity with adaptations of knowledge). But it is difficult for students who are not accustomed with these activities. It is also difficult for teachers who have to find adequate situations permitting these go and return between manipulations and proofs and who have to manage at the same time the learning of the new knowledge as well as the learning of software's competencies. This kind of studies has to be completed by some results on individual different students' attitudes when working on computers (Vandebrouck, 2008).

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PROFESSIONAL DEVELOPMENT FOR TEACHERS OF MATHEMATICS: OPPORTUNITIES AND CHANGE

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The RECME research was set up to develop understanding of ‘effective’ Continuing Professional Development (CPD) for teachers of mathematics by looking at a large number of initiatives adopting a variety of models, taking a non-interventionist, non-participatory approach. In addition to building a ‘big picture’, it also aims to develop an in-depth understanding of the individual initiatives by looking at the structure and organisation and at the responses of individual teachers to their CPD. The paper develops and uses an analytical framework to help us understand one particular initiative and the learning and teacher change of individual teachers participating in this initiative. We conclude with a discussion of the factors contributing to the effectiveness of the CPD.

Keywords: Professional development, mathematics, teachers, CPD

INTRODUCING RECME

In 2006 the National Centre for Excellence in the Teaching of Mathematics (NCETM) was set up in England in order to build a coherent infrastructure to support the continuing professional development (CPD) needs of teachers of mathematics. In 2007 the NCETM funded an eighteen month research project, Researching Effective CPD in Mathematics Education (RECME). The aims of the project include the characterisation of different types of CPD for teachers of mathematics and the investigation of the factors contributing to ‘effective’ CPD. In order to understand the range and scope of CPD opportunities existing in the UK, the project team researched a sample of thirty initiatives representing different models of CPD in mathematics education, run by a variety of providers, in different locations, and aimed at about 250 teachers of students in pre-primary, primary, secondary, further and adult education settings.

RECME is an ongoing project and has not yet produced comprehensive findings or recommendations. These are due by March 2009. However, most of the data for the project has been collected and this paper introduces a framework for the analysis of the data and uses it to analyse the data from one initiative.

THEORETICAL FRAMING AND METHODOLOGICAL DECISIONS

We adopt a broad sociocultural perspective which suggests that all human activity, including the learning of teachers, is historically, socially, culturally and temporally situated. This suggests that the experiences and contexts of teachers will have a major

influence on their learning and implies a need to pay attention not only to the situation, the opportunities and the context of sites of learning (in our case initiatives of professional development), but also to the individuals (teachers of mathematics) taking part in professional development.

Data collected

For each initiative we asked the leader/ coordinator for data concerning the form and structure of the professional development. We also observed at least one professional development meeting and took observation notes. The data we collected included dates of meetings, structure of meetings, number of participants, duration of the CPD, what takes place in meetings, funding/costs, support and communication structures, recruitment procedures and leaders of the meetings. For some initiatives not all this data was applicable.

With the help of the leaders/co-ordinators, we identified two teachers from each initiative. We visited these teachers in their classrooms and observed them teaching mathematics in order to develop understanding of the context in which they work, and interviewed them after the observed lesson. The interview data included questions about professional background, perceptions of their professional identity, thoughts on the observed lesson, influence of the CPD on the way they teach, motivation to take part and remain involved in the CPD, their CPD histories and how they felt about the CPD.

Analytical framework

An initiative of professional development can be described in terms of the content, context and processes in which participants engage (Harwell, 2003). There is a wide range of different models of CPD (see for example Kennedy, 2005) but most CPD aims to provide opportunities for teachers to become involved in processes of learning and change. We suggest that different teachers, influenced by the contexts in which they work and their personal motives, beliefs, theories and experience, will perceive different opportunities, and these perceptions may shift over time.

The professional development of the individual teachers inevitably relates to the opportunities provided by the CPD initiative (Muijs, 2008), and may lead to learning and changes in attitudes and beliefs (actual PD). Teachers may also change their classroom practice, but it is possible that changes in classroom practice could also be influenced by other formal and informal learning. Changes in practice could lead to changed student behaviours and possibly improved student learning (Guskey, 2002), although once again there are other factors which might influence any changes that do take place. In turn, changes in student behaviour and learning could influence the teacher learning (Cooney, 2001), their perceptions of the opportunities and experiences offered by the CPD, and the opportunities and experiences they decide to take up.

Finally, a sociocultural perspective suggests that we also need to take into account the influences of the school and national context on the design of the CPD initiative (Bishop & Denleg, 2006; Cobb, 2008) and of the motives, beliefs, theoretical understanding and experience of the designers of the CPD (Rogers et al., 2007), the feedback they receive from the ongoing CPD, as well as the specific aims of the initiative (Goodall, Day, Lindsay, Muijs, & Harris, 2005)

Figure 1, below, provides a diagrammatic representation of the interrelationships of all these factors.

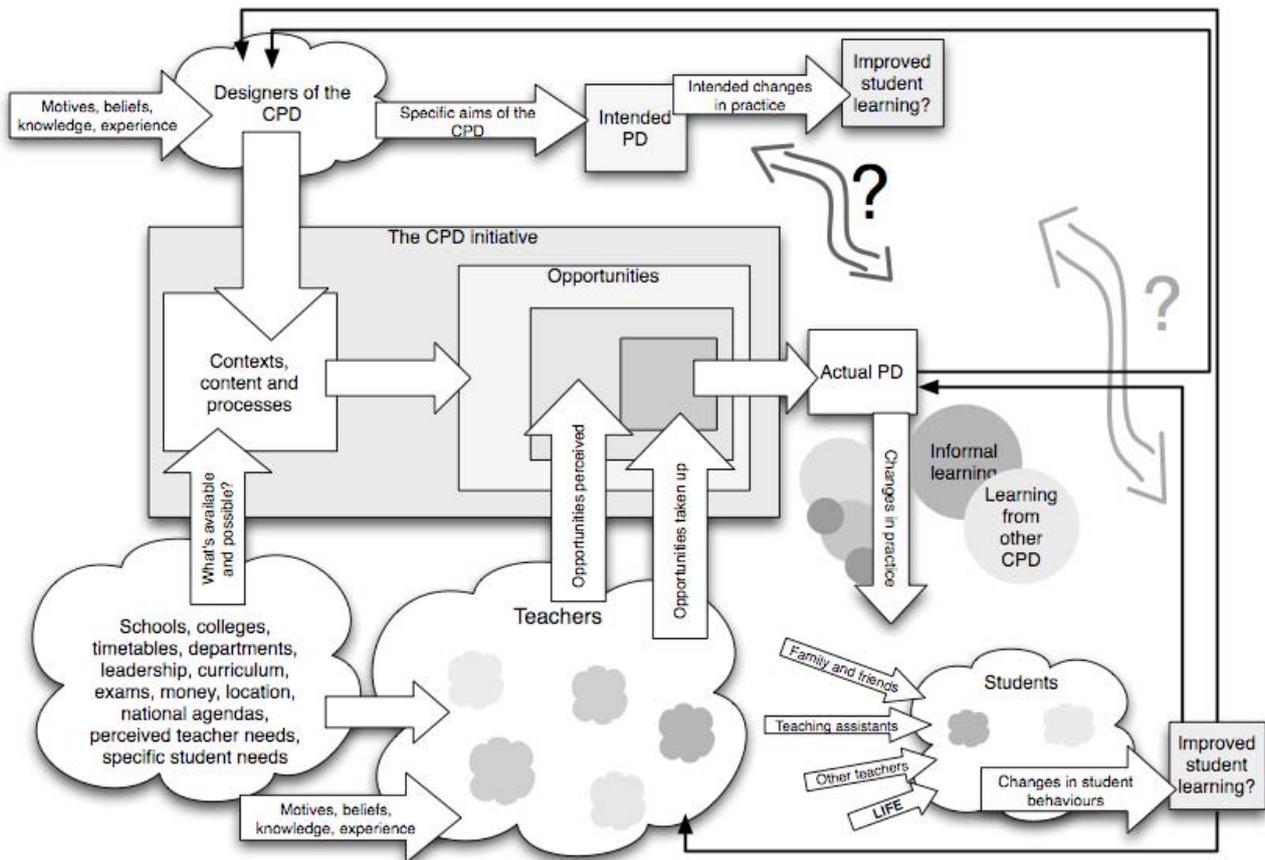


Figure 1: Understanding a CPD initiative

As with many analytical frameworks, this representation could be seen as ‘too neat’, yet the data is messy and complex. Further, it is a static diagram which cannot represent the ways in which the nature of the CPD may be dynamic and changing in response to feedback from teachers and their changing needs over time. However, we suggest that it provides a useful lens for understanding both the CPD initiative itself and the participation of individual teachers. In addition some of these arrows could, in many cases, be two ways.

Further it explicitly attends to the teacher professional development intended by the organisers of the CPD and the intended changes in teachers’ practice, and to learning and changes that do take place. This is important in our view, because both these can be seen to provide some ‘measure’ of the effectiveness of the CPD (Garet, Porter,

Desimone, Birman, & Yoon, 2001; Goodall, Day, Lindsay, Muijs, & Harris, 2005; T. R. Guskey, 2000; Thomas R. Guskey, 2003) (although we do recognise that the ultimate aim of the CPD is usually improved student learning).

CASE STUDY: ONE INITIATIVE AND TWO TEACHERS.

Context, content and processes of the CPD initiative

This initiative is run by a local authority mathematics adviser and a university-based teacher educator. The initiative is now in its third year; two cohorts have already completed the programme. The participants are all secondary school mathematics teachers who attend five separate day-long meetings over the course of a year.

During the meetings the course leaders initiate discussion, frequently asking the participants to discuss issues (for example, how they feel about group work in the mathematics classroom) and then to report back to the group. Frequently one of the course leaders notes down the points made on a flip chart and, when each small group has reported back, draws out some of the key points. During the meetings they also introduce new resources to the teachers and discuss how they might be used and hand out research papers and give the teachers time to read them and then lead a discussion about them. Much of the material they hand out focuses on questioning techniques and much of the discussion concerns using open questions and tasks rather than closed questions and tasks.

In addition, they introduce various classroom mathematics activities and ask the teachers to work in small groups to complete them. For example, one of these activities uses small cards with equations, graphs and co-ordinates of points printed on them, although some are left blank. The task is to decide how to group them, but importantly there is no correct or incorrect answer, and consequently can be seen as providing rich learning. Further, when these activities are used in the classroom, they provide opportunities for teachers to assess their students' prior knowledge. The teachers are asked to experiment in their classrooms between the meetings by using either this activity (suitably adapted for their particular circumstances) or some other activity designed by themselves. The activity they choose to use is called a 'gap' activity (because it is to be carried out in the 'gap' between meetings). There is no prescribed type of gap activity; the key point about the gap activity is that it represents something new for the teacher to try out in the classroom. Teachers are asked to bring some of the students' work from these gap activities to the next day meeting to form the basis of discussion.

Teachers are also asked to keep a journal. At the last day meeting, they are asked to make a presentation to the group, outlining how their practice has developed through the project.

Aims of the CPD

Although the course leaders state that ‘this project focuses on helping teachers to understand the underlying principles of assessment for learning and applying these to embedding effective practice in the classroom’ (www.nctem.org/recme), they told us that the actual content addressed in each of the days is, to some extent at least, informed and influenced by the work of the teachers both during the meetings and in the classroom, and by their concerns and questions. In order to be free to follow this flexible approach, the course leaders deliberately do not have any further documented specific aims.

However, they told us that their general aims are threefold and they see them as related and interdependent: to provide time for the teachers to reflect, to encourage teachers to put their learning into practice in the classroom and to engage the teachers with relevant research.

They also said that the course aims to create a community in which teachers meet, talk, share and learn from one another. The leaders have created a community web page where the teachers are able to share resources, thoughts and ideas, away from the face-to-face sessions.

Intended professional development (teacher learning)

The course leaders told us that they hoped that by providing the opportunities described above, participating teachers would be inspired to think more critically about their own practice and revise it accordingly, to pay more attention to how pupils learn mathematics, and to develop the confidence to allow pupils to follow their own directions rather than scripting their lessons in detail.

Intended changes in practice

The intention is that teachers will change their practice in the short term by experimenting with the gap tasks. In the longer term the course leaders said they hoped that teachers’ practice would change in three main ways:

- They would use more challenging and open tasks in the classroom, with less reliance on textbooks and closed questions, leading to more exciting and unpredictable lessons for the students
- They would reflect more on what happened in mathematics lessons, thinking more about what the learning had been rather than about how much material had been covered
- They would become more relaxed in their interactions with the students and develop more collaborative classroom practices.

The teachers: Barbara Bircher and Peter Millward

This section discusses the CPD experiences of Barbara and Peter, the two teachers who were invited to take part in the in-depth part of the research. It reports on what they said when our researcher interviewed them and on the observation of their lessons, and uses the framework developed above to structure the discussion. It begins by describing the backgrounds of the teachers and the contexts in which they work.

Barbara has been teaching mathematics in secondary schools since 1976 and is now subject leader for mathematics in her school. Peter is in his third year of teaching at a large comprehensive 11 – 18 school where he has overall responsibility for the first three year groups in the school (known as Key Stage 3 and culminating in a standardised national test).

Barbara became involved in the current CPD because she had heard a lot about the course, which is now in its third year, and she liked what she heard: the approaches she heard they promote are similar to the ones she believes in. She thought it would be valuable for someone in the department to attend and decided to go herself (rather than sending someone else from the department), because then she could cascade her learning to the rest of the department. She saw this as an opportunity for her to develop herself in order to ‘move the department forward’.

Peter said that he decided to take part in the CPD because a member of the senior leadership team asked him if he wanted to go. He said that much of the CPD he had previously experienced had taken place in school and ‘seems to be more about technical jargon than new stuff but that he chose to attend this CPD because he was looking for something with more mathematics.

Opportunities

In this section we report on those opportunities provided by the course that Barbara and Peter seemed to value. Both teachers mentioned the resources they had been introduced to, with Barbara saying that she valued having time to investigate them and Peter saying they were useful.

Barbara said that she values the time out of school to reflect and think and discuss, she enjoys having time to read. Peter also said he liked the fact that there was enough time for discussion and he seemed to value the opportunity to meet with other people in order to ‘stock up’ with ideas to try out in the classroom.

Peter did not mention the value of gap tasks, but he did say that, as a result of the course, he has to ‘push’ himself to try something out and this is the most useful thing about the course. Barbara told us that she had used most of the gap tasks with her classes and reported back on them. She said that knowing that she ‘had to’ report back on how she had found teaching these gap tasks meant that she had actually done them, and that otherwise she may not have. She said she enjoyed reporting back to

the group after doing a gap task. She said that the course had given her the opportunity to do what she believes is good maths teaching.

To Peter, the course leaders are very important; ‘they prepare the stuff, they help us along’. He says that they provide a link between the theory and practice in both his own classroom and what other schools are doing. The local authority advisor has a good overview of what happens in his local authority, and he says this is useful for the teachers.

Actual professional development

Barbara said that using the gap tasks had challenged her embedded practice of expecting the students to work in a predetermined direction and reawakened her awareness that ‘the obvious isn’t obvious’. She said that it has kept her interest in mathematics teaching and her desire to be a reflective practitioner continuing to improve. She said that the course had reminded her about what she really liked doing; teaching mathematics, adding that in recent years she has moved gradually away from her passionate interest in teaching, because of the pressures of school and management. Barbara said the course made her very excited and gave her the opportunity to do what she believes is good maths teaching. She finished the course wanting more. More specifically, she reported that she had learnt the value of sharing students’ work and of developing a classroom culture in which ‘it is ok to be wrong, as long as you are thinking about your learning’.

Attending the course had made her think about the direction she wanted to move in, in terms of her role in the school, and has provided her with clear ideas about the way she intends to develop the department.

Peter was much less forthcoming about telling us about his learning and changes in beliefs. However, he did report that the course ‘replenishes my enthusiasm’. He also remarked on a change in awareness:

‘I am more aware of what I am doing and thinking much more about what I am doing and why’.

Changes in practice

Both teachers reported that they had implemented some new teaching tasks as a result of the CPD. Barbara had tried some of the gap tasks and is now incorporating more open and investigative tasks in her everyday teaching. For example, she gave the class coloured paper and scissors and provided the students with instructions on how to create the shapes she wanted them to work with. Over the course of several lessons, the students investigated angles and lengths in the shapes, as well as tessellation properties.

Peter, on the other hand, did not use a gap task but told us that he has tried to integrate some of the ideas from the CPD into his normal practice, rather than relying

on the textbook too much. He has also used ideas for new tasks which came from another teacher in the group. For example, he asked a year 9 class to write a test and devise a mark scheme and he was very pleased with the work they produced. He was particularly pleased with the work one of his students produced. He said:

‘I will use this idea again - its fairly easy to setup, although grading is quite a challenge. It’s effective because it allows students to show what they have learnt and it always easily differentiates between students’ abilities. Answering a question on a test can be algorithmic, writing a challenging question (with a mark scheme) can show greater understanding’.

Both teachers reported that they used more open small tasks at the beginning of the lesson (sometimes called starter tasks in the UK). For example, Barbara said she might present a diagram and ask students to write a statement about it; she remarked that previously she would probably have asked a more direct question. She said she allowed them to make any points they wanted before she directed the discussion towards her main teaching points. She chooses some starter tasks in order to promote discussion, such as asking the students to find a number with exactly five factors, which led to a discussion of the fact that numbers with an odd number of factors are a special sort of number (square). She said that in the past she would probably have given the class a more closed starter such as ‘What are the factors of 16?’ Peter provided an example, saying he might say ‘The answer is a quarter, what is the question?’ and he said this provided the students with opportunities for creative thinking.

Barbara told us that in order to share students’ work she obtained a visualiser (a device which projects anything put under its lens onto a whiteboard) for her classroom. She now regularly shares student work in lessons. She also told us that because of her participation in the course, she has talked freely with her team about her own learning and she thinks this is good for the team. When our researcher spoke briefly to the second in charge in the department, he reported that the whole department had benefited from Barbara’s CPD because she shared new ideas with them and encouraged them to experiment in their own classrooms.

Peter says that since he has been doing this CPD his teaching has changed. He says that he tries hard not to talk to the students from ‘high up’ and that he likes to get down to them (physically). He has started to move away from writing the lesson objectives on the board, and now has primary and secondary objectives (skills-based and content-based respectively). Sometimes he leaves an objective blank and asks the students at the end of the lesson what it they thought it was. This is an idea that came from someone at the CPD.

THE INTERRELATED FACTORS CONTRIBUTING TO EFFECTIVE CPD

The discussion above provides some evidence that for both teachers some learning and changes in practice took place. In-line with the learning and changes the course

leaders intended (see page 5), both teachers took some risks, using more open and challenging tasks in the classroom, and developing more relaxed interactions with their students. Barbara appears to have developed confidence to allow pupils to follow their own directions more and she had begun to think more critically about her own practice. We argue that this demonstrates that, to some extent at least, the CPD was 'effective'.

This raises the questions of the factors that may have contributed to this effectiveness, and what barriers may have been present to reduce effectiveness. First, both teachers confirmed the importance of experimenting in the classroom as suggested in the literature (see for example, Guskey), and what is perhaps interesting is how the CPD is set up to encourage this experimentation. We suggest that teachers involved in this CPD felt they have to try something new in their classroom, because it is expected and because of the need to report back to the group. There was also some encouragement from the leaders' comment that attending the course gave permission to take risks. It is interesting that Barbara chose to do the gap tasks, whereas Peter decided to try something suggested by one of the other teachers participating in the CPD. This may demonstrate that, although it was expected to do something between meetings, it seems that the way the task was set up allowed a great degree of personal choice in the selection of gap tasks.

The differences between the gap tasks chosen by the two teachers may be explained by the differences in their experience and positions in their respective schools and by the culture of the schools. For Barbara, as an experienced teacher and head of department it may have been much easier to implement the gap task suggested by the leaders of the CPD, but as Peter told us, he was not able to experiment and try out new things in the classroom as much as he wanted (this was partly because of an intervention programme that has been put in place in his school to address the whole school emphasis on raising attainment).

Second, being part of the CPD group was important to both teachers. This does not surprise us, as again the literature suggests that working collaboratively may contribute to effective CPD. However, we are interested in what it was for the two teachers that they valued. What seemed to be important for Peter was having access to new ideas, whereas Barbara's emphasis was on the sharing of what she had done and the out-loud reflecting on it.

Thirdly, and again unsurprisingly (Borasi, Fonzi, Smith, & Rose, 1999; Day, 1999; Olson & Barrett, 2004), it seems that having time away from school to think and discuss was important to the teachers, although we cannot tell what contribution this discussion made to the professional development of the teachers. However, our suggestion is that they found it stimulating and enjoyable, and that this sort of discussion has an important role in retaining the interest and motivation of teachers.

As a final point, our observation of two of the meetings suggests that the participants enjoyed 'doing' the mathematics and our suggestion is that this is an important factor

contributing to 'effective' CPD. However, interestingly, neither teacher commented on the enjoyment they experienced when they were given the mathematical gap tasks to work on in the meetings.

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TEACHERS' PERCEPTIONS ABOUT INFINITY: A PROCESS OR AN OBJECT?

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The present study aims to examine elementary school teachers' perceptions about the notion of infinity. In particular, the two aspects of the concept- as a process or as an object- were examined through participants' responses. In addition, teachers' reactions during the comparison of infinite sets or numbers with infinite decimals were analyzed. Data were collected through a self-report questionnaire that was administered to 43 elementary school teachers in Cyprus. Data analysis revealed that the majority of teachers comprehend infinity as a continuous and endless process; thus, teachers confront difficulties and hold misconceptions about the concept.

Key words: infinity, teachers' perceptions, misconceptions, actual and potential infinity

INTRODUCTION

A major component of the research in mathematics education in the last decades has been the study of students' and teachers' conceptions and reasoning about mathematical ideas. Most of the research purported to examine the existence and persistence of alternative conceptions (preconceptions, intuitions) which diverge from the accepted mathematical definitions (e.g. Monaghan, 1986; Tall, 1992). The concept of infinity may be seen as a mathematical idea that causes various obstacles to learners due to the duality of its meaning, as an object and as a process (Monaghan, 2001). Thus the present study examines how primary school teachers conceive the notion of infinity in an attempt to define the notion, to provide suitable examples and to comprehend numbers or sets with infinite elements.

THEORETICAL FRAMEWORK

Definition of the concept of infinity

The notion of infinity constitutes an intuitively contradictory concept that has long occupied many philosophers and mathematicians. Concretely, infinity emerged as a philosophical issue in the work of Aristotle, who separated the concept in two different aspects- potential and actual- that correspond to the ways of looking at infinity- as a process or as an object (Sacristàn & Noss, 2008; Tirosh, 1999). According to Aristotle the potential infinity can be conceived as an ever lasting activity that continues beyond time, while the actual infinity as the not finite that is presented in a moment of time (Dubinsky et al., 2005). The former category of infinity appears as something that qualifies the process, whereas the latter category refers to an attribute or property of a set (Moreno & Waldegg, 1991).

The acceptance of potential infinity elicited a mathematical way of thinking that gave rise to great accomplishments in Greek mathematics - such as, the Eudoxus method

of exhaustion– but ruled out the possibility of developing an actual conceptualization of infinity (Moreno & Waldegg, 1991). In the 19th century, actual infinity through Cantorian set theory has profoundly contributed to the foundation of mathematics and to the theoretical basis of various mathematical systems (Tsamir & Dreyfus, 2002).

According to Galileo and Gauss, the use of actual infinity leads to inherent contradictions since it cannot be included in a logical, consistent reasoning. Due to the fact that the human brain is not finite, individuals cannot consciously focus on all the information at a given time- and therefore conceive infinity as an object- but they move between different aspects- and conceive infinity as a process (Tall, 1992). Usually, learners define infinity as "something that continues and continues" and not as a complete entity (Monaghan, 2001; Tirosh, 1999) or they conceive infinity using the limit notion, referring to a process of “getting close”, with the limit perceived as unreachable (Cornu, 1991). On the other hand, the concept of actual infinity ascribed to learners through the reference to large finite numbers or to collections containing more than any finite number of elements (Monaghan, 1986).

The construction of the N set

From the time that Aristotle introduced the two meanings of infinity- potential and actual- difficulties in the understanding of the set of natural numbers were provoked. For example, regarding the formation of the set of natural numbers, a simple, not finite process begins from number 1 and adds one in each step indefinitely without stopping. This results to a line of infinite sets ($\{1\}$, $\{1,2\}$, $\{1,2,3\}$, ...), which is an instance of potential infinity, a series of sets without end (Lakoff & Núñez, 2000). On the contrary, someone may consider the set of all natural numbers, without having the ability to enumerate all the elements of the set. By the encapsulation of the process, the object of $N = \{1,2,3,\dots\}$ is created, that corresponds to the set of natural numbers (Monaghan, 2001). That is an instance of actual infinity - a completed infinite entity (Lakoff & Núñez, 2000).

Comparing infinite sets

One of the misconceptions that appears in the comparison of infinite sets is the application of properties that apply only to finite sets. Tsamir and Tirosh (1999) mentioned that methods used by learners for comparing infinite sets are largely influenced by the methods they tend to use when comparing finite sets. As Galileo (1945) pointed, a finitist interpretation that prevails upon the comparison of infinite sets is the use of the inclusion idea: that a set and a proper subset cannot be equivalent (Sacristàn & Noss, 2008; Tirosh, 1999). For instance, every natural number has its square and vice-versa, which means that the set of natural numbers and the set of their squares are equivalent, although the set of squares is a subset of natural numbers. Such a conclusion is not consistent with simple logic since the whole and the part cannot be equivalent. Therefore, an individual, in an attempt to reinforce his/her beliefs that a set has a different cardinality from any of its subsets, uses the justification of “part-whole” (Singer & Voica, 2003) than the one-to-one

correspondence among the elements of sets that determines the equivalence between infinite sets (Tirosh & Tsamir, 1996).

Furthermore, many researchers (e.g. Tirosh, 1999; Tirosh & Dreyfus, 2002) explored the impact of different representations on the comparison of the same infinite sets. Researchers have focused on students' inconsistencies in relation to the concept of infinity using four different representational registers: horizontal, vertical, numeric explicit and geometric. Tirosh and Tsamir (1996) found that a numerical horizontal representation- in which the two sets are horizontally situated one next to the other- encouraged part-whole argumentation. On the contrary, the geometrical representation that is constituted of a schematic drawing of sets, triggered equivalent responses and "matching consideration" through a notion of pairing elements (Tirosh & Tsamir, 1996). It seems that geometrical representation prevents access to higher levels of conceptualisation and allows better understanding of one-to-one correspondence among the elements of infinite sets (Moreno & Waldegg, 1991).

Conceptualising the equalities $0.999\dots=1$ and $0.333\dots=1/3$

Various obstacles are presented with limiting processes that deal with the properties of the set of real numbers and of the continuum (Sacristàn & Noss, 2008). In particular, difficulties are observed during the comparison of irrational numbers which consist of infinite repeating and non-repeating decimals (Vinner & Kidron, 1985).

Many studies focused on the conceptualisation of the equalities $0.999\dots=1$ and $0.333\dots=1/3$ (Edwards, 1997; Monaghan, 2001). The majority of students tend to reject the former equality, on the ground that the two numbers have a negligible difference from one another (Monaghan, 2001) and with the limit being viewed as a boundary, rather than as the value of infinity (Cornu, 1991). With respect to the second equality, students seem to accept that $0.333\dots$ tends to $1/3$, as it may result by dividing 1 by 3, something unfeasible in the case of the equality $0.999\dots =1$ (Edwards, 1997). This happens because most students conceive number 1 more as an object, as an entity, while $0.999\dots$ is conceived as a process (Monaghan, 2001).

So far, several studies have examined learners' perceptions and misconceptions about infinity (Tsamir & Tirosh, 1999; Monaghan, 2001; Edwards, 1997). However, there is a lack of research studies that examine teachers' perceptions about infinity and this fact has served as a motivation to conduct this study. Namely, the purpose of the present study is threefold. Firstly, this study aims to examine the perceptions of elementary school teachers regarding the concept of infinity. In particular, the two aspects of the concept- as a process or as an object- are examined through the definition and participants' responses. Secondly, misconceptions that participants have during the comparison of infinite sets or numbers with infinite decimals will be discussed. Finally, the impact of different representations in the comparison of infinite sets will be investigated.

METHODOLOGY

Sample

The present study involved 43 participants, 25 pre-service and 18 in-service primary school teachers, 12 men and 31 women. The experience of in-service teachers in instruction varied from one to 32 years. In addition, 25 participants possessed a master degree and one of them was a PhD degree holder. It is worthy to notice that the participants were randomly selected from a seminar offered in Mathematics Education at the University of Cyprus during the fall semester 2007-2008, without taking into consideration if they were pre-service or in-service teachers.

Instrument

Data were collected through a self-report questionnaire (Figure 1), which took 20 minutes to complete. The questionnaire was comprised of four tasks that aimed to identify perceptions related to the concept of infinity.

<p>1. a) Please give a definition of the concept of infinity. b) Give two examples for the concept of infinity.</p> <p>2. How many elements are there in the set $S = \{-3, -2, -1, 0, \{1, 2, 3, \dots\}\}$?</p> <p>3. Which of the following sets has the bigger cardinality? Please justify your answer. a) The set of natural or the set of even numbers? b) The set $A = \{1, 2, 3, 4, \dots\}$ or the set $B = \{1, 3, 5, 7, \dots\}$? c) The set $A = \{1, 2, 3, 4, \dots\}$ or the set $B = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$?</p> <p>d) The set of squares $A = \left\{ \overset{1 \text{ cm}}{\square}, \overset{2 \text{ cm}}{\square}, \overset{3 \text{ cm}}{\square}, \dots \right\}$, or the set of numbers $B = \{1^2, 2^2, 3^2, \dots\}$?</p> <p>4. a) Is the equality $0.999\dots = 1$ true? Please justify your answer. b) Is the equality $0.333\dots = 1/3$ true? Please justify your answer.</p>
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Figure 1: The tasks of the questionnaire.

The first task aimed at eliciting teachers' perceptions about the concept of infinity. Participants were asked to report a definition for infinity and to present two examples that would involve the particular concept. The definitions were not coded as right or wrong answers according to formal mathematical concepts and notations, since the goal of the task was to address the underlying conceptions of infinity as a process or as an object.

The examples suggested by participants were grouped as mathematical or empirical examples according to their context. In particular, the examples that referred to mathematical concepts were categorized as mathematical examples. At the same time, the examples related to personal experiences or knowledge from real life were considered as empirical.

The second task examined teachers' understanding about the construction of an infinite set. Specifically, participants were asked to determine the cardinality of the set $S = \{-3, -2, -1, 0, \{1, 2, 3, \dots\}\}$, in which the infinite set of natural numbers appeared as an element of a different set. Moreover, the task attempted to investigate teachers' understanding about the construction of the N set as an entity or as a process.

The third task aimed to investigate the methods that teachers use during the comparison of infinite sets: the part-whole and the one-to-one correspondence. In addition, this task examined the impact of different representations in the selection of a criterion to determine the equivalence of infinite sets. The impact of four representations- horizontal, vertical, numeric explicit and geometric- were investigated in the comparison of infinite sets (Tirosh & Tsamir, 1996).

Finally, the fourth task included two sub-tasks that examined teachers' comprehension of the equalities $1 = 0.999\dots$ and $1/3 = 0.333\dots$ (Fischbein, 2001; Dubinsky et. al, 2005). The task aimed to observe the way teachers understand numbers with infinite digits and to compare the answers of the sample between the two equalities. The comparison was based on the different nature of the numbers, since the division of $1/3$ can result to $0.333\dots$, in contrast to 1 that can not be produced directly by $0.999\dots$

The questionnaire required teachers to complete the four tasks and to justify their responses. Quantitative data were analyzed with the statistical package SPSS using descriptive statistics. The justifications and the examples provided by the sample were analyzed using interpretative techniques (Miles & Huberman, 1984), as evidence of teachers' perceptions about the concept of infinity.

RESULTS

Task 1. Definition of the concept of infinity

Two out of three participants (72.1%) defined infinity as an endless process. Teachers used phrases such as: "it goes on forever", "it's a process that never ends", "it has no beginning and no end...always follows another number", "keeps going and increasing". The remaining teachers (27.9%) defined infinity as an object. In their own terms: "it is an infinite whole", "it is something countless", "it is a set with unlimited elements", "it is an undefined set".

The majority of teachers (79.1%) were able to provide two examples for the concept of infinity, either mathematical or empirical, while 11.6% provided only one. The remaining 9.3% of the participants were unable to provide at least one example. Specifically, 62.8% of teachers presented two mathematical examples and 86.1% provided at least one mathematical example. The mathematical examples that were provided can be grouped as: (a) sets of numbers (e.g. natural, odds), (b) infinite sequences and series, (c) numbers that can be expressed as an infinite sequence of decimal digits (e.g. $\sqrt{2}$, $1:3$), (d) geometrical examples (e.g. the set of straight lines through a point, the set of rectangles with perimeter 20 cm) and (e) trigonometric examples (e.g. the tangent of 90^0).

On the other hand, only 30.3% of the participants gave empirical examples. The empirical examples that were provided in their own words were: “sunrays”, “earth’s rotation about its axis” and “the number of a satellite’s tracks in the void”. Participants provided wrong examples for the concept of infinity using objects the quantity of which is a large finite number, as stars, universe, sounds, grain of sands, and the number $10^{10^{10}}$. In addition, it is worthy to notice that 2.3% of the participants did not provide any example at all. One interesting statement was the following:

“There are no specific examples for the concept of infinity. By the moment you define it, it stops being infinity any more”!

Task 2. The construction of the N set

In the second task, that referred to the cardinality of the set $S = \{-3, -2, -1, 0, \{1, 2, 3, \dots\}\}$, two different answers emerged. Even though it may seem to be striking, 38 out of 43 teachers (88.4%) considered the cardinality of the set S as infinity, while the rest of them (11.6%) considered that the cardinality is 5. The majority of the participants used explanations such as:

“Set S has infinite elements, since it is an overset of $\{1, 2, 3, \dots\}$ that is infinite.”

“The set consists of infinite elements, because this (showing the N set) is unlimited.”

“The cardinality of S is infinity because if you add 4 elements to infinity, you get infinity again: $\infty + \alpha = \infty$.”

“Elements included in S are: -3, -2, -1, 0 and all natural numbers.”

“S is an infinite set in its positive direction.”

Task 3. Comparing infinite sets

The third task aimed to investigate the way different representations influence the comparison of infinite sets. As expected, the geometric representation helped the comparison more than the others, since 76.7% of teachers realized that the two sets presented, had the same cardinality. The respective percentages of correct answers for the other representations were: 46.5% for verbal, 51.2% for horizontal, and 55.8% for vertical representation.

As Table 1 shows, the geometric representation facilitated the participants to understand the one-to-one correspondence among the elements of the two sets rather than the remaining representations. Nevertheless, none of the teachers showed a coherent reasoning that connects infinite sets to confirm their explanation.

Justifications	Representation			
	Verbal	Horizontal	Vertical	Geometric
1-1 correspondence	3 (7.0%)	3 (7.0%)	5 (11.6%)	13 (30.2%)
Part-whole	18 (41.9%)	17 (39.5%)	15 (34.9%)	6 (14.6%)
None	22 (51.2%)	23 (53.5%)	23 (53.5%)	24 (55.8%)

Table 1: Justifications for the comparison of infinite sets

Moreover, the geometric representation reduced the misconception “the whole is greater than the part” that in other cases causes false answers. Some indicative false answers using the “part-whole” justification are presented below:

“There are more natural numbers than odd numbers. Odd numbers are only a part of natural numbers.”

“Set $A=\{1,2,3,4,\dots\}$ has more elements than set $B=\{1,3,5,7,\dots\}$, because set A contains also even numbers.”

“Set $B=\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ has additional elements than $A=\{1,2,3,4,\dots\}$, since you can find many fractions between two natural numbers.”

Task 4. Conceptualising the equalities $0.999\dots=1$ and $0.333\dots=1/3$

Participants conceived the above equalities differently, providing three categories of answers. Specifically, 41.9% of teachers thought that the equality $0.333\dots=1/3$ is right in contrast with 4.7% that accepted the equality $0.999\dots=1$ as correct. The majority of the teachers (58.1%) used the concept of limit to confirm the correctness of the equality $0.999\dots=1$, while only 27.9% of them used a similar explanation for the equality $0.333\dots=1/3$. The difference between the two conceptions was supported by the following statement:

“ $0.333\dots=1/3$ because if you divide 1 by 3 you get 0.333... but you don't get 0.999... if you divide 1 by 1.”

A considerable number of participants answered that the two equalities are false (34.9% for $0.999\dots=1$ and 27.9% for $0.333\dots=1/3$). Some indicative false explanations offered by teachers regarding the equality $0.999\dots=1$ were the following:

“Number 1 will always be larger than the largest decimal number 0.999...”

“In daily life, the equality can be right due to rounding-up, but in mathematical contexts, the numbers 0.999... and 1 are different.”

“There is an infinitesimally small difference between the two numbers.”

“An equality is not right unless $a=a$ is valid.”

Teachers' explanations for the equality $0.333\dots=1/3$ were similar to the former ones.

DISCUSSION

The present study examined elementary school teachers' conceptions about infinity. Specifically, the aim of the study was threefold: to examine teachers' perceptions about the nature of infinity as an object or as a process, to investigate teachers' misconceptions during the comparison of numbers or sets with infinite elements and to discuss the impact of different representations in the comparison of infinite sets.

The majority of teachers comprehend infinity as an unlimited process as indicated by their responses on tasks 1, 2 and 4. This finding is in accordance with the work of many researchers (Tall, 1992; Monaghan, 2001; Tirosh, 1999) who stated that a person's comprehension regarding the notion of infinity is supported by the strength of his intellectual finite schemes that are mainly referred to the process that creates infinity than to the completed entity. The intuitive interpretation of infinity as potential constitutes a cognitive obstacle in the understanding of the concept and

therefore individuals confront difficulties and hold misconceptions about the concept (Fischbein, 2001).

Teachers mainly conceive infinity as a mathematical idea with limited applications to daily life. The fact that teachers quoted examples from various fields of mathematics (e.g. geometry, trigonometry, and series) indicates that the concept of infinity is presented throughout the mathematics curriculum. Although some empirical examples were provided, these included large finite numbers. According to Singer and Voica (2003), due to the human's disability in counting the grain of sands or in computing the number $10^{10^{10}}$, the person correlates them with the concept of infinity. Indeed, when an individual cannot observe something with his/her senses totally, then this thing is connected with the notion of infinity, which is by definition something unreachable.

The results of the study reveal the correlation between the definitions of infinity with its mathematical implications during the construction of an infinite set, as the \mathbb{N} set. Although teachers were expected to determine that set $S = \{-3, -2, -1, 0, \{1, 2, 3, \dots\}\}$ is identical to set $S = \{-3, -2, -1, 0, \mathbb{N}\}$, it seems that they couldn't perceive $\{1, 2, 3, \dots\}$ as a single object, as an entity. According to Dubinsky and his colleagues (2005), an individual is able to construct a completed idea for the concept of infinity after interiorizing repeating endless actions, reflecting on seeing an infinite process as a completed totality, and encapsulating the process to construct the state at infinity, understanding that the resulting object transcends the process.

Teachers' decisions as to whether two given infinite sets have the same cardinality depend on the specific representation in the problem (Tirosh & Tsamir, 1996). Geometric representation yielded one-to-one correspondence during the comparison of infinite sets and helped teachers avoid the justification "part-whole". The schematic drawing, in combination with the vertical representation, facilitated teachers to understand that infinite sets had the same cardinality. In contrast, the use of horizontal and verbal representations caused misconceptions of the form "part-whole" similar to those reported by Singer and Voica (2003). This particular finding shows that teachers give contradicting answers during the comparison of the same sets that are presented in different representations, not acknowledging that incompatible responses are not acceptable in mathematics.

Participants' responses about the equalities $0.999\dots = 1$ and $0.333\dots = 1/3$ confirm the results of previous researches (Monaghan, 2001; Cornu, 1991; Fischbein, 2001). Although the aim and the context of the two equalities were similar, they caused different answers. The equality $0.333\dots = 1/3$ was accepted as valid easier than the equality $0.999\dots = 1$ which reinforced the use of limit. As Edwards (1997) stated, $0.333\dots$ equals to $1/3$ because it might result from the division 1 by 3. Indeed, the number $0.333\dots$ can be constructed from a process, in contrast with $0.999\dots$ that is not intuitively or visually understandable (Dubinsky et al., 2005). For this reason, the concept of potential infinity is used in the first case, while in the second case there

is a mixed understanding of potential (0.999... as an infinite sequence of 9's) with actual infinity (object conception for the number 1).

The present study offers teachers an opportunity to consider the misconceptions related to the concept of infinity. If these misconceptions are reproduced during teaching, then students' misconceptions about the concept of infinity will be empowered and in turn become very difficult to overcome. The notion of infinity is related with important mathematical concepts, such as number configuration, number comparison and the numerical line, that are important for arithmetic and algebra. For this reason, teachers must be aware of the difficulties encountered regarding the specific concept, in an attempt to avoid "problematic" teaching. In addition, it is important for teachers to develop conceptual understanding of the notion of infinity that is to connect potential and actual infinity with concrete examples from real life (Singer & Voica, 2003).

Furthermore, the present study offers educators an opportunity to consider the abovementioned misconceptions and to propose ways to overcome them. In particular, academic programs offered to teachers should include mathematical knowledge regarding to infinity in combination with instructional approaches related to the concept. A proposed teaching approach could include the following steps: presentation with several typical tasks aimed at uncovering teachers' intuitions about the concept, discussion about infinity's applications in real life, introduction of the formal definition of infinity and the two aspects- potential and actual- and attempt to distinguish them in examples. Furthermore, students' difficulties for the concept, comparison of the intuitive beliefs in light of the formal definition, and explanation of the symbols and other representations of the concept may be presented. Thus, in the framework of the training program teachers could be exposed to opposing views of the concept that may be used to develop a more coherent appreciation of the formal definition and to the refinement of intuitions (Mamona-Downs, 2001). As Fischbein (2001) noted, appropriate teaching may help the learners to cope with counter intuitive situations while it makes them aware of intuitive constraints and of the sources of the mental contradictions.

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PERCEPTIONS ON TEACHING THE MATHEMATICALLY GIFTED

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The aim of this study is to describe and analyze the structure of the perceptions of elementary school teachers concerning mathematically gifted students. The study was conducted among 377 elementary school teachers, using a questionnaire of 21 statements on a 5-point Likert type scale. The results of the study revealed that teachers' perceptions regarding gifted students in mathematics can be described across four dimensions based on the following factors; teachers' needs, teachers' self-efficacy beliefs, characteristics of the gifted and the different services delivered to meet the needs of the gifted. Implications for teachers, researchers and policy-makers are discussed.

Keywords: giftedness, teachers' perceptions, teacher training, self-efficacy beliefs, special education

INTRODUCTION

Gifted students differ from their classmates. Therefore, differentiated instruction is required, in order to maximize their talents. However, according to Archambault et al. (1993), as well as Westberg et al. (1993), very few instructional or curricular modifications are made in regular elementary classrooms in order to enhance gifted students' abilities.

The present study purports to examine the perceptions of elementary school teachers regarding gifted students, with reference to mathematics. In particular, in this paper we firstly aim to confirm that teachers' perceptions can be defined across four dimensions which correspond to teachers' needs, teachers' self-efficacy beliefs, the characteristics of the gifted and the different services delivered to meet the needs of the gifted, as described in the model developed specifically for this study. Secondly, we intend to investigate the structure of teachers' perceptions about the ways to address the needs of gifted students, the characteristics of mathematically gifted students and the importance of the teacher in order to be able to provide the appropriate support and guidance to these students.

Investigating the views of teachers regarding gifted students is expected to provide valuable information on the aspects which are susceptible of improvements. In addition, this study could serve as a starting point for the development of inservice programs for teacher education concerning mathematical giftedness.

THEORETICAL FRAMEWORK

Characteristics of gifted students in mathematics

Mathematically gifted students are characterized by an expanded cognitive base and are more capable of exploiting knowledge in order to realize their objectives. A necessary trait of a teacher of the gifted should be the knowledge of their characteristics and needs, as stated by Kathnelson and Colley (1982). Several characteristics of mathematically gifted students have been discussed in previous studies. Maker (1982) pointed out three key areas in mathematics that gifted students differ from their peers; pace at which they learn, depth of their understanding and their interests.

Regarding the first area, gifted students are capable of providing answers with an unusual speed and precision (Heid, 1983), namely they are able to solve mathematical problems faster (Hettinger & Carr, 2003). Their ability in identifying relationships in subjects, concepts and ideas without previous related teaching (Heid, 1983), increases the pace at which they learn. The fact that gifted students are flexible in using different strategies and they are able to select the most suitable strategy for each situation in combination with the possession of complex metacognitive and self-regulative skills (Hettinger & Carr, 2003) proves the depth of their understanding. In addition, Johnson (2000) reported that mathematically gifted students give original explanations and have the ability to organize data, transfer knowledge and generalize ideas. It has also been observed that gifted students are often more interested and perform better in tasks that require mathematical reasoning than computational processes (Rotigel & Lupkowski-Shoplik, 1999). As far as their interests are concerned, gifted students prefer to discuss with adults and to be involved with professionals. They are more favorable to advanced issues than their classmates, e.g. mathematical proof, politics, space.

Nurturing gifted students

A number of methods have been proposed and developed to fulfill the needs of gifted students. Among them, enrichment activities, differentiation of teaching, flexible grouping, acceleration and increased use of technology are the most common ones. Research by Rotigel and Pello (2004) has shown that a combination of the aforementioned approaches is the best solution for the gifted.

Enrichment refers to the presentation of content in more depth, width, complexity or abstraction related to the curriculum delivered to all students (Florida Department of Education, Bureau of Exceptional Education and Student Services, 2003; Rotigel & Pello, 2004). According to Lewis (2002) and Renzulli (1976), new content is added to the curriculum, existing content is explored in more depth and the curriculum is expanded with additional tasks that require cognitive and research abilities.

Acceleration is defined as the practice of presenting content sooner or in a faster pace. Brody and Benbow (1987) argued that acceleration can be obtained in a variety of ways. For example, acceleration can be achieved in one or many subjects or by skipping grades. In addition, university courses offered to secondary education gifted students or early graduation from secondary education and early enrolment in a

higher institution may be considered as acceleration options (Brody & Benbow, 1987). Acceleration provides the appropriate level of challenge in order to avoid boredom from repeated learning and to decrease the time required to graduate from an educational level (National Association for Gifted Children, 2004).

Useful suggestions about ways teachers can use in their classrooms in order to differentiate teaching to fulfill the needs of gifted students are provided by Johnson (2000). In particular, Johnson (2000) pointed that gifted students need inquiry-based learning approaches that emphasize open-ended problems with multiple solutions, as an opportunity to show their abilities. To this end, the teacher should pose a variety of higher-level questions during justification and discussion of problems. Moreover, technology can serve as a means for the gifted student to reach the appropriate depth and width of the curriculum (Johnson, 2000).

Teachers' needs

There is a prevailing myth that gifted students do not need special attention since it is easy for them to learn what they need to know (Johnson, 2000). On the contrary, their needs require a deeper, broader, and faster paced curriculum than the regular one. Due to the complexity of giftedness, it is of great importance that teachers have specialized preparation in gifted education, namely in identifying and nurturing the mathematically gifted (Johnson, 2000; VanTassel-Baska, 2007). Not only strong pedagogical knowledge is needed, but also a strong background in mathematical content. Providing a more general framework, Jenkins- Friedman and her colleagues (1984) argued that an effective teacher should have five kinds of skills; managerial-facilitative, pedagogical, social-consultative, directive and planning and interactive skills.

In this direction, Gear (1978) observed that teacher effectiveness can be improved with specific training. VanTassel-Baska (2007), commented that teachers of the gifted need to be able to address multiple objectives at the same time, recognize how students might manipulate different higher level skills in the same task demand, and easily align lower level tasks within those that require higher level skills and concepts.

Despite all recommendations and efforts in providing appropriate support to gifted students, previous studies have shown that the majority of teachers have neither the time, qualifications nor sources to develop and implement a differentiated curriculum (Tyler-Wood et al., 2000). In addition, low teacher efficacy beliefs in meeting the needs of gifted students, their lack of relevant teacher training which is partially originated by the lack of preparation for this task during their graduate studies (Lee & Bailey, 2003), reveals the intensity of this phenomenon.

Teachers' perceptions regarding gifted students

Teachers' perceptions about teaching and learning have a powerful influence on the ways teachers act in the classroom and interact with their students (Bain et al., 2007).

Despite their importance, little is known about the current perceptions of individuals in teacher-education programs regarding the educational practices for gifted children (Bain et al., 2007). Particularly in the case of gifted students, there is a disparison between teachers' perceptions; on the one hand teachers are overwhelmed to work with gifted children and on the other hand they are negatively prejudiced towards them.

Regarding positive perceptions held by teachers about gifted students, Rothney and Sanborn (cited by Martinson, 1972) noted that teachers believe that the gifted will reveal themselves through academic grades and they need all existing content plus more. Therefore, teachers should add to the existing curriculum material requirements rather than delete anything. Studies conducted by Justment and colleagues (cited by Martinson, 1972) revealed that teachers experienced with special programs were generally enthusiastic to work with gifted students, since the experience with training programs produces more favorable attitudes toward gifted children (Martinson, 1972).

Nevertheless, teachers of the gifted often feel threatened by these students since they are sometimes confronted with students with more knowledge and abilities than themselves (Shore & Kaizer, 1989). In addition, the often stated misconception, as suggested by Bain and her colleagues (2007), namely that gifted children will find their way on their own, provides an alibi for educational system to continue neglecting their needs.

METHODOLOGY

Subjects

The sample consisted of 337 elementary school teachers. Table 1 presents demographical data of the study sample. The percentage of each category is presented in parenthesis.

Years of service	Men	Women	Total
1-10	39 (11.57)	174 (51.63)	213 (63.20)
>10	26 (7.72)	98 (29.08)	124 (36.80)
Total	65 (19.29)	272 (80.71)	337 (100.0)

Table 1: Sample demographic data

Data Collection

In order to collect data for this study, a questionnaire was administered to 337 elementary school teachers in Cyprus. The questionnaire consisted of 21 statements in a 5-point Likert scale with number 1 referring to the complete disagreement of the teacher and number 5 represented complete agreement with the statement. Participants indicated the degree that better expressed their opinion. In addition, empty space was provided to optionally add any remarks.

Data analysis

Data collected were analyzed in an effort to explore the perceptions of elementary school teachers regarding mathematically gifted students. In particular, the statements focused on four aspects; teachers' role, teachers' self-efficacy beliefs, ways to meet the needs of gifted and their characteristics. Given that on the theoretical part of the study several issues regarding mathematical giftedness have been highlighted, an effort was made to assess whether a theoretically driven model would fit to the data. To achieve this, confirmatory factor analysis was performed.

The statistical modeling program MPLUS (Muthen & Muthen, 2007) was used to test for model fitting in the present study. Three fit indices were calculated, before evaluating model fit: The ratio of chi-square to its degree of freedom (χ^2/df), the comparative fit index (CFI), and the root mean-square error of approximation (RMSEA). According to Marcoulides & Schumacker (1996), in order to support model fit, the abovementioned indices required to be verified. In particular, the observed values for χ^2/df should be less than 2, the values for CFI should be higher than 0.90, and the RMSEA values should be close to or lower than 0.08.

RESULTS

In this study, we hypothesized an a-priori structure of teachers' perceptions regarding the mathematically gifted and then tested the ability of a solution based on this structure to fit the data. The proposed model consists of four first-order factors: teachers' needs (F1; statements 15, 17, 18 and 21), teachers' self-efficacy beliefs toward teaching the mathematically gifted (F2; Statements 5 and 13), ways to meet the needs of these students (F3; Statements 9 and 20) and characteristics of gifted students in mathematics (F4; statements 1, 2 and 3) that form the second-order factor of teachers' perceptions concerning the mathematically gifted.

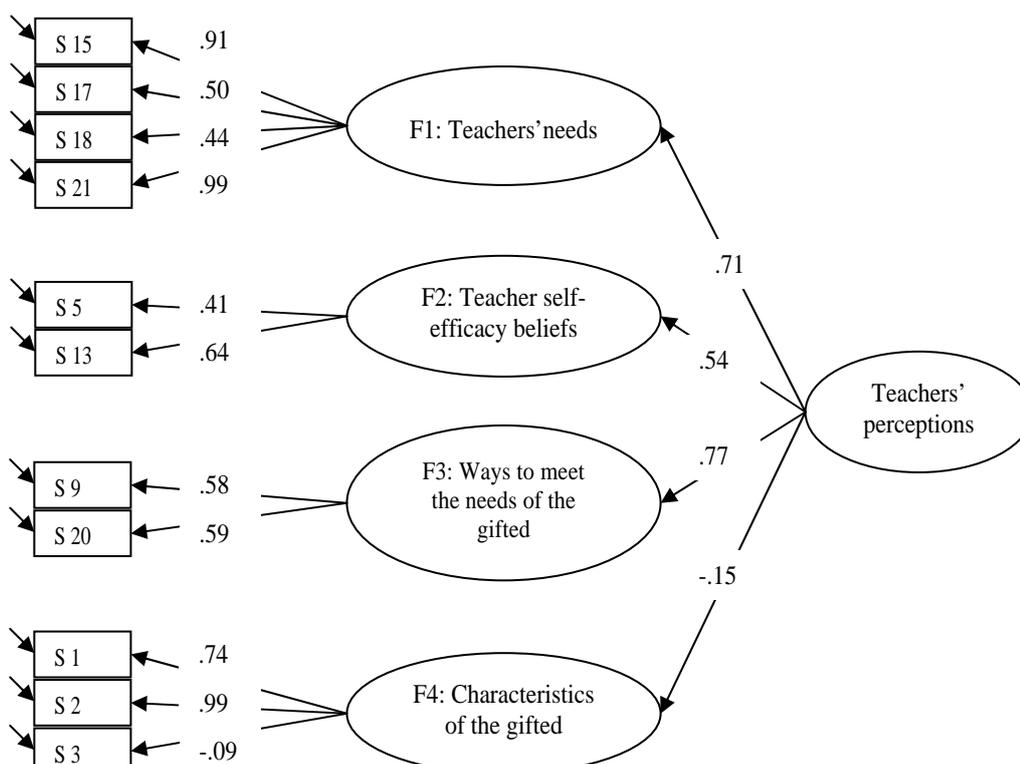


Figure 1: The structure of teacher perceptions about gifted students in mathematics.

Figure 1 presents the structural equation model with the latent variables of teacher perceptions regarding mathematically gifted students and their indicators. The descriptive-fit measures indicated support for the hypothesized model ($CFI=0.97$, $\chi^2=66.07$, $df=40$, $\chi^2/df=1.65$, $p<0.05$, $RMSEA=0.04$). The parameter estimates were reasonable in that almost all factor loadings were statistically significant and most of them were rather large (see Figure 1). Several statements were excluded from the model due to their low factor loadings compared to the remaining statements. The 11 statements included in the model are shown in Appendix 1.

In particular, the analysis showed that each of the statements employed in the present study loaded adequately only on one of the four factors (see the first order factors in Figure 1), indicating that the four factors can represent four distinct aspects of teachers' perceptions concerning gifted students in mathematics.

Teachers' comments that were written in the empty space provided are presented below to enhance the proposed model, after being categorized in the four factors formed by the abovementioned model.

Factor	Teachers' comments
1. Teachers' needs	<ul style="list-style-type: none"> - It is necessary for the teachers to receive training in teaching gifted students. Having a counselor in each school will be very helpful for the teachers. - The ideal is to have special teachers for gifted students in each school.
2. Teacher self-efficacy beliefs	<ul style="list-style-type: none"> - Gifted students might ask difficult questions that I will not be able to answer. I prefer not to have one in my classroom. - I am not aware of the criteria to identify a truly gifted child.
3. Ways to meet the needs of the gifted	<ul style="list-style-type: none"> - The Ministry of Education should send material for the gifted in order to differentiate their work. - The school should support gifted students, not only students who experience difficulties. They should be given opportunities to take advantage of their talents and experiences according to their interests. Challenging activities should be provided in order to avoid boredom. - It is difficult for them to follow a mechanical learning path. Thus, the learning process should conform to their personality and allow for creative activities. - Gifted students do not always prefer to have differentiated work. Sometimes they prefer to work like the others. Particularly in the first grades, they do not want to differ. - They should help low-ability students and facilitate teacher's work. - They can develop their talents out of school motivated and supported by their parents. - The fact that they have different potentials than those of their classmates, is enough. They do not need any other differentiation.

4.Characteristics of the gifted	<ul style="list-style-type: none"> - Being gifted does not mean being perfect in everything. Usually, you meet students gifted in one or more domains. - I think that nowadays it is difficult to talk about gifted students. Many children have special abilities-talents in specific domains. Intelligence is defined from various factors. A student may be gifted in mathematics, while another student may be gifted in art. I would say that there is no general giftedness. - In my opinion, the term “gifted” does not exist or it is used erroneously. - There are no objective criteria to define a student as gifted.
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Table 2: Teachers’ comments

DISCUSSION

Given the importance of the role of the teacher both in identifying and nurturing gifted students, the aim of this study was to examine the structure of the perceptions of elementary school teachers concerning gifted students in mathematics. The study reported in this paper provided evidence that teachers’ conceptions about mathematically gifted students can be described across four dimensions based on the following factors. Specifically, the first factor is teachers’ needs to appropriately cater this special group of students. The second factor refers to self-efficacy beliefs held by teachers, such as considering themselves able to provide adequate support to mathematically gifted students and help them realize full potential. The third factor is the different ways used during teaching to meet the needs of the gifted, i.e., providing them with more challenging activities than those of their peers. The fourth factor consists of the characteristics of the gifted; for instance, that gifted students prefer to reason than proceed to computational processes. The abovementioned structure suggests that teachers need to work not only on their knowledge regarding the characteristics of gifted students and the different approaches that proved to be useful in providing appropriate services, but also knowledge and skills required for the teachers to possess, as well as their self-efficacy beliefs. Based on this assumption, we could speculate that programs aimed at educating teachers in the domain of gifted education and more specifically in the field of mathematics, should focus on these four aspects.

The high factor loadings of the statements regarding the existence of counselors of the gifted in schools (S15 and S21) to the corresponding factor might be explained by the fact that teachers receive no guidance or training regarding educating the gifted. This is also reported in the remarks provided by teachers after completing the questionnaire. In Cyprus, there is no provision for gifted students stated in the mathematics curriculum. Therefore, the need for gifted education programs inside or outside the school boundaries is apparent. The teachers’ concerns about the absence of relevant support by the state is also evident by the factor loadings of F1 and F3 in the second-order factor which is the teachers’ perceptions. The results verify similar findings by Tyler-Wood et al. (2000) as well as by Lee and Bailey (2003).

It is evident from teachers’ remarks related to the ways of meeting the needs of gifted students, that although they are aware of various approaches, such as differentiation as suggested by Johnson (2000), enrichment discussed by Lewis (2002) and Renzulli

(1976), they also hold various misconceptions. In particular, a remark that was noted by a teacher is that gifted students should help low-ability students and facilitate teacher's work. Another view held by a teacher is that the fact that gifted students have different potentials than their classmates is already enough and they do not need any other differentiated teaching. The aforementioned perceptions contribute to the prevailing myth that gifted students do not need special attention since it is easy for them to learn what they need to know (Johnson, 2000). Another teacher pointed out that students can advance their talents out of school motivated and supported by their parents. It is also important to note that no teacher mentioned anything about the use of technology as a way of supporting mathematically gifted students as proposed by Johnson (2000).

The results reveal that teachers are also concerned about their efficacy. In fact, a teacher acknowledged the fact that he is not able to identify a gifted student, while another teacher stated that gifted students might ask difficult questions, thus embarrassing the teacher and causing negative attitudes towards the gifted. This remark enhances the findings of Lee and Bailey (2003), according to which teachers have low efficacy beliefs in meeting the needs of the gifted.

At the same time, the characteristics that distinguish mathematically gifted students do not seem to be of great significance to the teachers. This could be owed to the fact that teachers are more interested in providing suitable experiences and activities for their students, without being aware of their distinctive characteristics. This implies that whether teachers have high ability or gifted students in their classrooms, they treat all students in the same way. In order to successfully deliver the appropriate services to gifted students, teachers need first to identify them. Therefore, a solid understanding of characteristics observed in gifted children should be a requirement for teachers.

The present study extended the literature in a way that a model was validated examining the structure of teachers' perceptions concerning the mathematically gifted. The model proposed in this study offers teachers, researchers and policy makers a means to examine mathematical giftedness as it is experienced through the eyes of the teachers. From the perspective of teachers, the model may be used in order to acknowledge their lack of knowledge regarding behaviors that characterize gifted students and receive the appropriate support to feel confident to help mathematically gifted students realize their potentials. From the perspective of researchers and policy makers, it is likely that the model could serve as a starting point for the development of appropriately designed teacher training programs for the identification and nurture of the gifted. As a consequence, the change observed to teacher beliefs towards the gifted could be examined by researchers, as well as their shift in using various instructional approaches regarding mathematically gifted students. Finally, policy makers could exploit the results of this study by adding a special section in the curriculum for gifted students, acknowledging the fact that they have special needs that should be met.

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Appendix 1: The 11 statements included in the model.

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- S1** Mathematically gifted students solve problems faster.
- S2** A mathematically gifted student prefers to reason than compute.
- S3** Gifted students might have attitude problems.
- S5** I believe that I have the appropriate means to provide adequate support to gifted students.
- S9** Gifted students should be provided with more challenging activities compared to their classmates.
- S13** Having a gifted student in my classroom makes me feel very nervous.
- S15** It is important to have at least one specially trained teacher for gifted students in each school.
- S17** It is important to use identification procedures for gifted students.
- S18** University programs should include teacher training regarding teaching gifted students.
- S20** Acceleration of gifted students should be permitted through grade-skipping.
- S21** I believe that there should be counselors/mentors for gifted students.
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THE NATURE OF NUMBERS IN GRADE 10: A PROFESSIONAL PROBLEM

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Teachers who teach grade 10¹ in France have to ensure the continuity of the mathematics taught between Junior High and Senior High² without doing any systematic revision. It seems to be a difficult task as teachers have to elaborate on reprise gestures³ (Larguier, 2005) to go over knowledge already taught in Junior High while also introducing new knowledge. It is thus this problem of the profession (Cirade, 2006), which we analyze through direct observations of classes and data collected, about the way teachers tackle this. This study has allowed us to show some characteristic elements of this teaching problem. For example, the determination of the nature of numbers is a type of tasks between the two institutions; it can also be gone over as a reprise in various niches of the syllabus throughout the year. However, we show that teachers do not seem to take advantage of these opportunities.

Keywords: reprise, professional gestures, the filter of the numeric

A PROBLEM IN THE PROFESSION OF TEACHING THE NUMERIC

Going into grade 10 in France is a threshold to be crossed between Junior High and Senior High; it is an important passage between the two institutions. The mathematics syllabus states that in grade 10 students have to master the knowledge and know-how that most of them have already been taught in Junior High. A question then becomes central: the relationship between the *professionnal reprise gestures* and the knowledge and know-how. It takes us to the broader question of *interweaving* (Bucheton, 2009). We analyze the kind of gestures about the synthesis of numbers encountered during Junior High which must be done thoroughly during Senior High. We update the problems for teachers even though this part of the syllabus does not seem to be problematic for them to teach.

THEORETICAL FRAME

To study the question of the construction of numeric space in grade 10, we essentially use the framework of the “anthropological theory of didactics” which has been developed by Chevallard (2007) and studies concerning the numeric and the algebraic

¹ In France, there are two distinct institutions after primary school: “collège” for students aged 11 to 15 and “lycée” for students aged 15 to 18. The first Senior High class is called “seconde” and corresponds to grade 10.

² Junior High school will be used for “collège” and Senior High school will refer to “lycée”.

³ “Reprise” can mean to go over, to patch together, to interweave. We shall use the word “reprise” for reasons of economy.

in Alain Bronner's works (1997, 2007). Bronner developed a tool for the study of numeric space: the “filter of the numeric”. The function of this filter is to "pursue" the numeric, whether it is at a practical level or an institutional level. Various elements of a numeric space can be identified:

- **The objects:** the number systems, the set operators (taking the square root) and comparators ($< \dots$);
- **The types of practices** (exact calculation, approached calculation and mixed calculation) as well as the various institutional contracts of calculation;
- **The articulations and the dynamics** of the numeric domain with the other domains as well as the underlying contracts;
- **The rationales** (“*raisons d’être*” in French) of the numeric.

Analysis of the numeric domain is completed by the identification of the “mathematical organizations” of the numeric. Together they make up a numeric space. The observation of the numeric space also includes the “didactic organization” to say what is specifically numeric. We also take from Chevallard (1999) the notion of praxeology which is broken down into four elements: type of tasks, technique, technology, theory. It permits us to model a teaching task which we indicate by professional gestures. We also use the levels of didactic determination defined by this author (1999) to question the conditions and restrictions of various origins which weigh on the didactic choices of the teachers. These levels as defined by Chevallard are: civilization, society, school, pedagogy, discipline, domain, sector, theme, and subject.

The study of the *reprises* can be analyzed according to different criteria (Larguier, 2005). The principal criteria of all the *reprises* can be represented on an axis, the extremes of which are:

- on the one hand, systematic revisions which do not meet with the new knowledge required by the syllabus;
- on the other hand the *reprises* which link up with new knowledge. In other words, the new learning and knowledge are the continuation of the study which began in the previous classes. This first criterion can also vary between systematic revisions (a kind of repetition of the same), a form denounced by the official curriculum; and *reprises* in accordance with the syllabus which introduce something new.

The second criterion of analysis of the *reprises* concerns the mathematical contents institutionalized at the end of the learning experience. It involves the targeted mathematical praxeologies, in other words the mathematical organization. This establishes a connection with the objectives of the teacher with regard to the types of mathematical tasks which are given. These objectives are:

- techniques to be reproduced by imitation and without a justification, so that technologico-theoretical elements of the praxeology are missing;
- know-how only for action, legitimized only by explanations which do not allow for updating mathematical rationales. Technologico-theoretical elements of the

praxeology are then incorrect towards the epistemology of the discipline;
 - knowledge constituted with complete praxeologies that supposes that four elements of the praxeology are present and based on mathematical rationales.

This second criterion is called completeness of the praxeologies. It identifies the degree of completeness between two extremes: they are complete, and it seems that they are mathematically valid; otherwise they are incomplete.

METHODOLOGY

Our research on the teaching praxeology concerning the *reprises* of the numeric leans on the study of grade 10 with a particular methodology. It differs from usual methods in the didactics of mathematics; in fact the analysis of the teaching practices in the classes is not conditioned by the objectives and the expected behavior of the researcher. This would have been clarified by an analysis a priori according to the research project. Here, observation in class comes first, permitting discovery and access to the knowledge taught, without any interaction between the teacher and the researcher. From elements revealed to the researcher in the dynamic of the teaching, an analysis a priori is elaborated. This is done by taking into account the previous experiences of the students, the didactic memory (so called by Brousseau) of the class and the requirements of the syllabus. It is then possible to make parallels between this analysis a priori and the project of the teacher reconstituted by the researcher after the session. In the same way, parallels can be drawn between this analysis a priori and the analysis a posteriori of the observed session. The collected data by observing sessions in a class throughout the school year are completed by interviews with teachers and with some students representing various levels, as well as by all the written traces of the year (exercises, lessons, homework ...). Teachers and students only knew that the researcher was interested in the teaching of mathematics. They did not know about our interest for numerical domain. So the interviews with teachers and students were open and the focus of research was hidden. This condition was important to capture ordinary practices with the least possible influence of the researcher. Two experimented teachers (but not experts) agree to the researcher's presence in their classes, Mathieu in 2006 2007 and Clotilde in 2007 2008. This research follows a study in the framework of a Master 2 qualification (Larguier, 2005) which had made it possible to track down the difficulty of *reprises* at the beginning of the school year for novice teachers in grade 10, notably Rosalie.

THE PROBLEMATIC OF NUMERIC

In the document which accompanies the syllabus (June 2000) we found the following commentary concerning the sector “numbers” and the theme “nature and writing of numbers”: “We will make a summary of the knowledge encountered so far by the students and we will introduce the ordinary notations of the different sets. The students will have to know how to identify which numbers belong to which set”. So, the recognition of the nature of the numbers is a well-defined task in the syllabus and is faithfully followed by the teachers according to the researcher's observations. We

are going to develop our analyses concerning the following task: “recognizing which sets the given numbers belong to”. This type of tasks is emblematic of the numeric domain worked on at the beginning of the year during the resumption of the school year. It is also equally symbolic of the Junior High/Senior High link by allowing a *reprise* of former knowledge and at the same time working on completely new knowledge (like the nomination of sets). This type of tasks will be written as T, this represents an essential problematic to the numeric domain. This restriction is found at the level of the discipline in Chevallard's terminology.

In Clotilde's and Mathieu's classes many specimens of T are worked on in the first chapter. In general the justifications are not asked for. In Clotilde's workbook the following affirmations without any justification are found: $\sqrt{18}$ irrational or $1/3$ rational. The decision theory made in the relative class to this type of task T is incomplete. The technologico-theoretical block elements are absent, the expected response of the teacher rests on the numerous implicit elements which are certainly not shared by all the students.

The same observation concerning the incompleteness of the praxeologies relative to T was carried out on the 17th of September 2004 in Rosalie's class. We will take the same example which has been indicated and which concerns written numbers under the quotient form of two whole numbers. Rosalie does a particular study of two specimens $\frac{22}{7}$ and $\frac{103993}{33102}$ prompting this study with the fact that they are approximations of π . In other words, a cultural condition which is not based on a real mathematical problem.

For the first example, a possible technique known from Junior High, is to carry out the division of 22 by 7 in order to prove that the decimal writing of the number is unlimited and periodical. Rosalie expected this proof from the students as a relative technique to $22/7$, which corresponds to an interesting *reprise* to continue to work on the concept of decimal numbers as is seen in this extract:

A student wrote his answers on the board. Rosalie hears another student in the class:

Alexis: It's a rational number

Teacher: Why?

Alexis: Because it's a fraction and the decimal part is infinite

Teacher: How do we know that? It's best to write down the division because the calculator will always give a finite amount of numbers...of terms since it shows the

numbers it has on its screen. Now this one here (she points out “ $\frac{103993}{33102} \in \mathbb{R}$ ” written on the board by a student) who doesn't agree?

The proof for the first quotient $22/7$ is brought up orally, but it is not carried out effectively by the students, or the teacher. With the calculator experiment, Rosalie does not leave the students enough time to do it themselves. In doing this, she also avoids a debate which could have taken place on the nature of numbers displayed on the calculator screen. This certainly would have allowed her to consolidate the

necessary learning of this tool and the numbers in play (moreover, registered learning in the syllabus as one of the numeric themes). The mathematical decision theory linked with T is just a draft, it is not completely developed yet. We can therefore ask ourselves what is going to remain of this for the students. We equally make a hypothesis that the personal relationship between the students and the mathematical activity in general runs a risk of not conforming to the institutional relationship. Rosalie may let her students believe that it is enough to bring up a possible proof during a demonstration.

For the second example, the possibility of articulation with the new parts of the Senior High syllabus is interesting. Indeed, the two rational numbers $22/7$ and $103993/33102$ are both *idecimal numbers*⁴ (Bronner, 1997) but the choice of numerator and denominator for $103993/33102$ makes it necessary to change the technique compared to the previous example. The technique expected by Rosalie for the first number, to know the division “by longhand” of 22 over 7 cannot lead to the underlining of idecimality for the second number. The quotient obtained for the first number is 3,142857 while the length of the period from the second quotient is too big for the quotient to be calculated by longhand. We see a change of the didactic variable between the two tasks. We wonder if this is really what the teacher anticipated. Indeed, in the observed session, the fact that the second number is idecimal is not shown and is not even questioned:

Teacher: (...) Now this one (she points out $\frac{103993}{33102} \in \mathbb{R}$ written on the board by a student) who doesn't agree? Yohan, Kamel?

Kamel: I agree but it's also a rational number

Teacher: It is, that's true but the answer to the question lies in \mathbb{Q} . It's the \mathbb{R} of real and it's the \mathbb{Q} from quotient (she corrects what is on the board at the same time). But we suppose that Xavier is using the notations that he knows. Now the last one... (she points

out $\frac{167}{80} + \frac{\sqrt{10}}{3} \in \mathbb{R}$).

The study of the nature of numbers, beyond knowing whether a number is rational or not, is not made. There is not even a technique brought up contrary to what is brought up for $22/7$. Consequently there is no implementation of a new decision theory, it is avoided. A possible technique in grade 10 uses a theorem which is in the syllabus (optional). It is not available to the class at this moment of the year. The question of knowing if the number belongs to \mathbb{D} is thus left aside. In the second case, the demonstration of the *idecimality* of the rational number is not even brought up, it is simply completely avoided.

Nevertheless a decision theory corresponding to the syllabus could have been built into this class for task T. Here is the description: a possible technique in grade 10 is

⁴ Idecimal: in Bronner's terminology, following the model of rational/irrational, decimal/idecimal

to determine the irreducible fraction which is equal to the given quotient. In this case, Euclide's algorithm allows us to demonstrate that the numerator and the denominator are coprime, and that the given fraction is irreducible. The denominator has a decomposition in product of prime numbers $2 \times 3^3 \times 613$, it is not a product of powers of 2 and 5, the number is *idecimal*. This technique is possible only from grade 10 onwards, but it also uses tools which are taught in Junior High, like the idea of irreducible fractions. This also permits another way of conceiving the decimal number in the register of fractional writings (Duval, 1995). Therefore, it gives us the opportunity to really strengthen our knowledge of numbers. So, T is indeed in a moment of *reprise* in the numeric space, which allows us to connect past knowledge, and new knowledge.

The comparison between what could have been done with T and what was effectively done clearly shows what is avoided in the targeted mathematical organization. We wondered why Rosalie made these choices:

- Is it about a lack of reflection in the analysis of the session?
- Is it the decision about the mathematical theory regarding the syllabus which is seen as not being a suitable teaching form in this class?
- Does Rosalie anticipate that the technique is too difficult to set up and might discourage students at the beginning of school year? This technological element of the professional gesture was confirmed in an interview with her. She said that she does not want to put students off learning mathematics.

This observation brings to light one of the difficulties that teachers have in building numeric space. The work in this numeric domain assumes a very precise study of the mathematical decision theory in accordance with the knowledge of the students. Another symptom of the problem of the profession is probably the misunderstanding of teachers on these difficulties. It asks the following question: what is the knowledge necessary for teachers in order to achieve the process of didactic transposition between the reference mathematical knowledge and the knowledge to be taught (Bosch et al., 2005)?

But what are the *raisons d'être* of this emblematic task? What essential mathematical problem for the discipline motivates the mastery of decision theory linked to T? By asking these types of questions, we refer to Yves Chevallard who denounces the teaching of mathematics as being like a museum visit, or the traditional way of teaching answers, even when the original questions have been lost (Chevallard, 2000). He questions what motivates the calculation of numbers in order to express them under these particular forms. He makes us become aware of the problem which legitimizes this work in the numeric domain:

“We come to [...] a big problem in mathematics: how to recognize if two mathematical objects of a certain type are or are not the same object? How to know for example if $7 \times 5 - 8 = 23$? Or if $\frac{60}{84} = \frac{380}{532}$? Or again if $\frac{n(n+1)(2n+1)}{6} - \frac{(n-1)n(2n-1)}{6} = n^2$? There is one solution to this one generic, universal problem: to respond to the question asked. We

need to use a considered type of written system for the objects, where each of these objects has a writing expression and a written expression of its own. The calculation of the «canonic» writing of the objects to be compared therefore allows us to answer: so we have $7 \times 5 - 8 = 35 - 8 = 27$, which shows that $7 \times 5 - 8 \neq 23$. Similarly it comes from a part $\frac{60}{84} = \frac{4 \times 15}{4 \times 21} = \frac{3 \times 5}{3 \times 7} = \frac{5}{7}$, from another part $\frac{380}{532} = \frac{190}{266} = \frac{5 \times 19}{19 \times 7} = \frac{5}{7}$, meaning that we can positively conclude this time that we have equality $\frac{60}{84} = \frac{380}{532}$.

In this citation Chevallard wishes to show that the only question about numbers which is important is to know how to write a number in relation with its nature. Different kinds of writing are possible, and we have to know the canonic one, useful to compare and calculate with several numbers. So it is not the knowledge of the nature of the number that is important, but the knowledge of the canonical writing given for a type of number. This necessity is backed up by another necessity of mathematical work, which is the rule of the *institutional contract of calculation* (Bronner, 2007). For demonstration work in mathematics, we are obliged to use exact values. The following reasons explain then why it is important to know the exact values of trigonometric lines of particular angles such as: $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and why we keep this way of writing with a radical. We are going to further develop this example, various types of numbers appearing within the framework of trigonometry, a *reprise* of work on the numeric is then possible.

THE EMBLEMATIC TASK AND TRIGONOMETRY

In the part concerning irrational numbers we are going to come across “products” (Bronner, 2007) within the framework of trigonometry, but neither their appearance nor their nature is questioned. In Mathieu’s and Clotilde’s classes, the chapter on trigonometry was approached late in the year, for Mathieu from May 23rd, 2007 and for Clotilde from April 30th, 2008. By using our methodology, a work of comparative analysis was able to be carried out.

The comments of the syllabus of grade 10 state: “*The definition of $\sin x$ and $\cos x$ for a real x will be made «rolling up \mathbf{R} » on the trigonometric circle. We will make the link with sine and cosine of 30° , 45° and 60° ”.*

During Clotilde's lesson on May 16th, 2008, at the end of the sequence on trigonometry, she gives out a table which the students have to complete.

Exercice : On a donné les valeurs exactes du sinus et cosinus de quelques angles remarquables entre 0 et $\frac{\pi}{2}$.

Point								I	A	B	C	J				
x	$-\frac{5\pi}{6}$	$-\frac{3\pi}{4}$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos x$								1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0				
$\sin x$								0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1				

a. Retrouver le point qui correspond à chaque angle.
b. En déduire les valeurs exactes des cosinus et sinus de tous les angles du tableau.

This document presents an extraordinary showcase of numbers which appear in the numeric space of grade 10 with whole relatives, decimals, irrationals formed with the typical examples often used like π , $\sqrt{2}$ and $\sqrt{3}$. We observed that it does not become the student's responsibility to know that it is necessary to keep complex writings of these numbers, for example $\frac{\sqrt{2}}{2}$. If the teacher had given the responsibility of this question to the students, then he would have been able to carry out a *reprise* of the emblematic task T to justify the canonical writing of these numbers. But the awareness of the nature of the numbers is completely absent in this entire sequence even though it is very rich in respect to possible work on the numeric. The only justifications are under the form of conventional rules not referred to as necessities of the discipline. So, Clotilde does not accept the answer $\frac{1}{\sqrt{2}}$ and transforms it into $\frac{\sqrt{2}}{2}$ by arguing that: "as we already said we did not like the roots of 2 under the line of fraction, we write it like that".

Thus, teachers accustom the students to practices of exact calculation, which are governed by conventional rules only decided on by the teacher, while epistemological reasons support them. The institutional contract of calculation remains in this context of trigonometry entirely the responsibility of the teacher. Nevertheless, the underlying questions could be seen by the student as being an aspect of the mathematical work.

The numeric space elaborated in grade 10 is so enriched by new elements which are *operators* (Bronner 2007), namely the operators cosine and sine, generators of tables of real numbers containing many irrational numbers. These operators allow a production of numbers in a procedural way. The interest is centered on the way of obtaining the numerical values, and not on their nature. In the same way, there is no interest in the change of status of the number which must be seen as a variable of the

function cosine.

The *dynamic* implemented by both teachers is a *numerico-geometrical dynamic* (Bronner 2007). Numbers of various natures are generated by the operator cosine from the trigonometric circle and from the right-angled triangle. However, another dynamic remains implicit, it is an *inter-numeric dynamic*. This one could exist thanks to the numeric resumption of work at the beginning of the year linking with the symbolic task and the canonical writing of the numbers according to their nature. However, it would seem that this symbolic task is not exportable except the sector “Numbers” of the domain “Calculations and functions”. This place of trigonometry in grade 10 would allow the numeric to work, because irrationals come “naturally”. But, the awareness of the nature and the writing of these numbers is not the responsibility of the student. Nevertheless, it would be interesting to ask the question about the exact value of a number like for example $\cos 17$ and to make the students aware that the writing of the exact value is $\cos 17$, in the same way that the exact value of $\sqrt{34}$ cannot be written without using a radical. These examples could enrich the usual prototypes used as irrationals. Nevertheless, from the synthesis of numbers encountered in the vast *mixed-bag* of school, this type of number has been popular and can be reused as an example.

IDENTIFICATION OF A PROBLEM IN THE PROFESSION

We asked the question of the *reprise gestures* concerning the study of the nature of numbers by focusing our gaze on an essential problem in mathematics: writing numbers according to their nature. Obviously, this question takes its meaning only in the context of a problem. The most relevant register of writing is conditioned by the work to be done with these numbers. But what we also observed with the teachers was the absence *reprise* whenever the problem arose. The notions are only worked on as objects, the “*raisons d’être*” posed about the writing of the numbers becomes nothing more than a question of habit.

In the reality of our observations, the teachers introduce T to the students at the beginning of the year in a certain number of cases in accordance with the syllabus. They do this without taking into account the specific problems of the discipline, nor is it used later to pursue the study of synthesis relative to numbers. Nevertheless we have seen that a *reprise* of T is possible during the grade 10 syllabus (we have only quoted the case of trigonometry). Teachers do not see these new niches for reactivating this type of tasks no matter how essential it is to work on the numeric. Our study opens new ways for identifying specific teachers' knowledge in the matter of numeric domain. It is especially useful for the formation of teachers and the necessary practice of particular *gestures of interweaving*.

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A EUROPEAN PROJECT FOR PROFESSIONAL DEVELOPMENT OF TEACHERS THROUGH A RESEARCH BASED METHODOLOGY: THE QUESTIONS ARISEN AT THE INTERNATIONAL LEVEL, THE ITALIAN CONTRIBUTION, THE KNOT OF THE TEACHER-RESEARCHER IDENTITY

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The European project (PDTR)¹, which this paper deals with, is aimed at the development of research based methodologies for teacher training to promote new classroom approaches in the sense of PISA competences. After a short description of the Project, we present in some details the cultural choices, the work methodology and the outcomes of the Italian teams. Some reflections are made about the main problems involved, in particular on the intense attempts to clarify the meaning of the figure of the teacher-researcher, the true core of the Project. In a few final remarks we discuss the validity and the potentialities of the Project.

Key-words: European cooperation. Teachers’ professional development. Educational methodologies. Teacher-Researcher figure.

INTRODUCTION

The PDTR is a project finalized to induce in teachers structured view and knowledge of mathematics, in coherence with new pedagogical approaches and social needs, and to promote, by means of suitable classroom practices, motivation and sense-making in students involved in mathematical activities. A key idea of the project is that of *Teaching-Research*, based on the principle of inseparability of classroom practice and educational theory in the context of the action aimed at the improvement of learning. The intention is to build a formation and teaching path where instruction, research and professional development mutually support each other. The underlying hypothesis is that the involvement of teachers in “mentored” collaborative study within a research team and a familiarity with theoretical studies increase their awareness as school teachers, and bring them to change their beliefs, to conceive their professional development as a life-long process and to assume a scientific inquiring approach in their classroom pedagogy.

The central aim of the Project has been to initiate a process of transformation of the ways to teach mathematics, while respecting the standards and contents of national curricula. The main specific goals have been: a) introducing Teaching-Research into daily classroom practice, with special emphasis on the integration of mathematical and didactic knowledge; b) developing instructional research based materials, which improve students’ understanding and mastery of mathematical competences as

assessed by the OCSE-PISA tests, while, at the same time, increase their enjoyment of mathematics; c) promoting in teachers the attitude to give more weight to students' process of thinking than to formal skills and knowledge.

The Project has lasted three years: the first one mainly devoted to the study of general methodological-curricular choices, that be coherent with the approach to mathematical competences in OCSE-PISA tests; the second one centered on designing, implementing and analysing didactic experiments and producing shared materials; the third one devoted to a critical review and refinement of experimentations, and to the production of reports to be published. An additional task has been the study of the English language, to favour exchanges among participants.

THE ITALIAN CONTRIBUTION WITHIN PDTR

In Italy, many research projects were promoted by the National Research Council since the seventies, for the renewal of mathematics teaching. This implied the birth in several universities of the '*Nuclei di ricerca didattica*' (that is, groups formed by university and school teachers of all levels, working jointly) and contributed to the emergence of a new "bivalent" figure of teacher: the '*insegnante-ricercatore*'. Such a figure can be considered the result of a slow evolution of a motivated and able teacher through *stages of active involvement at different levels*, stages which can be said the steps of a *process of training to research*. This process, starting from simple experimentations, brings gradually the teacher to collaborate in the formulation of research hypotheses and in the theoretical analysis of research data, until to be able to autonomously realize a research project and to write scientific papers. This national frame constitutes the background of our cultural and methodological choices within PDTR, and of our way of conceiving the participants as *perspective teachers-researchers*, novice in research.

The two Italian (Modena and Naples) teams share not only this general framework, but also common research themes and a long habit of mutual collaboration. Therefore their work has been done along the same lines. Here, we want to report in some details three aspects of our activity: the theoretical and laboratorial work, the conduction of teaching experiments, the production of the final reports.

The work at theoretical level and the laboratory-based activities

We worked at three levels, facing: theoretical questions concerning mathematics education, with particular reference to the teacher figure; questions related to mathematical contents and questions devoted to a renewal of classroom practice.

We have taken inspiration from two related models of teacher, as *resonance mediator* (Guidoni, Iannece & Tortora, 2005) and as *decision maker* (Malara & Zan, 2002). In our view, teachers are influenced by important factors that the research should not neglect, such as knowledge, beliefs and emotions. Thanks to close contacts with Math Education ideas and theories, they can become more and more aware of all these components and to be able to possibly change them. For this reason we have

devoted special sessions to introduce teachers to selected literature samples, in order to clarify our theoretical reference framework. These include epistemological studies, mathematics oriented papers, and papers focused on didactic-methodological aspects and on classroom practices.

For what concerns didactic and methodological aspects, we have assumed a socio-constructive approach, with particular emphasis on studies about the mathematical discussion, the didactic contract and the classroom norms. A particular importance is also assigned to reflections on class processes, and to the role of teachers (and of their beliefs, actions, wordings, ...); for this we refer to Mason studies (see for instance Mason, 1998). Moreover, we have taken into account the linguistic and communication dimensions, as described by Pimm (1987) and Sfard (2000).

As to mathematical content we worked in Shulman's sense (1986). We privileged the arithmetic-algebraic field, directing our attention towards the competences promoted by the PISA tests.

For the renewal of classroom practice, we studied the units of the ArAl Project, which can be seen as models for socio-constructive teaching, and some protocols of classroom processes on them, highlighting the incidence of different variables in the process (teacher's behaviours, students' participation, affective relationships, gender issues).

The work related to teaching experiments and the methodology adopted

The work with teachers has been carried out in small groups and has been structured through: design and planning of teaching sequences, experimental setting in the classes, critical analysis of the enacted didactic processes, editing of reports for dissemination. The chosen themes concerned: a) problem solving, according to the theoretical framework of the PISA tests and with reference to the development of proportional thinking; b) the approach to the algebraic language as an instrument to represent relations, to interpret graphs, to solve optimization problems and to solve proof problems. Teachers were engaged in teaching experiments for at least two years, and in the second year the experiments were broadened and refined on the basis of the initially implemented ones. They involved students of school grades between 6 and 11, with a high concentration of grades 6-8.

In order to implement a given teaching sequence, we faced: a joint study of selected research papers on the chosen theme for the clarification of didactic key points and hypotheses to be tested; the construction (or adjustment) of tasks constituting the main steps of the path and the *a priori* analysis of pupils' potential difficulties. This work was not easy, due to: a) the need to combine the progressive development of the mathematical set of questions with curricular time constraints; b) the analysis of the difficulties of the tasks from both linguistic and mathematical points of view; c) the planning of discussions related to questions to be tackled and solved collectively.

In the classroom the teachers worked constructively, stimulating and orchestrating pupils' interventions, promoting reflections on what was gradually being carried.

They promoted verbalisation, by always inviting the pupils to write down their ideas, conjectures, reasons for their procedural choices, etc. Moreover, they (video) recorded classroom discussions, transcribed them, adding local and general comments on classroom processes.

The driven analysis of classroom processes and the birth of the ‘multi-commented diaries’

We carried out a complex activity of critical analysis of the transcripts, looking at the relationships between the knowledge constructed by students and the teacher’s behaviour in guiding them to such achievements. Our main aim has been to lead teachers to get a higher and finer control over their own behaviours and communicative styles and to observe the incidence of a critical analysis on both classroom processes and pupils’ behaviours and learning. This critical and reflective activity, based on the classroom transcripts commented by the teachers (shortly called diaries) developed along different moments of comparison between: the pair ‘teacher-mentor’; the teachers involved in the same teaching sequence; the whole team (teachers, mentors and the leader). Within some projects – due to participants’ different locations and therefore to the difficulty to meet – the diaries have been commented by at least three people: the mentor assigned to the teacher; the coordinating mentor; the head of the project. The diaries, so enriched by a multiplicity of written comments, reflect a variegated range of points of view and interpretations, which highlight crucial points of the process as well as critical elements in the teacher’s behaviour.

They allowed us to identify five key areas of teachers’ weakness concerning: *beliefs on cultural and/or educational issues*; *pedagogical content knowledge*; *bifurcation between theory and practice* (e.g. difficulties in realising what has been studied or planned, and in working on the basis of relational thinking); *linguistic issues* (massive use of operative linguistic expressions coming from the received model of teaching; difficult balance between colloquial language and language of scientific teaching; scarce attention to word paraphrases in view of an algebraic translation); *management of classroom discussions* (dialogues mainly between teacher and pupil; widespread prompting; yes/no questions; lack of attention to the development of ‘social intelligence’ in the classroom). But two issues seem to be crucial and dramatic at the same time: *the teacher’s language* in communication, often imprecise, not correct, full of slang expressions and rich in not always appropriate metaphors; *the conception of mathematics*, too often *operative*, where ‘calculate’ and ‘find’ often prevail over ‘represent’, and ‘do’ over ‘reason’ and ‘reflect’ (for more details see Malara’s contribution, in Czarnocha, 2008).

The reports editing

In the third year of the project teachers were asked to produce written report about their teaching experiments following the rules of the Mathematical Education community. This phase of teachers’ work turned out to be a true pivot toward the

acquisition of a *researcher behaviour*. In fact, teachers are used to report their classroom experiences within their own community, but this kind of “internal” communication, having its focus on students’ performances, leaves behind any information about one’s own role in the process and about the choices made for its development. In the first version of the report, almost all teachers applied this model of communication to the new situation, in spite of the attitude, developed in two years of participation in the project, to reflect on the influence of their own role in the development of a discussion, and more in general, on the relationship between teacher and pupils, with a special focus on the impact of their own knowledge, beliefs and emotions on the process itself (see next Section). The experts faced the problem, trying to change this communication praxis. Several individual and collective comparisons were needed to lead teachers to become aware they had to change their usual point of view and to include, in their writings, themselves as determinant components of the process itself. This way, by means of successive approximations, always mediated by interaction with the experts, teachers succeeded in writing their reports. Then these reports were reviewed by international reviewers before being accepted for publication (in the books edited within the project²).

From the point of view of the research training, this final phase has been crucial to attain project aims: the necessity of communicating lead teachers to make explicit for other people, but for themselves too, the key points of change in their classroom behaviours.

Reflections on the project spin-off for teachers

The project turned out to be a great opportunity for teachers to engage with a new way of conceiving and teaching mathematics and to reflect on their own conceptions and ways of being in the classroom. Teachers met major difficulties in transposing in their practice what they had learned at theoretical level, especially concerning the didactic-methodological aspects.

Here is a list of the main problems concerning the role of the teacher in managing class-based activities, in particular discussions: the problem of the language used, often misleading for the pupils; the problem of the pertinence and consistency of the indications provided at crucial moments of the discussion; the problem of listening to pupils and being unable to grasp the potentiality of interventions that diverge from predicted ones (especially when they come from pupils who are not viewed as leaders); the problem of a real social knowledge construction: the issue of sending back ideas to the class so that they might be validated and shared, the issue of institutionalizing knowledge, the issue of individual learning (the teacher often took for granted that pupils had understood or intuited something, only on the basis of reassuring ‘yes’ in chorus); the problem of checking that participation is actually collective (discussions often developed with the contribution of a few pupils and there were no interventions aimed at involving everybody).

Nevertheless, at the end of three years, several appreciable improvements can be noticed in teachers' classroom practice, as well as changes in their beliefs and a better awareness of their professional role. All this is also witnessed by the teachers themselves within their final essays. In the Appendix we will report a few excerpts from these essays.

THE INTERNATIONAL ACHIEVEMENTS. THE FIGURE OF THE TEACHER-RESEARCHER

At the international level, the Project did not fully meet our expectations. Many substantial disagreements emerged along the common work, concerning first of all different views about Math Education research contents and methodology, between Eastern and Western countries and, as a consequence, disagreements emerged on the way to conceive a teaching experiment. Therefore, only in the last year a first true international collaboration, a bilateral teaching project between Italy and Hungary, occurred (see Navarra, Malara & Ambrus, 2008).

The main points of difference concerned: variables to be observed (students vs the pair "teacher-students"); time (short vs long term experiments); types of intervention (simple proposals of PISA question vs insertion of suitable PISA problems into didactic paths designed for the whole year workplan); way to refine a teaching experiment (proposals of 'corrective tasks' for students vs critical analysis of classroom processes with/for teachers); and, *dulcis in fundo*, the figure of the teacher-researcher.

The question of defining what the word "(mathematics) teacher-researcher" means is by no means a rhetorical one and, well beyond the limited range of the Project, is of deep interest for the whole Math Education research community. Indeed, for some authors, the two domains of academy and school are incommunicable worlds, and therefore the unique possible concern of the teachers is their school-practice (Crawford & Adler, 1996). For others the two roles are still separate, even if there are teachers who are able to investigate about their practice; but it is very rare that a teacher can identify by himself a research question (Jaworski, 2003; Brenn quoted by Peter-Koop, 2001). Some other authors believe that the teachers can become true researchers, provided they frequent for enough time an academic environment (Malara & Zan, 2002).

One of the, so to call, side achievements of the project PDTR, but, in our opinion, a valuable one, has been that of trying to share a common view on this question, naturally arisen in order to achieve the main goals of the Project. So here we want to report some conclusions about it, reached at the end of several discussions and collaborative work, together with some reflections of ours. The question has received several interpretations and answers by the members of the PDTR staff, due to their different views deeply dependent on different theoretical frameworks and social and cultural traditions.

Moreover, the following related questions have arisen: “How do the double roles of teacher and researcher acting simultaneously in concrete situations accord to each other? How can the possible conflicts between the two roles, each embodying its own objectives and its own ethics rules, be managed? How can one harmonize the two roles in the different real situations or perhaps in the different phases of the work?” Of course, all the above questions are open ones. But the wide debate developed has given some contribution to them, witnessed by specific papers devoted to these items in the two books edited within the Project. It seems to us that they well represent the variety of positions.

The main task remained of reconciling the different views about the crucial point: when a teacher can be identified as a teacher-researcher. A shared conclusion has been that of recognizing some steps by which a teacher can become a teacher-researcher. Teachers teach following textbooks and external indications. Good (or excellent) teachers utilize natural skills and their own intuition to obtain good results from their students, following textbooks and other resources filtrated by their personality. A teacher-researcher adds to this a *personal aspect*: the habit to reflect upon one’s own teaching action and to utilize such reflections to interpret and to improve practice (one can also recognize this habit in a reflective teacher); and a *social aspect*: the readiness to face a matching, comparing one’s own actions with others’ actions, to identify and to clearly formulate research questions, to be able to communicate with other people according to the rules of an evolving scientific community. In particular, what surely characterizes teacher-researchers and distinguishes them from, may be, excellent teachers, is the capability to share ideas within a scientific community. This implies to follow some general and specific rules, for example to put well identified research questions into a general theoretical framework, to utilize experience and materials in order to argue about some well declared thesis, to accept criticism and to be continuously well disposed to changes.

We believe that to fix some minimal condition that characterize a teacher-researcher is necessary in order to satisfy the standard of a scientific community: in this sense it is important to have shared criteria to carefully distinguish an acceptable contribution for a research journal, from more freely written, though interesting, accounts of a teaching activity. At the same time we are aware that pretending to strictly satisfy those requirements as a necessary goal of the enterprise of forming reflective teachers or perhaps teacher-researchers could entail the risk of discouraging willing young teachers from realizing their urges for improving their professional behaviour. This recommendation has been one of the main points of discussion in the Project.

SOME FINAL REMARKS

The outcomes of the international meetings allowed us to understand the depth and the multiplicity of problems to be overcome, in order to achieve an effective collaboration between researchers belonging to different cultures. A necessary condition for such a collaboration goes through: *a real willingness of sharing*

problems; listening to others and taking into account the working and operating conditions of a certain group (in order to understand and to search for solutions after common studies and efforts).

In our opinion, the main result of the project might be considered *a deeper awareness of the problems that make an effective collaboration between Eastern and Western countries difficult*. By making these problems explicit, we might help others to overcome the rigid barriers we met. It is not an easy task, due to the weak common background, which makes actual interests often diverge.

NOTES

1. The Project PDTR (*Transforming Mathematics Education through Teaching-Research Methodology*) has been realized in 2005-2008 under the leadership of S. Turnau (Rzeszów University, Poland), with the help of B. Czarnocha and the expertise of H. Broekman, J. Mason, N.A. Malara. It has involved seven teams of mathematics teachers, apprentices in the craft of “teaching-research”, from Hungary, Italy, Poland, Portugal and Spain.
2. The two books (Czarnocha, 2008) and (Turnau, 2008) are downloadable from <http://www.pdtr.eu/index2.php>

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APPENDIX

Excerpts from the teachers' final global reflections revealing the impact of the project on them

NG (Primary school teacher). Thanks to PDTR project I have understood that my professional growth is still at the beginning, and it is a process that has never to be considered concluded. To sum up, in these three years I have learned to reflect on: cognitive processes (How have I done? How does my mind work when I learn? How does children's mind work when they learn? etc.); metacognitive activities of control (I have learned how to carry out this activity... I have used these strategies... such strategies allowed me to... Which structures or models do my pupils construct? How do they use these structures?...); the disciplinary structures on which I've been working with my pupils (above all arithmetical structures and “proportional thought”).

RF (Middle school teacher). Transcriptions, that have demanded time and energy, allowed for a self-evaluation of my own professionalism, a critical meta-reflection on my own way of managing collective discussions, on my way to send pupils' suggestions back to the class, to intervene and direct the discussion itself. After this process I got to a higher professional awareness: I became aware of the need to refine my capacity of grasping immediate feedback by pupils in a meaningful way, always keeping in mind the aims of the route I undertook. I also reached a higher awareness of the need for a careful control of didactic methods and of knowledge about the discipline. This has led an empowerment of my professional awareness on the pedagogical sensitiveness that needs to be used in order to favour pupils' cognitive, relational and affective increase.

MP (Middle school teacher). Through the training activities I actually saw the relevance of linguistic obstacles, which make the interpretation of texts with a mathematical content problematic well before their translation into the most typical languages of this discipline (numerical, algebraic, tabular, graphic). For many students this process implies an extremely hard move from a narrative context to a logical relational one. This aspect is often neglected in the ordinary mathematics

teaching activity, whereas it would require an in-depth reflection by teachers. ... Since the whole teaching sequence was video recorded, the careful analysis of the recordings strongly highlighted the main features of my *modus operandi* in the classroom. It is embarrassing and instructive at the same time to see yourself during the class, to find out that you did not grasp immediately the opportunities offered by students to guide the lessons towards fertile grounds for a discussion.

MB (Secondary school teacher). The *a-posteriori* analysis of my lessons sometimes meant realizing the inefficacy of my own didactical methodologies and behaviours. During this project of research on our own practice we had the possibility to learn to consider failures, not as negative events to be cancelled without trying to find a remedy, but as “launching pads” to bring ourselves into question. During this phase, the work with the mentor particularly helped me. The numerous pre and post class activities meetings and the crossed analysis of excerpts of class discussions represented a further source of reflections. Cooperating with the mentor gave coherence to my work, aimed at reaching prearranged objectives: the didactic ones, those related to the relationship to be established with my students and those correlated with the research on my practice. In these three years I gradually acquired more confidence in the tutoring-relationship with the mentor, who initially was an “uncomfortable” presence and quickly became an important reference.

SD. (Secondary school teacher). The relationship with the mentor and the coordinator must be particularly taken into account because, with their experience, they helped us in keeping the coherence between the path we planned and the objective of the project. Their advices concerned not only the theoretical framework of reference, but also the planning of the different phases of the path, the organization of the methodology of work in our classes and the *a priori* and *a posteriori* analysis of class activities. Thanks to this collaboration, I understood the importance of considering the didactic action as a set of measured choices of contents, proposals, methodologies and teacher’s behaviours. In this perspective, students’ contributions are interpreted as a resource, rather than a dreadful unforeseen event... Numerous aspects have made my participation in the Project significant, even if I am aware that I have only taken a little step in the professional development of a teacher, which is full of shades and potentialities.

WHY IS THERE NOT ENOUGH FUSS ABOUT AFFECT AND META-AFFECT AMONG MATHEMATICS TEACHERS¹?

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The role of affect in the teaching and learning of mathematics is widely recognised by researchers in the field of mathematics education, and a plethora of literature has been published on the subject. However, the related issue of meta-affect has been addressed only minimally. This paper aims to increase awareness of its importance within the community of mathematics teachers and mathematics teacher trainers. More specifically, it suggests how a meta-affective approach may be usefully adopted by mathematics teachers in the classroom as well to catalyse the personal and professional growth of current or future mathematics teachers.

Keywords: affect, awareness, belief, emotion, meta-affect.

Introduction

The realm of affect is an especially rich area of research in mathematics education. However, the impressive scientific achievements in both qualitative and quantitative terms have failed to adequately influence practice among mathematics teachers or moreover, to drive investigation into the application of scientific research to practical mathematics instruction in the classroom. To no avail, Burkhardt and Schoenfeld (2003) invited researchers to “make progress on fundamental problems of practice”. With twenty-five years of experience imparting in-service training for mathematics teachers and ten years of experience as a mathematics teacher trainer (in Italy a two-year postgraduate degree leading to teacher certification was launched ten years ago), the author has investigated the relationship between affect, meta-affect and changes in teaching practice among mathematics teachers. The adoption of a teaching methodology based on the resulting experience would appear to offer considerable promise.

Theoretical framework

McLeod (1992) identified beliefs, attitudes and emotions as the constructs upon which affect regarding mathematics is based. De Bellis & Goldin (1997) also recognised the role of values in this sense. Research into affect has evolved considerably since then, with growing investigation into the issues involved and a broadening of the theoretical background, to the point where multiple theoretical frameworks have emerged. We may thus address affect as a system of representation and communication (Goldin, 2002) in which beliefs, attitudes, emotion and values – the four elements in Goldin’s “*tetrahedral model*”- are viewed as a sub-domain; as a

¹ The author hopes the title doesn’t sound disrespectful to Schoenfeld (Schoenfeld, A. H.(1987). What’s all the fuss about metacognition?. In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189-215). Hillsdale, NJ: Lawrence Erlbaum Associates), who wrote the paper in question when asked to explain ‘metacognition’.

system “strongly, naturally and in a dynamical way” linked to cognition (Malmivuori, 2004); within a socio-constructivist framework (Op ‘t Eynde, 2004) or with an embodied cognition approach (Brown & Reid, 2004). The various theoretical frameworks highlight two elements which should attract the attention of researchers. The first of these regards the frequent appearance of the terms ‘metacognition’, ‘consciousness’, ‘awareness’, ‘self-awareness’ and ‘meta-level’ in relevant literature. An important step in developing the debate and research field would be taken by investigating the meta-levels of the four constructs, their theoretical collocation and their correlations with metacognition. Hannula (2001) offered an approach to the issue, but there remains much more to be learned. The importance of metacognition in the learning processes was first highlighted by Flavell (1976). LeDoux (1998) and Damasio (1999), by conducting investigations based on fMRI (functional Magnetic Resonance Imaging), CAT (Computerized Axial Tomography) and PET (Positron Emission Tomography), have demonstrated that the functioning of the cognitive and emotive systems are closely related. In light of these studies one might plausibly wonder whether the term metacognition still means anything, or what its role might be within the new scientific framework. Must it be accompanied by the term meta-emotion, must a new term be coined to comprise the two, or must yet other terms be coined? The second element to emerge from the theoretical frameworks of affect is how consistently they display links between affect and neuroscientific research (Schlögmann, 2003). This has made it possible to create a neuroscientific basis for the interdependence of affect’s four constructs, so frequently emphasized in research. It has also afforded clarification of other hotly contested issues, such as the nature of beliefs, which must necessarily be hybrid (i.e. Furinghetti & Pehkonen, 2002): that is, both cognitive and emotive. This supports author’s hypothesis (Moscucci, 2007) beliefs are the ‘best’ element, among the four constructs of affect, which to act on, and this is the reason why, in this contest, the author is particularly interesting in ‘beliefs’, which seem, together with emotions, to shape attitudes (Hart, 1989). The matter of defining ‘belief’ remains unresolved within the research field. Hence, here the term ‘belief’ will be taken to represent some sort of ‘primitive entity’, and every belief some sort of ‘axiom’ assumed as a result of personal experience; basically an affirmation which is accepted without proof. Furthermore, different mathematics-related belief systems (Schoenfeld, 1992; Leder, Pehkonen & Törner, 2002) are in some way all correlated. So we might say, by adopting terminology from algebraic structure language, that the individual’s beliefs regarding mathematics (although the choice of subject is inconsequential) do not make up a ‘set’ of beliefs but rather a ‘structure’ of beliefs. Researchers have not simply investigated the role of student beliefs in their learning processes, but also the role of the beliefs of mathematics teachers. As regards definitions, Richardson (1996) identifies teacher beliefs with their theoretical perspective of teaching methodology. This underlines the effect of teachers’ beliefs on their teaching practices. It would seem logical to deduce that teachers’ beliefs determine the quality of their practices (Cooney, 2001). However, almost twenty years ago, Cobb, Wood and Yackel (1990) noted that these influence

each other reciprocally, rather than in terms of ‘side of the implication’. The interrelations among teacher beliefs and student beliefs are equally complex and controversial (Beswick, 2005) and it appears currently impossible to hypothesize the entity of these relations, given that student beliefs have not been proven to be the product of teacher beliefs, nor *vice versa*. Nevertheless, although the theoretical issue has not been resolved, the impact of belief systems on the classroom behaviour of teachers has been recognised in numerous studies involving mathematics teachers (for instance, Pehkonen, 1994; Chapman, 1997, 1999).

From the realm of theory to didactic practice

As mentioned in the introduction, this proliferation of scientific research has failed to produce significant developments that may be of direct use to mathematics teachers in the classroom. And yet, such developments are sorely needed by mathematics teachers, students, school systems and indeed society in general. Thus any efforts to impact on the belief systems of teachers, and especially on any beliefs that are damaging to students, are more than welcome. Damaging ideas might be identified as ‘inefficacy beliefs’ (e.g. “A special inclination is needed to be good at maths in school”), in contrast with ‘efficacy beliefs’ in teaching mathematics, which have been investigated and illustrated (Philippou and Christou, 1998; 2002). The question to be answered is how to progress from inefficacy beliefs to efficacy beliefs and efficacy teaching practices. An approach addressing meta-affect may well prove useful. Goldin (2002) considers meta-affect as a key construct, “including affect about affect, affect about and within cognition that may be again about affect, monitoring of affect, and affect as monitoring”. The potential of meta-affect as a vehicle for the development of the professional profile of mathematics teachers has been confirmed throughout ten years of successful² mathematics teacher training carried out by the author with teachers undergoing training and already in service. Due to space restrictions, only in-service teachers will be considered here.

Towards a holistic approach to maths teachers affect

Fifteen or so years of training courses proved that, in spite of apparent success, the impact on classroom practice was undeniably disappointing, with the didactic practices of the teacher participants evolving only rarely. Few teachers could bear the prospect of giving up the “school mathematics tradition” (Cobb et al., 1992) (frontal lessons aimed at the introduction of the new technique, presentation of examples and setting of exercises), even if the main goal of the courses was precisely didactic quality. Indeed, within the Italian school system the proportion of failures in mathematics with respect to all academic subjects has been and continues to be

²Training is reputed successful when: 1) the participating teachers express their satisfaction with the training by means of their responses to a survey presenting questions in a 4-point Likert scale format; 2) the participating teachers begin to modify their teaching practices as suggested during the training course; 3) the modification of classroom practices by teachers produces positive effects (in the sense that the students benefit both in terms of their affect toward mathematics and their actual performance in this discipline).

telling: the Ministry sets this year's figure at 42%. Moreover, the 'discomfort' (lack of success but also, for instance, 'negative' emotions and 'inefficacy' beliefs towards maths) of Italians with mathematics was believed (and subsequently proven (Moscucci et al. 2005) to be an 'endogenous cause' (e.g. arising within the school system itself) of student dropouts. This alarming situation called for the creation of an intervention scheme based on the following principles: 1) teaching methodology and teacher affect are closely linked (this was contextualized above from a theoretical perspective); 2) dealing with beliefs as a purely psychological construct is limiting, as mathematics teachers work together with their colleagues within a social context that tends to perpetrate traditional, time-tested teaching techniques (Op 't Eynde, 2004); to consequently avoid marginalising teachers who attempt to update their approaches, the teacher educator needs to undertake group work as has been carried out during well-documented experimentation (Jaworski, 2003); 3) the teacher trainer must obviously make use of the same didactic methods that are presented to the teachers for use with their students. The outcome of these considerations was the creation of an intervention scheme (Moscucci, 2007), in which beliefs systems role was highlighted. Meantime, the author has understood the synergy springing out the contemporaneous work about emotions and beliefs. As has been repeatedly debated within the theoretical framework, the affect of an individual (be it a student or teacher) is a complex structure comprising closely-linked constructs. Therefore any effort to influence it must simultaneously address all the elements on which it is based. So, perhaps, the success of that intervention scheme is due to the global – we would say 'holistic' – approach to teacher affect.

Meta-affect: a 'tool' not enough used by mathematics teachers?

About thirty years have passed since Flavell (1976; 1979) published his metacognition research and the importance of this concept to the learning process has been proven and reported (for instance, Hartman (1998)). However, it is rare to meet mathematics teachers who make use of didactic techniques informed by the abundance of metacognition research. The big step in the field of metacognition might involve equipping maths teachers with tools of observation and intervention that could be applied first and foremost to themselves: "...increasing metacognitive activity through private reflection and shared conversations increases teachers' awareness of their subjective knowledge... beliefs are often challenged through this process, which lays the groundwork for the construction of new knowledge and for real change in teaching practice" (Hart, 2002). The training courses for mathematics teachers conducted by the author over the last ten years were structured by means of a method (Moscucci, 2007) that seeks to achieve meta-affective goals with the teachers prior to addressing discipline-specific issues. The distinguishing characteristic of this method is its emphasis on awareness (Marton & Booth, 1997): the teachers are put in a position to autonomously become aware of their own belief systems and emotions, without being obliged to openly declare their beliefs and emotions. There are two reasons for this. The first, as regards beliefs, is the well-known distinction between

“beliefs espoused and beliefs in practice” (Schoenfeld, 1989). What’s more, teachers often are not conscious or even aware of the beliefs underlying their teaching practice. The second regards emotions. Awakening the emotions that have accompanied teachers during the development of their professional capacity is extremely beneficial. The emotions experienced almost certainly influence their beliefs regarding mathematics learning and teaching. Even memories of what it was like to be a maths student as far back as primary school need to be evoked. Remembering is the first step. Then the emotion recalled must be elaborated to try to analyse its immediate impact and understand any eventual lasting repercussions. This means engaging teachers in ‘meta-emotive’ activity without attempting to place educators in the role of psychologist, but rather assisting teachers to self-analyse their memories. Let us briefly examine the close link between meta-emotion, meta-cognition and the awareness of beliefs. Emotion³ is a personal response to an event signalled by physical symptoms such as an accelerated heart rate, blushing and facial expression. With time (a matter of seconds or minutes) these symptoms lessen and eventually disappear. There is consciousness of the emotion, but awareness takes hold only as the intensity of the physical reaction diminishes and it again becomes possible to ‘think rationally’, as we say. If the emotion has been particularly intense or is part of a series of emotions related to a single situation (such as learning mathematics), it begins to generate thoughts regarding the emotion’s cause, origins, consequences and responsibilities. These spontaneous or subsequent thoughts may set off a chain of further thoughts as well as further emotions. The initial emotion and its related physical manifestations have only short-term effects, thus failing to directly influence an individual’s future. However, the resulting chain of thoughts and emotions may lead to the creation of certain beliefs that are known to be highly influential. Most beliefs are generated in this way. Thus awareness of this process is a fundamental step in controlling negative emotions, neutralising their impact on the present and re-elaborating the beliefs generated by them. When considering this process, a distinction must be made between maths teachers with a mathematics degree and those with a different degree (in Italy this is not only possible but predominantly the case with teachers of the grade 6-9 levels). With this latter group a greater effort must be dedicated to developing awareness of emotions, as such teachers often experienced difficulty with mathematics, as student, at school or at university. As also regards teacher attitudes, activities that develop awareness of them must be provided, and teachers can be left free to define ‘attitude’ as they wish. Awareness of one’s attitudes is intended as awareness of what teachers consider to be their attitudes toward mathematics both as a learner in the past and as a teacher presently. To give an example, the following activity frequently proves useful. Teachers are asked to put down in writing – informally, without attention to composition – how they perceive their attitudes. Then their students are asked to repeat the exercise anonymously by the researcher - trainer. The students may find it

³ When especially intense, the amygdale may come into play (LeDoux, 1995).

easier to express their opinions if they are provided with a guideline such as the beginnings of sentences to complete. The teachers observe the opinions expressed by their students and, following a personal analysis, are asked to put in writing their comments regarding both their and their students' tasks. As this brief description illustrates, this approach concentrates on beliefs and emotions, inasmuch as they are considered to shape attitudes, as underlined in the theoretical framework. The aim of this approach is to create a virtuous cycle between the re-elaboration of beliefs and emotions on one hand, and the adoption of non-traditional methods on the other (the non-traditional methods are, in certain cases, 'discovered' by the teachers in a socio-constructivist learning environment, in other cases by questioning their classroom practices). The first feeble attempts to make use of new methodologies and non-traditional disciplinary approaches produce initial resources that encourage teachers to progress in their development. The teachers begin to experience new emotions, thus they re-elaborate their beliefs, and recontextualise their previous emotions. This is how the virtuous cycle is catalysed. The awareness of one's own awareness represents another step toward quality in a teacher's meta-affective competence.

A short description of one experience

Of many cases observed, the following - chosen to give a 'hint'- offers elements to ponder as far as different teacher typologies are concerned. In 2005 the author was invited by the principal of a vocational school to set up and implement a three-year project aimed at reducing student failures in mathematics, which regarded over 60% of students (official data provided by the School Administration). The situation was in line with that of all schools of this kind, so it was actually no worse than average. Due to the lack of space, it is impossible to describe the details of the project. Briefly, it consisted in conducting activities based on meta-affect, as described in the previous section. The author worked with the teachers and the teachers worked with their students. As for subject teaching, the teachers were required to 'embrace' a socio-constructivist teaching methodology. The author personally met the students with special difficulties (three-four times -two hours- for each class involved) in order to diagnose their nature. The school's three mathematics teachers -all of them- were more or less of the same age, between forty and forty-five, while their psychological and professional profiles varied. One teacher, who will be called Victoria, was very cordial and outgoing, had a degree in mathematics, attended mathematics teaching conferences regularly, had previously participated in various innovative mathematics teaching projects and had always attempted to put into practice the developments presented in mathematics teaching journals. In spite of her efforts to improve her students' results, she had never been successful. She participated in the project with great expectations. Another teacher, who will be called Angela, had a degree in mathematics and was disappointed by the poor results and scarce interest of her students, to the point where she simply wanted to retire. Angela was more insecure than Victoria but sincerely wanted to help her students. Perhaps it was a sense of impotence that made her want to retire. Although without great hopes, she

participated in the project willingly. The third teacher, who will be called Bill, had a degree in IT and had taken the teaching job following a frustrating experience as an IT technician. He had acquired a reputation for strictness with the students. He commented that “his students didn’t work enough” or “lacked the basics”, and that “some of them simply couldn’t be helped”. He participated in the project only following the insistence of the principal. As questions came up during the initial meetings (What is the role of school in educating individuals? And what is the role of mathematics? What is ‘school mathematics’?), his interest seemed to grow. “The answers to certain questions should be obvious to a teacher while they may not be; most answers are simply rhetorical!”. The three teachers attended an introductory course (about 30 hours, as a whole), using the intervention scheme mentioned in the previous paragraph (Moscucci, 2007), during the month of September 2005, prior to the beginning of the school year. They worked as usual together with their mathematics-teaching colleagues, but in an atmosphere of “contrived collegiality” (Hargreaves, 2004), while in this new context they began to appreciate the value of ‘collaborative work’, undoubtedly benefiting from collaboration in “small groups”, as underlined by Santos (2007). They used the same methodology with their first -and second- year classes (involving more than 150 students). Throughout the year their work in class was supported by means of meetings with the author, every two weeks during the first three months of the year, later monthly, as well as long phone calls to provide emergency help. The author decided not to attend teachers’ lessons not to intrude a ‘strange’ element in the ‘classroom ambience’ and it was impossible to organize recording tools (but author’s meeting with the students in special difficulties). Unbeknown to the teachers and the author, the project was monitored by the principal through inspection of the attendance registers. At the end of the first school semester, appreciable improvements were noticed of the average final marks for the same level classes with respect to preceding years (data, and the following ones, from the Minutes of Class Meetings). The only change undertaken regarded the teaching methodology introduced in the project, so it is ‘highly’ likely that this was precisely the reason of these improvements. Victoria and Angela’s classes proved to be the most successful in the project, as, at the end of the first year, the number of failures in mathematics was reduced by about 90%. Angela also regained enthusiasm in her teaching. Bill encountered greater difficulty than his colleagues in applying the initial methodology focussing on meta-affect and the subsequent content methodology: while Victoria and Angela showed their enthusiasm for the activities suggested by the author, Bill always needed additional time to accept the proposals, and, above all, he was hesitant to update the activities in his classes. In any case his students achieved much better results with respect to previous years. Even if each teacher made up their own test, they were very similar except for insignificant details. Overall, at the end of the project’s first year, the only students to fail mathematics had also failed most other subjects and consequently had to make up the year. At the end of the year the school’s vice principal conducted a school-wide survey (completely unrelated to the project), and the results showed mathematics to be the students’

favourite subject. Undoubtedly the aspect of the project regarding course content played a part in the project's success, but it would have been impossible to even address course content without first eliminating the negative preconceptions towards mathematics of most students. In the third year of the project Victoria was transferred to a scientific high school renowned for its strictness and traditional methodology. The classes she adopted the method with achieved better results than all the other classes of the same year on a standardized test administered to all. In the last year of the project Angela suffered the lack of (mostly psychological) support from Victoria and lost some enthusiasm, but is still convinced of the method's validity. Bill seems to have become less strict and perseveres in trying to apply the method. The author has obtained such surprising outcomes as those described in this paper on many other occasions. Now she is planning to monitor wider experimentation in a vocational school. At present it seems important, at first, to spread a research hypothesis: the awareness of one's own belief systems accompanied by a personal reworking of the emotions felt during mathematics tasks, may be key in removing 'inefficacy beliefs' and 'recontextualising' past emotions so that they are innocuous in the present. Secondly, the author hopes other researchers, teacher trainers and teachers will try to adopt these teaching methods and schema so as to confirm or contrast the hypothesis.

Remarks

The positions of numerous researchers on meta-affect recognising its central role in affect, the relationship between meta-affect and metacognition revealed by neuroscientific research and the success of certain teaching methods based on meta-affective methodology should encourage researchers to investigate this subject from a theoretical perspective. After all, like many fields of education science, mathematics education displays distinct characteristics. In disciplines such as medicine or pharmacology, before a treatment such as pharmacological therapy can be applied, various levels of experimentation must be carried out. Instead, in the field of education it is possible and often especially effective to alternate research and the application of research outcomes to practice. Or better, this is a very fruitful way to proceed. This makes it particularly important to spread the use of practices with a high potential for success. The resulting discussion, rebuttal and development can only contribute to furthering research and increasing didactic quality.

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THE ROLE OF SUBJECT KNOWLEDGE IN PRIMARY STUDENT TEACHERS' APPROACHES TO TEACHING THE TOPIC OF AREA

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This study reviews the relationship between student teachers' subject knowledge in the topic of area and their approaches to teaching that topic. The research was carried out with four primary student teachers and examines the similarities and differences between the nature of their subject knowledge and their plans to teach the topic. In this paper results of two of the four student teachers are focused on to illustrate the contrasts in planning and subject knowledge. The intention is not to generalise relationships but to examine the phenomena presented. It raises questions related to the variables in connecting student teachers' subject knowledge and their knowledge of how to teach.

Key-words: subject knowledge; area; student teacher; approaches to teaching; understanding

INTRODUCTION

The importance of subject knowledge in the preparation of teaching activities is clearly recognised (Ball, Lubienski & Mewborn, 2001). If we see teaching fundamentally as an exchange of ideas it would seem evident that a teacher's understanding of a topic will impact on how the idea is 'shaped' or 'tailored' when presented in a classroom. As such "teaching necessarily begins with a teacher's understanding of what is to be learned and how it is to be taught" (Shulman, 1987, p.7). Shulman emphasised the transformation of a teacher's knowledge of a subject into 'pedagogical content knowledge' and consequent pedagogical actions by "taking what he or she understands and making it ready for effective instruction" (p.14). In this way mathematical content knowledge is 'intertwined' with knowledge of teaching and learning (Ball & Bass, 2003).

It is generally accepted that mathematics should be taught with understanding (Hiebert & Lefevre, 1986; Skemp, 1976). In the topic of area it would seem that children often rely on the use of formulae with little understanding of the mathematical concepts involved (Dickson, Brown & Gibson, 1984). They are unable to see the reasonableness of their answers and so are unable to monitor their use of these formulae. There is also evidence that student teachers have a similar reliance on formulae (Baturu & Nason, 1996; Tierney, Boyd & Davis, 1990).

It would seem that a student teacher with limited understanding of the mathematical topic such as area would not be effective in developing children's understanding. This study aims to investigate the impact of primary student teachers' subject knowledge on approaches to teaching the topic of area. As an interpretive study the

intention is not to generalise any relationship but to examine phenomena related to differences and similarities in the student teachers' understanding of the topic and in how they plan activities to teach the topic.

DEVELOPING UNDERSTANDING IN THE TOPIC OF AREA

Measuring area is based on the notion of 'space filling' (Nitabach & Lehrer, 1996). However, unlike children's other common experiences of measure such as length, the use of a ruler in measuring area is indirect. In this way instruction that focuses on procedural competence with measuring tools such as rulers "falls short in helping children develop an understanding of space" (p.473) and it is not surprising that many children confuse area and perimeter (Dickson *et al.*, 1984). Instruction that models the counting of squares on grids provides more success and may represent the notion of 'space filling'. However this does not represent the full nature of area. As Dickson *et al.* (1984) commented the possible restriction to a discrete rather than a continuous view of area measure might not lead to the notion of Π and the formula of the area of a circle.

Further to this, figures used as representations in the classroom often provide a static view rather than a dynamic view. That is, as a boundary approaches a line, the area approaches zero (Baturu & Nason, 1996). This may lead to misconceptions about the conservation of perimeter and area. The recognition of such a misconception goes back at least to the 1960s with Lunzer's (1968) notion of 'false conservation'. This false notion has more recently been cited by Stavy and Tirosh (1996) as an example of the intuitive rule 'more A, more B', in that as the perimeter increases so the area will increase. Alternatively the intuitive rule can be manifested as 'same A, Same B' in that the same perimeter will mean the same area.

It would seem that once introduced to the formulae, children have a tendency to use these regardless of the success of their answers (Dickson, 1989). Studies such as Pesek and Kirshner (2000) and Zacharos (2006) suggested that, where instruction involved procedural competence and use of formulae, children would insist on repeating strategies that caused errors and they often had difficulty in "interpreting the physical meaning of the numerical representation of area" (Zacharos, p. 229). Where instruction was based on measuring tools such as dividing rectangles into squares children demonstrated flexible methods of constructing solutions and often achieved more success. The studies suggested that the early teaching of formulae presented 'interference of prior learning' (Pesek and Kirshner) or 'instructive obstacles' (Zacharos).

Such 'interference' or 'obstacles' could explain why many children at the beginning of secondary school take algorithmic approaches to the solution of area measurement problems (Lehrer & Chazan, 1998). It follows that student teachers are likely to have a similar reliance on algorithms. If we refer back to Shulman's model of transformation and Ball and Bass's idea of 'intertwining' content and teaching

knowledge, then a student teacher's understanding of the nature of area would seem key to the way they would teach it. Studies that have examined student teachers' subject knowledge in the topic of area (Baturu and Nason, 1996; Tierney et al, 1990) found that student teachers often demonstrated a lack of understanding of how practical concrete experiences could relate to the use of formulae and how area measure evolves from linear measure. They were often uncertain about the reasonableness of their answers and were unable to explain how formulae were related. A study that has examined student teachers' lesson plans for teaching the topic of area (Berenson, Van der Valk, Oldham, Runesson, Moreira, and Brockman's, 1997) found that many student teachers represented the topic of area through the demonstration of procedures and use of formulae rather than focusing on the activities that would support understanding. What we do not know from these studies is whether the student teachers that planned to teach the topic through the demonstration of procedures were the students who demonstrated a lack of understanding of the topic.

THE STUDY

The four student teachers involved in this study had varied backgrounds in mathematics. At the time of the study they had completed the taught university based element of a one year Post Graduate Certificate in Education (PGCE) and they were about to start their final teaching practice. The student teachers had attended workshop seminars on the teaching of primary mathematics. All four student teachers had the same course tutor so would have followed the same content in their mathematics seminars. The student teachers were also reassured that the work for this project would not be used as part of their course assessment.

Clinical interviews were carried out with each of the student teachers to reveal underlying processes in their understanding (Swanson, Schwatz, Ginsburg and Kossan, 1981; Ginsburg, 1997). The first part of the interview examined the development of the student teacher's lesson plan and the second part of the interview involved the use of mathematical tasks to investigate the nature of their understanding in the topic of area. The mathematical tasks were equivalent with some standardisation of probing questions but further interrogation was managed flexibly in order to be contingent with the student teachers' responses. The interviews were audio taped and transcribed.

The use of lesson plans

Planning is central to teaching and the development of lesson plans is a key aspect of teacher training. Lesson plans provide a source of data in assessing student teachers' professional development. They can also provide useful cues in follow up interviews when the activities, explanations and questions used by the student teachers help to generate further descriptions (John, 1991, Berenson et al, 1997). Although lesson plans are limited to demonstrating the student teacher's 'espoused' theory of action

(Argyris and Schon, 1974) they can be seen as effective in indicating the student teacher's perceptions of teaching.

The student teachers were asked to plan a lesson to introduce the topic of area to a Y4 class (8 to 9 year olds). The student teachers were advised that they could use any sources they normally would to help plan the lesson. The only restriction being the ideas would be their own or their own interpretation of teaching ideas from other sources. The student teachers were questioned about the following:

1. How they had developed the activities
2. How they felt the activities would facilitate the children's learning
3. The instructions or explanations they intended to give
4. The questions they intended to ask the children
5. The difficulties that they felt the children would encounter

Area Tasks

The second part of the interview involved four tasks adapted from Baturu and Nason's (1996) and Tierney *et al.* (1990) studies to ascertain the subject knowledge of the student teachers.

Task 1 (Baturu and Nason, p.245) includes both open and closed shapes to test student teachers' understanding of the notion of area (see fig 1). Shapes G and F were included to test the ability to differentiate between area and volume, shapes J and K test the notion of area as the amount of surface that is enclosed within a boundary and shapes E, H and L test the understanding of area from a dynamic perspective.

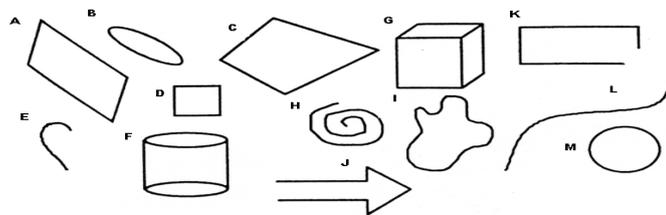


Fig 1: Task 1

Task 2 (adapted from Baturu and Nason) was designed to test the ability to compare areas, initially without the use of formulae (see fig 2). The student teachers were presented with two pairs of cardboard shapes. Dimensions were not given. Comparison by visual inspection alone would be inconclusive so the student teachers were asked to consider ways to compare area. This was used to determine if the student teacher was able to use measuring processes other than external measures and use of the formulae.

Pair A:

Pair B:

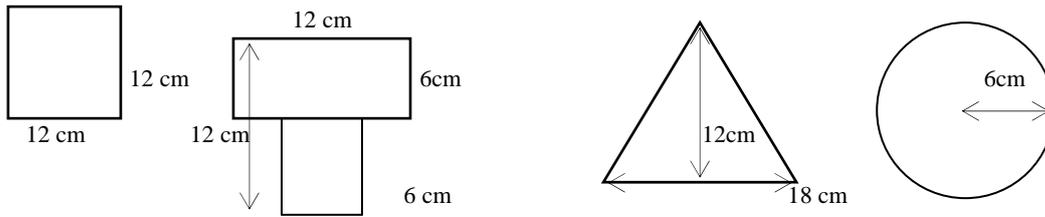


Fig 2: Task 2 (shapes not drawn to scale)

Task 3 (adapted from Tierney *et al.*) was intended to determine a dynamic view of area and the ability to consider changes in area and perimeter (see fig 3). The student teachers were given three cardboard shapes. Dimensions were not given.

1. a rectangle 9cm by 4cm
2. a parallelogram where the area is the same as the rectangle but the perimeter has changed (base 9 cm and height 4 cm)
3. a parallelogram where the perimeter is the same as the rectangle but the area has changed

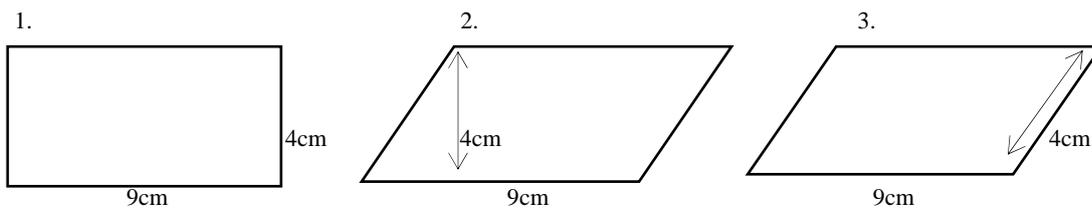


Fig 3: Task 3 (shapes not drawn to scale)

Task 4 (adapted from Baturu and Nason) aimed to test the correct use of formulae. It also tested for an understanding of the relationship with non-rectangular figures, including the use of the ratio Π (see fig 4).

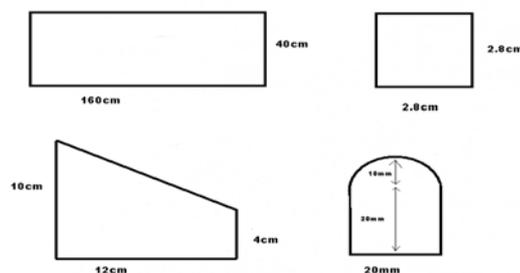


Fig 4: Task 4 (shapes not drawn to scale)

RESULTS AND ANALYSIS

In this paper it is presented the results of two of the four student teachers, Alan and Charlotte, are focused on to illustrate the contrasts in planning and subject knowledge.

Alan

Alan's highest qualification in mathematics was an 'A' level taken over 5 years ago. He felt that his confidence level was moderate to high. In his lesson plan he intended to model the use of the formula using a transparent grid over a rectangle and by, "thinking out loud", would state, "Find this side, this side and multiply together". He would then show the children how to check by counting the squares. He was concerned that the children might confuse area and perimeter and that they might add the lengths rather than multiply. In order to overcome this he would show how to use a ruler to measure the lengths and repeat the instructions from the introduction. He felt that he would have to tell the children what units to use and that the '2' means squared. Alan would continue the lesson with further practice of the formula with other rectangles and with shapes composed of rectangles. He suggested using a 'real-life' context by extending the use of units to square metres and finding the area of the classroom.

Alan's use of formulae and calculations in Tasks 2, 3 and 4 were quick and accurate. He used the formulae as a first resort in comparing areas of shapes in Task 2 and Task 3 rather than reasoning or comparing by placing the shapes on top of each other. Alan gave a clear definition of area related to the covering of surfaces. He was also aware of the relationships between formulae and the notion of Π as a ratio in finding the area of circles. He was able to consider the dynamic view of area with the parallelograms in Task 3 but did not identify the area of the open shapes as zero in Task 1.

Charlotte

Charlotte had obtained a grade C GCSE qualification in mathematics, the minimum entry requirement for a primary PGCE course, and she spoke of lacking confidence in mathematics. Charlotte stated that she found the lesson difficult to plan and had researched pedagogy based texts. Charlotte intended to introduce the topic with a large paper rectangle and ask, "How many children can fit onto this shape?" She would use these arbitrary units to determine the area of other shapes and then draw rectangles on the board and pretend that each child is a centimetre square. Charlotte felt that the activities would "lead naturally" to a definition of area as the "amount of space within a shape" and she intended to note the strategies that the children used. She also intended to set an activity to investigate the area of rectangles and changes in perimeter. She would encourage the children to talk together about the patterns they had found. Charlotte would ask, "What do you notice about the perimeter and

area of the two classrooms?” (sketches on the board) and “Can you draw different shaped rectangles with an area of 12 squares?”.

Charlotte’s notion of area from Task 1 seemed inconsistent. Although she stated that the area was the amount of space inside a shape she attempted to include some of the open shapes as those that had an area. She was uncertain as to whether the three-dimensional shapes would have an area, and if so, how to measure it. She was, however, secure in the relationships between the formula for the area of a rectangle and the area of a triangle and was aware of an activity to determine Π as a ratio. Charlotte was aware of the dynamic view of area from Task 3 and was able to compare the areas of the parallelograms with little difficulty. Charlotte made errors in using the dimensions and formulae for calculating areas in Task 4. She was also not aware of the correct units and confessed that she never knew when to use cm^2 or cm^3 .

ANALYSIS AND DISCUSSION

Performances on the mathematical tasks suggested that Alan had a good understanding of the nature of the topic of area. In particular Alan demonstrated quick and accurate use of formulae. In contrast Charlotte’s performance on the tasks demonstrated limited knowledge in the use of formulae and units. Her understanding of the nature of the topic of area appeared to be inconsistent.

Charlotte based her intended introduction to the topic of area on the counting of regions. Charlotte initially started with arbitrary units that would be used later to introduce the square unit. Charlotte was aiming to provide children with activities and problems that would help them realise the notion of area ‘naturally’. On the other hand, Alan’s lesson was focused on teaching the use of the formula. He was concerned that the children would not use the correct formula for area and he would articulate explicitly how to do this. There was an attempt to relate the use of the formula to ‘real-life’ by finding the area of the classroom.

According to the review of research above, Alan’s intended focus on the use of the formula from the start of his lesson might suggest a premature introduction that would create ‘interference’ or ‘obstacles’. However Alan was a confident mathematician who demonstrated accurate use of formulae and secure understanding of the nature of the topic. In contrast, the activities that Charlotte planned to use would be more likely to support children in developing a notion of area as ‘space filling’. This might reduce the children’s reliance on the use of formulae and consequently support their understanding. However Charlotte was less confident in mathematics and she demonstrated weaker subject knowledge.

Ambrose (2004) has suggested that student teachers may often believe that teaching mathematics is straightforward. They assume that, if they know the mathematics they need to teach, and then all that is needed is to give clear explanations of this knowledge. Further to this the student teacher may believe that the aim of teaching mathematics is to explain useful facts and skills to children to help them become

skilful and efficient in their use and to know when to apply them. The analysis of Alan's lesson plan indicates that he may have this belief of teaching. Stipek, Givvin, Salmon and MacGyvers's (2001) referred to this belief as a traditional 'knowing' orientation. They suggested that a shift away from such a traditional orientation towards an 'enquiry' orientation where mathematics is seen as a tool for problem solving, would be more effective. Analysis of Charlotte's lesson plan suggests that she may have been more inclined towards an 'enquiry' orientation.

In order to avoid the 'interference' or 'obstructions' that might become apparent by focusing on the procedures of area measurement we would want student teachers to move towards this 'enquiry' orientation. Stipek et al.'s empirical study indicated that teachers' beliefs about mathematics predicted their instruction. However they also suggested that less confident teachers were more likely to be oriented towards mathematics as 'knowing' due to lack of confidence in dealing with the questions that might be asked through an enquiry based approach. If we interpret Alan's orientation as 'knowing' and Charlotte's approach as moving towards 'enquiry' then this suggests an anomaly as Charlotte was less secure and lacked confidence in her knowledge of the content.

It could be said that as Alan used the formulae with particular ease and accuracy his aim was to support the children in developing such a use. Although he was able to realise relationships he did not see this as an important aspect of mathematics and hence he did not focus on this pedagogically. Charlotte's emphasis was not on ensuring clear explanations were given but that the children arrived at an understanding through the activities. She suggested that the children would use their own strategies and she intended to employ activities that would 'lead naturally' to their understanding. Could it be that Charlotte's lack of confidence and knowledge meant that she was uncertain of how to explain the mathematical ideas to the children? In this way she may have researched pedagogical approaches further. Or could it be that Charlotte's beliefs in the teaching of mathematics differed from that of Alan? Despite a lack of knowledge in mathematics, Charlotte's pedagogical approach may have been based on a belief that children develop understanding through active engagement in activities and that this belief has been carried over from her view of what is important in mathematics.

This is not to suggest that Charlotte would be more effective in teaching the topic. This study has not investigated how the student teachers responded to the children's learning in the classroom and Charlotte's misunderstandings are likely to inhibit her ability to develop the children's learning at some point.

CONCLUSION

Hill, Rowan and Ball (2005) have suggested that it is not knowledge of content but knowledge of 'how to teach' the content that is influential in considering teacher effectiveness. What remains a question is how this knowledge of 'how to teach' is

arrived at? Although this research does not provide any generalisable evidence it does raise questions regarding the nature of subject knowledge in relation to the knowledge of ‘how to teach’, and whether there may be other variables at play, such as orientations and beliefs about what is important in mathematics.

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DEVELOPING OF MATHEMATICS TEACHERS' COMMUNITY: FIVE GROUPS, FIVE DIFFERENT WAYS

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Developing a mathematics teachers' learning community is one of the in-service teacher training methods in the university. At the beginning of 2006, from the initiative of some teacher training educators, a mathematics teachers' community formed at Tallinn University. The aim of the project was to focus on two of the main problems in school mathematics: teaching percentages and functions. Although all the groups were given the same problem by the tutors, a different approach was used by each group. The article presents an overview of the division of task inside the groups at the end of the first stage of the whole process, and also in what way each group reached its final decision with the matter of how to teach percentages. It turned out that at this stage the workgroups had developed differently.

INTRODUCTION

By Wenger's (1998) theory, working in the *communities of practice* is one of the most common and natural ways of cooperation and it can be seen in every sphere of social life where there is communication between colleagues. The aim of the communication is to solve a certain problem, and in this solving process there occurs constant intercommunication between the group members and the participants learn from each other (Wenger, 1998; Olson & Kirtley, 2001). Communities of practice are mostly informal groups. In a well-formed community of practice people have to know each other well, which implies that the following qualities apply: (Q1) the members of the community know each other's abilities, (Q2) they can be set to work quickly, (Q3) there is a quick flow of information inside the community, (Q4) there is a fluent exchange of information, (Q5) there is a good grounding for finding new strategies, (Q6) the group finds original solutions to problems that have been solved already (Wenger, 1998; see also McGraw, Arbaugh & Lynch, 2001).

The mathematics teachers' learning community, as a part of the in-service teacher training method has, according to a number of researchers (e.g. McGraw, Arbaugh & Lynch, 2001; Goodchild & Jaworski, 2005; Olson & Kirtley, 2005), proved to be successful. The exchange of different opinions and views in the course of discussions gives the participants a chance to view the problems from different angles, and therefore it is instructive for every member (Olson & Kirtley, 2005). Jarowski (2005) points out the importance of disputes and constructive discussions inside the circle, as it is by this process that all conclusive decisions are made. Grossman, Wineburg & Woolworth (2001) also warn that at the initial stage of work the group is liable to

become a *pseudocommunity*, as discussions lack subject matter and the members reach agreement too easily when trying to find solutions.

A mathematics teachers' learning community (referred later as *MMM*-project [1]) was assembled at Tallinn University for the first time in 2006. The *MMM*-project was part of a wider project of enhancing mathematics teaching in Estonia (Hannula, Lepik & Kaljas, 2007). A preparation period of about seven months preceded the assembling of the *MMM*-project, during which mathematics educators at the university acquainted themselves with research on mathematics teachers' communities worldwide (e.g. Olson & Kirtley, 2005; Jaworski, 2005; Goodchild & Jaworski, 2005) and thereby planned the *MMM*-project. The project awoke great interest among mathematics teachers – there were 34 applicants (initially it was planned for 10 teachers), and all of them were invited. I was one of these mathematics teachers. The planning of the *MMM*-project and its initial stages has been described by Hannula *et al.* (2007).

The teachers participating in the *MMM*-project were divided into groups of 6 or 7 members (referred to as G1-G5 in the text) at random. I was a member of G1. At the first two seminars we discussed the problems which resulted from the teaching process of percentages. We worked in the groups only at the seminars, as more rather individual homework was given by the tutors (designing and mediating artifacts). At the third seminar in October 2006 the groups were given a collective task: to make a detailed schedule for 20-25 lessons, about teaching percentages for grades 6-9 (pupils aged 13-16), and producing worksheets for them. The present article focuses on the fourth seminar of the *MMM*-project, which took place seven weeks later where the groups presented their respective views on teaching percentages. Most of the groups (G1, G2, G4 and G5) also gave reasoning in their presentation of how they reached their conclusions, and how they divided the tasks between the group members. In principle, the fourth seminar also marked the end of the first stage of the project, as at the next seminar the groups had to present their completed work of teaching percentages, and then to start discussing a new topic.

This paper seeks answers to the following questions: (1) did any similarities occur in the division of task inside the groups and (2) on what did each group base their approach (the ideas given by the university mathematics educators, scientific articles, or the participants' own experience). According to Jaworski (2005), the approach is only taking shape at the first stage of the learning community's work. Therefore, it would be interesting (3) to analyse whether the groups, as learning communities, had acquired qualities of a solid community of practice at the end of the first stage of the *MMM*-project (Wenger, 1998), or if they were still pseudocommunities (Grossman *et al.*, 2001).

METHODS

In the study I analyze the division of labour inside the groups, the level of development of the groups at the end of the first stage of the *MMM*-project, and on

what were the groups approaches based, apart from each member's own thinking and experience. Unfortunately there isn't much authentic evidence of the division of task inside the groups. The participants were not interviewed about it by the tutors, although the results might have proved interesting, and there are no video recordings of the process of working in the group(s). One of the authentic materials is a video recording of the fourth seminar (hereafter Video), where the representatives (or a representative) of each group tell(s) about the work inside the group and what conclusions they have reached. Parts of this recording have been used as the material to warrant conclusions and as illustrative examples in the present article. At the Estonian mathematics teachers' annual conference (November 2-3, 2007) every group gave its view on how percentages should be taught at school, and each group also had an article about it in the proceedings of the aforesaid conference. These articles were another source that I could use. As the third source I used the teaching materials in each member's folder on the *MMM*-project's home page [2] and also in the folders of the different groups. In the autumn of 2008 I sent an e-mail to the participants of the *MMM*-project in which I asked them to explain the first stage of our project as they recalled it. In this paper I use excerpts of some of their answers to me.

By comparing the above sources it is possible to make some conclusions about the work inside the groups. I searched for certain similarities in the division of task. I also tried to specify the level of development within each group by seeking the qualities of Wenger's (1998) community of practice (see in the introduction, hereafter Q1 to 6). I got data based on each group's approach from their articles (used references), and from the video (the tutors' suggestions to groups).

RESULTS

Division of task

The majority of the groups (G2, G4 and G5) used division of task so that each member of the group had to prepare one subtopic in depth. The unified form and structure was either agreed upon earlier or at the fourth seminar during the group work.

"... We also divided the material by the topics so that each teacher could have one topic to think over more thoroughly ... what it might consist of. And this is exactly what all our members have been doing. And today we tried to unify a little ... what items to put down and where ..." (member of G2, Video).

"... On the basis of it we divided the lessons between us...who is taking what part of these lessons to analyse, and we realised that we had to put down worksheets for the pupils and worksheets (with answers, R. R.) for the teacher, and we agreed on what it should look like. And now we will start writing them, as we do not have anything else today," (member of G4, Video).

"... First we relied on our division of tasks as we had agreed earlier ... we had divided the topics between us as we had previously, and how many lessons might be

reasonable... Then we gave to every member of our group – we chose it ourselves – which topics for whom to analyse in detail. Each teacher ... or a colleague here can choose a topic to his own liking and then we write a program for pupils and for the teacher. We communicate by e-mail; we are trying to put our materials in the internet (*MMM*-project's home page, R. R.),” (member of G5, Video).

G1 compiled their own home page on how to learn and teach percentages, and how to go over the material, which refers to Q6 of Wenger's (1998) community of practice. This group had chosen a slightly different way of dividing tasks, although here also each member was responsible for a certain part of the whole work (Q1). One of its members had knowledge of the program *eXe-Learning*, which he used in making their home pages. There were two experienced teachers in the group with good teaching methods and they prepared the theoretical part. Others prepared exercises and searched for some tests in the web, and my task (as I was the member of G1) was to find visual material and suitable games in the internet.

“Visualization is very important and ... we had one member who specialized on this...” (member of G1, Video).

G1's teamwork can be characterized as very active. In other groups the report was made by one member and all the others were only listeners, whereas in G1 all the members took part in the discussion by reporting (Video).

The division of tasks is not clear in G3. There were two members who gave a report and one of them gave an overview of how he had taught percentages at school (Video). G3's folder on the project's home page is empty; there are some teaching materials in the group members' folders, but they do not follow the principles set for the group work.

Different approaches

The university mathematics educators gave all the groups the same task: to make (1) a detailed schedule of 20-25 classes and (2) worksheets to help pupils to understand percentages better. Yet every group had a different approach.

G1 did not give any detailed schedule of classes. Their group website was meant first and foremost for repetition, so that both the teacher and the pupil can go over the sub-themes (Pihlap, Aluoja, Kopli, Koppel, Lepik & Reinup, 2007; Video; *MMM*-project's home page).

“We had one more idea; we wanted to introduce something new, to do it this way as to put the picture and the text side by side, running simultaneously. So that those pupils who do understand the text perhaps do not need it, while others have difficulties with it and so the text keeps running alongside the picture.” (member of G1, Video).

The university mathematics educators gave G1 an idea to add to the homepage a test on the basic knowledge and skills of multiplicative thinking (Video). The group work of G1 on the *MMM*-project's home page and the sketch which they presented at the fourth seminar are very similar.

G2 gave a schedule of classes for teaching in different grades as suggested. The group presumed that the teacher would be using current textbooks and workbooks, and concentrated on making additional worksheets to them. G2 planned to present the most important items of their theory in a *PowerPoint* slideshow (Video). Their work on the *MMM*-project's home page is left unfinished and the group's folder is empty, although there is a lot of different teaching materials (*PowerPoint* slideshows as well) in the members' folders.

In G3 there were two teachers who had been teaching percentages in differing ways for a number of years. This explains why the approach in G3 was influenced by these two teachers.

“For the beginning I must say that it seemed to me that in other groups there have been attempts to teach percentages as it has been suggested; as to our group it is interesting to notice that we happen to have two teachers here (A and H) who have already practiced teaching in the way we advocate now. ... We have tried to have percentages together with fractions, or more precisely: finding a part. ... And now A, who practiced this in his class, is playing his videotape,” (H, member of G3, Video).

The presentation of this group's research work was the longest of all. The report was very interesting in my opinion, and full of subject matter. Yet, as mentioned before, one of the members of the group presented his own personal view of how to teach percentages (Video).

“And therefore I consider it very important that, namely, to begin with, I do not ask the pupils to do any operations, I take simple numbers and you will have to say quickly – three quarters, a half, one quarter or ten percent as well,” (A, member of G3, Video).

The group's article (Ojasoo, Kaasik, Lahi & Pärnamaa, 2007) is based mainly on the same report (Video). The group does not have a collective folder on the *MMM*-project's home page.

G4 based its work on Merrill's taxonomy (Gagne & Merrill, 1990; see Matiisen, Kalda, Kasendi, Tamm & Vahtramäe, 2007). The proportional number of classes was not fixed, and the work was divided into three major subdivisions: (1) immediate understanding (grade 6), (2) arithmetic/basic rules of calculation (grade 7), and (3) “life itself” (grades 8 & 9). On the given theoretical basis this group created entirely novel teaching material – different worksheets for pupils and for teachers (Q5 and Q6). The possibility to use current textbooks and workbooks was excluded (Video).

“As far as I understood we were given such a task ... we cast aside all schoolbooks and we have that batch, and the teacher goes in front of the class with that batch and the pupils will learn how to do percentages.” (member of G4, Video).

In an e-mail a member from G4 brought to mind the period when they had dealt with percentages in the *MMM*-project.

“I had read about and also practiced in my classes the heuristic approach that has been used in schools, and as it sounded interesting to my colleagues they were willing to try it. ... About specialised literature. It is difficult to tell now from which sources exactly. ...

Anyway, some articles written by our mathematics educators are among them.” (member of G4, from e-mail to R. Reinup, Sept.10th 2008).

The results of G4’s work in full are on the *MMM*-project’s home page in the group’s folder.

G5 based its program of teaching percentages on the official program for schools. The group members’ experience in teaching at school was their main starting point. In addition to this they read articles written by different researchers and thereby got an overview of the main problems teachers have when teaching percentages at school in Estonia (see Laanpere, Kattai & Sasi, 2007). The group decided to make some additional worksheets to complement the existing teaching materials. The new teaching materials were to be of help to teachers with little experience (Laanpere *et al.*, 2007; Video).

“We presume that we will use current schoolbooks and teaching materials as well. And when we are making those worksheets we will surely refer to the sources. ... Then each member in our group did some searching and found the teaching materials which have proved helpful in his work. Indeed, we have a number of different worksheets,...tests in our computers, games, and now we can see that they all prove useful.” (Member of G5, Video)

The work produced by G5 is on the *MMM*-project’s home page. However, it can be noticed that most of the teaching materials come from only one teacher.

Community or pseudocommunity

It is a rather difficult task to detect whether any learning community characteristic features can be found in any group (see also McGraw *et al.*, 2001). As I did not have any focused video recordings of the groups when working together at the seminars, there are no direct sources of what the work inside the group was like. It can be decided only indirectly whether we consider a group a learning community or a pseudocommunity, although videos, division of task inside the group, written materials, and above all the teaching materials in the groups’ folders on the *MMM*-project’s home page can be of help. This sort of complex analysis allows drawing some conclusions of the developing degree of the groups.

I have some difficulties when judging the work of G1 because I was the member of this group. There is not much material in the members’ folders on the *MMM*-project home page, but I know that all the members of the group sent their materials by e-mail to the member who created the groups’ home page, on which rather intensive correspondence took place, especially during the last week before the fourth seminar. The address of G1’s home page was sent to all the group members so that everyone could suggest any alterations to be made. Also, at the presentation all the group members were very active (Video). So the qualities of Q2, Q3 and Q4 appeared, and earlier we have referred to Q1 and Q6 in connection to G1. Due to the intensive interaction and the fact that all group members contributed, G1 can be considered to be a community of practice in the sense of Wenger (1998).

All the members of G2 worked hard, collecting teaching materials in their folders on the *MMM*-project home page (the biggest amount of materials compared with the other groups), however their processes did not converge towards a shared conclusion, and the group's folder is empty. On the video it can be seen that at the presentation at the fourth seminar the members of the group remain rather passive. Because of the passivity in producing their own material and in interaction, G2 can be considered to be in the developing phase as a group at the end of the first stage of the project. A weak developing degree of working communities at their first stage is also mentioned by Jaworski (2005).

G3 contained a very influential person and my understanding is that the other members in the group accepted his views about teaching percentages, without adding any or very little of their own. The analysis of the group members' folders on the *MMM*-project's home page affirms the assumption – their content was not in accordance with the group's explicated common aim as it was presented in a seminar meeting (*MMM*-project's home page; Video). Onward, when analysing the materials on the *MMM*-project's home page, it can be noticed that all the materials in the G5 folder mainly originated from only one group member, although at the initial phase all was planned differently (Video). In the work of G3 and G5 the qualities of a community of practice (in sense of Wenger, 1998) do not appear. Grossman *et al.* (2001) refer to the basic quality of a pseudocommunity is that the members of it “act as if they are already a community that shares values and common beliefs”. In my opinion these groups (G3 and G5) are not pseudocommunities in this sense exactly. In both cases there is some inherent discordance between the group's public report and the group's actual work on the *MMM*-project's home page. Yet, one might call them pseudocommunities as most of the work seems to be done by a single (or a couple of) member(s) and other members remained rather passive.

In my opinion G4 compiled a very interesting, complete and novel collection of teaching materials (Q6). According to the recollections the work process was very intensive (Q3, Q4).

“Common understanding developed among us on the grounds of everyday activities and experiences. We all had tried something new and we all could point out the benefits or weak facets of our experiments. As far as I know we all tried to put into practice most parts of other members' experiments in our schools. ... I have a sad story to tell, I cannot be blamed for having a small ego, and as a vice-principal (at a school, R. R.) I have acquired an ability to force my views upon others and I tend to do it in every situation. Therefore, I claim that I influenced other members of the group – but it's no use crying over spilt milk.” (member of G4, from e-mail to R. Reinup, Sept. 10th 2008).

Although this one member was concerned with having too much influence, the material produced did not originate from a single group member. Moreover, the material was produced in collaboration, not simply collected together. Therefore we can consider this group to have developed into a community of practice (Wenger, 1998).

SUMMARY

Every group of a learning community consists of different people and that's why every different group develops its individual face. One of the main aims with the communities is to gain a new quality through the cooperation of different members with different experiences (Wenger, 1998). In the first phase of the *MMM*-project the groups had to make new proposals and give their solutions to some problems that might help to improve the quality of teaching percentages at Estonian schools. The task set by the tutors was the same for every group, yet every group had a different approach.

There were certain similarities as to the division of task: each group member was responsible for one specific sphere (G1, G2, G4 and G5). The most typical division of task was the thematic approach (G2, G4 and G5). In group G1, taking into account each members' abilities, the participants divided tasks according to the contents of the task. This is a more sophisticated approach.

The group members relied on their own experiences when finding solutions to the tasks given to them, although in some groups (G3 and G5) it can be seen that the whole group relied on the experience of a couple of its members. During the whole project the tutors commented on the work inside the groups. G1 received a concrete suggestion from the tutors and the group took it into account. From references of the articles written by the groups it can be seen that G4 & G5 gained ideas from the literature.

There were no concrete proofs of how the communities developed. In my opinion G1 and G4 were the most highly developed groups. In G1 the group members understood each other's abilities well (the tasks were given to the most able members), and there was a quick flow of information (e-mails, supporting each other at the presentation); they found suitable strategies and original solutions (they made their own home page). The work in G4 can be characterised as a fluent exchange of experiences (most of it was put into practice by various members). They found suitable strategies (the work was based on Merrill's taxonomy) and they found an original solution (a set of worksheets). In the work of both groups G1 and G4 appeared to contain most of the qualities of communities of practice (Wenger, 1998), so I think that these groups can be called communities of practice. In the other groups the progress is somewhat questionable at the end of the first stage. G2 could not give a unified original solution, although there were a lot of teaching materials in the group members' folders. Generally, only two members of G3 put their views and experiences together and one of them presented it (based on the analysis of the Video). In G5 there was some cooperation formally, but the main author of the whole report is a single member of the group (based on the analysis of the group's folder on the *MMM*-project's home page).

The project with Estonian mathematics teachers confirmed Jaworski's (2005) presumption that in the first phase of the work the community is still developing.

“The reports we heard gave us lots of ideas to think over but they all did not have enough time to mature, and to put them into practice when teaching percentages at school. I am quite sure that the result here is rather a reflection of some former experiences than anything new, created in the course of the *MMM*-project.” (member of G4, from e-mail to R. Reinup, Sept. 10th 2008).

Some of the groups in the *MMM*-project developed more than the others, but participation (either actively or passively in the community’s work) was instructive for all its members.

“In my opinion, cooperation was the major driving force. An idea emerged, then someone made it clearer and someone else explained something. We all brought some worksheets; I was discussing my plans on my worksheet, but ideas began to spring up and everyone contributed – some gave more, some gave less. I am convinced that this sort of cooperation gave us lots of ideas and added willingness to achieve better results with pupils at school.” (member of G1, from e-mail to R. Reinup, Sept. 6th 2008).

Every idea needs time to mature. When comparing the teachers’ views during the whole *MMM*-project (from the beginning to the final phase), it can be noticed that during the project the participants developed a much more positive attitude in the subject (Kaljas, Kislenko, Hannula & Lepik, in press).

All five groups also presented their concepts and ideas worked out during the *MMM*-project at the Estonian mathematics teachers’ annual conference, which is one of the biggest mathematics teachers forums in Estonia. The large amount of teaching materials on the *MMM*-project’s home page is available to all mathematics teachers all over Estonia. Today the *MMM*-project has ended. The researchers can make conclusions and also start planning other projects of a similar kind in the future.

NOTES

1. In Estonian Meile Meeldib Matemaatika (*MMM*) – We Like the Mathematics
2. <http://zope.eenet.ee/mmmprojekt>

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FOUNDATION KNOWLEDGE FOR TEACHING: CONTRASTING ELEMENTARY AND SECONDARY MATHEMATICS

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This paper describes and analyses two mathematics lessons, one about subtraction for very young pupils, the other about gradients and graphs for lower secondary school pupils. The focus of the analysis is on teacher knowledge, and on the fundamental mathematical and mathematics-pedagogical requisites that underpin teaching these topics to these pupils. The claim is that, in the case of the elementary mathematics, the relevant 'foundation' knowledge is to teachers what Foundations of Mathematics is to mathematicians: invisible until it becomes necessary to know it: and that this very invisibility poses particular challenges to teachers of young children.

Keywords: teacher knowledge, subtraction, gradient, foundations of mathematics

INTRODUCTION

The complex and multi-dimensional character of mathematical knowledge for teaching is now better understood thanks to the seminal work of Lee Shulman (1986) and several subsequent studies. Mathematics teacher knowledge has also been analysed and discussed in several papers at earlier CERME meetings. Recurrent concepts in these discussions are subject matter knowledge (SMK) and pedagogical content knowledge (PCK). For mathematics educators, PCK is perhaps particularly interesting, in that it captures the notion of mathematical knowledge of a kind specific to the teaching profession. That is to say, it encompasses a large, and increasing, body of mathematical knowledge that would not be acquired in the process of learning mathematics for non-pedagogical purposes. The otherwise well-educated citizen does not need it, neither does the engineer, economist, biologist – or mathematician, for that matter. Instances of such knowledge include diverse representations of fractions, for example, or the Principles of Counting (in this latter case see, for example, Turner, 2007).

Another strand of CERME thinking on mathematical knowledge in and for teaching includes the examination of teaching episodes against different kinds of descriptive and analytical frameworks (see e.g. Ainley and Luntley, 2006; Huckstep et al., 2006, Potari et al., 2007). The Knowledge Quartet framework of Rowland et al. (2005) emphasises three ways in which 'Foundation Knowledge' becomes visible in the classroom, for example in the teacher's choice and pedagogical deployment of representations and examples. The underpinning Foundation Knowledge is rooted in the teacher's 'theoretical' background and in their system of beliefs.

[Foundation Knowledge] concerns trainees' knowledge, understanding and ready recourse to their learning in the academy, in preparation (intentionally or otherwise) for their role in the classroom. It differs from the other three units [of the Knowledge Quartet] in the sense that it is about knowledge possessed, irrespective of whether it is being put to purposeful use. [...] A key feature of this category is its propositional form (Shulman, 1986). It is what teachers learn in their 'personal' education and in their 'training' (pre-service in this instance). We take the view that the possession of such knowledge has the potential to inform pedagogical choices and strategies in a fundamental way. By 'fundamental' we have in mind a rational, reasoned approach to decision-making that rests on something other than imitation or habit. The key components of this theoretical background are: knowledge and understanding of mathematics per se; knowledge of significant tracts of the literature and thinking which has resulted from systematic enquiry into the teaching and learning of mathematics; and espoused beliefs about mathematics, including beliefs about why and how it is learnt. (Rowland 2005, p. 259)

The study of Potari et al. (2007) is unusual in this field (of teacher knowledge) in that it sets out "to explore teachers' mathematical and pedagogical awareness in higher secondary education and more specifically in calculus teaching." (p. 1955). The authors note the substantial body of work on teacher knowledge in primary or early secondary education, and assert that "teachers' knowledge in upper secondary or higher education has a special meaning as the mathematical knowledge becomes more multifaceted and the integration of mathematics and pedagogy is more difficult to be achieved." (p. 1955). The claim, then, is that the task of coordinating content and pedagogy becomes more complex as the mathematics becomes more advanced. This paper sidesteps that particular claim. Instead, I examine two lessons conducted with pupils whose ages differ by about seven years. One is at the beginning of compulsory schooling in England (Year 1, pupil age 5-6), the other in lower secondary school (Year 8, pupil age 12-13). The analytical framework is the Knowledge Quartet in both cases, and the focus is on Foundation Knowledge in particular. My claim will be as follows: that whereas from the mathematical point of view, the subject matter under consideration with the Year 8 class is significantly more complex than that in the Year 1 lesson, the PCK necessary to teach the latter well has something in common with Foundations of Mathematics in the mathematician's repertoire. Therefore it is difficult to conclude, in any straightforward way, which teacher has the more demanding task mathematically, where this ['mathematically'] is taken to encompass mathematical knowledge for teaching in the widest sense, as indicated by Shulman and made explicit by Ball et al. (2005).

The pattern in the following two sections will be to give a descriptive synopsis of the lesson first (i.e. to say what the lesson was about), followed by an account, necessarily selective, of the teacher Foundation Knowledge relevant to teaching this lesson.

YEAR 1 LESSON: SUBTRACTION

The teacher, Naomi, was in preservice teacher education. The learning objectives stated in her lesson plan are as follows:

To understand subtraction as ‘difference’.

For more able pupils, to find small differences by counting on.

Vocabulary - difference, how many more than, take away.

Naomi begins the lesson with a seven-minute Mental and Oral Starter¹ designed to practise number bonds to 10. In turn, the children are given a number between zero and ten, and required to state how many more are needed to make ten.

The Introduction to the Main Activity lasts nearly 20 minutes. Naomi sets up various ‘difference’ problems, initially in the context of frogs in two ponds. Her pond has four, her neighbour’s has two. Magnetic ‘frogs’ are lined up on a vertical board, in two neat rows. She asks first how many more frogs she has and then requests the difference between the numbers of frogs. Pairs of children are invited forward to place numbers of frogs (e.g. 5, 4) on the board, and the differences are explained and discussed. Before long, she asks how these differences could be written as a “take away sum”. With assistance, a girl, Zara, writes $5-4=1$. Later, Naomi shows how the difference between two numbers can be found by counting on from the smaller.

The children are then assigned their group tasks. One group (‘Whales’), supported by a teaching assistant, is supplied with a worksheet in which various icons (such as cars and apples) are lined up to ‘show’ the difference, as Naomi had demonstrated with the frogs. Two further groups (‘Dolphins’ and ‘Octopuses’) have difference word problems (e.g. I have 8 sweets and you have 10 sweets) and are directed to use ‘multilink’ plastic cubes to solve them, following the ‘frogs’ pairing procedure. The remaining two groups have a similar problem sheet, but are directed to use the counting-on method to find the differences.

Nine minutes later, Naomi calls the class together on the carpet for an eight-minute Plenary, in which she uses two large, foam 1-6 dice to generate two numbers, asking the children for the difference each time. Their answers indicate that there is still widespread confusion among the children, in terms of her intended learning outcomes.

Foundation knowledge: subtraction

Carpenter and Moser (1983) identify four broad types of subtraction problem structure, which they call *change*, *combine*, *compare*, *equalise*. Two of these problem types are particularly relevant to Naomi’s lesson. First, the change-separate problem, exemplified by Carpenter and Moser by: “Connie had 13 marbles. She gave 5

¹ The National Numeracy Strategy *Framework* (DfEE, 1999) guidance effectively segments each mathematics lesson into three distinctive and readily-identifiable phases: the *mental and oral starter*; the *main activity* (an *introduction* by the teacher, followed by *group work*, with tasks differentiated by pupil ability); and the concluding *plenary*.

marbles to Jim. How many marbles does she have left” (p. 16). The UK practitioner language for this is subtraction as ‘take away’ (DfEE, 1999, p. 5/28).

Secondly, the compare problem type, one version of which is: “Connie has 13 marbles and Jim has 5 marbles. How many more marbles does Connie have than Jim”. (Carpenter and Moser, 1983, p. 16). This subtraction problem type has to do with situations in which two sets (Connie’s marbles and Jim’s) are considered simultaneously - what Carpenter and Moser describe as “static relationships”, involving “the comparison of two distinct, disjoint sets”(p. 15). This contrasts with change problems, which involve an action on and transformation of a single set (Connie’s marbles). Again, the National Numeracy Strategy *Framework* (DfEE, 1999) reflects the tradition of UK practitioners in referring to the compare structure as ‘subtraction as *difference*’. We return to this point in a moment.

Carpenter and Moser go on to show that the semantics of problem structure, as discussed above, by no means determines the processes of solution adopted by individual children, although the structure might suggest a paradigm, or canonical, strategy. They describe six broad categories of subtraction strategy identified in the research literature. Some involve actions with concrete materials, others depend on forms of counting, yet others on known facts (such as 10-5) and derived facts (such as 11-5, derived from knowing e.g. 5+5). Most strategies with materials are associated with a parallel counting strategy. For example, *separating from*, the canonical strategy for the change-separate (‘take-away’) structure described above, involves constructing the larger set and then removing a number of objects corresponding to the subtrahend number. Counting the remaining objects yields the answer. The parallel counting strategy is called *counting down from*. The child counts backwards, beginning with the minuend. The number of iterations in the backward counting sequence is equal to the subtrahend. The last number uttered is the answer. Clearly, therefore, the child needs a suitable strategy for keeping track of the number of iterations; one way would be to tally them, typically with fingers. The *counting up* strategy involves a *forward* count beginning with the smaller number (subtrahend). The last number uttered is the minuend. This time, the number of iterations in the forward counting sequence is equal to the answer. Finally, Carpenter and Moser’s taxonomy of strategies includes *matching*, which is unusual in that it has no purely ‘mental’ parallel in the absence of concrete objects. The child puts out two sets of objects with the appropriate cardinalities. The sets are then matched one-to-one. Counting (or subitising) the unmatched cubes gives the answer. It is relevant to note here Carpenter and Moser’s finding with Grade 1 to 3 children that the matching strategy is very rarely used. The only exception to this rule was by Grade 1 children who had received no formal instruction in addition and subtraction. The majority of these children who successfully solved a compare-type problem did so by using a matching strategy. By Grade 2, matching had given way to counting up.

The National Numeracy Strategy *Framework* (DfEE, 1999) reflects typical Early Years education practice in recommending the introduction of subtraction, first as take-away, in Year R (pupil age 4-5), then as comparison in Year 1. One consequence of this Early Years initiation is the almost universal use of ‘take away’ as a synonym for subtraction (Haylock and Cockburn, 1997, p. 38). Another peculiarly-British complication is that the word ‘difference’ has come to be associated in rather a special way with the comparison structure for subtraction. It is not easy to be definite how and when this came about, but one useful reference is the teacher’s manual for the highly-influential *Mathematics for Schools* (Fletcher, 1971) primary text book series. The series was ‘new maths’ in spirit, tempered with typically-British pragmatism. In a section entitled *Comparison and ‘take away’*, Fletcher describes comparison in terms of matching the elements of two sets. Some elements of the larger set remain unmatched. Fletcher writes:

The cardinal number of this unmatched subset denotes the *difference* between the cardinal number of Set A and Set B. In determining a *difference* we compare a set of objects by matching its members with another set of objects. (p. 9, emphasis in the original)

It is clear that Fletcher is associating the word ‘difference’ with comparison in order to *distinguish* it from take-away, although the grounds for doing so are not made explicit. The same association can be seen in recent UK teaching handbooks, for example:

Story 2 introduces [...] the *comparison* structure. [...] When comparing two sets we may ask ‘how many more in A?’ or ‘how many fewer in B?’ or ‘what is the difference between A and B?’ (Haylock and Cockburn, 1997, p. 39).

Crucially, as we remarked earlier, the NNS itself refers to the compare structure as ‘subtraction as *difference*’. However, at the same time, the term *difference* is the unique name of the outcome of *any* subtraction operation, on a par with *sum*, *product* and *quotient* in relation to the other three arithmetic operations. There is evidence that these complexities, and others, present obstacles to the pupils throughout the lesson (Rowland, 2006).

YEAR 8: GRAPHS OF LINEAR FUNCTIONS²

The teacher, Suzie, had about 7 years’ teaching experience. The lesson begins with 10 minutes’ whole-class revision of fractions simplification e.g. $\frac{24}{6}$, $\frac{5}{25}$. Suzie then writes the lesson aims on a board:

Find the gradient of straight lines.

Use the gradient and the intercept on the y-axis to find the equation of straight lines.

² From the ESRC-funded T-media Project 2005-07, University of Cambridge. Principal Investigator Sara Hennessy

Suzie asks what ‘gradient’ means. She develops one response - “how steep” - in terms of steep hills. Other pupil ideas include: road, roof, a slide, swing frame, ski slope, stairs.

Suzie then writes on the board: “gradient = up/along”. She rolls the whiteboard to a squared section, and draws a line segment between two lattice points (4 along, 8 up). Suzie completes the triangle, using endpoints of the line segment, to show the horizontal and vertical increments. She says that the gradient is $8/4 = 2$.

Suzie then draws another line segment alongside the first. Its gradient is $3/6$. Some pupils say “2”. In response, Suzie asks: what is $3/6$? One girl asks: is it $1/3$? Suzie says: It is $1/2$. She asks which line (segment) has bigger gradient? She says that 2 is bigger than $1/2$. One pupil refers to the two completed triangles that Suzie has drawn, and asks if it’s about area [i.e. does the first line have bigger gradient because the first triangle has greater area?]. This phase lasts 15 minutes.

There is then individual/paired work for 15 minutes. Pupils share laptop computers and load the graphing software Autograph. Suzie distributes a worksheet. The sheet asks them to draw $y=x$, $y=2x$, $y=3x$ and find the gradients (and generalise). Then it shows graphs of two lines through the origin and asks for their equations. Finally, it asks for a prediction of the graph of $2x+1$, with Autograph check. Suzie circulates to assist pairs.

The lesson concludes with a short plenary. Suzie projects $y=x$ (from her laptop) on a screen and asks about the gradient. Likewise $y=2x$, $y=3x$. In each case it is calculated using a segment with one point at the origin. A boy says “the number before x is always the gradient”.

Then Suzie displays the graph of $y = 2x+3$. She picks the segment between $(-1, 1)$ and $(0, 3)$ to calculate the gradient. Suzie writes “ $y=2x+3$ ” and annotates “gradient” near the symbol ‘2’, and “cross the y -axis (intercept)” alongside ‘3’. Finally Suzie displays another line on the large screen, and asks “What is its equation?” She finds the gradient starting from $(0, 1)$. The intercept is 1. Suzie writes $y=3x+1$, and the lesson concludes.

Foundation knowledge: gradient

Some reflections of a *mathematical* kind on the nature of the ‘gradient’, a concept which occupied much of the lesson time, is prompted by the examples that Suzie drew on the whiteboard when she introduced the concept quantitatively. Her examples were of line segments, whereas gradient is an attribute of (infinite) lines. Indeed, the graphing software (Autograph) that they used later draws lines, not line segments. Fundamental issues to be understood and considered by the teacher, therefore, include:

- the gradient of a line is found by isolating a segment of the line;

- any segment yields the same ratio (this could be tested empirically: theoretically, it relates to similar triangles).

There also exists knowledge of an explicitly pedagogical kind – more PCK than SMK – about the teaching and learning of the concept ‘gradient’. This is accessible in part by didactical reflections related to the mathematical observations already made:

- some segments facilitate identifying the increases in abscissa (x -coordinate) and ordinate (y -coordinate) better than others;
- the increase in abscissa should be ‘simple’ (ideally 1) to facilitate calculation of the ratio (unless one uses a calculator).

There were few problems with finding the gradient of $y=mx$ because $(0, 0)$ could be taken to be one end of a line segment, and $(1, m)$ the other. However, $y = 2x+3$ was much more problematic. So was $y = 3x+1$, and it seemed that few pupils followed Suzie’s demonstration at the end of the lesson.

Beyond pure reflection, there is knowledge to be gleaned from empirical research. The iconic *Concepts in Secondary Mathematics and Science* study found “a large gap between the relatively simple reading of information from a graph and the appreciation of an algebraic relationship” (Kerslake, 1981, p. 135). In particular, the notion that proportional linear relationships hold in all segments of a line, and that lines are parallel if and only if they have the same gradient, was understood by very few pupils aged 13-15. In another study, Bell and Janvier (1981) identified what they call “slope-height confusion”, whereby slope as a ratio is not distinguished from the linear dimensions of a line. This resonates with the pupil’s question about area, although it is not the same. More recently, Hadjidemetriou and Williams (2002) have found that teachers tend to underestimate the difficulties experienced by children in answering graphical test items, not least because they themselves had the misconception the item was designed to elicit.

DISCUSSION

It is reasonable to claim that a particularly pithy concept (subtraction; gradient) lies at the heart of each of these lessons, and, from my observations, lies at the root of the pupils’ difficulty in learning what had been explicitly stated as the objectives of each lesson. This remark is not intended as a criticism of the two teachers involved, both of whom were committed to developing their teaching, and to the cause of mathematics teacher education. The complexity of the concepts would remain whoever was teaching them, and for other learners of similar ages. In both cases, there exists research evidence to suggest what can be expected of pupils (at the relevant ages) who have experienced instruction in these topics. This is useful in terms of anticipating the complexity of the material to be taught, and in terms of having realistic expectations of what will be learned, both because of and despite one’s best efforts.

What I find particularly interesting is the analysis of the concepts themselves. Some of this kind of analysis is achievable by ‘deep thought’, as it were, but in some cases it needs particularly insightful observational research (such as that cited on counting) to prise apart, or unpack, processes and skills that inevitably become automated, and therefore trivial, to adult users of those competences. The complexity of such skills necessarily becomes invisible to the educated citizen, yet it needs to be laid bare if they set out to teach them. My proposal here is that much elementary mathematics teaching is ‘difficult’, compared with teaching in the secondary grades and beyond, because the very concepts being taught, such as subtraction, lie somewhere beneath our conscious awareness, and our ability to analyse in pedagogically useful ways. Secondary and tertiary mathematics teaching is ‘difficult’ for different reasons, where teacher knowledge is concerned. In the case of Suzie’s lesson, for example, the teacher needs a good understanding of the defining characteristics of functions (e.g. Freudenthal, 1983; Even, 1999), which is ‘advanced’ knowledge in that it comes within the scope of undergraduate mathematics study. They also need a thought-out, connected understanding of the different ways in which functions can be *represented* symbolically and graphically, and how to navigate both within and between these two semiotic systems (Presmeg, 2006). Even (op cit.) found that this understanding could not be taken for granted in her prospective secondary teacher participants.

I liken much of the Foundation knowledge that underpins the teaching of elementary mathematics concepts – and this is where I arrive at the claim set out earlier – to the place of Foundations of Mathematics in mathematics itself, and in the world of the practising, so-called ‘working’, mathematician. Most mathematicians can get on with their work without the need to ask “But what is a set, a number, a line, a sentence, a theorem, ...” and so on. From time to time, particular individuals are motivated to ask, and to attempt to answer, such questions, for various reasons: out of curiosity, or in order to resolve paradoxes, or to explain why a proof cannot be accomplished. In some ways, it is easier to continue building up the edifice of mathematics than to dig down beneath it, to establish the foundations. In the same way, engaging with the foundations of mathematical ideas that educated citizens take for granted, in order to make them accessible to young learners, poses its own distinctive challenges. For more advanced mathematical topics, the challenge to teachers lies more in the complexity of the concepts, the extent of the prerequisite concepts, and the sophistication of the semiotic systems with which they are represented in mainstream mathematical practice.

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RESULTS OF A COMPARATIVE STUDY OF FUTURE TEACHERS FROM AUSTRALIA, GERMANY AND HONG KONG WITH REGARD TO COMPETENCIES IN ARGUMENTATION AND PROOF

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The article describes the conceptions and first results of an enrichment study to the international comparative study on the efficacy of teacher education, Mathematics Teaching in the 21st Century (MT21). The study focuses on the professional knowledge of future teacher students in three countries – Australia, Germany and Hong Kong – with regard to the mathematical areas of modelling and argumentation and proof. After describing the theoretical framework and the applied methodological approach some selected results with regard to argumentation and proof are presented.

Keywords: Education, Mathematical content knowledge, Pedagogical content knowledge, Proof.

Background of the study

Although teacher education has already been criticised for a long time, only rarely systematic evaluation and studies concerning the efficiency of teacher education and how future teachers perform during and at the end of their education can be found (for an overview on the debate see Blömeke *et al.*, 2008). Even in the field that is covered by most of the existing studies – the education of mathematics teachers – research deficits have to be stated: the research is often short term, of a non-cumulative nature, and conducted within the researcher's own training institution. Only recently more empirical studies on mathematics teacher education have been developed (cf. Chick *et al.*, 2006, Adler *et al.*, 2005).

In order to overcome this deficit the IEA (International Association for the Evaluation of Educational Achievement) currently carries out an international comparative study focusing on the efficiency of teacher education and the professional knowledge of future teachers called TEDS-M (Teacher Education and Development Study – Learning to Teach Mathematics). This study concentrates on future mathematics teachers and is conducted in 20 countries worldwide. We also refer to the COACTIV – study, another study on teacher education using similar conceptualisations of professional knowledge of mathematics teachers (see among others Krauss, Baumert & Blum 2008). Furthermore in order not only to develop a theoretical framework and adequate instruments for the TEDS-M study but also to offer a first research attempt to fill existing research gaps, a pilot study for TEDS-M was conducted called Mathematics Teaching in the 21st Century (MT211 [1]). This study also aimed to shed light on the important field of mathematics teacher

education from a comparative perspective. For this among others the knowledge and beliefs of future lower secondary teachers were investigated (for results see e.g. Blömeke, Kaiser, Lehmann, 2008, Schmidt *et al.*, 2008).

The study described in this paper is a complementary study to MT21 with the aim of gaining supplementary results basing on qualitative data as an addition to the quantitative data of MT21. This study is a collaborative study between researchers at universities in Germany, Hong Kong and Australia, using the theoretical framework and theoretical conceptualisation from MT21, but carrying out qualitatively oriented detailed in-depth studies on selected topics of the professional knowledge of future teachers, namely modelling and argumentation and proof, the latter being the theme of this paper. The study is only focussing on future teachers and their first phase of teacher education (for details see Schwarz *et al.*, 2008).

THEORETICAL FRAMEWORK OF THE STUDY

The initial ideas of MT21 are considerations about the central aspects of teachers' professional competencies and by this the related theoretical framework is also the theoretical basis of the supplementary study. Concerning the professional knowledge of teachers the study follows the ideas basically defined by Shulman (1986). He fundamentally distinguishes two domains, namely general pedagogical knowledge and content knowledge. The latter is further divided into three parts:

- subject matter content knowledge
- pedagogical content knowledge
- curricular knowledge

For the study these areas of content knowledge are further sub-divided. In the area of subject matter content knowledge for example with regard to Bromme (1995) mathematics as a school subject and mathematics as a scientific discipline are differentiated.

Beside these cognitive components furthermore also an affective and value-orientated component is taken into consideration. This component especially accounts for the epistemological beliefs, more precisely the beliefs towards mathematics itself and the beliefs towards teaching and learning mathematics. Again in accordance with the theoretical conceptualisations of MT21 (see Blömeke, Kaiser, Lehmann, 2008) the differentiation of different beliefs towards mathematics of Grigutsch, Raatz and Törner (1998) is basis of the study. Here four kinds of beliefs are distinguished with relation to mathematics:

- formalism-aspect of mathematics
- scheme-aspect of mathematics
- process-aspect of mathematics

- application-aspect of mathematics

Based on these theoretical distinctions concerning professional knowledge of future teachers the overall aim of our study is to answer the following questions:

- What kind of knowledge with regard to the described domains of teachers' professional knowledge do future teachers acquire during their university study?
- Which connections between the described domains of knowledge and the beliefs can be reconstructed within these future teachers?

In this paper from a mathematical content related perspective we concentrate on the area of argumentation and proof. Furthermore because of the limited space we only focus on the first question and describe some selected results. For a more detailed description of results related to the area argumentation and proof see Schwarz et al. (2008). For first results related to the second question with regard to the mathematical area of modelling see Schwarz, Kaiser, Buchholtz (2008).

Concerning the area of argumentation and proof we refer to specific European traditions, in which various kinds of reasoning and proofs are distinguished, especially “pre-formal proofs” and “formal proofs”. These notions were elaborated by Blum and Kirsch (1991): pre-formal proof means “a chain of correct, but not formally represented conclusions which refer to valid, non-formal premises” (Blum & Kirsch, 1991, p. 187).

Concerning the role of proof in mathematics teaching, Holland (1996) details the plea of Blum and Kirsch (1991) for pre-formal proofs besides formal proofs as follows: For him pre-formal proofs may be sufficient in mathematics lessons with cognitively weaker students, in other classes both kinds of proofs should be conducted. Pre-formal proofs have many advantages due to their illustrative style. In addition, pre-formal proofs contribute substantially to a deeper understanding of the discussed theorems and they place emphasis on the application-oriented, experimental and pictorial aspects of mathematics. However, their disadvantage is their incompleteness, their reference to visualisations, which require formal proofs in order to convey an appropriate image of mathematics as science to the students. The scientific advantage of formal proofs, namely their completeness, is often accompanied by a certain complexity, which may cause barriers for the students' understanding and might be time-consuming. However, there is no doubt, that treating proofs in mathematics lessons is meaningful with the aim of developing general abilities, such as heuristic abilities. The teaching of these two different kinds of proofs leads to high demands on teachers and future teachers. Teachers must possess mathematical content knowledge at a higher level of school mathematics and university level knowledge on mathematics on proof. This comprises the ability to identify different proof structures (pre-formal – formal), the ability to execute proofs on different levels and to know alternative specific mathematical proofs.

Additionally, teachers should be able to recognise or to establish connections between different topic areas. To sum up: Teachers should have adequate knowledge of the above-described didactical considerations on proving as well (for details see Holland, 1996, pp. 51-58). It can be expected that in addition to being able to construct proofs, teachers will need to draw on their mathematical knowledge about the different structures of proving such as special cases or experimental ‘proofs’, pre-formal proofs, and formal proofs and pedagogical content knowledge when planning teaching experiences and when judging the adequacy or correctness of their, and their students’ proofs in various mathematical content domains.

METHODICAL APPROACH

Based on the methodological approach of triangulation questionnaires with open questions and in-depth thematically oriented interviews were developed. This offers the opportunity to deepen the quantitative results of MT21 by means of this qualitative orientated data. The instruments are, as described above, restricted to the areas of modelling and argumentation and proof. The questionnaire consists of seven items that are domain-overlapping designed – as so-called ‘Bridging Items’. Each of the items captures several areas of knowledge and related beliefs on the base of the distinctions described above. In detail three items deal with modelling and real world examples, three with argumentation and proof and one is about how to handle heterogeneity when teaching mathematics. Furthermore, demographic information like number of semesters, second subject and attended seminars and teaching experiences – including extra-university teaching experiences - are collected. This questioning has been conducted with 79 future mathematics teachers on a voluntary base within the scope of pro-seminars and advanced seminars for future teachers at a German university. In Australia, 46 future teachers from two universities participated and in Hong Kong 84 future teachers from one institution.

Complimentary to this questionnaire an interview guide for a problem-centred guided interview was developed, which contains pre-structured and open questions (i.e., elaborating questions) on modelling and argumentation and proof. The questions are linked to the items in the questionnaire in the sense that they have the same theoretical base and cover the same sub-domains of teachers’ professional knowledge. The selection of the interviewees follows theoretical considerations and takes the achievements in the questionnaire into account. That means interviewees were selected according to an interesting answering pattern in the questionnaire or extraordinary high or low knowledge in one or more domains.

The evaluation of the questionnaires as well as of the interviews is carried out by means of the qualitative content analysis method by Mayring (2000). More detailed we apply a method of analysis that aims at extracting a specific structure from the material by referring to predefined criteria (deductive application of categories). From there, by means of formulation of definitions, identification of typical passages from the responses as so-called anchor examples and development of coding rules, a

coding manual has been constructed to be used to analyse and to code the material. For this, coding means the assignment of the material according to the evaluation categories. More precisely the method of structuring scaling (ibid.) is applied by which the material is evaluated by using scales (predominantly ordinal scales). Subsequently, quantitative analyses according to frequency or contingency can be carried out.

In the following one exemplary item of the questionnaire is described, which shows, how the different facets of professional knowledge – pedagogical content knowledge, mathematical knowledge and beliefs - are linked. A similar item is included in the interview, so that it is possible to connect the evaluation of the data on a rich data base.

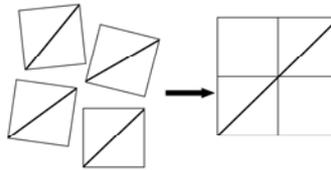
Read the following statement:

If you double the side length of a square, the length of each diagonal will be doubled as well.

The following pre-formal proof is given:

You use squared tiles of the same size. If you use four tiles to make one square, you will get a square with a side length twice the length of the squared tiles.

You can see immediately, that each diagonal has twice the length of the ones of the squared tiles because the two diagonals of two tiles are put directly together.



- a) Is this argumentation a sufficient proof for you? Please give a short explanation.
- b) Please formulate a formal proof for the statement above about diagonals and squares.
- c) What proof would you use in your mathematics lessons? Please explain your position.
- d) Can a pre-formal proof be sufficient as the only kind of proof in mathematics lessons? Please explain your position.

- e) Please name the advantages and disadvantages of a formal and pre-formal proof.
- f) Can the pre-formal and the formal proof for the statement about the length of diagonals in squares be generalised for any rectangle? Please give a short explanation.
- g) What do you think about the meaning of proofs for mathematics lessons in the secondary school?

Figure 1: Task from the questionnaire concerning argumentation and proof

SELECTED RESULTS

Both, part b) and part f) of the task described above lay their focus on the future teachers' mathematical content knowledge. Part b) does especially not require any mathematics at a university level but only knowledge about fundamental geometrical theorems (e.g. Pythagoras theorem) and abilities concerning elementary algebraic transformations and abilities in formulation proofs. The items was coded on a five-point-scale while both codes, +1 and +2, means a right solution (answers coded with +2 in addition have a comprehensible structure) and -2 means serious mistakes like circular arguments or just a rephrasing of the pre-formal proof while a formal one is required. Examples of future teachers' responses and a more detailed description of the different coding of different answers are not presented here because of the limited space. Related descriptions can be found in Schwarz et al. (2008).

The results are the following:

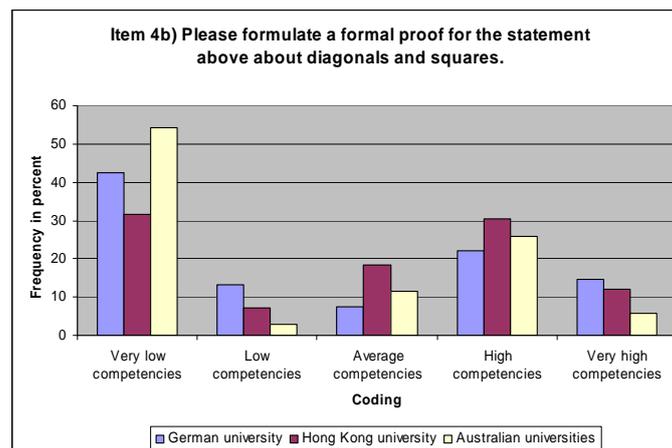


Figure 2: Results of item 4b)

One can see that for almost all institutions, the majority, in most instances, of future teachers in this case study were not able to execute formal proofs, requiring only lower secondary mathematical content, in an adequate and mathematically correct way.

Very similar results can be seen with regard to item f). Here also no university mathematics is needed but just an understanding of a proof suitable for lower secondary mathematics teaching. Again answers were coded on a five-point-scale with +1 and +2 meaning right solutions and -1 and -2 meaning wrong solutions. Then the results are the following:

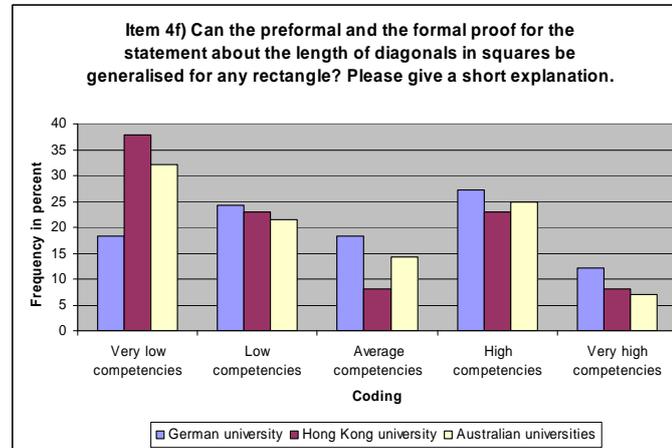


Figure 3: Results of item 4f)

Again, in most cases, the majority is not able to recognise and satisfactorily generalise a given mathematical proof.

In contrast, in all samples there was evidence of at least average competencies of pedagogical content reflection about formal and pre-formal proving in mathematics teaching with the exception of the Australian sample with respect to the sufficiency of pre-formal proof as the only type of proof in mathematics lessons. The related results are presented in a more qualitative way in the following paragraphs.

Preferences for pre-formal proving are evident, both with respect to mathematical content knowledge and pedagogical content knowledge. In contrast to the Hong Kong and Australian samples, there was a strong tendency in the German data for pre-formal proving to be incorporated into the pedagogical content-based discussion particularly with respect to problems of using proof with students of different abilities. In both the Hong Kong and Australian data, future teachers indicated a broad open-mindedness to various didactical conceptions but the pre-formal proof was perceived as an atypical part of mathematics teaching, possibly reflecting the use of alternate terms and conceptions for argumentation and proving that is not formal proof in the teacher education courses in these contexts. In both samples, mathematical content considerations tended to be the basis for didactical reflections.

With regards to affinity towards proving in lower secondary mathematics lessons Australian, Hong Kong and German students indicated a high to very high affinity to proving. It was assumed a higher affinity to proving would be expressed in more distinct pedagogical content reflection; however, the nature of these reflections differed with the samples. Future teachers in the German sample assumed dealing with proofs helped develop students' argumentation abilities especially with respect

to their own hypotheses rather than their completeness of mathematical theorems. The difficulties students might have with proving in the classroom also came to the fore. In contrast, the Hong Kong and Australian future teachers rarely mentioned difficulties students might have with proving. The responses of future teachers from both Hong Kong and Australia reflected a formal image of mathematics being reinforced through use of formal proofs in teaching and the practice of proving leading to the comprehension of mathematical theorems.

SUMMARY AND OUTLOOK

The paper describes first results of an additional study to the international comparative study on the efficiency of teacher education MT21. With regard to a theoretical framework distinguishing between different areas of teachers' professional competence results concerning future teachers' knowledge in different areas are presented restricted to the mathematical field of argumentation and proof.

As the presented additional study only focuses on future teachers, which means university students, no statements concerning the professional knowledge of practicing teachers can be made.

With regard to the further work to be done one of the next steps of the evaluation will be a more detailed distinction between different subgroups of the sample and the particular characteristics of their professional competence. For this evaluation the sample will be divided twice. On the one hand different school types the future teachers are studying for can be differentiated. On the other hand future teachers in different phases of their university studies, which means beginners or students at the end of their studies, can be distinguished. Besides that the results of the analyses of the interviews are to be linked to the results of the questionnaires. First results of these analyses can be found in Corleis et al. (2008). Finally the results of the additional study are to be related to the results of the main study MT21.

NOTES

1. The previous name of this study was PTEDS.

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KATE'S CONCEPTIONS OF MATHEMATICS TEACHING: INFLUENCES IN THE FIRST THREE YEARS

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In this paper I report on findings from a four year study of beginning teachers. The findings presented concern the conceptions of mathematics teaching for one of four case-study teachers and the influences on these conceptions. I present data from observations of lessons, interviews and written accounts that suggest Kate's conceptions of teaching became increasingly more consistent with a 'content-focused with an emphasis on conceptual development' view of teaching. Data are also presented which suggest that 'reflection' was the main influence on the development of Kate's conceptions both as an independent factor and in conjunction with the factors of 'experience' and 'working with others'.

INTRODUCTION

There is evidence that the conception of mathematics teaching held by individual teachers will contribute to the effectiveness of their teaching (Thompson, 1992; Askew, Brown, Rhodes, Johnson and Wiliam, 1997). The term conceptions is used here in the way suggested by Thompson (1992), as an inclusive term to include beliefs as well as other ideas such as mental images, concepts, meanings and preferences. Conceptions of mathematics teaching is clearly an area that needs to be addressed in any work which attempts to describe or influence the development of beginning teachers in relation to the teaching of mathematics. Assessing teachers' conceptions and the promotion of such conceptions that are believed to be positively influential in children's learning were seen as integral to my PhD study, an aspect of which I report on here.

Khus and Ball (1986) proposed four models of teachers' views about mathematics teaching, *a classroom-focused view*, *a content-focused with an emphasis on performance view*, *a content-focused with an emphasis on conceptual understanding view* and *a learner-focused view*. I used these models as a theoretical framework for the analysis of data collected in my study. Though I have analysed the data in relation to all four of Khus and Ball's models of conceptions of mathematics teaching, restrictions of space here only allow discussion in relation to the *content-focused with an emphasis on performance* and the *content-focused with an emphasis on conceptual understanding views*.

The aim of my study was to investigate the way in which beginning teachers' understanding of mathematics content knowledge needed for teaching might be developed through reflection using the *Knowledge Quartet framework*. This framework was used as a tool for identification and discussion of the teachers' mathematics content knowledge as evidenced in their teaching. The Knowledge

Quartet framework consists of four dimensions, *Foundation*, *Transformation*, *Connection* and *Contingency*. Details of this framework, and an account of how it was developed, may be found in the paper presented by Tim Rowland at the CERME meeting in Spain (Rowland, Huckstep and Thwaites, 2005).

Teacher's beliefs about mathematics and mathematics teaching were considered to be a component of mathematics content knowledge and are incorporated in the Foundation dimension of the Knowledge Quartet framework. Findings relating to the development of the Foundation aspect of one teacher's mathematical content knowledge were presented in a paper at the CERME meeting in Cyprus (Turner, 2007). The focus of the 2007 paper was on Amy, and drew on data from the first two years of the study. This paper focuses on the aspect of conceptions about mathematics teaching from within the Foundation dimension and presents findings relating to Kate over the full four years of the study.

THE STUDY

The study began with 12 student teachers from the 2004-5 cohort of primary (5-11 years) postgraduate pre-service teacher education course at the University of Cambridge. The numbers reduced, as anticipated, to 9 in the second year, then 6 in the third year and finally 4 in the fourth and last year of the study. All participants were observed teaching during the final placement of their training year, twice during the first year, three times during the second year and once in the third year of their teaching. These lessons were all video-taped. In the training year the video-tapes were the basis for stimulated recall discussions using the Knowledge Quartet framework to focus on the mathematical content of the lesson. During the first year of teaching, feedback using the Knowledge Quartet framework was given following the two observed lessons. Participants were then sent a DVD with a recording of their lesson, and a request to observe the lesson and write their reflections on it. In the second year of their teaching only minimal feedback was given following the lesson as I wanted to see how the teachers would independently make use of the Knowledge Quartet in their reflections. They were sent DVDs of their three lessons and wrote reflections on each of these, drawing on their previous training in using the Knowledge Quartet framework. Participants also wrote regular reflections on their mathematics teaching which they sent to me. Group meetings were held to discuss the mathematics teaching and participation in the project of participants. These happened at the end of the training year and the first year of teaching, and at the end of each term in the second year of teaching. In their third year of teaching each teacher was interviewed individually in the Autumn and Spring terms and a group meeting was held in the Spring term.

Case studies were built from observations of teaching, discussions following observed lessons, contributions to group meetings, written reflections and individual interviews. Data from transcripts of discussions following observed lessons and group interviews as well as from written reflections was all analysed using the

qualitative data analysis software NVivo. A grounded theory approach (Glaser and Strauss, 1967) was used which led to the emergence of a hierarchical organisation of codes into a number of themes. Analysis of data attributed to codes under the NVivo theme ‘beliefs’, and the Knowledge Quartet analysis of observed lessons were used to build a description of the participants conceptions of mathematics and mathematics teaching over the four years of the study. Analysis of data attributed to codes under the themes of ‘experience’, ‘reflection’ and ‘working with others’ allowed inferences to be made about the factors associated with changes to participants’ conceptions. Though data from all four case studies have been analysed in relation to changes in their conceptions of mathematics teaching, there is only room to report on Kate here.

Since in this discussion I hope to build a picture of the way in which the participants’ conceptions developed over time, it is necessary to refer to times at which different data were collected. To aid clarity, and achieve brevity in this, I will use the date of the year and a number only to identify the timescale. Table 1 is intended to help the reader place the data within this timescale.

Table 1: Notation used to indicate the timescale of data collected in the study

Notation used		Place in career
2004	Autumn term	Trainee teacher
2005/6(1)	Autumn / Spring term	First year in teaching post
2005/6(2)	Spring / Summer term	
2006/7(1)	Autumn term	Second year in teaching post
2006/7(2)	Spring term	
2006/7(3)	Summer Term	
2008(1)	Autumn Term	Third year in teaching post
2008(2)	Spring Term	
2008 (3)	Summer term	

FINDINGS

Analysis of teaching and of data coded under the heading ‘beliefs’, provided an account of Kate’s conceptions of teaching over the first three years of her career. In Kate’s lesson observed in 2004, the Knowledge Quartet code ‘reliance on procedures’ featured strongly and suggested a view which *emphasised performance*. Kate was teaching a lesson about doubling single digit numbers and demonstrated recording the doubling process by writing an addition in a witch’s cauldron with the answer in a bubble above e.g. ‘3 + 3’ in the cauldron and ‘6’ in the bubble. To record doubling of two digit numbers an extra bubble was added for the ‘tens numbers’ e.g. ‘23 + 23’ in the cauldron, ‘4’ in the tens bubble and ‘6’ in the units bubble. When questioned about this in the post-lesson reflective interview Kate suggested an amendment,

If I was going to do the tens and units, I should have asked for the units first ‘cus that’s what they know they have to start with, the most significant number which is tens.

Kate focused on a procedure reflecting the standard algorithm which suggested an emphasis on performance view of teaching. However in this same lesson, there was an indication that Kate was concerned to develop conceptual understanding. The Knowledge Quartet code ‘making connections between concepts’ was attributed when Kate made connections between doubles and near doubles and even and odd numbers and used pictorial representation to demonstrate why doubles must be even and near doubles odd.

In the lesson observed in 2005/6(1), Kate introduced the concept of multiplication by making the connection with repeated addition. She used a number of different representations modelling repeated addition to develop understanding of the concept of multiplication. However, when they came to do some problems themselves, the children were given specific procedures for calculating and recording. This lesson seemed to reflect a mixture of *content-focused views* of teaching with both an *emphasis on performance* and an *emphasis on conceptual development*.

The second lesson observed in 2005/6 did not feature ‘reliance on procedures’ and Kate made use of demonstrations to develop the children’s understanding of capacity and conservation. However, at the group interview in 2005/6(2), Kate suggested that she thought the children preferred an approach which emphasised procedures.

They really like doing boring things, they like doing number sentence things, they don’t like the other [problem solving] it’s more difficult, but they really like number sentences.

The three lessons observed in 2006/7 all demonstrated a concern for developing conceptual understanding rather than focusing on performance. In the lesson observed in 2006/7(1), there were no instances of ‘reliance on procedures’ and Kate used a number of demonstrations to build the children’s understanding of measuring using appropriate non-standard and standard units. In the lesson observed in 2006/7(2), Kate made use of a number of different representations to develop the difference conception of subtraction and also asked the children to explain their own strategies for completing the calculations. In the warm up part of the lesson observed in 2006/7(3), Kate set problems involving making the largest and smallest numbers on a spiked abacus using specified numbers of beads. This was designed to develop their conceptual understanding of place value. The main part of this lesson involved shading of different fractions on various grids. The way in which Kate introduced this, and the activities set for the children seemed to be aimed at developing a conceptual understanding of vulgar fractions.

The suggestion that Kate’s emphasis was on conceptual understanding in 2006/7, was supported by analysis of the NVivo coding of data. Kate had five instances of the code ‘conceptual understanding’ attributed to her data from 2006/7. In her reflective account written in 2006/7(3), Kate wrote,

Following the quite broad objectives of the new strategy, we have been trying to teach about data handling in quite a conceptual way and get children to think about the advantages and disadvantages of different ways to represent data.

Though observations of teaching and analysis of the NVivo data suggested Kate had moved towards an *emphasis on conceptual understanding view* of teaching, there were also a number of instances of her data from 2006/7 and 2007/8 which suggested she still held an *emphasis on performance view*. Two codes considered to reflect an *emphasis on performance view*, which featured strongly in her data, were ‘teaching different strategies’ and ‘need for structured work’. Five instances from reflective accounts written in 2006/7 were coded as ‘teaching different strategies’.

We have been looking at different addition strategies ... We had specific teaching sessions on some of these areas, then had some activities in which children were encouraged to choose a method for themselves. 2006/7(1)

This instance, and others like it, indicated that Kate felt she needed to give children a ‘toolbox’ of strategies from which to choose in order to perform calculations. Later in the year she seemed to have moved her position towards one in which she felt it was more helpful to focus on just some specific methods.

The week was structured around teaching a few particular methods, which is a little different from the approach we have often taken before when we have given the children opportunities to choose their own methods. 2006/7(2)

Kate’s move towards an approach involving teaching specific methods with which children can be successful seemed to reflect an emphasis on performance.

Data coded as ‘need for structured work’, suggested that Kate seemed more concerned that children achieved success in solving problems than that they developed a conceptual understanding. During the interview in 2007/8(2), we discussed the teaching of ‘word problems’. Kate indicated that she focused on getting the children to look for specific words in order to decide what sort of calculation was involved.

So, rather than understanding the concept behind the problem, it was ... we wanted the children to know what they could do, and that’s why I repeated the same lesson again. This time we approached it a bit differently and said ‘if you can spot one of these words, then you can work out for yourselves what it means and you will be able to do it’.

During the interview in 2007/8(2) Kate suggested that she recognised her teaching focused on achievement or performance rather than on developing conceptual understanding through exploration.

I don’t think that we do much open-ended, and that is perhaps a bit of a weakness in the way that I teach at the moment, because quite often, quite often in lessons I tell them what I want them to achieve.

Though Kate sometimes focused on performance in 2007/8, there was evidence from the lesson observed in 2007/8(2) that she continued to emphasise conceptual understanding. In this lesson Kate demonstrated the commutativity rule for addition before introducing the strategy of putting the bigger number first. She showed this by pinning two sets of differently coloured clothes pegs on a coat hanger to illustrate an

addition e.g. $2 + 3$, and then turned the coat hanger around to show the addition $3 + 2$. Kate did not simply tell the children the rule but demonstrated why it was the case. Later in the lesson, Kate demonstrated adding ten by moving down one row on a hundred grid. She asked the children why adding 10 to 23 gave the answer 33. Kate tried without success to get a response which showed an understanding of place value in relation to the layout of the grid. In the post-lesson reflective interview, Kate stated that she was unhappy that pupils had responded in this procedural way, and said that she would work on an approach directed at understanding why this procedure works.

Kate's data suggest that over the first three years of teaching her conceptions of mathematics teaching had encompassed elements of a *content-focused view with an emphasis on performance* and a *content-focused view with an emphasis on conceptual understanding*. All of Kate's lessons observed over the three years indicated that Kate was trying to develop conceptual understanding in her pupils, and this was supported by analysis of the NVivo coding of her data. Kate's later comments suggest that she was consciously trying to focus more on developing conceptual understanding. However these comments also suggest that she continued to be concerned that children were taught specific strategies, suggesting a view which emphasises performance.

The data discussed above presented a picture of Kate's conceptions of mathematics teaching over the first three years of her career. An analysis of data under the NVivo coding headings, 'experience', 'working with others' and 'reflection', gave some insight into the influences on these conceptions. Three instances of data under the heading 'experience' suggest that this was an influence on Kate's conceptions of mathematics teaching as *content-focused with an emphasis on conceptual understanding*. In her reflective account 2006/7(1) Kate wrote,

From last time we covered place value I realised that the majority of my year ones were not very clear on this concept. I wanted to make sure they understood the importance of tens and units on how we write our numbers.

During the interview in 2007/8(1), I asked what Kate thought had influenced the way in which her teaching had changed.

I think having done it before and knowing it works and sometimes I think when I have been teaching things, I have thought 'do I really understand this', or I have thought, 'I think I might be giving a misconception here or something', and then the next time I am really careful not to.

I would argue however that 'experience' alone did not influence Kate's conceptions of mathematics teaching. Rather, an examination of the three instances, demonstrate that it was Kate's reflection on her experience that influenced her conceptions of mathematics teaching. Phrases such as 'I realised', 'I have thought' and 'extrapolating in my head', all suggest active reflection.

There were several instances of data attributed to codes under the heading 'working with others' that suggested this too influenced Kate's conceptions of mathematics

teaching. Some such instances seemed to suggest that her colleagues had a view which emphasised on performance, while Kate's view was more one which emphasised conceptual understanding. In her reflective account 2006/7(2), Kate wrote,

Various materials suggest you should use them [empty number lines] in a 'come and show me how you are going to use this in your own way' kind of approach. However my colleague believes that we should only be teaching counting on along the empty numberline because that is what the children will be taught in year three.

Kate seemed to be in a dilemma because she was concerned with conceptual understanding while her colleague seemed to focus on content of the school curriculum. Two instances from the interview in 2007/8(1) suggest that Kate's 'enculturation into a community of practice' (Lave, 1988) involved exposure to views which emphasised performance. In the first of these, Kate's use of the term 'we', suggested that an emphasis on performance had resulted from shared planning.

We are trying to work on getting them to have skills of the physical, and the sort of organisational skills of recording their maths and they sort of need a structure to do it in.

In the second instance Kate was replying to my question about whether she ever talked to other people about reflections on her teaching.

Yes, occasionally. I think I would say, 'they found that really difficult, the numbers were too high and they didn't get a chance to work on the process because they were using those numbers', or 'that was really quick and they could have done another'.

This suggested that Kate saw her conversations about mathematics teaching with colleagues as being focused on the performance of the children rather than their conceptual understanding.

There were a number of instances of data under the heading 'working with others' that suggested Kate had an *emphasis on conceptual understanding* view of teaching. However, these did not necessarily suggest that Kate's colleagues had been influential in developing this view. In her reflective account 2006/7(1), Kate discussed a difference of opinion about a planned investigation.

The person planning for our team had planned for the children to investigate the question 'do all rectangles have four sides'. When this was first suggested it struck me as a rather trivial question, but as I continued to think about it I thought it was not a very good question at all because it suggested there was something intrinsically 'rectangular' about the examples they would be spotting which would allow them to recognise them as rectangles without taking into account their four-sidedness.

I haven't discussed this with my colleagues as I didn't want to be awkward, but I made a note to myself to keep my eyes open at planning meetings so I can politely say something straight away if I am uncomfortable with the mathematical ideas behind our planning in any other cases!

Kate focused on the conceptual appropriateness of the task despite the influence of her colleague, rather than because of it. During the interview in 2007/8(1), I asked Kate whether she ever talked to her colleagues about issues such as the use of representations in her teaching.

Not as often as we should because nobody wants to do the planning again. Um, I guess I would just use the other representation rather than discussing it with anybody.

This instance suggested that Kate did not automatically take on the ideas of her colleagues, but considered their conceptual appropriateness and changed them in her own teaching if she thought it necessary. Kate's 'enculturation into her community of practice' seemed to have been mediated by critical reflection. Kate engaged in the process Wenger (1998) referred to as *critical alignment* in such a way that she developed a view of teaching that continued to be strongly *content-focused with an emphasis on conceptual development*, despite this not seeming to be the general view of her community of practice.

The factors of 'experience' and 'working with others,' seemed to have had some influence on Kate's conceptions of mathematics teaching. However, both these factors also involved the mediation of reflection. Reflection also emerged as a separate heading in the NVivo coding process and Kate had a greater number of instances of her data attributed to codes under the heading of 'reflection' than to 'experience' and 'working with others' taken together. Codes under the heading 'reflection' which related to conceptions about mathematics teaching included, 'changed thinking', 'justification of teaching', 'questioning own teaching', 'suggested improvements' and 'judgements about effectiveness'.

Some of the instances of Kate's data coded under the heading 'reflection' suggested a view of teaching that emphasised performance. In her reflective account written in 2006/7(1), she focused on how well the children had performed on the tasks.

They seemed much more prone to making mistakes [in subtraction than addition] such as being one out because of counting the one they started on. They found taking away using number lines really tricky and were quite unreliable at taking away using objects.

Though such comments focused on the children's performance of tasks there were also suggestions in them that Kate was thinking about why they had difficulties. Similarly, some comments made during the interviews in 2007/8, focused on children's performance on tasks but also mentioned understanding. For example,

The year ones did a sheet of number sentences ... that was a bad choice of sheet because it was an 'empty box' sheet and we hadn't been doing any empty boxes ... they still got it wrong because they didn't understand what it was asking them ... but I understood why they did it. So, it was OKish because they were quite purposefully engaged ...

Though this instance suggested Kate focused on engagement rather than learning, it also indicated that she had given some thought to children's conceptual difficulties.

There were few instances of data that suggested Kate focused only on children's performance without in some way considering their conceptual understanding.

In her reflective accounts Kate made several comments which explicitly demonstrated her concern with the conceptual understanding that had, or had not been achieved through her teaching. For example,

In the first lesson we did several activities which involved putting numbers into order and then went on to positioning numbers on a numberline for their independent activity, but I think this activity had more to do with place value than ordering numbers as they had to work out how many tens marks to count along and then think about the units. 2006/7(2)

Kate also made a number of comments during the interview in 2007/8(1) which suggested she held a view of teaching which emphasised conceptual development.

The children thought that triangles would have a line of symmetry but the one we tried didn't. In retrospect I wish that we had discussed that a bit more because it would have been interesting to get all the triangles out of the box and compare them.

Data from the heading 'reflection', suggested that Kate's had a strong view of mathematics teaching as *content-focused with an emphasis on conceptual understanding*. Though, this does not necessarily suggest a causal link between reflection and her view, it can be argued that reflection did influence Kate's conceptions. Kate wrote these reflective accounts because of her involvement in the study. The kind of thinking she engaged in was therefore prompted by the requirement to reflect on her teaching using the Knowledge Quartet. During the interview in 2007/8(1), Kate confirmed that this framework had influenced her thinking,

The first few things I would be thinking of are the organisational things, and then I try to think 'did they learn anything' and 'was the learning alright even if the organisation wasn't' kind of thing. So, I think it is useful to have some kind of structure to help you know what you need to know and what they need to know and how to learn it.

Later in the interview, Kate reiterated that the structure provided by the Knowledge Quartet helped her reflect on whether or not her teaching had been effective in promoting understanding.

I think what I have said and how I have explained things, I am more aware than I would be if I didn't have such a clear idea of what I was looking for.

Summary and implications

Analysis of longitudinal data from one case study of a beginning teacher has given some insight into the conceptions of mathematics teaching held by that teacher, as well as insight into the influences on those conceptions. Though finding about Kate's conceptions and the influences on them are inferential, the use of the Knowledge Quartet framework for the analysis of lessons, and the systematic analysis of all data from interviews and reflective accounts, gives a strong basis for these inferences. It is

reasonable to suggest that Kate has developed a view of mathematics teaching that is increasingly *content-focused with an emphasis on conceptual understanding* and that the development of this view has been influenced by reflections on her teaching supported by the Knowledge Quartet framework. ‘Experience’ and ‘working with others’, have also been influential in developing Kate’s conceptions of mathematics teaching. However, reflection was an important mediator in these two factors. There is evidence, not discussed here, that Kate had also moved towards a *learner-focused view* of mathematics teaching. The direction of development of Kate’s conceptions is one which we might wish to replicate in other beginning teachers. If so, it would seem that finding ways of encouraging the sort of reflection on mathematics teaching that Kate has undertaken over the first years of her career, is an idea worth pursuing.

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PRE-SERVICE TEACHER-GENERATED ANALOGIES FOR FUNCTION CONCEPTS

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The purpose of the present study was to determine pre-service teacher-generated analogies in teaching function concepts and then to discuss them in terms of the content validity – whether analogies used are epistemologically appropriate to illustrate the essence and the properties of the functions as well as the structural relations between the analogues and the targeted concepts. The videotaped data of five pre-service teachers' were collected from their microteaching during "Practice Teaching in Secondary Education" course. Results revealed that pre-service teachers did not consider too much on their analogical models. So they generally failed to make effective transformations between the analogies and the target concepts.

Keywords: Function, analogy, pre-service teacher, content validity, teacher training

INTRODUCTION

What distinguishes a mathematics teacher from mathematics major is "the capacity of a teacher to transform the content knowledge he or she possesses into forms that are pedagogically powerful and yet adaptive to the variations in ability and background presented by the students" (Shulman, 1987, p. 15). In order to move from the personal comprehension to preparing comprehension of others, some combination of the following processes: preparation, representation, instructional selections, adaptation and tailoring to students' characteristics are proposed (Shulman, 1987). For representation of the selected sequence, teacher makes use of appropriate analogies, metaphors, examples, demonstrations, explanations, etc.

Analogies constitute one crucial component of the teachers' pedagogical content knowledge that they need most to transform subject matter into forms that could be grasped by the students of different ability and social background. Analogies are heuristic tools that enhance imagination and creativity in terms of making causal relations between the unknown and the well-known concepts (Gentner, 1998). By developing mental models students have the opportunity to access to a wide range of conceptual explanations and transformations that facilitate capturing similarities and making parallels between the concepts in areas other than mathematics and the concepts in different contexts within mathematics itself. Therefore, this article focuses on pre-service teacher-generated analogies in teaching function concepts. Function concept is central for secondary school curriculum and advanced

mathematical topics taught at school and university level. Further, the function concept is considered to have a unifying role in mathematics that provides meaningful representations of real-life situations (Lloyd & Wilson, 1998). Hence, the use of analogies is very common in the teaching of functions.

Pedagogical content knowledge (PCK) refers to “the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction” (Shulman, 1987, p. 8). That is, PCK is a key aspect to address in the study of teaching. To use an example in our context, pedagogical content knowledge refers not only to knowledge about functions, but also to knowledge about the teaching of functions with analogies. To teach functions with analogies teachers should transform the subject matter for the purpose of teaching and give arguments about it. That is, they should consider the characteristics of the function concept, choose or construct well constructed analogies, and consider the similarities and differences between the different aspects of the function concepts and the analog concepts. Therefore, the study reported here is related to pre-service teacher pedagogical content knowledge. Since the process of learning is influenced by the teacher, it is therefore important to understand how teachers explain what a function is to students, what they emphasize and what they do not; and what ways they choose to help students understand.

The present study contributes to a growing body of research in the field of function by examining pre-service teacher generated analogies to determine the analogies and the target concepts and then to discuss them in terms of the content validity – whether source analogies used are epistemologically appropriate to illustrate the essence and the properties of the functions as well as the structural relations between the analogues and the targeted concepts. More specifically, we posed two main research questions for this study: (1) How do the pre-service teachers manage with the analogies they introduce? and (2) Are these analogies relevant?

Task analysis of the lessons of the pre-service teachers provides less experienced mathematics textbook authors and teachers with guidelines on how to form and use analogies effectively in teaching functions. A careful examination of an analogy is a prerequisite to using it effectively in instruction. When teachers and authors use an analogy, they should anticipate analogy-caused misconceptions and eliminate them by forming epistemologically appropriate analogies and by mapping the similarities and differences between the different aspects of the function concepts and the analogies constructed. The present study directly responds to a need among mathematics educators for insight into the nature of analogies in function concepts and guidance on how to construct ones that are pedagogically effective.

THE STUDY

Context and Participants

The study was conducted with all pre-service teachers (PT1, PT2, PT3, PT4 and PT5) taking “Practice Teaching in Secondary Education” course that was offered in Master of Science without Thesis Program at Middle East Technical University during 2005-2006 fall semester. One was male and four were female. Three of the participants (PT2, PT3, and PT5) had experience in teaching mathematics at an institution where additional courses out of school were offered and other two had experience in teaching mathematics as a private tutor. Three graduated from mathematics department (PT2, PT3, and PT4), and attending to the Master of Science without Thesis Program and rest were continuing previous mathematics teacher education program to get their bachelor degree. Master of Science without Thesis Program is a certificate program to teach mathematics at secondary school level (grades 9-12). All these students were the total number of the students in their second term.

“Practice Teaching in Secondary Education” course involves practice teaching in classroom environment for acquiring required skills in becoming an effective mathematics teacher. In this course pre-service teachers spend their six class hours in real classroom environment at an arranged public secondary school, and two class hours at the university. In that two hours period at the university, pre-service teachers presented sample lessons one by one to their colleagues and the instructor.

At the beginning of the course, function topics covered at the 9th grade and triangles topics covered at the 10th grade were assigned to each participant to be presented in a 30 minutes period at the university, to provide an effective flow of lesson and to cover all topics relevant to functions and triangles. Each participant prepared three lesson plans about assigned topics to be presented at the classroom. Two of those presentations were on functions and one on triangles. Additionally, they also did teaching two times at the school with presence of the instructor (the first researcher) and the classroom teacher. At other times they did teaching at the school when the classroom teacher allowed them to do. Teaching at the university and the school constituted 30 percent of the course grade. Lesson plans constituted 15 percent of the course grade.

While preparing the lesson plans, they mainly focused on objectives, materials, teaching techniques and the development process in the lesson.

The Design and the Analysis

The study used a case study approach with naturalistic observation. The data were drawn from the observation of five pre-service teachers’ microteaching on functions conducted in two hours period at the University Class. Topics about functions involved function concepts, operation on functions, composite functions, and types of functions (constant, identity, greatest value, partial, and signum functions). In order to provide flexibility, they were not restricted to use any specific method in their

presentations. During some presentations, the use of analogy method aroused. The use of analogy, however, mostly did not appear in the lesson plans. The courses were presented in three different sequences: 1) analogies, definition or rules, and solving examples, 2) definition or rules, analogies, solving examples, and 3) definition or rules, and solving examples. This indicates that analogies appeared either while exemplifying definition or rules or making introductions to the topics. In the Methods of Science and Mathematics Teaching courses the history of and some misconceptions about functions had been included but not theories and applications of analogy. All presentations and discussions were video-taped and transcribed.

Literature about epistemology of the functions (e.g. Cooney & Wilson, 1993; & Harel & Dubinsky, 1992) and the guidelines in the Teaching with Analogies Model developed from task analyses (Glynn, Duit, & Thiele, 1995) provided a conceptual base for the data analysis. Content analysis (Philips & Hardy, 2002) was conducted to discern meaning in the teacher's written and spoken expressions. Lessons were fully transcribed and considered line by line whilst annotated field notes were used as supplementary sources. The first phase of data analysis included detecting analogy-based teaching instances and identifying source analogies and the target concepts. The subsequent phases embraced in-depth examinations of spotted cases in accord with 'content validity – whether analogies used are epistemologically appropriate to illustrate the essence and the properties of the functions as well as the structural relations between the analogues and the targeted concepts. The validity of the analysis was achieved by utilizing multiple classifiers to arrive at an agreed upon classification of analogies and their target concepts as well as their epistemologically appropriateness.

FINDINGS

Data indicate the key analogical models used in teaching function, composite function and types of function concepts particularly while defining or explaining them. The analog and target concept matching was summarized in Table 1.

Analog (Familiar Situation)	Target (Mathematics Concept)
1. Function machine	Function concept, Composite function
2. Posting a letter	Composite function
3. Packing-Unpacking a present to a friend	Inverse function
4. A perforated pail	Identity function
5. Age	Partial functions, Greatest value function
6. Watering a tree	Greatest value function
7. The shelters in the apartment	Greatest value function
8. Eating a cake	Greatest value function

Table 1: Analog and the target concept relations

Here three analogies are presented and discussed because of the space restriction.

Posting a Letter Analogy

“Posting a letter” analogy was given by one of the participants [PT5] during the composite function lesson provided by [PT2]. This analogy was provided to make clear the definition and the explanations. As seen in the dialog, [PT5], however, did not focus on what the inputs and outputs for f and g are. As a result of that, one of the participants [PT1] got confused and then asked “But I can write letters to two different people?”. This question reveals the importance of developing relationships among analogies and target concepts. Thereupon, the instructor posed questions as such “What is the domain in each case?”, Is it people or letters?, etc. If we consider the “writing a letter” analogy, then the function $f: A \rightarrow B$ is composed by f (writing) to an argument x (people) with an output (letters). This analogy could be given for not being a function because the univalence or single-valued requirement, that for each element in the domain there be only one element in the range, is not supplied in this analogy.

I think about posting a letter example. Let’s take the action of taking the letter to the post office as f function and the letter to be posted as x . Different people’s letters may arrive to the same address. For example my siblings’ letters would arrive to my family’s address too. There occur two actions here. The first operation is “I take the letter to the post office.” And the second operation is “The postman takes the letter to my family.” We name the first action as f and the second action as g . The composite of the actions is $g \circ f$. In the end the arrival of the letter requires the composite of two actions. [PT5]

“Posting a letter” analogy could be an example for composite function provided that the functions $f: A \rightarrow B$ and $g: B \rightarrow C$ are composed by first applying f (posting a letter to the post office) to an argument x (letters) and then applying g (posting letters from the post office to their arrival points) to the result (letters at the post office). Thus $g \circ f$ is the arrival of the letters to their addresses. It must, however, be mentioned that every letter written must have been posted as for each x in A , there exists some y in B such that x is related to y . Otherwise, a binary relation could not be met.

A Perforated Pail Analogy

“A Perforated Pail” analogy was constructed to remind identity function. When someone put something into the bore pail, it will fall dawn as it is. For all input, the output will be the same again. As seen below, [PT2] brought up some examples such as putting a pencil or shoe in the bore pail. She mentioned that the pail does not make any operation on the material. However, the size of the hole on the pail must be big enough for the materials to pass through. If it is not, then this could violate the total condition of being function. Furthermore, the hole on the pail should not give any damage to the material while passing through since identity function is a function that always returns the same things used as its argument. She, however, did not mention the breakdown point of this analogy.

Think about a bore pail.... We put a pencil in it and then we get a pencil again. Or, we put a shoe in it and then we get the same shoe. The pail does not make any operations on them. You get what you put. Then what we called that function: The identity function. [PT2]

The identity function of f on A is defined to be that function with domain and range A which satisfies $f(x) = x$ for all elements x in A . In the case of “Perforated Pail” analogy, while the function $f: A \rightarrow A$ is composed by applying f (putting materials to the bore pail) to an argument x (materials) with an output $f(x)$ (materials).

Function Machine Analogy

[PT2] used “Function Machine” Analogy to remind function concept and to introduce Composite function. First, she drew a function machine figure together with the explanation as such “You have a raw material named x [began to draw Figure 1] and you have a machine that gives output. You put x to this machine and this machine gives you the output as $f(x)$ ”.

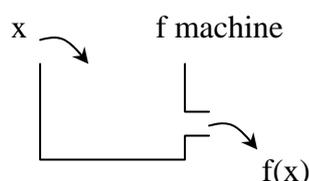


Figure 1: Pictorial analogy for function concept

To exemplify this further “Mixer” analogy - where banana and milk are input and the milkshake is output - was constructed. This, however, is not an appropriate analogy for functions of one variable. “Mixer” analogy can be an example of functions of several variables. When she was asked to make clear what the domain of the function mentioned in the analogy is, she could not make a connection to the function with two variables. One possible explanation for this inappropriate analogy is not considering the function as $mixer(milk, banana) = milkshake$. Further, the instructor expressed that “washing machine” analogy is appropriate for functions of one variable. In this analogy, inputs are dirty clothes, process is cleaning and the outputs are clean clothes.

While introducing the composite function, she first stated that “composite” is a kind of operation like addition and subtraction but operation with different rules. Taking into account the previous function machine figure, she extended the figure to be a pictorial analogy (see Figure 2) for composite function by pointing out that “In the f machine x turns out to be $f(x)$ and then we put $f(x)$ in the g machine. So we get $(g \circ f)(x)$ composite function”.

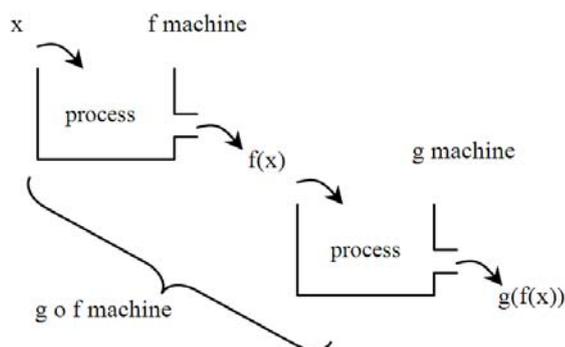


Figure 2: Pictorial analogy for composite function

However, the “washing machine” analogy that was given for functions could have been extended to composite functions. In the case of “washing machine” analogy the functions $f: A \rightarrow B$ and $g: B \rightarrow C$ can be composed by first applying f (washing process in washing machine) to an argument x (dirty clothes) and then applying g (drying process in a dryer) to the result. Thus one obtains a function $g \circ f: A \rightarrow C$ defined by $(g \circ f)(x) = g(f(x))$ for all x .

CONCLUSION

The present findings suggest that analogies need to be carefully thought out to be effective in order not to cause any confusion. The analogical models constructed by the pre-service teachers in the present study were analyzed in terms of whether the analogies constructed are epistemologically appropriate to illustrate the essence and the properties of the functions as well as the structural relations between the analogues and the targeted concepts. While mapping the analogies to the target concepts, the important things are the similarities as well as the break down points between them. The way the pre-service teachers used analogies could fall short of contributing to the students to develop epistemologically correct and conceptually rich knowledge of function due to two reasons. First, the source analogues were epistemologically inappropriate to illustrate the essence and the properties of the functions. Second, the analogies were epistemologically appropriate to illuminate the function concept, yet the teacher did not establish the mappings between the two.

In general they spontaneously followed the three steps: i) selecting an analogy (ii) mapping the analogy to the target (iii) evaluating the analogical inferences. Even the analogical models help students to visualize the newly learned symbols, concepts, and procedures, pre-service teachers need to know and show where the analogy breaks down and carefully negotiate the conceptual outcome. PTs should articulate the similarities and differences between the analogy and the target concept while they are presenting an analogy, and also should be aware of the limitations of the constructed analogy.

In the sense of these findings, it can be concluded that pre-service teachers' knowledge about the use of analogies were insufficient, and participants of the study were weak in transforming knowledge and developing sophisticated ideas in the process of teaching functions. In line with that, pre-service teachers did not consider too much on their analogical mappings and they were not able to construct the adequate relationships between the analogies and the target concepts along with the processes of mapping the analogical features onto target concept features. The difficulty appeared while developing sophisticated ideas in the process of teaching did not occur in giving mathematical definitions, rules, and procedures. For example, function was defined correctly as "f is a relation from set A to set B. If each element in set A correspond only one element in set B, then this relation is a function."

One of the limitations of the present study was that pre-service teachers were restricted to present function concept. May be if they were more flexible in the topic selection they would choose another mathematics topic in which they are more capable, thus they would generate more productive analogical models.

IMPLICATIONS

In teacher preparation courses, student teachers should be asked to generate their own analogies in different contexts of mathematics. This kind of courses could provide them an opportunity to constitute an available repertoire of analogies (Thiele & Treagust, 1994) and to create analogy-enhanced teaching materials. In addition, this array of experiences could allow them to discuss, model, and justify their interpretations of the concepts and to provide different approaches to the teaching of the concepts. The analogies discussed here will help pre-service and in-service teachers develop a sound relational knowledge of the function concepts as well as consider carefully on their analogical mappings to construct epistemologically appropriate ones and to map the similarities and differences between the analogies and target concepts. Discussing the analogies reported here with pre-service and in-service teachers could deepen their understanding of function concept as well as functions pedagogy to offer perspectives on a sound generation of analogies.

In the light of the discussions of the teacher generated analogies, mathematics textbook authors and teachers can develop productive analogies for various mathematical concepts. Carefully crafted analogies can serve as initial mental models for the introduction and presentation of newly learned mathematical concepts.

As a result of this investigation, a further study was planned to describe the multiple analogical models used to introduce and teach grade 9 function concepts. We examine the pre-service teacher's reasons for using models, explain each model's development during the lessons, and analyze the understandings they derived from the models.

Teachers should engage their students in a discussion in which the limitations of the analogy are identified.

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TECHNOLOGY AND MATHEMATICS TEACHING PRACTICES: ABOUT IN-SERVICE AND PRE-SERVICE TEACHERS

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Abstract: This article examines the practices of in-service and pre-service teachers in technology based lessons by exploring three dimensions: the students' tasks, the students groups' management and the discourse of the teacher. Regularities emerging from two case studies about in-service teachers are compared to results of a larger study about pre-service teachers. The article shows that what characterize teachers' practices in technology environments is not the same in the two populations of teachers and thus suggests some propositions for the design of training strategies seeking to improve the practices of novice teachers.

Key-words: technology, teaching practices, ordinary teachers, pre-service teachers

INTRODUCTION

For the last decade constraints and difficulties encountered by mathematics teachers integrating technologies has been an ongoing issue. Indeed the contrast between the technological development and the weakness of the integration of computer technologies in classrooms despite the abundance of governmental funding, questions necessarily researchers (Artigue, 2000), (Ruthven, 2007). Some researches have considered the role of teachers in the classroom use of technology throughout a holistic approach examining thus the influence of key factors on their activity (Monaghan, 2004); others have investigated teachers' ideas about their own experience of successful classroom use of computer-based tools and resources (Ruthven & Hennessy, 2002); others have shown discrepancies and variability in the ways teachers use technology in their mathematics classrooms (Kendal & Stacey, 2002). Research about student teachers' practices and their determinants in technological environment is nevertheless rather rare. It stresses particularly the problems that student teachers have to overcome such as their lack of familiarity and confidence with technology or their need to make explicit the connections between technological and paper-and-pencil work (O'Reilly, 2006). Furthermore, it stresses the growing awareness that technology-based lessons require extra time for planning and for teaching.

In this paper, I want to contribute to the research on issues related to teaching practices in technology environments and issues related to teacher education in these environments. I will do so by relying on the results of three research projects that I have carried out in the last four years. I will firstly present two case studies about teachers' practices in technology-based lessons taken from the two first researches. Secondly, I will highlight regularities that emerge from these studies. I will finally try to cross these findings with results of a third research about pre-service mathematics

teachers using computer technologies and conclude by issues about teacher education arising from this synthetic view on the three studies.

CASE-STUDIES

The two case studies that I am presenting here have been carried out in two researches: the first one on the characterization of the practices of 'ordinary' teachers using dynamic geometry (Abboud-Blanchard, 2008); the second one on the analysis of the activity of volunteer teachers using exercise-bases (Artigue *et al.*, 2006). In these two studies the main issue is to characterize teacher's activity in a technology-based lesson according to three polarities in complete interaction: the tasks proposed for students' learning (cognitive pole), the management of the students' groups (pragmatic pole), and the discourse and the interaction with students (relational pole).

Framework and Method

These studies use methods and concepts developed within the general framework of the two-fold approach which combines both a didactical and an ergonomical perspective in analysing the factors that determine the teacher's activity as well as that of students prompted by the teacher in class (Robert & Rogalski, 2005). Within this framework, analyzing lessons takes into consideration the fact that there are two main types of channels used by the teacher in classroom management: the organization of tasks prescribed to students (cognitive-epistemological dimension) and the direct interactions through verbal communication (mediative-interactive dimension). Furthermore, the authors (*ibid*) differentiate task from activity: task is what is to be carried out; activity is what a person develops when realising the task.

For each of the two case-studies I will first report on the *a priori* analysis of the students' tasks and what they are supposed to undertake in terms of initiative and use of knowledge already acquired and actually needed to execute the tasks. Secondly, I will present the lesson in progress, that is to say, what really happened in the classroom by underlining the teacher's aids and by studying the features of his/her discourse. The teacher intervenes often to provide assistance to the students sometimes modifying their activities. Robert (2007) defined two types of aids, depending on whether they modify the activities scheduled or they add something to the students' action. The first, "procedural help", deals with the prescribed tasks by modifying activities with regard to those planned from the presentation of the task. It corresponds to indications that the teacher supplies to the students before or during their work. The second, "constructive help", adds something between the strict activity of the student and the (expected) construction of the knowledge that could result from this activity. The analysis of the teacher's discourse provides more information about how he/she contributes to model students' activities. This analysis has been undertaken by using a methodology constructed by Paries (2004) who adapted tools used in psychology, notably the functions of scaffolding defined by Bruner (1983) who regarded interaction as the major form of assistance provided by adults for cognitive development. Thus, Paries studied the role of discourse in the

mediation of cognitive development and defined functions of the mathematics teacher's discourse by specifying the manner in which he/she intervenes gradually in details in the students' work. Paries distinguishes two groups of functions:

- The “cognitive functions” linked to the task, to the realisation of the task and to the mathematical content. These functions are: introduction of a task or dividing a task into sub-tasks, assessment, justification and structuring.
- The “functions of enrolment” apparently independent from the task, at least in their formulation, but can have an impact on its realisation. They allow the teacher to maintain communication. These functions are: engagement, mobilization of the student's attention and encouragement.

William's case

William has volunteered to participate in a government project to use exercise-bases with his grade 10 students (first year of the upper secondary level - aged 15/16 years). I chose to present here this study because William's case could be considered as representative of those of the other teachers engaged in the project.

He is a regular user of technology in both his personal and professional activities. He sees the use of exercises-bases in the classroom as facilitator without neither change in the approach of mathematics contents nor change in teaching practices: this software is just an additional mean that will be added to (and not replace) usual practices.

Students' tasks

The observed session is a training one and it took place in the computer room; students were assigned by groups of two to a computer. William's discourse was recorded; a remote cordless microphone was attached to the teacher. An observer was present in the classroom. William has chosen in the exercise-base a module of exercises-generator concerning two tasks: (1) To find the reduced equation of a straight line. The straight line is drawn on the screen with two of its points A and B in an orthonormal cartesian system; students have to find the values of m and p in the equation $y = mx + p$. (2) To solve systems of two linear equations (first degree equations with two unknowns)

In both cases, students must make their calculations on paper and give the two numbers solutions to the software that validates them in terms of true / false.

These two tasks are similar to paper-and-pencil tasks; the only difference is that each student can train at his/her own pace.

The development of the lesson and the teacher's help

During the lesson William tries to check up the work of every group of students with some regularity, and even when he moves at the request of a student, he quickly control the work of other students along his path. Despite this, students put a lot more time than what William had planned (half an hour per task). This gap between the

planned and actual time resolution leads William to ask students to move to the second set of exercises, although a few students have still some difficulties with the first exercises.

Among the interactions with students, I note only four collective ones which concern particularly the management of the session; the rest are individual (per group to a computer) interactions. Some aids are related to the handling of the software: they consist primarily to explain how to switch from one exercise to another or to resolve a technical problem. They are usually brief, local, and allow the student to continue the resolution. The individual help concern mostly mathematical resolution; they are of various kinds and are often procedural help: - controlling the resolution and calculations; - validating an answer or helping find the error (often at the request of students); - structuring the resolution or asking students to do it.

The frequency and variety of these mathematics aids show that the execution of the mathematical tasks seems to require a strong mobilization of the teacher.

To sum up, I notice that William, who is at ease in a technology environment, succeeds in providing students effective aids for handling computers and exercise-bases software. The class gives the impression of "functioning" in a satisfactory manner, all students work and progress. Nevertheless, the teacher is highly mobilized on the mathematical level; the majority of students cannot progress in the resolution without his help. So, despite an "illusion" of autonomy of students, the presence of the teacher seems indispensable.

The functions of discourse

I will not detail here the study of the teacher's discourse, because of the restricted length of this paper; I will rather give some significant percentages of the functions of discourse. I note first a small percentage (9%) of the functions of enrolment. Everything indicates that students are "supported" by the technology environment and work without needing to be constantly motivated by the teacher. The function of structuring occupies 21% of the total, because when helping students, William first begins by helping them bring "order" in their calculations. This is also due to the desire that students work more quickly because the time doesn't progress as William has planned (see above). The function of assessment occupies a high percentage (47%) because the software provides validation only in terms of true/false for the solution given by the student. The students are therefore responsible for the control of calculations but they seek constantly the teacher's help for this assessment. This requires the teacher to take over the function of accompanying the resolution and control of progress, and interpretation of the results not validated.

In addition to these results on the functions of discourse, I note that the functions succeed in a similar order with each group of students. Indeed when the teacher comes to see a group: he assesses or takes a stock of the situation of resolution, sometimes he structures it, and then he gives a sub-task to the students to execute until he comes back. This phenomenon of repeating the same succession of action in

each group with aid substantially similar implies a strong mobilization of the teacher which is 'non-economic' in terms of classroom management.

Anna's case

Anna is an 'ordinary' teacher not engaged in any innovation or research project. She has an episodic use of technological tools with her students that one wouldn't qualify as significant use. I present here her case because she corresponds to what we, in the research project, consider to be an average teacher representative of ordinary teachers. The lesson studied here is about space geometry in a grade 9 class (fourth year of middle school - aged 14/15 years). It takes place in the computer room with the use of dynamic geometry software; students are assigned by groups of two or three to a computer. The lesson observation was videotaped. The camera was at a rear corner of the classroom. A remote cordless microphone was attached to the teacher. No observer was present in the classroom. The topic is the section of a pyramid by a plane parallel to the basis, and Anna uses a ready-to-use session designed by the software developers.

Students' tasks

The figure downloaded by the students is a given cube ABCDEFGH in which they have drawn in a previous session: I, middle of [EF] and J, middle of [AB] and have also found the lengths JC and JD. First, the students have to draw the section of the pyramid IJCD by a plane passing by M, the middle of [IJ], and parallel to the basis JCD, getting thus two points N (middle of [IC]) and Q (middle of [ID]). This technological-task (t-task) is entirely guided by a set of manipulation commands and students only need to follow the instructions given in the worksheet provided by Anna. Secondly, they have to examine, with the software commands, the triangles JCD and MNQ. The aim here is that students get to see MNQ as the $\frac{1}{2}$ reduction of JCD. Once done, tasks that follow are mathematical-tasks (m-tasks): to calculate the areas of triangles MNQ and JCD, to calculate the volume of IMNQ and IJCD to compare these two volumes. These m-tasks are complex and require a certain number of adjustments such as taking initiatives (to construct a height in a triangle in order to calculate its area) or operating a change of frames (when comparing the two volumes) that consists in introducing the comparison of two numbers in a geometrical frame. Therefore, t-tasks are designed to be simple, guided and quickly executed in order to get a stronger focus from the students on m-tasks. The latter are more complex and require time to be carried out.

The development of the lesson and the teacher's help

Globally, I note that students are often in an autonomy-mode and for very long moments. When she is present, Anna divides the task into sub-tasks to be immediately executed by students, in a bid to allow them to pursue quickly their work. The teacher's collective interactions are rare and mostly concern the management of the session.

The assistance of the teacher consists almost exclusively in procedural help, simplifying the students' activities. The division of tasks into simple sub-tasks is clear: sometimes Anna nearly dictates the work to do and at times she even takes herself the mouse to accomplish some sub-tasks. Often, when the teacher is interacting with a group, students only follow her instructions, or even finish a sentence that she begins. I might here underline that the teacher stays with every group a very short time and thus her assistance allows the students to pursue their work on their own. One can wonder if dividing the task is somehow a way for Anna to be efficient. Still, Anna did not succeed to meet her objective; students were too slow in the construction of the section of the pyramid. She had prepared simple t-tasks in order to help the students to start quickly the mathematical activity. Perceiving during the lesson that these tasks took more of time than expected, she tried to accelerate their execution by doing the work herself or by coaching students step by step in the execution.

The functions of discourse

As in William's case I only give here some significant percentages of the functions of discourse. I first observe that the functions of enrolment have a low percentage (7%) which might be explained by the fact that the mobilization of the students' attention and the engagement in tasks is supported by the technology-environment itself. I notice also that structuring accounts for an important rate among cognitive functions (28%). As stated above, Anna is aware of the slow execution of the tasks and tries, by this mean, to accelerate the pace. As for the cognitive function of the introduction of sub-tasks, the high percentage (21%) is coherent with the analysis of the m-tasks. These tasks are complex, need adjustments, and on top of that, students' work progresses slowly. Assessment stands at 35% and corresponds to interactions with groups of students and not to collective interactions. Actually, after the start (collective phase), the class splits into several 'mini-classes' (groups of two or three students per computer) which function separately and to which the teacher talks independently from the remainder of the class. Besides, certain functions of the discourse apparently succeeded in these 'mini-classes' in this same order: assessment, structuring and introduction of a sub-task.

Regularities emerging from the two case-studies

Despite of the different contexts and profiles of the two teachers and also the different nature of the software used, a number of regularities emerge from the two studies, I want to emphasize these in this section. I will do so in order to highlight what actually is characteristic of a technology-based lesson led by in-service teachers. I will also illustrate continuities between these findings and those of some researches mentioned above, to suggest that a number of results may be more widely transferable.

On the cognitive level, in the two cases the exercises chosen by the teachers, in technology environment, are similar to the ones that would be proposed in pencil-

and-paper environment; the resolution of mathematics tasks is identical to what could be proposed in non-technology environment. This result is close to what Kendal and Stacey (*ibid*) underline about CAS (Computer Algebra Systems).the mathematical knowledge and skills stay globally within the range of those expected in non-technological environment. Indeed, the teacher has, on the cognitive level, a practically similar activity as in a non-technology environment. In the open environment of dynamic geometry we see that Anna has chosen a ready-to-use sequence where all the questions of the exercise except one, are feasible in a pencil-and-paper environment. In the environment of exercise-bases, William has also chosen training exercises used in pencil-and-paper environment. The content of the interventions of the two teachers when it comes to mathematical tasks is therefore identical to what they would have said or done in non-technology environments since there is no reference to the specificity of technology environment in these interventions. This can be traced to some indications provided by Ruthven and Hennessy (2002) about teachers who initially view technology through the lens of their established practice, and employ it accordingly. This fact certainly favours the connection of these sessions with the rest of learning process and helps to explain why for these teachers this connection is not perceived as problematic.

On the pragmatic and relational levels, firstly I note that the work in computer room generally entails that students must be in groups of two or three per machine. Consequently, there is a 'class split' in several 'mini-classes' working relatively independently, and a quasi disappearance of collective phases except the collective time management. The teacher is not able, in certain cases, to generalize the supply of certain indications given only to some students whereas they could be useful to all the others. Artigue *et al.* (*ibid*) encountered the same features notably the fact that individual interactions substitute for collective interactions and that institutionalisation phases are nonexistent because of the different 'trajectories' of students. Besides it, for each of the mini-classes, the teacher adapts to what students are doing and to their current reasoning, whereas in pencil-and-paper lessons, it is more often that the students have to adjust themselves to the teacher's project (Abboud-Blanchard & Paries, 2008). This appears to be an important element of the management of a technology-based lesson which differentiates it from a non-technology one. Moreover, the analysis of the interactions showed similarities in the successions of the functions of the discourse among the mini-classes. Secondly, as to the aid provided to students, I observe that the teacher focuses on local mathematical aid without decontextualization. There is a clear majority of cognitive functions of the discourse that operate as help, but mainly procedural help. This type of support is partly motivated by the teacher's concern about the progress of the students' work, in order to have all the tasks prepared for the session completed. This echoes a strong trend of teaching practices in the computer room underscored by several researches (Monaghan, 2004). Other characteristics seem to be related to specificities of the environment and enhance the previous difficulties. Indeed, not all the students handle the software with ease, thus the teacher has to provide technical help which is not

common in a mathematics course. Thirdly, in individual interventions that predominate, the rate of interventions of enrolment is much weaker than what is generally observed in non-technology class sessions (Paries, 2004). The functions of enrolment are rarely present in the discourse of the teacher; they seem to be taken in charge by the software. The teacher has also to 'share' with the computer certain functions of enrolment, which disturbs the usual management of the class.

Thus, the teacher's role in technology based-lessons seems to be essential according to the pragmatic and the relational poles. Indeed in the two case studies students' tasks were enough guided, one could *a priori* expect to see the teachers a bit observers (rather than actors) of their students' learning. The analysis shows that this is not the case; teachers are very present and very engaged in the students' work.

ISSUES ABOUT TEACHER EDUCATION

As member of a research team investigating the uses of technology by pre-service teachers, I studied the professional dissertations made by these teachers in which they report about technology-based lessons that they prepared and carried out in their classes (Abboud-Blanchard & Lagrange, 2006). The data come then only from what the teachers themselves reported and not from class observations.

The main result that I want to highlight in this paper is the focus of these dissertations on the preparation of students' mathematical tasks, while the teacher's activity is overlooked. Aspects of the teacher's role are very rarely questioned; they are rather mentioned as "events" in the body of the reports and in the conclusions. Indeed, the learning activities are often document-based, students being assigned tasks based on a written document that teachers deliver at the beginning of the session. In such classroom documents, tasks are organised as a series of subject-based questions, with instructions on how to use the software. Furthermore, in the development of lessons reported in the dissertations, it seems that the teacher has a marginal role in the technology-based lessons carried out and reported by pre-service teachers. For example, at the beginning of a typical lesson, the pre-service teacher provides guidance to the students on manipulating the software and makes sure that they understand the assignment. Then the students work on their own in the computer room and the teacher's activity is limited to individual help to manipulate the software. My hypothesis is that the teacher's marginal intervention can be explained – at least partially - by the prescriptive nature of the tasks. Another reason may be that pre-service teachers transfer part of their role to the computer, a kind of 'joint partnership'.

Comparing results about pre-service and in-service teachers

My aim in this section isn't to make a detailed comparison of the two first case studies and the study of pre-service teachers. A direct comparison wouldn't be relevant notably because of the differences of the methodologies used. I'm rather presenting here a synthetic approach of the three studies focusing on the results relative to the three poles developed above: cognitive, pragmatic and relational.

In the studies on the activity of in-service teachers I showed that the cognitive pole isn't what seems to be problematic for these teachers in technology-based lessons. What differentiates the teacher's activity in these lessons with the same in non-technology ones are mainly the management of students (pragmatic pole) and the interactions with students (relational pole). Thus what makes a technology-based lesson 'works' with experienced teachers seems likely more related to the pragmatic and relational poles than to the cognitive one. Whereas the study of the practices of pre-service teachers shows on the one hand that they focus on the cognitive pole and they neglect the two other poles, and on the other hand that they report their non-satisfaction of how technology-based lessons took place. Moreover, when we ask pre-service teachers about their experiences of technology-based lessons they most frequently reflect on difficulties related to time management of the session and also to preparation work to set up the tasks of students. They also underline that the teacher is no longer the only holder of knowledge. However such reflections tend to remain at a general level and do not seem to provoke pre-service teachers into making propositions for a more suitable integration of technologies in mathematics teaching. This also reveals that despite of their increasing awareness of the specificity of technology environments in preparation work and class work; it does not necessarily lead to a wider reflection about real integration of technology in their practices.

Can we take advantage of this awareness to develop an approach of teacher education programs? During discussions within the WG12 of CERME 5 (Carillo *et al.*, 2007) it seems that there was a consensus among participants on the fact that awareness is necessary for reflection and on promoting reflection as a means of professional development. Seeking to improve the practices of novice teachers, this last pattern can be used for the design of training strategies such as the analysis of video episodes of experienced teachers using technologies with a special focus on the role of the teacher and his/her interactions with the students. Such analysis would help pre-service teachers to bridge between a focus on the preparation of students' mathematical tasks and another on their own activity during the lesson in order to help them overcome the state of didactic tinkering and go further to a successful integration of technologies in mathematics teaching and learning.

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TEACHERS AND TRIANGLES

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During a workshop about triangles designed for in- and pre-service basic-school teachers, a diagnostic test was applied. The results are analysed in terms of several variables: the teachers' sex, the level at which they work, their occupation (namely, in- or pre-service teachers), and their professional experience. An important impact of the latter was found in the decrease of incorrect answers obtained.

FRAMEWORK

Shulman (1986) characterised the types of knowledge that he considered enabled teachers to carry out their practice. He proposed three categories: mathematical content knowledge (MCK), curriculum knowledge (CK), and pedagogical content knowledge (PCK).

There have been several discussions about Shulman's categories. We want to mention two in particular. The first one is a discussion both about the exact meaning of MCK. Some researchers stress that within MCK there is a difference between the knowledge of the formal academical discipline and the scholar subject (see e.g. Bromme, 1994). The former is the knowledge that professional mathematicians develop, and the latter is the mathematics that teachers must teach.

The second discussion is about how much MCK is a valid variable in understanding teachers' practices and designing teachers' education. There has been a variety of researches that show that "teachers' mathematics knowledge is generally problematic in terms of what teachers know, and how they hold this knowledge of mathematics concepts or processes, including fundamental concepts from the school mathematics curriculum. They do not always possess a deep, broad, and thorough understanding of the content they are to teach" (da Ponte & Chapman, 2006, p. 484). According to some authors, these researches are important because of several reasons. On the one hand, they allow to understand how "elementary teachers' understanding of subject matter influences presentation and formulation as well as the instructional representations that the teacher uses" (Sánchez & Ilinares, 1992, quoted by da Ponte & Chapman, 2006, p. 434). On the second hand, they have prompted "studies centred on describing student teachers' beliefs and knowledge as determining factors in their learning processes [... and have also] provided information used to prepare research-based material for use in teacher education and to develop research-based teacher education programmes." (Ilinares and Krainer, 2006, p. 430). On the other extreme, some authors question MCK's importance, because "the academic mathematical knowledge may not be 'naturally' a helpful instrument for the teacher in the school practice, since some of its values and forms of conceptualizing objects conflict with the demands of that practice". (Moreira & David, 2007, p. 38). They stress that to

help students to think mathematically, teachers need to understand student thinking, and thus the comprehension about the cognitive processes of the students becomes more important than MCK itself

While we are aware that many variables may qualify the importance of MCK, such as teachers' beliefs and practices, the cognitive processes of students, etc., we sustain that teachers should at least have a solid understanding of the contents they must teach. This does not always happen in Mexico, and in order to explain why, we must make a brief exposition of the Mexican situation about teacher training. Teachers receive their training not in universities but in "Escuelas Normales", which they attend after 6+3+3 years of regular schooling. There are "Escuelas Normales" (ENP) for student teachers who will become Primary school teachers (i.e, grades 1-6), and other "Escuelas Normales" (ENS) for those who will teach at the Secondary level (grades 7-9). At the ENP it is taken for granted that during those 12 years of previous schooling they have learnt all the mathematics they will ever need to teach, and that all they need to know about teaching mathematics is PCK; at the ENS student teachers have some courses focused on MCK. (Another situation in Mexico is the fact that there is not an assessment or a diagnostic about teachers' MCK with results widely spread). Thus, if teachers enter the ENP with misconceptions or deficiencies, these are not solved there, and the dragging of misconceptions and deficiencies becomes, through teachers' practice, a vicious circle. One of the well-known consequences of this process is that Mexico is always among the countries that obtain the lowest results in international assessments of students' performance, like PISA and TIMSS.

While other countries do not share the extremely low results in PISA and TIMSS, teachers' misconceptions and deficiencies are not exclusive of ours. For example, Hershkowitz & Vinner (1984, quoted in da Ponte & Chapman) investigated the processes of concept formation in children, through the comparison of students' learning and elementary teachers' knowledge of the same concepts; they found that one of the factors that affects the students' learning is the teachers' conceptions.

With respect to MCK, Llinares and Krainer (2006) acknowledge the importance of detecting student teachers' misconceptions but propose that it be done within the frame of student teacher's learning. They suggest that it is important to study the relationship between student teachers' conceptual and procedural knowledge, and for this teachers should know about children's mathematical thinking. One method they propose for the study of the mentioned relationship is the use of open-ended questions based on vignettes describing hypothetical classroom situations where students propose alternative solutions to some mathematical problems. This kind of tasks have also been used by Empson & Junk (2004), who suggest that some of the teachers' answers are influenced by a disconnection between teachers' MCK and their understanding of children's thought, with the consequence that they precipitate to correct mistakes without establishing a contact with what the student is thinking.

Presently, there is not a unified theoretical perspective on the researches about MCK and its relation to teachers' training and professional development. It has been suggested that "future work should include a focus on understanding the knowledge the teachers hold in terms of their sense making and in relation to practice [... and that there is a] need to pursue the theorization of teachers' mathematical knowledge, framing appropriate concepts to describe its features and processes, and to establish clear criteria of levels of proficiency of mathematics teachers and instruments to assess it." (da Ponte & Chapman, 2006, p. 467).

The work we are presenting here fits da Ponte and Chapman (2006) and Llinares and Krainer (2006) characterisations, a difference with the last ones being that we investigate not only pre-service teachers but in-service teachers as well. Our principal goal is to study in- and pre-service teachers' mathematical content knowledge, but not in an isolated manner. As other researchers (see for example Prestage and Perks, 2001), we are also interested in understanding how teachers obtain, maintain and organise their mathematical content knowledge. It is worth mentioning that we are aware that mathematical content knowledge should not be separated from the other two kinds of knowledge. With this in mind, we designed some workshops that will be described below.

METHODOLOGY

TAMBA: Workshops on Basic Mathematics for in- and pre-service teachers

Within a broader project that combines research with professional development, we designed a set of workshops called TAMBA (Talleres de Matemáticas Básicas). The workshops are offered as modules that can work independently or as a set. Each one is centred on one specific mathematical content linked to the elementary school curriculum in mathematics. They all have a duration of 2-4 hours, and a common structure: they start with a short paper-and-pencil diagnosis, which is immediately commented with the participants, followed by an activity designed to raise a cognitive conflict, which takes most of the workshop's time. After it, several issues are discussed in the group: the mathematical topics and the pedagogical difficulties, including the children's most frequent misconceptions. The workshops are video taped. The design of both the diagnosis and the activity is based on our previous knowledge of the population to which each workshop is directed, and on the specialised literature.

Geometry in TAMBA

One of TAMBA's workshops is called "coloured triangles". After the diagnosis, which will be described below, the activity is centred on the unicity of the triangle's area whatever the side used as "base" (this topic follows from the item 3 of the diagnosis). Depending on the teachers' cognitive level on the subject, a

demonstration is presented, and then the MCK and PCK issues of item 3 are discussed in the group.

The diagnostic evaluation has three items. In Item 1, four sets of three measures are given, and the participants are asked to say if a triangle can be built with them and, if not, why (two are possible and in the remaining two the triangle inequality is not accomplished). In Item 2, three triangles are given with measures for the sides and heights, and the participants are asked to say if the measures are possible or not, and why (two of the figures are not possible, because some heights are larger than a side from the same vertex). In Item 3, a hypothetical conversation between three girls who must calculate a triangle's area is presented, where they all make different mistakes and do not agree on the calculation, and the teacher is asked to write what s/he would say to the girls.

The teachers' answers to the written evaluation were analysed and classified according to their correctness and the kind of geometrical criteria used. The results, focused from a geometrical point of view, are being presented elsewhere. Here only the broad categories are briefly described. Teachers' ideas were classified as correct or incorrect; in the second case, several misconceptions were identified: about the triangle inequality, the base and/or height, the Pythagorean theorem, or other geometrical misconceptions. Within each of these broad categories, some finer subcategories were identified. In addition, the amount of items answered by each of the participants was registered, as well as the amount of ideas that s/he expressed clearly.

Implementation

The described workshop has been given twice. In 2007 it was offered to 36 teachers at the Conference of the Mexican Mathematical Society in the city of Monterrey (MR), and in 2008 it was offered to 31 teachers in a Teachers' Centre in Mexico City (MC). Table 1 summarises the characteristics of the participants in both workshops:

	SEX			LEVEL			OCCUPATION				EXPERIENCE		
	F	M	N/A	Primary	Secondary	N/A	In-service	Pre-service	Other*	N/A	n	Mean \pm SD in years	N/A
MR	22	9	5	20	9	7	16	7	4	9	22	17.9 \pm 10.6	0
MC	29	2	0	20	3	8	5	24	1	1	7	17.7 \pm 9.6	7
Total	51	11	5	40	12	15	21	31	5	10	29	17.9 \pm 10.2	7

* "Other" occupations are pedagogical consultants (PC) and experts in Special-Education Teachers (SET).

Table 1

The main difference between both groups is that there were more in-service teachers in Monterrey and more pre-service ones in Mexico City. In addition, all of the pre-service teachers in Monterrey were of the secondary level, whereas in Mexico City 15 of the pre-service were of the primary level and 2 of the secondary level (7 more did not answer that question). Another difference is that in Monterrey the participant teachers were highly interested in Mathematics Education, and had applied for and obtained funding to participate in the Conference (which was given for teachers with high scores in a national assessment), whereas in Mexico City the participants were regular attendants to a Teachers Centre located in a low-income zone.

RESULTS

For each participant, the percentage of items answered was calculated, as well as the percentage of those that had clear arguments. Then the total amount of ideas expressed was figured, each idea was classified according to one or several of the categories above mentioned, and the quantity thus obtained for each participant in each category was expressed as a percentage of the total amount of ideas expressed. Finally, for each category averages were calculated taking all of the participants (see Table 2) or diverse groups of them.

	Items answered	With argument	Correct ideas	Incorrect ideas	Misconceptions				
					Triangle inequality	Base	Height	Pythagorean th.	Other
All participants	80.0%	71.6%	27.8%	62.0%	15.0%	6.5%	8.6%	5.5%	8.1%

Table 2 [1]

As Table 2 shows, the average participants answered most of the items, and, when they did, mostly expressed their ideas with clear arguments. However, only a small percentage of these ideas were correct. Among the misconceptions, those about the triangle inequality were the most frequent.

In the following sections, these results will be analysed according to the recorded experimental variables: venue, sex, level, occupation, and teaching experience. Each time the arithmetic means are reported and analysed, although no statistical inferential analysis is carried out, the samples being neither representative nor large enough.

Venue

The 36 participants of the workshop held in Monterrey (MR) and the 31 of Mexico City (MC) differed in all of the variables considered. Table 3 shows the results obtained by teachers in both venues.

	Items answered	With argument	Correct ideas	Incorrect ideas	Misconceptions				
					Triangle inequality	Base	Height	Pythagorean th.	Other
MR	91.9%	77.3%	45.2%	42.2%	9.2%	6.0%	5.8%	8.4%	2.7%
MC	66.2%	64.9%	7.6%	85.1%	21.8%	7.0%	12.0%	2.2%	14.3%

Table 3

The teachers in MR obtained better results from all points of view: they answered more items, and expressed better their reasoning (more answers with argument). They had six times as many correct ideas and about half of the incorrect ideas expressed by their counterparts in MC; also, MR teachers had fewer responses classified in all but one of the different detected misconceptions. The largest differences were in the misconceptions about the triangle inequality, where MC teachers more than doubled their MR counterparts, and “other” geometrical misconceptions, where MC teachers made five times as many mistakes as MR participants. The one exception is the incorrect uses of the Pythagorean theorem, where MR teachers had in average 8.4% answers as opposed to only 2.2% of MC teachers. All this, as will be shown later, is related to the different characteristics of the participants in both venues.

Gender

There were also differences among the 62 teachers who reported their sex: In general, the 11 male respondents had better results than the 51 female participants did. Table 4 shows this.

	Items answered	With argument	Correct ideas	Incorrect ideas	Misconceptions				
					Triangle inequality	Base	Height	Pythagorean th.	Other
F	77.6%	66.5%	8.4%	84.1%	21.4%	7.8%	11.8%	2.2%	14.8%
M	86.9%	75.0%	41.7%	46.7%	10.4%	5.0%	5.9%	9.5%	2.5%

Table 4

The male teachers answered more questions in average than the female, and were slightly better in expressing their reasoning. Men had more of the correct ideas and fewer incorrect ones, and scored lower in all of the misconceptions, again with the exception of misuses of the Pythagorean theorem. This apparent gender effect will be commented later on.

Level

Only 52 of the 67 participants declared in which level they work or study. Their results are shown in Table 5.

	Items answered	With argument	Correct ideas	Incorrect ideas	Misconceptions				
					Triangle inequality	Base	Height	Pythagorean th.	Other
P	78.1%	74.6%	25.3%	63.8%	20.2%	5.5%	9.4%	6.9%	7.3%
S	86.2%	77.4%	40.1%	52.7%	11.6%	8.1%	5.0%	8.0%	4.6%

Table 5

Generally speaking, the 12 teachers of the Secondary level had results that were only slightly better than those of the 40 of the Primary level: more items answered as an average, more responses with argument, more correct ideas, and fewer incorrect ones. However, it is noticeable that the distribution of misconceptions found is not homogenous: Secondary level teachers have fewer answers with misconceptions about the triangle inequality, the height and other errors, but have more answers with misconceptions about the triangle's base and the Pythagorean theorem.

Occupation

Of the 67 participants, 57 declared if they were in-service teachers (21), pre-service teachers (31), or if they had other occupation (5 were PC or SET). Table 6 shows the results for the first two categories.

	Items answered	With argument	Correct ideas	Incorrect ideas	Misconceptions				
					Triangle inequality	Base	Height	Pythagorean th.	Other
In-	86.9%	78.8%	31.5%	58.9%	15.4%	7.2%	10.1%	13.1%	1.5%
Pre-	68.3%	60.9%	17.1%	72.8%	17.9%	5.0%	5.7%	3.1%	12.1%

Table 6

In-service teachers had better results than the pre-service ones: more items answered, more answers with argument, more of the correct ideas, and fewer incorrect ones. However, in-service teachers scored higher than pre-service ones in three of the identified misconceptions: about the triangle's base and height, and about the Pythagorean theorem.

Experience

Of the 36 participants who were in-service teachers, PC, or SET, 22 declared their teaching experience. Their results are shown in Table 7.

	Items answered	With argument	Correct ideas	Incorrect ideas	Misconceptions				
					Triangle inequality	Base	Height	Pythagorean th.	Other
1-10 yrs	81.9%	80.3%	16.2%	76.2%	18.2%	6.1%	11.7%	22.0%	6.2%
11-20 yrs	98.3%	86.5%	31.3%	60.3%	13.1%	13.1%	18.3%	5.6%	5.2%
>20 yrs	90.9%	78.1%	53.2%	34.1%	9.0%	5.1%	6.5%	0.0%	4.2%

Table 7

Teachers with more years of experience have a tendency towards better results, and teachers with less experience towards worse results, in almost all aspects. However, teachers with between 11 and 20 years of teaching experience have more answers classified as misconceptions on base and height than the other two groups.

Overall, the teaching experience does have a marked influence on a decrease in incorrect ideas, as the graph of Figure 1 shows (in it the value for 0 years is the average for all student teachers). The correlation coefficient between teaching experience and percentage of incorrect ideas is $r = -0.51$.

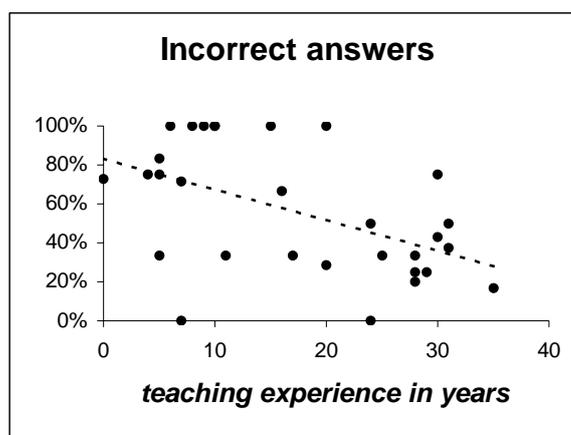


Figure 1

Language and didactical competence

Another characteristic of the responses to the diagnosis given by the participants is the quality of the language used and of the didactical explanations provided in the hypothetical situation of Item 3. Although we do not have here the space to show the analysis that we carried out, we want to state some of the findings. Many answers are based on orders or assessment, which reflect the disconnection described by Empson & Junk (2004) between MCK and the understanding of children's thought. It is also evident, as was stressed by Boero et al. (2002), that the natural language can provoke difficulties in the acquisition of concepts. Finally, some teachers, particularly of the Secondary level, have an attitude that could be expressed as "*I know so much that you cannot understand me*".

ANALYSIS AND CONCLUSIONS

Two considerations must be taken into account. Firstly, we must stress that if a teacher does not manifest a misconception, this does not necessarily mean that s/he does not have it; it could also be that in his/her expression the misconception just did not show. Secondly, although no hard facts can be deduced of the information obtained from this study, the results we have shown can be interpreted in terms of possible tendencies that could be investigated in a next step of the research.

It would seem that, with respect to MCK relating triangles, male teachers, secondary school teachers, in-service teachers and highly experienced teachers obtain better results than their counterparts do.

The gender effect that we found in these results could make sexists happy. However, in the group of teachers that participated in the two workshops, 62% of the female teachers were pre-service ones, and among the male teachers the percentage was 20%; thus, the gender effect could be confounded with the variable “occupation”. The other groups with better results were to be expected: teachers of the Secondary level receive more mathematical training in ENS, and in-service teachers have dealt with the teaching (and are thus more in contact with the students’ way of thinking, in accordance with the findings of Empson and Junk’s, 2004), and even more so as their teaching experience increases.

As for the differences between the obtained results in the two venues, the better results of MR can be related to two factors. The first factor is that, as Table 1 shows, in MR there were more Secondary level teachers (25% vs 10%), and more in-service teachers (44% vs 16%): two of the three “better” groups (with no differences on the fourth variable, the teaching experience). The second factor, which could be of even more importance, is the difference in the ways that teachers arrived to the workshops. MR teachers were highly interested in mathematics and its teaching, and also had good scores in a national assessment, whereas MC teachers did not share this characteristics and were regular attendants of a teachers’ centre in a low-income part of the city.

It can be interesting to comment on the cases that stray from the reported tendencies, which relate to misconceptions about the triangle’s base and/or height, and about the Pythagorean theorem. We carried out an analysis using the fine-categories in addition to the broad ones about base and height described and used in this paper, which we do not have here the space to present. However, this analysis shows that some of the misconceptions can be linked to didactical strategies (where the informal and potentially incorrect use of mathematics serves a didactical purpose), and that modern teacher training is slowly (and partly!) fighting some misconceptions about base and height, through fewer prototypical examples in the textbooks for student teachers. As for the misuses of the Pythagorean theorem, there are more answers with this classification in two of the three “better” groups (Secondary, in-service). One possible interpretation of this is that the groups with a higher level in general also have some idea about the existence of the Pythagorean theorem and, approximately, what it is about. (It could also be that more recently trained teachers have heard about the theorem). However, all of the teachers who pretended to use this result did it in one of several incorrect ways; this relates to Hershkowitz (1990) characterisation of misconceptions that increase as the students advance throughout their schooling.

The effect that the teaching experience has in decreasing (but not nullifying!) the amount of incorrect answers is something that must be valued in professional

development programs. When the teacher (and particularly the Primary school one) starts her/his practice, s/he must deal not only with the students' difficulties in the learning of mathematics, but also with her/his own deficiencies in MCK, which in turn have the effect of not only perpetuating but also aggravating their students' misconceptions. The professional practice can help in dealing with both the students' learning difficulties and the teacher difficulties in MCK, but if s/he had more support with MCK, the pedagogical difficulties would be easier to handle. Therefore, we coincide with Bromme (1994) in that MCK must be understood as the scholar subject, and we assert that it is something that must be attended to, diagnosed and solved, both in initial training and in professional development.

NOTE

1. The 71.6% of ideas with argument is 100% minus the answers without clear argument: 10.1% that were potentially correct and 18.3% that were incorrect. The 100% of ideas is formed by correct ones, plus those that were potentially correct but without clear argument, plus the incorrect ones, including those without argument. The same calculations were carried out for the other tables.

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MATHEMATICS TEACHER EDUCATION RESEARCH AND PRACTICE: RESEARCHING INSIDE THE MICA PROGRAM

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The paper describes an ongoing collaborative work between department of mathematics and department of pre-service teacher education, aimed at connecting research and practice in the development and study of mathematics teacher education. The work draws from learning experiences of future teachers through the designing and implementing Learning Objects in department of mathematics. The focus of research is to address the need for a better understanding of how future teachers of secondary school mathematics are shaped by didactic-sensitive activities during their undergraduate mathematics education.

Keywords: future teachers; mathematics needed for teaching; research and practice; innovative undergraduate mathematics program, mathematics teacher education

Introduction

In their preface to a special issue of *Educational Studies in Mathematics*, titled “Connecting Research, Practice and Theory in the Development and Study of Mathematics Education,” Even and Ball (2003) highlighted the need for addressing the gap between theory and practice, the divide between mathematics and mathematics education, and the divide between mathematicians and mathematics educators in the study of mathematics education. As they noted, there are emerging efforts to build collaborations and connections focused on the issues of practice in order to develop and study mathematics education. It is this sort of sensitivity to building connections and collaboration in addressing issues of practice and research that underpins our research. The central focus of our research is to address the need for a better understanding of how future teachers of secondary school mathematics are shaped by didactic-sensitive learning experiences during their undergraduate mathematics education (Mgombelo & Buteau, 2008a, 2008b). The research draws from learning experiences of future teachers in a non-traditional core undergraduate mathematics program called “Mathematics Integrated with Computers and Applications” (MICA) (Ben-el-Mechaiekh, Buteau, & Ralph, 2007; Ralph 2001). Among other things, MICA, launched at our institution in 2001, integrates computer, applications and modeling where students make extensive use of technology in ways that support their growth in mathematics (Ralph & Pead, 2006). Previous work describing MICA student learning experiences is reported in Muller and Buteau (2006); Buteau and Muller (2006); and Muller et al. (in press). Our focus in this paper is to describe our ongoing collaborative work aimed at connecting research and practice in the development and study of mathematics teacher education.

The rationale for our research is based on epistemological and practical grounds.

Mathematics teacher education is premised on the assumption that one has to be educated in mathematics in order to be able to teach it. This assumption highlights the well known problem of divide in mathematics teacher education between mathematics and teaching. From an epistemological perspective, the question is how mathematics and teaching could be integrated in mathematics teacher education. An initial characterization of this integration comes from Shulman's (1986) work on pedagogical content knowledge. Recently, Ball and Bass (2002) elaborated on pedagogical content knowledge and used the term *mathematics knowledge for teaching* to capture the complex relationship between mathematics content knowledge and teaching. This is the epistemological ground for our research.

In practice, any mathematics teacher education program has to contend with questions of how much mathematics and how much method or educational study should comprise such programs, and then whether and how these programs should integrate or separate out opportunities to learn mathematics and teaching (Adler & Davis, 2006). Answers to these questions are reflected in a wide spectrum of variations of programs, opportunities, and learning activities for future teachers (Mgombelo et al. 2006). In addition, there are also lessons from mathematics teacher education research and practice. With regard to secondary school teacher education, many teachers still struggle with teaching school mathematics for understanding even though their knowledge of mathematics may be adequate (Kinach, 2002). This points to mathematics needed for teaching.

Following Ball and Bass's (2002) work on mathematics for teaching there has been recognition that mathematics teacher education is an important area of study in departments of mathematics (Conference Board of Mathematical Sciences [CBMS], 2001; Davis & Simmt, 2005). For example, the 2001 report from the CBMS on "The Mathematical Education of Teachers" has two main recommendations for ways in which mathematics departments can attain these goals:

First, the content and teaching of core mathematics major courses can be redesigned to help future teachers make insightful connections between the advanced mathematics they are learning and the high school mathematics they will be teaching. Second, mathematics departments can support the design, development, and offering of a capstone course sequence for teachers in which conceptual difficulties, fundamental ideas, and techniques of high school mathematics are examined from an advanced standpoint (p.123).

It is with this sort of understanding that some departments of mathematics have made ongoing and emerging attempts to reform their programs to provide meaningful experiences for future teachers (Bednarz 2001; CMS 2003; Muller & Buteau 2006; Pesonen & Malvera 2000). This points to the need for research to investigate whether and how these attempts impact future teachers' learning of mathematics needed for teaching (Bednarz 2001). More importantly, as we noted earlier, for this research to be meaningful and productive, collaboration among mathematicians and mathematics educators is crucial (Even & Ball, 2003; Mgombelo & Buteau, 2006). We are

addressing this need for research and collaboration in our research. We, a mathematician and a mathematics educator, are interested in collaboratively extending our understanding of how future teachers of secondary school mathematics are shaped by their experience of designing so-called *Learning Objects* in the MICA program. In the following section we describe the MICA program and what we learned from reflections on practice regarding the students' learning experiences.

Learning from Practice: The MICA experience

In 2001, our institution launched its innovative core undergraduate MICA program based on guiding principles (a) to encourage student's creativity and intellectual independence, and (b) to develop mathematical concepts hand in hand with computers and applications. MICA also strives to strengthen the concurrent mathematics teacher education program. It exposes future teachers to a broad range of mathematical experiences rather than to a deep concentration in one or two areas. Future teachers also make extensive use of different software programs such as Maple, *Journey Through Calculus* (Ralph, 1999), Geometer's SketchPad, and Minitab, all of which nurture the understanding of mathematics.

In addition to a revision of all the traditional courses under the above-mentioned guiding principles, three innovative, core project-based courses, called MICA I - III, were introduced in which all students learn to investigate mathematics concepts by designing and implementing interactive computer programs, so-called *Exploratory Objects* (Muller et al., in press), from year one. As their final projects in MICA courses, students individually (or in groups of two) complete an original interactive computer program on a topic *of their own choosing*. These projects can be (a) exploratory (e.g., testing his/her own conjecture; see *Structure of the Hailstone Sequence* Exploratory Object, (MICA Student Projects, n.d.); (b) an application (e.g., modeling or simulation; see *Running in the Rain* Exploratory Object, MICA Student Projects); or (c) didactic, i.e., so-called *Learning Objects* (LO). The latter, generally designed by future teachers, are innovative, interactive, highly engaging, and user-friendly computer environments that teach one or two mathematical concepts at the school level. For example, a 9-task adventure with Hercules covering (Grade 4) perimeter and area; a journey through MathVille for learning the (Grade 9) exponent laws; or a fourfold Pythagorean Theorem plate-form offering (i) a review of right angles and triangles, (ii) an exploration of the theorem, (iii) a game to practice, and (iv) a five question test with applications, are all projects designed by first-year future teachers (see respectively *Hercules and Area* LO, *Exponent Laws* LO, and *Exploring the Pythagorean Theorem* LO, MICA Student Projects).

Overall, observations and reflections on students' experiences of designing LOs and Exploratory Objects indicated that the experiences promoted positive student learning experiences. Muller et al. (2008) summarize these experiences:

We suggest that the students develop the following skills: (a) to express their mathematical ideas in an exact way; (b) to self-assess their mathematics; (c) to

realize their creativity in mathematics and in communicating their understanding of mathematics; and (d) to become independent in mathematical thinking. We also suggest that students are exposed to the opportunity (a) to concretize personalized original mathematics work, and (b) to identify with their future profession. Finally, our observations lead us to suggest that students develop a personal relationship with the activity of designing and implementing an ELO; indeed, students seem to demonstrate a strong engagement and ownership in the activity, and exhibit much pride of their ELO (p.4).

These reflections prompted a pragmatic collaborative project between the Department of Mathematics and the Department of Pre-Service Education which involved LOs designed by MICA students and teacher candidates enrolled in pre-service education elementary mathematics methods course (Grades 4 to 8) (Muller et al., in press). Pre-service teacher candidates were asked to use LOs to learn or review the involved mathematics in the Object and to write their reflections on their experience. Their overall experience was positive as they appreciated the LOs and commented on their high regard for the first-year MICA student LO designers. Some teacher candidates who self-identified as having math anxiety, thought that the LOs provided a safe environment for them to re-learn mathematics.

Reflecting on MICA student learning experiences as well as pre-service teacher candidates' experiences of using the LOs, we started to focus on the MICA future teachers' experiences of designing and implementing LOs. It was clear to us designing and implementing LOs involves mathematical didactics work. Interesting empirical questions started to emerge: In what ways do future teachers experiences of designing and implementing LOs promote their learning of mathematics needed for teaching? What aspects of designing and implementing LOs prompt such a positive experience? How do these future teachers' learning experiences through designing and implementing LOs differ from their learning experiences in other traditional activities? These questions led us to focus on the suggested future teachers' development of a "personal relationship with the activity of designing and implementing [a] Learning Object" (Muller et al. 2008). We postulated that future teachers' behaviour, in terms of dedication, pride, ownership, and engagement with the activity could be a key to the future teachers' positive experiences and their learning of mathematics needed for teaching. This pointed to an in-depth investigation to explore the impact of future teachers experiences of designing and implementing LOs on their learning (Mgombelo & Buteau, 2008a).

Researching inside MICA: Learning Mathematics Needed for Teaching through the Designing and Implementing of LOs

The purpose of our research is to explore how future teachers of secondary school mathematics are shaped by their didactic-sensitive learning experiences during their undergraduate mathematics education. Our research is guided by the following questions: (a) Does the experience of designing and implementing LOs promote

future teachers' learning of mathematics needed for teaching? (b) In what ways do designing and implementing LOs provoke future teachers' awareness of their own learning of mathematics as well as what does it mean for students to learn mathematics? Guided by previously mentioned postulate (that ownership, dedication, engagement of the activity, and pride are key for the positive learning experience) we are interested in probing deeper into these future teachers' experiences in order to capture the qualitative aspects of their learning of the mathematics needed for teaching. The goal in our research is not to measure this impact in terms of how much do future teachers know mathematics needed for teaching. Our focus in the research is on future teachers' "knowing." Given the complexity of this kind of research we initially conducted a pilot –small scale study (2006-07). The goal of the pilot study was to gather first evidence of future teachers' experiences as well as to inform the design of a large scale study.

Guided by the above postulate our pilot study was framed by Mason and Spence's (1999) work on "knowing-to act" as a kind of knowing that requires awareness. Building on Gattegno's (1970) work on awareness, Mason (1998) further elaborates on the relationship of "knowing-to act" and awareness in mathematics teacher education. Mason developed three forms of awareness: "awareness-in-action," which involves a human being's powers of construal and of acting in the material world; "awareness-in-discipline," which is awareness of awareness-in-action emerging when awareness-in-action is brought into explicit awareness and formalized; and finally, "awareness in counsel," which is awareness of awareness-in-discipline involving becoming able to let others work on their awareness-in-discipline. To put this into a mathematics perspective, awareness-in-action might be exemplified by an act of counting numbers (1, 2, 3) without being aware of the underlying notions such as one to one correspondence. Awareness-in-discipline emerges when one becomes aware of this one to one correspondence in counting. Finally, awareness-in-counsel emerges when one is able to support others develop their awareness of counting as well as develop their awareness of the notion of one to one correspondence. Mason's levels of awareness served as analytical/interpretive tool for analyzing data

Data were collected from detailed questionnaires, journals, and focus group discussions that involved 4 future teachers enrolled in the MICA program, 4 teacher candidates in the Department of Pre-Service, and 1 practicing teacher. In order to probe MICA future teachers' experiences deeply in terms of awareness, questions and prompts in the questionnaires and journals were open-ended. The roles of the Pre-service teacher candidates and the practicing teacher in the research were to facilitate data collection through focus group discussion and not to act as research subjects.

All data from questionnaires, LOs, and transcripts from videos were analysed according to the interpretation of themes guided by the postulate that ownership, engagement in the activity and pride were key for positive learning experiences and by using Mason's three forms of awareness as outlined in the conceptual framework..

Using Mason's levels of awareness we identified which levels were engaged as well as ways in which they related to experiences of ownership, engagement and pride. Our analysis of data further elaborated on three prospective teacher behaviour aspects, ownership, engagement, and pride. We briefly elaborate these aspects.

Ownership

As noted earlier in this paper, prospective secondary school teachers can perform a number of school mathematics tasks without problem. Using Mason's (1998) forms of awareness, we could say these future teachers have awareness-in-action of mathematics needed for the tasks. Yet (as noted) if you ask future teachers how they would explain a mathematics concept or skill to someone who is learning for the first time, most of them would respond by rule-based explanation (e.g., negative times negative is positive in case of integers multiplication). These future teachers would be attending to content of their awareness-in-action and not their awareness of their awareness-in-action. As Mason notes, the behaviours to which awareness-in-action play a role can somewhat be trained without explicit allusion to awareness. We found a different scenario with the experience of designing and implementing LOs. This experience seems to prompt future teachers to take into account their own experience of learning the mathematics in order to generate ideas on how to design their LOs in ways that will make sense for the user's learning of mathematics in question. It is this future teachers' attention to their learning in order to bring to awareness their awareness-in-action that we refer to as *ownership*. This is exemplified by the following prospective teacher's response to the questionnaire question on why she chose the topic for her LO.

My MICA I Learning Object [...] dealt with explaining and practicing multiplication.... I chose this topic because in Grade four I was very, very behind on my multiplication. I could not do the calculations in my head, and I was stuck on the first sheet of questions my teacher would give us... Since it is something I struggled with and something that I have to overcome to become a Math major, I thought it would be a great idea to develop a program that could allow students to practice without just doing the same questions over and over. I also included different ways of thinking about what multiplication means (Mgombelo & Buteau, 2008a)

It underlines that this prospective teacher attended to her own learning of multiplication or own awareness in action of multiplication. The prospective teacher in the above response did not want to design a program based on multiplication routines and rules but instead wanted to include the different ways of thinking about what multiplication means – this involves awareness.

Engagement

Awareness-in-discipline arises when we become aware of awareness-in-action. According to Mason (1998), the term “discipline” means encountering both facts and techniques as well as habits of thought, types of meaningful questions, and methods

of resolving those questions. Our analysis of the data indicates that through the designing and implementation of LOs, future teachers engage with mathematics in terms of both aspects outlined above by Mason. Our analysis further indicated that future teachers' experiences of designing and implementing LOs tend to elicit the need to explain and attend to different representations and meanings of mathematics concepts, a very important aspect of mathematics for teaching (Ball & Bass, 2002; Davis & Simmt, 2005). We distinguish *engagement* as another aspect of learning mathematics needed for teaching. Engagement with mathematics is recognized in the way future teachers use games, graphics, and colors in their LOs in order to engage students in a meaningful way. These future teachers attended to different representations or meanings of mathematics concepts such as grid or area models of multiplication as revealed in a response from a prospective teacher questionnaire below.

I learned how to keep instructions short and simple, and how to gear a lesson towards your audience. I learned to think about the audience I was trying to reach and what would be engaging to them. I added in Bart Simpson and made it as bright and colorful as I could. I learned multiple ways of explaining multiplication. (Mgombelo & Buteau, 2008a)

We see from the above response from the prospective teacher questionnaire, that she "learned to think about the audience ...and what would be engaging to them." It is through this experience that she learned multiple ways of explaining multiplication. It is worth to note that this experience involves both future teachers' own engagement with mathematics as well as their audience's (students') engagement as revealed in the above response.

Pride

In order to sustain ownership and engagement in mathematics activities in the way we have described here, future teachers have to invest themselves in the activity (in terms of energy, emotion, interest, etc.). In addition to investing themselves, they need to have a sense of purpose and accomplishment. We have identified this investment as *pride*, the third aspect of future teachers' learning of mathematics needed for teaching. Here is an example from a prospective teacher's response that supports our claim.

You're always thinking about ideas and ways to improve your project while you are in class, watching television [...] (Mgombelo & Buteau, 2008a)

We can see clearly from the above quote how much personal energy, or in other words, dedication, this prospective teacher invested in the project. Our small scale study addressed the need to know about the impact of designing and implementing LOs on the learning of mathematics needed for teaching. It strongly suggests that the experience of designing and implementing LOs promotes future teachers' learning.

Conclusions: Further Research and Practice Collaborations

Our work underscores the importance of collaboration between mathematicians and mathematics educators in connecting practice and research in mathematics teacher education. From our pilot study further empirical questions emerged: What aspects of the designing and implementing LOs prompt such a positive experience? In what ways do prospective teachers' learning provoked by designing and implementing LOs differ from other traditional learning tasks? These questions have led to a larger-scale, collaborative research project (involving some 30 MICA future teachers candidates each followed over two years) that will thoroughly investigate the students' "repositioning" in terms of engagement, ownership, and pride, with respect to mathematics and mathematics didactics when realizing their MICA final projects (the LOs) compared to more traditional mathematics activities. We are also interested in exploring the characteristics or features of the learning activity (of designing and implementing a LO on a topic of their own choosing) that promote learning. A theoretical framework has been thereafter developed to guide this comprehensive study (Mgombelo & Buteau, 2008b). It mainly relies on Brousseau's (1997) work on theory of didactic situations; Mason's (1998) work on knowing-to act as previously discussed; and on positioning theory.

Our work has been extending on the connection between research and practice in many different ways. First, a collaborative Learning Object project building on Grade 5 students' ideas from a local school (Buteau et al. 2008) has been completed. The project involved the principal, 2 teachers, and Grade 5 students from the elementary school, as well as a mathematics student, pre-service teacher candidates, and both co-authors from our institution. The principal commented,

From day one, our Grade 5 students were extremely motivated and engaged in developing this tool that will be used by students from other schools. (Buteau et al., 2008, p.28)

A second connection yielded in the ongoing integration of MICA Learning Object use for didactical assignments in the Methods course at our institution. In addition, Mgombelo's informal observations about MICA pre-service students with stronger dispositions towards learning versus non-MICA pre-service students led her to reflect on the design of the course. This naturally leads to asking what is it exactly in the MICA education program that seems to promote this disposition - a question that points to our long-term research program. Thirdly, the research has been guiding Buteau's reflections on her teaching practices of the MICA I course and on the MICA activities (e.g., the description of the student development process of designing and implementing Exploratory and Learning Objects, (Buteau & Muller 2008), thus pointing back to the LO activity attributes that might promote learning mathematics for teaching.

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COGNITIVE TRANSFORMATION IN PROFESSIONAL DEVELOPMENT: SOME CASE STUDIES

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Professional development programmes for in-service teachers constitute a complex task. We intend here to shed some light on the conditions that may entail a cognitive transformation in the involved teachers, building on our personal experience in these programmes and some case studies.

Keywords: Professional development, in-service teachers, metaphors, cognitive modes.

INTRODUCTION

In this paper I report on some didactic phenomena (in the sense of Margolinas, 1998) arising in our work in professional development for in-service primary teachers, at the University of Chile. These phenomena are related to the cognitive transformations that emerge in the being of the involved teachers, as well as researchers, under favourable circumstances, depending on “the time, the place and the people” (see Mason, 1998). Our work could be described as “theory-guided bricolage” in developmental research (Gravemeijer, 1998; Freudenthal, 1991), with the caveat that a detailed theory is not put forth first, because it rather grows out of the ongoing process. This approach to professional development or enhancement for in-service teachers is inspired by my former research on the fundamental role of metaphors and cognitive modes in the teaching-learning process (Soto-Andrade 2006, 2007). It involves “researching from the inside” (Mason, 1998), and it requires an embodied first-person approach (Varela, Thomson & Rosch, 1991), in an enactive perspective (Masciotra, Roth & Morel, 2007).

After recalling the fundamental components of a tentative theoretical framework, I set down below my main research hypotheses and proceed to report on some concrete examples of activities and germs of didactical situations (Brousseau, 1998), involving metaphors and switches in cognitive modes, that we have worked out with teachers. Translated quotes of several teachers’ testimonies and reports are also included, as case studies. These give preliminary experimental evidence to support our hypotheses

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and suggest further research along these lines.

THEORETICAL FRAMEWORK

Nature and Role of Metaphors

It has been progressively recognized during the last decade (English, 1997; Lakoff & Núñez, 2000; Presmeg, 1997; Sfard 1994, and many others) that metaphors in mathematics education are not just rhetorical devices, but powerful cognitive tools that help us to build or grasp new concepts, as well as to solve problems in an efficient and friendly way (Soto-Andrade, 2006). We use conceptual metaphors (Lakoff & Núñez, 2000), that appear as inference preserving mappings going “upwards” from a source domain into a target domain, enabling us to understand the latter, usually more abstract and opaque, taking a foothold in the former, more down-to-earth and transparent in terms of our previous cognitive history. Metaphors are “met-befores”, as Tall (2005) says.

Cognitive modes

A cognitive mode is defined nowadays as one’s preferred way to think, perceive and recall, in short, to cognize. It shows up, for instance, when trying to solve problems. Flessas and Lussier (2005) gave a first operational description of what they call the 4 basic cognitive modes (“styles cognitifs” in French), combining 2 dichotomies: verbal – non verbal and sequential – non sequential (or simultaneous), closely related to the left – right brain hemisphere dichotomy and to the frontal – occipital dichotomy. We so obtain the *sequential-verbal*, *sequential-non verbal*, *non sequential-verbal* and *non sequential – non verbal* cognitive modes. They emphasize that effective teaching of a group of students, who may display a high degree of cognitive diversity, requires teachers supple enough to tune fluently to the different cognitive modes of the students.

An example: check that you have the same number of fingers in your hands by using the 4 basic cognitive modes (see Soto-Andrade, 2007, for more examples).

In what follows I adhere mainly to the framework laid by Lakoff & Núñez (2000), Presmeg (1997) and Sfard (1994, 1998) for metaphors, Flessas & Lussier (2005) for cognitive modes, Brousseau (1998) for didactical situations and to the research paradigms of Mason (1998), Varela et al., (1991) and Masciotra *et al.* (2007).

PROBLEMATICS

Professional development and enhancement for in-service teachers is a complex issue. In Chile, significant funding and human resources have been invested by the Ministry of Education, for more than two decades, to address this issue, but results have been rather scanty. Our students continue to perform poorly in international assessment tests like TIMMS or PISA, and also in national assessment tests like SIMCE [1]. Increasing evidence shows that after a typical 2 week intensive summer workshop, where they learn some more mathematics and design a couple of teaching modules, most teachers revert to their former inadequate teaching practices.

Under closer scrutiny, we have observed that most of our in-service primary teachers are unfamiliar with metaphors and cognitive modes, or visualization, in their practice. They are “frozen” in the verbal - sequential cognitive mode, unaware of this and also of the fact that their teaching is shaped by unconscious and misleading metaphors, like the acquisition metaphor (Sfard, 1998) or the container-filling or gastronomic metaphor (Soto-Andrade, 2006). They have special trouble in creating “unlocking metaphors” for the not specially gifted.

The urgent question is: How to promote a real change in the teaching practices of in-service teachers, in the short or mid term?

RESEARCH HYPOTHESES

Our main research hypothesis is that metaphors and cognitive modes are key ingredients in a meaningful teaching-learning process. Moreover the deepest impact on this process is usually attained by metaphors that involve a switch from one cognitive mode to another.

We claim that competences regarding multi-modal cognition and use and creation of metaphors and representations are trainable and that measurable progress can be achieved in a one semester course. This, in spite of the fact that most of our teachers report that their initial training included no use of metaphors and privileged just one cognitive mode: the usually dominant verbal-sequential one.

We hypothesize that explicit work on metaphors and transits between cognitive modes will foster teacher’s deep understanding of elementary mathematics. Furthermore, it will affect their professional practice in the classroom, in particular enabling average students to understand and handle mathematical objects and processes that would otherwise be within reach of only a happy few.

RESEARCH BACKGROUND AND METHODOLOGY

The background for our experimental research consisted in 5 classes (called “generations” in what follows) of in-service primary school teachers, of 30 teachers each, enrolled in a professional development programme, implemented by the University of Chile, on behalf of the National Ministry of Education, stretching from 2006 to 2008. This programme aims at “general” primary teachers, who are interested in enhancing their mathematical training, and certifies their mathematical proficiency after a 15 months period, where they must complete the requirements for 4 modules (numbers and data processing, geometry, ICT in education and problem-solving, 450 hours in all). They must also complete a 75 hour Seminar Project, which includes experimenting and theory-driven practice in the classroom.

Teachers applied for admission to this programme, with the support of their schools, and were selected according to their performance in a TIMMS like test, based mainly on mathematical contents pertaining to the curriculum of primary school. Selected teachers are usually highly motivated; they come to the University after hours, typically from 6 PM to 9:15 PM, at least twice a week, plus an intensive 2 week

summer workshop. Gender distribution is 90% female, 10% male, on the average. Ages range from 25 to 60, even 68 in one case (see below).

This sort of programme opens up hitherto unknown possibilities for deeper work with teachers. In particular, as coordinator for the Numbers Module (160 hrs approx.) and advisor to the seminar projects of 6 teachers in each generation on the average, I had the opportunity to test several activities and a-didactical situations in work sessions with the teachers. This module aims mainly at reviewing the mathematics as well as the didactics of numbers, specially elementary integer arithmetic, fractions, ratios, decimal and binary description of numbers. Work sessions were interactive, with teachers usually working in small groups of four on the average.

The underlying idea for this module was to open up the opportunity for the teachers to have a first hand experience of problematic and challenging situations to be tackled, eventually “bare handed”, where important mathematical objects or processes could emerge. So their experience would be an antidote to the usual cookbook recipe approach. Methodology consisted in observing the teachers, as they carried out various activities, with non intrusive guidance and support, recording their reactions, in video in some cases, and asking them to write reports on their work, besides communicating it orally to the whole class. After completion of the programme, I asked them to write a short report in the first person on their cognitive and affective experience, in the spirit of “researching from the inside” (Mason, 1998).

My viewpoint was that just recording contents taught plus results of post-tests administered to teachers provides a rather shallow understanding of their learning process. Instead, I tried to foster group work, monitoring the course of their work during sessions, by circulating and interacting with the groups, as a means to fathom their cognitive profiles and processes. This was complemented with the results of tests and challenges. The first-person report mentioned above also provided further insights into the process they had undergone. So my approach relies mainly in case studies rather than hard statistical evidence, emphasizing qualitative rather than quantitative assessment (see below however quotes on SIMCE [1] scores)

EXPERIMENTAL ACTIVITIES AND PRELIMINARY RESULTS

I comment here on some concrete albeit paradigmatic examples of the activities carried out, together with excerpts of the teacher’s reactions to them.

Example 0: Do you have an innate approximate number sense?

To make them feel the contrast between verbal-sequential and non-verbal non-sequential cognitive modes, we began with some experiments aiming at activating their innate approximate number sense or “numerosity” in the sense of Dehaene (1997), Lakoff and Núñez (2000), Pica *et al.* (2004), Halberda, Mazzocco and Feigenson (2008). For instance, they were asked to tell whether there were more yellow dots or blue dots in a random array of dots of both colours shown just for 200 ms (Testing your Approximate Number Sense, 2008). Our fifth generation of teachers scored here an impressive average of 95%, much higher than the statistical average

success of only 75% (as it was the case in a class of average Master in Science students in our Faculty). This suggests that primary school teachers tend to have a significantly better approximate number sense than random adults.

Example 1: How to keep track of your lamas?

An 8 year old aymara shepherd is in charge of a herd of lamas (more than 40, it seems) at some barren place in the highlands in the north of Chile. But he is tired and would like to take a nap... How could he check that when he wakes up there are no lamas missing? He has no palm device, no paper and pencil, not even small stones, or sticks or a knife; just his bare hands. Moreover he does not know how to count calling numbers by their name. How could he manage to register the number of lamas in sight before going asleep and to recover it when waking up?

Every generation of teachers engaged in group work, in groups of 4 to 5, to discuss how to tackle the problem. As a supporting aid, we simulated the lamas with a bunch of coins on the plate of an overhead projector. Usually, after half an hour or so, in one or two groups, the idea emerged of using the phalanges of their fingers, thumb excluded. The idea spread quickly and finally all groups rediscovered the classical method of non-verbal counting by dozens still used in the Middle East and Far East, where you touch with your right thumb the 12 phalanges of your right hand, say, one by one, and fold one finger in your left hand to register each complete round of 12 (Ifrah, 2005, p. 74). Most did that from little finger to index, but some did it from proximal to distal phalanges (the classical way) and others, the other way around. So they learned how to count non-verbally up to 60, using their fingers and they applied this successfully to the simulated herd of lamas on the overhead projector. They also related this with the ubiquitous emergence of the dozen and 60 in human cultures.

This example may be looked upon as an implementation of realistic mathematics education (Gravemeijer, 2007; Freudenthal, 1991). The underlying hypothesis and motivation for this activity is that it is important to practice and get the feeling of non-verbal arithmetic before engaging into classical arithmetic. So our idea was to prompt the teachers to go back to the non-verbal sequential mode in the context of counting. Their reactions to this sort of activity were stronger than expected:

My (programme) experience was totally significant in the most strict sense of the expression. It brought to me important changes in my way to approach lessons, in my professional practice and personal interests. But not everything was a “rose garden”... After the first lessons I was quite disappointed, because this course didn’t make any sense to me. My expectations were to learn “more mathematics”, fill in my gaps and not to debate endlessly about why, what for and how. I was even more disappointed with the Numbers Module, with metaphors! I didn’t understand anything: I expected to solve hard arithmetical problems, to design endless exercise lists to calculate with fractions or decimals, to learn more and better algorithms, and it turned out that we were exposed to questions I had never asked myself: How do indigenes in the Amazonas do arithmetic, although they have no language for numbers? How can a shepherd boy know how many

lamas he has if he doesn't know how to count? How could you teach counting to a little child, in a clever way? There, I had a cognitive break: I asked our teacher for an explanation of the aim of his lessons (I am now ashamed about that) and he kindly explained to me what he was after... (Evelyn, 32, 8 years of practice, 1st generation).

Example 2: Who has more marbles?

John and Mary have a bag of marbles each, all of the same size. How can they tell who has more marbles?

I invited the teachers, organized in small groups (3 to 4 each), to figure out other approaches than the usual sequential-verbal one (counting the marbles in each bag). Usually in less than half an hour they found at least one procedure for each cognitive mode (Soto-Andrade, 2007). The two pan balance for the non-verbal non-sequential mode emerged easily; also the idea of pairing off the marbles, without counting them, for the non-verbal sequential mode. Verbal - non sequential approaches took longer to appear (weighing simultaneously both bags in digital scales and reading off...).

Example 3. Registering quantities with dice.

The indigenes in an Amazonian village want to keep track of the quantities of seeds stocked for next year. How could they register quantities up to thousands if they have just a handful of dice at hand and they have not invented zero yet?

After half an hour work on the average, in small groups, the teachers find out, and begin even to do arithmetic in dice-system! They report to understand now much better the decimal system and try this activity with their pupils, with encouraging results. Among others, Gina (49, 25 years of practice, 4th generation) reported:

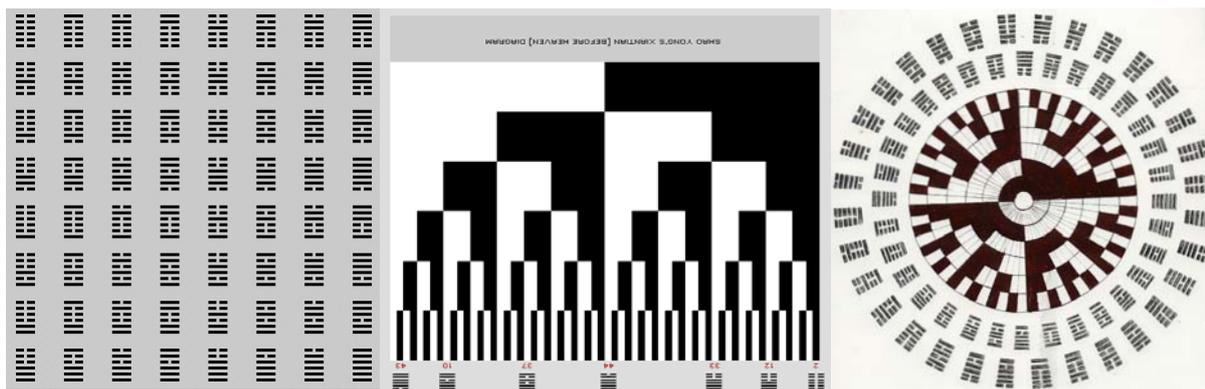
This experience was very important to me, because you were able to “un-structure” my mind and take away my fear of numbers. Now I see that this fear came from a dull teaching, full of cookbook recipes, that never gave me the opportunity to enjoy discovering the way to solve problems all by myself. Numbers was my favourite subject in this programme, it allowed me to fly, to play, to err and not to feel silly...

Example 4: The number sequence, otherwise...

Is it possible to represent the numerical sequence 0, 1, 2, 3, up to 63, let us say, in a non verbal and non sequential way?

Teachers usually get to the point of discovering the given sequence, written in binary way, in Shao Yong's square (below left), and then of encapsulating it in a single image. (Soto-Andrade, 2007). In the first generation, 5 out of 30 teachers, after 30 minutes work in small groups, came up with diagrams equivalent to Shao Yong's Xiantian (“Before Heaven”) or its inverted form (shown below, center, as in Marshall, 2006). Notice the underlying binary tree! In the 2nd generation, 6 out of 30 teachers, rediscovered Xiantian and, most remarkably, one of them, Ofelia (68, 50 years of practice), draw all by herself a circular version of Xiantian diagram (below right). In her own words:

The Numbers Module shattered all my schemes. For the first time, my brain, architected for algorithmic work, began to have a glimpse of a tiny light (showing the way) to working metaphorically, to solving a problem in different ways, to looking for different paths to reach the same target, not just be satisfied because I got there. I must confess that during the first weeks I was not able to fathom where we were heading to! When I first met a sequence of I Ching hexagrams, sincerely I was barely able to tell what I was looking at! So I never imagined that some weeks later I was going to be able to rediscover one of the oldest binary trees, Shao-Yong's circular Xiantian. Later I spent hours trying to solve problems using different cognitive modes...



Here the teachers have the possibility of transiting from the usual verbal sequential mode (the given sequence) to the non-verbal sequential mode (iconic hexagram binary representation) and then to non-verbal non-sequential mode (Xiantian). When interviewed, they unanimously reported having understood, in this unexpected way, for the first time the binary description of numbers.

Example 4. Brownie's walk

Random walks provide a nice way to introduce probabilities. Instead of the well known drunkard's walk, we introduced to teachers with no previous training in probability a puppy called Brownie (a baby incarnation of Brownian motion), who escapes randomly from her home in the city when she smells the shampoo her master intends to give her. The stepwise description of her random walk can be tackled by rudimentary means, even by simulation, or with the help of efficient metaphors, like the Solomonic metaphor or the pedestrian metaphor (Soto-Andrade, 2006). In the first one, Brownie splits into 4 pieces, each going to each cardinal direction, and so on... In the second one, a pack of Brownies (a power of 4 preferably) runs away from home, dividing themselves equally into four packs at each corner, and so on... The latter has the virtue of allowing the teachers to work with natural frequencies, in the sense of Hoffrage, Gigerenzer, Krauss & Martignon (2002), avoiding fractions up to the last minute. We have here also an integrative problematic situation, involving geometry, arithmetic and algebra, besides randomness.

After engaging in activities of this sort, teachers reported:

Cognitive metaphors simply surprised and fascinated me. I had learned with the traditional, mechanical system, and in that way I was teaching my students. Now, I learned about cognitive modes, how to reach every one of my students, and how, with the help of a metaphor, I succeeded in making mathematics closer, friendlier and more reachable. I got so convinced that I chose Numbers for my Seminar Project and I modified radically my professional practices. I wanted to prove that metaphors and these new approaches would give good results, not just for the emotional atmosphere in the classroom but also for “hard” tests. And indeed, my K-4 2007 class got the *first place in the country*, in the SIMCE assessment test [1], increasing by 25 points the previous score, up to 328 points, with no previous training for the test! (Evelyn, 32, 8 years of practice).

I took advantage of this way of working to carry several activities to my classroom, using various metaphors, which made the students enjoy more my lessons, learning more easily. I transferred all this to my pupils. And this year 2007, our K-4 classes, taught by my colleague Lily (also a student in this programme) and myself increased dramatically their SIMCE score [1], from 281 to 304 points (former SIMCE scores for this grade, since 2002, were 287 and 282). This happened with no special training for the test, contrary to the case of many other schools; the students had just the regular lessons with us (Gina, 49, 25 years of practice).

I had certain expectations: this program would deliver knowledge to me, besides methodologies to apply to my pupils. But you broke my schemes. What I expected did not happen. What you achieved was to take me out from my “pigeonholing” and to make me think further. If we as teachers are rigid and un-imaginative, hardly will we be able to have our pupils free their imagination and become enchanted with mathematics. This is badly needed, that's why they reject maths so much. I have questioned my way of interacting with my pupils and the way of structuring my lessons (Karem, 32, 6 years of practice, 4th generation).

CONCLUSIONS AND DISCUSSION

Observation of the teacher's performance shows that even those who never had this sort of experience before were able to activate less usual cognitive modes, to transit from one to another and to take advantage of new metaphors to understand better and to efficiently solve problematic situations. In particular, after some prompting, a high percentage of them were able to switch from their dominant verbal-sequential cognitive mode to a non-verbal or non-sequential one. These findings support our optimistic hypothesis that *cognitive flexibility*, i.e. the ability to approach the same object through various cognitive modes and transiting from one cognitive mode to other, is trainable, even for in-service teachers and that it is facilitated by group work.

However, their first person reports suggest that we had sub-estimated the magnitude of the cognitive shock they experience during the first weeks of our programme. It is interesting to note that testimonies of older and younger teachers are surprisingly alike in this respect. The same holds for their reactions thereafter and changes in their professional practice, as reported above. As a typical example, we recall a 50 year old

teacher, Yihecika, from our 3d generation, saying at his final Seminar presentation: “I am very moved, because I am an old teacher doing new things!”. At least in the case of these primary teachers, this disproves the hypothesis that changes in cognition and professional practice are out of reach for older teachers.

A rather unexpected outcome of the work carried out with our in-service teachers is the dramatic improvement of their student performance, in several cases, in traditional standardized multiple-choice tests like SIMCE [1]. We may notice that the *relative* improvement was approximately the same for Evelyn and Gina (25 and 23 points resp.) albeit *absolute* scores differed noticeably (328 and 304 resp.), as it is on the average the case between fully private schools and state supported private schools in Chile. Although our programme is intended for teachers in service at state-owned or state supported private schools, Evelyn has been teaching at a fully private high income school for 2 years because she was fired from her previous teaching job at a state supported private school right after completing her professional development programme (as it is the case of roughly 10% of our teachers!). On the other hand, Gina teaches in a low income state supported private school whose explicit aim in mathematics was to reach sometime the threshold of 300 points.

In conclusion, we have gathered some new positive experimental evidence related to this “theory-oriented bricolage”, that appears to entail significant cognitive transformations in the being of the teachers (Mason, 1998) and as a consequence, changes in their classroom practice and performance of their students, even measured in traditional ways.

1. SIMCE is a national assessment test, applied to K-4 every year and to K-8 every two years. It is much closer in spirit to TIMMS than to PISA. SIMCE national average score in mathematics for K-4 stagnates at 246 in 2006 and 248 in 2007. Standard deviation is about 50 points. In mathematics only 26% of the students attained the advanced level, whose threshold is 286 points.

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WHAT DO STUDENT TEACHERS ATTEND TO?

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The ability to notice key features of teaching is seen as part of student teachers' pedagogical content knowledge. The study shows what student teachers focus on when they have no experience of guided observation of lessons either in reality or on video and when they are not directed by the educator. Some preliminary findings from a wider study are presented which are in line with other existing research: namely, that the student teachers neglect the subtleties of the introduction of the mathematical content.

Keywords: pedagogical content knowledge, ability to notice, student teachers, videos

THEORETICAL FRAMEWORK

The notion of pedagogical content knowledge (or PCK) was first introduced by Shulman. The teacher needs understanding of the material he/she is teaching, but he/she also needs the “knowledge of the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others” (Shulman, 1986). He/she needs to be aware of topics with which pupils might have difficulties and of their common misconceptions and misunderstandings. Bromme (2008) claims that PCK can also be seen in the ways the teacher “takes into account pupils' utterances and their previous knowledge”. An (2004) stresses four aspects of the effective teacher's activity in the classroom which are part of PCK: building on students' mathematical ideas, addressing and correcting students' misconceptions, engaging students in mathematics learning and promoting and supporting students' thinking mathematically.

Thus, in my opinion, part of PCK is the *ability to notice*. In order for the teacher to take into account the pupil's utterance and build on his/her understanding, he/she has to notice the importance of this utterance in the first place, put it into the appropriate context, interpret it and only afterwards use it. According to Sherin and van Es (2005), *noticing* involves a) identifying what is important in a teaching situation, b) making connections between specific classroom interactions and the broader concepts and principles of teaching and learning that they represent, c) using what teachers know about their specific teaching context to reason about a given situation. This study is mainly concerned with the first aspect of noticing.

The (student) teachers' ability to notice is important for the development of what Mason and Spence (1999) call *knowing-to*: “Knowing-to is active knowledge which is present in the moment when it is required.” They distinguish this kind of knowledge from knowing-that, knowing-how, and knowing-why. Knowing-to triggers the other types of knowing and thus its absence blocks “teachers from responding creatively in the moment” (ibid). While Mason and Spence mostly

concentrate on the way knowing-to develops in pupils (e.g., while solving problems), they also touch on educating teachers to be able to know-to: “We propose that knowing-to act in the moment depends on the structure of attention in the moment, depends on what one is aware of. Educating this awareness is most effectively done by labelling experiences in which powers have been exhibited, and developing a rich network of connections and triggers so that actions ‘come to mind’”. (ibid)

In the same spirit, Ainley and Luntley (2006) propose the term *attention-dependent knowledge* for the knowledge that enables teachers to respond effectively to what happens during the lesson. It can only be revealed in the classroom. The analysis of videos can help us to label such events when this kind of knowledge is at play.

To sum up, the ability to notice seems to be an important component of the (student) teacher’s PCK. This ability can be developed, among others, by analysing videorecordings of the teaching of others and our own (e.g., Sherin & van ES, 2005; Star & Strickland, 2008; Muñoz-Catalán, Carrillo & Climent, 2007; Hošpesová, Tichá & Macháčková, 2007). Most of the studies confirm that (student) teachers must learn what to notice. Santagata, Zannoni and Stigler (2007) found out that “more hours of observations per se [...] do not affect the quality of preservice teachers’ analyses” and on the other hand, Star and Strickland (2008) claim that the ability to learn from observations of teaching “(either live or on video) is critically dependent on what is actually noticed (attended to)”.

The study presented here is a part of a wider study aimed at exploring how student teachers' ability to reflect on their own teaching and the teaching of others can be developed and what the characteristics of this development are. Here, I will restrict the questions to:

What do the student teachers focus on in a pedagogical situation, on their own, that is, without any expert drawing their attention to important moments?

How deep are their observations?

How do their evaluations of the same moment differ?

METHODOLOGY

The participants of the study are student teachers, future mathematics teachers of pupils aged 11 till 19. They are in their 4th or 5th year of study. In particular, the students whose work is dealt with below were in year 4 and had one term of the Mathematics Education (or ME) course previously (partially not taught by me). From now on, “students” will be used for student teachers and “pupils” for pupils taught in the observed lesson.

In order to answer the research questions, we need to put students in a situation in which they will be confronted with a mathematics lesson but in which an educator’s influence is minimal. The first, obvious, type of data are received from *individual students* who are asked to write unstructured reflections about a video recording of the whole mathematics lesson. They watch it at home. However, a discussion

between students can perhaps lead to a richer analysis. Thus, the second type of data is gathered from *pairs of students* who are asked to analyse a lesson on video. They do it at school, in an empty office, without the educator's presence, and they are being video recorded. In order to find out their immediate reactions, they are asked to stop the video whenever they feel that something deserves commenting on and to say the comment aloud to each other.

The collected data are organised in two ways: a) According to the lesson observed: the same videos of teaching have been used repeatedly so that reactions from different students are received. b) According to the type of origin, i.e., individuals' reflections, pairs' discussions, my teaching (videorecordings of the ME course in which video analyses are sometimes used), teaching practice (students' descriptions of didactical moments which they consider to be important when they observe lessons; their very choice and evaluation of these moments can be of importance).

The data collection still proceeds. In this article, I will restrict myself to the data connected to one particular lesson (see below) which was analysed by 3 pairs of students and 4 individual students. Their list follows (pseudonyms are used). In parentheses, the students' study results are given, received as a weighted average of their marks from mathematical courses during their first 3 years of study at the Faculty (1 is the best mark): A – 1, B – (1, 2), C – higher than 2.

Pairs (video recordings, transcripts, written reflections): John (B) and James (C), Molly (A) and Mark (B), Lota (A) and Meg (A)

Individuals (written reflections): Zina (B), Jack (B), Lance (C), Paul (B).

The students were told that they would be given a recording of an Australian mathematics lesson from Grade 8 from TIMSS Video Study 1999 and that the topic was the division of a quantity in a given ratio. The lesson in question was used on purpose – I believed that there was a lot to be noticed and, on the other hand, to be missed. Moreover, I supposed that the students would feel more interested in a foreign lesson.

The students were also given the teacher's preparation and self-reflection (written by her after viewing the video recording of her own lesson) and pupils' worksheets. They watched the video in English with the Czech subtitles. Pairs of students could write a reflection if they wanted (to complement their discussion while viewing the video), while the individuals were obliged to write a reflection. It was an unstructured reflection. They were told that they could write whatever they wanted or felt important.

In the data analysis, I had in mind six key moments which, in my opinion, were important from the point of view of the mathematical content and its presentation in the lesson. Their short description together with my perception from the lesson in question follows.

1. Manipulation. The division of a quantity in a given ratio is introduced using the model of cubes and boxes. This should help pupils to build an image of the whole process.

Comment: The pupils first work with cubes and create ratios such as 1 : 2, 5 : 8, etc. Then they work with empty boxes. When solving problems, they are asked to first model the situation and only then to calculate.

2. Block versus box. While blocks are counted as separate individuals, the empty boxes stand for a certain unknown number (or amount). Each must contain the same number (or amount). The letters a , b in the ratio $a : b$ stand not only for a certain number of things but also for groups of (or boxes full of) things.

Comment: The pupils are asked to imagine that there is a certain number of things (or a certain amount of money) in each box and to solve problems such as divide 210 dollars in the ratio of 2 : 5. The teacher often refers to the boxes and asks, e.g., how many things are in one box (when looking for a unit quantity). The pupils are asked to actually move boxes on their desk to the left or right according to the ratio.

3. Relationship between the ratio and quantity. In order for the division of a quantity in a given ratio to have integer answers, the whole quantity must be divisible by a unit quantity.

Comment: The teacher wants the pupils to think of their own story problems with ratios but she realises that there might be a problem if they do not see the relationship in question. She probably thinks that a non-integer answer would add to the cognitive burden and unnecessarily lead the pupils away from the idea of ratios. She, therefore, asks them whether they see this relationship. The pupils seem not to know what to do so the teacher points to the already solved ratios and to the numbers which she deliberately chose. When one girl says that the quantity must be “easily divisible”, the teacher picks her idea up and explains the relationship. The question remains whether this important idea could have been found by the pupils themselves when trying to think up (and solve) their own story problems.

4. Simplifying ratios. We know from the teacher’s reflection that the pupils should know about simplifying ratios from the previous lesson.

Comment: In the classwork, the need to simplify ratios does not arise. When the pupils work on posing problems, the teacher moves around and check them. A pupil has a ratio of 4 : 6 and the teacher says that “it would be better as 2 : 3, because we like simple ratios”. After a minute, she can see another pupil with a ratio of 6 : 3 and this time, she does not mention this possibility. There is no comment on simplifying ratios later during the classwork.

5. Two methods. The unitary method is based on finding the unit and then multiplying it by the numbers in the ratio. The fraction method enables us to calculate each share by multiplying the quantity by a fraction, i.e., given $a : b$, quantity q , then the first share is $a / (a + b)$ times q , etc.

Comment: The teacher demonstrates the fraction method on 3 examples written on the board and previously solved by the unitary method. In my opinion, it is rather quick and the pupils do not have any opportunity to actually try it. No wonder that, when asked to vote which method they prefer, they vote for the unitary method (which they used throughout the lesson).

6. Pupils' problem posing (or PP). When asked to pose their own problems, pupils are encouraged to think about the matter more deeply and the teacher can assess to what extent they understand it and where the problems lie. It is usually motivating for them. In my opinion, it is advisable to ask pupils to solve the problems, too, as it makes them focus on the mathematical part as well as the context.

Comment: The teacher asks the pupils to think of their own question with a ratio and then talks about making a "story". This might have contributed to most pupils producing a story without a question.

The problem posing activity enabled the pupils to grasp the difference between the two types of task: to look for a ratio, and to divide a quantity in a given ratio. The pupils apparently mixed the two types together and the teacher became aware of this fact only on the basis of this activity (based on her reflection).

The above six key moments were the springboard from which I started the data analysis. All the data were uploaded to the software *Atlas.ti* as separate documents. The documents were coded first using the six items (their names were used as the code names) and then open coded in the sense of Strauss and Corbin (1998), analysing a whole sentence or a paragraph rather than line-by-line because, especially in the pair experiment, one idea was spread in students' several utterances.

During the coding process, five more codes emerged as important for some students. Thus, I tracked them in all the reflections.

7. Involvement of pupils. It shows to what extent the pupils are actively involved in the construction of new knowledge (as far as we can say that from the video recording only!) and other mathematical work in the lesson. It involves two free codes: Pupils' activity and Pupils' understanding.

Comment: It is difficult to generalize, but at many stages of the lesson I have the impression that the pupils are not given enough time to think the questions over and find the solutions themselves, but rather that they are given the solutions by the teacher immediately. They are almost never encouraged to explain their thinking or strategies, but rather the teacher offers the explanation and corrects their mistakes.

8. Elaboration – consequences. It involves the elaboration of the observed teaching practice in terms of its possible consequence for the pupils' understanding or for the flow of the lesson. (See below for examples.)

9. Elaboration – their teaching. It concerns the elaboration of the observed teaching practice in terms of its possible connections with the students' future teaching practice.

10. Alternatives. It means suggesting an alternative action to what actually happened.

11. General perception. It means a general perception of the lesson based on the codes Chaotic versus calm, Teacher's personality, Teaching method, Appraisal / Criticism of the teaching practice, Classroom environment, Empathy for the teacher.

PRELIMINARY RESULTS

The results will be first presented in the form of two tables and then discussed.

Explanation: “+” – the student mentioned the item (it will sometimes be briefly given in what way), “x” – it did not appear. T stands for the teacher, Ps for pupils. In item 7, | means a reference to pupils' potential understanding. In item 10, | means a reference to the mathematics of the lesson, * to the organisation of the lesson.

Pairs	John + James	Molly + Mark	Lota + Meg	
1. Manipul.	+ no elaboration	+ “good idea”	+ good for Ps, they “see it”	
2. Block/box	+ consider them the same	x	+ see the difference	
3. Ratio vs. quantity	x	x	+ T should simply say it as a rule	
4. Simplify	x	x	x	
5. Two methods	x	+ very quick, voting nonsense	+ not 2 methods but a different notation, T should've stressed the common properties; voting nonsense	
6. Pupils'PP	+ consider it nonsense	+ good, story vs. task	+ good	
7. Involv. of pupils Ps' unders.	+ T shows the methods, explains where there is a mistake	+ pupils are only passively involved	x 	
8. Conseq.	x	+ PP – T can see how Ps understand	x	
9. Teaching	x	+ “I tried to imagine myself in T's shoes.”	+ “What to do with quick pupils?”	
10. Altern.	****	** * *	**** *	
11. General perception	chaotic, no system, T lacks organis. skills, no bird's view eye, doesn't care what Ps do, doesn't understand what Ps say	T is calm, does not get angry, no emotions, Ps comfortable with the work	T changes activities	
Individuals	Zina	Jack	Lance	Paul
1. Manipul.	x	+ good for Ps, but not enough time	+ good for	x

		for solution	understanding	
2. Block versus box	x	x	+ can see the difference	x
3. Ratio vs. quantity	+ it is the key question	+ thinks that Ps found it	+ Ps should have found it in PP	x
4. Simplify	x	x	x	x
5. Two m.	x	x	x	+ good
6. Pupils' PP	+ good	+ good	+ good	+ good, but above Ps' abilities
7. Involv. of pupils Ps' under.	+ Ps discover the knowledge for themselves	+ not enough time for own discovery of knowledge	+ Ps not involved enough	
8. Conseq.	+ PP – good for cooperation, application of math. in reality, motivating	+ PP – good for Ps' understanding	x	+ PP – breaks stereotype of problem solving
9. Teaching	x	x	x	+ “I have tried PP with pupils.”
10. Altern.	*	x	***	x
11. General perception	T leads Ps from concrete to abstract knowledge, towards relationships, waits till Ps find knowledge themselves	a calm lesson, probably too calm	too noisy, more discipline is needed	x

If the item is missing from the students' reflections, we can presume that they did not notice it or did not attribute any importance to it.

DISCUSSION OF RESULTS

Discussion of individual items

Manipulation was seen as important for the mathematical content of the lesson 4 times out of 7, however, only Lance and one pair could see the difference represented by blocks and boxes. Despite the teacher's frequent reference to it, John and James consider them the same and from their discussion we can infer that they are lost in the mathematical part of the activity. This aspect, which I see as important for the development of pupils' knowledge of ratio, was not mentioned at all 4 times out of 7.

The question about the relationship between the quantity and ratio was noticed 4 times out of 7 but another mathematical item about simplifying ratios was not addressed at all. The “two method” item was only mentioned 3 times and in 2 of

them, the vote was rejected as nonsense on the grounds that the pupils did not have time to actually try it.

Pupils' problem posing was commented on by all students and mostly judged positively. John and James have another view but they do not give any reason for it.

The students made interpretative comments, too. In most cases they commented upon the problem posing activity and its advantages. The reason why they actually thought about this type of activity deeper might be that it was novel for them. In Czech schools, problem posing by pupils is quite rare. Only 3 times, the students elaborated a little on what they saw from the point of view of their (future) role as teachers.

In many cases, the students suggested alternative actions for both the organisational and mathematical aspects of the lesson, often after a critical remark about what actually happened in the lesson.

Comparison of reflections

Two pairs stand out in the quality of reflection. At the one end of the spectrum, John and James made a lot of critical remarks but only suggested alternatives to the organisational aspects of the lesson. They probably did not give much thought to the mathematical part (except for frequent comments at the beginning of the lesson that "it makes no sense what the teacher does") and did not think about the types of tasks the teacher used. Their dialogue is mainly descriptive without any elaboration of what the event might mean. They are extremely critical about the lesson and, of the 10 students are the only ones to make critical comments on the personality of the teacher and her skills.

At the other end, Meg and Lota also did not understand at first where the teacher was heading with modelling but after much effort and discussion, they grasped it. They comment on nearly all mathematical items. They make the most references to pupils' possible understanding and suggest the most alternatives, most of which are for the mathematics of the lesson. Their level of reflection is deeper than the boys' one. I believe that, among others, their content knowledge might have influenced this difference. While Meg and Lota have A's, John has B and James has C. Their insufficient knowledge of mathematics and thus inability to see where the teacher was leading the pupils might have influenced their appraisal of the lesson.

Finally, quite surprisingly for me, there are opposing views concerning the same items. While Jack believes that the pupils discovered the relationship between the ratio and quantity themselves, Lance suggests otherwise as he points out that the pupils should be allowed to discover it when posing problems.

The involvement of pupils in the development of knowledge is also differently judged. While Molly and Mark, and Lance (and indirectly also John and James) think that the pupils were rather passive and the teacher did the explanation, Zina believes that the pupils were actively involved and Lance suggests that the teacher wants them to be more involved but that allows them little time.

The general impression from the lesson differs widely. John and James, quite understandably considering the above, see the lesson as chaotic, with no system, and have little empathy for the teacher. Molly and Mark as well as Jack consider the lesson calm and the pupils comfortable with the work. For Lance, there is little discipline and too much noise in the lesson.

It might have been illuminating to let the students discuss their opposing views to see on what grounds they put their claims. As it is, we have little information as to the reasons for the discrepancies.

Star and Strickland (2008) also studied preservice teachers' uninfluenced responses to a lesson on video, thus it seems appropriate to compare their results with mine. They let the students watch the video and take notes and then asked them questions concerning 5 aspects of the lesson which they should answer based on their memory and notes. (They did not look, however, into how the students interpreted the events.) The five aspects were: Classroom environment, Classroom management, Communication, Tasks (refer to the activities pupils do in the class; it includes my code Pupils' problem posing), Mathematical content (it includes my codes Manipulation, Block versus box, Relationship between the ratio and quantity, Simplifying ratios, Two methods). The first three dimensions are not among my codes as the students did not mention them. My remaining codes concern interpretation and, as such, cannot be put into the five categories.

Star and Strickland (*ibid*) found that without any training, the investigated student teachers were good observers of Classroom management, quite attentive to the category of Tasks and did least well on Classroom environment (in my study, the students hardly mentioned it, too) and Mathematical content. The authors say that "preservice teachers largely did not notice subtleties in the ways that the teacher helped students think about content" and "the mathematics of the lesson and the students' understandings of that mathematics were not noticed [...], either in the initial or in the second viewing of the video" (p. 118). This is echoed in the preliminary findings of my study where the mathematics of the lesson was rarely attended to.

FUTURE WORK

In order to answer my research questions, more analysis is needed. While doing the open coding, the elements of the following stage of analysis, that is axial coding, gradually emerged and some categories began to be assembled. Clearly, some codes are connected with the mathematics in the presented lesson only (e.g., Two methods) while others are more general (e.g., Alternatives). Some codes are closely tied (e.g., Alternatives and Elaboration – their teaching). In my further work, the various types of data for different lessons will be coded. It is assumed that during this process some categories will emerge which would help me to concentrate on some of them not in one type of data or in the data tied to one particular lesson, but more generally. It may also be valuable to compare reflections received from individuals and those from

pairs. Does a discussion between students influence the depth of their considerations? This will also be the focus of my future work.

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THE MATHEMATICAL PREPARATION OF TEACHERS: A FOCUS ON TASKS

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In this article we elaborate a conceptualization of mathematics for teaching as a form of applied mathematics (building on Bass's idea of characterizing mathematics education as a form of applied mathematics) and we examine implications of this conceptualization for the mathematical preparation of teachers. Specifically, we discuss issues of design and implementation of a special kind of mathematics tasks whose use in teacher education is intended to promote mathematics for teaching.

The notion of *Mathematics for Teaching (MfT)* (Ball & Bass, 2000) describes the mathematical content that is important for teachers to know and be able to use in order to manage successfully the mathematical issues that arise in their practice. According to Ball and Bass (2000), this specialized kind of mathematical knowledge, referred to as *Mathematical Knowledge for Teaching (MKfT)*, is important for solving the barrage of “mathematical problems of teaching” that teachers face as they teach mathematics: offering mathematically accurate explanations that are understandable to students of particular ages, validating student assertions, etc.

In this article, we focus on the following research question: What kind of learning opportunities might mathematics teacher education programs design to effectively support the development of prospective teachers' MKfT? To address this question, we elaborate a conceptualization of MfT as a form of applied mathematics and probe the implications of this conceptualization for the mathematical preparation of teachers, with particular attention to the nature of mathematics tasks that might be important for use in mathematics (content) courses for prospective teachers. To exemplify the constructs we discuss in the article, we use data from a research-based mathematics course for prospective elementary teachers in the United States.

CONCEPTUALIZING MATHEMATICS FOR TEACHING AS A FORM OF APPLIED MATHEMATICS

In thinking about the problem of teachers' mathematical preparation, we found useful Bass's (2005) suggestion of viewing mathematics education as a form of applied mathematics: “[Mathematics education] is a domain of professional work that makes fundamental use of highly specialized kinds of mathematical knowledge, and in that sense it can [...] be usefully viewed as a kind of applied mathematics” (p. 418). Given that mathematics education makes use of specialized knowledge from several other fields in addition to mathematics (psychology, sociology, linguistics, etc.), we propose that the characterization “form of applied mathematics” be used to refer specifically to the mathematical component of mathematics education, notably MfT.

The conceptualization of MfT as a form of applied mathematics calls attention to the domain of application of MfT (i.e., the work of mathematics teaching) and the

specialized nature of “mathematical problems of teaching” (Ball & Bass, 2000). In particular, the conceptualization has two important and interrelated implications for the mathematical preparation of teachers, which are aligned with existing research and theoretical accounts in the area of MKfT.

First, the conceptualization implies that the mathematical preparation of teachers should take seriously the idea that “there is a specificity to the mathematics that teachers need to know and know how to use” (Adler & Davis, 2006, p. 271). This idea relates to broader epistemological issues about the situativity of knowledge (e.g., Perressini et al., 2004) and to research findings that different workplaces require specialized mathematical knowledge by their practitioners (e.g., Hoyles et al., 2001).

Second, the conceptualization implies that the mathematical preparation of teachers should aim to “create opportunities for learning subject matter that would enable teachers not only to know, but to learn to use what they know in the varied contexts of practice” (Ball & Bass, 2000, p. 99). In other words, it underscores the importance of the development of a “pedagogically functional mathematical knowledge” (ibid, p. 95), which can support teachers to solve successfully mathematical problems that arise in their work. The characterization of MKfT as “pedagogically functional” helps clarify further the meaning we assign to the term “applied mathematics” in the proposed conceptualization of MfT. Specifically, our use of this term refers to mathematics that is (or can be) *useful for* and *usable in* mathematics teaching (the domain of application), and thus, important for teachers to know and be able to use when they teach mathematics (i.e., when they function in the domain of application).

Acceptance of the conceptualization of MfT as a form of applied mathematics necessitates that mathematics courses in teacher education design opportunities for prospective teachers to learn and use mathematics from the perspective of a *teacher of mathematics*. How might these opportunities be designed in teacher education?

Given the central role that mathematics tasks can play in individuals’ learning experience in classrooms, we considered fruitful to begin to address the question above (which is a reformulation of our research question) by conceptualizing a special kind of mathematics tasks that we call *Pedagogy-Related mathematics tasks* (*P-R mathematics tasks*). These tasks are intended to embody essential elements of MfT as a form of applied mathematics and support mathematical activity that can enhance the development of prospective teachers’ MKfT.

“PEDAGOGY-RELATED MATHEMATICS TASKS”: A VEHICLE TO PROMOTING MATHEMATICAL KNOWLEDGE FOR TEACHING

Feature 1: A primary mathematical object

Like all other kinds of mathematics tasks, P-R mathematics tasks have a *primary mathematical object*. This is intended to be the main focus of prospective teachers’ attention and to engage them in activity that is primarily mathematical (as opposed to pedagogical). The mathematical object of a P-R mathematics task can take different

forms such as validation of a conjecture or description of the mathematical relationship between two methods for obtaining the same mathematical result.

Feature 2: A focus on important aspects of MKfT

Like most other kinds of mathematics tasks used in mathematics courses for prospective teachers, the mathematical object of a P-R mathematics task relates to one or more mathematical ideas that have been suggested by theory or research on MKfT as being important for teachers to know (see, e.g., Stylianides & Ball, 2008). In our work with prospective teachers we pay special attention to such ideas that are also *fundamental* (Ma, 1999) and *hard-to-learn* for both students and teachers.

Feature 3: A secondary but substantial pedagogical object and a corresponding pedagogical space

The defining feature of P-R mathematics tasks is that they have a *secondary pedagogical object*. This object is substantial (i.e., it is an integral part of the task and important for its solution) and situates the mathematical object of the task in a particular *pedagogical space* that relates to school mathematics and, ideally, derives from actual classroom records. The pedagogical object and the corresponding pedagogical space of a P-R mathematics task help engage prospective teachers in mathematical activity from the perspective of a *teacher of mathematics*.

Consider for example a P-R mathematics task whose mathematical object is the development of a proof for a conjecture. The pedagogical object of this task could be a teacher's need that the proof be appropriate for the students in his/her class. The corresponding pedagogical space could be a description (scenario) of what the solvers of the P-R mathematics task might assume the students in the class to know in relation to mathematical content that is relevant to the task. Thus the solution of the task cannot be sought in a purely mathematical space, but rather in a space that intertwines content and pedagogy. As a result, the task can generate mathematical activity that is attuned to particular mathematical demands of mathematics teaching.

Next we discuss four points related to feature 3 of P-R mathematics tasks. First, the pedagogical object/space of a P-R mathematics task, and especially its connection to (actual) classroom records, can embody the ideas of "situativity of knowledge" and "pedagogical functionality" that we discussed earlier in relation to MfT as a form of applied mathematics. Specifically, the pedagogical object can support development of mathematical knowledge that is applicable in a particular context (pedagogical space) within the broader work of mathematics teaching.

Second, the pedagogical space of a P-R mathematics task determines to great extent what counts as an acceptable/appropriate solution to the task, because it provides a set of conditions with which a possible solution to the task needs to comply. This is important, because, almost always in teaching, a purely mathematical approach to a "mathematical problem of teaching" does not address adequately the different aspects of the pedagogical space in which the problem is embedded.

Third, given the complexities of any pedagogical situation, it is often impractical (if not impossible) to specify all the parameters of the situation that can be relevant to the mathematical object of a P-R mathematics task. This lack of specificity can be useful for teacher educators who implement P-R mathematics tasks with their prospective teachers: teacher educators can use the endemic ambiguity surrounding the pedagogical space in order to vary some of its conditions and create opportunities for prospective teachers to engage in related mathematical activities within the particular pedagogical space. The variation of conditions of the pedagogical space (and the mathematical activities that can result from this variation) can offer prospective teachers practice with grappling with the barrage of mathematical issues that arise (often unexpectedly) in almost every instance of a teacher's practice.

Fourth, the pedagogical object/space of a P-R mathematics task have the potential to motivate prospective teachers' engagement in the task by helping them see and appreciate why the mathematical ideas in the task are or might be important for their future work as teachers of mathematics. According to Harel (1998), “[s]tudents are most likely to learn when they see a need for what we intend to teach them, where by ‘need’ is meant intellectual need, as opposed to social or economic need” (p. 501; the original was in italics). In the case of prospective teachers, a “need” for learning mathematics may be defined in terms of developing mathematical knowledge that is useful for and usable in the work of teaching. By helping prospective teachers see a need for, and thus develop an interest in, the material that teacher educators engage them with, teacher educators increase the likelihood that prospective teachers will learn this material. This is particularly useful in relation to material that prospective teachers tend to have difficulty to see as relevant to their future teaching practices.

EXEMPLIFYING THE USE OF P-R MATHEMATICS TASKS IN A MATHEMATICS COURSE FOR PROSPECTIVE TEACHERS

General description of the course

The course was the context of a *design experiment* (see, e.g., Cobb et al., 2003) that we conducted over a period of four years and that aimed to develop practical and theoretical knowledge about ways to promote prospective teachers' MKFT. It was a three-credit undergraduate-level mathematics course for prospective elementary teachers, prerequisite for admission to the masters-level elementary teaching certification program at a large state university in the United States. It was the only mathematics content course in the admission requirements for the program,¹ and so it was designed to cover a wide range of mathematical topics. The students in the course pursued undergraduate majors in different fields and tended to have weak mathematical backgrounds. Also, given that the students were not yet in the teaching certification program, they had limited or no background in pedagogy.

¹ The students who are admitted to the teaching certification program take also a mathematics pedagogy course, but the focus of this course is on teaching methods.

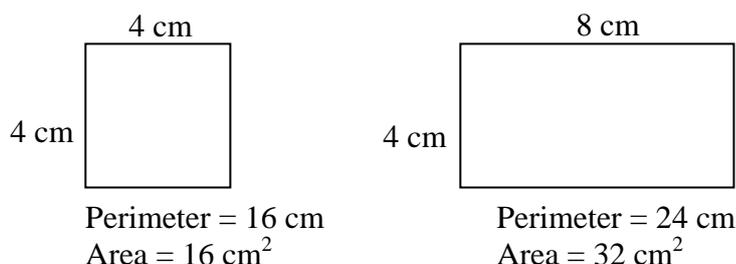
The most relevant aspect to this article of the approach we took in the course to promote MKfT is the design and implementation of task sequences that included both P-R mathematics tasks and *typical mathematics tasks*, which embody only features 1 and 2 of P-R mathematics tasks. A common task sequence in the course began with a typical mathematics task that engaged prospective teachers in mathematical activity from an adult's point of view. The P-R mathematics task that followed described some pedagogical factors that prospective teachers needed to consider in their mathematical activity. To satisfy feature 3 of P-R mathematics tasks about situating prospective teachers' mathematical activity in a pedagogical space, we used a range of actual classroom records such as video records or written descriptions (as in scholarly publications) of classroom episodes, excerpts from student interviews or textbooks, etc. Less frequently and when actual classroom records were unavailable, we used (similar to Biza et al., 2007) fictional but plausible classroom records.

An example of a task sequence and its implementation in the course

We illustrate the use of P-R mathematics tasks in the course with a task sequence that included a typical and a P-R mathematics task. To develop this and other task sequences in the course we followed a series of five research cycles of implementation, analysis, and refinement over the years of our design experiment. In this article we use data from the last research cycle that involved enactment of the course in two sections; these sections were attended by a total of 39 prospective teachers and were taught by the first author. Specifically, the data come from one of the two sections and include video and audio records of relevant classroom episodes, and fieldnotes that focused on prospective teachers' small group work.

The focal task sequence aimed to promote prospective teachers' knowledge about a possible relation between the area and perimeter of rectangles, with special attention to the ideas of generalization and proof by counterexample, which are considered important for elementary mathematics teaching (see feature 2 of P-R mathematics tasks in relation to Stylianides and Ball, 2008). The task sequence is an adaptation of an interview task used by Ma (1999) and developed originally by Ball (1988).

Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that as the perimeter of a rectangle increases, the area also increases. She shows you this picture to prove what she is doing:



1. Evaluate mathematically the student statement? (underlined)

2. How would you respond to this student?

Although question 1 refers to a student statement, it is essentially a typical mathematics task because the prompt asks prospective teachers to evaluate mathematically the statement, without asking (or expecting) them to take account of the fact that the statement was produced by a student. Question 2, on the other hand, is a P-R mathematics task because it introduces a student consideration that prospective teachers need to consider in their mathematical activity. The *mathematical object* of this P-R mathematics task is to evaluate mathematically the underlined statement, which is essentially what the prospective teachers were asked to do in question 1 (a teacher would need to know about the correctness of the statement before deciding how to respond to the student who produced it). The *pedagogical object* of the task is the teacher's need to respond to the student who produced the statement. The *pedagogical space* is the (fictional) scenario in the task with a student announcing enthusiastically to the teacher a mathematical "discovery," which was supported by a single example in the domain of the corresponding statement. Although an appropriate response to question 1 could say that the statement is false and provide a counterexample to it, an appropriate response to question 2 would need to include more than that. Specifically, from a pedagogical standpoint, it would be useful and important for the student's learning if the teacher did not just prove her statement false, but also helped her understand why the statement is false and the mathematical conditions under which the statement is true.

The prospective teachers in the course worked on the two questions first individually, then in small groups, and later in the whole class. The whole class discussion started with the teacher educator asking different small groups to report their work on the task, beginning with question 1 (all prospective teacher names are pseudonyms).

Andria: We said that it [the student statement] was mathematically sound because as you increase the size of the figure, the area is going to increase as well.

Tiffany: We thought the same, because as the sides are getting bigger... [inaudible]

Stylianides: Does anybody disagree? [no group expressed a disagreement]

Evans: I agree. [Evans was in a different small group than both Andria and Tiffany]

Stylianides: And how would you respond to the student?

Melissa: I think it's true but they haven't proved it for all numbers so it's not really a proof.

Andria: I think that you don't have to try every number [she means every possible case in the domain of the statement] to be able to prove it because if the student can explain why it works like we just did, like if you increase the length then the area increases. [pause]

Stylianides: Yeah, so it's impossible to check all possible cases [of different rectangles].

Meredith: I'd say that it's an interesting idea, and I'd see if they can explain why it works.

As the excerpt shows, all small groups believed that the student statement was true, but at the same time they realized that the evidence the student provided for her claim

was not a proof (see, e.g., Melissa's comment). As a result, the prospective teachers started to think how they could prove the statement and what they could respond to the student. For example, Andria observed that it would be impossible to check every possible case. Also, both Andria and Meredith pointed out that the student needed to explain why (i.e., prove that) the area of a rectangle increases as its perimeter increases. Yet, the teacher educator knew that the statement was false, and so he probed the prospective teachers to check more cases and see whether they could find an example where the student statement failed. All small groups found quickly at least one counterexample to the statement and concluded that it was actually false.²

The prospective teachers did not expect this intuitively "obvious" statement to be false, so they became motivated to work further on question 2. The teacher educator gave them more time to think about this question in their small groups. The excerpt below is from the whole class discussion that followed the small group work.

Natasha: We said that the way that they [the students] are doing it, where they're just increasing the length of one side, it's always going to work for them but if they try examples where they change the length on both sides that's the only way it's going to prove that it doesn't work all the time. So you should try examples by changing both sides.

Stylianides: What do you think about Natasha's response? Does it make sense? [the class nodded in agreement] So what else? What else do you think about this?

Evans: You can kind of ask them to restructure the proof so that it would work.

Stylianides: What do you mean by "restructure the proof"?

Evans: Like once they figure out that it doesn't work for all cases they could say it's still like... if they saw it and if they revise it like the wording or just add a statement in there that if they can come up with a mathematically correct statement...

Stylianides: Anything else? [no response from the class]
I think [that] both ideas [mentioned earlier] are really important. So when you have something [a statement] that doesn't work, then it's clear that this student would be interested to know more. For example, why it doesn't work or under what conditions does it work because, obviously, some of the examples that the student checked worked. [...]

Natasha and Evans proposed two related issues that the elementary teacher in the task scenario could address when responding to the student: why the statement is false and the conditions under which the statement would be true. Based on our planning for the implementation of the task, the teacher educator would raise these issues anyway, because, as we explained earlier, a teacher response to the student that would consist only of a counterexample to the statement would be mathematically sufficient but pedagogically inconsiderate. The fact that the two issues were raised by prospective

² The prospective teachers had opportunities earlier in the course to discuss the idea that one counterexample suffices to show that a general statement is false.

teachers instead of the teacher educator is noteworthy, because Natasha and Evans had no teaching experience and also the issues they raised were requiring further mathematical work for themselves and the teacher education class. Take for example Evans's contribution, which raised essentially the following new mathematical question: Under what conditions would the statement be true? It is hard to explain what provoked Natasha and Evans's contributions, but we hypothesize that the pedagogical object/space of the P-R mathematics task played an important role in this. Specifically, we hypothesize that the need to respond to a false but plausible student statement made the prospective teachers think hard about related mathematical issues and how to "unpack" them in pedagogically meaningful ways (Ball & Bass, 2000; see also Adler & Davis, 2006).

Following the summary of the two issues as in the previous excerpt, the teacher educator engaged the prospective teachers in an examination of the conditions under which the student statement would be true. A more detailed discussion of the prospective teachers' work on the task sequence is beyond the scope of this article.

To conclude, our discussion in this section exemplified the idea that the application of mathematical knowledge in contextualized teaching situations can be different than its application in similar but purely mathematical contexts. Although the mathematical objects of the typical and P-R mathematics tasks in the sequence were the same (namely, the mathematical evaluation of a statement about a possible relation between the area and perimeter of rectangles), the pedagogical space in which the P-R mathematics task was embedded changed what could count as an appropriate solution to it, thereby generating mathematical activity in a combined mathematical and pedagogical space.

CONCLUDING REMARKS

Although the primary object of P-R mathematics tasks is mathematical, their design, implementation, and solution require some knowledge of pedagogy. This requirement derives primarily from the pedagogical objects of P-R mathematics tasks, which, although secondary to the tasks, determine to great extent what counts as acceptable/appropriate solutions to the tasks and influence the mathematical activity (to be) generated by the primary objects of the tasks. For example, the design of the P-R mathematics task that we discussed earlier used knowledge about a common student misconception regarding the relation between the area and perimeter of rectangles. Furthermore, successful implementation and solution of this task required appreciation of the pedagogical idea that a mere counterexample might be a limited teacher response to a flawed but plausible student statement.

The pedagogical demands implicated by the design, implementation, and solution of P-R mathematics tasks make it reasonable to say that instructors of mathematics courses for prospective teachers need to have, in addition to good knowledge of mathematics, knowledge of some important pedagogical ideas. This requirement might be hard to fulfill in contexts such as the North American where mathematics

courses for prospective teachers are typically offered by mathematics departments and are taught by (research) mathematicians. However, if such knowledge is agreed to be essential for teaching MfT to prospective teachers, then the field of mathematics teacher education needs to find ways to support the work of instructors of mathematics courses for prospective teachers. One way might be to offer instructors access to what we may call *educative teacher education curriculum materials*. This is the teacher education equivalent of the notion of educative curriculum materials, i.e., curriculum materials that aim to promote teacher learning in addition to student learning at the school level (see, e.g., Davis & Krajcik, 2005).

The pedagogical aspects of P-R mathematics tasks raise also the following question: Would it make sense to promote MKfT in mathematics courses designed specifically for prospective teachers, or would it make more sense to promote it in combined mathematics/pedagogy courses, which, by definition, pay attention to both pedagogical and mathematical issues? The idea of promoting MKfT in combined mathematics/pedagogy courses may be attractive to some given the potential of P-R mathematics tasks to intertwine mathematics and pedagogy. Yet a possible decision to eliminate mathematics courses designed specifically for teachers in favor of combined mathematics/pedagogy courses might create different kinds of problems. In their examination of different types of tasks in formal assessments used across a range of mathematics teacher education courses in South Africa, Adler and Davis (2006) reported the concern that in combined mathematics/pedagogy courses the mathematical and pedagogical objects lose their clarity and that evaluation in these courses tends to condense meaning toward pedagogy.

The conceptualization of MfT as a form of applied mathematics that we elaborated in this article highlights the idea that, irrespectively of whether MfT is promoted in specialized mathematics courses or combined mathematics/pedagogy courses, prospective teachers' learning of MfT should not happen in isolation from pedagogy. P-R mathematics tasks can facilitate the integration of mathematics and pedagogy in prospective teachers' learning: although these tasks make mathematics the focus of prospective teachers' activity, they situate this activity in a substantial pedagogical space that shapes and influences the activity. Future research may explore ways in which to facilitate the integration of mathematics and pedagogy from the opposite direction, i.e., by making pedagogy the focus of prospective teachers' activity and having mathematics play a secondary but substantial role in this activity. Towards this end, one can reverse the relative importance of mathematical and pedagogical objects in P-R mathematics tasks to coin the twin notion of *Mathematics-Related pedagogy tasks*. Specifically, these tasks can be defined to have a primary pedagogical object (with a corresponding pedagogical space) and a secondary but substantial mathematical object, and can be used to generate activity that is predominantly pedagogical (as opposed to mathematical in P-R mathematics tasks).

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PROBLEM POSING AND DEVELOPMENT OF PEDAGOGICAL CONTENT KNOWLEDGE IN PRE-SERVICE TEACHER TRAINING

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The paper focuses on problem posing as the possible method leading to development of pedagogical content knowledge of mathematics education in pre-service training of primary school teachers. In the background there is our belief that this knowledge is of utter importance for quality of the education process. Using samples of (a) problems posed by teacher students, (b) students' assessment of the problems posed, (c) students' opinions on the significance of "problem posing" in teacher training, we will demonstrate how we employed problem posing in pre-service teacher training. We start from the belief (proved in our previous work) that an analysis of the posed problems is a good diagnostic tool; it gives the opportunity to discover the level of understanding as well as the causes of misconceptions and errors.

Keywords: mathematics education, teacher training, content knowledge, problem posing

INTRODUCTORY REMARKS: MATHEMATISATION OF THE SOCIETY AND MATHEMATICAL LITERACY

On many different occasions we come across the signs of an increasing importance of mathematics in contemporary life, the opinion that the society is being "mathematised". We must understand mathematics if we are to be able to understand the world that surrounds us. That is why the need of mathematical literacy is more and more emphasized. These trends also impact the focus of the research in the field of didactics of mathematics (e.g. the central topic of PME 30 conference in 2006 was "Mathematics in the centre").

We understand mathematical literacy as functional. It begins with the ability to understand a mathematical text, the ability to recall mathematical terms, procedures and theory, to master the necessary mathematical apparatus and with the ability to apply it, to solve problems. However, in our view to be mathematically literate also means to "understand mathematics", to perceive it as an abstract discipline. Development of mathematical literacy triggers perfection of the ability to reason, of critical thinking, it teaches how to apply mathematics efficiently. To be functionally mathematically literate means to see the mathematics that surrounds us; to see the questions and problems arising both from real and mathematical situations. In order to educate mathematically literate pupils we need professionally competent teachers.

In our previous work we have been focusing on the potential of a qualified

pedagogical reflection and we have showed that it is one of the possible ways of development of professional competence of primary school teachers (Tichá, Hošpesová, 2006). In this paper we show that problem posing represents another possible way. We also show the potential of problem posing in diagnosis of the teacher-students' subject didactic knowledge.

THEORETICAL FRAMEWORK

Professional competence and content knowledge

The calls for development of mathematical literacy make demands on professional competences of the teacher. In our previous research, especially the need for a good level of *subject didactic competence* appeared very strongly, i.e. the knowledge of mathematical content and its didactic elaboration as well as its realization in school practice (Tichá, Hošpesová, 2006). It corresponds with the following generally accepted Shulman's idea: if teaching should become a profession, it is necessary to aim at creating a *knowledge base for teaching* which encapsulates, in particular, *subject-matter content knowledge, pedagogical content knowledge, and curriculum knowledge* (Shulman, 1986). It is the knowledge of mathematical content that most authors place in prominent positions on their lists of items of knowledge required from teachers (e.g. Bromme, 1994; Harel, Kien, 2004). The need of solid niveau of subject didactic competence is extremely demanding for primary school teachers. Especially if we realize that the content of mathematical education at primary school level is a system of propaedeutic to many fields (arithmetic, algebra, geometry, ..., functions, statistics, ...). Yet these teachers are not specialists in the subject – on the contrary, they must master many more subjects than mathematics.

What is often emphasized is the need to create an “amalgam” of the components of the teacher's education. “The two basic elements of teacher knowledge are mathematics and pedagogical knowledge. When these two elements are separated and remain at a general level, mathematics teaching does not share the characteristics of ... a good teaching. The blending of mathematics and pedagogy is necessary for developing mathematics knowledge for teaching.” (Potari et al., 2007, p. 1962). In other words “... mathematical experiences and pedagogical experiences cannot be two distinct forms of knowledge in teacher education.” (Potari et al., 2007, p. 1963).

Problem posing as a way to refinement of competences

Opinions on the employment of problem posing

Our existing experience indicates that one of the beneficial ways of improving subject didactic competences of pre-service teachers of mathematics is development of the ability to pose problems (and the related activities). Already Freudenthal and Polya emphasize the significance of activities aiming at problem posing as a part of mathematics training. The same need is referred to by many others (Silver, Cai, 1996; English, 1997; Pittalis et al., 2004 etc.). Apart from “problem solving” (in the sense

of “learning mathematics on the basis and through problem solving”) they emphasize the need and significance of development of the ability to pose problems. There is an agreement among many authors that “problem formulating should be viewed not only as a goal of instruction but also as a means of instruction. The experience of discovering and creating one’s own mathematics problems ought to be a part of every student’s mathematics education” (Kilpatrick, 1987, p. 123).

Teacher educators show and stress links between problem posing and problem solving, and problem posing and mathematical literacy (competence). That is why stress is on the inclusion of activities in which students generate their own problems in mathematics education. At the same time most literature points out that the treatment of issues regarding problem posing has by no means been satisfactory so far. For example, Christou *et al.* (2005) bring forward the fact that “little is known about the nature of the underlying thinking processes that constitute problem posing and schemes through which students’ mathematical problem posing can be analysed and assessed” (p. 150). And Crespo (2003, p. 267) adds “... while a lot of attention has been focused on teacher candidate’s own ability to solve mathematical problems, little attention has been paid to their ability to construct and pose mathematical problems to their pupils.”

Problem posing in the frame of grasping of situations

We started to pursue the issue of problem posing while studying the process of grasping situations (Koman & Tichá, 1998). What we understand by grasping situations is the search for questions and problems growing from a mathematical or “non-mathematical” situation, i.e. also problem posing. We define problem posing similarly to a number of other teacher educators as (a) creation of new problems or (b) re-formulation of a given problem, e.g. by “loosening the parameters of the problem” (by modifying the input conditions), by generalization, on the basis of the question “What if (not)?”, etc. The problem may be worded or re-worded either before its solution or during the solving process or after it. We perceive the process of problem solving as a dialogue of the solver with the problem, we ask: How to begin? How to continue at the point reached? The solver reacts to the “behaviour, response of the problem”, chooses a particular strategy, creates an easier problem, changes the conditions of the assignment to be able to continue.

Our experience from work with teacher students (and also with 10-15 year old students) confirms that their effort to pose problems guides them to deeper understanding of mathematical concepts and development of their mathematical and general literacy. Problem posing enriches both the teaching and the learning.

TEACHER STUDENTS AND PROBLEM POSING (INVESTIGATION)

The focus of the investigation: goals and questions

In our ongoing research we look for the ways leading to development and refinement

of professional competences of both teacher students and in-service teachers. We try to show if and to what extent “problem posing” and “the level of subject didactical competence” and also “mathematical knowledge” influence each other, i.e. in presently research we look for answers to the following questions: How rich knowledge base (general as well as specific, mathematical) is needed for proficiency in problem posing? How does systematic application of problem posing contribute to development of subject didactical competence / mathematical knowledge?

The topic of the investigation: translation between representations of fractions

We believe that problem posing can be regarded as a translation between representations e.g. as posing problems that correspond to a given calculation (Silver, 1994). The incentive to this focus was investigations that confirm the great significance of utilization of various modes of representation for the development and deepening of the level of understanding. Many authors (see e.g. Janvier, 1987; Tichá, 2003) stress that the level of understanding is related to the continuous enrichment of a set of representations and emphasize the development of the student’s capability of translation between various modes of representation.

One of the key topics of mathematical education in primary school is the foundation of the base for understanding the relations between a part and the whole. In the process of division of the whole into equal parts, the preconception of the concept of fraction is formed. The concept of fraction is one of the most difficult concepts in mathematics education at primary school level. The subject matter is difficult not only for pupils and teacher students but often also for in-service teachers who face problems regarding both the mathematical content and its didactic treatment. That is why we paid so much attention to this topic in teacher training. The core of our work lay in the construction of the concept of fractions and in posing problems with fractions. We focused on formation of preconceptions and intuitive perception of fractions, on problem solving and the potential of problem posing.

The procedure and findings of the investigation

The stress in the course of didactics of mathematics for primary school teacher students was continuously on problem posing, thus on the development of the students’ proficiency in problem posing (the seminar was attended by 24 teacher students). One of the components of the work in the course was realization of an investigation whose aim was to show the students that problem posing can also be employed as a diagnostic tool, thus which on the basis of the problems posed it is possible to investigate the level of understanding as well as the obstacles in understanding and misconceptions.

The investigation was carried out in several steps: posing problems corresponding to a given calculation; individual reflection on the posed problems; joint reflection on a chosen set of the posed problems; evaluation of the activity “problem posing”.

Posing problems corresponding to a given calculation

The students were assigned the task: *Pose and record such three word-problems to whose solution it is sufficient to calculate $1/4 \cdot 2/3$.*

The problems were posed during work within one of the last seminars. What is satisfactory is the immediate finding that problem posing competence can be developed in appropriate conditions; teacher students who attended the course in which stress was put on the development of proficiency in problem posing were able to pose several problems. On the contrary students who came in contact with problem posing more or less haphazardly were not able to pose any problems if asked to do so. Some of the latter even did not understand what the point of the activity was and refused to pose any problems – in their opinion they should only solve such problems that were assigned to them and had been formulated by somebody else. The same can be observed in mathematics education at schools.

Reflection on the posed problems

A database of the posed problems was formed (without giving the author's name); each of the participants had access to the database. The participants of the course assessed the suitability and correctness of the posed problems that they had chosen themselves.

Then the lecturer selected a triplet of problems posed by one student. This triplet was then assessed and analyzed by all participants (the lecturer found this triplet of problems very interesting and asked their author for permission to use them in the subsequent work). The following step was joint reflection; joint assessment of individual problems, comparison and justification of opinions.

The following triplet of problems was chosen

1. *There was $2/3$ of the cake on the table. David ate $1/4$ of the $2/3$ of the cake. How much cake was left?*
2. *There was $2/3$ kg of oranges on the table. Veronika ate $1/4$ kg. How many oranges remained (kg)?*
3. *The glass was full to $2/3$. Gabriel drank $1/4$. What part of the glass remained full?*

In advance, the lecturer went through the problems with their author. It was only in this dialogue that the student began to consider correctness of the posed problems. (It is interesting that all students began to ponder over correctness of the posed problems only after being asked to do so. However, to our gratification the students generally found and corrected their mistakes themselves.) Let us quote an extract from the dialogue between the student (S) and the lecturer (L).

S: Here (*she points at problems 2 and 3*) I don't count a part of something, I reduce, take away. ... Actually I don't know what I meant by it.

L: What could you have meant?

S: Something like this (*she sketches an illustration – a circle*) – I divide in into

quarters and take away one. But, somebody could understand it that he drank a quarter of the glass. Well, I posed only one correctly. ... I should have checked.

L: How would you have checked?

S: Well, it seems I should have calculated it somehow. Or have somebody else to calculate it. Somebody who is better at it.

Samples of student assessment of the triplet of the posed problems

The **third problem** can be, according to some students, accepted on the condition that its wording is modified / supplemented; the given wording is regarded by many as confusing. However, the students only stated that it was confusing, they did not specify why or where.

The **first problem** was evaluated by a majority of the students positively. But the arguments of some of the evaluators reveal misconceptions: *If we have $\frac{2}{3}$ of a cake, we can eat $\frac{1}{4}$, but the denominators do not equate. If he ate $\frac{1}{3}$ out of the $\frac{2}{3}$, then they would. It would be possible in real life but it is not correct mathematically.*

This statement was illustrated by a picture (Fig. 1) and by the word problem: *There are $\frac{1}{4}$ of all pupils present in class A today and $\frac{2}{3}$ of all pupils present in class B. If we multiply the number of pupils from both classes present today, what will the result be?*

Another student wrote and claimed: *The problem is correct. David ate $\frac{1}{4}$ out of $\frac{2}{3}$ of a cake ... = $\frac{1}{4} \cdot \frac{2}{3} = \frac{1}{6}$ of the cake.*

However, the student supplemented his statement with a picture (Fig. 2) that testifies his wrong interpretation of the whole ($\frac{1}{4}$ and $\frac{2}{3}$ out of the same whole).

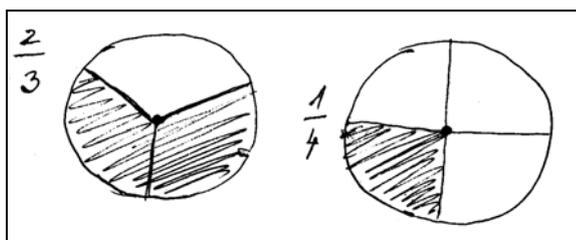


Fig. 1

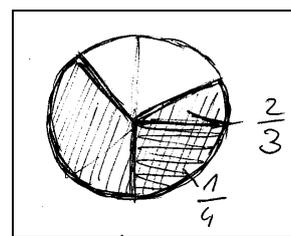


Fig. 2

When assessing the **second problem**, the students stated that this problem did not meet the condition from the assignment. However, their justification reveals that the conceptions of the evaluators themselves are also incorrect. Several illustrating examples of such evaluation follow.

- *Problem 2 is incorrect. There was $\frac{2}{3}$ kg of oranges = $\frac{2}{3}$ out of one (out of $\frac{3}{3}$). Veronika ate $\frac{1}{4}$ kg – but out of what? Out of $\frac{2}{3}$? of $\frac{1}{3}$?*
- *Number two is incorrect. From the total $\frac{2}{3}$ kg of oranges, she ate $\frac{1}{4}$ kg. She ate $\frac{1}{4}$ but it does not say out of what.*
- *The second word problem isn't correct; it's not a suitable problem. I am not*

interested in the number of oranges but their weight. This wording would require that the oranges should be cut to pieces.

What does the students' production show?

The subsequent joint reflection on the posed problems was of utmost benefit both to the participants and the lecturers. It enabled the students to become aware of their own weaknesses and it pointed to the teacher educators what they should focus on. Some of the findings follow.

The individual assessment and especially the following joint reflection show that many students do not have any idea of “what is in the background” of a particular simple calculation that they perform mechanically. They are not able to place it into a specific real life context. They did not pose problems in accord with the given calculation (what become transparent here are obstacles as far as multiplicative structure is concerned). A considerable proportion of the students posed additive problems corresponding with the calculation $1/4 + 2/3$.

What comes to surface is the students' difficulty as far as interpretation of fractions is concerned. The offered formulations show that when assessing the second problem they for example do not realize that they understand and interpret the fraction alternately as an operator and as quantity “*she ate 1/4 kg*” vs. “*She ate 1/4 but it does not say out of what.*”).

If the students were asked to pose more than one problem, we could observe stereotypical nature of these problems. Students often set their problems either only into discreet space (sets consisting of isolated elements) or only into continuous space. We could also observe a monotony of the motives: marbles and cakes (those are the models most often used in our textbooks).

What do the students think of problem posing?

The students were also asked to express their opinion on these, for them often unusual, activities. Let us present here several statements from individual reflections which illustrate how the students perceive “problem posing”.

- *I have problems with word problems. To pose a word problem on my own ,..., was extremely difficult. The difficulty is not in posing a problem, but in being able to solve it myself. It was toil and moil for me.*
- *What I personally found most difficult was to ask the question correctly, when I posed the third one, I could think of no further questions and that's why I only managed to pose the most banal ones.*
- *As soon as I came to understand the assignment of this task, I was immediately full of various ideas ... I was delighted because I love discovery ... that there were no limits.*
- *My first reaction was that of fear. However, I started from what first came to my mind – a simple problem and then I began to toy with it. It was very pleasant to look*

for and discover various combinations...

In the discussion the students indicated that it was easy to formulate a great number of problems of the same type but it was difficult to formulate a sequence of problems (cascade) of a growing difficulty or a problem for whose solution it was necessary to connect various pieces of knowledge or problems in which the role of the fraction alternates (i.e. various sub-constructs of fractions, ..., Behr et al., 1983).

CONCLUDING REMARKS ON THE BENEFIT OF PROBLEM POSING AND ON THE PERFORMED INVESTIGATION

Our experience from work with teacher students (and also from our long-term cooperation with in-service teachers) proves that poor level of pre-service mathematical training is pervasive and the flaws are difficult to overcome (Hošpesová & Tichá, 2005; Hošpesová *et al.*, 2007). Problem posing is in our opinion one of the beneficial possibilities.

The detection of a change in the “nature, climate” of work in the seminar

It seems to us that problem posing contributed to a change in approach to work in the seminar – the students gradually overcame their fears or anxiety and many of them gained self-confidence.

The character of the problems posed by the participants also changed. Before their participation in the seminar they posed simple, “textbook-like” problems, predominantly drill. The wording of the problems was often erroneous and the problems were uninteresting and demotivating from mathematical point of view. Many of the problems had no solution, despite the author’s intention.

After the course finished, a great variety of assignments of the problems could be observed (including charts, graphs etc.). There were also problems enabling different ways of solution and problems demanding explanation, reasoning, argumentation, allowing different answers with respect to the solver’s interests.

It turned out that it is not enough to demand from the students to pose a problem if one is to detect the quality of their understanding. It is crucial that it should be possible to assess the posed problems individually and/or collectively. This certifies the need to carry out joint reflection. If the authors are given the chance to assess the problems of each other, their insight into the situation deepens and the ability to handle reality, i.e. to “see mathematics in the world surrounding us” develops.

The benefit for students

The analysis of the posed problems makes the participants map the level of their own notions and concepts, understanding, various interpretations and makes them realize possible misconceptions and erroneous reasoning. It is an impulse for work on themselves (reeducation).

It was confirmed that the result of inclusion of problem posing into the curricula is

better approach to problem solving. It stimulates the use of various representations, construction of knowledge nets, development of creative thinking, improvement of attitude to mathematics and increase in self-confidence.

The benefit for teacher educators and researchers

From the point of view of teacher trainers and researchers problem posing provides an opportunity to get an insight into natural differentiation of students' understanding of mathematical concepts and processes and to find obstacles in understanding and misunderstandings that already exist.

Our belief that problem posing supplemented with reflection is the path to development and enhancement of subject didactical competence, i.e. of pedagogical content knowledge was confirmed.

Open questions

There still exist many questions which ask for deeper investigations, e. g. How can be the benefit that problem posing brings to its authors and the shift in their (pedagogical) content knowledge detected and measured? Which teacher's and/or student's competences are developed? What conditions are essential for introduction of problem posing? What help and guidance can be offered when incorporating problem posing?

NOTE

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SUSTAINABILITY OF PROFESSIONAL DEVELOPMENT

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This contribution addresses the issue of sustainable impact of professional development projects. It claims for widening the scope from evaluations of short-term effects to analyses of long-term impact. For that, the contribution discusses various types of effects and possible levels of impact. In particular, an overview concerning factors promoting the impact of professional development projects is provided. A case study that analysed the impact of an Austrian professional development project three years after its termination is introduced. The paper closes with further research questions that emerged from this study.

Key-words: professional development, sustainable impact, promoting factors, case study

INTRODUCTION

The quality of teaching and learning represents a recurring key issue of research. In particular, teachers are considered to be playing a central role when addressing this topic: „Teachers are necessarily at the center of reform, for they must carry out the demands of high standards in the classroom” (Garet, Porter, Desimone, Birman, & Yoon, 2001, p. 916). Various types of professional development projects are offered to support and qualify these teachers. The expected effects of such projects by both the facilitators and the participants are not only related to the professional development of individual teachers to improve teacher quality, but also to the enhancement of the quality of whole schools, regions and nations. The desideratum of all such projects providing teachers support and qualification is to enhance the learning of students. As Kerka (2003) states, “Funders, providers, and practitioners tend to agree that the ultimate goal of professional development is improved outcomes for learners” (p. 1). This strategy, to achieve change at the level of students (improved outcomes) by fostering change at the teachers’ level (professional development), is based on the assumption of a causal relationship between students’ and teachers’ classroom performance: “High quality professional development will produce superior teaching in classrooms, which will, in turn, translate into higher levels of student achievement” (Supovits, 2001, p. 81). Similarly, Hattie (2003) states, “It is what teachers know, do, and care about which is very powerful in this learning equation” (p. 2). Ingvarson, Meiers, and Beavis (2005) sum up: “Professional development for teachers is now recognised as a vital component of policies to enhance the quality of teaching and learning in our schools. Consequently, there is increased interest in research that identifies features of effective professional learning” (p. 2).

TYPES OF EFFECTS

The expected outcomes of professional development projects are not only focused on short-term effects that occur during or at the end of the project, but also on long-term effects that emerge (even some years) after the project's termination (Peter, 1996). Effects that are both short-term and long-term can be considered to be sustainable. So sustainability can be defined as the lasting continuation of achieved benefits and effects of a project or initiative beyond its termination (DEZA, 2005). As Fullan (2006) points out, short-term effects are “necessary to build trust with the public or shareholders for longer-term investments” (p. 120). Besides these short-term effects also long-term effects need to be considered; otherwise the result could be to “win the battle, [but] lose the war” (ibid.). Hargreaves and Fink (2003) state, “Sustainable improvement requires investment in building long term capacity for improvement, such as the development of teachers’ skills, which will stay with them forever, long after the project money has gone” (p. 3). Moreover, analysis of sustainable impact should not be limited to effects that were planned at the beginning of the project; it is also important to examine the unintended effects and unanticipated consequences that were not known at the beginning of the project (Rogers, 2003; Stockmann, 1992).

SUSTAINABLE IMPACT

Evaluations and impact analyses of professional development projects are formative or summative in nature; in most cases they are conducted during or at the end of a project and exclusively provide results regarding short-term effects. These findings are highly relevant for critical reflection of the terminated project and necessary for the conception of similar projects in the future. But apart from and beyond that, an analysis of sustainable effects is crucial: “Too many resources are invested in professional development to ignore its impact over time” (Loucks-Horsley, Stiles, & Hewson, 1996, p. 5). This kind of sustainability analysis is often missing because of a lack of material, financial and personal resources. “Reformers and reform advocates, policymakers and funders often pay little attention to the problem and requirements of sustaining a reform, when they move their attention to new implementation sites or end active involvement with the project” (McLaughlin & Mitra, 2001, p. 303). Despite its central importance, research on this issue is generally lacking (Rogers, 2003) and “Few studies have actually examined the sustainability of reforms over long periods of time” (Datnow, 2006, p. 133). Hargreaves (2002) summarises the situation as follows: “As a result, many writers and reformers have begun to worry and write about not just how to effect snapshots of change at any particular point, but how to sustain them, keep them going, make them last. The sustainability of educational change has, in this sense, become one of the key priorities in the field” (p. 120).

Zehetmeier (2008) summarises the literature concerning the sustainability of change and provides a case study of four teachers from one school, analyzing the impact of a professional development project three years after its termination. For that, he

develops a theoretical model which allows analysing both the various characteristics of the project, the different levels of impact, and the factors promoting or hindering the sustainability of impact (see also Zehetmeier, in prep.).

LEVELS OF IMPACT

When analyzing possible effects of professional development, the question of possible levels of impact arises. Which levels of impact are possible and/or most important? How can impact be classified? Recent literature provides some answers to these questions; the following levels of impact are identified (Lipowsky, 2004):

Teachers' knowledge: This level can be defined in different ways, for example, referring to content knowledge, pedagogical knowledge, and pedagogical content knowledge (Shulman, 1987), or attention-based knowledge (Ainley & Luntley, 2005), or knowledge about learning and teaching processes, assessment, evaluation methods, and classroom management (Ingvarson et al., 2005).

Teachers' beliefs: This level includes a variety of different aspects of beliefs about mathematics as a subject and its teaching and learning (Leder, Pehkonen, & Törner, 2002), as well as the perceived professional growth, the satisfaction of the participating teachers (Lipowsky, 2004), perceived teacher efficacy (Ingvarson et al., 2005) and the teachers' opinions and values (Bromme, 1997).

Teachers' practice: At this level, the focus is on classroom activities and structures, teaching and learning strategies, methods or contents (Ingvarson et al., 2005).

Students' outcomes: Many papers highlight that students' outcomes are related to the central task of professional development programmes: namely to the improved learning and knowledge of the students (Kerka, 2003; Mundry, 2005; Weiss & Klein, 2006).

Zehetmeier (2008) points out that the complexity of possible impact is not fully covered by this taxonomy. For example, results of an impact analysis in the context of the Austrian IMST project (Krainer, 2005, 2007) show that the project made impact also on students' beliefs or other – non participating – teachers' practice. In particular, the findings of this analysis demonstrate that the taxonomy of levels of impact (see above) needs to be extended (Zehetmeier, 2008): The categories knowledge, beliefs, and practice are suitable to cover the impact in the teachers' level. But also on the levels of pupils, colleagues, principals, and parents all three categories (knowledge, beliefs, and practice) are respectively necessary to gather possible levels of impact. Moreover, in addition to these in-school levels, also beyond-school levels need to be considered when analyzing the impact of professional development projects: e.g., other schools, media, policy, or scholarship. These results lead to a grid of possible levels of impact (Zehetmeier, 2008, p. 197):

	In-School Levels					Beyond-School Levels			
	Teachers	Pupils	Colleagues	Principals	Parents	Schools	Media	Policy	Scholarship
Knowledge									
Beliefs									
Practice									

FOSTERING FACTORS

What are the factors that promote and foster the impact of professional development projects? Literature and research findings concerning this question point to a variety of different factors. To give an overview, in the following section Borko's (2004) four elements of professional development projects are used to organize and classify these factors: *participating teachers*, *participating facilitators*, *the programme itself*, and the *context* that embeds the former three elements.

Within the element of *participating teachers* the following factors are fostering the impact of professional development programmes: If the teachers are involved in the conception and implementation of the programme, they can develop an affective relationship towards the programme by developing ownership of the proposed change (Clarke, 1991; Peter, 1996). They can be empowered to influence their own development process (Harvey & Green, 2000). Teachers should be prepared and supported to serve in leadership roles (Loucks-Horsley et al., 1996). An "inquiry stance", taken by the participating teachers, also fosters the sustainability of impact (Farmer, Gerretson, & Lassak, 2003, p. 343): If teachers understand their role as learners in their own teaching process, they can reflect and improve their practice. Cochran-Smith and Lytle (1999) also use this notion for describing teachers' attitude towards the relationship of theory and practice: "Teachers and student teachers who take an inquiry stance work within inquiry communities to generate local knowledge, envision and theorise their practice, and interpret and interrogate the theory and research of others" (p. 289). Altrichter and Krainer (1996) recommend a reorientation of professional development programmes from "teachers to be taught" towards "teachers as researchers" (p. 41) and refer to Posch and Altrichter (1992) who state: „The most important part of teacher professional development takes place on site: by reflection and development of the own instructional practice and by school development" (p. 166).

Similar to the teachers, also the *participating facilitators* of the professional development programme should take a “stance of inquiry” (Ball, 1995, p. 29) towards their activities. They should reflect on their practice and evaluate its impact (Farmer et al., 2003). The facilitators’ knowledge, understanding, and their image of effective learning and teaching also foster the initiative’s impact (Loucks-Horsley et al., 1996). The development of mutual trust between the facilitators and the participating teachers represents a further fostering factor (Zehetmeier, 2008).

The *programme itself* should fit into the context in which the teachers operate, and provide direct links to teachers’ curriculum (Mundry, 2005). It should focus on content knowledge and use content-specific material (Garet et al., 2001; Ingvarson et al., 2005; Maldonado, 2002), and should provide teachers with opportunities to develop both content and pedagogical content knowledge and skills (Loucks-Horsley et al., 1996; Mundry, 2005). Moreover, an effective professional development programme includes opportunities for active and inquiry-based learning (Garet et al., 2001; Ingvarson et al., 2005; Maldonado, 2002), authentic and readily adaptable student-centered mathematics learning activities, and an open, learner-centered implementation component (Farmer et al., 2003). Further factors fostering the effectiveness and sustainability of the programme are: prolonged duration of the activity (Garet et al., 2001; Maldonado, 2002), ongoing and follow-up support opportunities (Ingvarson et al., 2005; Maldonado, 2002; Mundry, 2005), and continuous evaluation, assessment, and feedback (Ingvarson et al., 2005; Loucks-Horsley et al., 1996; Maldonado, 2002).

Lerman and Zehetmeier (2008) highlight that community building and networking represent further factors fostering sustainability. This claim is supported by several authors and studies, even if the categories used to describe these activities are sometimes different: Clarke (1991), Peter (1996), and Mundry (2005) point to cooperation and joint practice of teachers, Loucks-Horsley et al. (1996) and Maldonado (2002) highlight the importance of learning communities, Wenger (1998) and McLaughlin and Mitra (2001) identify supportive communities of practice, Arbaugh (2003) refers to study groups, and Ingvarson et al. (2005) stress professional communities as factors contributing to the sustainability of effects. In particular, providing rich opportunities for collaborative reflection and discussion (e.g., of teachers’ practice, students’ work, or other artefacts) presents a core feature of effective change processes (Clarke, 1991; Farmer et al., 2003; Hospesova & Ticha, 2006; Ingvarson et al., 2005; Park-Rogers et al., 2007; Zehetmeier, 2008).

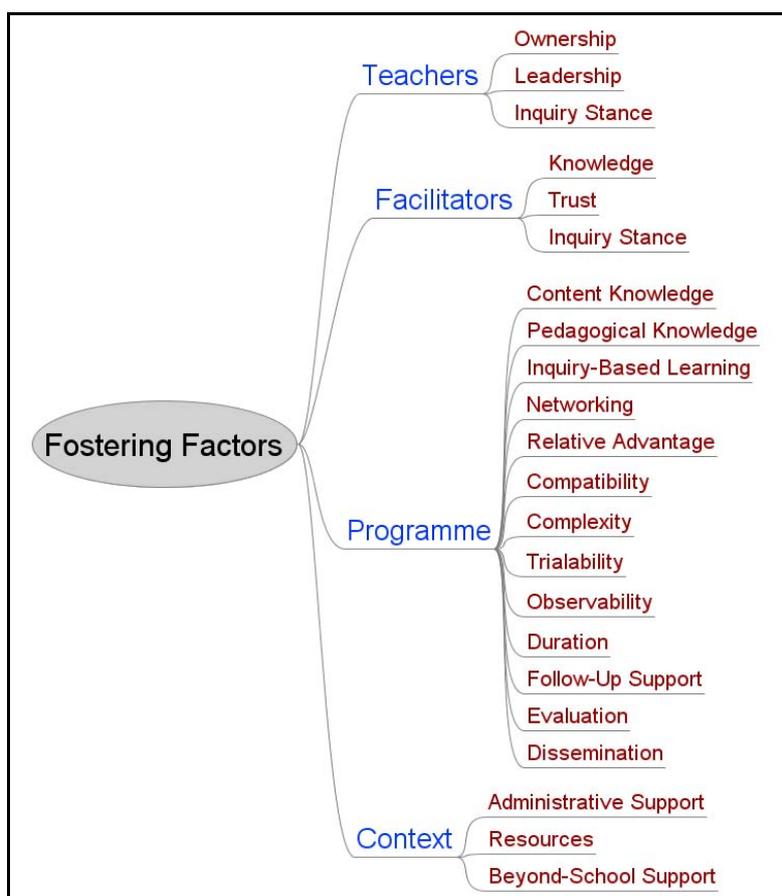
The dissemination of innovations or innovative teaching projects is another factor that fosters the sustainability of professional development programmes (Zehetmeier, 2008). E.g., teachers participating in the Austrian IMST project (Krainer, 2005, 2007) write down and publish reflective papers or project reports. As Schuster (2008) shows, teachers’ writings have a positive impact on their reflection skills and knowledge base. The dissemination of good practice projects and ideas requires a

structural framework that allows teachers to publish or actively present their projects and results. E.g., the Austrian IMST project created a web-based wiki where some hundreds of project reports written by Austrian teachers can be easily accessed. Moreover, an annual nation-wide conference is set up, where teachers can share their projects, ideas, and results. A professional development programme aiming at sustainable impact should provide these possibilities for dissemination even after the programme is terminated. Otherwise the possibility of dissemination along with the involved advantages for teachers' professional growth is likely to fade away (Zehetmeier, 2008).

Rogers (2003) highlights that the diffusion of an innovation depends on different characteristics: Relative advantage, compatibility, complexity, trialability, and observability. Fullan (2001) describes similar characteristics (need, clarity, complexity, quality and practicality) that influence the acceptance and impact of innovations. Relative Advantage includes the perceived advantage of the innovation (which is not necessarily the same as the objective one). An innovation with greater relative advantage will be adopted more rapidly. Compatibility and need denote the degree to which the innovation is perceived by the adopters as consistent with their needs, values and experiences. Complexity and clarity include the teachers' perception of how difficult the innovation is to be understood or used. Thus, more complex innovations are adopted rather slowly, compared to less complicated ones. Trialability denotes the possibility of participating teachers to experiment and test the innovation (at least on a limited basis). Innovations that can be tested in small steps represent less uncertainty and will be adopted as a whole more rapidly. Quality and practicality make an impact on the change process. High quality innovations that are easily applicable in practice are more rapidly accepted. Observability points to the claim that innovations which are visible to other persons (e.g., parents or principals) and organisations are more likely to be rapidly accepted and adopted.

The *context* which embeds teachers, facilitators, and the programme itself, is of particular importance regarding the sustainability of innovations and change processes (e.g., McNamara, Jaworski, Rowland, Hodgen, & Prestage, 2002; Noddings, 1992; Owston, 2007). Teachers need administrative support and resources (McLaughlin & Mitra, 2001). School-based support can be provided by students and colleagues (Ingvarson et al., 2005; Owston, 2007), and in particular by the principal (Clarke, 1991; Fullan, 2006). To foster sustainability not only at the individual (teacher's) level but also at the organisational (school's) level, Fullan (2006) proposes a new type of leadership that "needs to go beyond the successes of increasing student achievement and move toward leading organizations to sustainability" (p. 113). In particular, these "system thinkers in action" should "widen their sphere of engagement by interacting with other schools" (p. 113) and should engage in "capacity-building through networks" (p. 115). Support from outside the school (e.g., by national or district policies) is also an important factor fostering the programme's impact (McLaughlin & Mitra, 2001; Owston, 2007).

The following figure sums up and illustrates these factors that promote and foster the impact of professional development projects:



FUTURE RESEARCH

Impact analysis that combines and compares various cases and bigger samples could help answering the following questions (see also Zehetmeier, 2008):

- Do different professional development projects make different sustainable impact? Are there any patterns of impact?
- Does a professional development project show different sustainable impact on different participating teachers? Are there any patterns?
- Are there any hierarchical structures within the different levels of impact? Does one level require another one to occur?
- Are there any factors that promote certain levels of impact in a particular way?
- Are there any “universal” factors fostering sustainable impact?

Upcoming impact analyses dealing with these and similar questions appear to be necessary and promising; from the perspective of both scholarship and practice.

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A COLLABORATIVE PROJECT AS A LEARNING OPPORTUNITY FOR MATHEMATICS TEACHERS

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This paper analyses the evolution of Maria, a mathematics teacher involved in a long term collaborative project together with a researcher and two other teachers. The study aimed to understand teaching practices and to develop richer classroom communication processes. It follows a qualitative-interpretative approach, with data gathered through recording of meetings and interviews. We discuss to what extent this project became relevant for the professional practice of Maria. The results indicate the potential of collaboration to understand communication phenomena in the classroom, putting practices under scrutiny and developing richer communication interaction patterns between teacher and students.

Key-words: Mathematics communication; collaboration; professional development.

INTRODUCTION

The possibilities of collaboration between teachers and researchers as a research strategy are receiving increasing attention. Collaboration is an opportunity to combine joint work with individual input, taking advantage of the potential of different individuals building a common experience (Hargreaves, 1994). In this paper, we take collaboration as an experience shared by a set of people who identify a common interest and establish and implement a working agreement, providing mutual support and challenging each other. This perspective defined a collaborative project involving a researcher (the first author of this paper) and three mathematics middle school teachers, whose purpose was to understand classroom communication, putting practices under scrutiny and developing richer communication processes.

Our research question enquires what are the influences, if any, of a collaborative project on the conceptions and practices of a teacher regarding classroom communication. It links concerns emerging in recent research on collaborative work, (Boavida & Ponte, 2002; Jaworski; 1986) and classroom communication (Alro & Skovsmose, 2004; Lampert & Cobb, 2003; Sherin, 2002). Here we restrict the scope of analysis to Maria, one of the teachers. In particular, we discuss to what extent the project became relevant to her professional practice. First, we discuss the meaning of collaboration in educational research and how communication was understood within the project group. Then, we present the methodology and analyse the “case” of Maria.

Finally, we end with a discussion concerning issues that arise in collaboration as a research strategy in mathematics education.

BACKGROUND

Collaboration. Collaboration plays an increasing role in educational research. In a collaborative project, participants may take advantage of working together (Kapuscinski, 1997), but often tensions emerge along such a process. They arise, for example, from the different attitudes teachers and researchers maintain towards practice, planning, motivations or use of knowledge (Kapuscinski, 1997; Olson, 1997). There are, of course, a variety of collaborative structures, and corresponding different degrees of individual commitment – as indicated, for example, by Clift and Say (1988), Day (1999), Goulet and Aubichon (1997), and Wagner (1997). A number of aspects, however, are recognised as consensual as characteristic of any true collaboration. One of them is that the relationships between the participants should not be hierarchical. Mutual support requires some sort of egalitarian base (Boavida & Ponte, 2002). There are, of course, different roles, a difference which, moreover, should be made clear in the group, but all roles must have similar relevance. Another element concerns diversity, understood as an added value to collaboration, which should be assumed as such by the group (John-Steiner, Weber & Minnis, 1998).

In a collaborative context, participants do not waste time to promote what they believe to be their own image (Fullan & Hargreaves, 1991). Disagreements are frequent and welcome, given that discussions are centred in values, purposes and practices. A considerable effort is required to build a collaborative culture, which always supposes an effective personal development. In particular, it requires making explicit some common objectives inside the group. Each participant must be aware of her/his own role in the way he/she relates to the others and cares about such a relationship (Drake & Basaraba, 1997). Teachers' involvement in a project depends on how they perceive its relevance, namely to practice, as well as on the way decisions are made inside the group (Bonals, 1996). Essential to the success of a collaborative project is also the ability to carry on reflection exercises together (Day, 1999). To develop such an ability to think critically with others requires some degree of maturation in dealing with doubt and incertitude (Fernandes & Vieira, 2006).

The benefits of collaboration are well documented in the literature. Fullan and Hargreaves (1991) and Maeers and Robison (1997), for example, mention how it helps teachers to feel less isolated and impotent. It is also a factor for change in educational practices, namely when the collaborative experience is made public (Olson, 1997). Active involvement in a collaboration and sharing of concerns and experiences promotes personal and professional development (Lafleur & MacFadden, 2001) as it leads to increased self-knowledge. Collaboration increases the self confidence of every participant (Maeers & Robison, 1997).

Collaborating along a reasonable period of time is not an easy task. Collaborations are fragile, by definition, requiring balances that often are difficult to set up and maintain (Fullan & Hargreaves, 1991; Olson, 1997). Therefore, planning and flexibility, dialog and negotiation, are essential to any collaborative project. Finally, managing expectations, emotions, personal differences, becomes fundamental whenever a collaboration is to be maintained.

Communication in the mathematics classroom. Several authors underline the importance of communication processes in the mathematics classroom (Bishop & Goffree, 1986; Ponte & Santos, 1998; Yackel & Cobb, 1998). Communication is a social process along which participants interact, sharing information and mutually constraining their activity and evolution. It concerns not only an heterogeneous set of interactive processes evolving in a classroom but also their contexts, underlying denotations and expressive resources. Such a perspective includes two issues clearly identified in the literature (Ponte, Boavida, Graça & Abrantes, 1997) in the study of communication in the mathematics classroom: (i) *continuous interaction* between the actors in a classroom, and (ii) *negotiation of meanings* understood as the processes such actors set to share their own ways of making sense of mathematical concepts and procedures, and their evolution and relation to the formal curriculum contents.

Mathematical learning requires a stepwise construction of a reference framework through which students construct their own personal account of mathematics in a dynamic tension between old and newly acquired knowledge. This is achieved along the countless interaction processes taking place in the classroom. Of especial import are the interactions between students and teacher, which simultaneously constrain and are constrained by the kind of lesson. For example, in a learning context in which the teacher stresses exposition and solving exercises, he/she tends to control the whole process. In other contexts he/she may assume instead the role of a coordinator. The nature of the questions posed by the teacher is particularly relevant, leading to the development of communication and reasoning skills (Barrody, 1993).

It is widely recognised the fundamental role that the teacher plays either in enabling or limiting the communicative processes within the classroom (Barrody, 1993; Lappan & Schram, 1989; Pimm, 1987). Such a role makes itself explicit from the outset, for example, when selecting challenging tasks or encouraging students to express and argue their own views (Lampert & Cobb, 2003; Ponte & Santos, 1998), or else when resorting to tasks and educational materials that put the focus of the lessons on mathematical ideas, conjectures or intuitions, instead of calculations and procedures. The teacher is also responsible for creating an atmosphere of self-esteem and mutual respect, so that students feel comfortable to participate, as well as for structuring the classroom discourse.

METHODOLOGY

This paper reports a study, qualitative and interpretative, based on a case-study design (Yin, 1989). This is part of a broader research project involving three case studies developed within the context of a long term collaborative project on communication in the mathematics classroom (Martinho, 2007). The project involved a researcher, the first author of this paper, and three mathematics teachers, Maria being one of them. This group was initiated by the first author who invited a teacher with whom she had already collaborated, who later invited two other teachers to join. Along a year and a half, the project involved regular meetings devoted to a variety of tasks, namely, analysis of documents, lesson planning and review, free debates on communication issues, and project planning and evaluation. Each teacher selected a number of lessons to be observed and recorded by the researcher, and finally these lessons were discussed in group meetings. Data gathering for this research study was based on two semi-structured interviews and on the recordings of group meetings and the researcher's field notes. The aim of the interviews was to get a deep understanding of the way the teacher reasoned about her own communication practices. The focus was on creating a friendly environment to allow a natural flow of conversation about the topics of interest. The recordings of group meetings and the researcher's field notes provided complementary data about the teacher activity, concerns and reflections at each moment. Data collection and analysis were carried simultaneously during collaborative work, mutually influencing each other. The research adopted the interactive model of analysis (Huberman & Miles, 1994).

The project started in 2004, with regular working meetings taking place every fortnight (in a total of 25 meetings), along the whole academic year of 2004/05. From September 2005 onwards, meeting periodicity changed to a weekly basis. Even today, after the formal closing of the original project, the group still meets every week, including now two more teachers. All of them, except the researcher, work in the same middle school.

RESULTS AND DISCUSSION

Maria. Maria is 52 years old and counts 31 years as a teacher. She is married and has two children, already grown up. She assumes her work with professionalism and commitment. For 6 years she served as a school principal and is quite active in a trade union. She has an accurate sense of public service and citizenship. In general, Maria is resolute, determined, and always exigent with herself. She concluded a bachelor degree in chemical engineering in 1974. Becoming a teacher was not her first professional option; only later, she completed another degree on teaching biology and geology. At present, she teaches mathematics and natural sciences. This background may explain her main concern as a mathematics teacher: to provide evidence of the usefulness of this subject.

Maria feels some difficulties in several mathematical topics (she often says that she is not a mathematician) and this clearly influences her teaching practice. She has a deep respect for mathematics as a wonderful world that, however, she is not able to master easily. Mathematics, in her view, is a network of abstractions, concepts and methods, tightly connected. Therefore, she fears that her way of teaching, emphasizing a detached view of each concept or sub-area, may not contribute to make mathematics an interesting and motivating subject for her students. Therefore, she seeks possible links among the topics she teaches, but recognizes her difficulties in improving her practice just by herself. In the group meetings she eagerly took notes of any observation seeming profitable regarding mathematical connections. To some extent, this feeling of inability in giving a unified view of mathematics was challenged (and altered) during the collaborative work.

Maria within the collaborative project. From the outset, Maria played an active role in the project, assuming the group as a personal learning experience. Among the topics addressed she mentions the joint discussion of lessons and their previous planning. In such a context, she said, “it becomes easier to try new experiences” (M15, January 05)¹. Moreover, in several occasions she values the importance of group discussions: “The interest of this sort of work, even if not to learn a lot of new things, is to put us thinking and to raise new questions” (M23, June 05).

We describe several influences of the project on Maria’s communication conceptions and practices. First, she acknowledges how fundamental it is to recognise one’s own communication failures so that effective change becomes possible. She values the group discussion of past lessons as a step in building such awareness: “I guess what matters to identify communication problems in the classroom is to be able to identify failures. Often, the daily routine is so pressing that we are unable even to recognise them” (M25, July 05). She also points out that it is too easy to blame students when a lesson fails, instead of recognising communication problems. For Maria, the role of discussions in small groups became increasingly clear: “Only when we meet in a small group, like this, and begin to ask what’s going wrong, one becomes aware of difficulties in communicating with our students” (M25, July 05).

Second, Maria also emphasizes that our joint work helped in breaking the daily routine of isolated teachers which tends to obfuscate the real problems. Among these problems she underlines how difficult it is to respect students in their heterogeneity:

We talk to the average student, forgetting those with extra difficulties or kids with different ways of making progress. We still plan lessons in a sort of canonical format that is the format we have rationalised from our previous experience as students ourselves. In the absence of sharing experiences and mutual questioning, we still go on the same way. (M25, July 05)

¹ (M15, January 05) stands for the transcript of the 15th group meeting, hold on January 2005.

Third, Maria focused in some particular elements of her practice. For example, she was challenged to address the issue of students working in groups in mathematics classes. She had already some experience of group work in natural sciences classes, but wondered how this could be done in mathematics. Along the project she tried a number of experiences with group work, allowing students to work by themselves and discussing results afterwards. The project was most helpful in modifying her initial conviction that this requires much more time than conventional lectures to cover about the same contents. She used to say:

Sure, these steps [group work] help students to build deeper mathematical insight. My doubt is: and time? (...) How much time can one devote to discovery, building insight, mastering mathematical reasoning? My dilemma is: build mathematics or follow [successfully] the national curriculum. (M18, March 05)

Later she comments on an experience carried out on a statistics unit: “It took five lessons; normally I need less than that for this topic” (M22, May 05). But she acknowledges the fact that this activity was a training experience for herself. Training for developing more careful lesson plans and a few routines enabling her to “waste less time”, or, as she notes, “to use the available time with increased quality” (M24, July 05).

Finally, we observed her effort to take into account in her own practice the main concerns shared in the project group. For example, she indicates that she does a serious effort to reduce the number of interventions she has in the classroom: “A number of things inside my own mind are already working. For example, reminding me: let’s see what they think, what they say” (M22, May 05). She became more attentive to what her students say. Similarly, she sought her students to listen more carefully to each other. She points out episodes illustrating her greater willingness to give more time for students answering and reasoning in class: “before [the project]”, she commented, “I used to guide their answers, suggesting a possible way of handling the question straight away” (M21, May 05). Note that she recognized that such an attitude “was made possible because of the discussions within the group” (M21, May 05). Moreover, she said “my concern with negotiation of meaning increased as a consequence of our work. Now I require students to give proper and detailed explanations and raise themselves new questions” (M22, May 05).

The project was lived by Maria as an opportunity to think about the impact of communication issues in the classroom and their relevance as a source of common difficulties in teaching. This is further illustrated by her comment in the last meeting of the academic year 2004/05: “A fundamental issue is to be aware that several daily difficulties in our professional life are related to communication” (M25, July 05). And, later, she wrote concerning the work developed:

(...) Discussing together what classroom communication effectively means, studying a few theoretical papers as well as experience reports, our own availability to share our classes with others, to reflect in a critical way about our own practices, all this made the

project sessions a true opportunity of professional development. Several connections were built at different levels (pedagogical, scientific, didactical), giving to this group a real sense of what needs to be changed and how. (June 06)

An indicator of the relevance this collaborative project had for the three teachers involved was the decision they took to extend it behind the initial closing data: The group still goes on at the moment of writing. Quite recently Maria wrote in an email concerning group planning for 2008-09: “I am completely available for this project. Actually, it is an irreplaceable space for sharing, knowledge building, and friendship” (September 08).

Maria always supported the project with enthusiasm and a pro-active attitude: sharing plans, discussing suggestions, inviting others to assist to her lectures. She never neglected the possibility to discuss a lesson, sharing her own thoughts and taking care to make explicit the strategies used and her motivations underlying them. The project influenced her practice with respect to the sort of discourse and interactions with students, but mainly, as she stresses, in what concerns her ability to bring variety to her lessons and relationships with students. Maria understood this project as a personal challenge, not always easy to follow. But she was always willing to share: “I have to wait so much, until Friday, to tell you...” And this led another teacher in the group to comment: “This group is our therapy” (May 06).

CONCLUSION

The purpose of this study is to illustrate how a collaborative project can influence its participants and have an impact on their practices. Maria was chosen as the focus of this paper since she was the teacher who was most influenced by the project. Probably that happened because she took such a decision from the outset: To be open to the group influence and look into it in a positive, pro-active way. We shall now extend the discussion to the group level.

The focus of this research was communication in the mathematics class, a broad theme that may include a variety of issues and experiences. As it developed, however, it became clear to the researcher that a collaborative research entails the need for never avoiding or ignoring the questions raised by the participants or the issues that they think are most relevant, even if this implies taking less obvious “routes”.

Allowing others to come into their own classroom as well as sharing and discussing their experiences had a deep significance to all the teachers involved in this project. Maria was no exception. But this did not evolve without concern, and the feeling that something that used to be “private” was now made available to others. A number of fragments of a discourse seeking auto-justification provide evidence that collaboration is a process that extends itself in time. As underlined by several authors (e.g., Fullan & Hargreaves, 1991), mutual support in the group is essential to get through, or at least to control, our own difficulties and vulnerabilities. Just as it happened with Maria, the project helped all the others to increase self-confidence,

reducing the feeling of impotence and solitude. This role, which is central in a collaborative project (Maeers & Robison, 1997), was recognised by all the participants, with different degrees.

A collaborative project is a social construction. As such, it entails the need for all participants to share their different ways to approach a situation or experience (John-Steiner, Weber and Minnis, 1998). The relative heterogeneity of participants made mutual influence possible and played an important role in the perception that the group has of its own development.

For the researcher, this was a rich experience, namely as an opportunity to approach very closely school reality and the way it is experienced by teachers. Nothing is given once and for all, and so sometimes she felt tired, unable, almost lost. But progress was made because in the group we have always felt that, in spite of difficulties, we needed to go on because it was exactly from our disagreements that we evolved as a group.

This research study shows that, even with a highly motivated group, changing is always slow. The steps to undertake cannot be too large. Often, the researcher felt that her attempt to propose a number of experiences and activities was fruitless: What is really necessary is that every teacher in a collaborative group takes the group objectives as his/her own.

Along the project, Maria assumed herself the role of researching her own practice and provided evidence of how that entailed changes in her professional practice. This seems to be consistent with related research (e.g., Fernandes & Vieira, 2006) which shows that collaborative work fosters an attitude of serious enquiry about the teacher's own practice. As a consequence, Maria considers herself now more able to challenge her students, to develop their autonomy and to explore their mutual interactions in the classroom. She feels them more active and responsible towards their own learning. She is confident about her stance, but keeps saying that to make changes effective one needs a reflexive attitude and time to mature.

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REFLECTION ON PRACTICE: CONTENT AND DEPTH

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ABSTRACT

This text is based upon an ongoing investigation with the main goal of studying the professional development of primary school teachers, specifically the ability to reflect, within a continuous training program.

This study follows a methodological approach of a qualitative type, comprising case study, with recourse to interviews, participant observation and documental analysis.

A first analysis of the written reflection of one of the participants, included in the reflection portfolio, points, in terms of content, towards less spreading of the themes approached, the ones considered the most significant being subsequently extracted and correlated. A greater depth in the reflection is also noted, with the teacher having concern to justify her statements, present a critical analysis of her role and rethink her practice.

Key-words: Professional development, mathematics' teacher, teacher training, reflection, practice

INTRODUCTION

Reflection is one of the activities most frequently considered to contribute to the professional development of teachers, since it may be presented as a means to improve classroom practices.

The Program for Continuous Training in Mathematics for Primary School Teachers, launched by the Board of Education and the Board for Science Technology and Higher Education, has been under development in Portugal since the academic year of 2005/2006. This program aims at an improvement in the teaching and learning of Mathematics as well as developing a more positive attitude towards this branch of knowledge. It involves conducting group training sessions, classroom supervision sessions and one final plenary meeting for a final appraisal of the program. Participant evaluation is undertaken through the elaboration of a portfolio, over the duration of the program. Contents of this program include the nature of the tasks, namely problem solving, and the use of physical resources, in which manipulative materials are included.

This paper is based in an ongoing investigation, whose goal is to study the professional development of primary school teachers through participation in the program. Specifically, we aim here to answer the following question: (i) In what way does the teacher's ability to reflect evolve throughout the training program?

THEORETICAL FRAMEWORK

“The professional development of teachers, both inside and outside the classroom, is the result of their reflection and participation in training opportunities which improve and increase their development and progress.” (National Council of Teachers of Mathematics, 1994, p. 175). Reflection is an activity which may contribute towards the teacher’s professional development. The term reflection is, however, polysemic. To Dewey (1933), in the field of education, the “active, persistent, and careful consideration of any belief or supposed form of knowledge in the light of the grounds that support it, and the further conclusions to which it tends, constitutes reflexive thought” (p.7), appearing as an activity thoughtfully and directly connected to practice. Zeichner (1993), although stressing that terms such as reflexive practitioner and reflexive teaching have become slogans for teaching reform and teacher training, attributes a strong personal angle to reflection, considering that there are no recipes to teach the teacher how to reflect. Schön (1983) also contributes in clarifying this concept, considering three kinds of reflection: in action; on action and upon reflection in action.

Addressing teacher training programs, Lee (2005) finds differences in the content and depth of the reflection undertaken by future teachers. Specifically he identifies the following as factors related to the depth of the reflection: personal context, professional experiences encountered and ways of communicating.

To Day (2001), just conceiving the existence of reflection as a means of learning does not demonstrate the depth, reach and goals of the process, as “good teachers are technically competent and reflect upon matters pertaining to the goals, the process, the content and results” (p. 72).

One of the contexts which may be supportive in producing reflection is the one involving portfolios. Written reflection is one of its basic components, particularly if one is examining documented teaching, and is focused on what the teacher and the student have learned (Santos, 2005; Wolf, 1996). Reflection is, thus, “the critical heart of the record” [contained in the portfolio] (Lyons, 2002).

Summing up, this study considers that reflection helps to looking backwards and rethinking one’s own practices (Muñoz-Catalán *et al.*, 2007; Oliveira & Serrazina, 2002), although it is possible to find idiosyncratic differences in the process of reflection (Hospesova *et al.*, 2007). Moreover, reflection as analytical thought is above all associated with unsolved problems (Dewey, 1933), or rethinking meanings previously associated with educational situations.

INVESTIGATION METHODOLOGY

This work takes place in a natural environment, in which the researcher is also the leader of a working group made up of nine teachers. We have chosen to adopt a qualitative methodological approach (Bogdan & Biklen, 1994), undertaking three

case studies (Gall, Borg & Gall, 1996), with the help, in data gathering, of semi-structured interviews, participant observation and documental analysis.

Initial, intermediate and final interviews have as a main goal the gathering of data pertaining to the participant teachers, on the basis of the issues under consideration. Interviews, after each class has taken place, are related to points emerging from the experimental classroom activity. Group training sessions and classroom supervision sessions were observed. Interviews and observations undertaken were fully audio taped and transcribed. Documental analysis focused on the records included in the portfolios (planning, material used, student production and reflections), in the field notes about supervision sessions and in the reflections about group training sessions

In her portfolio, Sara, one of the participants, has included three reflections on tasks tried out in the classroom during the course of the program, although she was only compelled to include two. In this paper we present the analysis of the first reflection, which took place in December 2006, and of the third, in April 2007.

To address the presentation of written reflections to be included in the portfolio, guidelines, followed in the training program, were provided, consisting of the following points: 1. Activity goals; 2. Activity description; 3. Reflection on the activity, including four aspects: (i) activity planning; (ii) evaluation of what the students might have learned with the activity; (iii) importance of the activity for the teacher; and (iv) the teacher's future perspectives regarding Mathematics.

Analysis of information gathered started after completion of the training program and consisted of organizing and interpreting data, considering the problem under investigation, theoretical framework and the empirical work which had taken place. Specifically, fields of analysis considered were content and depth (Lee, 2005). Regarding content, we have defined as categories for analysis the ones included in the guidelines. Regarding depth, we have considered: (i) Confrontation with one's own practice (identification and description of what one considers important or problematic); (ii) Interpretation (why does one perform the way one does?); (iii) Putting into perspective (confrontation of action with what one thinks and feels about it) and (iv) Reconstruction (what ought to be kept? What can be different? what can be changed, why?)

TEACHER SARA'S WRITTEN REFLECTION

Sara is around forty, and has twenty to twenty five years of professional experience. She has a Primary School Teacher's degree and the Scientific and Pedagogical Training Complement for Primary School Teachers, which bestows a license level degree.

Sara tells us she has always liked mathematics. Although she considers herself as having enough knowledge to teach she has invested time in keeping herself up to date through attendance at training sessions and programs.

Regarding the sort of tasks she planned and put into practice in the field of Mathematics, before attending the program, Sara said she *sometimes* uses problem solving. She states that she is aware of not using a lot of materials in the tasks she puts forward, relating this idea to the need to keep up with the program:

I am, I am aware I don't use much. I think we are rather limited concerning time because we are always concerned with keeping up with the program and then we may get one day behind, which we may need later. [initial interview]

Specifically, regarding reflection upon practice, before attending the program Sara explains she did not reflect much and that she had never made a written reflection:

Also, it is not that one completely overlooks it. But, when returning home, one puts school somewhat aside because we must also support our family a bit (:...) Perhaps, after several activities, I sit down and reflect a bit to myself. Not on paper, but to myself [initial interview]

The first reflection she presents in her portfolio is based on the students solving the following problem: Francisco raises chickens and rabbits. He has in all 16 heads and 48 legs. How many chickens and how many rabbits does Francisco own? The third one relates to constructing and identifying geometrical figures using the *Tangram*.

Sara has respected the guidelines in both reflections. Specifically, in point 3 – Reflection on the activity – of the written reflection that she produced, and related to the item – *activity planning* – she begins by making reference to what she considers essential to someone who solves a problem and stresses the difficulties to the one proposing it (speech 1). She presents, succinctly, the goals of the task she has put forward (speech 2):

1. Interest in the problem and its ownership by the one who solves it are essential. The hardest step for the one presenting it, might be to choose the problem or even to make it up.
2. When presenting the problem to the students I wished them to explore the context, gather data and find differences [Sara's portfolio 1st reflection]

The third reflection begins with her expectations in relation to the fulfillment of the task, regarding her previous knowledge of the class.:

As I was aware that the tangram had already been used in the classroom, I was led to think that free activities and the relationships between the pieces had already been explored. So, I started the class aware it would be a noisy class, but that it would be easy to reach the projected goals within the time allotted. [Sara's portfolio 3rd reflection]

She mentions some flaws regarding planning, especially regarding the sequence of the proposed activities:

In the course of the class I noticed that planning had some flaws, namely regarding the order of activities. I came to the conclusion that I should have started the class with a deeper exploration of the tangram.

Activity 2 should have taken place more towards the end of the class, because they were very worried about drawing, which caused it to last for a long time and some of them only managed it with help. [Sara's portfolio 3rd reflection]

Concerning the item – *evaluation of what the students might have learned with the activity* – in the first reflection she identifies what she considers to be the main concern of students during the activity and explains her reaction regarding that concern (speech 1) She also mentions the students' reactions regarding difficulties felt in the beginning of the task; she tries to account for them and explains her way of reacting in face of the situation (speech 2):

1. During the course of the class I noticed a huge concern of the students to place the data and perform an operation. I read the problem once more and showed them that the results were not dependent on adding or subtracting these figures.

2. I noticed they were having trouble with starting the task on paper. They asked a lot of questions such as “I did not understand this here”, I guess to call for the teacher's attention, to see if they could get a little help. At first, the idea was not to interfere or help the students but due to the number of requests I finally decided to lend a little hand [Sara's portfolio 1st reflection]

As a matter of fact, at the beginning of the task, just after Sara had handed over the problem's instructions, some comments were heard: “I know the operation!”, “It's too much!”, and “I already know the problem!” While she read the problem aloud some students interrupted with questions: “What are heads?”, “What are chickens?” Sara explained: “16 heads means 16 animals”. And she asked: “How many legs does a chicken have? And a rabbit?” After the reading she informed them: “Each one of you does it as you want” The students tried to solve the problem individually, always requesting the assistance of the teacher and even of the researcher.

She noticed that although the students remained restless and constantly requested the teacher's assistance they started designing their strategies. Sara moved about the room in order to see the work the students were performing. After some time Sara asked some students to explain their ways of solving the problem on the blackboard. One of the students made the following sketch:



He began by making 16 circles and made a dividing slash in the middle and counted the “number of chicken” and the “number of rabbits” making a jot over each circle and simultaneously explained his reasoning.

Another student made drawings. She started by drawing a child and two sets of eight animals some with two feet others with four. In the end she explained her reasoning to the colleagues. Another student drew an animal with four legs, another with two, and so forth, up to a total of 16 animals.

Only the students who had come up with the correct answer were asked to come up to the blackboard.

In the course of her reflection, besides identifying the solving procedure used by most students, she comments on it and stresses a strategy used by just one student:

I realized that many students started by dividing the number 16 in two groups and then added the legs. I think that choosing this method is related to the 8 multiplication table, which we had studied recently and was on the board.

I was sorry Cláudio couldn't go up on the blackboard to show the method he had used to solve the problem. He did not come up with 8 rabbits and 8 chickens because he got lost in counting but his representation was different and interesting. [Sara's portfolio 1st reflection]

Sara also mentions time management, specifically lack of time to communicate the different solving procedures used. "I think I gave too much time to individual solving, which did not allow the children to go up on the blackboard to explain their reasoning and to check for the existence of diverging results".

She mentions that "not many of the students managed to come up with valid reasoning to get to the result one wished for" and she points out, justifying this, that the students felt some difficulties in problem solving, although there was some development in competencies (speech 1). She also indicates the main learning outcomes the students achieved (speech 2):

1. I noticed the students felt some difficulties in solving these sorts of problems, perhaps because they were not used to them, even so, there was a development of competencies which led to the building up of personal strategies. Problem solving placed the students in an active learning attitude, both by giving them the possibility of constructing notions as an answer to the questions raised, and by urging them to use the acquisitions made and to test their efficacy.

2. They have learned to show curiosity and the taste for exploring and solving simple problems;

To solve situations and daily problems using representations and schemes;

They have learned to make simulations of real life events [Sara's portfolio 1st reflection]

In the third task, Sara began by giving some information about the origin and use of tangrams. The students listened attentively. Many of them said they had already worked with that material. After distributing the tangrams among the students, these at once started building free figures. Sara passed around a work sheet with the instructions for the task. Some students remained interested in figure building. Sara asked a student to read the introductory text about tangram and she read the first questions in the work sheet. "1. Which is the tangram's original shape? 2. In how many parts is it divided? and 3. Which geometrical shape does each component represent?" The students recorded their answers in their work sheets. Next, Sara

asked the students to perform the second task indicated in the sheet: “using all the elements, build and record the figures built: a) a square; b) a rectangle and c) a right triangle”.

Several students mention not understanding what they are supposed to do. Others say: “I can’t make it” and ask the teacher’s help. Others advance on their own and solve the problem. Some students also show difficulties in recording the results and concentrate on this point, failing to advance in building the various figures requested. Many appear seriously worried about not being able to perform the task and some give up. Several students find it difficult to know what a right triangle is.

In her reflection, and regarding this point, she correctly evaluates the mathematical output of students, indicating learning outcomes achieved:

They memorized geometrical figures and defined them regarding the number of sides;

They learned that one of the seven elements of the tangram is called a parallelogram;

They were able to find out that you can build squares out of the several elements of the tangram;

They have learned that you can build a lot of figures with the tangram.

There were also learning acquisitions in other areas such as Portuguese Language, because besides having to communicate they also had to read and write. And they also learned some trivia, for instance, that the Chinese tangram is not the only one [Sara’s portfolio 3rd reflection]

She identifies, justifying this, two particular cases of students which surprised her when performing the task:

Two students surprised me, one for the better, one for the worse. Hélia surprised me for the worse because she has shown she is a participating student who likes to commit herself to solving the activities and in this particular class she needed a lot of help to solve the activities I put forward;

Pedro surprised me for the better because he showed himself to be more committed in solving the activities, did not interrupt the class as often, and managed to solve what was asked of him [Sara’s portfolio 3rd reflection]

Regarding the item – *importance of the activity had for the teacher* – in her first reflection Sara only presents a brief remark:

For me, as a teacher, it was an important class, as it allowed me to see that children felt a lot of difficulties in translating real and everyday language into Mathematical, symbolic language [Sara’s portfolio 1st reflection]

In her third reflection, she explains in a detailed way how important the activity had been for her, connecting it to the learning outcomes achieved by the students:

One of the factors which either contributed to or made some students’ learning difficult was the fact that it was an individual task, as it became complex for me to provide

answers to all requests as quickly as possible, which was what they wanted. Even so, this activity was very important for me, as I think I left the students motivated to work with the tangram, a material with which many mathematical themes or contents can be associated [Sara's portfolio 3rd reflection]

As a matter of fact, Sara was widely called on by students, either to help them build shapes or to draw them. She tried to answer all requests, by giving them some clues but, mostly, by reminding them that they had to try to build the shapes themselves. It was apparent that Sara experienced some difficulty in providing assistance to all the students, as, on one hand, the class was made up of over twenty students and, on the other, as she repeatedly mentioned during the activity, she wanted the students themselves to find out the answer.

Regarding the item – *the teacher's future perspectives regarding Mathematics* –, in her first reflection Sara presents future valuation of problem solving:

I think that in this class one must pay more attention to problem solving because it will help them to develop reasoning and prepare them for a future where they can more easily develop personal problem solving strategies and to, step by step, assume a critical attitude in face of the results [Sara's portfolio 1st reflection]

In the third reflection, she presents future classroom work perspectives, showing a definite interest on resorting to the use of manipulative materials:

Although it is a large and noisy class I would have no qualms about proposing a similar activity. I think it would be very useful for these children to work more with manipulative materials as they allow mathematical abilities to develop and to broaden knowledge in every area. They also allow imagination, reasoning and communicative skills to develop. [Sara's portfolio 3rd reflection]

Throughout the academic year, Sara has tried out problem solving more often, for instance using problems originating from the National Examinations.

Regarding this matter, in her final interview Sara stated there had been some changes in her teaching practice compared to the program's beginning and pointed out some aspects she had started placing more value on:

There have been several changes from the beginning of the program because I started giving more value to verbal interactions and the nature of the tasks put forward, to value learning more and to value reflection much more [final interview]

Also, regarding the use of manipulative materials, after attending the training program she greatly stressed their use, namely with regard to awareness of their capabilities:

I learned I can use known material such as the tangram and the geoboard, to teach concepts with I never formerly associated with them (...) We came into contact with new materials and with how to work with already known ones such as the tangram and the

geoboard but which were underused, which we had in the classroom but which we did not use as they could be used [final interview]

FINAL CONSIDERATIONS

Regarding the written reflections presented, although she always based herself upon the guidelines, Sara does not reason in both of them in the same way, either with regard to content or to depth.

With regard to content, there are some distinguishing aspects which naturally arise from each task's specificity, for instance: expectations regarding the noise to be naturally experienced while performing a task involving manipulative materials. However, in the first reflection, the diversity of themes approached within each category is very large. For instances, in item – *evaluation of what the students might have learned* – Sara highlights the students' main concern within the development of the task, identifying her own reaction and ways of handling the situation as well as the students' reactions. She also identifies solving procedures used by the students, difficulties felt and main learning outcomes of the students.

In her third reflection, there is a more restricted range of subjects approached. However, in general, she covers the main items of the guidelines and, essentially, focuses on her role in what she identifies as having developed below or against expectations. She specifies the aspects approached, directing them in a sustained way towards her students and towards more specific mathematical acquisitions. She tries to explain her statements in length.

Concerning the depth in her first reflection, there are contents which are only briefly touched upon (for instance, communication of the problem solving procedures), there are others in which she presents some justification for certain events (for instance, students' difficulties concerning problem solving). Thus, the first reflection is marked by confrontation with her own practice, some interpretation and very little putting into perspective, thus focusing on a retrospective dimension. In her third reflection, it seems possible to state that Sara has by now absorbed that which was fundamental to obtain from the activity undertaken, showing some distance from the specific items mentioned in the guidelines. She establishes connections among different items and always tries to account for her statements. She reflects upon the described points, showing her role in the development of the task and rethinking her future practice. She thus shows herself as having reached the level of appropriation and some approximation to the level of reconstruction, situating herself, in consequence, in a prospective dimension.

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DEVELOPING MATHEMATICS TEACHERS' EDUCATION THROUGH PERSONAL REFLECTION AND COLLABORATIVE INQUIRY: WHICH KINDS OF TASKS?

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Abstract. After the reprise of a model of intervention for the training of mathematics teachers (both initial and in-service) developed after experiences carried out in a cooperative modality (Pesci, 2007a), several tasks are presented for encouraging the development of disciplinary, didactic, and relational competences of the teachers. The theoretical framework related to these tasks puts in evidence the reasons of their choice: the importance, for teachers, of collaboration in sharing personal experiences, difficulties, and resource, the importance of autobiographical reflection, of reflection on one's own classroom practices, and of epistemological reflection on the disciplinary contents. The connection to the debate about tasks which is developing considerably in relation to the education of teachers (Javorski, 2007) is underlined.

Key-words: mathematics, teachers, cooperation, collaboration, tasks.

INTRODUCTION

This paper has two goals, that of developing and specifying the model of intervention on teachers delineated in the contribution at Cerme5 (Pesci, 2007a) and that of explicitly connecting the model to some crucial ideas for the education of mathematics teachers which the literature is highlighting with growing intensity. How do I intend to reach the two goals? By supplying examples of tasks for teachers which, on the basis of the mathematical contents proposed, on the didactic modalities adopted and on the requested personal reflections, make evident their theoretical motivations and their connection to the debate delineated by Jaworski (2007) and synthesized by Watson and Mason, in the same special issue of JMTE. More specifically, this paper foresees a brief look back at the model of intervention on mathematics teachers already outlined (Pesci, 2007a) and the description of some tasks for teachers which have the goal of promoting personal reflection on their own relationship with mathematics and the encouraging of epistemological reflection on specific mathematical contents. Then there are the synthesis of the theoretical background for the choice of such tasks, with reference to related literature, and some suggestions for future research.

A MODEL OF INTERVENTION ON MATHEMATICS TEACHERS

The main issues of the model described in Pesci (2007a) are summarized shortly in this section, with the aim to make evident the frame in which the following tasks

should be placed. The model was developed in the framework of situated cognition and distributed cognition:

The frame of reference is that of social constructivism, which emphasises discussion, negotiation of meanings, collaboration, and development of positive personal relationships (Ernest, 1995, Bauersfeld, 1995) and the concept of cognition is that formulated both as “situated cognition” (Nunez, 1999) with relevance to the context, and as “distributed cognition” (Crawford, 1997) with relevance to interrelationship and to sharing. (Pesci, 2007a, p. 1946).

The model was based also on cooperative modality, which gives special importance to relational and social aspects: in their different interpretations, all cooperative models share their explicit attention to both disciplinary dimension and social one. The goals to be reached along the educational process are not placed only at the disciplinary level but also at personal and social ones, with a special attention to the quality of the relationships established amongst people (Johnson, Johnson & Holubec, 1994, Cohen, 1984). At the base of the model (interpreted both for students and for teachers), there was, therefore, the idea of a co-construction of knowledge, a social construction, with the principles that for several decades, even with different accents, pervade the most diffuse teaching-learning models. At the centre of the learning process, managed by an expert, there are the learners and the inter-relationships (between learners and with the expert) with the consequent emphasis on the role of language and on the phases of discussion, argumentation, confutation, comparison, and sharing. What is suggested by teaching-learning cooperative models is also coherent to what is underlined by neuroscience (Damasio, 1999) and by epistemology (Polanyi, 1958): in each process of building or revisiting knowledge it is necessary, as a matter of fact, to keep track of the close connection between emotion, sentiment and cognition. This is valid not only for the students, in class, but also for the teachers, in their training meetings. In each training intervention, therefore, there was a special attention to the affective-relational aspects.

With reference to relational and social aspects, I consider essential that a meaningful intervention on mathematics teachers (a) could give time and space to their reality as teachers in that precise moment of their professional history through the autobiographical discourse; (b) could constitute a direct experience of what is proposed, with wide possibility of dialogue with the other participants; (c) could be, in each case, attentive to the modalities of communication. (Pesci, 2007a, p. 1952)

The main goal, in planning meetings for teachers, was to promote their personal reflection, taking account of disciplinary, didactic and relational aspects:

The basic idea is that of creating, in each encounter, occasions for personal reflection and for dialogic inquiry, with the same spirit stressed in the project *Learning Communities in Mathematics* (Jaworski, 2004), where the main objective is that both researchers and practitioners are engaged in action and reflection for mutual growing. (Pesci, 2007a, p. 1952).

The following tasks for mathematics teachers are examples of how it could be possible to foster their reflection and inquiry on the three different and essential aspects of their competence: *disciplinary*, *didactic* and *relational*.

EXAMPLES OF TASKS FOR MATHEMATICS TEACHERS

Autobiographical reflection. Every time that it is possible, in particular when the training meeting foresees more than one session, I organize the initial phase with the teachers starting with their personal relationship with mathematics, both with reference to their own history as *student* and to their own history as *teacher*.

In the first case, I propose answering several written questions, which have to do with their recollection of a pleasant episode (and respectively an unpleasant one) during a mathematics lesson, referring to all of their pre-university scholastic life. Sometimes I turn to the request for an opportune metaphor, such as “to do mathematics was like entering a jungle, or a challenging game, or a long marathon, etc.”, described in Pesci (2006).

In the second case, the activity of reflection on one’s own “history” as a mathematics teacher can come about through a choice of metaphor or with the request to complete a questionnaire of this kind:

From my “history” as a teacher

An episode to remember

An episode to forget

A moment of change

A wish that came true

A wish that didn’t come true

In both cases, the task, by its nature, is individual, but I usually invite the participants to share within their own group (of 4-5 people), if they want, the interpretation of the task or some experiences, both before writing and at the conclusion of the writing. The only recommendation is that, in each case, there is a period of silence, during which each person can collect his own thoughts and write calmly. To this aim, it is essential that, right from the beginning, each commits to observing the others attentively, being aware of when it is opportune to intervene with their own contribution or give space to the intervention of another person or remain silent.

The personal reflections which are asked for are of various natures and, obviously, depend a lot on the characteristics of the group itself. For example, in a group of teachers who have been in-service for several years, but were not yet confirmed, it came out that more than a third of the participants (there were about 60) highlighted, as a ‘wish not yet come true’ that of didactic continuity. It is clear that the same kind of wish does not appear anymore with teachers in regular service for years. It is not important, in this context, to list the different kinds of responses collected. Instead, it seems interesting to observe, at least, these two facts:

- the tasks of an autobiographical nature, followed by the sharing of personal experiences, have as a consequence to immediately orientate teachers' attention toward the other members of the group, reducing the attention which, at the beginning of the activity, everyone has toward the presenter of the training, and encouraging the perception of the others' resources, at the level of disciplinary competence and interpersonal qualities. When the activities are carried out together, it is, without a doubt, the most productive starting point;
- to put into play the one's own memories and one's own history is unusual but manages to capture the participants in an absorbing way: the result is a sort of requalification of the way of being present at the training event. Often I perceive in the teachers, also during the following activities, a less superficial, more meaningful, and more profound, involvement, as if the autobiographical connotation were able to give greater strength and authenticity to the actions that they share.

Reflection on one's own classroom practice. Amongst the tasks proposed to the teachers to encourage their reflection on their own classroom practice, I'll quickly cite two examples connected to two different kinds of experiences. During a cycle of seminars on how to confront the difficulties in mathematics, in the secondary school, it came out that all the participants (about 20) had already adopted specific strategies to help students overcome difficulties in mathematics. Therefore, I held it to be opportune to dedicate an entire meeting to the specific reflection on such strategies, inviting each one to respond to some questions, amongst which were the following:

You have already adopted specific strategies to help your pupils overcome difficulties in mathematics: choose, in the case of several strategies, the one which you hold to have been the most effective and describe how you realized it in class, according to the following chart:

- a) Strategy used
- b) With what frequency?
- c) With pupils of which classes?
- d) Briefly describe how you develop such strategy in class
- e) For which mathematical contents did you turn to such strategy?
- f) Which are, in your opinion, the strong points of such a strategy?
- g) Which are, in your opinion, the weak points of such a strategy?

Naturally, it was only the beginning of a longer path, certainly not exhausted in one meeting. Still, I noted that the participants were not used to reflecting on the methodology of their own practices, but they were almost exclusively worried about the mathematical content to develop in class. For example, it came out that whoever had tried to make the young people work in groups, had not structured the activity in any way, not foreseeing specific roles for the pupils and not planning sufficient time and adequate space for the activity. Even the mathematical questions were chosen without specific motivations. Analogously, whoever had proposed a learning experience of peer tutoring, had not programmed any form of collection of the work

carried out, neither for the pupil in the role of the teacher, nor for the one in the role of the pupil. Not having a clear idea that a key element for success, in these cases, is precisely the awareness of the importance of *setting*, they did not share with the pupils the methodology of the activity to be carried out and they did not put the right emphasis on it. The results, in fact, were not satisfactory.

In another case, following experiences conducted in classes with the cooperative learning modality, after a rather long period (more than a year), I had foreseen with the teachers specific instances of reflection on the perceived effects (positive or negative at the disciplinary or relational level) on the pupils and on themselves. Several questions and several results, which are not necessary to take up here, are described in detail in Pesci (2007a). Here, I would like to put in evidence some general observations, also in relation to what I noted during the seminars on the difficulty of learning cited before.

The modality that I put into effect with the teachers is usually that of sharing and discussion in small groups (4-5 people) before the general discussion and debate. I noted that this encourages, in a decisive way, the participation of everyone. Each one, in the small group, feels more welcome, safer, and freer therefore to express their own difficulties, their own fears, their own experiences and desires. Realizing that a fear (for example, that of not being up to maintaining control of the class) or a difficulty (for example, that of managing the time in class well) is common to others, gives greater strength to each one in the search for and sharing of the best strategies for confronting them. The requested reflections on the practices of the teachers go on to involve their acting in class and out of class and the sharing with colleagues shapes itself as an important occasion of comparison and growth. The relational competences of the teachers, specifically the ability to communicate with their colleagues, to share resources, and to confront together the obstacles has, without a doubt, a central role in the building of a team of prepared, reflective and able to change teachers (Dozza, 2006). In other words, it seems necessary to give time and space to such activity of personal reflection.

Reflection on specific mathematical contents. I will describe briefly two different situations as examples of the tasks proposed to the mathematics teachers for reflection on their teaching discipline. The first is more appropriate for a single intervention, which can be completed in one meeting and the second is more appropriate for starting a longer activity, which can be developed in successive meetings. Both tasks have the characteristic of a simple enough presentation, which makes the teachers curious and therefore easily involves them, but continuing on they become more complex. These tasks are therefore right to be confronted in collaboration, in the direction of the discovery of their didactic values and of the variety of the mathematical themes from which to choose possible developments. Besides, both the tasks could be proposed to the students, the first example starting

from the upper classes of the primary school and the second starting from the secondary school.

The first problem is placed in ZxZ (the “pointed” plane) and proposes a search for isosceles triangles with the oblique side assigned AB , limited to those with all three vertices in points of ZxZ . The investigation, apparently very simple, proves to be quite demanding, both for the geometric questions and the arithmetic questions involved. Besides, it can be developed with questions of isoperimetry, of equiextension, and of congruence between the triangles found, going on to weave together, in a single context, the use of arithmetic and geometric competences and of argumentative and demonstrative procedures. It is evident, therefore, that also the discussion about the didactic value of the problem turns out to be quite full and interesting.

The second problematic context looks at Euler’s formula and its validity; to be explored in several models proposed concretely or drawn on the blackboard. It is well known that in simpler cases, for example for regular polyhedra or for convex polyhedra, it is easy to count *faces*, *corners*, and *vertices* and immediately to verify the validity of the well known numerical relationship $V - S + F = 2$. In more complex situations, instead, one encounters some difficulties. It is necessary to clarify, on the one hand, which are the figures in the space that can be considered “polyhedra”, and on the other hand, which are the elements in the space that can be considered “faces” or “vertices” or “corners”. The two questions are obviously connected and it is well known how much they are not banal, as is highlighted by the historic reconstruction of the attempts to demonstrate Euler’s formula described in Lakatos’ book (1976). When this activity is proposed to the teachers, it usually turns out to be evident how it is right for encouraging collaboration and the sharing of resources. It has to do, in fact, with an investigation that is not taken for granted, with an obligatory end, but rather open to further reflections, of a theoretical type or also an epistemological one (Pesci, 2007b).

THEORETICAL FRAMEWORK FOR THE CHOICE OF TASKS

In this section there are the basic ideas which constitute the theoretical framework for the choice of the kinds of tasks described.

a) On the cooperative methodology to put into effect with the teachers, I have already described the theoretical references in the second section of this presentation. Here, I would add some reflections which could clarify better the features of the model proposed. It is important to remember that, in general, when one speaks of the shared principles of the models of social construction of knowledge, one has not yet arrived at outlining a standard didactical procedure, because for this it is necessary to choose the fundamental values which one intends to promote. As Ernest (1995) observes, standard didactical procedure is defined in each case on the basis of the *values* which one intends to promote. To define better the model of intervention experimented with

mathematics teachers, I would like to stress that the fundamental *value* that I chose to promote is the *collaboration* amongst all the participants (teachers and didacticians) at the educational moment. The goal is that of more easily arriving together at a higher result than that which each one could reach alone, whatever the proposed task could be. The term *collaboration*, here, could be interpreted as a synonymous of *cooperation* in reference to the fact of sharing the urgency to develop, in a symmetric way, both the cognitive-disciplinary and the affective-relational competences of the subjects. But here the term *collaboration* has a more general meaning: a positive inter-relationship amongst the people involved, not necessarily connected to a specific modality of acting in groups. The *collaboration* amongst the participants (teachers and didacticians) has the following goals: to encourage the sharing of personal experiences, of resources, of difficulties, and to encourage reflection on the mathematical contents, on their epistemological meaning, on their classroom practice, and on their own professional history. In short, the *collaboration* with peers, interpreted at the level of teachers, seems the most efficient road for covering the role of teacher, which lies within the competence in projecting the educational path and the reflection-evaluation of the processes activated.

I would like to add one last characteristic of this model. The interaction between equals, in a climate of positive *collaboration*, implies a particular *setting*, that is the organization of time, space, and modes of interaction which allow the progressive evolution of the disciplinary and relational competences. All that is a privileged environment also for the well-being and for the mental health of the participants (Dozza, 2006). Trust in oneself, generosity in the welcoming and helping of the others and the recognition of oneself in the others, contribute to affirming and enriching one's own identity in the community to which one belongs, supporting the development of personal potentialities.

b) Autobiographical reflection, by means of the use of metaphors or narrations of meaningful episodes from one's life, turns out to be a preferred tool for accessing the deepest parts of self, allowing that decentralization which is necessary to be able to tell about oneself (Barker, 1987; Darrault-Harris & Klein, 1993). The narration of self was rediscovered in the last 10-15 years as an educational modality which is important for both students and teachers (the first direct references to the autobiographical practice in adults' education can be found in French studies, i.e. Pineau, 1983, the Italian studies have been developed mostly starting with Demetrio, 1996). Amongst the objectives that can be pursued, there is fundamentally the reflection on one's own experience, in particular, on its attributive implications and on the causal links to the events of one's history. This allows the recognition that the narration of oneself is not a simple report of events, but rather a reinterpretation of them, in the light of the present. Telling about self means giving meaning, coherence, and continuity to one's various experiences and also encourages the definition or the reformulation of one's identity. Autobiographical reflection, elaborated for oneself, but also communicated to and shared with others, encourages a positive development

of interpersonal communication, the recognition and re-evaluation of personal facts and characteristics, the ability to listen to oneself and understand oneself, and a consequent openness to listening to and welcoming of others. So, it seems that autobiographical activity emerges as a fundamental tool in the work with teachers, a work which has at its centre the teachers in their totality, personal and professional at the same time.

c) The tasks of the disciplinary type proposed in the preceding section are, on the basis of the experiences carried out, particularly appropriate for developing epistemological reflection on mathematics in an *inquiry* style (Javorski, 2004), in a climate of investigation of mathematics which could be transferred to the class. With reference to this I would like to link to a question proposed by Watson and Mason (2007, p. 213).

We question whether tasks need to be structured in ways which require ‘inquiry’ or whether instead ‘inquiry’ is the mindset with which teachers, and ultimately their students, need to approach all tasks.

I would say that both things are necessary. A task must be interesting enough to stimulate involvement and action. It must be open enough, that is, appropriate to being developable in several ways and therefore with personalized in-depth study. In other words, the task has to be *generative* of several different possibilities of development (as Borasi well described in the 21 examples showed in detail in her book, 1996). Besides, the structuring of the environment in which the task is proposed must be adequate, in the sense that it must foresee times, materials, and attitudes which can fully support the investigative activity. In other words, the *milieu* (Brousseau, 1997), in which a task and the following activity take place, has to be suitable for the intended work. It is still evident that also the attitudes of the participants in the investigation must be appropriate, that is, ready to participate in the activity, allowing themselves to be involved in the problem and putting into play their own time and their own resources. The two aspects (the characteristics of the task and the attitude of the one who confronts it) turn out to be, in my opinion, strongly intertwined and they influence each other in turn. A task which does not have the characteristics cited cannot give rise to *inquiry* and on the other hand an appropriate task, proposed in an unprepared *milieu* for the inquiry, will not be developed and unlikely will not become object of research.

d) The last observation that I would like to propose is relative to the general sense of a training experience proposed to the teachers, with the modalities and by means of the tasks described. As shown also by the analysis conducted by Watson and Mason (2007, p. 208):

Tasks are often designed so that teachers can experience for themselves at their own level something of what their learners might experience and hence become more sensitive to their learners. The fundamental issue in working with teachers is to resonate with their experience so that they can imagine themselves ‘doing something’ in their own situation,

through having particularised a general strategy for themselves ... their professional choices of actions are the manifestation of what they have learned or are learning.

It is precisely in this direction that I develop each intervention on the teachers. I am convinced that a training meeting can be effective in the measure in which it can be set up, for the participants, as *metaphor* of experiences of living in class; a *metaphor* therefore understood not as verbal construction, but as life experience (Pesci, 2003, 2005, 2006, Fabbri & Munari, 2000).

CONCLUSION

The model of intervention on teachers and the tasks here described put an explicit accent on the necessity to intertwine *disciplinary*, *methodological* and *relational* aspects for teachers' professional preparation, without leaving out a special care for structuring an adequate *setting* for the intervention itself. A theoretical frame for this complexity can not be simple and, of course, it could be different from that here described. It could be the occasion for further investigation and analyses, for instance in the direction: a) to formulate different models which could describe the same complex "scenario" of mathematics teachers' professional education; b) to elaborate specific and adequate instruments of analysis of teachers' interaction, at the different levels of competences involved by the model proposed. A final observation refers to the importance the model puts on the necessity to take account of teachers' personal biographies (their personal stories, their preferences, their expectations). I believe this is a feature not yet explored in depth for teacher education (see for instance the review about the common assumptions related to mathematical tasks in teacher education in Watson & Mason, 2007). Such orientation could be of interest for research, with possible fruitful resonance from perspectives of teachers' educators.

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THE LEARNING OF MATHEMATICS TEACHERS WORKING IN A PEER GROUP

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The research described in this paper is part of a study in which we will follow mathematics teachers during a certain period and describe the development of their practical knowledge. Teachers' practical knowledge is their knowledge and beliefs that underlie their actions. In this study we are focused on what teachers know and believe about learning and teaching statistical investigation skills. Concept maps and semi-structured interviews are used to represent and archive teachers' practical knowledge. In addition, a system of four categories is developed which, in our view, is appropriate for exploring mathematics teachers' practical knowledge. The results show that although changes in practical knowledge occur within a year, not all changes are due to working together in a peer group.

INTRODUCTION

Because of educational changes teachers should be able to learn permanently, individually as well as together with fellow teachers. This study reports on the learning of mathematics teachers from the same school, collaborating in a peer group for a longer period. The area of interest is the development of teachers' practical knowledge by collaborating in a peer group in order to achieve an educational design in statistics for students in lower secondary school. By creating an environment in which teachers can learn and develop, they have an opportunity to revise their practical knowledge by using each others expertise. The researcher guides the meetings, but the teachers are making the final decisions in order to create ownership. This kind of professional development is new to the teachers involved. During the peer group meetings, teachers are developing a research task for students which also will be implemented and evaluated. The research task is aimed at students doing statistical investigations about a theme of their own choice. Implementing research tasks is one of the goals of mathematics education in The Netherlands.

THEORETICAL BACKGROUND

Learning of experienced teachers in a peer group

A considerable amount of current research on teaching and teacher education focuses on teacher collaboration. Teacher collaboration is presumed to be a powerful learning environment for teachers' professional development (Meirink, Meijer & Verloop, 2007). However, empirical research about how teachers actually learn in collaborative settings is lacking. Learning in collaborative settings stimulates teachers to use the expertise of colleagues for improving their own teaching practice, and therefore adjust, enlarge or change their practical knowledge (Borko, Mayfield, Marion, Flexer & Cumbo, 1997). Borko et al. (1997) mention: "We believe that

teachers would learn best by actively constructing new assessment ideas and practices based, in part, on their existing knowledge and beliefs, and sharing ownership of the workshop content and processes”. Furthermore, learning in a peer group is more intense when people with different ideas and opinions cooperate (Putnam & Borko, 2000). Verloop, Van Driel & Meijer (2001, p.453) mention that exploring teachers’ practical knowledge can be relevant in consideration of educational changes. In certain educational innovations teachers were only the executors instead of also the developers (see Van den Akker, 2003). To commit ownership in this study, teachers are developers and implementers of an educational design for learning and teaching statistics for students of the 7th grade of secondary school. Teachers afterwards evaluate the implementation of the design. Because they work together we expect an increased teacher learning, leading to more in-depth practical knowledge.

Development of practical knowledge

The research presented in this paper is focused on the development of teachers’ educational goals and practical knowledge of mathematics teachers when they collaborate in a peer group. The term *knowledge* as well as the term *beliefs* may frequently be found in studies about teachers’ cognitions. The concepts that these terms refer to are often not easily distinguishable. On the other hand, to explore and analyse the learning of teachers, the term practical knowledge is frequently found in studies about teachers’ cognitions (Kagan, 1990; Pajares, 1992) In most studies, only one term is used to refer to both knowledge and beliefs. Kagan (1990) states that: “Readers should note that I often use beliefs and knowledge interchangeably (...)”. Pajares (1992) also pretends that knowledge and beliefs are not distinguishable. He states that teachers’ beliefs are personal values, attitudes or ideologies and knowledge is a teacher’s more factual proposition, sometimes formal and sometimes practical. Meijer (1999, p.22) puts forward that: “Taken together, teachers’ knowledge and beliefs are a huge body of personal theories, values, fractional propositions, and so forth, that is to be found in teachers’ minds, and that teachers can, sometimes more easily than other times, call up and make explicit”. In this study, following Pajares (1992) and also Meijer (1999), teachers’ beliefs and teachers’ knowledge are viewed as inseparable. This will be referred to as teachers’ practical knowledge.

In this study we developed and used a system of four categories which, in our view, are the most appropriate for exploring mathematics teachers’ practical knowledge. Statements of teachers will be classified into the named categories. These categories are derived from the categories used by Meijer (1999, p.61) and Van Driel, Verloop & De Vos (1998). The categories will be described and explained below.

1. Educational philosophy

The category ‘Educational philosophy’ includes the vision of teachers on education in general, what motivates him or her to teach. Teacher’s educational philosophy can deviate from, for example, his actions in the classroom and does not need to

correspond with reality. This category is an extension of the categories used by Meijer (1999). Meijer used the category 'Student knowledge', this are thoughts about students in general, which is part of the category 'Educational philosophy' in this study. Teachers' educational philosophy is of great importance on his actions and thoughts.

Teachers' former experiences in the classroom have a strong hold on their educational philosophy, just like experiences with professional development and consultation between fellow teachers (see Meijer, 1999). Ernest (1989) mentions that the mathematics teacher's mental contents or schemes includes the vision on mathematical knowledge, beliefs concerning mathematics and its teaching and learning. Ernest states that educational changes only can take place when teacher's deep-rooted beliefs about mathematics and about the learning and teaching of mathematics will change. We expect to find particularly deep-rooted beliefs in this category, and therefore we expect the fewest changes in practical knowledge.

2. Learning and teaching statistics

This category includes teachers' practical knowledge of school mathematics, in particular of statistics. Within the scope of pedagogical content knowledge (PCK) also specific perception of statistics, learning difficulties and learning strategies of students within the domain of statistics are gathered in this category. Knowledge of teaching statistics is therefore also part of this category. This category is a combination of the categories 'Subject matter knowledge', 'Curriculum knowledge' and 'Knowledge of student learning and understanding' in the research project of Meijer (1999).

Next to practical knowledge, teachers need understanding of the subject matter content to teach a subject (Sowder, 2007). Shulman (1986, p.25) mentioned: "Where the teacher cognition program has clearly fallen short is in the elucidation of teachers' cognitive understanding of the subject matter content (..)". He thereby introduced the term pedagogical content knowledge (PCK). Verloop et al. (2001, p.449) indicated that PCK can be considered as a specific form of teachers' knowledge due to the focus on students and on subject matter. The category 'Learning and teaching statistics' is strongly related to teachers' working together in a peer group on the educational design and its implementation in the classroom. The teachers in this study are not used to working in a peer group. We therefore expect important changes in this category.

3. Student activities

This category describes teachers' practical knowledge about students in the first class of secondary school and students in general, their activities during the lessons of this course and their learning activities. A direct relation with the subject matter (statistics) is not necessary. This category is an extension of the category 'Knowledge of purposes' used by Meijer (1999).

Together with the category ‘Learning and teaching statistics’, this category is expected to be strongly influenced by teachers’ collaboration in a peer group. We expect a connection between the objectives of the design formulated by the teachers, how important teachers think research tasks are in math classes and the student activities during the course.

4. Teacher activities

On the one hand this category describes teachers’ practical knowledge of the use of materials during the math classes and the practical knowledge of statistical research assignments. On the other hand this category contains teachers’ practical knowledge of designing, implementing and evaluating lessons in statistics and teachers’ role during the implementation. This category is a combination of the categories ‘Curriculum knowledge’ and ‘Knowledge of instructional techniques’ by Meijer (1999).

Research questions

The main question presented in this paper is: How does the practical knowledge of mathematics teachers develop as a consequence of designing, implementing and evaluating an educational design (altogether this is called the intervention) for learning statistical investigation skills by working in a peer group?

The main question can be determined by answering three basic subquestions:

1. What is the practical knowledge of the participating teachers prior to and after the intervention?
2. What are the changes in practical knowledge of the participating teachers during the intervention?
3. Which are possible causes of changes in practical knowledge?

METHODOLOGY

In this study four mathematics teachers of the same school are collaborating in a peer group. During the seven peer group meetings they are developing an educational design in statistics for students in lower secondary school. After the implementation of the design, the last peer group meeting serves to evaluate the design in order to improve the content.

In the study presented in this paper, we use two of the three instruments Meijer (1999) used, completed with three other instruments. The instruments below were used in this study and are at the same time provided with an explanation:

1. A questionnaire about teacher background variables

Just like Meijer, Verloop & Beijaard (1999) we use a list with questions about the teacher’s background. There are patterns that indicate that it is of crucial importance how a teacher deals with his or her experience, training, and consultation with colleagues.

2. Two *concept maps* by each teacher referring to the teaching and learning of investigative skills: one concept map was drawn before the intervention (this is called CM[0]). The other concept map was drawn afterwards (CM[1]).
Explanations by the teachers about their concept maps, directly after the drawing of the concept maps. The explanations of the teachers are all recorded on tape and are used as an additional source of information to the concept map.
3. *Semi-structured interviews*. Like the concept maps we had two interviews: one before (Int[0]) and one after the intervention (Int[1]).
The interviews were held immediately after the explanation of the concept map, in one session.
4. *Registrations and evaluations* of all seven peer group meetings. All peer group meetings are recorded on a voice recorder and evaluated through written evaluation forms filled in by each teacher.
5. *Observations* of the lessons taught within the project. All the nine lessons of all the teachers were observed and recorded on videotape.

The first source of information gives an idea of teacher's experiences with teaching investigative skills during the past years. This will be used for an explanation of the teacher's development. The next two sources of information will be used to determine changes in practical knowledge of teachers. The fourth source of information serves to find causes for the observed changes or to indicate professional development. The fifth source of information serves as a validation-check and is meant to see if teachers 'teach as they preach'.

The combining and analyzing of data from the different sources of information was a procedure with six phases (Morine-Dershimer, 1993; Meirink et al., 2007). In this paper not all the phases will be described, only phase four, where we look at the similarities and the changes in practical knowledge by first comparing CM[0] with CM[1] and Int[0] with Int[1] and after that divide teachers' statements and answers over the named categories. To describe possible changes in practical knowledge and to find out what causes these changes, we use two interesting cases. The first case is a less experienced teacher, Ann, and the second case is an experienced teacher, Bart. The names of the teachers mentioned here are fictitious.

RESULTS

Case Ann

Teacher background variables

Female, 48 years old, ten years of experience in adult education and three years of experience in grades 7-10 of secondary school. Little experience with implementation of research tasks.

Changes in practical knowledge

Below, in table 1, a list of differences in pre- en post-concept maps and in pre- en post-interviews from Ann is presented. The differences are divided over categories and the instrument concerned is also specified in table 1. There is also a list of similarities, but this list will not be given here. We will focus on the differences, because the differences are more interesting.

Category	Differences
Educational philosophy	<ol style="list-style-type: none"> 1. These students are too young to state a hypothesis (from CM[1]). 2. "How did I learn it myself?" (from CM[0]).
Learning and teaching statistics	<ol style="list-style-type: none"> 1. The introduction assignment was not applicable, there was no relationship between variables (from Int[1]) 2. Nowadays you need a computer for presenting and processing data. (from CM[0]) 3. Statistical concepts should come up for discussion during the introduction (from Int[1]). 4. Evaluating the process with students is important (from CM[1]). 5. Implementation of statistical research requires a systematic routine (from CM[0]).
Student activities	<ol style="list-style-type: none"> 1. Some children could not work together at all (from CM[1]). 2. Students can ask each other critical questions about their posters (from CM[1]).
Teacher activities	<ol style="list-style-type: none"> 1. The role of the teacher is to guide the students (from CM[0])

Looking at the differences in table 1 it is obvious that the differences in the category 'Learning and teaching statistics' are dominantly present. This is partly a consequence of the used methods. The focus question of the concept maps is 'Learning and teaching statistics' and the interviews are also focused on the learning and teaching of statistics. Furthermore, the differences are mainly caused by Ann's basic assumption. Before the implementation of the educational design, in CM[0], she noticed "to be blank". Afterwards, in CM[1], she changed her basic assumption and noticed that the implementation of the design was the most important. Ann's teaching experiences in the past play an important role, enforced by experiences during the implementation of the educational design. However, Ann's research experiences do not play an important role anymore, though this was often a success (see CM[0]). During the evaluative peer group meeting it becomes clear that Ann still is enthusiastic about the educational design, although she proposed a few revisions like more interest in students working together and adjust the introduction assignments. Ann composed the student groups herself. She mentioned that she would do that again, because she is convinced that students have learned a lot by this way of working. Observations of lessons show that Ann is a good coach. She encourages her students to reflect on choices made and she is able to revise her goals if necessary. Repeatedly, she succeeds in creating a good atmosphere, in which students are able to work undisturbed.

Case Bart

Teacher background variables

Male, 47 years old, eighteen years of experience in teaching in secondary schools. In the past, he implemented two small research tasks, of which one was a statistical task.

Changes in practical knowledge

Table 2 below shows a list of differences in pre- en post-concept maps and in pre- en post-interviews with Bart.

Categories	Differences
Educational philosophy	<ol style="list-style-type: none"> 1. "Students understanding of the subject matter is very important. I didn't mention that because I haven't the impression that they really understood what they were doing" (from CM[1]). 2. "In any case, in my view students must have learned enough. There has to be a sufficient amount of data, the result has to be satisfactory and the teamwork should be good"(from Int[1]). 3. The factor time is important: "How labour-intensive is it?" (from Int[1]).
Learning and teaching statistics	<ol style="list-style-type: none"> 1. Strengthen that which is in the newspaper and on tv. Bart mentions: "That did go wrong. I couldn't make that clear either" (from CM[1]). 2. In CM[1] Bart is focused on students: "You now know what it was. You do not know that in advance. I automatically focus on the students. That is correct. intended or unintended" (from CM[1]). 3. Statistics in the observation period is not really hard: "We use the chapter Statistics to catch up in time" (from CM[0]).
Student activities	
Teacher activities	<ol style="list-style-type: none"> 1. "I found the teaching part rather awkward. In fact, I had no time left because of the method we used. Perhaps therefore I skipped it unintended" (from CM[1]).

Looking at the differences in table 2 it is obvious that the amount of differences in the category 'Educational Philosophy' and the category 'Learning and teaching statistics' are the same. It is remarkable that there are no differences in the category 'Student activities', while Bart is focused on the students during the construction of CM[1]. During the construction of CM[0] he also focuses on the teacher by adding the term 'teaching'.

The differences are mainly caused by the experiences of Bart preliminary to the implementation of the educational design. Bart is skeptic about students working in a team, because he experienced teamwork as unsatisfying. He thinks the lessons are more chaotic and that he loses control. However, lesson observations give another impression. Bart's lessons are well prepared with clear explanations and a great deal of structure. From the explanation of CM[1] and during the evaluative peer group meeting, it appeared that Bart doubts whether students learnt the statistical concepts sufficiently and if it would be better to use a more didactic teaching method. At least, that will save him a lot of time. It is remarkable that, although Bart does not believe in teamwork, he once again would choose for students working in teams. Next time,

he will choose smaller groups (two students) and let students compose the groups themselves.

CONCLUSIONS

To get an accurate insight into teachers' practical knowledge and its changes, the construction of concept maps combined with the semi-structured interviews give important information. The classification used here gives a structural description of the practical knowledge of Ann and Bart. It turns out that this knowledge of both Ann and Bart is deep-rooted; it is derived from former experiences and confirmed by implementing the educational design (see Ernest, 1989). The category 'Learning and teaching statistics' embodies the most similarities in practical knowledge, but also the most differences. The practical knowledge in the category 'Learning and teaching statistics' depends highly on the experiences perceived during the intervention. Besides, the changes in this category are probably due to the experimental design. Even though he had a less positive experience before the implementation of the design, Bart's ideas about teamwork do not change. He maintains his opinion that direct instruction is more effective than teamwork. On the other hand, Ann could adjust the goals easily during the lessons. She was more flexible and she showed more persistence during the selected trajectory (see Pajares, 1997). Both Ann and Bart, however, were willing to make concessions during the peer group meetings. They experienced the interest of combining each other's ideas and constructing an educational design to which everybody could commit.

In a follow-up study it would be interesting to look at the different roles teachers play in peer group meetings. Ann, for example, appeared to be a leader, highly committed and motivated. Bart appeared to be a follower, trusting the ideas of Ann (Shamir, 1991). We also need to look more closely at the categories involved in this study. It is difficult to categorise teachers' statements. Furthermore we may need to use sub-categories or rename existing categories.

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USE OF FOCUS GROUP INTERVIEWS IN MATHEMATICS EDUCATIONAL RESEARCH

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In my doctoral work I studied three mathematics teachers in lower secondary school in Norway and how they interpreted a curriculum reform, L97 (Hagness & Veiteberg, 1999). This study included methods as focus group interviews and individual interviews with teachers, teachers' self estimations and classroom observations (Kleve, 2007). In this paper I discuss how I used focus group interviews both for the purpose of obtaining information from teachers about their mathematics teaching, about their beliefs about teaching and learning mathematics and also for the purpose of validating the whole research and its findings.

Keywords: Mathematics, Ethnography, Beliefs, Focus groups, Curriculum reform

RESEARCH METHODS FITTING INTO AN ETHNOGRAPHIC APPROACH

If one wants to find out something, one “goes out and has a look” (Pring, 2000, p. 33). In my research I wanted to find out how teachers interpreted the curriculum and how they implemented it in their classrooms. I therefore decided to enter the mathematics classrooms to investigate teachers' practice, and to have focus group interviews with the teachers to find out what they said about L97, their own teaching practice and about mathematics teaching and learning.

I conducted an empirical study using research methods fitting largely into an ethnographic style of inquiry. The study was a case study of mathematics teachers' interpretation of the curriculum reform L97, both in terms of what they said about it and in terms of their classroom practice. Focus of the study was how teachers' practices were related to their beliefs about teaching and learning mathematics.

I chose methods of data gathering in line with methods suggested in the literature to carry out research with an ethnographic approach (Bryman, 2001; Eisenhart, 1988; Walford, 2001; Wellington, 2000). I used focus group interviews, individual interviews with the teachers, classroom observations, estimation form and teachers' own writings about ideal teaching. All these research methods provided me with data to analyse with regard to teachers' teaching practice and their beliefs about teaching and learning mathematics. Use of focus group interviews which this paper is about, was thus one of several research methods I used in addressing teachers' beliefs.

I used focus groups both for the purpose of selecting teachers for my study and as a research method. I contacted the school leader of a community outside Oslo. The teachers who participated in the first meeting were selected by her. None of these teachers became part of my further study. The next two focus groups were conducted with teachers from three different schools in another community. They were selected by their headmasters whom I had contacted. Four of these teachers became part of the

whole study and participated in the fourth focus group meeting which took place after the classroom observations. The process by which the teachers for my study were selected is beyond the scope of this paper. However, it is outlined in Kleve (2007).

Focus groups contain elements of two research methods: it is a *group interview* and the *interview is focused*. The members of a focus group are invited because they are known to have experience from a particular situation which in this case was teaching mathematics. A focused interview is to ask open questions about a specific situation (Bryman, 2001).

According to Krueger (1994) focus group interviews are useful in obtaining information which is difficult or impossible to obtain by using other methods. Using focus groups generally means that the researcher can intervene into the conversation and pose questions to probe what somebody just has said. According to Bryman (2001) the use of focus groups has not only a potential advantage when a jointly constructed meaning between the members of the group is of particular interest. Participants' perspectives are revealed in different ways in focus groups than in individual interviews, for example through discussion and participants' questions and arguments. However, Bryman pointed out possible problems of group effects in a focus group situation that must not be ignored. I experienced such group effects and I realise the importance of treating group interaction as an issue when analysing data from the focus groups.

TEACHERS' BELIEFS ABOUT MATHEMATICS TEACHING

In my study I use the term belief, and I look upon teachers' beliefs about teaching and learning mathematics and about L97 as cognitive constructions highly influenced by socio-cultural factors such as teacher's own experience and the school context, and also influenced by the teacher's knowledge in mathematics and about mathematics teaching. The insight I can get in my research into teachers' beliefs is through what the teachers say and write and through my interpretations of what I have observed in their classrooms. I do not look upon beliefs as something that can be directly observed. Through the use of different theoretical lenses, my conceptions about teachers' beliefs have to be inferred from what they say about what they are doing in the classroom; what they say they think about their practice; what they say they think is good mathematics teaching and what they say about L97.

It has been important for me both to study teachers' beliefs about teaching and learning mathematics and also what I observed them doing in their classrooms. Thompson (1992) wrote that in order to understand teachers' teaching practices from the teachers' own perspective, understanding teachers' beliefs with which they understand their own work is important. I do not see a teacher's beliefs and his/her practice as a cause-effect issue, but rather as a reflexive process. A teacher's beliefs are influenced by his/her practice and the interactions in the classroom are again influenced by the teacher's beliefs. A teacher's practice can both act as a reinforcement of his/her beliefs but also as an incitement for change.

One component of the teacher's interpretation of the curriculum is what s/he did in the classroom, the enacted curriculum (which is also influenced by incidents in the classroom, students' interactions, behaviour, and so on). The other component is what the teacher said in focus groups and in conversations, what s/he wrote and his/her responses to an estimation form. It was the relation between these two components I studied. It is the latter I term teachers' beliefs.

ANALYSING DATA FROM FOCUS GROUPS

A challenge in using focus groups was to what extent I was able to interpret the meanings lying behind and looking through the words the participants were saying and from that make inference about the teachers' interpretation of the curriculum. In analysing the data from my focus groups it was important for me to be aware of the different levels of information the data give. On one level teachers speak from their inner thoughts and meanings, struggling to express what are really inside their heads, they speak from their individual constructions they have perceived viable in their own practice. On another level they speak from what they know as a teacher and what they say is deeply embedded in social practices of being a teacher, and thus socio-culturally rooted. A third level can be rhetoric: The teacher knew who I was, and could try either to express what s/he was thinking I wanted to hear or since s/he knew what the curriculum said, s/he could express that or s/he could challenge that. In such cases the teachers would respond to me and who I am rather than to who they are. When analysing what teachers said in focus groups it is important to be aware that the teachers' views were revealed in different ways than in individual conversations. What they said could be a way of positioning themselves rather than trying to express their inner thoughts. Information revealed that way illuminates other aspects of teachers' beliefs than aspects illuminated through use of other research methods. Krueger & Casey (2000) encourage use of questions leading persons to speak from experience rather than wishes for or what might be done in the future. That increases the reliability since it focuses on particular experience from the past.

What the teachers said in focus groups conducted before classroom observations was not influenced by my presence in their classrooms and individual interviews. In that respect data from these focus groups provided me with information about teachers' beliefs and practice which went beyond what was obtained through the other research methods. On the other hand, data from the focus group meetings were also valuable for the purpose of triangulation and supporting the other sources of data from the teachers' utterances (individual interviews, self-estimation, writings and questionnaire). I audio recorded and transcribed the discussions that took place in these groups. Below I present some findings from these meetings which highlighted issues from perspectives of L97.

FOCUS GROUPS AND TEACHERS' BELIEFS

I will now present an analysis of the third focus group (FG3), which was conducted before I started the classroom observations.

The teachers participating in this focus group were the four teachers in my study: Alfred, Bent, Cecilie and David. In addition Petter, Kari and Tom, one of my former students, participated. For this focus group I had prepared the following questions for discussion:

- What in your opinion is important competence for mathematics teachers?
- In what way do you relate your work to L97?
- Has L97 inspired you to try out new activities in your mathematics teaching?
- What is the greatest challenge in your work as a mathematics teacher?
 - What have you succeeded with?
 - What do you think you have not yet accomplished?

I started with the first question explicitly, and aspects of other questions were addressed as part of the discussion. However, there was no time to discuss the last two parts of the fourth question. What the teachers felt they had succeeded with and what they found they had not accomplished, were issues explicitly discussed in the fourth focus group meeting later in my study. An analysis of this focus group interview (FG 4) is presented in the final part of this paper.

Focus groups from a socio-cultural perspective

How does what participants say reflect meanings of the group or society more widely? How does what they say reflect aspects (including criticism) of the political and cultural society, of dominant groups influencing the official educational discourse (Lerman, 2000), of their own school situation as a teacher or the one they had as a student themselves? Or how does what they say reflect aspects of the curriculum?

To illustrate this I will provide an example from FG3 which shows use of rhetoric. David knew who I was; he knew I was a teacher educator; he knew I had carried out courses for teachers in relation with the curriculum reform. Therefore, I conjecture David thought I wanted to hear nice things about the curriculum. Based on his understanding of what L97 said, he challenged it. This could have been because he wanted to position himself within the group, but it could also have been because he really meant that L97 is not a good curriculum for the mathematics subject. Yet another way to interpret what he said and why can be that he did not really know what the curriculum was saying, and he wanted to react reluctantly to it from the very beginning. In the quotation below, Petter (P) indicated he was sceptical to L97. David (D) then said (sarcastically?): “there are some nice pictures in it”. That illustrated how teachers argued for or against a new curriculum, how they interpreted it. The language (also what was not said) was a mediating tool in the exchanges of meanings. Petter was the most experienced teacher in the group and had a special role here. He indicated something to which David responded and it illustrates how what they said was deeply embedded in the socio-cultural setting in the group and their experience. (I is me)

- I: L97, how well do you know it? P, you seem dying to say something...
- P: Yes, I feel I am getting hot-headed when you mention L97.
- D: There are some nice pictures in it (sarcastic?)
- I: Now we have talked very much about how L97 is weighting the mathematical topics. But what about the working methods it initiates? Do you have any opinions about that?
- D: Read the newspaper, many interesting writings about it there.
- [There had been written many critical articles in the newspaper about L97 recent days]
- I: But what do you mean?
- D: I am critical to the correct pedagogical view we are served from above. I am not sure if it is right.
- I: Can you say some more about it?
- D: I believe that maybe pupils learn most if they have a teacher, who knows their things, is enthusiastic, finds teaching being fun, who is a good motivator, and good in making the pupils function together. I really believe that the learning outcome becomes better than than if the students have lessons outdoor, working schedules and so on. I dare being that old fashioned, I think so.
- P: One must be allowed to disagree with L97? Or?
- D: Disagree, and say it over and over again, everywhere you are
- I: I want to know what your disagreement is about. What is the pedagogical view coming from above?
- D: I think it implies knowledge's loss of flavour. Projects where pupils find something on the internet print it out and read it with a few replacements of words in front of the whole class.
- I: Is that what L97 says?
- D: No, but that is what happens.

My experience with Petter and David, and to a certain degree also Alfred (he was not so outspoken as the other two) in this focus group was that they were supporting each other with regard to a kind of ignorance towards L97. They had been teaching mathematics for many years, and they expressed their frustration of how the "old" kind of mathematics, especially algebra, was not in the curriculum any more to the extent they wished. Their mutual support in these views expressed in the focus group can be looked upon as communication of a rhetorical kind.

Next I will provide an example of how what teachers said in the focus groups reflected aspects of their experience as a teacher. Reflecting on the utterance from Bent below, he talked from a socio-culturally related everyday experience. Bent

offered us something about the way he operated in the classroom. He spoke from his experience as a teacher, and what he had learned from this experience. From the quotation below it may be hard to understand what he meant, which demonstrates his struggle to express his experience. He said that teaching from the board could start off from a simple level. However, very soon what was presented from the board became too difficult for some students whereas others wanted to proceed even further. This illustrates the challenge of having students with different abilities in the same class. He said:

I think a typical course, when you shall start with a new topic, is to teach from the board in the beginning and to start with something simple and then build it up to a certain level, and to work on tasks parallel to that. At a certain level you just have to stop the lecturing and separate. Some disappear far up and some remain on that level if they have at all reached the level they should. After that it is almost impossible to deal with teaching.

Below I will discuss how Bent went beyond his experience and offered us some of his reflections on his teaching.

Aspects of teachers' confidence

When studying the transcripts, which I had imported into NVivo, I noticed how the teachers expressed differing degrees of confidence throughout the discussion. Bent suggested the ability to motivate the students, and the importance of having mathematical knowledge to get an overview of the subject oneself, as competencies for a mathematics teacher. He used the expression "I am trying to ..." when relating these competencies to his own practice: "I am trying to relate to practical issues, trying to make a relation to real life in a way, however I don't always manage". He was "trying to" make the students see the relevance in what they worked with; he was "trying to" convey the mathematics' intrinsic value, especially when it was not so easy to relate the mathematics to students' everyday life. He also said that he was trying to be enthusiastic. His use of words when speaking from his classroom practice revealed that he was not sure if he succeeded in doing what he thought was important, but he was trying. Continuing the quotation from Bent above, he went beyond his everyday experience in saying something about the issues that arose for him when he operated in certain ways, and his thoughts about it. Bent also revealed some of the "weaknesses" he perceived in himself as a teacher. He had tried out something but through what he said he demonstrated awareness that this might not have been the right thing.

Then you have to walk around giving tasks. Last year I optimistically tried MUST tasks, OUGHT tasks and MAY tasks, that they should try to stretch themselves, but I didn't succeed in making it work. It turned out to be that they did what they had to (MUST) (agreement in the focus group), and some just tried OUGHT. But if they had homework in other subjects, they chose the less challenging way. So then it was easier to do as P says, give many tasks and rather reduce for those who need it. It is easier to put pressure on those who need challenges.

By saying this Bent also demonstrated that he had reflected on his own practice as a teacher. Being able to put his weaknesses as a teacher on the spot like this and sharing it with me and the other teachers in the group, I do not interpret as lack of self confidence but rather as reflecting a teacher who had faith in himself and had self confidence enough to be able to see his own teaching from more than one point of view. He had been able to step aside to consider his own teaching.

Bent also offered us his reflections on different levels of students' learning of mathematics, in which the other teachers consented, but without any further discussion. Bent said: "I have a feeling that they learn on different levels". He said that on one level they learn to solve a problem theoretically and perhaps manage to solve a similar problem in a same kind of context: "you have learned it in one setting on one level". He said:

The next level is being able to carry out what you have learned theoretically for example about symmetries, and applying that when searching for and finding symmetrical patterns in a carpet: Going out looking in math-morning [which was the project work he talked about], having to apply it, then you learn and experience on a higher level.

He called this an "application competence". On yet another level you learn by expressing a problem orally. He said: "Formulating a problem for others is yet one level of learning".

When Tom said he felt that he did not know how to make students understand, especially those with "two"¹ in mathematics, David responded:

I believe you'll have to live with that as a teacher. It is classical. You can work with some students throughout three years and they do not see /understand /remember the difference between $2x+2x$ and $2x \cdot 2x$. Even if you stand on your head and invent all possible variations you can think about there will still be some I believe [who will never manage], regardless of how clever you are as a teacher.

By saying this David demonstrated confidence as an experienced teacher. He spoke from his own experience as a teacher, an experience he knew that Tom did not have. This utterance also reflects a view that not all mathematics is for everybody, and that you cannot put the responsibility for this (the "two-students" not understanding or remembering) on the teacher. Through his long experience as a teacher, David had learned to accept this and he was now telling that to Tom who was a younger and less experienced teacher.

Cecilie also demonstrated self-confidence when telling about how she was handling the issue that students with different abilities in mathematics were placed in the same class. She had mixed two classes and grouped them according to interest in mathematics. She expressed her disagreement with Tom who had said that clever

¹ He referred to getting the grade (mark) 2 in mathematics which is the lowest passing grade. 6 is the best grade.

students will always manage, and she recommended the other teachers to group the students according to abilities (“interests”) the way she was doing.

The above discussion about aspects of teachers’ confidence demonstrates how such information can be obtained through the use of focus groups. The way in which teachers expressed their confidence in own teaching practices highlighted issues of their teaching practices and informed my investigation of how they responded to a curriculum reform.

Mathematical focus

To highlight issues of my study of teachers’ mathematics teaching, it was useful to study what aspects of mathematics they talked about in the focus group. One significant aspect throughout the conversation in the focus group was that *algebra* was the mathematical focus teachers mentioned most frequently when expressing their meanings and exemplifying from their teaching. David referred to algebra several times and was very concerned about algebra having been toned down in the new curriculum and said that he put more weight on algebra, equations and functions than L97 suggests. He also said that he would keep doing it because some students would need it for further studies. David said he was not so eager to force all work within mathematics into an everyday context: “I am more concerned that mathematics is a ‘logical and playing subject’. When the students have done a huge algebra task and say ‘YES I have managed’, that makes me happy”.

Bent also referred to algebra when expressing the importance of the mathematics’ intrinsic value. He expressed the value in itself of having the knowledge to solve an algebraic task or equation. Furthermore, Bent talked about having carried out a project work in mathematics which had been very successful. L97 encourages interdisciplinary project work and also project work within each subject. It was one of the latter in mathematics Bent referred to.

Cecilie mentioned algebra together with mathematics history as exciting topics to work with in her teaching of mathematics.

With regard to my study, what the teachers said in this focus group and how they said it gave me information about how the teachers responded to L97 in terms of what they were saying about it and what they were saying about their own classroom practice. The focus groups highlighted key issues and gave me a starting point for working with each of the teachers, Alfred, Bent, Cecilie and David, who became part of my further study.

FOCUS GROUPS FOR THE PURPOSE OF VALIDATING THE RESEARCH

The last focus group I had with the teachers who had been part of my study took place towards the end of my work with them. I have chosen to comment briefly on my findings from Focus group 4 for the purpose of cross case-analysis and also to illuminate and validate my findings from the rest of my study with the teachers.

I had asked the teachers to prepare two issues to share with the group; first, one issue they felt they had succeeded in carrying out as a mathematics teacher and one issue they felt they not yet had accomplished. They found the task difficult. However, after a few minutes discussing and reflecting on the difficulty of the task, Cecilie volunteered to start with hers. She felt she had succeeded in challenging and motivating the clever students, which is in accordance with what she had expressed in our conversations. The task she felt she had not yet accomplished was enabling the students to copy out their written work in mathematics clearly. Bent responded by expressing that more important for the students than the written presentation of mathematics is for them to understand when to multiply and when to divide in working it out. This emphasises Bent's focus on students' conceptual understanding which I also found through my work with him in the classroom and in our conversations.

Bent chose to present issues from two of the lessons I had been observing with regard to what he felt he had succeeded in and what he not yet had accomplished. His presentation of the issues revealed that he had been reflecting on these lessons. About the fraction lesson he said that he felt he had succeeded to a certain extent. However, he could have done more with it. With regard to the use of concrete materials, he expressed a disappointment that the effect had not been as intended. It had however been better in the other 9th grade class he was teaching. He thus expressed a feeling of having succeeded with the use of concrete materials in that class (in which I did not observe). This suggests that the complexity of the classroom and the classroom discourse often influence the outcome of an activity, and thus the enacted curriculum which is jointly constructed by the teacher and the students and the materials used.

Presenting what he felt he had been successful with, David said: "I have managed to make them cleverer in doing percentage calculations". This emphasises how he looked upon himself as conveying mathematics to the students and that students' learning is dependent on the teacher's ability to explain. When he was asked by the others in the group how he had done it he said: "It is just to explain as well as possible". This emphasises further how he looked upon explaining as the most "efficient" teaching strategy, which also characterised his teaching. However, he also offered an elaboration of how he had done it which revealed that he as a teacher was consciously systematic when presenting mathematics for his students. He said:

I have been very systematic with percentage types 1, 2, 3, 4, 5. Therefore, when one of the types turns up, I refer to the type. Number 1 is like "3 students absent how many percent?" Then it is in connections with changes, then having to calculate backwards, and then comparing two numbers.

David's systematic way of preparing the mathematics to be taught was a feature in his teaching.

With regard to what he had not yet accomplished, David focused on kinds of errors students made, especially how they used the equal sign wrongly, and he also

supported Cecilie in her suggestion: how to enable students to copy out mathematics in a lucid written way which clearly showed how they had solved the task.

What was said in this last focus group emphasises my findings from the analysis of the individual teachers: Cecilie felt she was successful in her work with the clever students, but had difficulties enabling students to present written mathematics with a clear overview; Bent reflected upon both success and not-yet-accomplished aspects of the issues presented; and David felt success in explaining and had not yet found out how students could avoid making errors. For detailed portraits of the three teachers see Kleve (2007).

This last meeting provided me also with information beyond what I had observed in the classroom, and what I had talked with the teachers about in the conversations. Bent offered his reflections around his work with fractions and use of concrete materials. Cecilie shared her difficulties with enabling students copying out their written work clearly, in which David supported her. By challenging David about *what* he had done to make students become good in percentage calculations we were initiated into a systematic way of preparing his teaching. This demonstrates that the use of focus groups provide researchers with information beyond what can be obtained otherwise.

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ANALYSIS OF INTERACTIONS IN A COLLABORATIVE CONTEXT OF PROFESSIONAL DEVELOPMENT

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We consider that the processes of interaction in a collaborative context of professional development have a significant influence on the degree of involvement of one of the participating teachers, and modulate the influence the context exerts on her professional development. We present an instrument for the analysis of interactions, which was developed in the course of this research and which aims to capture the dialogical nature of the discourse through three defining features distributed across six columns: the unit of information (utterance); the co-participants (the teacher and Interactant); and the contexts providing the sense of each contribution (Episodes, Action and Nature of the action). We also include a column for Content to complete the analysis with the epistemological input of each contribution to the discourse.

Keywords: analysis of interactions, collaborative context, professional development, dialogical approach, mathematics education.

INTRODUCTION

This paper is part of a longitudinal study researching the professional development, in terms of mathematics teaching, of new entrant into primary teaching participating in a collaborative research project (PIC) (Muñoz-Catalán et al., 2007).

The collaboration is composed of two experienced primary teachers, three researcher-trainers, and Julia, the subject of the study (from her first year of teaching onwards). The group meets once a fortnight for three hours, during which tasks are carried out with the aim of deepening understanding of our own classroom practice, as well as the learning and the teaching of mathematics from a problem solving perspective. Until now, this project had remained the background to our studies, constituting a privileged source for data gathering (Climent & Carrillo, 2002). In the case of Julia, however, given the relevance that this project has proved to have for understanding her professional development, the analysis of Julia's interactions within the group has emerged as a key element for understanding not just the what, but also the how of said development. We believe that in and through the interaction, Julia goes about constructing her interpretation of the suggestions, critiques and knowledge brought into play, an interpretation which moulds the formative potential of the PIC.

So as to analyse Julia's interactions in the group, we have devised an instrument which is presented in this paper, and which we refer to as IMDEP (the Spanish acronym for Instrument for the analysis of Teacher's Interaction in a context of Professional Development). It has been devised during the research process

following our methodological perspective of allowing the data to speak (Strauss & Corbin, 1998), and consonant with our dialogic perspective of the discourse (Linell, 2005).

A DIALOGIC APPROACH TO THE ANALYSIS OF THE DISCOURSE

We consider that knowing implies an interaction with the object of knowledge, through which the subject interprets and reconstructs the meanings in play in the process. Following G. H. Mead and J. Dewey (in Corbin & Strauss, 2008), knowledge is created through action and interaction, for which reason we attribute a relational nature to it. According to this perspective, we can identify cognition with communication in that the interaction is an essential requirement for each to develop. While communication necessarily requires an interpersonal exchange, cognition can occur in solitary activities such as reading, in which the interaction is with the text. Communication and cognition, then, are two aspects of the same phenomenon, and are dialogically interlinked (Linell, 2005).

Our interest in Julia's construction of meaning activities within the group led us to approach the analysis of interactions with a dialogic conception of discourse (Linell & Marková, 1993, Linell, 1998, 2005). We recognise that people's responses to others' actions depend on the meaning they attribute to them. From this perspective, human dialogue is more than the sum of individual discourse acts; it is a sequence of activities with the aim of establishing mutual understanding on the topics under discussion. In this sense it is a question of shared activities, coordinated amongst all the members and mutually interdependent (Linell & Marková, 1993; Marková & Linell, 1996). The semiotic mediation acquires a key place in the communication, which "*may be understood as some kind of abstract third party in the dialogue*" (Linell, 2005, p. 10).

The relation between discourse and its context is one of interdependence: a particular discourse derives a large part of its sense from the specific context, but at the same time "*these contexts would not be what they are in the absence of the (particular) discourse that takes place within them*" (Linell, 2005, p. 7). This interdependence is established at two levels: on one hand, the specific time and place in which the interaction takes place (situation); on the other, the sociocultural praxis governing the specific situation. This is what Linell (2005) refers to as the double dialogicality of discourse.

Following the dialogical approach (Linell, 2005), the principle features we can attribute to conversation are interaction, context and the joint construction of meaning, semiotically mediated.

THE INSTRUMENT FOR ANALYSING INTERACTIONS: IMDEP

We can understand professional development as defined by an increased awareness of the factors bearing upon educational phenomena and contributing to a better understanding of one's own practice (Krainer, 1999). Practice becomes a source for development when the teacher becomes actively involved in the process of questioning their own practice (Jaworski, 1998), and develops a critical, reflexive attitude. In this conceptualisation, reflection becomes medium and referent of the development (Climent, 2005; Llinares & Krainer, 2006).

Analysing Julia's interactions in the PIC allows us to focus on her construction of meaning within the frame of shared construction. Our focus, then, is not on the result of this social construction, but rather the individual processes of construction within the said social construction. We concur with recent studies, such as Llinares & Krainer (2006) point out, in considering contextual and organisational elements as key to accounting for teachers' learning.

This analysis leads to a better understanding of how reflections deriving from the group influence individual understanding and performance. The features of Julia's contributions to the discourse provide clues to the meanings which she attributes to the joint understanding under negotiation at each stage of the conversation.

Development, structure and features

This instrument emerged during the research process in close relation with the data (Strauss & Corbin, 1998). Our focal point was Julia, and hence our analysis of interactions centred on her contributions to the discourse. In the same way that dialogical properties can be attributed to a single contribution to the discourse, without considering previous and subsequent contributions (Linell & Marková, 1993), so can they equally be applied to the set of contributions by a single member, namely Julia.

Audio recordings are made of all the PIC sessions and fully transcribed, recording the contributions of all members. The transcription does not include gestures, but provides a verbatim record of all spoken language, along with all information concerning the discourse relevant to our understanding. The presence of the researcher in the PIC sessions ensures a better interpretation of each contribution, given that the dialogue is constructed in and through the processes of interaction and in relation of interdependence with the contexts.

With respect to analysing Julia's contributions to the discourse, we were interested in recording to whom they were directed, in what moment of the session, the form in which the action was expressed, its nature and the content it conveyed. These concerns became questions which guided the close inspection of the data, and which resulted in the instrument below:

Utterance	Episodes	Julia	Action	Interactant	Nature of the action	Content

Instrument for analysing Julia's interactions in the discourse in a training context

The instrument aims to capture the dialogical nature of the discourse, and covers the three key elements felt to be intrinsic to all the interaction: the unit of information (the column labelled *Utterance*), the co-participants (*Julia* and *Interactant*), and the context which provides the meaning of each contribution (*Episodes*, *Action*, *Nature of the action*). An additional column, *Content*, was added in the interests of linking the sociological aspect of each intervention to its epistemological contribution to the dialogue.

We consider the contribution as the basic unit of interaction, equivalent to the turn with respect to dialogue (Linell, 1998). A numerical code was assigned to each of Julia's contributions, indicating the order in which each appeared in the discourse. This code is the content of the *Utterance* column.

The columns *Julia* and *Interactant* refer to the co-participants in the communicative exchange under analysis at any particular moment. Each contribution must be understood in its sequential environment (Linell & Marková, 1993) as it is dependent on previous and subsequent contributions. As a result, we understand the participant at in each turn to be both emitter of their own contribution and receiver of the previous contributions of others (including those not specifically directed at them). Nevertheless, when we broke the group interactions down into contributions during the analytical process, we identified two types of operational interlocutors for each of them: the person originating the contribution, that is Julia in all cases so far as this study is concerned, and the addressee of the contribution, whom we designate with the generic label interactant (whether the group as a whole or some member(s) of it).

The transcript for each session was also analysed from the point of view of content, with units of information being identified. The code for these units corresponding to each contribution comprises the column *Julia*. Whilst it might be observed that this column could be substituted for that of *utterance*, given that it is essentially a new way of codifying the same contribution, each column nevertheless fulfils different analytical aims: the *utterance* column focuses on each contribution from a discursive perspective; the *Julia* column locates Julia's contributions with a view to analysing their content and so serves as a bridge between analysis of the interactions and analysis of the content (both at different moments of analysis, but subsequently integrated into a joint interpretation).

We now turn our attention to the third item we have highlighted as key to the processes of interaction – the context (as reflected in the columns *Episodes*, *Action* and *Nature of the action* in the instrument).

We are aware of the variety of factors which influence and interact with each other at each moment of the interaction. Strauss & Corbin (1994) represent this influence as a

conditional matrix, formed by concentric circles corresponding to distinct aspects of the world: “*In the outer rings stand those conditional features most distant to action/interaction; while the inner rings pertain to those conditional features bearing most closely upon an action/interaction sequence*” (p. 275). Out of all the circles we are interested in those that are most germane to each session and at each moment of the interaction. This leads us, on one hand, to structure each session into *Episodes*, and on the other, to consider the sequential environment, that is, the simultaneous dependence of each utterance on the adjacent contributions (*Action* and *Nature of the action*). The activity frame (represented in *Episodes*) and the sequential environment together comprise the double contextuality of each contribution (Linell & Marková, 1993; Linell, 1998).

We define *Episode* as any segment the session can be divided into which coincides with a change in activity or in the aim of the work being undertaken. In the case of an episode being particularly long, or involving various self-contained discussions, we then divide it into sub-episodes, consistent with Schoenfeld’s (2000) procedure for video analysis.

The *Action* column refers to the kind of response Julia makes to previous utterance, emphasising the responsive nature of each contribution. Given that the actions are defined by their contextual relations, we conceive the action as an inter-action (Linell & Marková, 1993). Four different actions emerged during the course of analysis:

<u>Respond</u>	The act of reciprocating appropriately to what has been asked, including those questions expressed in an indirect way.
<u>Ask</u>	The act of questioning another in order to ascertain their opinion or knowledge of some topic; indirect questions are also included.
<u>Answer</u>	The act of replying to statements directed specifically to her.
<u>React</u>	The act of providing a response to a statement which is not specifically directed at her. This category includes both responses which contribute to the overall communicative goal in hand and those which are autonomous.

Table1. Principle actions deriving from the analysis

Although we consider that all contributions imply an active interpretation on the part of the emitter, this latter can adopt a role which is receptive with respect to others’ turns, that is a *responsive role* (when responding or answering), or one which impulses or promotes new turns, that is an *initiator role* (when asking and reacting). Hence, these inter-actions provide an indication of the degree of initiative and the role adopted by Julia in the unfolding of the discourse.

The column *Nature of the action* seeks to capture the communicative function of each contribution to the discourse. Although we recognise the multifunctionality of these (Linell & Marková, 1993), we have generally chosen the one (or ones) which we consider best capture Julia’s role in the discourse dynamics at each specific point.

Unlike the *Action* column, here we realise an interpretative rewriting of each contribution, headed by the verb which better describes its function in the discourse. A list of the verbs which emerged during the course of the analysis was compiled, from the definitions of which we then selected the usage applicable to Julia's contributions (see appendix).

Below is an extract from the table for analysing interactions, corresponding to a PIC session in which a video of Julia's practice is analysed.

Int.	Episodes	Julia	Action	Interactant	Nature of the action	Content
62	Continuing the analysis of G7, begun in the previous session	S8. 78	Answers	Researcher-trainer (R) 1	Agrees that the activity was difficult and that the pupils were tired and did not yet have the left/right distinction fully assimilated.	Difficulties that she associates with the activity
63		S8. 79	Responds	R2	Points out the objectives of the worksheet	Objectives of the worksheet
64		S8. 80	Reacts	R1	Points out that besides taking the objectives from the book, as other teacher affirms, she also adds her own.	Objectives of the worksheet
65		S8. 81	Asks	R1	Understands what he is asking about.	
66		S8. 81	Responds	R1/Inés (experienced teacher)	Evades direct answer. Explains other occasions in previous years when she had tackled the topic.	

Example of the use of the IMDEP instrument

Given that it is an instrument for analysing interactions in a context of professional development, an analysis of the discursive dynamics of the interactions is insufficient without the addition of the epistemological contribution of each turn to the discourse. For this reason we have included the *content* column, in which we briefly outline what each contribution deals with, like a signpost for later interpretation.

THE INFLUENCE OF THE PIC IN PROFESSIONAL DEVELOPMENT THROUGH THE ANALYSIS OF INTERACTIONS

The PIC, as a collaborative environment structured according to the principles of professional development rather than training (Ponte, 1998), exerts its influence through the joint pursuit of professional activities through means of debate and reflection. In this context, Julia was not required to assimilate the knowledge and information transmitted by others, but rather to participate in the collective

construction of meanings which takes place in the interaction – a construction which is assimilated by Julia via a new personal interpretation.

Julia's processes of assigning meaning are mediated by various factors and are produced in and through the interaction. Some of these factors are inherent in Julia herself, others are characteristic of the PIC and its members, but all of them operate concomitantly with others which arise in and are determined by the interaction. It is in the interaction that the role of Julia within the group is defined, along with the degree of confidence she establishes with each member, the image she has of them and they of her, and so on, aspects which influence how Julia accepts the reflections, opinions, suggestions and critical analyses about her practice. In short, we consider that the processes of interaction determine the extent to which Julia is involved in the group and hence, mediate the role which the PIC has in her reflection and professional development.

Our instrument of analysis provides us with information on:

- At what points in the session Julia tends to contribute and the degree of involvement towards her professional development within the group.

- Whether she tends to act on her own initiative or in response to others' turns explicitly directed to her; that is, the way in which her role develops during the course of the interaction (initiatory or responsive).

- Whose critical comments she receives best and whose she seems not to accept; likewise, towards whom she shows a greater interest in knowing their thoughts or opinions. What features characterise the contributions of these members such that these reactions happen.

- After or before whom she usually contributes and why.

- Depending on the episode or activity to be done, what functions predominate in Julia's contributions; in addition, the relation between the function of her actions and the people to whom they are directed.

- The relation between the actions and the nature of the contributions and the episodes framing them. For example, whether there is a difference in Julia's contributions when a video of herself, or of the other teacher, is analysed.

- The relation between the characteristics of her contributions and the content under discussion at any moment. What kind of content would she seem to give more importance to according to the predominating function or action.

It can be seen from this perspective that the analysis of interactions allows us access to the meanings which Julia constructs and which she attributes to the various contributions at each point in the conversation, providing us with clues as to how the PIC shapes her professional development. Consequently, we feel that the interactions are the means through which Julia develops in the group and in turn the point of reference by which we as researchers gain access to how the PIC exerts its influence.

CONCLUSION

This paper presents the instrument for the analysis of Teacher's Interaction in a context of Professional Development, which has been developed in the course of the research we are conducting. The IMDEP shows itself to be a useful tool for accessing and understanding the meaning that Julia constructs at each point of the interaction, with a view to gathering clues to the role that the PIC plays in her professional development. We have explained the theoretical grounding of the instrument, both from the perspective of our epistemological position and from our dialogical conception of discourse (Linell, 1998, 2005).

The IMDEP represents a contribution in three senses: first, our interest does not lie with the communication between students working on groups or between the teacher and students as is usually the case in the research literature (Bjuland, 2004; Cobb et al., 1997), but rather it lies in the interactions between educational professionals in a context of professional development. Secondly, the adoption of dialogical approach to the analysis of interaction tends to involve an interest in the joint construction of knowledge taking place in the group, in place of the attribution of meaning of one member participating in the group, as is our case. Finally, we aim to establish a relation between the interactions arising at each point of the communicative flow of the PIC and the extent of its influence on professional development, which allows us to gain insights into how social contexts operate upon it.

We intend to continue deepening in the analysis of interactions in contexts of professional development and making improvements to our instrument. In future papers we hope to illustrate and discuss examples of how the IMDEP is helping us to understand how the PIC is having an influence in Julia's professional development.

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APPENDIX: NATURE OF THE ACTION (ORGANIZED BY ACTIONS)¹

RESPOND		
Accept	Confirm	Inform
Clarify	Disagree	Show openness
Analyse	Explain	Propose
Offer idea	Express doubt	Reaffirm
Agree	(Re)formulate	Reject
Joke	Indicate	Recognise
Express lack of knowledge		
Deny		
Evade response: Avoid an awkward question or one to which the addressee lacks a reply (assigned together with Offer idea, Agree, Explain and Reaffirm)		

ASK		
Know: Hear or obtain information about something		
Understand: comprehend		
Question	Request confirmation	Propose
Provoke		

REACT AND ANSWER

Accept	Express doubt
Clarify	Express surprise
Analyse	Reformulate: Reduce a proposition to clear and simple terms.
Offer idea	Point out: Briefly give information or an opinion.
Agree: State truth or appropriacy of previous affirmation or proposition.	Inform
Joke: Express own idea humorously, point out nonsensical aspect of some previous utterance, or respond ironically to an utterance.	Show openness: Display a favourable attitude towards carrying out a proposed or an assigned action.
Comment on	Request confirmation: Request further proof of veracity of an idea or the acceptance of a suggestion, idea or proposal.
Confirm	
Correct	Propose
Question: Challenge the basis of an affirmation, suggesting the reasons and foundations.	Reaffirm: Ratify what has been said. Explain one's own response, arguing in favour of a position which appears not to be accepted or shared by the others.
Disagree	Reject
Explain	Recognise

¹ Only the verbs with a particular nuance in the context of this paper, or which can have several meanings, are defined here.

ADAPTING THE KNOWLEDGE QUARTET IN THE CYPRIOT MATHEMATICS CLASSROOM

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This paper builds on the work carried out by colleagues on using an empirically-based conceptual framework, the Knowledge Quartet, as a tool for the analysis of mathematics lessons taught by preservice teachers in the UK. This framework categorises situations from classrooms where mathematical knowledge surfaces in teaching, and was used with the aim of understanding what relationship can be observed between Cypriot preservice teachers' mathematical knowledge and their teaching. In particular, in this paper I suggest that the framework needs to be supplemented in order to incorporate the interpretation of mathematics textbooks by teachers. I illustrate this by giving examples from lessons taught by participants in my study.

Key-words: Teacher Knowledge, Knowledge-Quartet, Textbook

INTRODUCTION

The object of the study discussed is based on the classic distinction by Shulman (1986) between two aspects of teachers' mathematical content knowledge, Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). PCK includes the representations, examples and applications that teachers use in order to make the subject matter comprehensible to students. SMK consists of substantive and syntactic knowledge (Schwab, 1978). Substantive knowledge focuses on the organisation of key facts, theories, and concepts and syntactic knowledge on the processes by which theories and models are generated and established as valid.

From a variety of perspectives, research in the field of preservice teachers' knowledge focuses on their SMK and PCK. Some researchers have investigated preservice teachers' understanding of different topics in mathematics (Ball, 1990; Philippou and Christou, 1994; Rowland, Martyn, Barber and Heal, 2001) and others have focused on investigating the relationship between SMK and PCK and teaching (Rowland, Huckstep and Thwaites, 2004; Hill, Rowan and Ball, 2005) and have suggested that content knowledge might affect the process of teaching. These studies have shown that preservice teachers' substantive knowledge of mathematics was significantly better than their syntactic knowledge, and this was reflected in their teaching.

In Cyprus, concern among policy makers about students' achievement in mathematics has grown recently, and many attempts have been made to improve the instructional practices in public primary schools. Attempts of improving mathematics teaching in Cyprus have focused on learners and the curriculum, rather than focusing on teachers. Research on teacher knowledge has been neglected in the Cypriot literature. The few

studies in this field (e.g. Philippou and Christou, 1994) focused on investigating aspects of Cypriot preservice teachers' substantive and syntactic knowledge of mathematics and have shown that the participants were poorly prepared to examine different mathematical concepts and procedures conceptually. However, if we want to understand better what goes into teaching mathematics effectively, the challenge is to identify the ways in which preservice teachers' knowledge of mathematics, or lack of it, is evident in their teaching. No one type of knowledge functions in isolation in teaching and thus, research in the field of teacher knowledge should focus on understanding the relationship between the different kinds of their knowledge. The identification of this relationship will help teacher educators to assess teacher preparation programmes, and to improve them where necessary. The study reported in this paper was carried out in the context of my ongoing doctoral study which is centred on understanding the relationship between Cypriot preservice teachers' SMK and PCK to teaching. In particular, the focus of this paper is on reporting results related to one of my research questions. I discuss whether the original conceptualisation of the Knowledge Quartet was relevant and adequate in the analysis of teaching in the Cypriot primary mathematics classroom.

THE STUDY

My approach to investigating the relationship between Cypriot preservice teachers' mathematical knowledge and teaching involved a mixed-methods approach. My study entailed four data collection methods. First, a questionnaire was designed to examine Cypriot preservice teachers' SMK of mathematics. 104, final year university students, following a teacher preparation programme, completed the questionnaire. It aimed to collect information about the participants' beliefs about mathematics and its teaching, and their substantive and syntactic knowledge of it. As a part of the questionnaire the participants were asked to respond to ten mathematics items that assessed their SMK. The aim of the interview questions was firstly to clarify the questionnaire data and second to gather some information about the interviewees' PCK of mathematics. The interview questions proposed two hypothetical scenarios that were relevant to teaching mathematics, representing real classroom situations which a teacher might encounter while teaching mathematics. The interview tasks provided information about what teachers know and believe about mathematics, and also about the knowledge and skills that they draw on in making teaching decisions.

While these interview tasks represented real situations in the mathematics classroom, their context remained hypothetical, and did not provide information on what teachers actually do in the classroom and how their knowledge of mathematics influences their teaching decisions in classroom where they interact with their students. This kind of information was provided by observing participants teaching mathematics in the classroom. Five of the interviewees were chosen to be observed while teaching mathematics. In Cyprus a large part of the teacher preparation programme (a four

year university course) is spent in teaching in schools under the guidance of a school based mentor.

For the observations I used a framework that emerged from observing several lessons that were taught by preservice teachers in England (Rowland et al, 2004). This framework is called the *Knowledge Quartet* and is a tool that can be used in order to describe the ways in which SMK and PCK are revealed through teaching. As a part of my study I also evaluated the adaptability of the framework in the Cypriot classroom.

Finally, the data from the questionnaire, interview and observations were compared with data from the analysis of mathematics textbooks in Cyprus. Textbook analysis provided information on what policy makers consider desirable knowledge for teachers. However, what is considered desirable knowledge for teachers is often different from the knowledge that teachers use in and reveal through practice. A comparison of these two kinds of knowledge is considered to be helpful in modifying and improving teacher preparation programmes.

The combination of four methods and their integration during the interpretation phase provided strong inferences and produced a more complete understanding of the relationship between participants' content knowledge and their teaching. In the remainder of this paper I will focus on just one aspect of the study described here, and discuss issues related to the adaptability of the framework in the context of the Cypriot classroom.

THE KNOWLEDGE QUARTET

At the CERME meeting in Spain, Tim Rowland presented a paper (Rowland, Huckstep and Thwaites, 2005) about the Knowledge Quartet and suggested that this can be used as a tool for classifying ways that preservice teachers' knowledge comes into play in the classroom. At the following CERME meeting in Cyprus Fay Turner (Turner, 2007) also presented a paper about the Knowledge Quartet and explained how she is currently using the framework as a tool for professional development with a group of early career teachers.

The Knowledge Quartet consists of four dimensions, namely, Foundation, Transformation, Connection and Contingency. Foundation consists of trainees' knowledge, beliefs and understanding of mathematics. Transformation concerns knowledge-in-action as demonstrated in the act of teaching itself and it includes the kind of representation and examples used by teachers, as well as, teachers' explanations and questions asked to students. Connection includes the links made between different lessons, between different mathematical ideas and between the different parts of a lesson. It also includes the sequencing of activities for instruction, and an awareness of possible students' difficulties and obstacles with different mathematical topics and tasks. Finally, Contingency concerns teachers' readiness to respond to students' questions, to respond appropriately to students' wrong answers

and to deviate for their lesson plan. In other words it concerns teachers' readiness to react to situations that are almost impossible to plan for.

Below, I argue that when adapting the framework in the Cypriot mathematics classroom, this needs to be supplemented by consideration of the use and interpretation of mathematics textbooks. I give three examples from lessons taught by participants in my study to illustrate this.

ADAPTING THE KNOWLEDGE QUARTET IN THE CONTEXT OF THE CYPRIOT CLASSROOM

When adapting the Knowledge Quartet it was not assumed that the knowledge used by Cypriot and English teachers is the same. Therefore, as part of my study I evaluated the adaptability and the validity of the Knowledge Quartet. In this section I describe the appropriateness of the Knowledge Quartet in the context of the Cypriot classroom, and explain that the framework needs to be expanded by adding a new code in the Transformation dimension.

For the most part, I found that the Knowledge Quartet could be used successfully to analyse mathematics lessons in the Cypriot mathematics classroom, in understanding how participants' SMK and PCK were related to their teaching. In particular, the issues raised for attention in lessons observed in the UK were also observed in the Cypriot mathematics classroom.

In my analysis of the lessons, I identified all the situations that I thought were significant with respect to participants' mathematical knowledge. The Knowledge Quartet proved to be comprehensive in describing most of the teaching episodes that were considered important for the purpose of my study. With reference to the 'Foundation', 'Connection' and the 'Contingency' dimensions, the codes proposed in the original study could be used to describe all the situations I thought were significant in understanding the relationship between participants' content knowledge and their teaching. For example, participants' ability to anticipate students' difficulties and obstacles, to hear and respond appropriately to students' thinking, to choose appropriate examples and representations, and to make connections between different mathematics concepts, were significant issues in understanding the ways in which their content knowledge came to play out in their teaching. In addition, issues related to participants' awareness of students' conceptions and misconceptions about a mathematical topic, their decisions about sequencing activities and exercises, or interrupting a classroom discussion to obtain clarification, or their decision to use a student's opinion to make a mathematical remark, were significant in identifying the relationship between participants' knowledge and teaching.

It was also clear from the data that Foundational knowledge underpinned the other three dimensions. In general, the application of teachers' knowledge in the classroom always rested on their Foundational knowledge, which was acquired in the academy in preparation for their role in the classroom.

On the whole the Knowledge Quartet was found to be a valid tool for analysing the lessons observed in the Cypriot classroom. However, an additional issue that proved to be significant in the analysis of my lessons was the use of mathematics textbooks, in particular how activities in the textbooks were adapted. Here, textbooks refer both to students' book and the teachers' guide. In the original study a code 'adherence to textbooks' was classified in the Foundation dimension of the framework. This code was used to describe episodes where teachers accepted textbook as authority for what and how to teach. However, the ways in which teachers adapted textbook activities are not addressed in any of the existing publications about the use of the Knowledge Quartet as a tool for observing mathematics lessons in the UK. This is not surprising, since the use of textbooks is not a common practice in the English primary school mathematics classroom. In contrast, the textbook is central and always present in the mathematics classroom in Cyprus.

All the participants in my study considered the textbook as the main resource both for their planning and teaching. However, they all combined it with other resources, and included their own developed activities. The participants adapted the textbooks in very different ways. For example, there were cases where participants modified the textbook material in ways that made the lesson more meaningful and interesting for their students. However, in some instances participants were not sure how to adapt the textbook activities appropriately, modifying them in ways that altered their focus. This suggested that the ways in which preservice teachers used the textbooks was important in understanding how their knowledge came into play in their teaching.

The above led me to conclude that when adapting the Knowledge Quartet for observing lessons in Cyprus, and indeed in many other countries, there is a need to take careful account of these differences. Thus, issues related to the adaptation, modification, and interpretation of the textbook material are important in analysing a mathematics lesson in Cyprus. Having presented the appropriateness of the dimensions of the Knowledge Quartet in the context of the Cypriot classroom, I provide some examples from the lessons observed to demonstrate how the participants in my study used the textbook activities.

ADAPTING THE TEXTBOOKS: SOME EXAMPLES FROM THREE PARTICIPANTS

The lessons observed took place during the students' placements in school. These lessons were analysed using the four dimensions of the Knowledge Quartet. In this section, I give some examples related to how three participants (Rita, Elsa and Christiana) used the mathematics textbooks. Christiana chose to do additional courses in mathematics in her undergraduate teacher education course, and was classified in the group with a 'high' SMK score (this was assessed in the questionnaire, see page 2). Elsa was classified in the group with 'low' SMK score and Rita in the group with 'medium' SMK score. Neither of them chose to do additional courses in mathematics during their training. In general, the results showed the positive influence of strong

SMK in the effective use of textbooks. Christiana elaborated upon the textbook in ways that made her lesson more meaningful and interesting for students. She was able to draw on her own understanding and use appropriately textbook activities and extends them to promote students' conceptual understanding. In contrast, Rita and Elsa seemed to have problems in understanding the textbook suggestions due to their lack of SMK. In many instances they could not understand the mathematics targeted by textbook activities, and so could not make much of them. Therefore, it becomes clear that in order to use textbook activities appropriately, teachers need to understand their content.

Not understanding the mathematics targeted by the textbook

Rita's lesson on multiplication by four offers an example of how she interpreted one of the activities in the textbook in ways that altered its focus. Figure 1 illustrates this activity.



Mr Michalis has recently opened a new restaurant. He has 50 square tables in the restaurant. Each table can seat 4 customers. On Sunday night 36 customers went for dinner. By 23:00 half of them had left. One hour later all the other customers left and the restaurant closed.

1. How many tables does the restaurant have?
2. How many tables remained empty on Sunday night?
3. How many customers were in the restaurant just after 23:00?
4. Show on the clock the time that the restaurant closed.
5. On Monday ten friends went to the restaurant for lunch. Mr. Michalis needed to put tables together so that ten friends could sit next to each other. How many tables were needed?

Figure 1 Textbook Activity (2nd Grade, Students' Book, Part B, p.87)

In addition, in the teachers' guide it was clearly stated that:

intentionally some information is not given [...] students should think of all the possible answers to the questions asked, taking into consideration that each table can seat 1,2,3 or 4 customers (Grade B, Teachers' Guide, p. 103)

Rita seemed not to take into consideration what was suggested in the teachers' guide. She used a rather 'traditional' approach in solving the problem. She read the problem to her students, and did not leave them much time to think, before leading them towards the answers. More importantly, when dealing with question two of the problem she seemed to take for granted that exactly four customers were sitting at each table and said:

36 customers were in the restaurant. There were four people at a table. Thus, 36 divided by 4 will give us the number of tables that were full.

Rita's approach to solving the problem focused on procedures, required a single answer, and focused on relatively few skills. However, the focus of the problem was meant to provide students with the opportunity to explore a number of possible solutions. Rita showed a desire to develop conceptual understanding in several instances in her lessons, however, it seems that in this case her beliefs about good mathematics teaching could not be implemented because she did not understand the problem solving intention. I can infer from my post-observation discussion with Rita that she changed the focus on the activity due to her lack of understanding. In this discussion I asked Rita if she could think of an alternative way of solving the problem and she was adamant that she could not. Her answer suggested that she might not have read the teachers' guide. However, the aims that were proposed in her lesson plan were exactly the same as those proposed in the teachers' guide, so it seems that she did read the guide, but that her reading was superficial, and for some reason she missed some of the information provided. It could be argued that she followed the teachers' guide rather mechanically, moving through activities without understanding their focus. In this case her problems in understanding the teaching suggestions in the guide might stem from insufficient understanding of the problem.

Another example, of not understanding the suggestions in the textbook occurred in Elsa's lesson on the parts of a circle. In this lesson Elsa tried to define the different parts of a circle. Table 1 shows the definitions that she proposed alongside the definitions that were suggested by the teachers' guide.

The definitions that Elsa gave to her students were mathematically incorrect. Even though she used the activities proposed in the textbooks she did not use the suggested definitions. It seemed that her understanding of the different parts of a circle is limited. Below I provide an extract from our post-observation discussion to support my argument:

- Elsa: Generally, I think that everything went well. However, my impression is that students were confused about the chords.
- MP: What do you think confused them?
- Elsa: Uh, I think that the definition of a chord is confusing itself. To be honest, I am confused myself. On the one hand, according to the definition provided in the textbook, a chord does not pass through the centre. On the other hand, the teachers' guide mentions that the diameter is the biggest chord. I think this is very confusing.

The extract above indicates that Elsa's understanding of the parts of a circle was limited. She seemed not to be aware of the correct definitions of different parts of a circle, and, due to her limited understanding, was unable to follow the suggestions included in the textbook. It was likely that Elsa chose not to use the definitions as

suggested in the textbook because she believed that these were too difficult for her students. In trying to make these easier for her students, she made it more difficult.

	Elsa's definitions	Definitions suggested by teacher's guide
Diameter	Is a straight line that starts from the beginning* of the circle and reaches the end of the circle passing through its centre	Each chord that passes through the centre of the circle. A straight line passing through the centre of a circle and connecting two points on the circumference
Radius	It is a line that starts from the centre and reaches the end of the circle	A straight line segment connecting the centre of the circle with a point on the circumference
Chord	Is a line that starts from the beginning of the circle and reaches the end but does not pass through the centre	A straight line segment connecting two points on the circumference
Circumference	The 'round -round' * of a circle	Not included

* This is the exact translation for Elsa's definition from Greek, which in effect means the boundary of a circle

Table 1: Defining the parts of a circle

In general, in mathematics definitions should be inclusive. However, Elsa's definition of the chord was exclusive. Her statement 'does not pass through the centre' excludes the diameter which indeed is a chord. In contrast the definition of the chord in the teachers' guide was inclusive. In addition, it was clearly stated that the diameter is the biggest chord. Therefore, it can be argued that her problem in understanding the definition proposed in the textbook stemmed from her limited understanding of the topic. This was indicated by her tendency to refer to the 'beginning' and the 'end' of a circle, meaning points on the circumference.

Elaboration upon the textbook: making activities more meaningful and interesting for students.

An example of developing the textbook material is offered by Christiana's activity illustrated in Figure 2. The version of the activity as proposed in the students' book is also presented. Both activities have been translated from Greek. It is clear that in her modified version of the textbook activity Christiana put emphasis on developing students' conceptual understanding. I consider Christiana's version to be an

improvement because she elaborated on the textbook activity in a way that made it more meaningful to her students, by helping them to explore division and multiplication as reverse operations.

A FACTORY PRODUCING JAM

The students in Philippos' class visited a factory producing jam. The jam was bottled and then packed into large boxes. Each box could hold 50 bottles. On that day the production was 9250 jars of jam. How many boxes were needed for packing the jars? The table below shows the production of jam for each day of the week. Fill in the information in the table provided

Days	Jars for each day	Jars in each box	Number of boxes
Monday	24 500	50	
Tuesday	18 900	50	
Wednesday	11 750	50	
Thursday	21 600	50	
Friday	12 600	50	

Textbook activity (4th grade, Students' book, Part C, page 33)

Christiana modified the activity and asked her students to fill in the information in the table presented below

Days	Jars for each day	First filling		Second filling	
		Number of jars in each box	Number of boxes	Number of jars in each box	Number of boxes
Monday	24 500	50		100	
Tuesday	18 900	50		100	
Wednesday	11 750	50		100	
Thursday	21 600	50		100	
Friday	12 600	50		100	

Then the students were asked to write down their observations relating to the numbers of boxes needed for the first filling and the second filling

Figure 2: Elaborating textbook activities

CONCLUSION

In general the Knowledge Quartet was comprehensive in the classification of teaching situations in which participants' mathematical knowledge surfaces in teaching. Issues related to the interpretation of textbooks were not addressed by the framework, however were important in analysing mathematics lessons in a Cypriot classroom. This suggests that when adapting the Knowledge Quartet for observing lessons in Cyprus, and indeed in many other countries, there is a need to take careful account of possible differences between the context in which the framework was originally developed, and the context in which this is applied.

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PROFESSIONAL KNOWLEDGE IN AN IMPROVISATION EPISODE: THE IMPORTANCE OF A COGNITIVE MODEL

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One approach towards improving teacher performance is that of classroom practice. In this paper, taking a cognitive perspective, we present a system for modelling teacher performance. We demonstrate the process of construction of this model with reference to a brief lesson episode involving teacher improvisation, which took place in the first cycle (the first four years) of primary school in Portugal. Included in the model are the cognitions made evident by the teacher as well as the relations between them.

Keywords: Improvisations, cognitions, modeling the mathematics teaching, practice, primary school

The teaching process can be analysed from various theoretical perspectives and focus on very different aspects, amongst them the teacher and their performance. With respect to classroom practice, the teacher's decisions are influenced not only by the particular context, but also, and we believe fundamentally, by his or her cognitions.

With the aim of understanding what happens in the classroom from the point of view of the teacher, in terms of both their actions and their cognitions, we decided to focus on performance, in particular the relations between the teacher's actions, cognitions and the type of communication used. The teaching-learning process is far too complex to permit a single, all-encompassing analysis, however, and hence we recognise the need for developing a model which allows it to be simplified for a more fruitful analysis. The model we developed to fulfil this aim was based on Monteiro (2006), Monteiro, Carrillo & Agued (2008), Schoenfeld (1998a, 2000) and Schoenfeld, Ministrell & Zee (2000). We denominate it a 'cognitive model', because it focuses only on certain of the elements comprising the system it models, in this particular case, the cognitions of the teacher with respect to their classroom practice. With this model we try to study some dimensions of professional knowledge and some relations amongst them. We hope this paper helps consider the common analysis of lessons by focussing on a limited number of variables as beneficial for researchers, trainers and teachers working in collaboration.

In the next sections we are discussing the cognitions and the kinds of communication. For the purpose of this paper, teacher's action should be identified with his/her performance in the classroom when dealing with their students' knowledge building.

The cognitions

Following Artz & Thomas-Armour (2002), we understand by cognitions all those cognitive constructions – beliefs, knowledge and goals – which each individual

carries with them, the study and analysis of which, along with the relations among them, offers valuable contributions for both research and classroom practice, which can be understood as the ultimate aim of research.

As teachers we can have goals over the short, medium and long term. For Schoenfeld (1998b), goals can be simply something which one aims to attain, and can be explicit or latent, and can likewise be pre-determined or emerge during the teaching activity (Aguirre & Speer, 2000). We believe that such emergent goals especially occur in unplanned situations, particularly those which the teacher have not anticipated. We concur with Saxe (1991) that each individual – and specifically here a teacher - has the capacity to construct, adapt, model and remodel such goals in accordance with his or her own personal and professional development.

As was noted in respect of goals, so too does research into beliefs offer great potential for both theory and practice. The more we can learn about the influence of teachers' beliefs on their teaching, the deeper our understanding (Aguirre & Speer, 2000). In this study the instrument used to undertake the analysis of teachers' beliefs was that of Climent (2002). Climent presents a set of indicators of primary school teachers' beliefs (i.e., first six years in Spain) with respect to beliefs on methodology, mathematics, learning, and the roles of pupil and teacher.

Concerning our focus on professional knowledge, of particular relevance is the work, still in progress, of Ball, Thames & Phelps (submitted) which adapts Shulman's (1986) formulation for the components of professional knowledge. Further, some incorporations, namely certain descriptors from Park & Oliver (2008), are also included.

Ball and colleagues (Ball, 2003; Ball, et al., submitted), following Shulman's (1986) classification, introduce the notion of mathematical knowledge for teaching. They divide content knowledge and pedagogical content knowledge each into three categories. Content knowledge, they consider to be formed by horizon knowledge (HK), common content knowledge (CCK) – i.e., typical 'schoolboy' mathematics – and specialised content knowledge (SCK). Pedagogical content knowledge (in Shulman 'curricular knowledge'), they likewise divide into three types, each a variant of content knowledge: teaching (KCT), student (KCS), and the curriculum (KC). Hence, they maintain that teachers should have a specific professional knowledge, so that in addition to a knowledge of 'how to do' – that is, common mathematical knowledge (CCK) – they should also have a knowledge of 'how to teach to do'. Thus, for example, beyond knowing how to calculate the difference between two numbers (CCK), it is necessary for the teacher to possess an understanding which allows him or her to perceive and identify not only the students' mistakes but also the source of these mistakes, which becomes much more complex (SCK). Likewise, they should also be familiar with alternative procedures for dealing with content, so that they can easily meet the needs of their pupils. Equally, a knowledge of how the various mathematical topics relate to one other and the way in which the learning of a

particular topic develops as one moves up the school (HK) is essential for the effective teacher.

As an integral part of methodological and curricular content knowledge identified by Shulman (1986), Ball, et al. (submitted) consider that teachers should possess a composite knowledge of teaching and specific content (KCT). This corresponds to the type of knowledge to which the teacher resorts in situations that are related to the organisation of different ways the students explore mathematical contents, such as: determining the sequencing of tasks, choosing examples, and selecting the most appropriate representations for each situation. Park & Oliver (2008) also include the specific strategies for teaching the content in question.

Regarding knowledge of content and students (KCS), Ball et al (submitted) relate this to the need for the teacher to anticipate what the students think, their difficulties and motivations as well as listening to and interpreting their comments. Park & Oliver (2008) include here the knowledge of the possible wrong conceptions, motivations and interests of the students, as well as their needs.

Kinds of communication

The way in which the teacher communicates with others (their students in this case) provides a great deal of information about him or herself and how they regard the whole process of teaching – including body language, level of anxiety, etc. The type of communication the teacher employs is in direct relation with the cognitions they hold, in that the way the teacher chooses to communicate reflects the way they view the teaching process. With different forms of communication, so the actions are distinct and quite possibly the underlying teaching views themselves.

We adopt the classification of Brendefur & Frykholm (2000), with some adaptations introduced by Carrillo, Climent, Gorgorió, Rojas & Prat (2008). Brendefur & Frykholm (2000) propose four types of mathematical communication: unidirectional, contributive, reflexive and instructive.

Unidirectional communication is associated with a form of teaching in which the teacher takes the principal role, requiring the student to do no more than faithfully repeat what he or she has heard. With respect to contributive communication, the student is afforded some participation in the classroom discourse, although the interactions which take place are by and large of a corrective nature and do not go very deeply into the content. The key feature of reflexive communication is that the interactions between the teacher and students act as triggers for subsequent investigative work. We agree with Carrillo et al. (2008), that development of students' mathematical comprehension is best achieved through such inquiry-based activities. Instructive communication, is similar to reflexive communication, but aims also to shed light on the matter in hand, bringing about an integration of students' ideas – progress and/or difficulties – made explicit or intuited by the teacher or by the students themselves.

The context and modelling process

The remainder of this paper is dedicated to presenting and discussing the modelling of an episode in which the teacher reviews content through dialogue. This occurs in a 4th year class given by a teacher of 18 years experience. The episode is taken from a wider research project on professional development studying the relationships between teachers' beliefs, knowledge, goals and actions. It combines a case study with an interpretative methodology whereby there is minimal intervention on the part of the researcher. Data collection – audio and video recordings of the teacher – was conducted in situ. Brief informational talks were also used before and after each lesson to gather lesson previews – lesson image – and to clarify some inferences. The video recordings provided a record of the teacher-students interactions, and enabled lessons to be viewed and analysed, as many times as required.¹ That wider research project involves a collaborative work between the researcher (first author of this paper) and two primary teachers. The collaborative work started after the first phase of data collection. It was focused in the teacher's practices mainly by discussing some situations they consider to evidence good practices and others they want to improve their **teaching**.

The first stage of the modelling process involved the transcription of the audio recordings, followed by the video (Illustration 1). Transcription also included an initial division of the lessons into episodes, defined by triggering and terminating events and associated with specific goals. Subsequently, when all the lessons pertaining to the same phase (of three in total) had gone through this procedure, there began the process of identifying the indicators of beliefs (Climent 2002), content, specific goals, type of episode, type of communication, means of working, resources used, and the teacher knowledge required for implementing the episode (Ball, et al., submitted; Park & Oliver, 2008). Also determined at this point, was whether or not the episode formed part of the lesson image (cf. Table 1).

The action sequences identified correspond to routines, scripts or action guides, and improvisations (Monteiro, 2006; Monteiro et al., 2008; Schank & Abelson, 1977; Schoenfeld, 2000; Schoenfeld et al., 2000; Sherin, Sherin & Madanes, 2000). A routine is any kind of action independent of context, executed routinely; scripts, or action guides, are specialisations of routines, but conceptually dependent. Improvisations correspond to all those actions undertaken by the teacher in response to an unexpectedly arising event.

In this study the definition of improvisation has a wider sense than that of the researchers mentioned above², and distinguishes two types that can arise in class. The distinction concerns the relation pertaining (or not) between the events/actions and the contents. Thus, either the action is related to the content under consideration at that moment (or which has been, or is to be, dealt with), or the action has no relation

¹ The recordings also allowed the teacher to prepare reports and to reflect more fruitfully on the various interactions between the participants through repeated viewings.

² They only consider situations in which the actions are unconnected to the contents. We consider that improvisations correspond to the set of teacher's actions in response to all unexpected events.

with the teaching contents, focusing only on administrative questions, student conflicts or general management issues. We call the first type (concerned with the teaching activity) ‘content improvisations’, and those of the second (concerned with classroom management) ‘management improvisations’.

It should be noted that content improvisations constitute episodes which do not form a part of the lesson image and which necessarily have emergent goals. Because such episodes have not received prior consideration, the teacher’s cognitions come very much more to the fore since their response is so much more intuitive. Content improvisations are consequently one of the points in which cognitions are most in evidence.

A teaching episode and its analysis

In this section we present a transcript of an episode from the first of a series of four lessons aimed at introducing the concept of ‘a thousandth’. Given that the transcript illustrates a goal in emergence, the episode cannot be considered to form part of the lesson image. The extract shows the teacher taking the opportunity presented by a student doubt to revise, via a whole-class dialogue, the difference between squares and rectangles through reference to the lengths of the sides.

- 246 S This isn’t a rectangle, it’s a square . . .
- 247 T Is this shape a rectangle or not?
- 248 S No!
- 249 T So, why isn’t it a square, Tiago Luís?
- 250 S Because the sides aren’t the same length.
- 251 I thought it was a square, Miss.
- 252 (Inaudible)
- 253 T Paulo quiet.
- 254 What features does it have it have to be a square?
- 255 S It has to have the sides the same length.
- 256 T The sides all the same.
- 257 Ss (Inaudible, everybody speaking at the same time)
- 258 T (Puts hand up)
- 259 Quiet, quiet, put your hands up.
- 260 (T points to one of the sides of the square)
- 261 Paulo, if this side is twenty-five squares long, and this side is . . . how many?
- 262 Ss Forty!
- 263 T Forty . . . so, is it a square?
- 264 S No!
- 265 T Why not, Paulo?
- 266 S Because the sides aren’t the same length.
- 267 T Exactly.
- 268 S To be a square it has to be twenty-five by twenty-five.

Illustration 1 – Transcript of an excerpt from the first, of a series of four, classes aimed at introducing the concept of ‘a thousandth’, corresponding to an improvised content revision dialogue (T: teacher; S(s): student(s))

This excerpt corresponds to the ninth episode in the first lesson of the first phase of work [I.1.9]. The triggering and terminating events coincide with the start end of the

transcript. The teacher's emerging goal is to revise the difference between squares and rectangles in terms of the lengths of their sides. The communication type she employs is contributive, with the students working in a large group (the whole class).

The coding within the square brackets indicates that the lesson takes place during the first phase (pre-collaborative work) and corresponds to the ninth episode of the first lesson [I.1.9]. The left-hand box provides information on the specific category to which each indicator of beliefs belongs (in brackets) in addition to the goal and knowledge which have been identified, the triggering and terminating events, the type of episode and whether or not it forms part of the lesson image. The right-hand boxes record the sub-episodes ([I.1.9.1], [I.1.9.2]) along with their specific goals and the kind of dialogue involved.

[I.1.9] Dialogical revision of content - difference between squares and rectangles - in a contributive way, with the whole class (246-268)³

Forms part of the lesson image? No.

Triggering event: T asks whether shape is a rectangle or not.

Indicators of beliefs⁴:

TT30 (**Teacher's role**) – The teacher is the one who validates ideas raised in class, questioning students, whose replies lead to self-correction (in reality veiled correction, stage-managed by the teacher).

TR16/TT16 (**Learning**) – The student interacts with the material and the teacher, the latter being the mediator between material and student. The interaction produced between teacher and student is unequal, with the flow teacher-student being stronger than the contrary.

Goal: Revise difference between squares and rectangles (length of sides).

Knowledge:

CCK (**Common Content Knowledge**) – Knowing the difference between squares and rectangles (in terms of the length of the sides).

SCK (**Specialized Content Knowledge**) – The teacher gives evidence of an incorrect use of the classification of polygons (using a disjunctive classification implying that the set of squares is separate from that of rectangles)

KCT (**Knowledge of Content and Teaching**) – The teacher considers contributive dialogue appropriate for the revision of the difference between the length of the sides of squares and rectangles.

KCS (**Knowledge of Content and Students**) – The teacher considers that the students show difficulties in considering squares as specific cases of rectangles (246-250), (254-256)

(GAP: the teacher does not perceive this difficulty of considering squares as rectangles as she uses disjunctive classifications and an incomplete definition of squares focused exclusively on the properties of the sides (forgetting the rhombus), which could generate erroneous conceptions (256).)

Type of episode: Content improvisation.

Terminating event: T considers that the students' doubt has been

³ Classified numbering of transcript.

⁴ This episode reveals beliefs concerning methodology (TR3, TR5), the role of the teacher (TT26/29, TT30) and learning (TR16/TT16, TT14), where TR denotes Traditional Tendency and TT Technological Tendency.

[I.1.9.1] T holds a dialogue with the group, and contributively revises the difference between the relative lengths of the sides of squares and rectangles (246-260)

Interactive dialogues (246-260)

Specific goal: Revise the difference between the relative lengths of the sides of squares and rectangles.

[I.1.9.2] T holds a dialogue with the group, and contributively clarifies that, by virtue of its sides not all being the same length, the shape cannot be a square (261-268)

Interactive dialogues (261-268)

Specific goal: Clarify that a rectangle is different from a square as one has sides of all the same length and the other does not.

Table 1 – Modelling of an episode corresponding to the ninth episode in the first of four lessons introducing the concept of a thousandth

This episode did not form part of the teacher's lesson image as it arose from a student comment. In the course of enacting the episode the teacher employs two actions which, from the analysis we have carried out until now, form the basis of all the revision episodes, independently of the resource(s) used, the form of work and the type of communication. It should be noted that, for this type of episode, these actions do not have to occur in the same order as in this specific case and that these are the only two kinds of actions the teacher does when she wants to implement this specific type of episode in this particular manner.

Relations between cognitions

The evidence for the teacher's cognitions is obtained from their actions, the kind of communication which occurs, the form of work of the students and the resources used. The table below illustrates the relations observed between the actions and cognitions in respect of the specific goal in this case. Some of the teacher's knowledge (to the right of the table) are relevant to the whole episode while others are specific to particular actions.

Indicators of beliefs/contributive language	Actions	Knowledge/contributive communication	
TT30 (Teacher's role) – The teacher is the one who validates ideas raised in class, questioning students, whose replies lead to self-correction (in reality veiled correction, stage-managed by the teacher).	T holds a dialogue with the group, and contributively revises the difference between the relative lengths of the sides of squares and rectangles (246-260).	KCS (Knowledge of Content and Students) The teacher considers that the students would have difficulties in considering squares as specific cases of rectangles (246-250), (254-256).	CCK (Common Content Knowledge) – Knowing the difference between squares and rectangles (in terms of the length of the sides). KCT (Knowledge of Content and Teaching) –
TR16/TT16 (Learning) – The student interacts with the material and the teacher, the latter being the mediator between material and student. The interaction produced between teacher and student is unequal, with the flow teacher-student being stronger than the contrary.	T holds a dialogue with the group, and contributively clarifies that, by virtue of its sides not all being the same length, the shape cannot be a square (261-268).	SCK (Specialized Content Knowledge) – The teacher gives evidence of an incorrect use of the classification of polygons (using a disjunctive classification implying that the set of squares is separate from that of rectangles).	The teacher considers contributive dialogue appropriate for the revision of the difference between the length of the sides of squares and rectangles.

Table 2 – Relations between actions and cognitions with respect to the revision of the difference between squares and rectangles, in terms of the lengths of their sides, via a contributive whole class dialogue.

The actions of revising and clarifying the content are underpinned by beliefs related to the role of the teacher (TT30) and to the learning process (TT16). The cognitions identified show that the teacher regards herself as the only person with the capacity/ability to validate the information mobilised in class. In viewing her role in this way, she conditions the interactions between other elements of the process of learning, thus preventing a balance being reached among them and making it impossible to achieve a triangle of learning, as advocated by Pinto & Santos (2006). These actions/beliefs are linked to each other in such a way that together they form the basis of all revision episodes.

The knowledge identified, as well as the gaps in knowledge, are specific to the situation and the context, and so cannot be generalised, not even for this teacher.

Possibilities for initial and in-service teacher training

This type of analysis may also be of use in initial teacher training as the starting point for an approximation between theory and practice. It would mean that researchers and teachers “speak the same language”, using the same codifications; in doing so, a great degree of collaboration is needed.

This type of analysis (by student teachers in their teaching practice), although based on the experience of others, may lead to an awareness of their own cognitions, of the way they relate and influence one another. This awareness would help the development of a critical, as opposed to submissive, attitude during their teaching practice; merely observing the mentors does not necessarily lead to learning (Brophy, 2004). It is important, then, that the time spent in schools by trainee teachers as observers or assistants should be given careful consideration and attention.

In the sphere of in-service training, this type of analysis can be effected by the teacher him or herself, who, in watching recordings of their lessons, will be able to reflect upon their own practice (Schön, 1983, 1987). This reflection, accompanied by discussion and critical exchanges with colleagues and researchers, can be considered a first step towards sustained professional development (Climent & Carrillo, 2003; Jaworski, 2006) aimed at improving professional competence through qualified professional reflection (Hospesová, Tichá & Machácková, 2007).

We selected content improvisations as the focus of our analysis because, when they occur, the teacher “is working without a safety-net”. They are unforeseen situations not subject to advanced planning, and consequently all the teacher’s cognitions come into play in their purest form, faithfully reflecting their mode of acting and their position with respect to the process and intervening elements. It will be in these situations that, in the initial stages of training programmes as in professional development, significant information can be obtained which can contribute to the development process, enriching discussion and leading to a self-awareness of one’s professional attitude. These situations can permit access to what Tomás Ferreira (2005) terms ‘teaching modes’, underlining the relationships between their dominant

classroom interaction, teacher's key beliefs and in this case, also their professional knowledge.

This analysis and understanding are very important now that there exists in Portugal a Programme of In-service Training in Mathematics for teachers of the 1st and 2nd cycles of Basic Education with a supervision component (Serrazina et al., 2005). One of the ways of achieving some of the goals of this programme – deepening teachers' mathematical, pedagogical and curricular knowledge and encouraging a positive attitude in teachers towards mathematics and the capabilities of the students – could involve the analysis and discussion of teachers' classes, through the use of this cognitive perspective and of the model.

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CERME 6 – WORKING GROUP 11

APPLICATIONS AND MODELLING

TABLE OF CONTENTS

Introduction.....	2042
<i>Morten Blomhøj</i>	
Mathematical modelling in teacher education – Experiences from a modelling seminar	2046
<i>Rita Borromeo Ferri, Werner Blum</i>	
Designing a teacher questionnaire to evaluate professional development in modelling	2056
<i>Katja Maaß, Johannes Gurlitt</i>	
Modeling in the classroom – Motives and obstacles from the teacher’s perspective	2066
<i>Barbara Schmidt</i>	
Modelling in mathematics teachers’ professional development	2076
<i>Jeroen Spandaw, Bert Zwaneveld</i>	
Modelling and formative assessment pedagogies mediating change in actions of teachers and learners in mathematics classrooms	2086
<i>Geoff Wake</i>	
Towards understanding teachers’ beliefs and affects about mathematical modelling.....	2096
<i>Jonas Bergman Ärlebäck</i>	
The use of motion sensor can lead the students to understanding the cartesian graph.....	2106
<i>Maria Lucia Lo Cicero, Filippo Spagnolo</i>	
Interacing populations in a restricted habitat – Modelling, simulation and mathematical analysis in class.....	2116
<i>Christina Roeckerath</i>	
Aspects of visualization during the exploration of “quadratic world” via the ICT Problem “fireworks”	2126
<i>Mária Lalinská, Janka Majherová</i>	
Mathematical modeling in class with and without technology.....	2136
<i>Hans-Stefan Siller, Gilbert Greefrath</i>	
The ‘ecology’ of mathematical modelling: constraints to its teaching at university level	2146
<i>Berta Barquero, Marianna Bosch, Josep Gascón</i>	

The double transposition in mathematisation at primary school	2156
<i>Richard Cabassut</i>	
Exploring the use of theoretical frameworks for modelling-oriented instructional design	2166
<i>F.J. García, L. Ruiz-Higueras</i>	
Study of a practical activity: engineering projects and their training context	2176
<i>Avenilde Romo Vázquez</i>	
Fitting models to data: the mathematising step in the modelling process	2186
<i>Lidia Serrano, Marianna Bosch, Josep Gascón</i>	
What roles can modelling play in multidisciplinary teaching.....	2196
<i>Mette Andresen</i>	
Modelling in environments without numbers – A case study.....	2206
<i>Roxana Grigoras</i>	
Modelling activities while doing experiments to discover the concept of variable.....	2216
<i>Simon Zell, Astrid Beckmann</i>	
Modeling with technology in elementary classrooms.....	2226
<i>N. Mousoulides, M. Chrysostomou, M. Pittalis, C. Christou</i>	

INTRODUCTION

APPLICATIONS AND MODELLING

Morten Blomhøj, Roskilde University

The red thread in the programme for working group 11, *Applications and Modelling*, was to identify and discuss different theoretical perspectives found in research on the teaching and learning of mathematical modelling. Particular emphasis was placed on the relation between research and development of practices of teaching. The presentation: *A survey of theoretical perspectives in research on teaching and learning of mathematical modelling* were given by Morten Blomhøj and Gabriele Kaiser to set the scene for the working group. And the work was ended with a closing panel discussion with Javier García, Gabriele Kaiser, and Hugh Burkhardt as panellists and Susana Carreira as moderator. In addition Hugh Burkhardt gave a historical perspective on the field by his presentation: *The challenge of integrating modelling in mathematics teaching practices – a historical view* by.

The presentation and discussion of the papers was structured according to five themes: (1) Teachers' professional development for teaching and assessing mathematical modelling, (2) The role of ICT in teaching and learning mathematical modelling, (3) Researching the teaching and learning of mathematical modelling within the Anthropological Theory of Didactics (ATD), (4) Researching the teaching and learning of mathematical modelling within the framework of Realistic Mathematics Education (RME), (5) Researching the teaching and learning of mathematical modelling under the Models and Modelling Perspective (MMP). Each theme was introduced shortly and rounded off with a general discussion. The proceedings is organised in accordance with the thematic structure of the programme and include the 19 papers presented and discussed at the conference.

Theme 1 was introduced by Katja Maass and Geoff Wake and included six papers.

In the first paper, Rita Borromeo Ferri and Werner Blum present and discuss their experiences with modelling seminars as a way of integrating the teaching and learning of modelling in mathematics teacher education. As a basis for their design of the modelling seminars the authors have identified four main competencies related to the teaching of mathematical modelling: (1) Theoretical competency, (2) Task related competency, (3) Teaching competency, (4) Diagnostic competency. It is argued that mathematics teacher education should support the development of such competencies and include experiences with modelling activities in school practices.

Katja Maaß and Johannes Gurlitt write about the problem of how to evaluate teachers professional development in the teaching of mathematical modelling. Based on the authors experience from the international LEMA project the paper discusses the challenges related to the design and application of an evaluation questionnaire for teachers participating in a professional development project.

The paper by Barbara Schmidt is also related to the LEMA project. She analyses - also by means of questionnaires - the motives and obstacles experienced by the teachers for including realistic modelling activities in their teaching. According to the regulations of mathematics teaching it should include realistic modelling activities. However, different institutional and educational factors seem to form obstacles for this ambition. The findings suggest that it is possible to identify types of teachers that experience motives and obstacles for realistic modelling differently.

The paper by Jeroen Spandaw and Bert Zwaneveld reports on the development of a text book for secondary mathematics teacher education. One of its objectives is to further the coming teacher professional development for teaching mathematical modelling. The paper discusses issues such as the teachers' dispositions for modelling, educational goals for teaching modelling, design aspects, testing in modelling, the role of domain knowledge, and computer modelling. The paper also reflects on the relationship between mathematics, teaching of mathematics and modelling, and on the role of modelling in the Dutch mathematics curriculum.

The next paper is concerned with formative assessment in relation to mathematical modelling activities. Using a Cultural Historical Activity Theory perspective, Geoff Wake argues that modelling activities and related pedagogies and in particular the quest for formative assessment in relation to learners modelling processes have the potential to bring about a significant change in classroom activity for learners and teachers; and that such changes might support the learning of mathematics for more students and better prepare them to apply mathematics. This paper is also related to the LEMA project.

In the last paper of theme 1 Jonas Bergman Ärlebäck reports on a study on teachers' beliefs and affects about mathematical modelling. Five different domains of beliefs are identified as important for if and how teachers will include mathematical modelling in their teaching: (1) the nature of mathematics, (2) real world (reality), (3) problem solving, (4) school mathematics, (5) applying, and applications of, mathematics. Two teachers' beliefs are analysed according to these five domains.

Theme 2 was introduced by Morten Blomhøj and included four papers each presenting concrete cases of ICT supported modelling activities.

Maria Lucia Lo Cicero and Filippo Spagnolo in their paper report from an experimental project with three upper secondary classes that have been working with motion sensors and computers to produce graphs for different motion phenomena. From pre- and post-tests and analyses of the classroom interactions it is argued that students developed modelling competencies and that the modelling activities can enhance the students' mathematical and physical understanding of important concepts such as Cartesian graph, function, derivative, velocity and acceleration.

In the second paper Christina Roeckerath presents and analyses a simulation software package that can support the students' modelling and analyses of different types of biological interactions between species such as predator-prey, competition or parasitism. It is argued that such modelling activities can provide the students with an insight into the interdisciplinary relationship between mathematical modelling and theoretical population biology, and support their learning of biology.

Mária Lalinská and Janka Majherová discuss in their paper different aspects of visualization in relation to projectile motions modelled by secondary students using a spreadsheet and a graph drawing software. The motion of fireworks is used as a situational context to set the scene for the modelling activities and it is argued that ICT-supported modelling activities allow the students' to experience and understand better the mathematical and physical elements involved in the phenomena.

In the last paper of theme 2, Hans-Stefan Siller and Gilbert Greefrath present and analyse in detail modelling cycles in which technology is integrated by means of handheld or computer based software. The potentials in different types of software (CAS, DGS and SP) for supporting the students learning of modelling and mathematics are discussed and illustrated with the example of modelling "dangerous road intersections".

Theme 3 was introduced by Berta Barquero and Javier García and included five papers.

The first paper by Berta Barquero, Marianna Bosch and Josep Gascón introduces the metaphor of ecology and the notion of levels of didactic determination from ATD, and show theoretical constructs can be used to better understand the institutional constraints that hinder the large scale implementation of mathematical modelling activities. The theoretical ideas are exemplified through an analysis of "applicationism" - a notion used by authors to capture the set of beliefs that guides applications of mathematics in traditional mathematics teaching.

In the paper by Richard Cabassut it is argued that mathematical modelling activities can be analysed as a double didactical transposition within the framework of ATD. Real world problems and related techniques undergo a transposition when used in mathematics teaching similar to the transposition that mathematical concepts, techniques and theories undergo. This transposition process is analysed with respect to the modelling cycle, and examples of mathematisation tasks from the LEMA project are used to illustrate the elements in the transposition process.

García and Ruiz-Higueras in their paper illustrate how the ATD can be used as a theoretical framework for designing mathematical modelling activities for teaching. A design - also from the LEMA project - for 4-5 years children is presented and analysed to illustrate the theory based design process. The experiences from the

implementation of this design show how even very young pupils can be involved in rich and meaningful mathematical modelling activities.

The paper by Vázquez reports about the ATD based design of modelling activities for engineering students. The processes of transposition of the praxeologies involved in a particular modelling task – the modelling of a motor – are analysed, and it is argued that in order to understand the technologies linked to the students techniques, it is necessary to take in account the different disciplines involved.

Serrano, Bosch and Gascón in their paper analyse from a ATD perspective the mathematisation process in the modelling cycling process. A modelling task for university students on forecasting the sale of a given product from an empirical time serie is used as an example. The experinces show that a modelling activity initiated with a real-situation can lead to mathematising that affects both the system and the model and that challenge the students' modelling competency and their learning of important mathematics.

Theme 4 was introduced by Mette Andresen and included two papers and a poster by Simon Zell and Astrid Beckmann.

In the first paper Mette Andresen presents a long-term research and developmental project concerning mathematical modelling in a multidisciplinary context in upper secondary teaching. A course of lessons based on the Vioxx case is used to illustrate the different levels of reflection in the students' modelling work in this context.

The paper by Roxana Grigoras deals with the modelling of real world phenomena where no numerical data are given. In the case studied, lower secondary students' are trying to make sense out of a picture of the surface of the planet Mars. In this very open modelling activity the students use a number of fundamental mathematical ideas. The activity is analysed using RME as a theoretical framework.

Theme 5 was introduced by Nicos Mousoulides and included only one paper. Here N. Mousoulides, M. Chrysostomou, M. Pittalis and C. Christou present and discuss a case where a class of 11-years students worked with the fresh water shortage problem in Cyprus. It is a real life problem, the students' used relevant technology (Google Earth and spreadsheet) and they were in fact able to compare, judge and reflect on the different models developed. The activity was design and analysed within the framework of MMP.

MATHEMATICAL MODELLING IN TEACHER EDUCATION – EXPERIENCES FROM A MODELLING SEMINAR

Rita Borromeo Ferri and Werner Blum

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Mathematical Modelling has recently become a compulsory part of the mathematics curriculum in Germany. Hence future teachers must have a strong background about different aspects of modelling and also about appropriate methods how modelling can be taught. That means that the content and the methodology of university courses on modelling have to include all these aspects. In our paper, we will report on university seminars on modelling for students in their fourth year of study. Among other things, the students had to write a “learning diary” over the whole semester. The results give interesting insights in students’ learning processes of modelling, their progress and their problems during the semester and their considerations about teaching modelling.

INTRODUCTION

Although mathematical modelling is now a compulsory part of the mathematics curriculum in Germany and one of the main competencies within the national Educational Standards, it is not at all guaranteed that pupils will be taught by teachers who have a sound knowledge of modelling. One reason for this is the fact that modelling has normally not been taught in teacher training courses at University, because modelling is not contained explicitly in the curriculum for future math teachers in Germany. However, there is no doubt (see, e.g., Krauss et al. 2008) that teachers have to be experts in modelling themselves in order to be able to teach students effectively and that their thinking has to be shaped towards creating rich classroom environments that enable students to be actively involved in modelling (Chapman 2007).

In the last few years, a lot of empirical studies have dealt with the question of how modelling can be taught in school (see, e.g., Maaß 2007 or Blum & Leiß 2007) or how students at University can be sensitized for modelling through complex modelling tasks (see Lingefjaerd & Holmquist 2007, Blomhoj & Kjeldsen 2007 or Schwarz & Kaiser 2007). The results of these studies opened new ways of thinking about modelling and the way it can be integrated in school mathematics in a profitable way. However, the question of how these aspects can be integrated in teacher education still remains open. Two of the main questions are:

1. How can future teachers be prepared in university courses for teaching modelling at school, which contents and which methods are appropriate?

2. How do students' processes of learning and understanding develop during such courses, what are their main difficulties and problems, and how can progress be observed?

In this paper, we will report on such a modelling seminar which has been taught at the University of Hamburg by the first author and with similar features at the University of Kassel by the second author, and with which we have tried to tackle these two questions. Our guiding principle for the conception of this seminar was: If we want our students to teach modelling in an appropriate way (with a correspondence between content and method, cognitive activation of pupils, reflection on learning and integration of summative assessment) we as lecturers have to conceive our own teaching in exactly the same way (correspondence between content and method, cognitive activation, reflection, summative assessment).

CONCEPTION, GOALS, CONTENT AND STRUCTURE OF THE SEMINAR

The main basis for our data collection was a modelling seminar for students in their fourth year of study at the University of Hamburg. In this course altogether 25 future teachers from all school levels were participating, including teachers for students with special needs. (The authors' experiences from other modelling seminars showed that this kind of mixture builds a good basis for discussions and is important for arguing that modelling is suitable for all kinds of school levels and types.) The course took place once a week for 90 minutes over one semester that means 14 lessons altogether. According to the meaning of a "university seminar", the students were expected to be actively involved in all activities and to cover a major part of the course by their own presentations. In the following we will describe more precisely the conception of this seminar and the way the students were observed over the semester. Mathematical Modelling as a subject in teacher education may, of course, be structured in many different ways because it is a vast field and contains a lot of important aspects (see Blum et al. 2007). In our considerations for planning and structuring a modelling seminar in a new way, the content and the methods should fit to each other. This is also a challenge for the lecturer. Concerning content, we regard the following competencies concerning modelling as particularly important:

- (1) Theoretical competency (knowledge about modelling cycles, about goals/perspectives for modelling and about types of modelling tasks)
- (2) Task related competency (ability to solve, analyse and create modelling tasks)
- (3) Teaching competency (ability to plan and perform modelling lessons and knowledge of appropriate interventions during pupils' modelling processes)
- (4) Diagnostic competency (ability to identify phases in pupils' modelling processes and to diagnose pupils' difficulties during such processes)

We did not include an "Assessment competency" (that is the ability to construct and mark tests appropriate for modelling). This competency is, of course, very important

for in-service teachers but can, in our view, not be expected from students who have not enough experience in assessment.

These four competencies were the basis for the structure of the seminar. The seminar was subdivided in the following five parts, also in order to have an appropriate balance between more theoretical and more practical phases:

Part 1 (Theory): Theoretical background about modelling (3 lessons)

Part 2 (Practice): Solving and creating modelling problems (3 lessons)

Part 3 (Theory and Practice): (1) Students analyse transcripts of pupils' work on modelling problems; (2) What are modelling competencies;* (3) Types of teacher interventions while modelling; (4) Methods how to teach modelling in school (4 lessons)

Part 4 (Presentations): Groups of students present their own modelling tasks and how pupils in school solved these tasks. (3 lessons)

Part 5: Last lesson – reflection of the whole work over the semester

*At the end of this part there was an intermediate evaluation of the seminar on the basis of a questionnaire.

One important goal of the seminar was that students do not only solve or construct modelling tasks but also learn methods how they can teach modelling. For us as lecturers it seemed important not to merely say which methods could be useful, but to integrate them directly into the work in the seminar. We decided to use teaching strategies from the field of “Cooperative Learning” (see e.g. Johnson & Johnson 1999, Kagan 1990), also because the first author had good experiences using this while teaching modelling at school. We think that Modelling as the content and Cooperative Learning as a teaching strategy fit together very well also at university seminars. Research has shown (see Johnson & Johnson 1995) that cooperative learning techniques promote pupils' learning and academic achievement, increase pupils' retention, enhance pupils' satisfaction with their learning experience, help pupils develop skills in oral communication, develop pupils' social skills, and promote pupils' self-esteem. Several studies on modelling made clear that modelling is better done as a group activity (Ikeda, Stephens & Matsuzaki 2007), also because this supports discussions about mathematics or extra-mathematical aspects, trains argumentations and gives the chance to profit from group synergy. That is why in the first lesson of the seminar the students had to build so-called “basis-groups” of five people who were supposed to work together over the whole semester; altogether there were six such groups. However, working in groups is only under certain conditions more productive than competitive and individualistic efforts. Those conditions are (Kagan 1990): Positive Interdependence, Face-to-Face-Interaction, Individual-&Group-Accountability, Interpersonal-&Small-Group Skills and Group Processing. So we had to take care that all group activities fulfil these conditions. We combined these activities with the content-parts of the seminar:

Part 1: Students had to know about different *directions* in the discussion on modelling and different *modelling cycles* (see e.g. Kaiser, Sriraman & Blomhoj 2006 and Borromeo Ferri 2006 as literature which was given to the students). They learned this content with the activity “Jigsaw”: Each group member is assigned some particular material to learn and later on to teach to his group members (in this case each student had one direction of modelling as his particular topic, e.g. realistic modelling, and in the second round one version of the modelling cycle). Students with the same topic worked together in “expert-groups”, so the basis-groups were divided. After working in these expert-groups, the original basis groups reformed and students taught each other. So at the end of this part the students had learned this content mostly on their own. It was, of course, important for the students that they also could ask all kinds of questions, especially in the last lesson of this part, and that we reflected both the theory and the activity Jigsaw.

Part 2 started with the question “What is a good *modelling task*?” For that we used the activity “Think-pair-share”. This involves a three step cooperative structure. During the first step, individuals thought silently about a question posed by the instructor. Individuals paired up during the second step and exchanged their thoughts. In the third step, the pairs shared their responses with the entire group. After that the basis-groups solved the modelling task “Filling Up” (“Tanken”, see Blum/Leiß 2007). For a better understanding we showed the students a possible solution process by means of the seven-step modelling cycle that we ourselves use in our work (Blum & Leiß 2007, Borromeo Ferri 2007), in order to help them to understand which part of their solution can be regarded as a real model or a mathematical model and so on. The six groups had then time for sharing ideas for their own modelling tasks which they had to construct and to test in school. For that “creating part” we used the method “RoundRobin Brainstorming”: One person of each group was appointed to be the recorder. A question or an idea was posed with many answers, and students were given time to think about the answers. After the “think time,” members of the team shared responses. The recorder wrote down the various answers of the group members. The person next to the recorder started and one person after the other in the group gave an answer until time was called. At the end of this part, the groups had finished creating their modelling tasks and in addition they had learned how they could do a subject-matter analysis of the problem. Similar to the first part, we discussed questions and reflected the used methods for potential uses in school.

Part 3 contained a lot of interesting aspects of modelling. So we started each aspect with a short theoretical input and the students then had an activity on their own. Concerning aspect (1), the basis-groups worked on the transcripts of *pupils’ solution processes* to the modelling task “Lighthouse”, and we had a discussion afterwards especially about the distinction of the phases while modelling. Before we started with our input for aspect (2), we used the method “silent writing conversation”. Every group got a big sheet of paper. In the middle of the paper they were to write “*modelling competencies*”. The students had to do a brainstorming about what

modelling competencies could be, however without saying a word. So they had to comment the products of the other group members also in a written way. After that we gave information on modelling competencies and had then a discussion in the plenum, mainly about how teachers can support modelling competencies and how they can assess these in school. Like before, we started aspect (3) with an activity, this time “Inside-Outside-Circle” before we gave a theoretical input about the meaning of “*intervention*” and “*self-regulated learning*”. The activity “Inside-Outside Circle” follows the principle that all students are integrated in the learning process. So the students form an inner and an outer circle. Those in the inner circle look outside, those in the outer circle look inside. Then the whole group was asked: “What do you think a teacher has to know when teaching modelling so as to be able to intervene appropriately in case of students’ difficulties?” The students stood opposite to each other and discussed this question in pairs. After five minutes, the outside circle moved on and students in new pairs exchanged their thoughts. The same was done with the second question, thus addressing aspect (4): “What do you think are good or bad methods for teaching modelling?” We closed this lesson with a discussion and a reflection about the five activities of cooperative learning we had so far during the seminar and how they fit to the contents of the seminar. Simultaneously this was meant to be a meta-reflection on different levels: 1. the students had to think about each method and about teaching them in school in connection with modelling; 2. we as lecturers had to reflect whether the chosen activities were useful to teach the contents of the seminar.

In *part 4*, all groups presented their *modelling tasks* and their *experiences* they had, in the meantime, gathered in school with these. Because of the participation of future teachers for all school levels, also the presented experiences were from primary to upper secondary school. The final *part 5* rounded off the seminar with a *summary* of all aspects.

Taking into account the rather elaborate conception of the seminar, we liked to know if the students felt sufficiently well-equipped to teach modelling at school. Furthermore we were interested in students’ individual learning and understanding processes and how these develop during such a course as well as in their main problems and difficulties.

METHODS OF ANALYSING LEARNING PROCESSES

Reflection was a major issue for the students in the seminar, because thinking over one’s own actions generally deepens the understanding a lot. To get insight into the thinking and learning processes, the students had to write a “learning diary” (see e.g. Gallin & Ruf 1990). One important goal of a learning diary is to write down one’s individual learning story. It also helps stabilising the competencies related with the contents. For the goals of our explorative study, the “learning diary” was the adequate instrument to stimulate reflections on students’ own learning processes over a long

time. Interviews could have been an alternative, but not with a whole seminar. The organization of a learning diary looks mostly as follows: write down the date, the topic of the lesson and the activity; write down why you had to do the activity; look back and think about where you are in the learning process. The students in the seminar had to do this in a similar way concerning their learning of the topic of modelling. In the last five minutes of each lesson, the students had time for writing their reflections into their learning diary. At the end of the semester, all diaries were collected in order to analyse them with respect to understandings and problems referring to a) the different parts on the content, b) the methods used, c) the way how the seminar was taught, and d) the students' own reflections on teaching and learning modelling in school. So we coded (Strauss/Corbin 1990) and categorized statements of the students according to these four aspects to get an overview and to find patterns. In addition, we analysed each diary with regard to hints concerning the learning process of the individuals.

RESULTS

Most of the students knew from their first semester course a little bit about modelling and what it means, but that was only a small part of the lecture. So 18 from 21 students wrote in their reflections after the first lesson that they had not known that modelling is such a big field.

“In this lesson I got a first insight in the theme “modelling”. There it became apparent for me that this theme is very wide and does not only exist of the modelling cycle I know from my first semester course.” (Katrin, 2nd of April 2008)

Not unexpectedly, dealing with part 1 was not easy for the students. To distinguish between different directions and then again between different modelling cycles was a high demand for them, what the reflections clearly show. But the method Jigsaw was a helpful strategy for the students to help each other and to become more clarity about the content. Anyhow the students felt that this strong theoretical part was helpful to get appropriate background knowledge.

“Sometimes it was not easy to understand one direction of modelling in the expert-groups, because of the shortness of the text. But this method [Jigsaw] is perfect! Everyone of the group has to explain something and so we discussed till I understood it better.” (Swetlana, 9th of April 2008)

A progress in the learning process of the students could be reconstructed in Part 2. All students reflected that they understood the seven-step modelling cycle finally through the modelling task “Tanken” which we presented to them in a detailed manner after they solved this problem. Furthermore they felt that now the background from part 1 will help them to create an own modelling task, so for them theory and practice linked together here.

“It was good, that we went through the modelling cycle with an exemplary task. Thereby one became aware how complex a modelling task can be [...]. Now it will be easier for us to create our own modelling task.” (Sarah, 16th of April 2008)

“Slowly I understand the modelling cycle better. Working with the “Tanken”-Task helped me to distinguish several steps of the modelling cycle.” (Alexander, 16th of April 2008)

When analyzing the reflections on part 2 it became very clear that creating modelling tasks is as important for learning and understanding modelling as solving modelling problems. The students had to think over the school level in which they wanted to test the problem, how complex the task should be, how much time the pupils would probably need, and so on. Helpful for them was the method used in this context.

“It was good that we were to create our own modelling task in our basis-group. However we recognized that this will be a difficult undertaking. But the method RoundRobin was exactly adequate to get helpful suggestions from other basis-groups.” (Anna, 23rd of April 2008)

Thus the three lessons of part 2 were once again a linking between theory and practice for the students, and a progress in their process of understanding could be reconstructed especially concerning the modelling cycle. Furthermore they had to deal with the question of authenticity and complexity while creating their own modelling task. The students were confronted with a lot of aspects of modelling in part 3 as described above. We have no space to go more into detail here, but we try to summarize the important points. Analyzing transcripts of pupils’ modelling processes in aspect (1) was helpful for the students to distinguish several modelling phases.

“The transcripts of the pupils helped me in some part to distinguish several modelling steps.” (Heidy, 7th of May 2008)

Modelling competencies and beliefs were interesting for the students. Most of them liked the question of how modelling competencies could be supported. They commented that one lesson was not enough for this content and that they would like to know more about this topic.

“The silent-writing-conversation was very fruitful at the beginning concerning the meaning of modelling competencies. Of high interest for me was the question of how modelling competencies can be supported. This is especially for a teacher an important question.” (Jan, 21st of May 2008)

Starting aspect (3) with the method Inside-Outside-Circle was for all students a good start for the topic of teacher interventions. Most of the students started to reflect more about themselves as a teacher personality and also liked to have more time for this topic.

“After the discussion in the Inside-Outside-Circle I think that a teacher must be well prepared when he has a modelling problem for his lesson, because he has to analyse and to diagnose his pupils quickly to help them.” (Carolin 21th of May 2008)

“Today I learned a lot about different kinds of teacher interventions, firstly theoretically and then practically through group work with a case study of a teacher. But I take much more out of this lesson today: The case study showed me how invasive a teacher can intervene, so that this intervention is restricted only to the content. But I will look to myself how I intervene to correct my interventions.” (Andreas 21th of May 2008)

The reflection of the methods (aspect 4) was very constructive, because the students learned the methods on their own through the seminar. So they were able to decide about advantages and disadvantages. All of the students agreed also that these methods can be integrated while teaching modelling, but they have to be practiced.

“It is good that we are learning not only modelling as a subject in this seminar, but also the methods how we can teach this at school!” (Katja 28th of May 2008)

Testing the modelling task at school and then presenting the results in part 4 was particularly important for the learning processes of the students. Whereas the processes of understanding of the students concerning modelling partly stumbled in part 3 because of the diversity of the aspects, part 4 stood for their progressives. The reflections indicated that they learned and understood more about what modelling means on a theoretical level and also how to teach it.

“Today my group and I had our presentation. I think it was good! [...] Overall the testing was helpful for me as a teacher, because I could see where pupils had problems while modelling. Also to get the self-awareness to walk between the small level of intervention and reservation was important for me. Furthermore it showed me that the task should be phrased precisely and to allow enough extra time.” (Benjamin 4th of June 2008)

“Testing the modelling task in grade five was important and helpful for my understanding of modelling and the practical transformation in school. [...] It was good to have a chance testing modelling problems at school.” (Birgit 18th of June 2008)

Summary of the results

We summarize our results concerning the two questions at the beginning. First we asked how teachers can be prepared in university courses for teaching modelling at school, which contents and which methods are appropriate. On the basis of our experiences, we are sure that in general a balance between theory and practice must be given. Both should be connected by means of an appropriate teaching strategy, which must be reflected in the seminar. Of course, the contents of such a seminar may vary, but according to our experiences, the following contents are well suited for such a seminar (see the competencies referred to at the beginning): (1) Knowledge about modelling cycles, goals/perspectives and types of tasks; (2) Solving, creating and analysing modelling tasks; (3) Planning and practising modelling lessons; (4) Diagnosing actual modelling processes of pupils.

Second we asked how do students' processes of learning and understanding develop during such courses, what are the main problems and how can progress be observed. We decided that students had to write a learning diary to help us to answer that

question, also in combination with the evaluation of the seminar. These were the main problems of the students: to understand several directions of modelling and the distinctions between modelling cycles in the literature; to distinguish phases of the modelling cycle in general and also analysing transcripts of pupils' modelling processes; subject-matter analyses of modelling problems; and finally dealing with the question of authenticity while creating a problem. Progress of the students concerning these difficulties could be reconstructed mostly when, pragmatically speaking, they linked theory with practice. Reflecting these developments during the seminar helped the students, undoubtedly, on their way to become competent teachers of mathematics.

In conclusion, we would like to emphasize once again the necessity that university students who are to become mathematics teachers must have vast opportunities to deal with mathematical modelling both on a theoretical and on a practical level, including experiences with modelling at school. This will not only contribute to preparing them to be competent teachers for mathematical modelling but will also contribute to further develop their understanding of mathematical subject matter and of mathematics as a discipline (Lingefjaerd 2007).

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DESIGNING A TEACHER QUESTIONNAIRE TO EVALUATE PROFESSIONAL DEVELOPMENT IN MODELLING

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LEMA is an international project to design a professional development course for modelling. In order to measure the effects of the course, an evaluation questionnaire was developed and pilot tested. Based on theoretical background about modelling, this paper outlines the challenges of the design process, presents reliability and validation data, and exemplifies each scale with a few sample items.

Keywords: Modelling, professional development, international approach, design of an evaluation questionnaire, empirical study

INTRODUCTION

Researchers, practitioners and policy makers in mathematics education agree that educationist aim should be to enable students to apply mathematics to their everyday lives (PISA, OECD, 2002) and contribute to the development of active citizenship. However, modelling is still rare in day-to-day teaching around Europe. LEMA (Learning and Education in and through Modelling and Applications) is a trans-national European Project¹ that tackles the problem at teacher level by developing a common course of professional development in mathematical modelling. The aim of this paper is to provide an approach to the evaluation of professional modelling development in different national contexts and settings that is theory-based and driven by analysis of needs.

Teachers' knowledge and beliefs about the nature of the subject, their views on how to educate the subject and their self-efficacy beliefs about teaching the specific subject influence how they design or select tasks, plan, implement and evaluate their lessons (e.g. Brickhouse, 1990). Thus, to successfully implement mathematical modelling in their actual classroom practice, teachers need to (amongst others) (1) know the key concepts of mathematical modelling, (2) change their beliefs about the nature of mathematics education (if not appropriate for modelling), and (3) increase their awareness of their own competency to implement mathematical modelling in their actual classroom practice (self-efficacy).

THEORETICAL BACKGROUND

Mathematical modelling means applying mathematics to realistic, open problems. There are many descriptions of modelling processes (Blum & Niss, 1991; Kaiser-

¹ Partners of LEMA: Katja Maass (Coordinator) & Barbara Schmidt, University of Education Freiburg, Geoff Wake, University of Manchester, Fco. Javier Garcia Garcia, University of Jaen, Nicholas Mousoulides, University of Cyprus, Ödon Vancso & Gabriella Ambrus, University of Budapest, Anke Wagner, University of Education Ludwigsburg, Richard Cabassut, IUFM Strasbourg. The project is founded by the EU.

Messmer, 1986). They vary according to the described modelling cycle, the relevance given to the context and the justifications seen for modelling in mathematics lessons (Kaiser & Shiraman, 2006). In this study, we follow the description of the modelling process in PISA (Prenzel et al., 2004), albeit restricting it to context-related problems.

Modelling competency is the ability to carry out modelling processes independently. It comprises the competencies to carry out the single steps of the modelling process as well as competencies in reasoning mathematically and metacognitive modelling competencies (Maaß, 2006). Similar distinctions have already been made by Kaiser and Blum (1997) and indirectly by Money and Stephens (1993), Haines and Izard (1995) and Ikeda and Stephens (1998) in setting up assessment grids.

Modelling lessons: A trans-national project, which aims at developing a common, research-based professional development intervention for Europe, faces the challenge presented by partners having different theoretical backgrounds related to the teaching of modelling situated in their different national contexts. Thus, we sought to identify where common ground and indeed differences might be used to enrich such a project. For example, the English partner adopts a socio-cultural approach – drawing particularly on ideas of the development of learner identity and using Cultural-Historical Activity Theory (CHAT) (Engestrom & Cole, 1997). Research of the Spanish team, on the other hand, relied upon the Anthropological Theory of Didactics (Chevallard, 1999). Finally, the German partner's position is located in the international discussion on modelling and focuses on authentic tasks showing the usefulness of mathematics.

Drawing upon all of these approaches, a theory-based design emerged as follows: The German partner provided descriptions of the modelling cycle and focused on authenticity, both aspects being used to design tasks and support didactical development in the classroom. With relation to the Spanish partner, authenticity can also be seen in the search for questions that are crucial for students as social individuals. The English partner's perspective that engaging with mathematics can be considered a social activity provides teachers and researchers with a range of new learning and pedagogical models.

In short, the theoretical approach – related to the teaching of modelling – used in this study is a synthesis of a variety of theoretical backgrounds. This allows for and ensures that combined expertise guided this trans-national project.

Professional development of teachers: When considering teachers' competencies in teaching, we follow Krauss et al. (2004) and Shulmann (1986) by distinguishing professional knowledge (content knowledge, pedagogical content knowledge, pedagogical knowledge), beliefs, motivational orientation and competencies in reflexion and self-efficacy.

Empirical studies of teachers' professional development (e. g. Tirosh & Gerber, 2003; Wilson & Cooney, 2002) show that professional development interventions lead to changes if the courses are long-term, with embedded phases of teaching and reflexion, and if further factors which might have an impact on teachers' possibilities

to teach modelling (such as the framework, the head of the school, parents, teacher's own beliefs) are taken into account.

Teachers' beliefs about the nature of mathematics and its education are believed to have a major impact on if and how a teacher employs innovation in everyday teaching. According to Pehkonen and Törner (1996), beliefs must be understood here as being composed of a relatively lasting subjective knowledge of certain objects as well as the attitudes linked to that knowledge. Beliefs can be conscious or unconscious, whereas the latter are often distinguished by an affective character.

Kaiser (2006) showed that innovations required by the curriculum are interpreted by the teacher in such a way that they fit into his or her existing belief system. Grigutsch, Raatz & Törner (1998) classified beliefs about mathematics into various aspects: the aspect of scheme (fixed set of rules); the aspect of process (problems are solved); the aspect of formalism (logical, deductive science); and the aspect of application (important for life and society).

Teachers' self-efficacy beliefs in this context can be described as teachers' beliefs in their capabilities to organize and execute mathematical modelling activities in their planning and classroom practice (see Bandura, 1997, Bandura, 2006). Self-efficacy is a future-oriented belief about the level of competence a person expects he or she will experience in a given situation. Self-efficacy beliefs influence thought patterns and emotions that enable actions and effort for reaching goals and persist in the face of adversity. "The self-assurance with which people approach and manage difficult tasks determines whether they make good or poor use of their capabilities. Insidious self-doubts can easily overrule the best of skills" (Bandura, 1997, p. 35).

Considering pedagogical content knowledge about modeling as an external measure of learning success, both beliefs about the nature of mathematics and the education of mathematics and self-efficacy beliefs touch the individual's own perception and motivational aspects potentially relevant for application in the actual classroom setting.

DEVELOPMENT OF THE COURSE

Analysis of needs: In order to design a suitable professional development course for teachers, we first conducted a needs analysis to assess teachers' mathematical beliefs, their use of tasks, and their attitude towards given modelling tasks. Altogether, $N = 563$ teachers from the partner nations participated in the needs analysis. The measurement instrument included items about beliefs, which were rated on a 4-point rating scale ranging from 1 (strongly disagree) to 4 (strongly agree). Additionally, items related to the tasks teachers use (e.g. tasks practising basic skills vs. problem solving tasks) and three concrete modelling tasks were given to the teachers (all related to the same context, but differing in their task openness). Teachers were asked how likely they were to use each task and to justify their response.

Results revealed that teachers gave high rating scores for belief items in the process dimension (e.g. mathematics allows you to solve problems: $M = 3.49$) and the application dimension (e.g. mathematics is useful in everyday life: $M = 3.5$). Conversely,

they gave relatively lower scores for the formalism and scheme dimension (e.g. mathematics is a fixed body of knowledge: $M = 2.44$). However, when asked which tasks they would most likely use in their lessons, the majority selected “tasks that practised basic skills” ($M = 3.48$) as opposed, for example, to “problems with other than one solution” ($M = 2.38$). Accordingly, when asked if they would use any of the given modelling tasks, the closed tasks proved to be very popular, while the more open versions drew less enthusiasm. Commonly cited detractors for the open tasks were perceived difficulty and class time constraints.

In terms of designing an evaluation questionnaire, the analysis of needs also made clear that the more related the questions are to day-to-day teaching (e.g. related to concrete tasks), the more objections become evident.

The course of professional development: Based upon the findings of the needs analysis, the following considerations were given particular attention: First, we addressed the teachers’ concerns and difficulties in using modelling tasks by providing further information about the benefits inherent to each modelling task. We also addressed different ways to assess and support students in their development of modelling competencies.

Based on the needs analysis and the synthesis of various theoretical backgrounds, we developed the professional development course into five key aspects (modules):

(1) *Modelling*: To implement modelling in lessons, teachers need background information about this concept (sub-modules: What is modelling? Why use it?) (Blum & Niss, 1991).

(2) *Tasks*: When it comes to planning lessons, teachers need to learn how to select appropriate tasks for their students and anticipate the modelling outcomes. An emphasis was placed on authentic tasks (sub-modules: Exploring, Design, Classification and Variation) (see e.g. Maaß, 2007, Burkhardt, 1989, Galbraith & Stillman, 2001, Kaiser-Meßmer, 1986)

(3) *Lessons*: Research has shown that group work, discussion and working independently all support the development of modelling competencies (sub-modules: Methods, Using ICT, Supporting the Development of Modelling Competencies, Exercising Mathematical Content Through Modelling), (see e.g. Tanner & Jones, 1995, Maaß, 2007, Ikeda & Stephens, 2001)

(4) *Assessment*: If modelling is implemented in lessons, it also has to be evaluated. Assessment should be used not only for grading but also for supporting learning through feedback (Williams & Black, 1998) (sub-modules: Formative Assessment, Summative Assessment, Feedback).

(5) *Reflexion*: As outlined above, reflexion about implementation in lessons and dealing with challenges is crucial for the success of professional development courses (sub-modules: Implementation, Challenges).

Piloting: This course was piloted and evaluated in all 6 partnership countries. Piloting took place in 2008 and comprised 5 days. Implementation, however, was quite different. For example, in France the training was given as a one-block course in January 2008, addressing teachers teaching students aged 6-8 years. In Spain, the course contained two blocks in April and May. Finally, in Germany the course consisted of 5 separate days from January to November and addressed primary and secondary teachers at the same time.

The main question was how such a course, which was piloted under different conditions and in different national contexts, could be evaluated. We did not consider students for evaluation because this seemed to be almost impossible given the huge variety of students concerned (age 6 to 16) and the given national contexts. Focusing on teachers, we used questionnaires and interviews and exemplary videos. Questionnaires and interviews give insight into teachers' point of view and so provide information about teachers' intentions and thus about necessary preconditions of the change of day-to-day teaching. Here, we will focus on the teachers' questionnaire.

DESIGN OF THE QUESTIONNAIRE

Instrument Development and Field Testing: The questionnaire was prepared to assess all teachers taking part in the course (6 countries, 10-40 teachers per country). To measure possible knowledge gains and belief changes we implemented a pre-post-control-group design. The development and testing of the instrument took place in 5 steps.

Step 1: Establishing rationales guiding the design: First, items should mirror the theoretical background and key-aspects of the modules of the professional development. Thus, the questionnaire sought information about the pedagogical content knowledge, beliefs, and self-efficacy about mathematical modelling as well as beliefs about mathematics and its education. Items covered these categories and all five modules. Second, considering the target group and their understandable preference for a short questionnaire, our aim was to find a balance between a reasonable length and what would still provide a reliable assessment. Third, careful guidelines were developed to improve compliance in filling out the questionnaire and the quality of the implementation of the questionnaire (i.e. we provided information regarding the necessity of an evaluation for further improvement and emphasized that it was the course and not the teachers that was being evaluated) .

Step 2: Procedure and materials preparation: Considering the first rationale of Step 1, the scale construction was based on established scales wherever possible. The scale of belief items about the nature of mathematics and its education was based on Grigutsch, Raatz and Törner (1998). Items were related to four aspects of beliefs (see above). Learners rated their beliefs on a 5-point scale, ranging from strongly disagree to strongly agree.

Based on Bandura's method for measuring self-efficacy beliefs (Bandura, 2006), we designed a self-efficacy scale assessing efficacy beliefs related to modelling on a

100-point scale, ranging in 10-unit intervals from 0% ("cannot do at all"), to 100% ("highly certain can do").

For the assessment of the pedagogical content knowledge, we decided to use questions in an open format, the main consideration here being sensitivity for measuring knowledge provided in the course. Teachers were supposed to show their active knowledge, as this is probably the knowledge which they use in teaching. The open questions used for this knowledge assessment were rated by two independent, trained raters considering the amount and quality of arguments based on an expert solution.

Step 3: First Item Refinement – A Small Tryout: First, we conducted a small pilot study. The tryout instrument was administered to a group of 7 teachers and 3 teacher trainers. In addition to filling out the questionnaire, participants were asked to comment on the items they found misleading or difficult to understand. Consequently, items that were mentioned as being misleading were adapted. Items where the answers of teachers showed a lack of focus were reformulated. Considering the target group, the initial questionnaire was too long (time needed > 60 min). Thus, we analyzed the questionnaire for time-efficiency and possibilities to omit certain items. For example, the first questionnaire contained an open item referring to beliefs about the different areas where mathematics can be useful. As this was not directly linked to the content of the course, we omitted this item due to time vs. diagnostic value considerations. The first question "What is modelling?" was moved to the end, because some teachers were unable to answer it and therefore became discouraged right from the beginning.

Step 4: Second Item Refinement – Expert Questioning: To improve the content validity of the items, the questionnaire was submitted to 10 modelling experts, each with more than 5 years' teacher education. They were asked to evaluate whether the item statements would adequately deliver information about the proposed modelling curriculum.

As a result, certain questions were reworded, for example the rather general "What is modelling?" became "Name as many characteristics about modelling tasks as possible". In addition, it was moved back to its original location at the beginning of the questionnaire, so that examples of modelling tasks given in other parts of the questionnaire would not influence one's response to this question. To address possible feelings of discouragement among participants, we decided to provide the following introduction: *Whether or not you have already heard of or know anything about mathematical modelling and modelling tasks, it does not matter here. We simply want to know the starting point for the teacher training course.* This introduction also served the purpose of informing participants that they were not going to be tested. Another useful lesson gained from the modelling experts was to clarify the intention of the items related to beliefs by focusing on the beliefs about mathematics education and to omit items related to beliefs about mathematics itself. Again, the questionnaire was shortened. For example, the original questionnaire included three suitability rat-

ings of different tasks, that took almost 20 minutes to answer but only comprised three single Likert-type scales accompanied by short comments.

Step 5: Testing and Item Selection: After conducting the above-mentioned revisions, we conducted a pilot study with prospective teachers, including 24 experts in modelling and 23 novices in modelling, to simulate pre- and post-testing. This testing targeted the following research questions:

1. How reliable is the pedagogical content knowledge-scale, the beliefs about modelling scale and the self-efficacy scale?
2. How good is the interrater-agreement between two independent raters scoring the open format knowledge questions?
3. Is the developed scale able to differentiate between novices (without experience in modelling and experts (in modelling)?

The first two questions about psychometric properties of the scales can be answered as follows. The reliability of the aggregated pedagogical content knowledge score was good (Cronbach's $\alpha = .83$). Forty percent of the open format questions were co-rated by a second rater (for 18 of the 47 participants), and the interrater agreement, shown by the intraclass correlation coefficient (ICC) was good ($ICC_{2,2} = .91$). Thus, only one rater coded the rest of the protocols. The reliability of the aggregated beliefs about modelling scale was good (Cronbach's $\alpha = .87$). The reliability of the aggregated self-efficacy belief score was high (Cronbach's $\alpha = .96$).

To answer the third question of whether the scales were able to differentiate between novices and experts we used a one-factorial ANOVA to analyze the data. An alpha level of .05 was used for all statistical tests. As an effect size measure, we used partial η^2 , qualifying values $<.06$ as small effects, values in the range between .06 and .13 as medium effects, and values $>.13$ as large effects (see Cohen, 1988). Results of the analysis of variance showed that the experts had significantly higher knowledge scores about modelling than the novices, $F(1,41) = 23.22$, $p < .001$, $\eta^2 = .36$ (large effect). The analysis also showed that the experts had significantly higher scores related to beliefs than the novices, $F(1,34) = 13.97$, $p < .001$, $\eta^2 = .29$ (large effect). Last, the analysis revealed that the experts had significantly higher self-efficacy beliefs about modelling than the novices, $F(1,35) = 6.68$, $p < .014$, $\eta^2 = .16$ (large effect).

These findings provide evidence that lead to the conclusion that also from a quantitative point of view, the developed questionnaire shows good reliability and construct validity. We also surveyed 8 practicing teachers with the questionnaire and found that descriptively they scored close to the novices concerning modelling skills.

In order to address concerns about further shortening the questionnaire, items that did not seem absolutely necessary for measuring the pedagogical content knowledge, beliefs about mathematical modelling or self-efficacy were screened for discriminatory power and difficulty. In other words, if the items were too general or too easy, they

would not be able to measure improvement. The final questionnaire contained the following sections: biography, beliefs about mathematics lessons, pedagogical content knowledge and beliefs related to modelling, and self-efficacy. The following examples provide a closer look at some of these sections:

Beliefs: We used items based on Grigutsch, Raatz and Törner (1998) but with a focus on school mathematics, for example – each of them with a 5-point Likert scale.

	School mathematics in my lessons from my point of view as a teacher	Strongly disagree				Strongly agree
5.1.1	School mathematics is a collection of procedures and rules which determine precisely how a task is solved.					
5.1.2	School mathematics is very important for students later in life.					
5.1.3	Central aspects of school mathematics are flawless formalism and formal logic.					

Pedagogical content knowledge: Within this section we addressed the following aspects: modelling, reasons for and against modelling, tasks, methods and assessment. All items were related to a corresponding modelling task because, as the needs analysis clearly showed, being as concrete as possible was paramount to getting valid results. Most of the questions had an open format. For example:

Imagine you are teaching children whom you regard the right age for this task. The following 5 questions are all related to the task below and all connected with each other.

It is the start of the summer holidays and there are many traffic jams. Chris has been stuck in a 20-km traffic jam for 6 hours. It is hot and she is longing for a drink. How long will the Red Cross need to provide everyone with water?

Imagine you are teaching children whom you regard the right age for this task.	not very likely				very likely
6.1 Please x one to show how likely you are to use this type of task	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>	<input type="radio"/>

6.2 Give as many reasons as possible (pros and/or cons) and mark them as such (+/-).

To evaluate the concept of assessment, a student's solution was given to the teachers and they were asked to provide written feedback. Further, teachers were asked also about methods they would use in a specific situation, all in relation to the task given.

(7) Self-efficacy: The scale was based on Bandura (2006) and included the following sample items that had to be rated on a 100-point scale, ranging in 10-unit intervals from 0% ("cannot do at all"), to 100% ("highly certain can do"):

I feel able to teach mathematical content using a modelling approach.

I feel able to develop detailed criteria (related to the modelling process) for assessing and grading students' solutions to modelling tasks.

FINAL NOTES

This paper exemplified the development process of designing a questionnaire evaluating the success of a professional development course on mathematical modelling. The greatest challenge was accommodating the participants' preference for a short questionnaire and evaluating the multifaceted aspects of the course as accurately as possible.

The results of the evaluation will be finalized in September 2009. The final evaluation questionnaire is available on request from the first author.

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MODELING IN THE CLASSROOM – MOTIVES AND OBSTACLES FROM THE TEACHER'S PERSPECTIVE

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Modelling is not only written into educational standards throughout Germany; other European countries also stipulate the integration of reality-based, problem-solving tasks into mathematics at school. In reality, however, things look quite different: in many places maths lessons are still dominated by exercises in simple calculation. So why? What is stopping teachers from introducing modelling? What would motivate them? In order to explore this issue in depth, a supplementary empirical study was conducted as part of the EU Project LEMA¹. This paper intends to introduce the project, the development of the questionnaire and the survey design. Finally, first results will be presented.

THE LEMA PROJECT TEACHER TRAINING PROGRAMME

Within the framework of LEMA (Learning and Education in and through Modelling and Applications), a concept for a further training course for teachers on the theme of modelling and reality-based teaching was developed, piloted and evaluated. The aim was for teachers to become familiar with contemporary didactic and methodical concepts. They should acquire a basic knowledge of mathematical modelling and reality-based tasks in the school context, and after the training, they should be aware of why modelling should be learnt in maths lessons and how their pupils can learn it. In other words, they should know which subject matter, teaching forms and methods are most suitable for supporting pupils in their learning, at which point in the lesson modelling can be introduced and how a basic knowledge can be secured. In addition, practical concepts for putting together and evaluating and grading tasks for class tests should be acquired. A further aim was to be able to analyse, modify and describe the learning potential inherent in modelling tasks, and to be able to develop tasks which take into consideration the heterogeneity of school classes².

The course content was designed for about five days of further training. The modular structure of the course allows for a choice of content and is flexible in terms of the length of the training. Furthermore, it is conceived in such a way that teachers from all types of schools and of all academic abilities can take part. In Germany, two parallel training courses were to take place on five days spread out over the year (Jan. 08 – Nov. 08). There should be about two months

¹ LEMA = Learning and Education in and through Modelling and Applications. Coordinator: Katja Maaß Pädagogische Hochschule Freiburg. Participant countries: DE, EN, FR, ES, HU, CY

² www.lema-project.org

between each day of training so that the teachers participating in the course have the opportunity to integrate the contents of the training into their lessons.

BASIC THEORY

Mathematical modelling generally refers to using mathematics to solve realistic and open problems. At the same time, the exact definition varies depending on the aims, which model of the modelling process is being used and the nature of the context assigned to a modelling task (Kaiser-Messmer 1986, Kaiser & Shiraman 2006).

Obstacles to the integration of modelling

In day-to-day school life, modelling still plays a much smaller role than one would wish (Burkhard 2006, Maaß 2004). It appears that at the moment teachers see more obstacles to using modelling than advantages. Blum (1996) has divided these obstacles into four categories: organisational, pupil-related, teacher-related and material-related.

Organisational obstacles: With this Blum (1996) is referring mainly to the short amount of time – 45-minutes – teachers have for class.

Pupil-related obstacles: Modelling makes the lesson too difficult and less predictable for pupils (Blum/Niss 1991, Blum 1996). Pupils can have difficulties carrying out individual steps or even the whole modelling process (Maaß 2004). Standard calculating tasks are more popular with some pupils because they are easier to understand and to solve the problem one simply has to apply a certain formula. This makes it easier for pupils to get good grades in mathematics (Blum/Niss 1991).

Teacher-related obstacles: There appears to be a variety of obstacles for the teachers. The literature on this issue refers repeatedly to the time aspect. Teachers need more time to update tasks, to adapt them to the needs of the respective class, and to prepare them in detail (Blum/Niss 1991). In addition, there are obstacles in relation to the actual lessons: teaching becomes more demanding and more difficult to predict (Blum 1996). Furthermore, a teacher requires other skills and competencies in order to be able to deal with a changed approach to teaching. The latest literature also refers to teachers' beliefs about – or attitudes to – mathematics teaching as being an obstacle to innovation in the classroom (Pehkonen 1999, Törner 2002). Blum (1996) emphasises the fact that teachers do not view modelling as mathematics. Moreover, some teachers do not consider themselves competent enough to carry out modelling tasks when the context is taken from a subject area they did not study (Blum/Niss 1991, Blum 1996). In addition, a significant aspect of the perceived obstacles is the question of how to assess performance, as teachers feel overwhelmed by the increasing complexity of this process (Blum 1996).

Material-related obstacles: Teachers often simply do not know enough modelling examples which they feel would be suitable for their lessons, or they select excessively detailed materials. (Blum/Niss 1991, Blum 1996).

Motivations for integrating modelling

Though there are several arguments against modelling, one can counter these arguments with numerous good reasons why modelling should be integrated into mathematics lessons, despite the existence of the obstacles as described above. A comprehensive representation of these reasons can be found in Blum (1996, p.21 ff.), Galbraith (1995, p.22) and Kaiser (1995 p.69).

The offer-and-use model Figure 1 shows an attempt to integrate influences on the quality of teaching into a more comprehensive model of the effectiveness of a lesson.

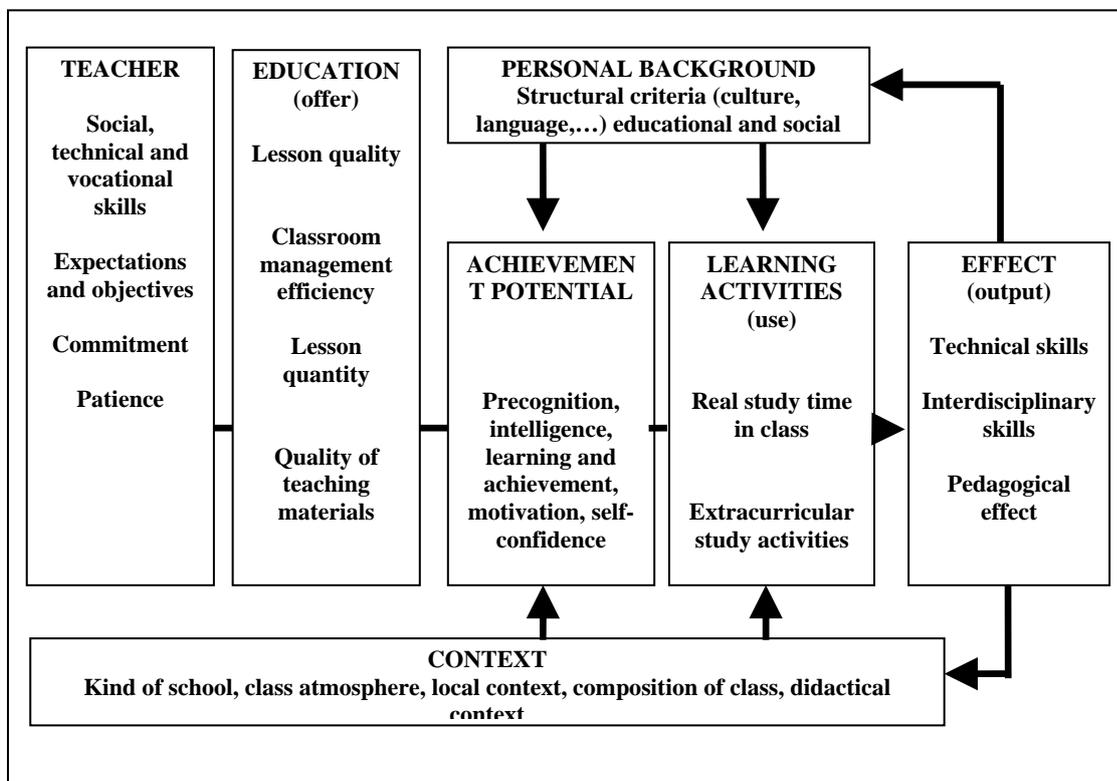


Figure 1: Offer-and-use model; Source: Helmke (2006)

As well as characteristics of the lesson, the model also includes characteristics of the teacher's personality, the classroom context, the individual personal background requirements and the achievement potential and learning activities of the pupils. This model represents a theoretical basis for the obstacles and motives for modelling. At the same time, the model should serve as a basis for systematically organising the reasons for motives and obstacles so as to indicate in which areas of the model the relevant motives and obstacles are to be found. For example, the interviews produced a first indication that the motives belong to the pupil domain and the obstacles with the teachers.

RESEARCH QUESTIONS

The previous section set out some arguments against modelling. However, these are based almost exclusively on experience and have not been subjected to empirical analysis.

This suggests the need of some kind of instrument with which to measure or assess empirically the arguments against modelling. In order to ensure the resulting point of view is not one-sided, this instrument should also analyse the arguments *for* modelling. This has the additional advantage that not only the deficiencies are revealed, but that solutions are also presented and made available. Therefore, the central questions for the survey are:

(1) What are the obstacles and motives? (2) Which obstacles and motives appear meaningful in terms of their being put into practice? (3) Which changes in the obstacles and motives can be identified during training? (4) Can in the process certain types of teachers be identified? (5) Is there a rubric for the offer-and-use model which seems to be especially relevant?

How these questions might be answered is presented in the following.

METHODOLOGY

Survey for the study: To find out which aspects teachers view as obstacles and motives for modelling, quantitative and qualitative methods were applied. Amongst other things, a questionnaire was designed with the aim of ascertaining the obstacles and motives (see next section).

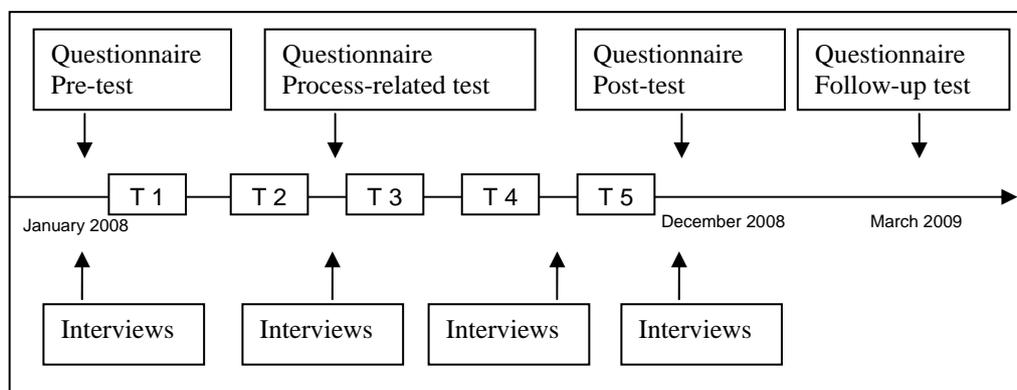


Figure 2: Study schedule

In addition, guided interviews were to be conducted. The advantage of using questionnaires is that a very large number of subjects can be used and that the questionnaire can be highly standardised (Oswald 1997). Only then can the desired generalisation of data be achieved. Significance tests can be applied to test hypotheses and develop a general statement (Bortz & Döring 2006). However, questionnaires are also limited in that data acquisition is not based on a process but mainly only focus on specific points. A further disadvantage lies in the reduction of the information: due to the pre-defined answer format of the questionnaire, the possibilities available to the survey subjects when providing their comments are limited.

Therefore, ideal is the additional use of interviews (Flick 1995). This allows the subjects the opportunity to express their answers in a more open form (v. Eye 1994). Using a set of interview guidelines, the interviewees are granted as much space to provide their own descriptions as possible. Where something is not clear, this type of interview affords the researcher the chance to ask again, to rephrase the question or to explore in more depth spontaneously and associatively things the interviewee might say (Hopf 1995). A central element to research questions is also that in addition to ascertaining obstacles and motives, the interviewer can enquire as to the background behind the arguments.

Study design: The questionnaire was to be implemented at four points in time: pre-test, post-test and follow-up test, as well as a process-related test in the middle of the further training). Four different survey dates were chosen so as to be able later to discover a possible development curve or teacher types. At the same time, additional individual interviews should be conducted with six teachers chosen randomly. So far, the results of the pre-test and process-related test questionnaires are now available for this study. The first and second interviews of the selected subject group are also available. More data will be generated by the end of the year.

Random sample: The random sample includes teachers from two further training courses with a total of 52 participants and a corresponding control group of 47 subjects. The allocation to experimental or control group was random.

The random selection of the teachers for the interviews was based on the results of the pre-tests. This meant that three teachers were selected who saw many obstacles to modelling and three who instead saw many motives for modelling.

Finally, table 1 is intended to show which assessment tools were chosen, their basic structure, their usage during the study period and a brief description of the respective random sample.

QUESTIONNAIRE DEVELOPMENT

To lay the foundations for the study, a questionnaire was developed whose purpose it was to throw light on the obstacles and motives for the teacher regarding modelling in mathematics lessons.

To be able to guarantee this, a three-stage design was developed.

Questionnaire development: The first items were developed from the subjective theories of researchers (deductive item construction). For this, the *obstacles* described above were restated as items. Furthermore, items were also formulated from the identified *motives*. To guarantee the authenticity of the items, the “natural” polarity of the obstacles and motives were retained in the items. The result was a preliminary questionnaire which included a total of 65 items. The answer format corresponded to a 5-level Likert scale (Rost 1996), which ranged from “applies completely” to “does not apply at all”. As the items named on the questionnaire were not expected to prove complete, additional open questions

were integrated which allowed the subjects to add any obstacles and motives for modelling which were not mentioned. With the help of these open items, together with the evaluation and optimisation of the closed items, the aim was to create a second and third test version of the questionnaire. This was necessary in order to be able to change the phrasing of items with ceiling effects, thereby minimizing the effect. At the same time, it was important to check the changed items once again in another test version in order to ensure that all ceiling effects were eliminated. If for the third test version no changes can be made to an item, it is removed from the questionnaire. Another reason why the three test versions are necessary is that the open question format generates new items which also have to be checked in a test version for ceiling effects.

The questionnaire was tested on 240 mathematics teachers in three runs. In the end, the questionnaire included 120 items.

Item polarity: The effects of item polarity are a source of controversy in the literature (Bühner 2006, p. 66f). On the one hand, some people are of the opinion that negatively expressed items confuse (e.g. “I am not often sad”). On the other hand, the tendency to say yes should be counteracted. Questionnaires with positive and negative items influence both factors and validity. Other studies have proven, however, that item polarity has only a limited effect on studies (ibid. p.66f). Due to these contradictory points of view regarding item polarity, in this study the natural polarity of the items was retained. This means that a high level of validity for the questionnaire is assumed, as the items in their natural polarity are less ambiguous and clearer. Thus the questionnaire includes both positively and negatively formulated statements about the research topic.

Forming categories: The aim was to organise the 120 items into categories. At the same time, the categories should be formed from the items (inductive categorisation). The first indications for categories were provided by Blum’s classification (1996) as illustrated above. In addition, the items were repeatedly analysed together as a whole, so as to check for more possible category indicators. In so doing, a great deal of flexibility and openness was extremely important. Through this dynamic process new categories of content were constantly being discovered and others rejected. In addition, a categories validation was carried out by an expert rating, whose task it was to check if the categories were consistent in terms of content.

In the end, the items generated 23 categories. In conclusion, the categories were assigned the aspects of the offer-and-use model (fig.1) so as to give them a theoretical base (deductive approach). These are described in the following.

FIRST RESULTS

In developing the questionnaire, the areas in which teachers see obstacles and motives for modelling were indicated. As the data collection is still incomplete, a final evaluation can not yet be given. Instead, it is more important that the categories be seen as a first indicator of to which areas the various obstacles and

motives can be assigned. Thus the intention of the following is to outline the categories and to assign them to areas in the offer-and-use model. In addition, the established categories should be supported by quotes from the interviews.

The *teacher personality* area includes all categories which have to do with the personality of the teacher. Categories could be identified which confirm the obstacles found in the literature and described above. For example, there are obstacles in terms of the context of a modelling task. Some teachers appear to be held back by the unfamiliar contexts in modelling [“...how on earth am I supposed to know that? I didn’t study biology! I’m certainly not going to add a task to that.”]. Another obstacle appears to be the amount of preparation time needed [“I recently had a really good idea for a modelling task. I spent three hours working on it until I was satisfied with it. I simply can’t do that for every lesson. After all, I have 6 teaching hours to prepare for every day.”]. The belief of some teachers that modelling makes the lesson too difficult for the pupils could also be confirmed [“The pupils had no idea what they were supposed to calculate. This isn’t surprising when so much information is missing!”].

However, it is worth noting that these same aspects represent not only obstacles but also motives. For example, some teachers appear to regard an unfamiliar context as a challenge [“What’s really exciting is what I learn myself in the process!”], and others see in modelling an opportunity to gain time in terms of the preparation [“I just cut out a newspaper article, think of a suitable question to go with it and I’m finished.”], apparently holding the opinion that modelling requires less time to prepare. For this area new aspects could also be discovered which have so far not been mentioned in the literature. According to some teachers, modelling appears to require an increased level of flexibility [“I do try to think about which ideas the pupils could come up with, but it’s not possible to predefine all the directions they could go in. Sometimes they ask questions I don’t know the answers to myself, and suddenly the lesson takes a quite different direction to the one planned.”] The role of the teacher, which changes when using modelling tasks, was regarded by these teachers as a positive role [“The pupils only really call on me when they’re lost. Otherwise I can just take a back seat and observe them; the atmosphere is very relaxed.”].

In the area *lesson quality* two categories from the literature could be confirmed: some teachers criticize the fact that there is insufficient availability of materials.[“At the moment we are looking at functions, and for this I took the task with the bridge. And then another one ... and another. But I can’t always do bridge tasks; it’s too boring for the pupils. But there aren’t any other tasks for functions.”]. In addition, one’s ability to plan the lesson is negatively affected as it is more difficult to predict the way in which the lesson is going to go with modelling. Moreover, three new categories could be assigned to this area: first, teachers appear to regard modelling as being very complex [“The tasks are just too complex for the pupils; they feel really overwhelmed.”]; second, as well as the time factor being a problem in terms of the preparation for the lesson, time

was also cited as an issue for the actual lesson, as some teachers feel that modelling tasks are very time-consuming [“I haven’t done any modelling recently because quite simply there isn’t the time. When I decide to use modelling tasks, I need more than an hour. Perhaps two, or even better, three. But I don’t have the time.”]; third, concerning methods, both positive and negative aspects could be named, with some teachers holding the view that modelling tasks offer a huge variety of methods [“I can apply absolutely loads of methods; and besides, the pupils are then much more motivated.”], whereas others held exactly the opposite view, claiming that modelling tasks are in terms of methodology extremely difficult to design [“I have no idea which methods I should use for these tasks.”].

In the area *individual personal background*, the category ‘pupil motivation’ could be corroborated. Here, too, as corroborated by the literature, there appear to be two forms of this aspect. Several teachers hold the opinion that pupils are more motivated when doing modelling tasks [“The pupils find the practical work in modelling tasks really interesting. Then they’re fully motivated and have much more fun.”], while others claim that standard, traditional calculating exercises are more popular [“The pupils come to me and ask when we can do normal tasks again.”]. Three further categories could be established: some teachers believe that when doing modelling tasks pupils are more creative in their thinking and calculating [“The pupils have really good ideas that even I wouldn’t have come up with.”]; some teachers are convinced that modelling tasks lead to greater independence in the pupils [“The pupils work much more independently.”], which they view as being a highly positive aspect; and there is the question of the difference in abilities within one class. Here, again, opinions go in two opposite directions. A section of the teachers hold the view that modelling should not be applied in a class where there is too big a difference between the various abilities [“The weaker pupils freeze up even more and the stronger pupils are bored because there isn’t much calculating to do.”], while the others would appear to disagree with this view, arguing that it is exactly then that modelling should be used [“The weaker pupils tend to get lost less and are also more motivated. The stronger pupils can try out new ideas, taking more and more parameters to make the calculations more complex.”].

The area *context* stands for the basic conditions. The influence of colleagues and parents plays a significant role. And here, too, it appears to go in two different directions, which can also be found in the literature. Concerning the cooperation with colleagues and/or parents, the experience of teachers seems to be either good [“I asked the parents at parents’ evening to work out one of the modelling tasks, and after that they thought it was really good!”] or bad [“The parents? They don’t support it at all! They want me to set tasks like the ones they had at school.” Or: “My colleagues are all quite old and they’re not going to change things in their classes now. If I start talking about modelling tasks, they just

smile at me patronisingly. So there is no cooperation at all.”], both sides obviously having a very different effect on the use of modelling.

The area *effects* describes effects which can be attained from the long-term use of modelling. Here, all of the motives named in the literature and described above could be confirmed. Teachers appear to be aware of the positive effects modelling seems to have. It was also corroborated that teachers consider the measuring of performance as regards modelling somewhat problematic, as it would seem to be more complex [“I found it really difficult to assess the results. One of the pupils perhaps only guessed but got the right result; the other carried out a really complicated calculation but made a mistake. How can I assess that fairly?”]. A new category is the efficiency of the lesson. Some teachers see a more efficient lesson through modelling [“It is quite simply more efficient, because every pupil can contribute to these tasks. The pupils are all constantly occupied when they are modelling. And besides, they can remember the content of the lesson much better when they are actively involved, for example when they have had to measure the playground.”], while others claim to see quite the opposite. [“I can’t really afford to do modelling in my lessons, as it means giving up so much of the exercises.”]

This list shows that as well as the reasons for and against modelling named in the literature, further relevant aspects are to be found. It is interesting that the very same aspects that are viewed positively by some teachers are viewed negatively by others.

PERSPECTIVE

By the end of the year, the data collection from the questionnaires and interviews will be completed. This should provide more information on the obstacles and motives, also highlighting any changes that occur to said obstacles and motives in the course of the further training. The question is whether in the process it will be possible to identify certain types of teachers.

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MODELLING IN MATHEMATICS' TEACHERS' PROFESSIONAL DEVELOPMENT

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One of the chapters of the new Dutch handbook of didactics of mathematics, which is currently being written by a team of didacticians, concerns mathematical modelling. This handbook aims at (further) professional development of mathematics teachers in upper secondary education. In this paper we report about the issues we included: dispositions about modelling, goals, designing aspects, testing, the role of domain knowledge, and computer modelling. We also reflect on the relationship between mathematics, teaching of mathematics and modelling, and on the role of modelling in the Dutch mathematics curriculum.

INTRODUCTION

In this paper we describe how the subject of mathematical modelling is treated in the new Dutch handbook of didactics of mathematics, which is to appear within the next few years. The intended audience of the handbook consists of students in teachers' colleges as well as mathematics' teachers in upper secondary education who want to learn about teaching modelling as part of their professional development. We try to bridge the gap between educational research and teaching practice by bringing together results, scattered about the literature, thus making them accessible to (future) teachers. We highlight those topics which our post graduate courses for teachers have shown to be most urgent for their practical needs.

Many maths teachers are not familiar with modelling or do not want to spend time on modelling in math' class. Therefore we first address the question what modelling is (not) about and why it should be included in the mathematics curriculum. Next, we cover briefly some essential issues concerning the teaching of modelling.

We focus on the non-mathematical aspects of mathematical modelling, since the didactics of the necessary mathematics is dealt with in other chapters of the handbook. Furthermore, experience with professional development courses for teachers shows that these non-mathematical aspects of modelling deserve very careful consideration as they are often ignored.

We restrict ourselves to the question of how to apply known mathematics to non-mathematical problems. In particular, we do not discuss modelling as a tool to learn mathematics.

MODELLING, MATHEMATICS AND TEACHING OF MATHEMATICS

Goal of modelling

As mentioned above, we restrict ourselves to the application of mathematics using mathematical models to non-mathematical problems. Looking through the eyes of a scientist, it is our goal to understand the relations between the variables of our context. Mathematics is an important tool to achieve this goal. Scientists use mathematical models to experiment with variables and possible relations between them and answer specific questions, such as: Which percentage of Rhine water ends up in the ecologically important and sensitive Waddenzee? Of course, everyday life can also be a source of interesting problems, for example: Does it pay out to drive across the border to fill up the car? Students tend to think models are copies of reality (Sins, 2006). It is important that they learn that models are made to answer specific questions and that the same context can lead to completely different models, depending on the question. A steel ball can be modelled as a point mass or a sphere or a conductor or a lattice or a free electron gas, depending on the question to be answered.

Modelling cycle

We describe the modelling process by a simple version of the modelling cycle. We start with a problem, which is to be solved using tools from mathematics. In the first stage the problem is described in terms of relevant non-mathematical concepts. During this stage one typically has to make some choices about (simplifying) assumptions. The result of this stage is a *conceptual model*. This conceptual model is then translated into a *mathematical model*, which can be analyzed mathematically. The actual translation of the conceptual model and the original question into mathematics may also be subject to certain choices. Next, the mathematical solution is translated back into the context and language of the original problem. We call this *interpretation*. Finally, one *validates* the solution. If necessary, one starts the modelling cycle all over again, adapting one or more of the steps.

Role of mathematics in modelling

The role of mathematics in modelling can vary considerably. It can be elementary or advanced. Sometimes computers are needed to aid mathematical analysis. The mathematics may involve calculus, algebra, geometry, combinatorics or some other field. The modelling problem can be well-defined with clear-cut data, a specific question, a standard mathematical model and ditto solution. In such problems mathematics and context science merge into a very potent mixture. The interplay between mathematics and context is then especially fruitful with techniques like dimensional analysis, where mathematical algebra is applied to physical units. The famous theoretical physicist Wigner was quite right when he spoke about “the unreasonable effectiveness of mathematics”! Conversely, a physical concept like velocity can be helpful to learn a mathematical concept like the derivative.

There may be several possible models and it is not always clear *a priori* which one serves our purpose best. If one doesn't have a complete theory describing the relevant phenomena, one usually fills the gaps by posing simple (e.g. linear) relations. For such models validation is a main point of concern. Most models are not built up from scratch anyway, but emerge as refinements and combinations of existing models.

Applications of mathematics in maths education

Mathematics started as an applied science, dealing with practical problems in trading, measurement, navigation, etcetera. The separation of theoretical mathematics from the empirical sciences is a relatively recent phenomenon, brought about by the development of non-euclidean geometry around 1800. In the middle of the nineteenth century mathematical education followed this trend and its focus shifted from applications to logical reasoning. Since then, the emphasis has swung back and forth between pure and applied mathematics (Niss, Blum & Galbraith, 2007).

Mathematics' education should pay attention to both sides of mathematics. However, many students consider mathematics as a theoretical, abstract subject, which hasn't much to do with reality (Greer, Verschaffel & Mukhopadhyay 2007). They have a blind spot for applied mathematics and the role of mathematics in the sciences or daily life. If students never learn how to apply mathematics, then their mathematical knowledge is indeed useless. Furthermore, it is counterproductive if common sense, intuition and reality are not used to aid mathematical understanding.

Modelling and the Dutch mathematics teaching programs

Non-mathematical contexts have played an important role in parts of Dutch mathematics education since 1985. Since 1998 all mathematics programs for secondary education involve modelling. The experiment which preceded the introduction of the new program indicated that assessment of open modelling tasks was a major problem and was avoided by many teachers. The modelling tasks in the national exams, too, paid little attention to conceptualization, interpretation and validation (De Lange, 1995). To counteract this deficit, the Freudenthal Institute in Utrecht started organizing modelling competitions for schools where these aspects *do* play an essential role.

All these efforts have partially paid off: PISA shows that Dutch students perform well on modelling related tasks. On the other hand, Wijers & Hoogland (1995) and De Haan & Wijers (2000) mention in their evaluation reports of the above mentioned modelling competitions that many students' papers lack in mathematical substance. Students tend to neglect relevant concepts and work by trial and error. Sins (2006) also laments the lack of conceptual thinking and understanding of the purpose of modelling. Future maths education should address these weaknesses more effectively.

PROFESSIONAL DEVELOPMENT

Ongoing professional development is obligatory by Dutch law since 2006. Since many maths teachers in upper secondary education have only scant knowledge of applications of mathematics, post graduate courses for teachers should fill this gap.

We use Schoenfeld's description of complex tasks like modeling (Schoenfeld, 2008, based on his work on problem solving 1985 and 1992). The essence of this framework is as follows. Anyone who takes up a complex task like mathematical modelling starts with certain *knowledge* (not only mathematical knowledge like facts, algorithms, skills, heuristics, but also domain knowledge), *aims* and *attitudes* (opinions, prejudices, preferences). Parts of these are activated, one makes *decisions* (consciously or not, depending on one's familiarity with the problem), one adjusts aims and designs a plan. During the execution of the plan one *monitors* the progress on several levels, going back and forth between the stages of the modelling cycle. Metacognition thus plays an important role in modelling.

We address the issues of aims and attitudes in the sections *Goals*, *Authenticity*, *Dispositions* and *Epistemological understanding*. Knowledge aspects are dealt with in the sections *Domain knowledge*, *Authenticity*, and *Computer modeling*. We conclude with a discussion of decisions and monitoring in *Monitoring and Assessment*.

Goals of teaching modelling

Modelling isn't easy. It takes a lot of time and is difficult to assess (Galbraith, 2007a) and (Vos, 2007). So why should we take up modelling in mathematics education? First, students have to learn how to apply mathematics, to prepare them for their further education and their jobs, as well as for everyday life. (It might improve their understanding of mathematics as well.) Modelling can help to achieve this (Niss, Blum & Galbraith, 2007). Second, modelling shows that mathematics is useful to scientists as well as practical problems solvers. Third, modelling is useful for students to make their picture of mathematics more complete: it is not a set of ancient, irrelevant algorithms, but an interesting, important, creative, still developing part of science, society and culture (Blum & Niss, 1991). Finally, modelling may help to counteract naïve conceptions like the illusion of linearity (De Bock, Verschaffel & Janssens, 1999; Greer & Verschaffel, 2007).

Authenticity

According to Galbraith (2007b): "Goals and authenticity are in practice inseparable, as the degree to which a task or problem meets the purposes for which it is designed is a measure of its validity from that perspective." Palm (2007) also emphasizes the importance of authenticity. He describes an experiment where two different tasks are distributed randomly among 160 Swedish school children. Mathematically, the tasks are identical: to determine how many busses are needed if 360 students have to be transported and each bus can hold 48 students. One version consisted of just this

question, the other was much wordier, paying attention to other aspects of the school trip as well. The second, more authentic version was solved correctly by 95% of the students, whereas the first version was solved correctly by only 75% of the students! Greer & Verschaffel (2007) and Bonotto (2007) also describe how lack of authenticity can hamper students to use common sense in maths class. Authenticity is also beneficial for motivating students. Lingefjård (2006) found that students are interested in problems concerning health, sports, environment and climate. Van Rens (2005) showed that mimicking scientific research practice in the class room, including writing papers and peer review, enhances motivation and improves the quality of the students' work.

Dispositions about modelling

Abstraction and generalization belong to the core business of mathematicians. Model building, on the other hand, depends critically on the characteristics of the context and the specific research question. This tension (Bonotto, 2007) between mathematics and modelling makes many maths teachers and students feel uncomfortable (Kaiser & Maass, 2007). In their opinion there is no place for modelling in the mathematics curriculum, which should be devoted to “proper” mathematics. We know, however, that even students with solid mathematical knowledge are not necessarily able to use this knowledge outside mathematics (Niss, Blum & Galbraith, 2007). In the minds of many students and teachers there is no connection between the subjects taught during maths class and the topics taught next door by the physics or economics teacher. We are not just talking about superficial problems like different notations, conventions or terminology, but also about deeply rooted opinions about mathematics and reality.

Greer, Verschaffel & Mukhopadhyay (2007) argue that students are trained to expect that problems in maths class are always solvable, that solutions are unique and that reality can be ignored. Students even think that using non-mathematical knowledge is forbidden (Bonotto, 2007). As Schwarzkopf (2007, 209-210) put it:

The students do not follow the logic of problem solving, but they follow the logic of classroom culture.

This obviously impedes successful modelling in teaching of mathematics.

Understanding what modelling is about is strongly related to dispositions about modelling (Sins, 2006). He distinguishes between three levels. At the lowest level a model is considered a copy of reality. Students at the intermediate level understand that models are simplified representations of reality constructed with a specific goal. Different goals may lead to different models. At the highest level attention shifts towards theory building: Models are constructed to develop and test ideas. Sins experiments show that a higher level of epistemological understanding leads to better models. Students at the highest level use their domain knowledge to analyze the relevant variables and the relations between them. Most students, however, are at the middle level. They try to reproduce measurement data by varying the parameters one

by one. They ignore domain knowledge, reason superficially and consequently produce poor models.

Epistemological understanding

Sins (2006) investigated the influence of epistemological understanding of modelling on the quality of models made by students. He advises to make the goals of a modelling task explicit: what do we want to *understand* or which problem do we want to solve? He proposes that the teacher presents reasonable models to his students who have to analyze and improve them. This way students learn about the tentative nature of models: They are not perfect copies of reality, since they often depend on choices, approximations and incomplete information. Furthermore, this adjusting of existing models and iteration of the modelling cycle gives a fairer picture of the modelling process as performed by experts, who of course have lots of standard models at their disposal and rarely start from scratch.

It is not sufficient to just talk about modelling with students. Indeed, students who model themselves perform significantly better on modelling skills such as using various data, recognizing the limits of applicability of a model and adjusting models (Legé, 2007). However, even if students have a sound epistemological understanding of modelling, in very open modelling tasks they still do not always understand what is given, what is asked and how to attack the problem.

Domain knowledge

Modelling typically concerns extra-mathematical contexts. As a consequence, the maths teacher may find himself in an awkward position, since he cannot be an expert in all possible modelling domains, such as the natural sciences, computer science, economics, arts, sports or other specific (not necessarily scientific) contexts.

The same holds for students. We know, however, that lack of domain knowledge leads to poor models (Sins, 2006). So it is essential to choose a modelling context where students' lack of domain knowledge is not an issue. Furthermore, the teacher has to encourage the students to actually use their domain knowledge. Finally, the teacher has to be familiar with the modelling problem himself. In particular, he has to be aware that a problem can lead to several different models.

Computer modelling

Computers can be useful to in modelling, especially when the mathematics gets complicated. Using a graphic modelling tool it is easy to modify a model, run simulations and display the results graphically. The representation of a model in such a tool reminds one of a concept map in the sense that it indicates the relevant variables and the relations between them.

In Löhner (2005), who summarized claims and results from the literature on computer modelling, we find that computer simulations make validation and adaption of models very natural. It facilitates exploring the limits of validity of a model.

Unfortunately, it also facilitates the superficial ad hoc modifications and data fitting behaviour Sins (2006) warns against. Löhner (2005) finds that students who work with computer models over a longer period of time tend to start working in a *top down* fashion and develop a more mature, qualitative attitude towards modelling, although one shouldn't expect too much in this direction. Simulation results may lead students to new research questions. Computer modelling is challenging and motivating for students, as long as the models are not too complicated and the software is easy to use. It also helps to turn abstract, theoretical models into something more concrete, which makes it easier to discuss these models. Finally, experimenting using computer modelling helps students to understand and remember the phenomena and associated theory.

Monitoring

Monitoring the modelling process of a group of students can be very difficult. Different students make different and often implicit assumptions and simplifications, have different goals and use different data and notations. This makes monitoring the modelling process of a group of students very difficult if not virtually impossible (Doerr, 2007). It is thus very important to force students to make all of the above explicit. The teacher can make life easier by inserting go-or-no-go-moments at certain points of the modelling cycle. However, even if everything is written down neatly, it can still be difficult for teachers and students to compare different modelling results. Are the differences due to different conceptualization or to mathematical errors? This problem can be moderated by discussing and comparing the various conceptual models with the whole group. Monitoring becomes much simpler if consensus is reached about the data, the goal and notations. This also facilitates understanding and comparing the different results, which in turn improves motivation and understanding (Van Rens, 2005; Bonotto, 2007).

If modelling is new to students it is advisable to have them record their modelling process in a pre-structured log. In this log they have to describe all data, assumptions, etcetera. The log can also be very useful for assessment.

Assessment of modelling

One of the main obstacles when teaching modelling is evaluation. The goals of modelling can not be assessed as objectively as is customary in education of mathematics (De Lange, 1987). Maths teachers who take the non-mathematical aspects of modelling seriously have to come to terms with this lack of objectivity. To reduce the subjectivity one can use a team of assessors (Antonius, 2007; Vos 2002; Vos 2007) and weighted lists of evaluation criteria. One can search for *rubrics* on the internet and adapt them to the assessment at hand. One can use the modelling cycle to generate evaluation criteria: conceptualization (analysis of the original problem, data, relevant concepts, data, variables, relations, simplifications, modelling goal), mathematization, mathematical analysis (completeness, correctness), interpretation, validation, conclusions, adaptations. Other criteria which are mentioned by experienced

assessors of modelling are general impression, readability, representation and originality. A common pitfall is to overestimate appearance, so it remains necessary to study and evaluate *thoroughly* the technical contents of students' work (De Haan & Wijers, 2000).

De Lange (1987) argued that traditional written tests are not suited very well to test higher skills like modelling. He mentions several alternatives, which may be more appropriate, like group work, home work, essays or oral examinations. Vos (2007) argues, however, that alternative tests like observation, interviews and portfolio's are often too time consuming and too subjective. She investigated experimentally how teamwork can indeed reduce subjectivity. Furthermore, she shows how alternative, laboratory like tests using manipulative materials can lead to valid assessment of modelling skills. These results are confirmed by Antonius (2007), who adds, however, that this kind of assessment levels out the differences between strong and weak students.

Above we emphasized the importance for teachers of taming excessive divergence for monitoring the modelling process. Similarly, assessment is facilitated by posing authentic "convergent" modelling tasks (Niss, 2001):

Mathematical modelling involves the posing of genuine, non-rhetorical questions to which clear and specific answers are to be sought.

CONCLUSIONS

To prepare teachers for mathematical modelling teachers' colleges have to take into account (apart from the necessary mathematics and their didactics) the lessons learned from literature about the role and goals of modelling in science and mathematics education, the modelling cycle, dispositions, authenticity, epistemological understanding, domain knowledge, computer modelling, monitoring and assessment. Unfortunately, empirical research on modelling education is mostly restricted to short term teaching experiments. To design effective modelling education it is necessary to gain more experience and to systematically carry out *longitudinal* research into the effects of teaching modelling.

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MODELLING AND FORMATIVE ASSESSMENT PEDAGOGIES MEDIATING CHANGE IN ACTIONS OF TEACHERS AND LEARNERS IN MATHEMATICS CLASSROOMS

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This paper explores how modelling and associated tasks and pedagogies can bring about a refocusing of the nature of assessment which it is argued, when viewed through the lens of Cultural Historical Activity Theory, appears to currently adversely mediate the object of activity in many school mathematics classrooms. An international professional development programme in mathematical modelling has been designed with formative assessment as a key theme. Drawing on data resulting from classroom activity developed from this programme I argue that modelling undertaken with a formative assessment approach can bring about a significant change in classroom activity for learners and teachers that might better prepare students to apply mathematics.

INTRODUCTION

A Cultural Historical Activity Theory (CHAT) analysis of classroom activity suggests that in almost all classrooms, at least in England, the collective activity of teacher and students is mediated to a large extent by “rules” of assessment and “performativity” which ultimately focus on learners’ qualifications. Whilst this is not necessarily clearly discernible on any particular day in any particular classroom, recent research (Williams et al, 2008) points to an all pervasive culture of “performativity”. Systemic measurements of performance and accountability are seen to drive the curriculum in our classrooms to the extent that this can be detected in classroom discourse with teachers making regular reference to the demands of assessment and examiners. In terms of CHAT, assessment and performance measures are part of, and lead, the “rules” that mediate the activity of the classroom activity system with its object of learning mathematics. These rules have culturally and historically evolved affecting, for example, the texts and pedagogic instruments used by teachers and learners. They also help mould and define the expectations of what lessons in mathematics should be, in the sense of Brousseau’s didactical contract (1997).

The current culture is, therefore, such that a teacher’s enactment of the curriculum does not necessarily match his or her espoused beliefs about the nature of the subject they teach, and consequently how it should be taught and learnt (see for example Tobin and McRobbie (1997)). Boaler (1997) documents case studies that illustrate how, in England, this has led to a narrowing of professional practice and risk taking, leading to a normative cultural script (Wierzbicka, 1999) where many lessons comprise of an initial period of transmission by the teacher of key mathematical ideas

or rules and procedures followed by a period where students practise these. This is not only detrimental to learning where shallow or surface learning dominates at the expense of deep learning and understanding (see for example, Entwistle (1981)), but can also be responsible for a narrowing of participation. As Brown et al (2008) report this can lead to situations, when students are asked about their likely future participation in mathematics beyond the compulsory curriculum (to age 16 in the UK) to responses such as

“I hate mathematics and I would rather die.”

This paper explores, how, taking a mathematical modelling approach to classroom practice that incorporates formative assessment introduces a range of new mediating instruments allowing teachers and learners to refocus their classroom actions. The work reported here resulted from classroom experiences that emanated from the work of a professional development programme as part of an EU funded Comenius project, Learning and Education in and through Modelling and Applications (LEMA). Central to the approach advocated by this programme is the focusing of classroom activity on modelling with teachers and learners becoming fully involved with formative assessment practices. An overview of the framework that guided the development of the programme in this respect is outlined in the next section before some resulting classroom experiences are described and analysed in terms of CHAT.

ASSESSMENT FOR LEARNING

Following a thorough review of research relating to assessment, Black and William (1998a) claimed that focussing on formative assessment, i.e. assessment with the purpose of informing teacher and learner about learner progression, raises student attainment. Thus the assessment *for* learning movement, as it became, conceptualised assessment as crucially providing feedback at all stages of day-to-day classroom activity and promoted this in favour of summative assessment, or assessment *of* learning, where the focus is on measuring outcomes, often being used to give grades. In follow-up studies that involved teachers and their pupils working with researchers Black and Wiliam (1998b) clarified the key areas that need to be considered if classroom assessment practices are to be effective in improving learning. These are identified in the diagram of Figure 1 and outlined below.

This emphasises an overarching pedagogic philosophy in which teachers and students strive together to ensure that, as a community, they will use their monitoring, at every stage, of the mathematical modelling taking place in their classroom to inform them of how to improve students' learning. Fundamental to this is the clarifying of learning objectives so that all know what it is they are trying to achieve.

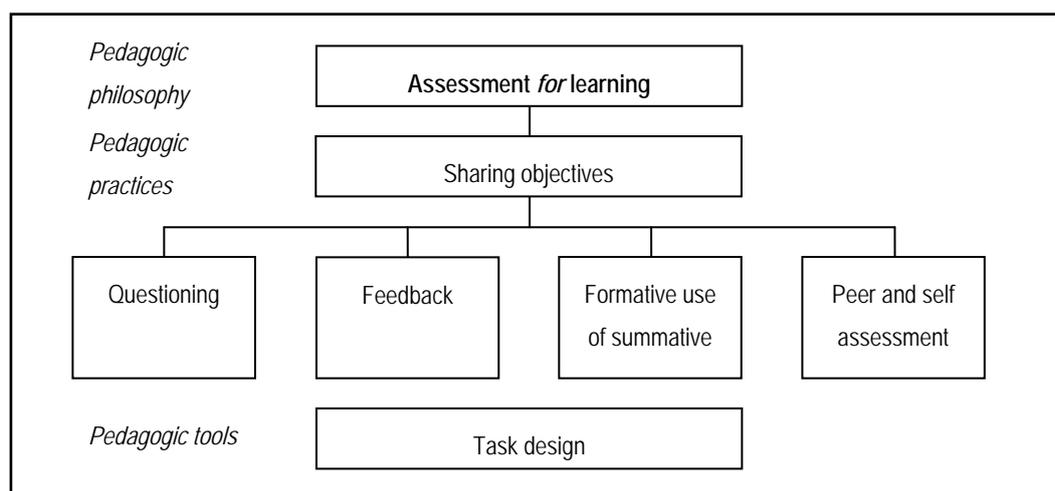


Figure 1. Schema illustrating key aspects of formative assessment

In terms of mathematical modelling this requires that students understand the overall nature and aim of modelling and the key sub-competencies they need to acquire. In supporting assessment *for learning* four key underpinning aspects of classroom activity were identified by Black and Wiliam:

(i) *Questioning*. Classroom discussion between teacher and students and between students is crucial in the learning of mathematics (see for example, Ryan and Williams, 2007) and fundamental to this are the questions that teachers pose. In summarising research in this area Tobin (1987) points to findings that suggest that the time between a teacher asking a question and intervening, perhaps to re-phrase the question, (often referred to as “wait-time”), is in many classrooms very short, and if lengthened leads to more effective learning. However, he points out that it is the quality of the question that is crucial in opening up opportunities for thinking and consequently learning.

(ii) *Feedback*. How teachers best give feedback to students to scaffold their learning (in the sense of Vygotsky) is always an issue of concern but this is possibly even more problematic when developing new pedagogic practices such as those associated with mathematical modelling. The research in this area that informed the development of good practice in formative assessment is clear in suggesting that the best feedback focuses on the task, is given immediately and is given orally rather than in writing. An important study by Butler (1988) reached the conclusion that as soon as teachers give a grade for a piece of work their comments about how to improve are ignored and that feedback that comprises of comments about how to improve instead of grades is more effective in raising student attainment.

(iii) *Formative use of summative assessment*. Much work has been done in developing ways in which such summative assessment of mathematical modelling can be carried out: see for example many of the bi-annual proceedings of the ICTMA. Whilst this has had little impact on summative assessment that leads to qualification at a national level, the frameworks and structures that have been developed may well provide suitable structures to inform formative assessment in classrooms.

(iv) *Peer & self assessment.* Of course, learning is most effective when the learners themselves have a clear understanding of what it is they are trying to achieve, can measure their progress against clear objectives and know how to proceed to achieve their aims. Hence, the important focus on clarity of learning objectives. Peer assessment, where students assess each others' work, provides a valuable direct source of feedback for students, often using a language and given in a manner they readily understand, and also allows them to start to reflect on their own work and learning.

In addition to these important pedagogic practices one further key area that needs to be considered is the design of the tasks that are used. Here, where assessment is being refocused and considered as being an integral part of daily classroom activity, the tasks students are asked to engage with are therefore absolutely critical. If, for example, the teacher wants students to focus on their ability to interpret from mathematical model to reality, the tasks used need to be designed to allow a range of possible and appropriate interpretations to be made by the students being taught. On other occasions other particular modelling sub-competencies or meta-cognitive awareness of the modelling process as a whole may need to be the focus of attention of classroom activity, requiring tasks to be designed accordingly.

A MODELLING CLASSROOM

Here I describe some detail of a lesson that was designed to involve students in mathematical modelling incorporating formative assessment approaches. Due to restrictions of space I focus on just two aspects of the lesson particularly related to formative assessment practices: namely teacher questioning and peer assessment. The lesson was one of a sequence taught by both the teacher of the class and researcher following the teacher's partial attendance at the LEMA professional development programme in England, which the researcher had led following his work as part of the development team. The lessons were developed using materials and approaches advocated by the programme, and in the particular lesson outlined here the intention was to involve students in peer assessment as a prelude to future self assessment. The students were aged 13-14 and in an upper mathematics set in a comprehensive school catering for students of all abilities (aged 11-18), in a town in the north west of England. The teacher started the lesson by introducing its objectives (Figure 2a). The emphasis of the first objective was on the development of good communication skills about mathematical modelling rather than on the mathematics itself; additionally the remaining objective of the lesson was for students to focus on their assessment of their own modelling activity and that of their peers. Following this the teacher reminded the class of the sub-competencies of mathematical modelling to which they had previously been introduced, and which had been clarified using the schema of Figure 2b. This is based on that used as the theoretical basis of the PISA study (OECD, 2003); here it has been adapted to highlight processes that are used in developing a solution to a modelling task as the

“modeller” moves from one key stage to the next. The teacher highlighted these suggesting that the students might wish to think about them when making a poster of their “solution”.

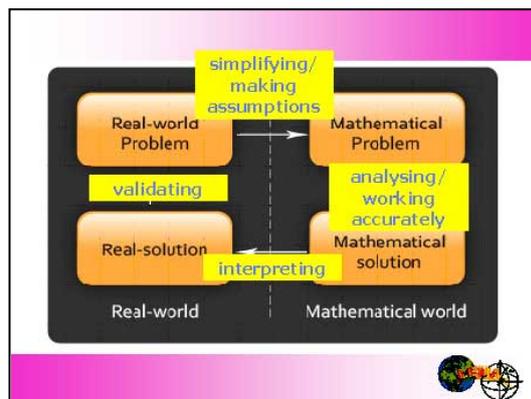


Figure 2a. Lesson objectives.

Figure 2b. Schema outlining modelling cycle

Finally in this introduction to the lesson the teacher set the task:

In a school playground there are two trees: one is small and one is large. There is also a straight wall.

A group of pupils organise a race: each pupil starts at the small tree; then has to touch the large tree; followed by the wall; before finally running back to the small tree.

Where is the best place for a pupil to touch the wall?

The pupils started to tackle the problem, working in groups of four or five: as the lesson was shorter than usual, the pupils had only about half an hour to complete their work and poster. The teacher circulated the room as the groups worked. Here I illustrate the teacher’s interactions with one group. He approached their cluster of tables and discussed where they had got to.

Teacher: OK, what’s your group doing?

Pupil 1: Going for the middle point of the wall (gesturing to a diagram of the situation)

Teacher: And you think that’s the solution?

Pupil 1: Yeah

Teacher: How could you convince somebody that’s the solution?

Pupil 1: I don’t know.

Pupil 2: Does it have to be in a triangle [referring to the path taken by someone in the race]

Teacher: [reflecting the question to other members of the group] Does it have to be in a triangle?

Pupil 3: Yeah, because there are three points....

Pupil 2: Yes, that’s the only way you can do it.

Teacher: Well, I suppose somebody could run

Pupil 2: If the wall was there, then they could just go like that [pointing to a sketch diagram]

Teacher: [indicating to the rest of the group Pupil 3's sketch with a section of wall lying along a straight line joining the two trees] oh right, so if the wall was there.... so the first thing you are doing is making some assumptions. So you have to say what your assumptions are: you've assumed everything is in a straight line [indicating this on Pupils 3's diagram] and you've assumed that it's like that [indicating the triangle path on Pupil 1's diagram]. What is it you actually want to....

Pupil 1: Find out where the wall is.

Teacher: Right, so at first you have to decide what the situation looks like.....

The teacher continued circulating the room encouraging groups as they worked on the problem and towards the end of the period completing their posters which explained what they had done to arrive at their solution. Following this the teacher focused the whole group on the second objective of the lesson: "To think about assessing our own and each others' work". This was "operationalised" by adopting the pedagogic practice of asking each group to consider the poster of another using pink sticky notes to identify up to 3 positive features of the poster being considered and 3 or fewer features where there could be improvements using yellow sticky notes (see Figure 3 below). As these early attempts demonstrate much of the feedback focused on issues relating to communication ("Not enough diagrams") and aesthetics ("Cool trees! [referring to drawings] and "Colourful"). In many ways this was a disappointing outcome, but this was the first time the class had been asked to take part in such formative assessment processes, and in a lesson a week later students gave slightly more attention to issues of mathematical content but there still remained room for there to be more of a focus on the processes involved.

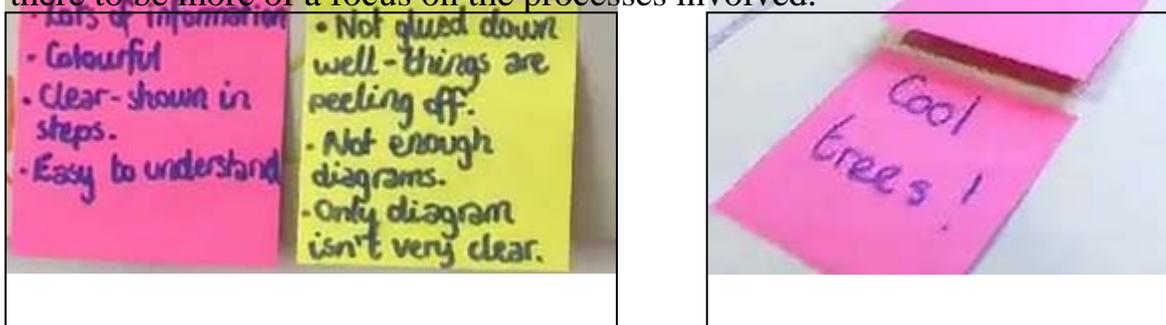


Figure 3. Peer feedback on modelling task.

DISCUSSION

In the brief extracts with which I illustrate a modelling lesson here we observe activity that is very different from the normative script of lessons that I describe earlier and which a recent nationwide inspection report corroborates as being the norm (Ofsted, 2008). Consider, for example, the interaction of the teacher with the

group of students, where the teacher prompts discussion and problem solving rather than “transmitting” rules and procedures. I now consider how Cultural Historical Activity Theory might enlighten our thinking about the nature of such lessons and highlight potential areas of conflict for teachers who attempt to follow such approaches.

CHAT builds on the fundamental thinking of Vygotsky, who suggested that the action of a subject is mediated by ‘instruments’ which may include artefacts and tools, or in the case of communicative action, as is often the case in classrooms, by cultural tools, concepts and language genres (see for example, Engestrom, 1995). This is indicated by the top triangle in the schema of Figure 4.

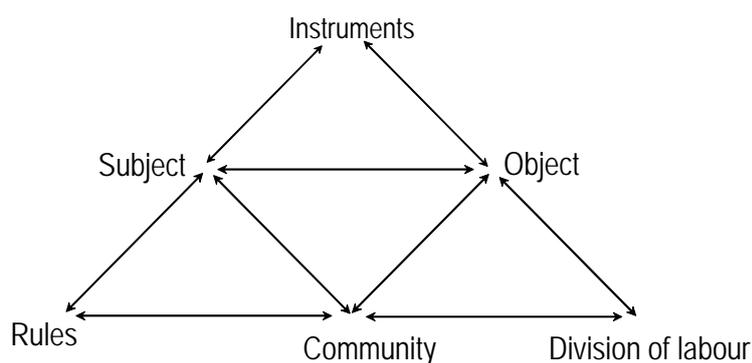


Figure 4. Schema of activity system

Leont’ev extends thinking to take account of the communal nature of activity: the schema of Figure 4 thus indicates the additional nodes of mediation in a culturally-mediated and historically-evolved Activity System. These indicate the importance of the ways in which the division of labour and associated norms/expectations/rules mediate the subject’s activity in relation to the community.

I suggest that in the modelling classroom which attempts to involve formative assessment practices there is a shift in the attention of both teacher and students to view assessment in terms of informing learning and this in turn considerably alters the dynamics of the learning community. Highly visible in bringing about this refocusing are the pedagogic tools that the teacher employs. Crucial in this regard is the use of a rich modelling task, but equally important are (i) the sharing of learning objectives that in this case (at an early stage of the students’ development as mathematical modellers) focus on the object of the activity (the learning of mathematics), (ii) the teacher’s decision to involve groups of students in working on this, (iii) their need to develop a poster communicating their solution together with their way of working and (iv) the peer assessment activity which clearly refers back to the shared learning objectives.

Greater insight might be gained into the nature of the classroom activity by exploring further Leontev’s (1978) theoretical development of Vygotsky’s thinking in which he explores the nature of a subject’s action in relation to the communal activity and the

manner of the operation that achieves this. He suggests three parallel hierarchies shown schematically in Figure 5.

ACTIVITY	--	ACTION	--	OPERATION
COMMUNITY	--	SUBJECT	--	INSTRUMENTS
MOTIVATION	--	GOAL	--	METHODS

Figure 5. Schema illustrating the nature of the action of a subject in relation to communal activity.

Thus in terms of classroom mathematical activity we need to understand how things are “normally” for the subject and how the modelling classroom differs from this. In both classrooms the activity has as its object the learning of mathematics: normally this is motivated for the community, as I suggest earlier, by the pressure to perform well in summative assessment and with institutional measures of performance having a major influence in defining goals related to achieving high grades in national assessments. This has led over time to a use of a restricted range of instruments: in particular, reflecting the highly structured nature of the summative assessment (Wake, 2008) the texts used involve students in practice exercises that in the main involve students in the recall and use of instrumental understanding (Skemp, 1978). Equally pedagogic practices are in general restricted with “the teacher doing most of the talking, emphasising rules and procedures rather than concepts or links with other parts of mathematics” (Ofsted, 2008 p. 20), and with teacher talk constituting “a substantial proportion of pupils’ time for learning mathematics” (ibid. p. 20). Thus, the actions of teacher and student might to a large extent be considered as active and passive respectively.

In the modelling classroom, however, the introduction of new instruments (tasks and pedagogic practices) brings about a change of motivation and goals. On these occasions the motivation for teacher and learners, as encapsulated in the learning objectives of the lesson illustrated here, has been, perhaps only temporarily as I shall discuss below, re-focused on the students’ learning. This alters the nature of the actions of both teacher and learners: *both* are now active with learners struggling to solve a task and make reflective judgements about their ability to do so using new rules of assessment that focus on process as opposed to outcomes. At this early stage of this class of students working on modelling the *operation*, the method by which the action is instrumentally accomplished, requires careful attention by both teacher and students. The introduction of new instruments for use by both teachers and learners destabilises their usual ways of operating, introducing new challenges for all. Thus the development provides a ‘break-down’ in the usual routine of the classroom activity which now becomes the focus of attention and hence conscious action. Previously we (Williams and Wake, 2007) and others (eg Hoyles et al, 2001) have recognised this in workplace activity. Here in classrooms, I propose, this as a useful

way of deliberately provoking a means of mediating changes in the *actions* of teachers and learners.

Finally, a word of warning! The developments in classroom activity arising from the LEMA programme, such as described here, are in many ways encouraging, demonstrating the potential to enrich the learning experience of students of mathematics. The claim by Black and Wiliam that a focus on formative assessment practices will ultimately lead to increased attainment in summative assessment is helpful to teachers working in a system where measurement of performance is so pervasive. However, bringing about the necessary changes in teacher and student actions involves teachers, either individually or as a collective, in considerable risk taking: when all around are following the “safe” option there is a great deal of pressure to conform to the “norm”. Additionally, as Hodgen (2007) points out the simple messages often associated with “assessment for learning” are not necessarily sufficient in allowing teachers to make the shift. Perhaps programmes of professional development such as that developed by LEMA will help in this regard. However, it seems unlikely that teachers will be able to sustain developments in such a way unless summative assessment is realigned to support this. Elsewhere, (Wake et al, 2004) our research has shown that attention needs to be paid to each mediating node of an activity system if curriculum development of this sort is to be effective: in paying such attention there needs to be alignment of purpose and an awareness of how each part of the system interacts with each other.

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TOWARDS UNDERSTANDING TEACHERS' BELIEFS AND AFFECTS ABOUT MATHEMATICAL MODELLING

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Work in progress on a framework aiming at capturing teachers' beliefs about mathematical models and modelling is presented. It is suggested that the belief structure of mathematical models and modelling as perceived by teachers fruitfully might be explored as partly constituted of the teachers' beliefs about the real world, the nature of mathematics, school mathematics, and applying and applications of mathematics. Some aspects of the suggested framework are explored using two case study interviews. It is found that the two teachers do not have any well formed beliefs about mathematical models and modelling, and that the interpreted beliefs structure of the teachers contain inconsistencies which are made explicit within the framework. The empiric findings also suggest some modifications of the framework.

INTRODUCTION

Since the mid 1960s gradually more emphasis has been put on mathematical modelling in the written curricula documents governing the content in Swedish upper secondary mathematics courses (Ärleback, in preparation). In the latest formulation from 2000, *using and working with mathematical models and modelling* is put forward as one of the four important aspects of the subject that, together with *problem solving, communication and the history of mathematical ideas*, should permeate all teaching (Skolverket, 2000). Indeed, it is stressed that “[a]n important part of solving problems is designing and using mathematical models” and that one of the goals to aim for is to “develop their [the students’] ability to design, fine-tune and use mathematical models, as well as critically assess the conditions, opportunities and limitations of different models” (Skolverket, 2000). However, as noted by Lingefjärd (2006), “it seems that the more mathematical modeling is pointed out as an important competence to obtain for each student in the Swedish school system, the vaguer the label becomes” (p. 96). The question naturally arises what mathematical models and modelling are and mean for the different actors in the Swedish educational system. Ärleback (in preparation) concluded that the governing curricula documents, the *intended curriculum* (Robitaille et al., 1993), do not give a very precise description of the what a mathematical model or mathematical modelling is, but rather describe the concepts in an implicit manner as exemplified above. Therefore, focus is turned to teachers who interpret and realize the intended curriculum, and thereby have a big impact on which mathematical content and what view of mathematics students in classrooms are exposed to. One way to try to understand part of the process of what ends up in the classroom, the *(potentially) implemented curriculum* (ibid.), is provided by studying teachers’ beliefs.

The question of how teachers’ knowledge, beliefs and affects towards the learning and teaching of mathematics influence and relate to their practice is a highly active

field of research (Philipp, 2007). Thompson, acknowledging the dialectic nature between beliefs and practice, argues that “[t]here is support in the literature for the claim that beliefs influence classroom practice; teachers’ beliefs appear to act as filters through which teachers interpret and ascribe meanings to their experience as they interact with children and the subject matter” (Thompson, 1992, p. 138-139). Indeed, the six authors of the chapters on teachers’ beliefs in the book edited by Leder, Pehkonen and Törner (Leder, Pehkonen, & Törner, 2002) all infer a strong link between teachers’ belief and their practice, working from a premise that could be expressed by “to understand teaching from teachers’ perspectives we have to understand the beliefs with which they define their work” (Nespor, cited in Thompson, 1992, p.129). In particular in connection with mathematical modelling, while discussing four different categories of mathematical beliefs, Kaiser (2006) concluded that depending on the mathematical beliefs held by a teacher, it is more or less likely that they build up obstacles for introducing applications and modelling in their mathematics teaching. Furthermore, Kaiser and Maaß (2007) looking at “what are the mathematical beliefs of teachers towards applications and modelling tasks?” (p. 104), found that for the group they studied, applications and modelling did not play a significant role in their beliefs about mathematics and mathematics teaching. The investigated teachers rather created/modified and adapted application-oriented beliefs in line with their existing mathematical beliefs.

In a research project aiming to design, implement and evaluate sequences of lessons exposing students to mathematical modelling in line with the present governing curricula carried out in collaboration with two upper secondary teachers, initial individual interviews was held with the participating teachers. The purpose being first to provide information about the teachers’ background and their views and beliefs on the nature of mathematics, about their teaching, views on problem solving and mathematical modelling, as well as their opinion for the reasons and aims for mathematical education. Secondly, the interviews also intended to end up in a common understanding and agreement of key concepts among the researcher and the two teachers, laying the foundation for the collaboration project. The aim of this paper is partly theoretical in that we seek to develop a framework trying to capture and conceptualize beliefs about mathematical models and modelling and relate these to other types of beliefs studied in the literature. Nevertheless, it also aims to provide background about the two teachers participating in the research mentioned above and hence to feed in to the bigger analysis of that project.

BELIEFS, BELIEF STRUCTURES AND BELIEF SYSTEMS

Reviews on research on different aspects of beliefs in connection to mathematics knowing, teaching and learning often conclude that there is a great degree of variation of the involved concepts and their meaning used by different scholars (Leder et al., 2002; Pajares, 1992; Philipp, 2007; Thompson, 1992). The motive with the following small theoretical exposé is to establish the vocabulary used in the paper and to relate some of different concepts used in the literature.

As a point for theoretical departure we start from the work, and use the vocabulary, of Goldin (2002), who defines beliefs as one out of four “subdomains of affective representation[s]” (p. 61), distinguishing between *emotions*, *attitudes*, *beliefs*, and *values*, *ethics* and *morals*. More specifically, beliefs are “multiply-encoded cognitive/affective configurations, usually including (but not limited to) prepositional encoding, to which the holder attributes some kind of *truth value*” (p. 64, emphasis in original). For an individual, a collection of mutually reinforcing or supporting non-contradictory beliefs taken together with the individual’s justifications for this constitutes a *belief structure*. Törner (2002) argues that beliefs generally are about something and introduces the notion of this something as a *belief object*, to which a set of beliefs, the *content set* is associated, which can be seen as the analogue of Goldin’s beliefs structures. Other scholars often refer to similar constructs as *belief systems* or *cluster of beliefs*, but in Goldin’s framework, a *belief system* is an “elaborated or extensive belief structure that is socially or culturally shared” (Goldin, 2002, p. 64). This terminology makes it easy to talk about and distinguish between beliefs held by an individual contra shared beliefs within a community, as well as the dialectic and tension between these types of beliefs.

Many authors deepen their discussion on beliefs drawing on Rokeach (1968) or Green (1971), or a combination of the two, introducing different *dimensions* of beliefs. Rokeach talks about a *dimension of centrality* for the individual, where a *central belief* is a belief which is non-contradicting within a persons’ belief structure, whereas beliefs with some disagreeing features are less central for the individual. Green on the other hand introduces the construct of *psychological centrality* and uses *peripheral* and *central* to describe beliefs that the individual holds more or less strongly. Both Rokeach and Green argue that the more central a belief is, the harder it is to change it. Green also talks about *quasi-logicalness*, which captures the fact that some beliefs only are in consensus within a belief structure provided that a non-standard and personal logical explanation is provided. In connection to quasi-logicalness Green also proposed to differentiate *primary* beliefs from *derivative* beliefs. Returning to Goldin’s framework of beliefs, part of the dimensions above are captured by the notion of *weakly-* or *strongly-held* beliefs. The two factors determining to what strength a belief is held are *importance for the individual of the belief being true* and the *degree of certainty the truth-value of the belief is attributed*.

MATHEMATICAL MODELLING

The literature on the aims, use and results of different approaches to incorporate and use mathematical modelling in the teaching of mathematics has steadily been growing since the beginning of the 1980s. The theoretical perspectives invoked display a great variety (Kaiser & Sriraman, 2006) as does the research methods used to explore this vast field of research; see for examples the recent 14th ICMI study (Blum, Galbraith, Henn, & Niss, 2007) and the published proceedings from ICTMA 12 (Haines, Galbraith, Blum, & Khan, 2007).

Mathematical modelling is often perceived as a multistep or cyclic problem solving process using mathematics to deal with real world phenomena. The student or modeller is supposed to use his mathematical modelling skills or modelling competencies (Maaß, 2006) to work through the steps, stages, phases or activities of the process. In this paper mathematical modelling refers to the complex and cyclic-in-nature problem solving process described for instance by Blum, Galbraith & Niss (2007), here illustrated in figure 1.

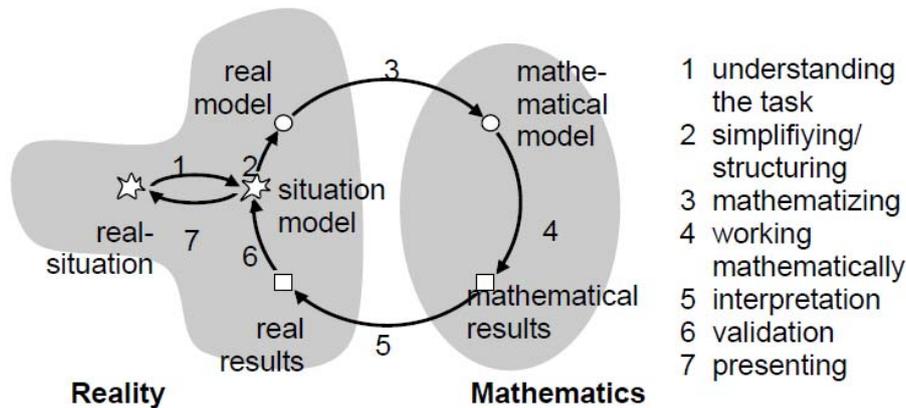


Figure 1. The modelling cycle from Borromeo Ferri (2006, p. 87)

It should be noted that this is only a schematic, idealised and simplified picture of the modelling process. For instance, in an authentic modelling situation the modeller normally jumps between the different stages/activities in a more non-cyclic, but rather unsystematic, manner (Ärlebäck & Bergsten, 2007).

A SUGGESTED BELIEF STRUCTURE OF SOME ASPECTS OF MATHEMATICAL MODELLING

In setting out to investigate teachers' beliefs about mathematical models and modelling it is important to be explicit and specific about what object the beliefs should be about. Using the terminology of Törner (2002), the *belief object* under study in this paper is defined to be *mathematical models and modelling as perceived by upper secondary mathematics teachers*. For clarification we stress that the focus at this stage in the research process is not on the teachers' beliefs of the teaching and learning of mathematical models and modelling.

The literature review suggests the importance and influence on teachers' practice of their beliefs about mathematics and its teaching and learning. Hence, the validity of the framework suggested here steams both from analyzing the view taken on mathematical modelling in this paper and from research on mathematical beliefs of various sorts. A teachers' belief structure of mathematical models and modelling is suggested to be constituted of the beliefs of the following (*sub-*)*belief objects*:

Beliefs about **the nature of mathematics**. This is without question the most general of the constituting sub-belief objects, assumed to serve as a primary and central belief in the belief structure of modelling. The perspective taken on the nature of mathematics might radically change the interpretation and meaningfulness of fig. 1.

Beliefs about the **real world (reality)**. In our view, it is important that the problems used in connection with modelling to the greatest extent possible be from real problem situations in the real world. Different views, both philosophical and pragmatic, potentially influence the way one might think about mathematical modelling and models. In addition, how reality is perceived, especially in contrast to the nature of mathematics, can make a difference when it comes to the interpretation and validation of ones' modelling work. In fig. 1, beliefs about the real world might especially influence the phases 1, 2, 5, 6 and 7.

Beliefs about **problem solving**. In principle, depending on perspective, modelling is about problem solving or problem solving is about modelling (see Lesh & Zawojewski, 2007 for an overview). Regardless of which view adopted, the meaning of and role played by problem solving as a mathematical activity, seen as part of one's practise of one's mathematical knowledge and skill/competence might have important implications for how mathematical modelling and models are perceived. In connection to fig. 1, (mathematical) problem solving beliefs are important for the phases 3, 4 and 5.

Beliefs about **school mathematics**. Thompson (1992) concluded that the consistency between teachers' beliefs about the nature of mathematics and beliefs about the subject mathematics taught at schools are of varying magnitudes. Therefore, school mathematics beliefs are incorporated in the bigger belief structure to capture the potential influences they might have on other beliefs of the teachers.

Beliefs about **applying, and applications of, mathematics**. The application of mathematics is sometime synonymous with different views taken on modelling, and hence it is important to include beliefs about applying and applications of mathematics in the belief structure of mathematical models and modelling. Depending on point of view, beliefs about applications of mathematics are significant for phases 3 and/or 5 in fig. 1.

The five categories of beliefs above are suggested to constitute a way of describing the belief structure of mathematical models and modelling. This framework is initially based on the indicated links to the modelling cycle and will need empirical investigations to be further developed and validated.

This framework does not set up isolated beliefs but, by the discussion above, these beliefs are rather overlapping belief structures in themselves. Hence, an indication of the validity of the framework would be that the substructures display inner coherence, that is, display an inner quasi-logical structure. However, it is possible that taken all together as constituting the belief structure of mathematical models and modelling, incoherencies appear and then the question is which beliefs are more central, primary, and in line with official guidelines.

SOME EMPIRICAL FINDINGS

Although the empirical data used here was not collected primarily with the testing of the above framework in mind, due to its focus on teachers' views on mathematical

modelling, we see it as relevant for discussing the viability and usefulness of the framework. As a result, it may also point out directions for how to develop it further.

Method

The interviews with the two teachers (here called Lisa and Sven) in the projects briefly described in the introduction were partly structured around five mathematical problems to serve as a basis for the discussion and reflection. Three of these were standard text problems from a widely used textbook in Sweden, one the so called *Fermi Problem* studied in (Ärlebäck & Bergsten, 2007), and one was *The Volleyball Problem*, a so called *modelling-eliciting activity*, described in (Lesh & Doerr, 2003). The interviews were recorded, transcribed and analysed using what may be called a contextual sensitive categorization scheme based on the five sub-beliefs object in mind. Due to the nature of the data, beliefs about the real world and applications and applying mathematics surfaced only sporadically and can therefore not be fully accounted for here. To economize with respect to writing space, the accounts of the teachers' beliefs are here given mostly in narrative form.

Lisa

Lisa, 36 years old, has been an upper secondary teacher in mathematics and physics for 13 years and is now working in her second school going on her 5th year. She teaches on a 70% basis and the other 30% she spend on administration, marketing and teacher education networking. She became a mathematics teacher because it seemed to make a lot of fun and as far back she can remember she always enjoyed doing and thinking about mathematics.

Beliefs about **the nature of mathematics**: Lisa talks about mathematics as a *tool* and something that *develops and strengthens ones' thinking* (logic). She connects mathematics to *structuring* and *organizing*, and a number of times talks about *geometrical pattern, forms and shapes in nature* and *mathematics as an art form*.

Beliefs about the **real world**: Lisa's comments in the interview seem to imply that the most prominent consequences of working on real problems are that the numbers occurring in the calculation are messy and that the calculations should be preformed and answered using better accuracy (more decimals).

Beliefs about **problem solving**: For Lisa problem solving is about *solving puzzles* and she associates feelings of satisfaction and happiness with the success of solving a hard problem. Problem solving is for Lisa something that preferably takes place in a technological environment with free access to every source of information possible. She also stresses the importance for the problem context to be familiar to the students.

Beliefs about **school mathematics**: Lisa repeatedly states the importance for school mathematics to be experienced as an entity, *a well defined course*, but also comments on the written governing curricula documents as theoretically formulated and hard to understand both for students and teachers. Lisa regretfully confess that some areas of mathematics (such as ordinary differential equations) only are taught as a set of

procedures and recipes although the areas really have a great potential for making the subject more interesting and intriguing.

Lisa's direct talk about **mathematical modelling**: When asked about mathematical models and modelling, Lisa first seems to have a clear conception of what this means; without any time for consideration she says: "*Well, it might be a whole lot of things... a mathematical model... it might be that you describe a course of events or situation, or really just to make an assumption is a mathematical model, although a very simple one*". Then she retreats and only considers a made connection/relation to constitute the model, not an equation or an algebraic representation of the relationship, but changes her opinion on this and clarifies that a mathematical model does not have to be expressed in mathematical terms. Rather, it should be the need of the situation that decides which degree of mathematization to use. The goal however, she continues, should always be a formulation of the model using mathematical symbols and ways of writing. Lisa also draws parallels between modelling and generalizing, and gives numerous of examples of what she considers to be different types of models when discussing the problems. She considers all five problems except The Volleyball Problem to be about, and include different aspects of, modelling.

Sven

Sven, 58 years old, has been teaching mathematics and physics (and computer science and chemistry) at the upper secondary level for 33 years and has been working at four different schools and last changed workplace in 1981. He teaches on a 60% basis and plans/manages the school schedules the rest of his working hours. It was mere coincidence that Sven became a teacher, following his personal fascination of mathematics, which led to physics and later also to teacher education.

Beliefs about **the nature of mathematics**: Sven describes mathematics as a *pure, exact and axiomatic science*, enabling to *part right from wrong*. It is about *logic*, the *relations between different quantities*, and it has a central *aesthetic component*. He emphasises that "*knowledge of the tools open up for the realization of the beauty*".

Beliefs about **problem solving**: Sven talks about mathematical problem solving as an *exercise for the intellect*, as something decoupled from other subjects and contexts. When discussing the problems he carefully places them in a syllabus context; where, when, and how the topics touches in the problems are treated within the course.

Beliefs about **school mathematics**: When talking about school mathematics Sven expresses the importance to *learn to think logically* and to *prepare for learning in other subjects as well for higher education*. He thinks the aesthetic side of mathematics is something only a few students can appreciate and hence it plays only a minor role in the classroom.

Beliefs about **applying, and applications of, mathematics**: For Sven, application of mathematics is "*a tool used in other sciences; physics, chemistry and economics*".

Sven's direct talk about **mathematical modelling**: When asked to describe mathematical models and modelling Sven answers, "*Yes, well... no, I don't know...*"

turning to the five problems and try to use them helping him to form and formulate his perception of mathematical models and modelling. To begin with Sven talks about a model as something to use solving problem, a tool, but elaborates his thinking further: *“I think it [a mathematical model] is something you create... in a more or less obvious manner...and there can be more than one model to use to solve a given problem.”* Sven then describes different ways of working with a model; creating, using, and exploring it. He also strongly connects making assumptions and modelling, and considers all five problems used in the interviews as related to modelling. Sven also mentions that it is important for the students to learn to use and apply mathematics.

Discussion and conclusion

Although Lisa initially seemed to have a clear conception of mathematical models and modelling, it became clear throughout the interview that this was not the case. She rather, like Sven, had to make up and formulate her views as the interview went on. One explanation why neither of them had a clear conception of modelling might be the vague formulations found in the curriculum documents that provide no support and only circumstantial guidance. However, since they volunteered to participate in a research project about mathematical modelling, one could suspect that they had been doing some thinking about the project, and thus had some firm ideas about the central concepts. If they had, this was nothing that surfaced during the interviews. However, when talking about mathematical modelling, directly or indirectly during the interviews, the different categories of beliefs in the framework are touched on, as described above.

No flaws in the quasi-logic holding together the different sub-beliefs structure were detected in neither teacher's sub-beliefs structures. Sven for instance expressed the school mathematical belief that it is important for the students to learn to use and apply mathematics, and professed a similar belief about the application of mathematics. Lisa, when discussing The Volleyball Problem, on the other hand, strongly rejected it as a modelling problem since *“it is more about comparing advantages and disadvantages, structuring and organizing [than modelling]”*. This is in conflict with her beliefs about the nature of mathematics and a direct contradiction to what she said previously in the interview. One possible way to interpret this is that Lisa strongly held conflicting primary beliefs about the nature of mathematics on one hand, and mathematical modelling on the other.

Although the data was not initially collected for the testing of the suggested framework, the analysis indicates that it may be useful for exploiting beliefs about mathematical models and modelling, other professed beliefs, and relations between them. However, a thing to consider is to follow up the point made by Thompson (1992, p. 130-131), who lists a number of studies in mathematics education indicating the important impact teachers' beliefs about mathematics on the one hand, and about teaching of mathematics on the other, have on their practice. Including the teachers' beliefs on the learning and teaching of mathematics in general, and

mathematical models and modelling in particular, seems to be the next logical step. A perhaps as urgent dimension to add to the framework is to include more actively affective considerations, which Goldins' (2002) framework make possible.

If indeed beliefs can be seen as filters influencing the teachers' practice, it is important to try to get a better understanding of beliefs about mathematical models and modelling if we want teachers to integrate it more in their mathematics teaching. Kaiser (2006) concluded that "beliefs concerning mathematics must be regarded as essential reasons for the low realisation of application and modelling in mathematics teaching" (p. 399), and we believe, like (Törner, 2002, p. 80), that higher consciousness about one's beliefs lead to a higher degree of integration of the beliefs in ones' practice. A question that we feel needs priority is how beliefs are formed.

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THE USE OF MOTION SENSOR CAN LEAD THE STUDENTS TO UNDERSTANDING THE CARTESIAN GRAPH

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Abstract. *This paper shows the experimental results of a didactical lesson conducted in three classes of Upper Secondary School using motion sensor. It is an example of modelling practice, in which the students are involved in mathematics representations of real phenomena. Our research corroborates works about the use of MBL-tools, according to which the use of motion sensor allows the students to reading, understanding and interpreting kinematics graphs. Besides our analysis shows that the students acquire these competence respect to graphs of other type too. These results emerge from the implicative statistical analysis of the pre-test and the post-test and from the qualitative analysis of the lessons.*

Key words: *teaching, learning, Cartesian graph, motion sensor, modelling*

INTRODUCTION

This research work consists of the analysis of a didactical situation conducted in three classes of Upper Secondary School. The didactical activities were developed using a motion sensor to visualize, to understand and to interpret space-time and velocity-time graphs, representing moving bodies. Motion sensor is one of MBL-tools (Microcomputer Based Laboratory). In the late 1980's, these tools were produced by the project "Tools for Scientific Thinking" in the Center for Science and Mathematics Teaching at Tufts University. The central objective was to help students in order to recognize the connections between the physical world and the abstract principles presented in the classroom (Krusberg 2007). Motion sensor is used in Physics laboratory to study rectilinear motion of bodies moving in front of it.

Our research proves that not only the students improved reading, understanding and interpreting motion graphs but they also improved these graphing practices (Roth 2004 p.2) in other types of Cartesian graphs. We believe that this is an interesting result because learning mathematics means that a person acquires aspects of an intellectual practice, rather than just acquiring any information and skills (Roth 2004 p.7). These interdisciplinary activities give the opportunity to optimize available time in classroom and to increase the student's motivation.

We chose this argument of research because graphing practices are part of the mathematics curricula of all school levels. Moreover, they can become prerequisites

for other mathematical subjects. For instance, Cartesian graph is one register of semiotic representation of a function. Besides, graphing practices are central to scientific communication and to the scientific enterprise more broadly (Roth 2004 p.2). Moreover graphing practices have many applications in everyday life as the comprehension of an economy graph printed on a newspaper, the understanding of a temperature graph hanged on a hospital bed, etc.

RESEARCH QUESTIONS

Research hypothesis: *Motion sensor is a learning tool to reading, understanding and interpreting kinematics graphs.*

Research questions:

- 1) *Using motion sensor to reading, understanding and interpreting kinematics graphs, do students learn to reading, understanding and interpreting other types of Cartesian graphs and, in particular, function graphs representing a statistical phenomenon?*
- 2) *How can modelling activities aid for the understanding of Cartesian graphs?*

THEORETICAL FRAMEWORK

MBL tools collect physical data and allow visualizing them in tables and Cartesian graphs in real time (Thornton & Sokoloff, 1990). So MBL tools can facilitate the comprehension of abstract representations of physics phenomena and can give long lived conceptual understanding (Bernhard, 2001). Besides collected data can be manipulated, analyzed and fitted, studying the characteristics of the phenomena and testing the relationships between the variables. The efficiency of motion sensor compared to traditional methods for helping students to learn basic kinematics concepts has been proved by several researches, as Thornton & Sokoloff (1990), Redish et al. (1997), Liljedahl (2002), Arzarello & Robutti (2004). Our research wants moreover to show that when the students are involved in activities with sensor motion they become able in graphing practices, not only in kinematics field.

The idea of using motion sensor to improve graphing practices finds strong theoretical support in the cognitive theories *of the Embodiment of the mind*, for which «the detailed nature of our bodies, of our brains, and of our daily functioning in the world structures human concepts and reasoning» (Lakoff & Núñez, 2005, p.27). So it's fundamental in this kind of activity as the students can visualize and analyze in real time the graphs of bodies. Beside according to *Metaphorical Thought* «for the most part, human being conceptualize abstract concepts in concrete terms, utilising ideas and models of reasoning founded on a sensor-motor system» (Lakoff & Núñez, 2005, p.27). Particularly «the functions on the Cartesian plane are often conceptualized in terms of motion on a route» (Lakoff & Núñez, 2005, p.70) and motion sensor induces this type of conceptualization as the students see the graph constructed under their own eyes as *motion of a point that leaves a wake*. It can be

explained through a historical-epistemological analysis of the concept of function, which finds its origins in the ambit of kinematics and geometry.

This analysis shows that the representations of the function are: verbal, Cartesian, analytical and tabular (for numerical values). So the laboratory activity with sensor motion could be utilized as kinematics approach to the concept of function (Arzarello & Robutti 2004) because it allows studying all the representations of a kinematics function and to pass from one kind of representation to another. A representation cannot describe fully a mathematical construct and each representation has different advantages, using multiple representations for the same mathematical situation is at the core of mathematical understanding (Duval 2002). The representations of the function developed in different historical periods. Before tables of functions appeared (2000 B.C.), then geometrical representation (middle of the 14th C.) and later analytical form (17th C.) (Youschkevitch, 1976). Using motion sensor the chronological introduction of the representations of the function is respected (Piaget & Garcia, 1985). Besides it involves the students in a historical process that conducted to the function concept: modelling process. In fact it allows analyzing the motion of a body as a point in moving along a straight line respect to the reference point, studying all its mathematical representations (Gilbert, 1998).

In this activity the modelling is a transversal objective, reached by the study of other matters of the mathematics curriculum (Lingefjård, 2006). Modelling practise can be a way to increase *thinkers*, who can use their mathematics for their own and for society's purposes (Burkhardt, 2006). To conclude we want to point out that motion sensor is an artefact. As referring to mathematical meanings it may be seen as «tool of semiotic mediation» (Bartolini Bussi & Mariotti 2008). The role of the teacher becomes fundamental in the use of this tool to reach the graphing practices.

Some didactical considerations

To clarify the connection between the graphing practices in motion graphs and in any Cartesian graphs, we made the following comparison between competences:

C	MATHEMATICS	PHYSICS
C1	Reading the coordinates of a point of the graph	Reading the values of a kinematics variable in relation to the values of the temporal variable
C2	Reading the extremes and the size of intervals	Reading space and time of departure and arrival, the covered space and the spent time
C3	Distinguishing among increase, decrease and constancy of a function	Distinguishing between motion of approach, motion of separation and still bodies
C4	Individuating absolute maximums and minimums of a function	Individuating absolute maximum and minimum distance with respect to the position reference system
C5	Individuating relative maximums and minimums of a function	Individuating relative maximum and minimum distance with respect to the position reference system
C6	Confronting the different degrees of rapidity of increase or decrease of tracts of a curve	Confronting the velocity of differing tracts of motion
C7	Forming hypothesis and conjecture	Forming inferences on experimental data

EXPERIMENTAL WORK AND RESEARCH METHOD

The experimental work consisted of two laboratorial lessons¹ of two hours each one. It was led in three Italian classes² of Upper Secondary School (43 students). It is a homogeneous sample because before the experimental work they possessed the same competences in graphing practices and necessary prerequisites for this activity:

- Knowing the real number field and representing them on a straight line
- Representing points on the Cartesian plane
- Knowing motion concept and kinematics variables

The research methodology adopted is *Theory of Didactic Situations* by Brousseau (Brousseau, 1997). The laboratorial lesson was preceded and followed by the administration of a test, with the aim of evaluating the a priori and a posteriori students' behaviours. We made the qualitative analysis of the didactical activities analyzing the teaching/learning process through the analysis of the involved semiotic register. It refers to *APC space and Semiotic Bundles* by Arzarello (Arzarello & Robutti 2008). We made also a quantitative analysis of tests through *Statistical Implicative Analysis* by Gras (Gras et al., 2008). Cause of limited space, in this paper we show only the main results of our analysis.

Statistical Implicative Analysis

It is a non-parametric statistic, so it uses small samples and it is appropriate for this kind of research. We use the method of *implication* that establish the implication intensity between variables and the method of *similarities*, that classifies variables and groups them according to hierarchical levels (similarities) (Gras et al., 2008). Data were analyzed by using C.H.I.C.³ software that visualizes *implication graphs and similarity tree*, working on Excel tables. We studied the implication of the *students' behaviours variables* by tables like this:

	Behaviour 1	...	Behaviour <i>n</i>
Student 1			
...			
Student <i>m</i>			

The values of this table are 0 or 1, depending if a student doesn't follow or follows the behaviour that corresponds in the table respectively. We analyzed the similarity of the *students' variables* using the *supplementary variables* method (Spagnolo 2005), (Fazio & Spagnolo, 2008). Here we use the supplementary variables as models of

¹ Lessons was conducted by the teacher-researcher M. L. Lo Cicero in her curricular classes.

² 1. December 2007, 4th class of Classical Liceo (17 years), (*Liceo Classico "Scaduto"*, Bagheria (PA), Italy)

2. April 2008, 2nd class of Commercial Technical Institute (15 years), (*"Jacopo del Duca"*, Cefalù (PA), Italy)

3. May 2008, 4th class of Classical Liceo (17 years), (*Liceo Classico "Scaduto"*, Bagheria (PA), Italy)

³ Classification Hiérarchique Implicative et Cohésitive. Information regarding the software can be found at the following site of the A.R.D.M. (Association de Recherche en Didactique des Mathématiques): <http://www.ardm.asso.fr/CHIC.html>

student's behaviour, so the outcomes of our research depend from the similarities of the *students* respect to the *correct models of students' behaviour*. The correct models of students' behaviour are selected by combination of the correct behaviours. To obtain the similarity trees we used tables like this, with binary values:

	Student 1	...	Student m	model of student's behaviour 1	...	model of student's behaviour p
Behaviour 1						
NOT behaviour 1						
...						
Behaviour n						
NOT behaviour n						

Phases of the didactical activity

The phases of the didactical activity were the following ones:

1. Prediction, reading and comprehension of the graphs of rectilinear student's motion of three types:
 - a. Leaving motion from the sensor
 - b. Approach motion to the sensor
 - c. Still body with respect to the sensor
2. Prediction, reading and comprehension of various rectilinear student's motion, with leaving and approach with respect to the sensor.
3. Study of rectilinear uniform motions of a train on tracks.

During phase **1** the students made a reflection on the variables studied by the sensor. They observed and calculated space and time of departure and arrival, the length of space and the time spent. Not all the students immediately realized the relation between abscises and ordinates. After the study of the leaving motion the students correctly predicted the other types of graphs. In phase **2** the topics of the previous phase were consolidated for every piece of curve of leaving, approach or stilling. Also the maximum and minimum distance reached with respect to sensor was read. The students noted that the slope of every piece of curve depended on the corresponding velocity of the student. Then the students were asked to make a relationship between spatial intervals and temporal intervals about pieces of a curve and to make comparisons. Besides the students calculated the mean velocities and compared them and the observations about the slopes of the pieces of the curve with the graphs velocity-time. In the phase **3** they studied the analytical representation of a uniform rectilinear motion by the fit of the data. The students noted that this is a particular type of straight line equation.

After the laboratory activity, the students were involved in a metacognitive reflection about the development of the lesson. The students reconstructed the phases of the modelling process and reached the devolution of these processes (Brousseau, 1997). They realized that physics phenomena, belonging to everyday life, could be representable by mathematical representation. In particular, uniform rectilinear

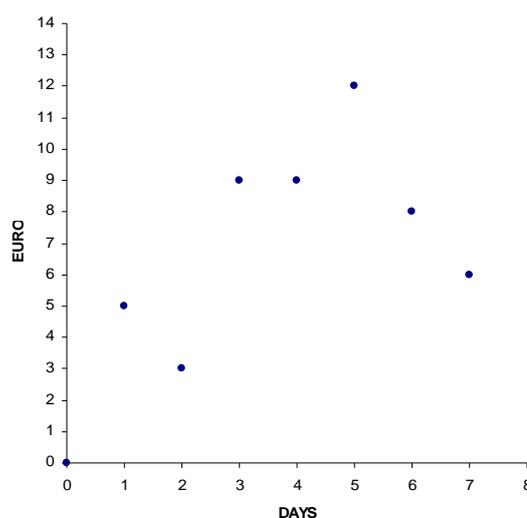
motion can be represented by algebraic equation, commonly studied in scholastic mathematics. So this modeling process was an occasion to realize that mathematics is a tool to read the existence of mathematics in our everyday life (Lingefjård, 2006), (Kaiser & Schwarz, 2006). During the didactical activity it was noted that motion sensor induces curiosity and desire of learning in students. They were encouraged to experiment several typologies of motion to compare the graphics produced with their own predictions. It was noted that the process of prediction is important to acquire the skill of *forming hypothesis on the base of experimental data*.

Test

A test was administered before and after the laboratorial lesson. The students worked individually, they were not allowed consulting books or notes. They had sixty minutes to accomplish the task. The test contained items concerning reading and understanding of space-time graphs representing motion of bodies, contextualized in real life. So the students had to interpret models of kinematics phenomena. The students' improvements in kinematics graphical practises were remarkable, so they corroborated our research hypothesis. Besides the test contained the following exercise (*Sara's test*) concerning reading of not kinematics graph:

Sara's dad decided to reward his daughter every time she got a good grade at school by giving her five euro, which she could decide to spend or save as she pleased; but this would be her only source of income. The adjacent graph shows the money Sara possessed on each day of a week. Observe it and answer the following questions:

- How many euros did Sara possess on the 4th day? (C1→COORD)
- On which day/days did Sara surely get a good grade? (C5→R-MAX)
- Knowing that on the second day Sara didn't get a good grade, how much money did she spend that day? (C2→INT)
- Could she have gotten a good grade on the 6th day? (Justify your answer.) (C7→HP)
- On which day/days did Sara possess the most money? (C4→A-MAX)



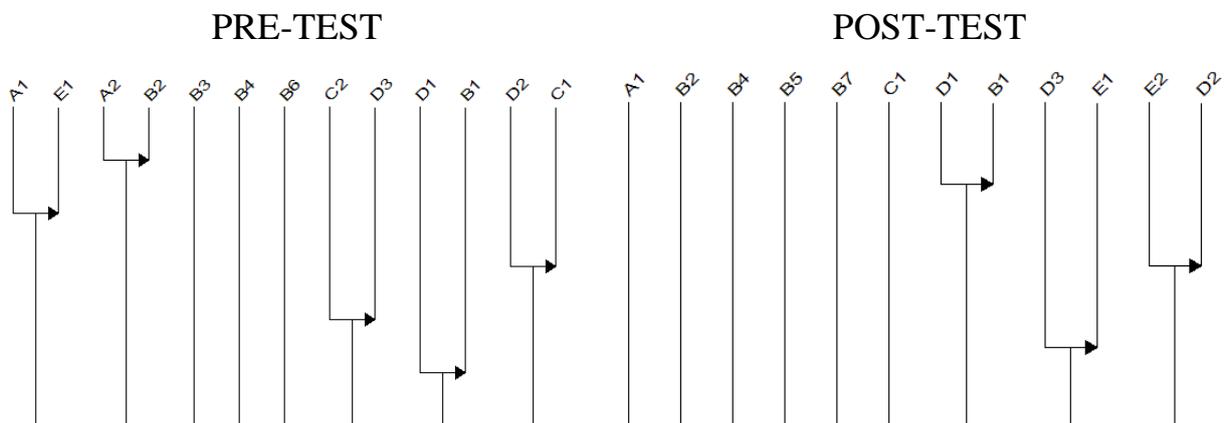
A priori analysis of students' behaviours of Sara's test

As it is indicated by *Theory of Didactic Situations*, we made an a priori analysis of students' behaviour in working out the test:

Q ⁴	A.	BEHAVIOURS
a.	9	A1: Correct reading of the value of the ordinate in correspondence with abscissa
	26	A2: Sum of the euro that Sara possessed in the first four days. Interpretation of the graph like earned money.
b.	1,3,5	B1: Correct identification of the days corresponding to the relative maxima
	5	B2: Confusion between the concept of relative maximums and of absolute maxima
	1,3,4,5	B3: Writing, besides of the days which correspond to the relative maxima, also, of the 4 th day, in which the euro remained constant
	3,4,5	B4: Writing of the highest values
	3	B5: Writing of the day corresponding to the major growth
	1,3,4,5,6,7	B6: Writing of all the days except the absolute minimum
	3,5	B7: Writing of the highest relative maxima
c.	2	C1: Correct identification of the width of the interval
	3	C2: Confusion between the concept of interval and of value of the coordinate. Wrongly interpretation of the graph like spent money
d.	Yes, ...	D1: Affirmative answer to the question <i>d</i> , justifying with the affirmation “she could have spent the earned money”: forming correct hypotheses on the base of experimental data
	No, ...	D2: Negative answer to the question <i>d</i> , justifying with the affirmations “she spent 4 euro” or “her budget would have become 13 euro”: not forming correct hypotheses on the base of experimental data
	Yes, ...	D3: Affirmative answer to the question <i>d</i> , justifying with the affirmation “because she earned 8 euro”: not forming correct hypotheses on the base of experimental data and wrongly interpretation of the graph like earned money
e.	5	E1: Correct identification of the absolute maximum
	3,4,5	E2: Writing of the highest values

EXPERIMENTAL RESULTS AND CONCLUSIONS

We classified the behaviours of the students in tables. Using *Chic* software we obtained the following implicative graphs of the student’s behaviours:



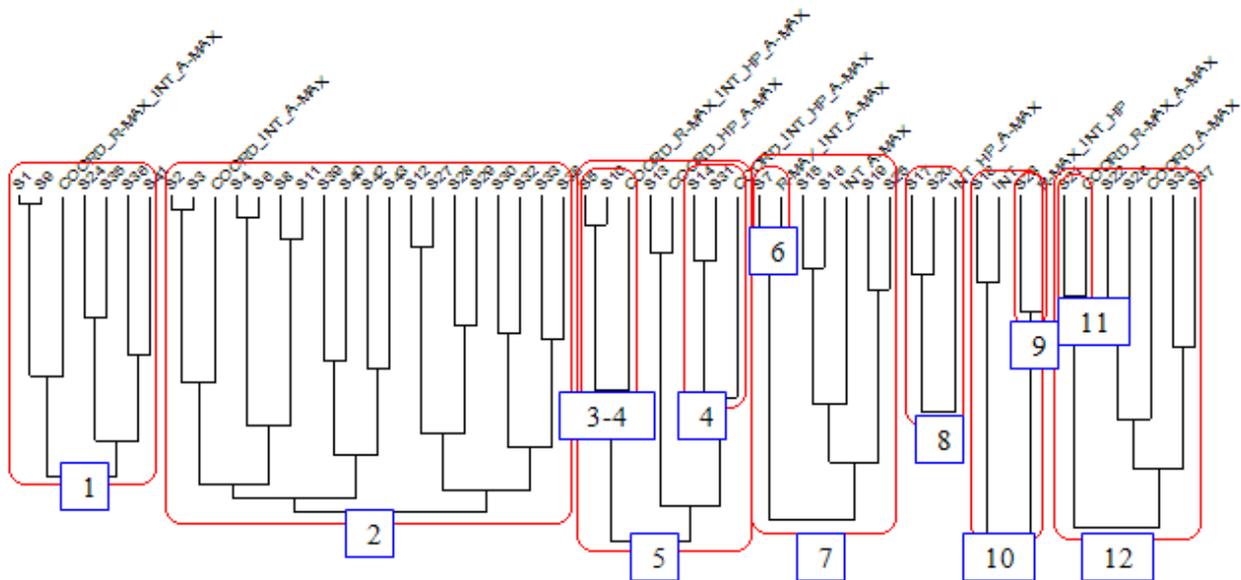
⁴ Q=Questions. A= Students’ Answers

In the implicative graph of the pre-test there is a strong implication of A2 towards B2: all the students that follow the behaviour A2 follow the behaviour B2 too. They represent two mistakes in reading of graph (reading of coordinates and relative maxima respectively). The implication $A1 \rightarrow E1$ inverts the expected implication between the reading of the coordinates and of the absolute maximum. It is due to the wrong interpretation of the graph like earned money in the answer *a*. D2 implicates C1 because the behaviour D2 includes the competence of reading of the width of intervals. The implication $C2 \rightarrow D3$ points out a wrong interpretation of the graph like spent money and earned money respectively. So the same students gave two wrong opposite interpretations of the graph. Since $D1 \rightarrow B1$, the students that form correct hypotheses on the base of experimental data are able to read relative maxima.

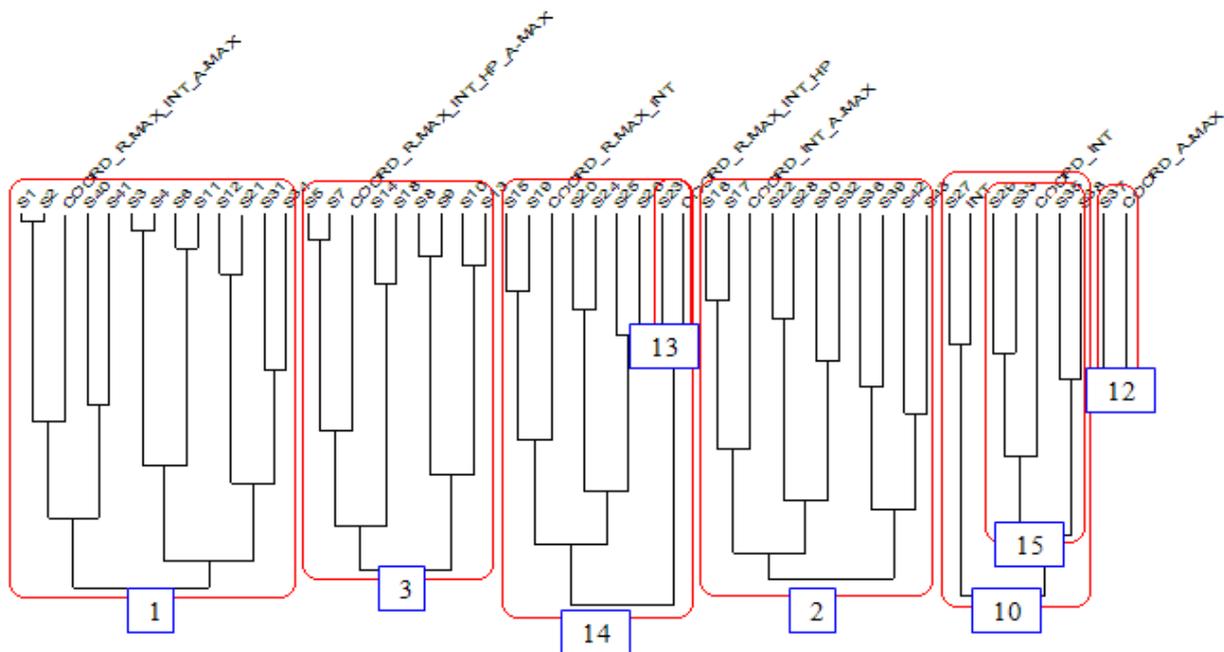
In the graph of the post test the implication $D1 \rightarrow B1$ is stronger than in the pre-test. Given that $E2 \rightarrow D2$, if the students don't read correctly the absolute maximum then they don't form correct hypotheses on the base of experimental data. Finally, since $D3 \rightarrow E1$, the students that don't form correct hypotheses on the base of experimental data and interpret the graph like earned money are however able to read absolute maximum.

We analysed the similarity of the variables *student* respect to the variables *models of students' behaviour*. Below we report the graphs obtained by C.H.I.C.:

PRE-TEST



POST-TEST



In these graph we can observe the improvements of the competences of each students. The general improvements for each competence are:

	COORD	R-MAX	INT	HP	A-MAX
N° correct answers, pre-test	34	11	37	8	41
N° correct answers, post-test	43	27	42	9	31

Below we report a table extrapolated by the similarity trees. It shows the numbers of the students that possessed 5 or 4 or 3 or 2 or 1 competences in the pre and post-test.

	5 comp.	4 comp.	3 comp.	2 comp.	1 comp
N° stud, pre-test	2 (group 3)	8 (groups 1,4)	24 (groups 2,5,6,8,9)	8 (groups 7,12)	1 (group 10)
N° stud, post-test	8 (group 3)	13 (groups 1,13)	16 (groups 14, 2)	5 (groups 12,15)	1 (group 10)

In particular, in the similarity trees we note that the group n. 3, representing the students that possessed all the competences, is increased by 6 students in the post-test. The group n. 1, representing the students that possessed all the competences except the forming hypotheses, is increased by 6 students in the post-test.

Conclusions

The experimental results show that a laboratory activity with the use of motion sensor develops the competences of the students in reading, understanding and predicting of kinematics graph. This tool allows studying the steps of the modelling process of the phenomena *rectilinear motion* and to make metacognition reflection of their own learning. Modelling activities aid for the understanding of Cartesian graphs because they are the bridge between the real phenomena and the mathematical

representations. According to the theory of *Embodiment* the students construct their knowledge observing the real phenomena and connecting it with its graphical and tabular representations. Our mind conceptualizes a function as a point that is moving on the plane and the use of motion sensor induces this kind of conceptualization. So, using motion sensor, the students acquire competences in reading, understanding and predicting Cartesian graphs not representing only a kinematics phenomena. In particular, our research shows improvements of the students in reading of the correspondence between abscises and ordinates, of maxima and width of intervals of a statistical function.

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INTERACING POPULATIONS IN A RESTRICTED HABITAT– MODELLING, SIMULATION AND MATHEMATICAL ANALYSIS IN CLASS

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This presentation will introduce an authentic modelling process for two interacting species which is well accessible to high school students. Based on an analysis of ecological systems, a simple conceptual model leads to simulation software tools and the derivation of a mathematical model. A wide range of systems, e.g. predator-prey, competition or parasitism can be investigated. The approach also allows independent modelling activities and in silico experimentation by students. As the presented modelling process builds on authentic research by Johannson and Sumpter (2003) it allows to give students an insight into current research of Theoretical Biology.

MODELLING

The importance of modelling in the teaching of mathematics is universally accepted. But often work with models in education consists only in the usage of formulas or the fitting of parameters. There is not much suitable teaching material about reality-based mathematical models. Some groundbreaking efforts were made by Sonar and Grahs (2001, 2002). Gotzen (2003) created well comprehensible, reality-based one-species-models for school use in his doctoral thesis. The two-species models presented in this paper are based on his work.

We will use the following modelling process:

1. Definition of the purpose of the model.
2. Analysis of the real situation.
3. Establish a conceptual model, from simplified description of the real situation.
4. Simulation software and mathematical model equations on the basis of the conceptual model.
5. Predictions and validation using the simulations and/or the mathematical models.

Except for some slight modifications these are the five modelling steps presented by Gotzen, Liebscher and Walcher (2008). See also earlier work of Schupp (1988). Results gained during the modelling process have to be compared to reality and the intention of modelling and thus must eventually be corrected. Hence the modelling process is rather a modelling-cycle as Blum and Leiß (2007) presented. Nevertheless the main idea of modelling will be comprehensible for students following the five steps.

We will introduce population models which are very suitable for educational use because of their relevance, authenticity and traceability for students.

- The models are relevant as they are built on current research of Theoretical Biology (Johansson & Sumpter, 2003).
- As the modelling process is based on capturing the most relevant features of a population development, observed on an ecological level, it ensures a strong biological foundation. This kind of modelling is called “bottom up” modelling. A detailed description of the advantages of “bottom-up” models and a separation from classical “top-down” models is given by Sumpter & Broomhead (2001).
- They are suitable for educational use because the whole modelling process is comprehensible with means of school education. Furthermore the models provide explanations of the observed phenomena and allow predictions.

The models are applicable in mathematical and biological classes in secondary school as well as in education at university (e.g. classes of Biomathematics).

The software and a workbook, which gives all necessary instructions and allows self-contained work of students, are allocated for free use in the internet (Roeckerath, 2008).

PURPOSE OF THE MODEL

We want to derive a bottom-up model of two interacting species which is capable to give information about their development over time. The model shall capture the main important ecological patterns and phenomena affecting the development of the species. Thus we are looking for a model, which gives the size of each population at every generation.

THE ECOLOGICAL SYSTEM

The basis of the modelling process must be an analysis of the ecological system in order to capture the main important structures concerning the development of both populations.

We look at two interacting species which share a restricted habitat. The populations have non-overlapping generations. This ecological phenomenon is common for insects and annual plants and means that at every time there is only one generation alive. Thus parents and children never live together. Parents distribute their offspring randomly over the entire habitat. The offspring is during the first development state (nearly) not able to move (eggs, larvae, seeds).

Individuals interact with individuals of their own as well as with individuals of the other species. These phenomena are called intra- respectively interspecific interactions and affect the individuals' ability of reproduction.

We want to include several kinds of intra- and interspecific interactions appearing in ecology. In the following we want to capture them in formulating interaction laws.

Intraspecific Interactions

Intraspecific interactions appear mostly as competition for resources like food, territory or sunlight. The availability of such resources is mainly responsible for the ability of an individual to reproduce itself. We want to distinguish two kinds of intraspecific competition: exploitation and interference competition, which Nicholson describes as “scramble” and “contest” (1954).

Exploitation competition can appear when individuals share a restricted quantity of resources. In this case a high density causes a lack of resources which prevents individuals from reproducing. Ecological examples of this phenomenon are weakness because of hunger or lacks of breeding or germination areas. We capture the main idea in the interspecific law

INTRA 1. If there is a sufficiently high population density no individual will be able to reproduce.

In the case of interference competition individuals deal directly with each other. There is one dominant individual, which is able to gain enough resources and to reproduce, even if there is a high population density. Ecological examples are cainism, where cubs kill each other until only the strongest cub is still alive, or allelopathy, where plants spread poison into the ground in order to prevent other plants from growing. A simplified description of these phenomena gives the reproduction law

INTRA 2. There is a dominant individual which is able to reproduce even if there is a high density.

More detailed biological background concerning intraspecific interactions and concrete biological examples can be found in the article of Gotzen, Walcher, Liebscher (2006).

Interspecific Interactions

Interactions between individuals of different species can have a positive, negative or no influence on their development. There are many ecological examples showing these kinds of influences. For example an individual of a predator-population needs prey. Thus an interaction with individuals of the prey species will cause positive effect on the predator’s reproduction. A suitable reproduction law for positive influence is

INTER +. If an individual interacts with at least one individual of the other species, then it will be able to reproduce.

On the other hand interspecific interactions can also cause negative influence. A prey animal is only able to survive and reproduce if it will not be killed by a predator.

Also in the case of competition for resources between different species interspecific interactions have a negative influence on the reproduction. A simplified summary is the reproduction law

INTER -. If an individual does not interact with any individual of the other species, then it will be able to reproduce.

In eco-systems there can be populations which share a habitat and interact but one species is not affected by the other. For example huge plants which take daylight from small plants. There is an interaction, but the huge plants are not affected by the small plants. This is captured in the reproduction law

INTER 0. Individuals reproduce independently from the other species.

Using the specified interaction laws we will be able to describe a wide range of two interacting species. A concrete example is the following ecological system.

Example: Amensalism

There are two populations of plants which use the same resources. The first species shows the following dominant behaviour. It affects the second species negatively without any influence for it self. Thus it not affected by the second species. This ecological phenomenon is called amensalism. Within the species 1 obtains exploitation competition and within the species 2 interference competition. Using the interaction laws we can determine that species one follows **INTRA 1** and **INTER 0** and the species 2 follows the reproduction laws **INTRA 2** and **INTER-.**

FROM THE ECOLOGICAL SYSTEM TO THE CONCEPTUAL MODEL

After the ecological observations, we will now capture the main important structures affecting the populations' development in a conceptual model. A conceptual model is a (partly very strong) simplified description of the reality. The challenge is to distinguish the relevant and the irrelevant factors. The conceptual model is often only a caricature of the real system but it is clearly arranged and practicable.

The habitat is displayed on a field with a fixed number of sites. Each site represents an area of the habitat. As shown in figure 1(a) and 1(b) for each area the containing individuals are displayed by a dot in the corresponding site. To distinguish the different species the dots are differently coloured.

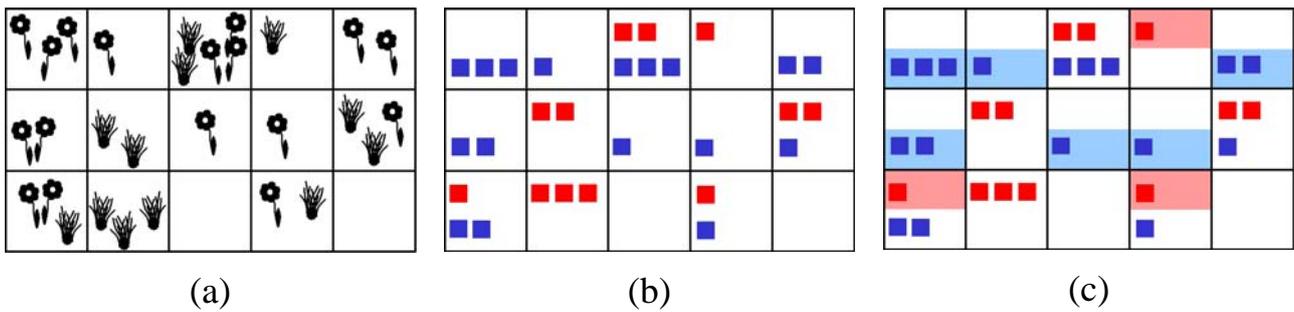


Figure 1: Conceptual model

Individuals displayed at the same site are close to each other and thus interact. Due to the non-overlapping generations the development of the real system can be described with discrete time-steps and it is only affected by the number of reproductions. A site provides enough resources for at most one reproduction per species. As parents deposit their offspring randomly somewhere in the habitat, for every new generation the concerning number of dots will be randomly distributed over the field.

Interaction laws

The sites provide a basis to comprise the concept of “high density” for the intraspecific, and the concept of “presence” for the interspecific interaction laws in the conceptual model.

We assume that we have a high density at a site, if it contains more than one individual. Using this understanding of density we can integrate the introduced interaction laws in our model.

INTRA 1. At a site there will be a reproduction for a species, if it contains exactly one individual of the same species.

INTRA 2. At a site there will be a reproduction for a species, if it contains at least one individual of the same species.

The concept of “presence” can easily be realized in the conceptual model. The other species is present, if there is at least one of its individuals. Thus we get the following interaction laws for the conceptual model.

INTER +. At a site there will be a reproduction for a species, if it contains at least one individual of the other species.

INTER -. At a site there will be a reproduction for a species, if it contains no individual of the same species.

INTER 0. At a site there will be a reproduction for a species, if it contains any number of individuals of the other species.

Now the means to determine if there is a reproduction for a species at a site are available: If a species follows the interaction laws **INTRA** and **INTER** then there is a

reproduction for this species at a site if and only if **INTRA** and **INTER** are both fulfilled at the site.

In order to get a species' population size of the next generation the reproduction laws must be applied at each site. Multiplying the resulting number of reproductions with the mean number of offspring per reproduction we get the population size of the next generation. The generation cycle repeats by spreading this number randomly over the field. On the basis of this conceptual model, software was created which simulates the development of the species.

Example: Amensalism

Species 1 follows **INTRA 1** and **INTER 0**

Species 1 will reproduce at a site, if and only if it contains exactly one individual of species 1 and an arbitrary number of individuals of species 2.

Species 2 follows **INTRA 2** and **INTER-**

Species 2 will reproduce at a site if and only if the site contains at least one individual of species 2 and no individual of species 1.

Figure 1(c) shows the evaluation of the field concerning the reproduction laws of species 1 and 2. A light blue respectively a pink mark of a box represents a reproduction of the blue respectively the red species.

FROM THE CONCEPTUAL MODEL TO THE STOCHASTIC MODEL

The simulation tools provide excellent observation and exploration possibilities to students. Furthermore it should be mentioned that in silico investigations using simulations are very common in modern biological research.

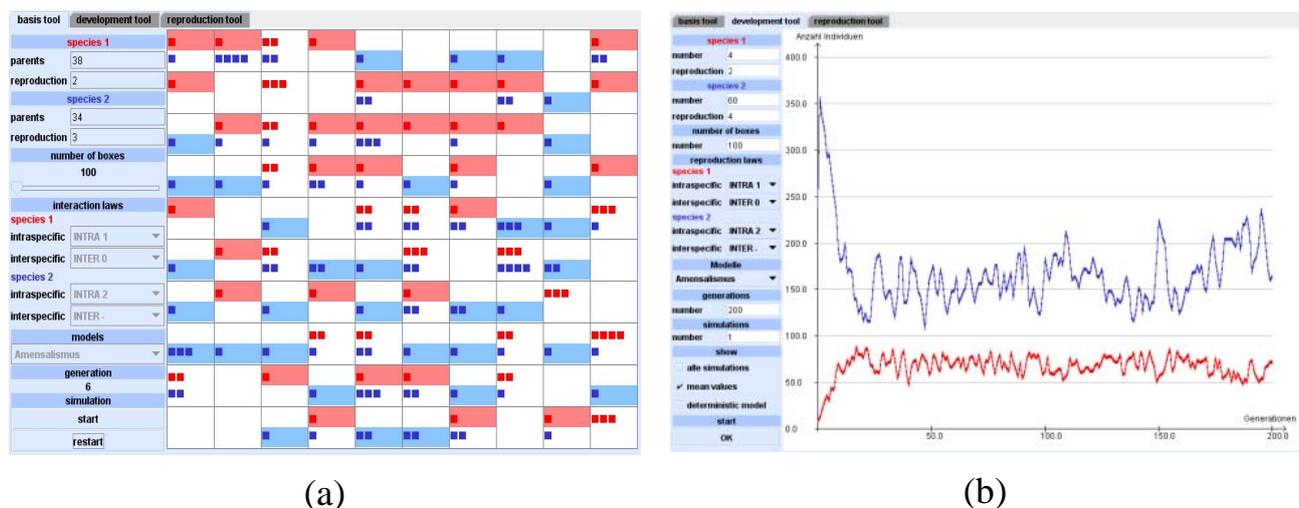


Figure 3: (a) The basic tool; (b) The development tool

Figure 3(a) depicts the graphical surface of the basic tool, which implements the simulation of the described generation-cycle. For each species students can enter the start sizes of the population, the mean number of offspring per reproduction and the intra- and interspecific interaction laws. Starting the first simulation the entered number of individuals will be spread randomly over the field. The program evaluates for each site and each species if there is a reproduction according to the selected interaction laws. Thus, the program computes the population sizes for the simulation of the next generation.

The development tool, pictured in figure 3(b), was created to get a better insight of the species' development. The tool simulates the development over a longer period of time and displays the resulting population sizes of each generation in a coordinate system. This offers a clear depiction of the long term development for both species.

Example: Amensalism

In figure 3(b) a simulation of the amensalism system (species 1: **INTRA 1 + INTER 0**, species 2: **INTRA 2 + INTER-**) is shown. In this case the two species are able to live in coexistence. Changing the parameters, students can determine values for the initial populations and the mean numbers of offspring per reproduction which cause an extinction of one species or which allows coexistence. Thus students are able to explore the biological role and of the parameters.

FROM THE CONCEPTUAL MODEL TO THE DETERMINISTIC MODEL

Using the conceptual model students are able to derivate a mathematical description of the systems. We define the number of individuals at a time t as $S_1(t)$ and $S_2(t)$. Due to non overlapping generations, the change of population size from generation t to generation $t+1$ exclusively depends on reproduction. The function of reproduction $R_1(S_1, S_2)$ respectively $R_2(S_1, S_2)$ indicates for species 1 respectively for species 2 how many individuals are able to reproduce, when S_1 individuals of species 1 and S_2 individuals of species 2 are randomly spread over the field. We define the number of mean offspring for each reproduction, as r_1 for species 1 and r_2 for species 2. Thus we get the following mathematical description of the population sizes.

$$S_1(t+1) = r_1 \cdot R_1(S_1(t), S_2(t))$$

$$S_2(t+1) = r_2 \cdot R_2(S_1(t), S_2(t))$$

To derive the whole mathematical description we need the reproduction functions. With the reproduction tool students can derivate reproduction functions via regression.

The reproduction tool, shown in figure 5(a), allows simulations of the functions $R_{1,S_2}(S_1)$, $R_{2,S_2}(S_1)$, $R_{1,S_1}(S_2)$ and $R_{2,S_1}(S_2)$, which determine the number of

reproductions of one species depending on a fixed number of individuals of one species and a variable number of individuals of the other species.

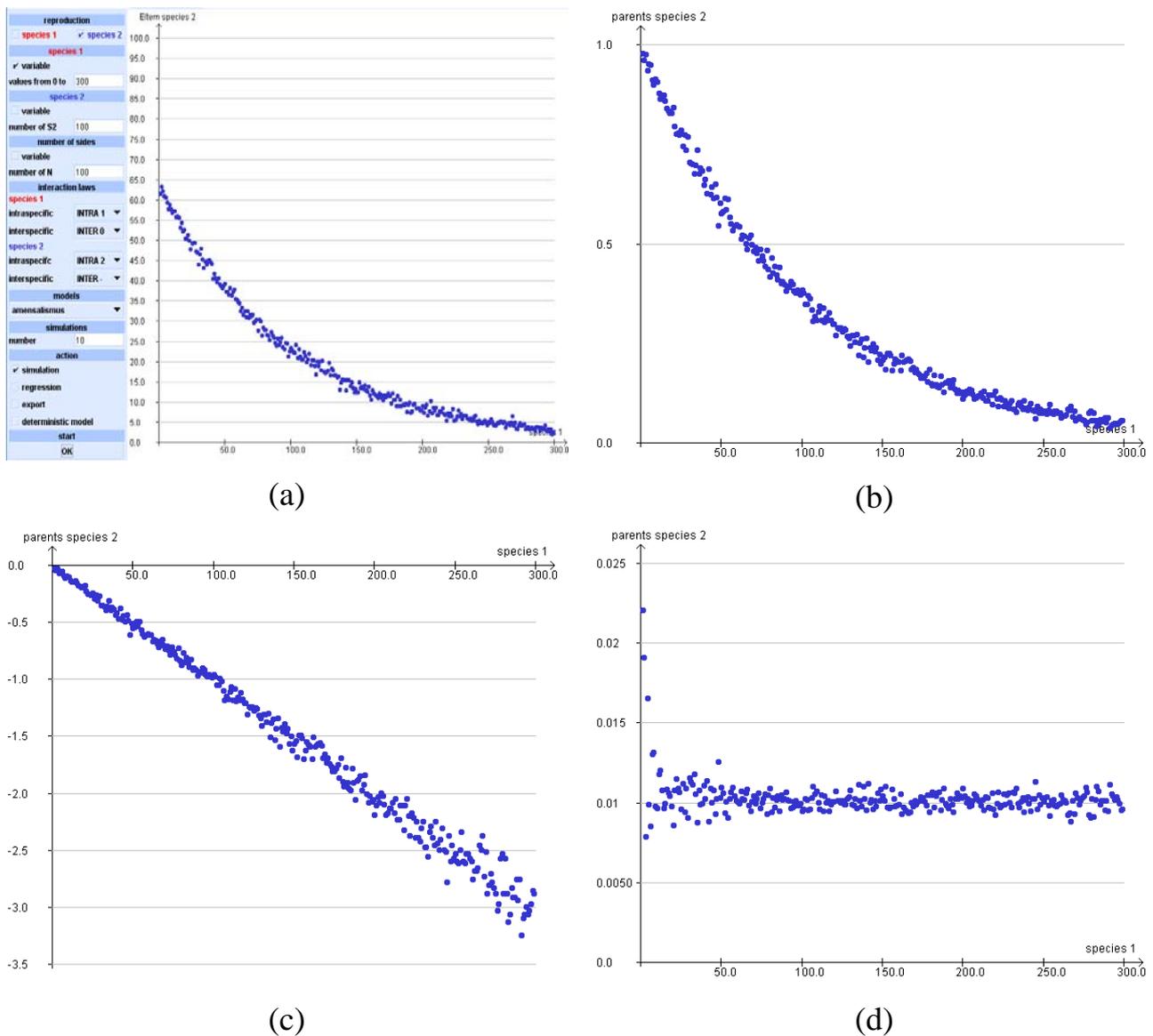


Figure 5: The reproduction tool: (a) Simulation of $R_{2,S_2}(S_1)$ ($\tilde{R}(S_1)$ simulation values); (b) $\tilde{R}^{(1)}(S_1) := \tilde{R}^{(2)}(S_1)/64$; (c) $\tilde{R}^{(2)}(S_1) := \log(\tilde{R}^{(1)}(S_1))$ can be approximated by a linear function; (d) $\tilde{R}^{(3)}(S_1) := -\tilde{R}^{(2)}(S_1)/S_1 \approx 0.01$

The tool provides the possibility to modify the simulated values in order to determine the reproduction functions. In the following $R_2(S_1, S_2)$ of the amensalism system with $N=100$ will be derived.

Example: Amensalism

Simulating $R_{2,S_2=100}(S_1)$ with the reproduction tool we get the graph $\tilde{R}(S_1)$ shown in figure 5(a). $R_{2,S_2=100}(S_1) = M e^{-KS_1}$ seems to be a proper approach to approximate $\tilde{R}(S_1)$. With the software the simulation values can be linearized in order to check if a

certain function is suitable to approximate them. Figure 5(b) shows the resulting graph $\tilde{R}^{(1)}(S_1)$ after dividing the simulation values by M , which is approximately 64. In the next step the logarithm will be applied to $\tilde{R}^{(1)}(S_1)$. As it is shown in Figure 5(c) the resulting graph $\tilde{R}^{(2)}(S_1)$ can be approximated by a linear function. This verifies that the approach is suitable. The constant $K = 0.01$ can be obtained by dividing $\tilde{R}^{(2)}(S_1)$ by S_1 , as it is shown in Figure 5(d).

In order to figure out how R_2 depends on S_2 , it has to be checked how the remaining constants in $R_{2,S_2}(S_1) = M e^{-KS_1}$ depend on S_2 . Determining $R_{2,S_2}(S_1)$ for different values of S_2 , shows that M depends on S_2 , while $K = 0.01$ remains constant. Thus, we obtain $R(S_1, S_2) = M(S_2) e^{-KS_1}$. If $S_1 = 0$, then $R_2(0, S_2) = M(S_2) = R_{2,S_1=0}(S_2)$. Using the tool $R_{2,S_1=0}(S_2) = L(1 - e^{-KS_2})$ with $L = 100$ can be determined. Thus $R_2(S_1, S_2) = L(1 - e^{-KS_2}) e^{-KS_1}$ is the reproduction function of species 2. With the derivation of the reproduction function of species 1 we can determine the following model equations for the amensalism system:

$$\begin{aligned} S_1(t+1) &= r_1 \cdot L \cdot S_1(t) \cdot e^{-KS_1(t)} \\ S_2(t+1) &= r_2 \cdot L \cdot (1 - e^{-KS_2(t)}) e^{-KS_1(t)} \end{aligned}$$

Detailed descriptions and instructions for many different systems can be found in the workbook (Roeckerath, 2008).

PREDICTIONS

A good model offers predictions for the real system. Many systems develop over time from different states into a relatively stable final state, the climax state, like coexistence of both species or extinction of one or both of them. They can also develop cyclic or even chaotic behaviour. As mentioned above for the amensalism system, students can use the development tool to explore which values of $r_1, r_2, S_1(0)$ and $S_2(0)$ yield to different kinds of systems' behaviours. Doing these kinds of predictions students are able to explore the ecological meaning of parameters.

As the derived models are dynamical systems in form of difference equations, next to the development tool students from a higher educational level can gain predictions with analytical or numerical investigations. A stable fixed point for example gives information about population developments which reach a climax state.

CLASSROOM USE

The models and tools offer various options for classroom use. They were tested successfully in a mathematics workshop for 12th grades students and a 13th grade biology class. During the workshop students worked independently with a workbook (Roeckerath, 2008). Most of them were able to derive the model equations for the modelled systems using the workbook. In the biology class the models were used to

introduce population dynamics and to do some in silico experimentation. A derivation of the model equations was not part of the lessons.

The introduced models give a realistic insight into scientific research and real mathematical applications. Authentic modelling processes are always complex. The introduced models cannot be discovered by students autonomously. But they convey the basic processes of “real” modelling. A reasonable use of the introduced models in education requires that the teacher tries to find a proper balance between leading students in certain situations and encourage them to explore and experiment independently.

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ASPECTS OF VISUALIZATION DURING THE EXPLORATION OF „QUADRATIC WORLD“ VIA THE ICT – PROBLEM „FIREWORKS“

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Abstract

This paper deals with several proposals for the modelling of physical phenomenon of projectile motion (angled-launched projectile) in the Earth gravitational field. The problem, which we solve in this article named “Fireworks”, is situated in the discipline intersection of mathematics, physics and informatics at the secondary schools. We compare the utilisation of the graphic calculator, the mathematical software WinPlot and spreadsheets in the solving process of this problem. It offers a large space for the unconventional approaches of teaching, for the use of information technologies and for the creation of interdisciplinary relations. The paper lays emphasis on the innovative process in mathematics teaching in Slovakia that incites stimulating discussions in this field new modern methods, ICT and e-learning.

Key words: modelling, quadratic function, graphic calculator, spreadsheets, teaching.

1 INTRODUCTION

*Some Mathematics becomes more important because technology requires it.
Some Mathematics becomes less important because technology replaces it.
Some Mathematics becomes possible because technology allows it.
Bert K. Waits [1]*

There are many arguments for and against the use of Information and Communication Technology (ICT) in mathematics teaching. This paper sets out some aspects of visualization, which is favourable to the exploration in mathematics learning.

The most considerable didactic aspects of the utilization of ICT in mathematics teaching are [3]:

- Aspect of *visualization* that relieves the conception of thinking process and keeps the learning process shorter,

- Aspect of *process simulation* that enables to create an adequate model on the basis of diverse input values (parameters) as well as to understand their hierarchy,
- Aspect of *interaction between an IC technology and a user* that represents one of the most important attributes of multimedia.

In the following text, we would like to focus primarily on the aspect of *visualization (demonstration)* in mathematics teaching. Problematic of the demonstration in mathematical research and also in mathematics teaching is considered to be one of the most important in the development of mathematical thinking. In relation to that, the literature remarks the notion of *visual thinking*. It is well known that the development of human cognition in the certain field relies on the groups of specific separate models of a future notion or knowledge [10].

Mental operations with the images can be complemented by real experimental manipulations and they lead to the concrete practice manipulation. In the frame of visual thinking, we can assert not only the *algorithms*, but also *heuristics*. *Visualization* represents one of the fundamental strategies in the field of creativity, discovering, inventions and abilities for problems solving. The importance of visualization is affirmed by the fact that the biggest part of brain cortex is aimed at *vision* and *visual analysis*.

Today there is no one to argue about the importance and significance of the development of visual thinking for the school mathematics. In spite of this, the visual methods of problems solving are moved at periphery and they are rare in school mathematics teaching. This reflexion is also underlined by the statement of contemporary mathematician and known popularizator of mathematics, Ian Stewart: „*Images transfer much more information than the words can transfer. Many years, we tried to unteach our students to use the images, because „they are not exact“. It is the sad misunderstanding. Yes, the images are not exact, but they help to think and we could not despise this aid.*“ [3]

The main objective of the innovative process in the mathematics teaching in our country is to show the pupils that the mathematics education is not purposeless. The mathematics is the science, which has various important applications in real life that are inevitable for the development of other scientific and technical disciplines. The process planted into the long term horizon must respect the pupils' mental abilities oriented at the discovering and the cognition of mathematical notion, the development of pupil's creativity, critical thinking and team-work, but also the need of scientific discussion in the class. The international comparative studies TIMSS and PISA show the actual deficiency of these pupils' abilities in Slovak school system [5]. That is the reason why in this paper we would like to offer one physical problem together with the possibilities of its solutions including the utilisation of mathematical knowledge and convenient ICT that is accessible to schools. First section outlines the central problem of this article named „*Fireworks*“. The next sections detail the ideas

of the central problem solution with the help of the mathematical software WinPlot, the graphic calculator TI 83+, and MS Excel (spreadsheets).

2 PROBLEM „FIREWORKS“

This part of article was inspired by the mathematics teaching at secondary schools in USA, especially by the implementation of IMP (Interactive Mathematics Program). The aim of this program is to teach the mathematics differently and to prepare a pupil in the constructive way to encounter the world where he lives. The objective is not to let the pupil receive the knowledge in the inactive way, but above all to let him experiment, search, ask, look for the answers, create and test his own hypothesis, consider, work in teams, share and communicate his ideas and inventions.

The principal topic of the following sections is a quadratic function, whose concept is presented from the several points of view (functional, algebraic and geometric). We consider the choice of the „Fireworks“ problem as very suitable, because it includes not only the mathematical problem, but also the physical problem, which the pupils are able to solve effectively by the aid of ICT [4].

Problem definition

High school football team has just won the championship. To celebrate this triumph, the young football players want to put on a fireworks display. They will use rockets launched from the top of a tower near the school. The height of the tower is 50 metres off the ground. The automatic mechanism will launch the rockets with the initial velocity 28 metres per second.

The team members want the fireworks from each rocket to explode when the rocket is at the top of its trajectory. They need to know how long it will take for the rocket to reach the top, so they could set the timing mechanism. Also, they need to know the best place for spectators to stay (they need to know how high the rocket will go).

The rockets will be oriented to an empty field and shot at an angle of 65 degrees above the horizontal. The team members also want to know how far from the base of tower will the rockets land, so that they can fence off the area.

Theoretical background (several formulations)

The problem includes the physical phenomenon of projectile motion named angled-launched projectile [6]. This motion consists of a uniform rectilinear movement in the direction of axes x with velocity v_1 and a vertical displacement with initial velocity v_2 . Vector of initial velocity v_0 and direction of projectile motion contain the angle α which is named *elevation angle*. The horizontal distance of the projectile range depends on this angle (the distance is biggest when $\alpha = 45^\circ$). The range distance depends also on the initial velocity v_0 .

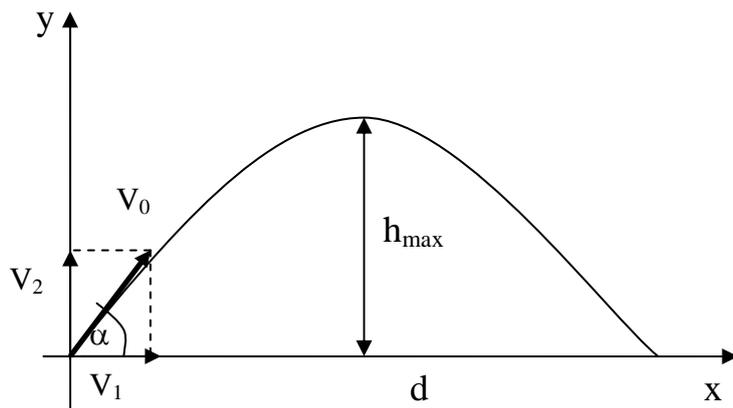


Figure 1 Scheme of angled-launched projectile

$$v_1 = \cos(\alpha) \cdot v_0$$

$$v_2 = \sin(\alpha) \cdot v_0 - g \cdot t$$

$$x = v_1 \cdot t$$

$$y = v_2 \cdot t - \frac{1}{2} \cdot g \cdot t^2$$

David is member of the football team. He is also high school student and he is good in mathematics and physics. He would like to help his team to solve the „Fireworks“ problem. He says that there is a function $h(t)$ that gives the relation between the rocket's height off the ground and the time t elapsed since launch. This relation can be represented by the equation (in metres and seconds): $h(t) = 50 + 28 \cdot \sin 65^\circ t - 5t^2$, $h(t) \cong 50 + 25t - 5t^2$.

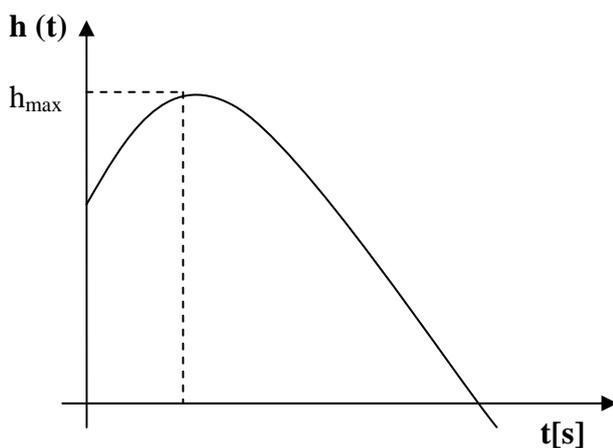


Figure 2 Sketch of the problem situation

We can probably see where the numbers 50, 28, 65 come from. The coefficient -5 in the quadratic component $-5t^2$ coheres with the force of gravity done by the relation: $G = \frac{1}{2}gt^2$.

David also says that it is possible to find a relation describing horizontal distance. The rocket travels with this function: $d(t) = 28t \cdot \cos 65^\circ$.

Again, t is the time (in seconds) since the rocket was launched and $d(t)$ is the distance (in metres).

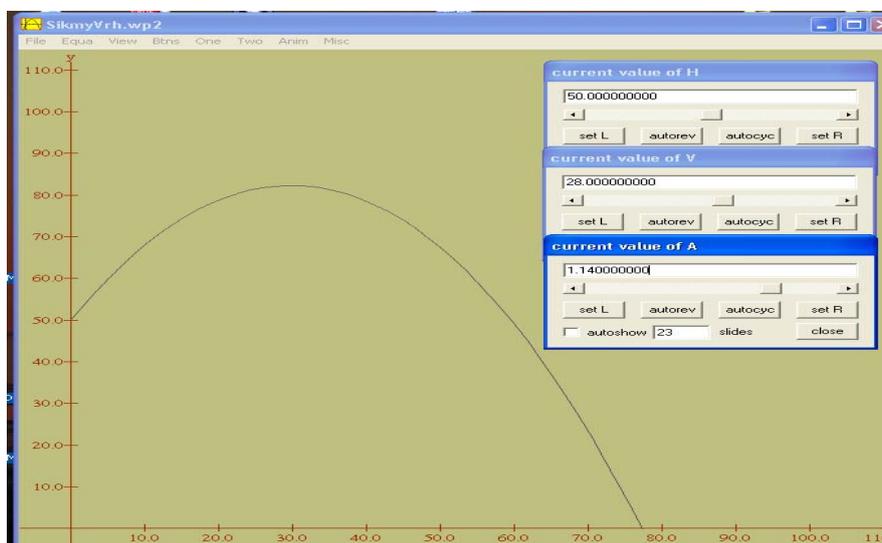
Tasks:

1. To draw a sketch of the situation.
2. To find answers to the partial questions of the football team players:
 - A) What time does the rocket need to reach the top of its trajectory (to find the point where does the function $h(t)$ reach its maximum)?
 - B) Where (horizontal distance of the rocket from the tower) does the rocket reach its maximum height?
 - C) How far (horizontal distance) from the base of the tower does the rocket land?

3 PROBLEM SOLUTION BY THE AID OF THE SPECIAL MATHEMATICAL SOFTWARE WINPLOT AND THE GRAPHIC CALCULATOR

Software WinPlot enables to create an interactive programme, which describes our „Fireworks“ problem. The pupil can change the parameters of the tower height, the initial velocity and the elevation angle. He can observe how these parameters influence the trajectory of the rocket motion. The image underneath represents the trajectory of the rocket since its launch from top of the tower until its landing. The ordered pair $[d(t), h(t)]$ express the coordinates of the projectile (fireworks rocket)

moving in the frame of its trajectory in terms of the time t .



Variable parameters:

H – Height of tower

V – Initial velocity

A – Elevation angle

Figure 3 Trajectory of the rocket's motion made in WinPlot

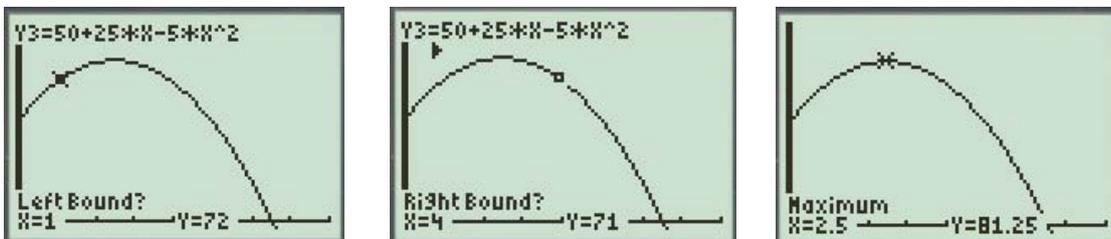
Solving of the „Fireworks“ problem by the aid of graphic calculator (TI 83+)

To solve the problem, we can chart the graph of the quadratic function $h(t) = 50 + 25t - 5t^2$, which represents functional dependence of the launched rocket height h on the time t , with the help of graphic calculator. The task is solved graphically [2].

In order to graph the quadratic function, firstly we have to insert its formula to the function editor **Y=** (a). We must also adapt the window editor to see the whole graph of the quadratic function (b). Then we can let the calculator draw the graph of the function (c).

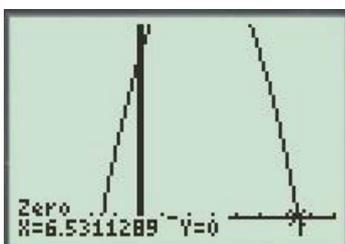


We can answer the question 2A) “What time does the rocket need to reach the top of its trajectory?” by finding the top, the highest point of the graph of quadratic function (it means to find a point where the function reaches its maximum). The calculator function 2nd [CALC] 4: maximum, enables us to count the maximum of the quadratic function $h(t)$ with the corresponding value of t , so we get e.g. the maximum height $h_{\max} = 81,25 \text{ m}$ and the time when the rocket reaches this height $t = 2,5 \text{ s}$.



In the task 2B) we have to find the place (horizontal distance of the rocket from the tower), where the rocket reaches its maximum height. We can calculate this position simply by putting the obtained value $t = 2,5 \text{ s}$ into the equation $d(t) = 28 * t * \cos 65^\circ$, so we receive: $d(t) = 28 * 2,5 * \cos 65^\circ$. Therefore, the place where the rocket reaches the maximum height off the ground level is approximately **29,6** metres far from the tower.

The following calculation will answer the last question 2C): where should we look for the area (place, point) of the rocket’s landing.



At first, we enumerate the time of rocket landing on the ground (it is one convenient positive root of the equation $h(t)=0$, or the intersection of the quadratic function graph with the x coordinate axis).

By this calculation, we obtain the time $t = 6,53$ s of rocket landing. Finally, we are able to take the time $t = 6,53$ s and insert it into the function pattern $d(t) = 28*t*\cos 65^\circ$. We acquire the distance of the rocket landing, $d = 77,28$ m far from the base of tower.

The graphs we have demonstrated by the aid of graphic calculator can offer the image and a lot of information about the rocket movement. However, they do not simulate the trajectory of this movement. For this purpose, it is better to use the interactive program made in WinPlot.

4 PROBLEM SOLVING WITH THE HELP OF SPREADSHEETS

During teaching, it is suggested to utilize a spreadsheet processor as a tool to model various possibilities that could occur and to analyze data. The spreadsheet processor and graphical processing of data can be used during education as tools to model and simulate the dynamic processes. These tools are known to the students as quite standard. By the application of spreadsheet programs' features, we can gain quantitative modeling tools, which are suitable for the use during elementary and high school education.

The spreadsheet programs allow us to use one of their important features – an ability to put calculations' results into the graphs [8]. An adequate example could be a modeling of mathematic functions x^2 , $\sin(x)$, $\cos(x)$ or modeling of the angled-launched projectile.

Using the formulas for individual parameters, we put values of α (an angle), v_0 (initial velocity) and constant g (gravitational acceleration) into the cells with absolute addresses. Then we generate a table of calculations for the sequence of values of time parameter t . With the help of functions, we find the values of the maximal height and distance of fall.

Together with the students, we can experiment with the model by changing the angle of throw or initial speed and then observe how it affects a trajectory. The results are visually displayed on the graph.

For the “Fireworks” problem, we use David's equations $d(t) = 28*t*\cos 65^\circ$ and $h(t) = 50+25*t-5*t^2$. We create a table of time values together with functions $h(t)$ and $d(t)$. Based on values we graphically represent relation of the time t to the height function h and distance function d . The students can use the graph to approximate the maximal values of height and fall together with the corresponding time moment. These values can be determined also by utilization of the spreadsheet calculator's function for the maximum.

t [s]	h(t)	d(t)
0	50,00	0,00
0,3	57,05	3,55
0,6	63,20	7,10
0,9	68,45	10,65
1,2	72,80	14,20
1,5	76,25	17,75
1,8	78,80	21,30
2,1	80,45	24,85
2,4	81,20	28,40
2,5	81,25	29,58
2,7	81,05	31,95
3	80,00	35,50
3,3	78,05	39,05
3,6	75,20	42,60
3,9	71,45	46,15
4,2	66,80	49,70
4,5	61,25	53,25
4,8	54,80	56,80
5,1	47,45	60,35
5,4	39,20	63,90
5,7	30,05	67,45
6	20,00	71,00
6,3	9,05	74,55
6,5	1,25	76,92
6,6	-2,80	78,10

Table 1 Values of variables

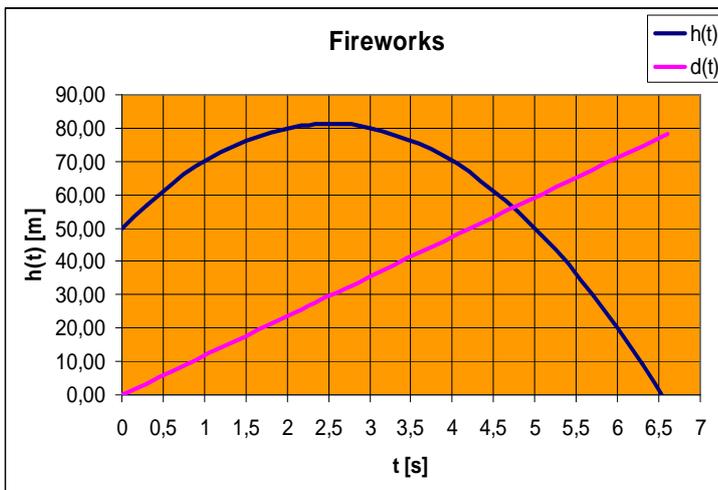


Figure 4 Graphs of functions in the MS Excel program

A representation of the calculations' results by the column of numbers together with the graph allows students to get a deeper insight into the observed phenomenon. The advantages, which result from this kind of spreadsheet programs' utilization in education, are the following ones:

- complicated, repeating calculations are cut down to minimum
- more models of "what happens, if" type can be checked out
- models can be tested by the greater amount of data
- it's possible to graphically represent the examined relations

5 CONCLUSION

Creation and application of the models for the purpose of real world's phenomena demonstration is the subject of teaching process. These models take a significant part in application of didactic principles of science, demonstration and activity [9]. Scientific knowledge is related not only to the content of teaching process, but also it represents the method of its acquirement. Modeling and simulation of the systems, as a scientific method, helps students to gain new information by examining the various systems, based on their models [7].

The graphic calculator, software WinPlot and the spreadsheet processor could be the appropriate tools for the creation of visual and graphically high implemented animation models. A very important function of the models is an enhancement of visual demonstration. A purpose of this demonstration is to create the conclusive ideas for the student. At the age of 12 years, when students acquire an ability to accomplish the formal operations and to think abstractedly, it is desirable to arouse their visual feeling of abstract representations and descriptions of real processes and devices. Various symbolic models, such as diagrams or graphs, can also be used during the mathematics teaching.

A model used for the didactic purposes helps us to demonstrate and discover all the significant features of examined phenomenon. It is appropriate for students not only to get prepared models of the reality, but also to create some themselves. Thus they have to reproduce a structure of the model and to reveal all of its features. Consequently they can improve it or work it over. As a result, the students can learn in a more creative way. This approach creates an area for the use of educational software and tools, which gives us an opportunity to teach the students a given topic with the help of ICT.

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MATHEMATICAL MODELLING IN CLASS REGARDING TO TECHNOLOGY

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Based on the well known modelling cycle we develop a concept of modelling in mathematics education using technology. We discuss the specifics of modelling with computers and handhelds and show some technical possibilities of software tools for mathematics classes. Exemplary we show different modelling cycles using technology based on the three major types of software tools for mathematics.

INTRODUCTION – MODELLING WITHOUT HELP OF TECHNOLOGY

The concept of modelling can be found as a basic concept in some areas of natural sciences, especially mathematics. Therefore it is not remarkable that this basic concept can be found in several curricula all over the world. In mathematics the concept of modelling and the application of real-life-problems in education has been discussed intensively over the last years – see for example Kaiser & Sriraman (2006, p. 304), Siller (2006). It is possible that students of all ages are able to recognize the importance of mathematics through such problems because real-life problems...

- ... help students to understand and to cope with situations in their everyday life and in the environment,
- ... help students to achieve the necessary qualifications, like translating from reality to mathematics,
- ... help students to get a clear and straight picture of mathematics so that they are able to recognize that this subject is necessary for living,
- ... motivate students to think about mathematics and computer-science in a profound way so that they can recall important concepts even if they were taught a long time ago.
- ... allow the teaching of mathematics with a historical background.

If we look at the concept of modelling (figure 1) designed by Blum & Leiß (2007) in mathematics education we will be able to find three important points:

- Design & Development:
Comparable to “Finding the real model” and to the step

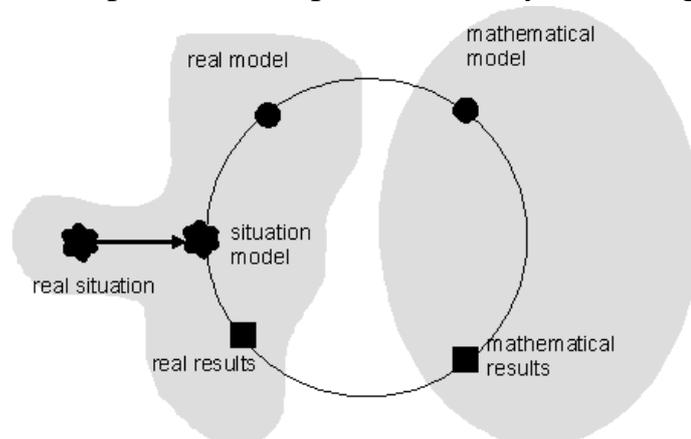


Figure 1: Modelling cycle of Blum & Leiß (2007)

of “Translation” – Real situation to real model by including the situation model.

- Description: Comparable to “Finding the mathematical model”.
- Evaluation: Comparable to “Finding (Calculating) mathematical results” and to the step of validating.

In curricula the usage of technology and the aspect of modelling very often is demanded. For example you can read in the Austrian curriculum (2004):

“An application-oriented context points out the usability of mathematics in different areas of life and motivates new knowledge and skills. [...]

The minimal realization is the acquiring of the issue of application-oriented contexts in selected mathematical topics; the maximal realization is the constant addressing of application-oriented problems, the discussion and reflection of the modelling cycle regarding its advantages or constraints. [...] Technologies close to mathematics like Computer algebra systems, Dynamic Geometry-Software or Spreadsheets are indispensable in a modern mathematical education. Appropriate and reasonable usage of programs ensures a thorough planned progress. The minimal realization can be done through knowing such technologies and occasional applications. In a maximal realization the meaningful application of such technology is a regular and integral part of education.”

So each of us has to ask where the usage of technology can be best implemented. The integration of technology in the modelling cycle can be helpful by leading to an intensive application of technology in education. We have thought about a way that the use of technology could be implemented in the modelling cycle. Our result can be seen in figure 2. The “technology world” is describing the “world” where problems are solved through the help of technology. This could be a concept of modelling in mathematics as well as in an interdisciplinary context with computer-science-education.

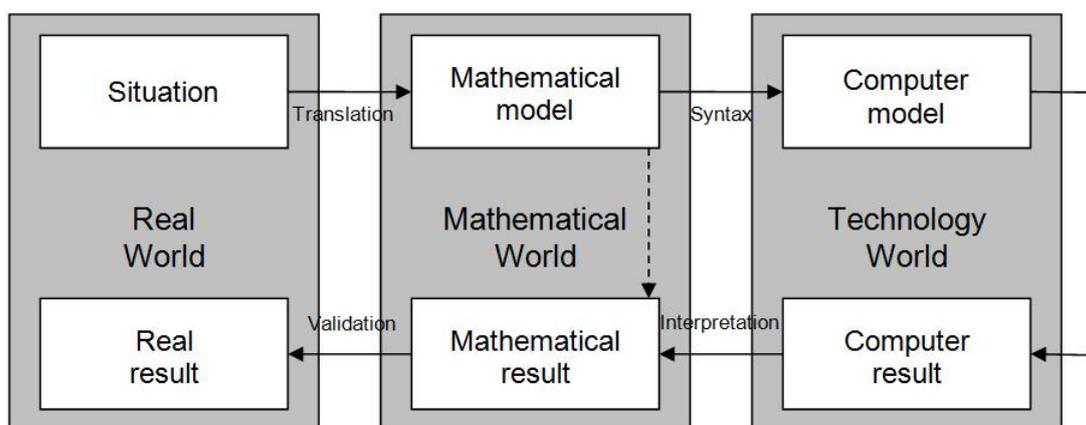


Figure 2: Extended modelling cycle – regarding technology when modelling

The three different worlds shown in figure 2 are idealized; they influence each other. For example the development of a mathematical model depends on the mathematical knowledge on the one hand, on the other it is affected by the possibilities given in the technology world. Using technology broadens the possibilities to solve certain mathematical models, which would not be used and solved if technology would not be available. At this point we want to mention, that successful modelling demands mathematical knowledge and skills in certain software tools.

Based on this graphical illustration we have to discuss the use of technology in terms of modelling in a more detailed way.

MODELLING WITH THE HELP OF TECHNOLOGY

Through the usage of computers in education it is easier to discuss problems which can be taken out of the life-world of students. Through such discussions the motivation for mathematical education can be effected because students recognize that mathematics is very important in everyday life. If it is possible to motivate students in this way it will be easy to discuss and to teach the necessary basic or advanced mathematical contents such as finding a function or calculating the local extreme values of a function.

Unfortunately a lot of teachers and educators prefer not to work with real-life problems. The reasons for this are manifold, e.g. teachers do not want to use CAS or other technology in class or the preparation for such topics is very costly in terms of time. There are however, lots of reasons to combine modelling and technology. Fuchs & Blum (2008) quote the aims of Möhringer (2006) which can be reached through (complex) modelling with technology:

- Pedagogical aims:
With the help of modelling cycles it is possible to connect skills in problem-solving and argumentation. Students are able to learn application competencies in elementary or complex situations.
- Psychological aims:
With the help of modelling the comprehension and the memory of mathematical contents is supported.
- Cultural aims:
Modelling supports a balanced picture of mathematics as science and its impact in culture and society (Maaß, 2005a, 2005b).
- Pragmatically aims:
Modelling problem helps to understand, cope and evaluate known situations.

As we can see the use of technology can help to simplify difficult procedures in modelling. In some points the use of technology is even indispensable:

- Computationally-intensive or deterministic activities,
- Working, structuring or evaluating of large data-sets,

- Visualizing processes and results,
- Experimental working.

With technology in education it is possible not only to teach traditional contents using different methods but it is also very easy to find new contents for education. The focus of education should be on discussion with open, process-oriented examples which are characterized by the following points.

Open process-oriented problems are examples which ...

- ... are real applications, e.g. betting in sports (Siller & Maaß, 2008), not vested word problems for mathematical calculations.
- ... are examples which develop out of situations, that are strongly analyzed and discussed.
- ... can have irrelevant information, that must be eliminated, or information which must be found, so that students are able to discuss it.
- ... are not able to be solved at first sight. The solution method differs from problem to problem.
- ... need not only competency in mathematics. Other competencies are also necessary for a successful treatment.
- ... motivates students to participate.
- ... provokes and opens new questions for further as well as alternative solutions.

The teacher is achieving a new role in his profession. He is becoming a kind of tutor, who advises and channels students. The students are able to detect the essential things on their own. Therefore we want to quote Dörfler & Blum (1989, p. 184): “With the help of computers (note: also CAS-calculators) which are used as mathematical additives it is possible to reach a release of routine calculation and mechanical drawings, which can be in particular a big advantage for the increasing orientation of appliance. Because of the fact, that it is possible to calculate all common methods taught in school with a computer, mathematics education meets a new challenge and (scientific) mathematics educators have to answer new questions.”

ENABLING TECHNOLOGY

The use of technology in mathematical education always depends on the enabling technology. For mathematical education there are many different hard- and software tools. The three major types which have emerged in this area are - see for example Barzel et al. (2005, p. 36):

- Computer algebra system (CAS): With the help of such a tool it is possible to work symbolically, algebraically and algorithmically.

- **Dynamic Geometry Software (DGS):** With the help of such software it is possible to create geometrical constructions interactively and work with digital work sheets.
- **Spreadsheet Program (SP):** With its help it is possible to organize and/or structure data for easier handling, calculating in tables and common analysis.

New developments in the area of technology try to combine these three aspects, although it is difficult to combine all three points and form unique software for each characteristic.

For example the CASIO Classpad, respectively the associated software package Classpad Manager, offers a real interactivity of geometry and algebra.

The simultaneous application of CAS and DGS of the Classpad is, in our opinion, also a useful application. With the help of CAS it is possible to calculate for example non-linear equations symbolically and at the same time the geometrical aspects can be shown through dynamical geometry.

For these purposes equations have to be transformed from the CAS-part to the geometry part. But this method is – until now – not as effective as it should be. After such a transformation the equations cannot be changed interactively. But this problem is not really important, because such examples can be handled easily with other tools, e.g. Geogebra. In Geogebra it is not possible to use a real CAS-part, but the interactivity can be done easily. And a new feature, which is currently available in a Beta-version, is the implementation of a spreadsheet-tool. With its help it is possible to combine interactivity with numerical solutions, calculated in a spreadsheet. To sum up there are several tools combining two or three of the major types CAS, DGS and SP.

Example

The following example which could be discussed with students can be found in everyone's life-world:

Dangerous intersection:

Two cars with different velocities are driving on two different streets towards an intersection where those streets meet. One car is going 60 kilometres per hour; the other has a velocity of 50 km per hour. Try to think about the situation at the intersection – is it possible that an accident can happen? It is given that both cars are running with the constant velocities towards the intersection.

The example can be discussed now under the aspect of different didactical principles:

- **Haptical discussion:** Students model the given situation, for example with some toy cars, and try to find a solution. This could be a starting point for cross-disciplinary teaching with physics (without computers).
- **Graphical discussion:** Students have to draw a chart or diagram of the given situation, and/or modify a given chart (with paper and pencil or DGS).

- Symbolical discussion: Students have to describe the situation for both cars with the help of a function or functional dependency (with paper and pencil or CAS).
- Numerical discussion: Students compute lots of data to solve the problem (with a scientific calculator or SP).

It is not that important which method students' first use to solve this problem. An important point is that students are working based on experience and the methods used are kept sustainable. But it is important for the students to see the different approaches for this problem. In our course we used the following problem:

A picture which describes the given situation visually can be found in figure 5.

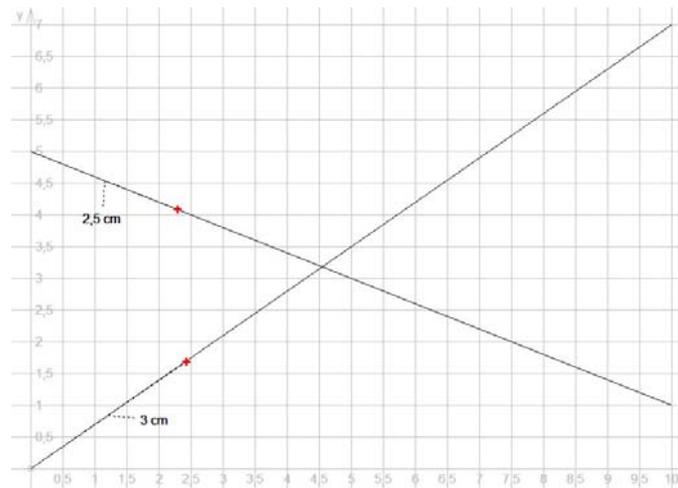


Figure 3: Graphical visualization of the problem

This problem – here in an adequate norm - can be solved in completely different ways. If we use the help of dynamical geometry software, we can move a point for the second car by moving the point for the first car automatically in the right scale and see what happens at the intersection. If we have a closer look at our concept “Modelling with the help of technology” and try to translate the steps which are necessary for solving the problem into our model, it could appear as presented in the following figure (figure 6):

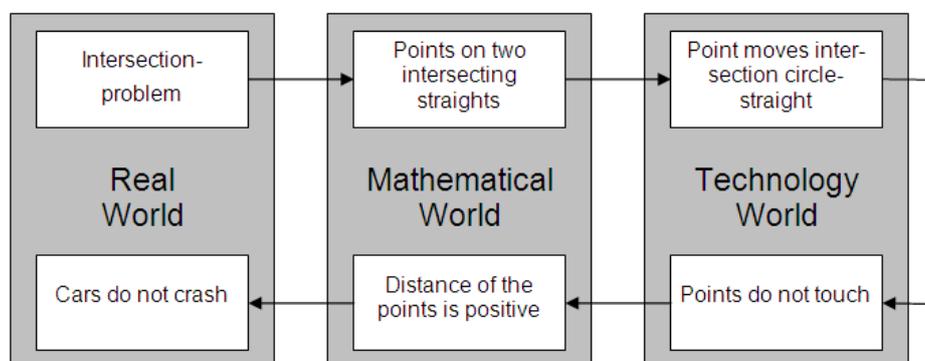


Figure 4: Extended modelling cycle for the problem “Dangerous Intersection”

Alternatively the solution can be calculated with the help of a CAS.

```

#1: f(t) :=  $\frac{t \cdot [10, 7]}{\sqrt{(10^2 + 7^2)}}$ 
#2: g(t) := [0, 5] +  $\frac{5 \cdot t}{6} \cdot [10, -4]$ 
#3: d(t) := |f(t) - g(t)|
#4: SOLVE( $\frac{d}{dt} d(t), t, \text{Real}$ )
#5: t =  $\frac{261540 \cdot \sqrt{149}}{1037531} + \frac{495300 \cdot \sqrt{29}}{1037531}$ 
#6: t = 5.647806845
#7: d(5.647806845)
#8: 0.2573238092
#9: f(5.647806845)
#10: [4.626863648, 3.238804553]
#11: g(5.647806845)
#12: [4.369880841, 3.252047663]

```

Note: For easier readability we have decided to present the solutions in the CAS in decimal notation.

Figure 7 shows the same mathematical model, but a different computer model in the technology world. A CAS works algebraically so we cannot use a geometrical construction to work on the mathematical model. Therefore we decided to use derivation and distance to solve the problem.

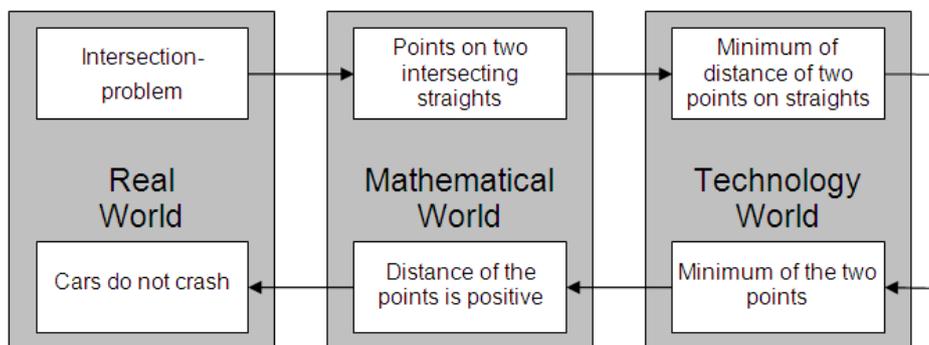


Figure 5: Extended modelling cycle with a different part in the technology world

The solution can also be calculated with the help of a spreadsheet. We will just document this possibility without discussing it. There will of course be a third model in the technology world.

t	x1	y1	x2	y2	dx	dy	d
5,6	4,6	3,2	4,3	3,3	0,3	-0,1	0,2608
5,61	4,6	3,2	4,3	3,3	0,3	0,0	0,2595
5,62	4,6	3,2	4,3	3,3	0,3	0,0	0,2585

5,63	4,6	3,2	4,4	3,3	0,3	0,0	0,2578
5,64	4,6	3,2	4,4	3,3	0,3	0,0	0,2574
5,65	4,6	3,2	4,4	3,3	0,3	0,0	0,2573
5,66	4,6	3,2	4,4	3,2	0,3	0,0	0,2575
5,67	4,6	3,3	4,4	3,2	0,3	0,0	0,2581
5,68	4,7	3,3	4,4	3,2	0,3	0,0	0,2589
5,69	4,7	3,3	4,4	3,2	0,3	0,0	0,2600
5,7	4,7	3,3	4,4	3,2	0,3	0,0	0,2614

Table 1: Worksheet in SP for the example “dangerous intersection”

These three possible solutions (by DGS, CAS, SP) are prototypes for student solutions which represent different mathematical concepts and models. In all three models the assumptions concerning the position of both streets, represented by straight lines, are equal.

The model designed with the help of dynamical geometry software uses only the implicit representation of parameterised straight lines. The main mathematical concept is studying the distance of two (moved) points in the plane. Designing the model as it is shown, presumes the understanding in analytical geometry and connections between the two moving points. The ratio of the velocities of both cars, idealized as points, influences the movement of one point depending to the other. The dynamical visualization allows pupils to experiment with the model (e.g. changing the position of both cars). Thereby possibilities for further developing the model are given (e.g. including the length of the cars).

The models designed by CAS and SP are using parameterised straight lines as algebraic expressions. The distance of both points can be calculated with the help of Pythagoras' theorem. In the CAS model the minimum is calculated with the help of differential calculus, whereas in the SP model the minimum has to be found numerically. One possibility of the CAS model is adding other variables (e.g. different velocities for the cars, changing the starting point of one car) for experimenting or developing the model further. Here more possibilities are imaginable. All of them are very ambitious.

TEACHER EDUCATION

The use of technology in mathematical education does not only depend on technology but also on the knowledge and beliefs of the teacher concerning the different types and usages of technology.

The work with computers in teacher training sets the stage for use in schools. At the beginning and at the end of the course “Computer for Mathematics in School” students who attended were asked about their opinion on the use of computers in class for education. 10 students were present in both interviews comparably. Every question (shown in the diagram of figure 9) had four possible answers: yes, rather yes, rather no, no. In certain cases the beliefs using computers in class changed after at-

tending this course. The topics of the course are the use of CAS, DGS and SP for mathematics in school.

The interviews show a possible change of beliefs while working on topics with computers in class. Some of the positive results concerning computer use can be seen in figure 9, whereas a small bar is closer to the answer “Yes” a bigger bar closer to “No”.

The students are asked to say what changes occur using computers in mathematics classes.

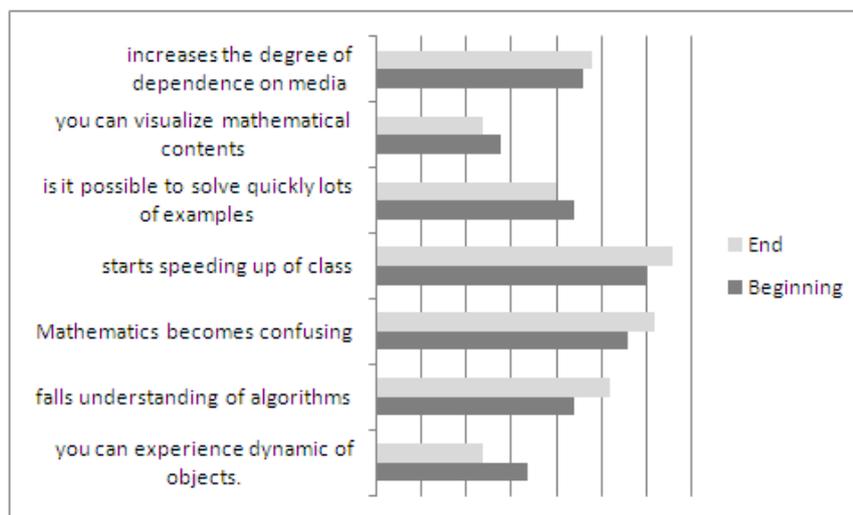


Figure 9: Results of the interview about changes by using computers

This first results show, that it would be interesting to have a closer look at the different strategies of students while modelling with a digital tool. For this research it is necessary to find more examples like “dangerous intersection” with relevance to real life and with different approaches.

Even in teacher education it would be a possible way to discuss examples like “dangerous intersection” by focussing on different computer models. To help the students to reflect upon the role of mathematical software in mathematical modelling processes criteria should be developed and applied (e.g. mathematical content, level of difficulty, possibilities of further developing).

Recapitulatory the use of computers in mathematical education can support and create understanding, in order to improve motivation. The role of technology in the modelling cycle has to be pointed out and examples in education have to be adapted and even created. To implement these points more research in this field needs to be done.

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THE 'ECOLOGY' OF MATHEMATICAL MODELLING: CONSTRAINTS TO ITS TEACHING AT UNIVERSITY LEVEL

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Considering the general problem of integrating mathematical modelling into current educational systems, the paper focuses on the study of the institutional constraints that hinder the implementation of modelling activities. The study of these restrictions and the way new teaching proposals can overcome them appear as an unavoidable step for the large-scale dissemination of mathematical modelling activities at all school levels. Within the framework of the Anthropological Theory of the Didactic, it is proposed the use of a hierarchy of levels of didactical determination as a frame to set and analyse from the more specific constraints, related to the usual way of organising mathematical contents, till the more generic ones, linked to the 'dominant epistemology' concerning the role of mathematics in experimental sciences.

Key words: ATD, mathematical modelling, constraints, conditions, applicationism.

1. THE PROBLEM OF INTEGRATING MATHEMATICAL MODELLING INTO CURRENT EDUCATIONAL SYSTEMS

Nowadays, there seems to be no doubt about the possibility of introducing students to a mathematical activity orientated towards the study of applied and modelling problems. This agreement is shared by many researchers in the field of mathematical modelling and applications, and supported by the new curricular orientations that have recently been introduced in our educational systems, thus trying to focus mathematical teaching more on the study of 'real life situations' than on systems of well-organised mathematical contents. Several investigations from different theoretical perspectives have shown that mathematical modelling activities can exist at school under suitable conditions, at all levels and related to almost all curricular contents.

Beside all the progress of establishing modelling as a normalized activity in some controlled processes of teaching and learning mathematics, the problem of the large-scale dissemination of these processes has recently been addressed as both an urgent and intricate task. Some authors have started pointing out the existence of strong limitations hindering the inclusion and permanent survival of mathematical modelling practices in the classroom. For instance, Blum *et al.* (2002, p. 150) depicts the situation as follows:

While applications and modelling also play a more important role in most countries' classrooms than in the past, there still exists a substantial gap between the ideals of

educational debate and innovative curricula on the one hand, and everyday teaching practice on the other hand.

Kaiser (2006, p. 393) seems to go in the same direction when she states:

Since the last decades the didactic discussion has reached the consensus that applications and modelling must be given more meaning in mathematics teaching. [...] However, international comparative studies on mathematics teaching carried out during the last years, especially in the PISA Study, have demonstrated that worldwide young people have significant problems with applications and modelling tasks.

Related to this state of things, Burkhardt (2008) emphasizes the existence of two realities: on the one hand, the good progress and encouraging results in research about teaching modelling and applications; on the other hand, the difficulties of its large-scale diffusion in the classroom. He states quite brutally (op. cit., p. 2091):

[W]e know how to teach modelling, have shown how to develop the support necessary to enable typical teachers to handle it, and it is happening in many classrooms around the world. The bad news? ‘Many’ is compared with one; the proportion of classrooms where modelling happens is close to zero.

To describe the difficulties encountered in the diffusion of modelling, many researchers use expressions such as ‘counter-arguments’ (Blum, 1991), ‘obstacles’ (Kaiser, 2006), ‘dilemmas’ (Blomhoj & Kjeldsen, 2006) or ‘barriers’ (Burkhardt, 2006), pointing out a new direction of research which moves from the problem of the design, implementation and analysis of modelling practices to the study of the conditions that affect the existence, permanence and evolution of these practices. In a research on teachers’ beliefs about mathematical modelling, Kaiser (2006) defines different teachers’ profiles to explain how some beliefs can become important ‘obstacles’ for the implementation of applied and modelling practices in teaching, because the nature of contextual and applied problems does not seem to be compatible with those beliefs. (p. 399). In the same direction, Blomhoj & Kjeldsen (2006, pp. 175-176) point out the existence of different ‘dilemmas’ that should be faced before widely incorporating the teaching of modelling. These dilemmas refer to: the understanding of mathematical modelling competency from a holistic point of view; considering mathematical modelling as an educational goal in its own right and the dilemma of teaching directed autonomy.

At a more general level, Burkhardt (2006, pp. 190-193) outlines and discusses the existence of ‘barriers’ that obstruct a large-scale implementation of modelling, such as the systemic inertia, the unwelcome complication of the ‘real world’ in many mathematics classrooms, the limited professional developments of teachers, the role and nature of research, and the development in education. To overcome these barriers and many others still unknown, he refers to some ‘levers’ (such as changes in curriculum descriptions supported by well-engineered materials to support assessment, teaching and professional development, etc.) that may show some

promise progress in this field. Michelsen (2006) points out an even more general barrier when he questions the common separate vision of scientific disciplines, and states that traditional borders between disciplines suppose a clear constraint for the development of applied activities (op. cit., p. 269):

The challenge is to replace the current monodisciplinary approach, where knowledge is presented as a series of static facts disassociated from time with an interdisciplinary approach, where mathematics, science, biology, chemistry and physics are woven continuous together.

This situation can be summarized in the formulation of the following didactic problem, which has to be located at the core of all research aiming to integrate mathematical modelling in teaching and learning practices:

What kind of *limitations* and *constraints* exist in our current educational systems that prevent mathematical modelling from being widely incorporate in daily classrooms' activities? What kind of *conditions* could help a large-scale integration of mathematical modelling at school?

Within the framework of the Anthropological Theory of the Didactic (ATD), most of the research related to mathematical modelling and teaching practices¹ (Artaud 2007, Bolea *et al.* 2004, Barquero *et al.* 2008, Barbé *et al.* 2005) takes into account the problem of the 'ecology' of didactic organisations, that is, the study of the conditions needed to implement teaching and learning activities and the constraints that hinder their normal evolution in a given educational institution. The origin of this ecological problematic, which was first applied to mathematical objects and practices before being enlarged to a wider institutional perspective, can be located in the study of the process of *didactic transposition* and its related phenomena (Chevallard 1985, see also Bosch & Gascón 2006).

In our research project on the study of a global modelling process at university level centred on the study of a population dynamics (Barquero *et al.*, 2008), we have observed the existence of different kinds of transpositive constraints that hinder the normal evolution of modelling practices in the classroom. We will develop this point further in the next section, preceding it by a short presentation of the 'levels of didactic determination', a key notion introduced by Chevallard (2002) that we will use as a frame to analyse the different kinds of conditions and constraints that affect teaching and learning processes.

2. CONSTRAINTS ON THE TEACHING OF MODELLING ACTIVITIES

2.1. Levels of didactic determination

¹ Several works within the framework of the ATD as Chevallard (1992), Chevallard, Bosch & Gascón (1997) have analyzed and described mathematical modelling activities from this approach. From ATD, it is assumed that doing mathematics consists essentially in the activity of producing, transforming, interpreting and arranging mathematical models.

Mathematics teaching and learning processes can exist because a lot of conditions make them possible: the existence of a social educational project, the choice of a set of contents to be taught, a school organisation with grades, syllabi, teachers and students grouped in classrooms, teaching materials, teachers' training programmes, etc. These conditions are also factors that, while allowing some things to happen, are also impeding others to take place. In the research and design of new teaching proposals, taking into account these conditions and constraints seems necessary if we do not want to have a set of 'ideal' didactic organisations unable to 'survive' under normal conditions, being, as Chevallard (2002, p. 42) put it, only a 'world on paper'. To study the 'ecology' of mathematical practices that exist (or could exist) in a teaching institution and the possible ways of constructing them (the didactic organisations), this author introduced a hierarchy of 'levels of didactic determination' that consists in the following sequence (Ibid.):

Civilization ↔ Society ↔ School ↔ Pedagogy ↔ Discipline ↔ Domain ↔ Sector ↔ Theme ↔ Subject

This hierarchy goes from the most generic level –Civilization– to the most concrete one – the subject or questions that are to be studied by a group of persons. We refer to the lower levels that go from the *discipline* to the *subject* as the *mathematical levels* if the considered discipline is *mathematics*. They refer to the fact that, when a teaching project has been decided on, the contents or the aim of this project should be located in a discipline (or different ones) and, within this discipline, it should be related to the different domains, sectors and themes that structured it in the considered educational institution. For instance, in Spain, a first year course of mathematics for science students at university level is usually structured into three domains: calculus, linear algebra and differential equations. Frequently, the domains are in turn divided into 'sectors', which contain different 'themes', to which every subject or question to study belongs. At secondary school level, the domains are different and can change over time, with each curricular reform: the classical division into 'arithmetic, algebra, geometry' first changed to 'numbers and measure, functions, geometry, statistics', and has now turned into 'change and relations, space and form, statistics, measure, number'. We consider these low levels (as) the 'specific' ones. They are a useful tool to analyse the constraints coming from the didactic transposition process and the concrete way this process organises teaching contents at school: from the division into disciplines and blocks of contents, until (till) the low-level concatenation of subjects.

The upper levels of determination refer to the more general constraints coming from the way Society, through School, organises the study of disciplines (pedagogical level). They concern the status and functions traditionally assigned to educational contents and the general way teaching and learning study activities are organised at school. In effect, there are a lot of common conditions for all disciplines that concretely affect what the teacher and students can do in the classroom. For instance, the amount of hours and sessions assigned to the teaching of a concrete discipline,

the possibilities for disciplines to interact more or less easily, the way students are grouped (by age, by level, by gender, etc.), the organisation of the school space, etc. All those conditions and constraints belong to the *school level*, while the *pedagogical level* refers to those only affecting the teaching and learning of ‘disciplines’. The way disciplines are grouped, valued, linked, diffused belongs to this level: the choice of an interdisciplinary way of studying questions or the way of presenting disciplines as independent. Very close to the previous levels, the *society and civilization levels* concern the way our society and civilization understand the rationale, functions, aims, etc. of school instruction.

The next two sections briefly introduce some of the institutional constraints encountered during an empirical investigation concerning to a local implementation of what it is called Study and Research Course (SRC) on population dynamics (see Barquero *et al.* 2008). As it is explained in this work, SRC are proposed as new didactical devices to teach mathematical modelling with a double purpose: to make students aware of the rationale of the mathematical contents they have to learn and to connect these contents through the study of open modelling questions. In more detail, our proposal for the teaching of modelling at university level (Barquero 2006 & *Ibid.*) consists in the implementation of a ‘mathematical modelling workshop’ that was run in parallel with the ‘usual lectures’ (dedicated to present the main contents of the course and exemplify them through carrying out some exercises on the blackboard). The workshop focused on the study of a population dynamics starting with the question of how to predict the evolution of the population given its size in some previous periods of time. To provide answers to this initial question and to the sequence of questions that followed it, the construction of different mathematical models was required. When studying the links between the questions and the answers provided by the models, new questions appeared that forced to broaden the previous models to more comprehensive, rich and complex ones, which made them continue with the process. At the end, this sequence of modelling activities covered most of the contents of a first-year course of mathematics for natural science students at university level.

Even though this local implementation was able to overcome some of these institutional constraints by setting up a set of suitable local conditions for the workshop², the large-scale implementation of such teaching proposals required the study of the real scope of these constraints in order to be able to introduce the appropriate changes at the appropriate level of didactic determination.

² For instance, the teacher of the workshop was a researcher in mathematics education and the teacher responsible of the course (the “lecturer”) was a mathematician who does his research in mathematical modelling and was participating in the educational research project. On the other hand, the implementation was developed in an annual course where a group of only 25 students were attending it. Moreover, its program was enough flexible to change of order the introduction of most of the mathematical tools that were required by the development of the workshop.

2.2. ‘Specific’ constraints at the lower levels of determination

If we consider the ‘low levels’ of determination (see figure 1) that are specific of the teaching and learning of mathematics, three main interconnected constraints have been made clear by the workshop experiment(ed). The first constraint is located at the *thematic level* and has been studied in other didactic investigations under the name of ‘thematic confinement’ (Chevallard 2002, Barbé *et al.* 2005). It (comes) stems from the fact that the prevailing culture in educational institutions tends to confine the teacher’s responsibilities at the strict level of the theme, without giving him/her the legitimacy to re-organise mathematical contents in a way different from the one imposed by tradition. In other words, teachers can (and have to) decide how to structure and sequence the themes, what subjects, problems and activities to include in each theme, how much time to spend on each one, etc. but they are rarely asked to decide on the choice of the themes or on the concrete division of mathematics into given domains and sectors. As has been shown by García *et al.* (2006), the problem of the disconnection of mathematical contents and the many efforts to solve it through modelling activities is related to this phenomenon of ‘thematic confinement’.

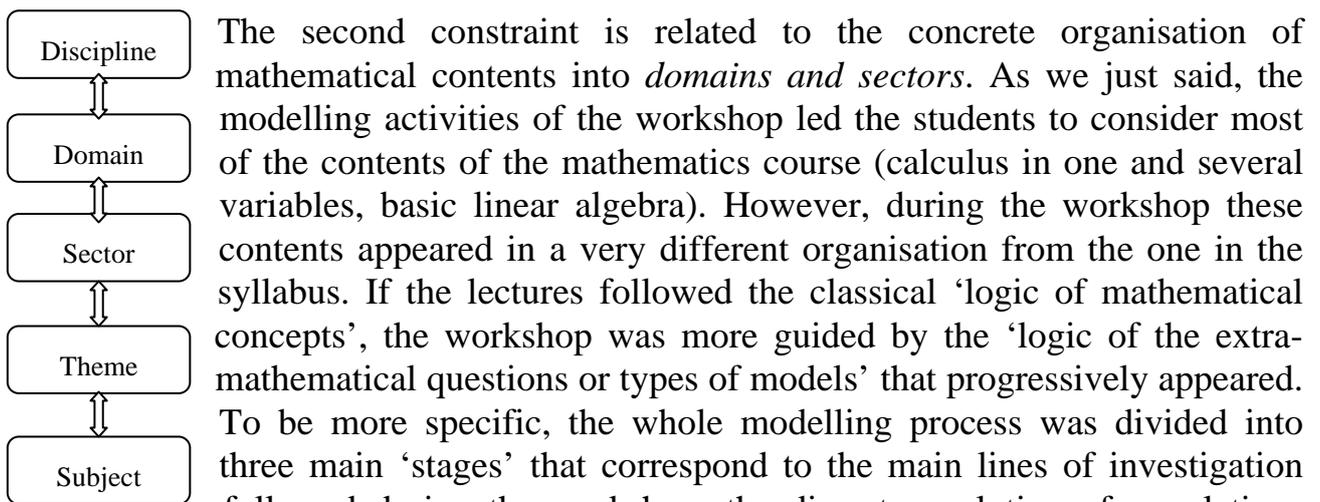


Figure 1

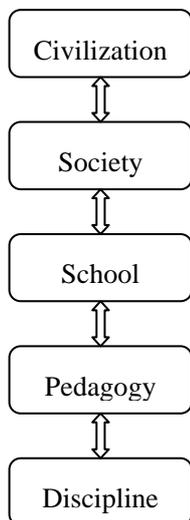
The second constraint is related to the concrete organisation of mathematical contents into *domains and sectors*. As we just said, the modelling activities of the workshop led the students to consider most of the contents of the mathematics course (calculus in one and several variables, basic linear algebra). However, during the workshop these contents appeared in a very different organisation from the one in the syllabus. If the lectures followed the classical ‘logic of mathematical concepts’, the workshop was more guided by the ‘logic of the extra-mathematical questions or types of models’ that progressively appeared. To be more specific, the whole modelling process was divided into three main ‘stages’ that correspond to the main lines of investigation followed during the workshop: the discrete evolution of populations with separate generations (discrete one-dimensional models: recurrent sequences); the discrete evolution of populations with mixed generations (discrete multi-dimensional models: transition matrices) and the continuous evolution of populations (differential equations). This forced the teachers to continuously work in a sort of ‘double curriculum’ project and it seems obvious that, in the long run, much more effort was needed to preserve the new organisation.

Finally, if we move to the *discipline level*, the running of the workshop showed the necessity of strongly modifying the traditional didactic contract that currently exists at universities. To carry out a modelling activity, it is necessary to break with the rigidity of the structure “theory lessons – problem lessons – exams” and to give the students some mathematical responsibilities that are usually assigned exclusively to

the teacher: addressing new questions, creating hypotheses, searching and discussing different ways of looking for an answer, comparing experimental data and reality, choosing the relevant mathematical tools, criticizing the scope of the models constructed, writing and defending reports with partial or final answers, etc. Thus, the teacher had to assume a new role of acting like the director of the study process instead of lecturing the students, which highlighted that the teaching culture at university level does not offer a variety of teaching strategies for this purpose.

2.3. ‘General’ constraints at the upper levels of determination

When we move to the most generic levels (see figure 2), the *pedagogical constraints* appeared when it was necessary to find a suitable timetable for the workshop, with long sessions of two or three hours instead of the usual classes of 50 minutes, as well as some computers available in the class. Organising the students’ work in teams, including the assessment of the teams’ work and its inclusion in the individual evaluation of the course also appeared as difficult obstacles to overcome.



Considering the *society* and *school levels*, by now, we have only studied those related to the ‘dominant epistemology’, that is, the way our society, the university as an institution and, more concretely, the community of university teachers (and students) have to understand what mathematics is and what its relation is to natural science. Our first hypothesis is that the widespread understanding of mathematics and its relation to natural sciences is what we can call “applicationism”. It may be depicted in the following way: a strict separation between mathematics and other disciplines (in particular natural sciences such as biology and geology) is established; when mathematical tools are built, they are ‘applied’ to solve problematic questions from other disciplines, but this application does not cause any relevant change neither for mathematics nor for the rest of disciplines where the questions to study appeared. For example, in the majority of the

Spanish university courses we have examined, the study of population dynamics is a subject located in the sector of differential equations under the label of ‘application’, as if some dynamics laws could exist without any mathematical tool to describe it and, in the same way, as if differential equations could independently exist without any extra-mathematical problem to solve. One of the main characteristics of this dominant epistemology at university level is that it extraordinarily restricts the notion of mathematical modelling. Under its influence, modelling activity is understood and identified as a mere ‘application’ of previously constructed mathematical knowledge or, in the extreme, as a simple ‘exemplification’ of mathematical tools in some extra-mathematical contexts artificially built in advance to fit these tools. To be more concrete, the main characteristics of ‘applicationism’ can be described using the following indicators:

I₁: *Mathematics is independent of other disciplines* ('epistemological purification'): mathematical tools are seen as independent of extra-mathematical systems and they are applied in the same way independently of the nature of the considered system.³

I₂: *Basic mathematical tools are common to all scientists*: all students can follow the same introductory course in mathematics, without considering any kind of specificity depending on their speciality.

I₃: *The organisation of mathematics contents follows the logic of the models* instead of being built up from considering modelling problems that arise in the different disciplines. All happens as if there were a unique way of organising mathematical contents and different ways of applying them.

I₄: *Applications always come after the basic mathematical training*: the result is then a proliferation of isolated questions that have their origin in the different systems. The first thing is to learn how to manipulate the mathematical concepts and later learn about their use. The models are built from concepts, properties and theorems of each theme independently of any extra-mathematical system.

I₅: *Extra-mathematical systems could be taught without any reference to mathematical models*, that is, there exists the belief that natural science can be taught without any mathematics.⁴

To empirically contrast to what degree 'applicationism' prevails in university institutions (see Barquero *et al.* in press), we used these indicators to analyse teaching materials (syllabi, textbooks' prefaces and curricular documents) and to design an interview and a questionnaire addressed to geology and biology teachers and students of a science faculty in Catalonia. The study was developed during the years 2007 and 2008. The analysis of about 30 syllabi of mathematics for natural science courses of 10 different Spanish universities mainly confirmed *I₂*, *I₃* and *I₄*. Some of the prefaces of the most recommended books for these courses helped to corroborate *I₁* and *I₅*. A good example is the case of Salas & Hille (1995) (our translation):

In this edition, you will find some easier applications to physics and, as extra chapters, some more difficult applications [...]. Despite the incorporation of more applications, this book is still a mathematics book, not a science book or an engineering book. It is about calculus and its main basic ideas are limits, derivatives and integrals. The rest is secondary; the rest could be left out.

The interview with a sample of 8 geology and biology teachers and researchers and the answers of 30 other teachers to the questionnaire showed the following results: Related to *I₁* and to *I₃*, up to 97% agreed that "Mathematics is introduced independently of geological or biological systems that could be modelled using

³ This indicator is more general than the other ones as it refers to a characteristic of mathematics as a discipline and not to the way it is taught.

⁴ This is an extreme indicator of the independence between mathematics and natural sciences (especially in the case of biology and geology) that is surprisingly widely shared to the point that, in most cases, people state that scientific systems could be studied without any mathematical tool.

mathematics” and that “the teaching of mathematics is more structured according to mathematical notions than to natural science problems”. Related to I_4 , up to 80% disagree that “mathematics is introduced only after its necessity has been shown and as a tool for the study of science problems”. Finally, the most worrying fact (related to I_5) is that almost 40% agree that in natural sciences degrees, mathematics could only be used to analyse the quantitative aspects of science phenomena.

3. CONCLUSIONS

Using this “ecological” metaphor, we can say that for modelling to be able to normally ‘live’ in a teaching institution, it is necessary to study the conditions that facilitate and the constraints that hinder the type of mathematical activities that can be carried out in this institution. In this sense, the Anthropological Theory of the Didactic appears (as) a priority line of investigation to study these institutional constraints that affect the teaching and learning of mathematical modelling in current educational systems. From the ATD, the study of this “ecology” needs to take into account the different levels of didactic determination, not only to reach the variety of constraints acting on the classroom activities, but also to know better at what level – that is, in what intermediate institutions (from the ‘mathematical lesson’ to the ‘Western civilization’ in our case) it is necessary to act in order to improve the conditions that make the large-scale development of this activity possible.

In order to carry out this study, it appears necessary to provide a general model of mathematical activity that integrates mathematical modelling into the other dimensions of mathematical practices. Researchers in mathematics education have to emancipate from the dominant epistemologies that are implicitly imposed by educational institutions to which we belong. With this purpose, it is important to set out an alternative epistemological model, that is, an operative definition of what mathematics is and what the main characteristics are of the different mathematical activities that exist in our social institutions. As well as, the integration of a description of mathematical modelling within a general epistemological model of mathematics that takes into account the institutional environment of this activity.

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THE DOUBLE TRANSPOSITION IN MATHEMATISATION AT PRIMARY SCHOOL

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This paper proposes a theoretical framework to analyse the articulation between real world and mathematical world in mathematisation at primary school. This paper is not a report of studies presenting a methodology and results. First we describe this theoretical framework based on Chevallard's anthropological theory of the didactic and on the mathematisation cycle proposed by PISA. Then we illustrate this articulation between real world and mathematic world by using the theoretical framework on some examples, from class or from teachers training, issued from the European project LEMA. In this illustration the teaching of mathematisation is the double transposition of the real world knowledge and of the mathematical one. We conclude by questioning the mathematisation through the double transposition problematic.

THE DOUBLE TRANSPOSITION

Real world and mathematical world

We will differentiate the real world and the mathematical world. “If a task refers only to mathematical objects, symbols or structures, and makes no reference to matters outside the mathematical world, the context of the task is considered as intra-mathematical” (PISA 2006, p.81). A possible construction of this world is axiomatic, on a deductive way. Of course the genesis of parts of the mathematical world is in the real world as shown by history. The plausible reasoning could be a reasoning used as heuristic to find a proof or a mathematical solution, but is not a mathematical reasoning to define or to construct a mathematical object, or to prove on a mathematical way. Jaffe and Quinn (1993, p.10) have proposed to set a new branch of mathematics where plausible reasoning will be used: “Within a paper, standard nomenclature should prevail: in theoretical material, a word like “conjecture” should replace “theorem”; a word like “predict” should replace “show” or “construct”; and expressions such as “motivation” or “supporting argument” should replace “proof”. Ideally the title and abstract should contain a word like “theoretical”, “speculative”, or “conjectural””. After a debate in *Bulletin of the American Mathematical Society* this idea was rejected. On the contrary, in the real world the plausible reasoning could be used to define or to construct objects and to validate solutions of a problem. We “focuse on real-world problems, moving beyond the kinds of situations and problems typically encountered in school classrooms. In real- world settings, citizens regularly face situations when shopping, travelling, cooking, dealing with their personal finances, judging political issues, etc., in which the use of quantitative or spatial reasoning or other mathematical competencies would help clarify, formulate or solve a problem” (PISA 2006, p.72).

The double transposition

Using the terminology of Chevallard's anthropological theory of didactics, we consider that the real world is an institution producing the knowledge of real world. In this institution, real world problems have to be solved, using techniques, justifications and validations from the real world. Some of these validations can use argumentations that are not allowed in a mathematical demonstration: pragmatic argument (it is validated because the action is successful), argument of plausibility (as above-mentioned), argument from authority (majority of people, expert ...). The mathematical world is another institution producing a mathematical knowledge (called the scholarly mathematical knowledge). In this institution, mathematical problems have to be solved, using techniques, justification and validations from mathematical world. The mathematisation can be considered as an object to be taught in France (Cabassut 2009), in Germany but not in Spain (Garcia et al. 2007). The process of didactic transposition "acts on the necessary changes a body of knowledge and its uses have to receive in order to be able to be learnt at school" (Bosch et al. 2005, p.4). Here we consider the knowledge of the real world institution and of the scholarly mathematical institution. The mathematisation teaching is the place of a double didactic transposition, one from real world into the classroom and the other one from the mathematical world into the classroom.

MATHEMATISATION CYCLE

Before illustrating this double transposition in mathematisation process, we will present a framework to analyse it. We adopt the mathematisation cycle used in LEMA¹ project. This cycle is inspired by the study Pisa (2006), itself inspired by the works of Blum, Schupp, Niss and Neubrand. As illustrated in the joined figure, we consider five processes in which different competencies are developed:

- setting up the model, what includes "identifying the relevant mathematics with respect to a problem situated in reality, representing the problem in a different way, including organising it according to mathematical concepts and making appropriate assumptions, understanding the relationships between the language of the problem and the symbolic and formal language needed to understand it mathematically, finding regularities, relations and patterns, recognising aspects that are isomorphic with known problems, translating the problem into mathematics i.e. to a mathematical model" (PISA 2006, p.96),
- working accurately within the mathematic world, which includes "using and switching between different representations, using symbolic, formal and technical language and operations, refining and adjusting mathematical models, combining and integrating models, argumentation, generalisation" (PISA 2006, p.96),
- interpreting, validating and reflecting, which includes interpretation of mathematical results in a real solution in the real world, "understanding the extent and limits of mathematical concepts, reflecting on mathematical

arguments and explaining and justifying results [...], critiquing the model and its limits” (PISA 2006, p.96),

- reporting the work: this process is more a transversal process which includes “expressing oneself, in a variety of ways, on matters with a mathematical content, in oral as well as in written form, and understanding others’ written or oral statements about such matters” (PISA 2006, p.97).

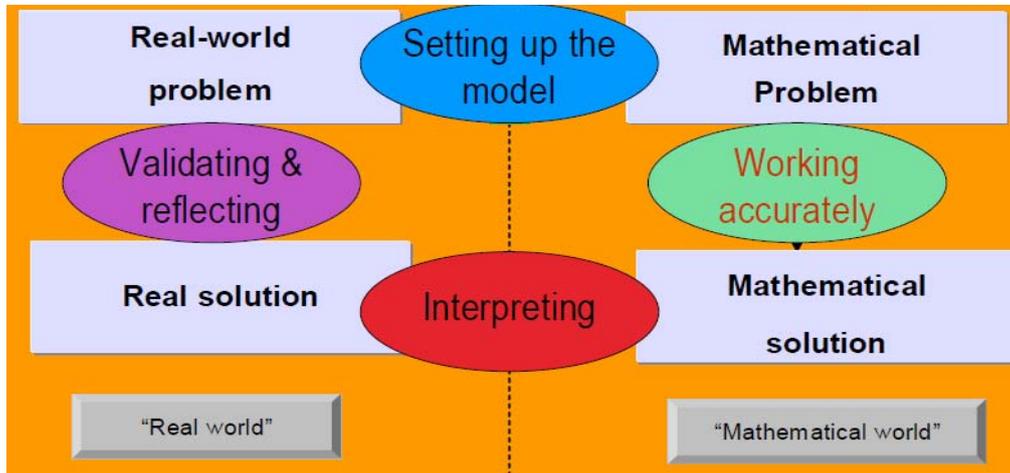


Figure 1: Mathematization cycle used in LEMA

We illustrate now the double transposition in modelling in the different steps of the modelling cycle. These examples are extracted from the European project LEMA¹. This project proposes a teacher training course on mathematization. The information from these examples is from French pupils' observations made when implemented in class. There are also observations done with French primary school teachers or with trainers for primary school teachers.

In these examples we mainly point knowledge and techniques of real world involved in the modelling process. We don't emphasize on knowledge and techniques of mathematical world that are generally well taken in consideration in the related literature.

SETTING UP THE MODEL

Non-mathematical model

The following task was proposed to a French class CP (1st grade: 6-7 years): The class will read a story in a pre-primary school class. How to organize this reading?

In a first-time the pupils must build a mathematical model of the real problem. A possible model is, knowing pupils' number in the class and the number of pages in the book, how to share among pupils the number of pages of the book with the same number of pages per pupil. This model was already practised in class and was suggested by pupils during the discussion. However, in the discussion that takes place in the classroom, some pupils propose a “volunteer” sharing model where pupils read if they are volunteers (for example because they like reading): the distribution of the

pages is done until there are no more. This model is not a mathematical model: the problem is solved on a pragmatic way. It is one reason why we have chosen the word “mathematisation” in place of “modelling”. With mathematisation, we clearly indicate that the chosen model has to be a mathematical one. For example (Maass 2006, p.115) suggests considering a real model before considering a mathematical model. **The teaching of modelling has to distinguish mathematical models and non mathematical ones.**

Non-mathematical arguments to choose a model

After discussion, guided by the teacher, it was decided to choose the model of equitable sharing of numbers of pages to read. The main reason of the choice is that this model is more equitable than the other: each pupil gets the same number of pages to read. The choice of this mathematical model is based on a non-mathematical argument (conception of equity: is it more equitable to force to read a pupil who doesn't like reading than to choose volunteers?). It was not proposed other models, like the equitable sharing of the number of words to read that would have shown the relativity of the concept of equity: is it more equitable to share a number of pages or a number of words? In this phase of choice of some models, arguments of choice could be mathematical or not: taking into account preferences (those who like to read), taking in account equity.

It may happen that the choice of a model is made because of a lack of knowledge of models used in real life, what we illustrate with the following example given in teachers training (Adjage, Cabassut 2008).

Figure 2 Berliner task²

<p>Anne is on holiday in the Black Forest. It is a special offer for a type of pastry called "Berliner" as you can see from the picture. The baker offers the cake € 0.80 each. If you were the baker, would you have proposed the same price on the poster?</p>		
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In this situation it is surprising that it is cheaper to buy a single Berliner and three times a bag of 3 Berliners, rather than to buy a bag of 10 Berliners. It is frequent in real life that buying in large quantities is not always cheaper than in small quantities. It is therefore certain that the models of proportionality or decrease in the price with the increase in the quantity purchased are not valid to explain the Berliner prices.

Maybe other models based on the laws of marketing and psychology, justify a price as 1.99€ below the psychological threshold of 2€ or 6.99 € below the psychological threshold of 7€. The trainer didn't know about the models used in marketing or psychology and have chosen the known proportionality model by lack of knowledge of other models. It looks us important to provide to teachers and trainers tasks resources where models used in the real life are described and discussions on the choice of these models are offered in order that the choice of models are done by conscious arguments more than by lack of choice. **The teaching of modelling has to distinguish mathematical arguments and non mathematical ones to choose a model.**

Choice of the data and hypotheses based on non-mathematical arguments

To complete the construction of the model requires data specifying the number of pupils who read and the number of pages to read. All pupils agree on the number of pupils who read by choosing the number of pupils in the class at the present time. It may be noted that this number could change with the day of the reading in the pre-primary school class. But no pupil has considered this problem. Different assumptions about the number of pages to read are made: a group counts all pages (even those where there is nothing to read), others exclude the front page with the title of book, the ones with the single word "end", or having only illustrations. The justification of these different choices is not based on mathematical arguments. **The teaching of modelling has to distinguish mathematical arguments and non mathematical ones to choose data and hypotheses.**

Model to build and model to reproduce

In the process "setting up the model", it has to be differentiated the case where the model is already known by the pupil and the case where the model is new and has to be built by the pupil. In the previous example the pupils have already met equitable sharing problems that they have often solved by using the distribution technique (every pupil receives one after the other an object from the set of objects to distribute so long there is a rest of objects). We have observed that in this example, some pupils have proposed quickly the equitable sharing model. Let us propose an example where the model is new.

Figure 2 Giant task³

The task was proposed to a group of French CM1 (grade 5: 10-11 years old).

What is the approximate size of silhouette, which can see only a foot? This photo² was taken in an amusement park.



Here pupils have not met the model of proportionality and from this point of view this may be a problem to discover this model.

If the students have a model, they must choose from the stock of available models which accords better with reality. What characteristics of the models must students identify? (And in this case in the study of models which characteristics are putting forward?) What elements of reality must students identify? (And in this case what studies of the reality must be developed by the students?). A part of the heuristic strategies to set up the model comes from the mathematical world (the stock of available models). Of course the real world situation brings also heuristic strategies.

If the students have not an available model, they should build it and make assumptions. What assumptions do they do? How to train pupils to do the "right" assumptions? Here the main part of heuristic strategies seems to come from the real world situation. Of course pupils can use analogies with mathematical available models to set up a model for a real world problem, even if these models are not the right ones for this problem. We see that there are articulations between strategies issued from the real world knowledge and strategies issued from mathematical knowledge of available models. Nevertheless some of the strategies are not specific to mathematisation problems and are more generally developed in problem solving at primary school with or without real world context (Ministère 2005, 7-17).

The teaching of modelling has to organize **the transposition of the knowledge of the mathematical models to reproduce**. Here the traditional process of didactic transposition can be used as suggested in (Artaud 2006 p.374): “the first encounter, the exploratory moment, the technical moment, the technological-theoretical moment, the institutionalisation moment, and the evaluation moment”. For the model to build, if this model is a future model to reproduce, we are in the first encounter or the exploratory moment of the previous case. If not, we have to specify **what knowledge of the real world and of the mathematical world has to be transposed to build a model**.

WORKING ACCURATELY

Working accurately takes place in the mathematical world and produces mathematical solutions of the mathematical problem. So we could think that there is no articulation between real world and mathematical world during this process. Let us come back to the previous example of reading task. Once the equitable sharing model and its assumptions (number of pupils and number of pages) identified, each group of pupils works accurately to solve the problem. Different techniques of distributions are proposed (one by one, two by two ...). Different representations of the situation are worked. Some pupils use cubes representing the distribution to distribute effectively the cubes. Other ones use drawings to represent the set of pupils and the set of the pages and to draw a connection between the two sets. These two techniques show relations with real world: action in the pragmatic technique and visualisation in the drawing technique. **How the mathematical solution is validated?** Is it true

because the action has a success (pragmatic validation) or because I see the solution on the representation (visual validation)? More generally we have shown in (Cabassut 2005) how **proofs in the mathematical world articulate mathematical arguments and extra-mathematical ones, especially by using pragmatic, visual, or inductive techniques.**

INTERPRETING

In the reading task, a mathematical solution has to be interpreted as a real world problem solution. The solutions represented by cubes or the drawings have to be re-interpreted in the real situation. This interpretation is fairly simple because the situation looks less abstract than in higher grades. More the mathematical model is abstract more the re-interpretation could present difficulties. (PISA 2006, p.97) points some competencies involved in the interpreting process: “decoding and encoding, translating, interpreting and distinguishing between different forms of representation of mathematical objects and situations; the interrelationships between the various representations; and choosing and switching between different forms of representation, according to situation and purpose [...] decoding and interpreting symbolic and formal language, and understanding its relationship to natural language; translating from natural language to symbolic/formal language; handling statements and expressions containing symbols and formulae; and using variables, solving equations and undertaking calculations”. **The use of semiotic representations, and specially the natural language, illustrates the articulation between real world and mathematical world.**

VALIDATING AND REFLECTING

Experimental control

In the case of the reading task, different solutions of the real problem are proposed related to the fact that different assumptions are made to take in account the rest of pages insufficient to distribute one page at every pupils. The common data are 49 pages to share between 17 pupils. In one group, fifteen students each receive three pages and two students each receive two pages. In another group, they add two more pages, the title front page and the last page with the words “the end”, and they distribute three pages to every pupil. Both solutions were validated in the class. In both cases it is possible to control the validity of the solution by playing the distribution in the class and by checking the results of the play.

No possible experimental control

For the giant task, it is not possible to check the giant’s height. There is no complete photo showing the complete giant and it is not possible to visit the amusement park situated abroad. The validation is made on a consensus criterion. As nobody opposes a critic and no contradiction is discovered, the solution is considered as valid. This way to valid is not specific to mathematisation. Lakatos (1976) has shown the same

phenomenon in the mathematical proofs. The validation will be partially based on non-contradiction. But the fact that nobody has discovered a contradiction doesn't mean there is no contradiction, as shown in the history of mathematical proofs. For giant task which is not a familiar situation, the validity is based on the lack of contradictions, which is not a mathematical deductive criterion but a plausibility criterion.

Assumptions and validity of the model

In the case of the giant task, a group of pupils has produced the following data. On the photo the pupils measure 1 cm for a man's foot and 7 cm for his height, what gives a ratio of 7 between both measures. The groups of pupils made the additional assumptions: in the reality an adult's foot is about 30cm and a adult's height is about 180 cm, what gives a ratio of 6 between both measures. With these data it is difficult to use a proportionality model to solve the problem. Here the difficulty is that, as the problem is opened, the pupils have to make additional assumptions to solve it. And it can occur that these additional assumptions are not compatible with a wished model.

In the same task, we can observe solutions proposed by two different groups. In the first solution, pupils measure 9cm for the giant's foot and 1 cm for the man's foot. It means that on the photo the giant's foot is 9 times bigger than the man's foot. They assume that in the reality the ratio is kept. They additionally assume that in the reality the man's foot is about 30cm. Therefore in reality the giant's foot is 9 times greater what gives $9 \times 30 \text{cm} = 270 \text{cm}$. But on the photo, the man's foot measure 1cm and his height 7 cm, which means that the man is 7 times taller than his foot, on the photo and by extension in the reality. They additionally assume that the giant has the same ration on the photo and in the reality. They conclude that the giant's height is $7 \times 270 \text{cm} = 1890 \text{cm}$.

In the second solution, the man's foot measures 1cm and the giant's foot 9 cm; therefore the foot of the giant is 9 times greater than the foot of man. It is assumed that there is the same ratio between the heights. As a man is about 180 cm, the giant's height is about 9 times taller than a man's height. They conclude that the giant's height is $9 \times 180 \text{cm} = 1620 \text{cm}$. Both solutions are validated even if they lead to different results because of different assumptions.

It is clear that this validation is similar to that of a conditional statement in the mathematical world: under this condition the conclusion is true, provided that the used reasoning is valid and that the applied theorems are true. In the real world, the role of theorems is played by assumptions like "the ratio on a photo is the same than the corresponding one in the reality" or "the ratio between size of the foot and height is approximately constant". Often such assumptions are valid in approximation or in very accurate conditions. They need a social knowledge of the real world. The teachers have to take in account if pupils have this social knowledge.

We have seen in the above examples that the validating and interpreting step can involve **arguments and techniques of mathematical world** (like hypothetical-deductive reasoning) and of **extra-mathematical world** (like experimental control).

AUTHOR'S POSITION AND IMPLICATION FOR RESEARCH

In the previous examples, we have illustrated in the whole mathematisation cycle that mathematical knowledge and techniques and extra-mathematical ones have to be transposed and interfere. Blum (2002) observes: "In spite of a variety of existing materials, textbooks, etc., and of many arguments for the inclusion of modelling in mathematics education, why is it that the actual role of applications and mathematical modelling in everyday teaching practice is still rather marginal, for all levels of education? How can this trend be reversed to ensure that applications and mathematical modelling is integrated and preserved at all levels of mathematics education?"

We have observed that a lot of resources don't take in account the double transposition problematic. We propose that teachers training and didactical research give more attention to the double transposition problematic in the mathematisation and try to answer the following questions. In a mathematisation task, what knowledge of real world and of mathematical world has to be transposed? What techniques, justifications and validations from both worlds have to be used? How different knowledge, techniques, justifications and validations are articulated and interfere between the two worlds? What effects on teachers' practice, on pupils' learning and on class didactical contract have these articulations and interferences?

NOTES

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2. Photos published with Katja Maass and Cornelsen's kind authorisation : from Maaß, Katja (2007): *Mathematisches Modellieren - Aufgaben für die Sekundarstufe I*. Berlin: Cornelsen Scriptor (copyright).

3. Photo published with Rüdiger Vernay 's kind authorisation and acknowledgment to the Problem Pictures website www.problempictures.co.uk.

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EXPLORING THE USE OF THEORETICAL FRAMEWORKS FOR MODELLING-ORIENTED INSTRUCTIONAL DESIGN

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Designing modelling processes adapted to school restriction and able to produce a wide, rich and meaningful mathematical activity is far from been unproblematic. That situation seems to be even more problematic if the focus is on Early-Childhood Education. In this paper we explore the possibilities that existing theoretical frameworks can bring us. First, some theoretical consideration about modelling, the lack of sense of school mathematics and the use of theories for instructional design are outlined. Second, the design of a study process under the control of the Anthropological Theory of Didactics is described. Finally, the real implementation of this study process with 4-5 years old pupils is reported, showing how very young pupils can be involved in a wide, rich and meaningful mathematical activity.

INTRODUCTION

Modelling is occupying a central position in the current educational debate, from policymakers and curriculum developers to researchers and teachers. Focusing on research, important efforts in many directions can be observed: students' modelling competence, instructional design, modelling pedagogy, teacher training and support, students' and teachers' beliefs, among others.

On the other hand, research in mathematics education is developing more and more sophisticated theoretical frameworks which aim to understand the complex relations existing in every teaching and learning process. In a simplified way, every theoretical framework can be considered as a model of some *teaching and learning reality*. Structuring and simplifying processes are therefore necessary: every theory focuses on some objects and relations whilst other objects and relations are pushed into the background.

There is an ample consensus about conceptualizing modelling as a cyclic process where a dialectic between an extra-mathematical world and a mathematical one is established, as described by Blum, Niss and Galbraith (2007). Many different versions of the well-known modelling cycle have been developed, depending on researchers' interests and backgrounds. Conceived as different models of the modelling processes, each version tries to capture some features of these processes and/or the modelling-based teaching and learning processes.

However, it seems that there is a gap between research in modelling and applications, on the one hand, and research in mathematics education, on the other hand. That leads us to explore how existing theoretical frameworks not explicitly developed from a modelling perspective could be used to enhance research in the so-called modelling and applications domain. Particularly, we will focus on modelling-oriented instructional design through the Anthropological Theory of Didactics. Moreover, in

this paper we will focus on early-childhood education levels, which are normally neglected in the existing research in modelling and applications.

MODELLING-ORIENTED INSTRUCTIONAL DESIGN

The process of designing modelling-oriented teaching sequences optimized to be used at school is far from being unproblematic. In our work with in-service teachers in a European training course in modelling and applications (LEMA project) one of their main concern was how to design interesting and authentic tasks, adapted to their school constraints and students' level and useful to develop the intended mathematical curriculum [1]. Although some teachers can show a great creativity to find real situations and problems and they feel able to adapt them for educational purposes, most of the teachers feel that it is a big, difficult and time-consuming work. Normally, anecdotic and isolated tasks are developed which address to some mathematical topics but these tasks fail in their intention of giving rise to a rich and wide modelling-based mathematical activity.

In the core of this situation a problem of didactic transposition (Chevallard, 1991) can be identified. Normally, real situations do not come themselves with interesting and crucial problems able to develop the desired wide and rich activity. We agree with Lehrer and Schauble (2007, p. 153) in that “models cannot simply be imported into classrooms. Instead, pedagogy must be designed so that students can come to understand natural systems by inventing and revising models of these systems”.

Our approach in this paper is that of *applications and modelling for the learning of mathematics*, as Blum, Niss and Galbraith (2007) state and, particularly, the use of a modelling approach to help students to provide meaning and interpretation to mathematical entities and activities (also called *educational modelling* by Kaiser et al., 2007). That agrees with the current Spanish curriculum which reacts again the traditional lack of sense of school mathematics and asks for a meaningful mathematical activity where mathematical topics from different mathematical domains are connected and integrated.

ATD AS A FRAMEWORK FOR INSTRUCTIONAL DESIGN

In the last years, a group of Spanish and French researchers have been exploring and developing the Anthropological Theory of Didactics (ATD from now on) as a reference framework for instructional design. The notion of *Study and Research Course* (Chevallard, 2006) as well as the basic assumptions of mathematics as a human activity linked this effort with modelling and gave rise to a new research agenda.

In brief, mathematics is conceived in the ATD as a human and social construction. Over centuries, mathematics praxeologies have been developed, refined, optimized, rejected, combined, etc. as new problems arose in many different domains: from daily life to natural and social sciences and, obviously, from intra-mathematical problems. In our modern societies, School has the responsibility of the diffusion of a part of this cultural heritage to young people so that they will be able to live and act as

responsible and democratic citizens. A common and traditional way of doing that is *showing* to the students already finished mathematic praxeologies, as artefacts they can *visit* and they should preserve. Chevallard (2006) call this the *monumentalistic* approach which directly relates with the lack of sense of mathematics at school.

Opposite to that approach and looking for students' sense-making, Chevallard (2006) advocates for a renewed school epistemology where interesting and crucial problems and questions are in the core giving rise to a meaningful mathematical activity.

The ATD has developed the notion of *Study and Research Course* (SRC) as a model to analyse and design school teaching and learning practices. What are the main characteristics of a SRC? In short: (a) a SRC should start from a *generative* and *crucial* question Q_0 [2]; (b) the *community of study* has to take the study of Q_0 seriously (Q_0 , and the situations where Q_0 is inserted, is not the excuse teacher uses to introduce some mathematics); (c) the study of Q_0 will give rise to answers (that is, praxeologies) but also to new questions Q_i , making the study process an open process and, to some extent, undetermined in advance; (d) as far as Q_0 or some Q_i can be extra-mathematical, not only "pure" mathematical answers and questions are expected through the study process but also *mixed mathematical praxeologies* (Artaud, 2007); (e) a SRC gives rise to a collaborative and shared study process, looking for good answers and for good questions (sometimes new answers are developed, sometimes already existing answers are found, depending on the *media* available in the community of study). Finally, it is expected that the community of study develops their own personal answer A^\heartsuit .

As far as Q_0 and some Q_i emerging from it are of extra-mathematical nature, the subsequent SRC can be considered as a wide modelling process. Indeed, as we reported elsewhere (García, Bosch, Gascón and Ruiz-Higuera, 2006), modelling can be reconceptualised as the progressive construction of a set of praxeologies of increasing complexity. The SRC is therefore a didactic device useful to develop and design wide, rich and meaningful modelling processes with educational purposes.

As far as mathematics will emerge through the process as needed answers for taken as seriously problems instead of an already existing construction, *living* elsewhere and brought to school ignoring the why, the lack of sense of school mathematics will be avoided. Therefore, the SRC is a didactic device useful to make modelling a reality at school fighting against the *monumentalistic disease*.

DESIGNING A STUDY AND RESEARCH COURSE FOR EARLY-CHILDHOOD EDUCATION: COLLECTING SILKWORMS

Institutional, pedagogical, curricular and epistemological background

In Spain, the early-childhood education is a non-compulsory educational level for 3 to 6 years old children (3 grades) although almost every child in this age attends to the school. It is not conceived as a kindergarten but as an educational level ruled by a national curriculum. Three are the main domains in this level: *self-knowledge and personal autonomy*, *knowledge of the environment* and *languages: communication*

and representation. Among the general aims of this level, three of them are of special relevance for our work: (b) *to observe and explore children's familiar, natural and social environment*, (f) *to develop communicative skills in different languages and forms of expression* and (g) *initiation into logical-mathematical skills* (MEC, 2007)

School activity has to be organized in a holistic and integrated way. Children's reality and near environment should be the starting point for every teaching and learning situation. Therefore, modelling could be present on every teaching and learning situation although it is not explicitly described in the national curriculum.

During this stage, pupils are supposed to develop quantification skills and the cardinal sense of numbers (measure of a discrete set) as well as languages and forms of expression to communicate about quantities. Pupils will develop numbers' cardinal sense through their activity in many situations where the measure of one or several discrete sets is necessary. Numbers (both the meaning and the signs) will emerge as models to deal with this quantification [3]. Validation and interpretation processes as well as communicative needs are crucial to make pupils' knowledge evolve from self-invented representation of quantities to numerals and numbers.

As Ruiz-Higueras (2005) describes, following basic works in Didactics of Mathematics developed by Brousseau and cols. in the University of Bordeaux, the question that should guide early-childhood reconstruction of numbers should be: *why do we need numbers and their representation?* Three would be, at least, the functions of numbers in this level: to measure a discrete set (from the set to the number), to produce a set (from the number to the set) and to order a set (to assign and locate the position of an element in an ordered set). Centred in the first and second function, school situations where numbers emerge as models to express the measure of a set, to verify the conservation of a set, to manage a set, to remember the quantity, to reproduce or produce a set of an already known quantity and to compare two or more sets has to be designed and implemented.

If the design process of teaching and learning situations takes care about the reality and authenticity of the situations considered, then modelling is an optimal pedagogical approach for teachers to develop teaching and learning situations concerning numbers and their representation in early-childhood education.

Design of the Study and Research Course

The Study and Research Course (SRC from now on) reported here has its origins in a real school situation lived by a teacher [4]. She was working with her 4 years old students about butterflies and she thought about introducing silkworms and the transformation process into *butterflies* (metamorphosis). It was spring and pupils are used to collect silkworms and to feed them with white mulberry leaves. So, it was easy to bring a box with silkworms into the classroom and observe its evolution. The teacher, in order to deal with the holistic and integrated principle, decided to make some mathematical work with this situation. She is used to work with a-didactic situations (in Brousseau's sense) and their students are used to face problems, to

develop different solutions and representations, compare them, formulate messages in mathematical codes (including self-invented codes), validate the solutions against the *milieu*, discuss about the problem and the different solutions, etc. Students are developing their knowledge of cardinal numbers during this school year and they have been working in many situations where they have to produce a number that measure a discrete set, to build a collection equal to a given number, to compare different collections, to express orally or in a written form how many elements are needed to complete or to reproduce a given collection (both with the collection in front of them and with the collection hidden). However, they do not always use the number as the best way to answer *how many* questions and, depending on the student, they can count up to 20 (or more) but many of this numbers are meaningless.

At the beginning, only an anecdotic an isolated activity (*if we've got N silkworms, how many leaves do we need to feed them?*) seemed to appear. But, as soon as we start working with the teacher and taking the situation seriously, a rich variety of praxeologies emerged.

Compared with other situations used by the teacher, two are the main characteristic of this one. On the one hand, it is a real and authentic situation (silkworms are in the classroom and have to be fed). On the other hand, it is a dynamic system: silkworms will turn into cocoons and, finally, moths (butterfly for pupils) will emerge and die. That means that there are, at least, three different collections to be controlled over the time. In terms of dynamical systems, each state of the system can be described with the vector $(t, n(t), c(t), m(t))$ where t is time, n is the number of silkworms, c is the number of cocoons and m is the number of *moths*. A conservative law rules the system: for every t , $n(t)+c(t)+m(t)=N$, where N is the original number of silkworms.

Working in this kind of systems is quite challenging for 4-5 years old pupils. Techniques to deal with time have to be developed and ways of organizing data are needed in order to record changes. That means that during the study process at school, not only praxeologies around cardinal numbers will emerge but also praxeologies concerning time measurement and data handling. Along the whole study process, silkworms will not only be the excuse teacher uses to introduce some mathematical work, but the centre of the process. Interpretation and validation will be dense during the study process.

IMPLEMENTING THE SILKWORM SRC AT SCHOOL

We will report in this section about the real implementation of the silkworm SRC in two different classrooms (4 and 5 years old students). Data have been taken from a self-report written by the teacher as she was developing the SRC and she was managing the study process at school. The study process took place in spring 2008. There is no space here to explain the study process in detail (both students' work and teachers' decisions). So, we will try to focus on the main issues of this process.

The study process started when the teacher was talking about butterflies in classroom and decided to link that topic with worms and metamorphosis process. She thought

that bringing some silkworms into the classroom (figure 1) could be very motivating for their students. No mathematical work was planned in advance but she quickly noticed that a rich mathematical activity could be developed from this situation.



Fig. 1. Silkworms at school.

The first problem arose earlier: silkworms have to be fed, how silkworms' feeding should be organised? Some restrictions into the system had to be introduced: first, a leave for each silkworm each day; second, new leaves are needed each day and third, taking the leaves from the mulberry tree it is not possible (it is dangerous!) but the gardener will do it for us if we ask him. That gave rise to a quantification activity (praxeology around numbers as cardinals):

the first and second restrictions were introduced in order to give rise to techniques dealing with cardinals and the comparison among different collections. These are really problematic situations for 4 and 5 years old students and numbers and numerals will emerge as the best models to deal with them (although some intermediate models, for instance, iconic representations, are also used). Although many pupils can recite the number names in sequence and they know numerals up to 9 or even more, many of them are not able to use them in context to measure a collection, to produce a new collection or to compare two or more given collections. For instance, the following dialog was recorded by the teacher:

Student: Teacher, we've got twenty-five silkworms minus two.

Teacher: Why? Can you explain it?

Student: Yes, there are twenty-five pupils in the class and, each day, two of us don't have a silkworm.

Teacher: Then, how many silkworms are there?

Student: Twenty-three.

Teacher: How do you know that?

Student: I don't know.

The silkworm activity offers a rich real situation to develop quantification skills. Moreover, as they have to write a message to the gardener with the leaves needed each day, communicative skills will be also developed.

Time is not a relevant variable for pupils yet, although the teacher asks pupils to write the date on the ordering-sheet. In the 5 years old classroom, students are quite engaged in silkworms care. The class was divided into groups which have to take care of the silkworms each day. A list of things to be done and a diary was made (figure 2): *1st, counting the silkworms; 2nd, cleaning the house; 3rd, bringing as many leaves as silkworms; 4th, filling in the diary; 5th, writing down the numbers; 6th, writing a letter to the teacher* (asking for new leaves). The teacher introduces also a table where pupils record the date, the name of the caring group, the number of

silkworms and the number of leaves.

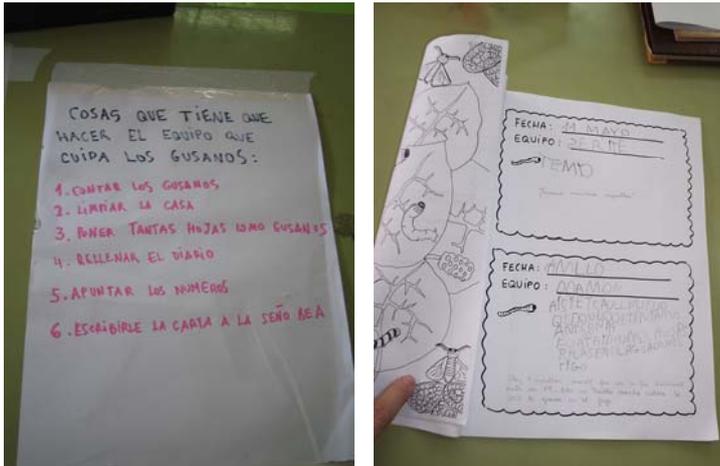


Fig 2. Things to be done and diary

Days were going by until the day cocoons appeared. That caused the first evolution of the mathematical activity (and, obviously, pupils' happiness). On the one hand, pupils decided to put the silkworms apart in other box because cocoons could be damaged when they had to clean the box and fed the silkworms. That caused the division of original collection in more than one collection and additive

strategies to control the whole collection were needed. On the other hand, time arose as an important issue: they needed to control time in order to measure how many time will pass until the moth emerges from its cocoon. The static system has changed into a dynamical system. Pupils' quickly asked for time control:

Student 1: When will butterflies emerge?

Student 2: Well, tomorrow.

Student 1: No, they will take more days.

Student 2: Yes? How many?

Student 1: Now, here (the student points at a day in the calendar).

Student 2: Well, we can count the days (in the calendar) and when the butterfly emerges we will know how many days are.

FECHA	EQUIPO	Nº GUSANOS	Nº CAPULLOS	Nº HOJAS
14 Mayo	SERPENTE	12	12	24
	DOFÍN	5	5	16

Fig. 3. Table to control system's evolution (1)

Teacher knows that time control can be excessively demanding for 4-5 years old students. She needs to introduce some *tools* in classroom in order to let students control quantity and time together. A table (figure 3) is introduced by the teacher (date, group name, number of silkworms, number of new cocoons, number of leaves and total amount of cocoons). It will emerge as a tabular model of system's variation and records its evolution.

From a mathematical point of view, the original praxeology about quantity is evolving and widening including time measurement and strategies to handle with data (obviously, adapted to 4-5 years old students). From now on, students activity can be characterized as a dialectic between

the system divided into different sub-collections (different boxes with silkworms and cocoons) and the tabular model (where system evolution is been recorded).

The day the first moth emerged provoked the necessity of calculating the time passed since the cocoon appeared. Again, this is a problematic task for 4-5 years old pupils. At this level it is usual to introduce some techniques to measure time working over calendars. Pupils are used to work with them and they can get some control on time passed or needed just counting on the calendar.

Student: It's been three days.

Student: No, I said ten days.

Student: Days have gone and they can't be counted.

Teacher: Yes, we can. Let's see, how can we know the days from my birthday?

Student: Well, we look for it in the calendar. We say one, two, three,... (some pupils went to the calendar in the wall and counted, pointing with the finger, since teacher's birthday).

Teacher: Now it is the same. Let's see, Antonio as responsible of the day, tell us how many days. First, you have to look for the day the cocoon appeared.

Student: It has to be one of the first cocoons because it is in the brown box.

Teacher: Ok. Antonio, look for the day in the calendar and count...

The result was twelve days. The next days the same activity gave rise to different results. That was interpreted as there were not a fixed number of days for the moth to emerge but a range. Pupils' interest decayed as they knew the days but they were interested in moths' care.

FECHA	CAPULLOS	MARIPOSAS NUEVAS	MARIPOSAS MUERTAS	MARIPOSAS VIVAS (MARIPOSAS VIVAS)
23 MAYO	23	0	0	0
24 MAYO	22	1	0	1
25 MAYO	21	1	0	2
26 MAYO	20	1	0	3
27 MAYO	19	1	0	4
28 MAYO	18	0	0	4
29 MAYO	16	2	0	14
30 MAYO	14	2	0	16
31 MAYO	4	4	2	17
1 JUNIO	0	11	7	18

Fig. 4. Table to control system's evolution (2)

As they knew that moths will die very soon, the teacher decided to repeat again the time-quantification activity with the collections: cocoons, new moths, death moths and moths alive (figure 4).

When all the moths died, the system was over and the activity around them finished. However, the class had lots of information about the system and its evolution. The models constructed during the study process recorded this evolution. An interpretation activity was introduced by the teacher in order to make these tabular models useful to recover information about a system that had passed by.

The teacher proposed the pupils to make a poster representing the collection in different stages: silkworms eating leaves, cocoons and silkworms and cocoons and moths (figure 5).

Students had to interpret data on the table and produce the corresponding collections. From a modelling point of view, the fact that the system will never be back again in

the classroom makes this interpretation activity completely crucial to summarize what happened and to talk about the system to somebody else. From an educational point of view, this is one of the most important moment of the study process: models can, to some extent, relieve the system and produce information about it even if the system will never be back again. The learning of time-quantity relations is one of the main aims at pre-school. During this final activity, students need to control time and quantity at the same time and the interpretation of the model is the key for that control.



Fig. 5. Reconstructing the system from the model

CONCLUSIONS

Designing modelling processes adapted to school restrictions, able to produce a wide, rich and meaningful mathematical activity is far from been unproblematic. In this paper we argue for the necessity of sophisticated theoretical frameworks for modelling-oriented instructional design. Moreover, for very young students, there is a lack of research concerning modelling-based teaching and learning.

We have described a process of study designed under the Anthropological Theory of Didactics and carried out by 4-5 years old pupils. First of all, the example shows how the theoretical framework allows us to control the design process and its real implementation. Secondly, the study process reported here shows how very young pupils can be involved in a wide, rich and meaningful modelling activity where different praxeologies of increasing complexity emerge as the system is evolving over time. Pupils use, learn and widen their mathematical knowledge as they want to take care of the silkworm collection and to know more about silkworms' transformation: quantification skills, additive and subtractive strategies, time-quantity relations and data handling procedures are brought into play. Finally, once the system has disappeared, models previously developed emerge as tools to reconstruct the system in every stage and to recover time and quantity information. Very young pupils are engaged in a modelling activity, producing and using models, a long time before they are able to really understand what modelling is and the role modelling plays in daily-life, society and science.

NOTES

¹ In Spain, the national and autonomic curriculums are not modelling-oriented. Although many teachers and textbooks are interested in applications and modelling, their main concern is to develop the mathematical topics listed in the curriculum.

² When a question can be considered as “generative” and “crucial”? It depends mainly on the institution where the study process will take place, the educational system and, finally, the society. School level and curricular constraints, the way the educational system is organized and the main aims of school within society need to be considered.

³ Number’s ordinal sense will not be considered in this paper.

⁴ We will like to thank to the teacher, Mrs. Blanca Aguilar (from “El Olivo” school in Torredonjimeno, Jaén, Spain), her effort and enthusiasm which made the experimentation possible.

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STUDY OF A PRACTICAL ACTIVITY: ENGINEERING PROJECTS AND THEIR TRAINING CONTEXT

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This paper deals with the question about place that should be given to mathematics in engineering training. In particular, we analyze a practical activity: engineering projects. This activity intends to reproduce the working context in industrial engineering. Our research is developed in the frame of the Anthropological Theory of didactics (Chevallard 1999). We use the Expanded Model of Technology (Castela, 2008) to analyze the engineering project. In this paper, we present the analysis of a task of modelling developed in the projects context

Background

What place should be given to mathematics in the training of engineers? Which contents should be approached in this training? How should it be articulated with other domains of the training?

These question have already been asked and treated in different institutions and different times. For example, Belhoste et al. (1994) who studied the formation given by the French *Ecole Polytechnique* between 1794-1994, have shown that different models of training have arisen during XIX century: Monge's model, Laplace's model and Le Verrier's model. These questions which underlie the establishment of training's models and the changes of model, from Monge to Laplace then to Le Verrier, are the fundamental questions of relation between science and application, relation between science and technology.

Nowadays, these questions are modified by the technological development, technology taking an increasing place in the engineers' work:

Before the advent of computers, the working life of an engineer (especially in the early part of his or her career) would be dominated by actually doing structural calculations using pen-and-paper, and a large part of the civil engineering degree was therefore dedicated to giving students an understanding and fluency in a variety of calculational techniques. For the majority of engineers today, all such calculations will be done in practice using computer software. (Kent, 2005)

In other words, the development of powerful software changes the mathematical needs because this software encapsulates some of the usually taught mathematics. Mathematics may even appear to be useless to some engineers.

During last years, various researches concerning the nature and the role of the mathematical knowledge in the workplace have been realized (Noss et al., 2000; Kent & Noss, 2002; Magajna & Monagan, 2003; Kent et al. 2004). These works point out the existence of gaps between the educational programs and the real

world in which the engineers work. For example, the institutional speech asserts that undergraduate engineers need a solid mathematical education, but the researches show that for graduate engineers mathematics is of little use in their professional work.

Once you've left university you don't use the maths you learnt there, 'squared' or 'cubed' is the most complex thing you do. For the vast majority of the engineers in this firm, an awful lot of the mathematics they were taught, I won't say learnt, doesn't surface again. (Kent and Noss, 2002)

In our research we intend to contribute to the analysis of the observed gaps and to investigate the role that educational practices and technology play in these gaps. We especially study how one innovative practice in a French engineering Institute intends to articulate theoretical and practical knowledge.

Theoretical Framework: The Anthropological Theory of Didactics ATD (Chevallard, 1999)

The general epistemological model provided by the ATD proposes a description of mathematical knowledge in terms of mathematical praxeology $[T/\tau/\theta/\Theta]$

The praxeology has four components: the first type of tasks T or problems T , the technique is a way to solve the problems, the technology is a theoretical discourse to describe, explain and justify the techniques and the theory is also a theoretical discourse to describe, explain and justify the technologies. The praxeology has two blocks:

Practical block or “know-how” (the praxis) $[T,\tau]$ integrating types of problems and techniques used to solve them

Theoretical block or “knowledge” (the logos) $[\theta,\Theta]$ integrating both the technological and the theoretical discourse used to describe, explain and justify the practical block. (Bosch, Chevallard & Gascón, 2002)

As part of ATD, study is seen as construction or reconstruction of the elements of a mathematical praxeology, with the aim to fulfil a problematic task. To represent finely these processes of construction or reconstruction, ATD offers a model of the study of mathematical praxeology. This model so-called: Moments of the study distinguishes six moments or phases. In this paper we only consider the moment of institutionalization: this moment has the object to specify what is "exactly" the worked out mathematical praxeology. It appears de facto that there are not kept in general in the technology "purified" the elements which are not justified or produced by a theory of empirical knowledge they are rather related to the concrete conditions than the usage of techniques.

Castela (2008) proposes that in the technology of praxeology there are two components: theoretical θ^{th} and practical θ^{p} .

“...the technology of technique is the knowledge orientated to the production of an efficient practice, which has functions to justify and legitimize the technique but also *to equip* and *to make easier* the implementation with it. Beside possible elements of knowledge borrowed from certain appropriate theories (we shall speak following “*the theoretical component*” of technology, noted θ^{th}) this knowledge appears in technology which, according to research domains, is qualified as operative, pragmatic, practical. Collective work was forged in experience; this *practical component* plays technology (noted afterwards θ^{p}) express and capitalize the science of the community of the practitioners confronted in the same material and institutional conditions with the tasks of type *T*, it favours the diffusion within the group.” (Castela, 2008, p.143)

There are six functions associated with the practical component of *praxéologie* θ^{p} :

1. **To describe** the technique. The verbal description of the series of steps that make up a technique is an important step in the process of institutionalization.

2. **To motivate** the technique and the different gestures which compose it.

To explain why, in which aims. It describes the aims expected by the technique and analyzes the effects, consequences, different gestures and the difficulties that its absence could provoke.

3. **To promote** the technique’s utilization. It considers that knowledge allows users to use the technique with effectiveness but also with a certain comfort.

4. **To validate** the technique: it works, it does what is said. Its main goal is to guarantee the technique, when this is used completely it produces a valid solution and the elements where it belongs achieve the expected aims.

5. **To explain** why does technique work? Is about being interested in the causes of effectiveness. Contrary to the second function, the objective is to detail the mechanisms that make that the technique and its components have the desired effect.

6. **To evaluate** the limits, conditions of effectiveness of the technique. The function of validation is positioned on the side of the truth and justified by a theory. In a practical context this function will consider the efficacy.

The institutionalization within different institutions

There are different institutions which maintain a report with a given praxeology. We shall differentiate the institutions with a function of production $P(K)$ of knowledge. And the user *UI* institutions of this praxeology, in sense where subjects of *UI* have to accomplish tasks of type *T*. The aim of $P(K)$ institutions

is to produce and validate the different components of praxeology. But, we asserts that to a praxeology used in a user Institution; this is a part of technology isn't justify for a theory. The technological knowledge validated by an institution $P(K)$ do not exhaust technology, which includes in general a component θ^p for which it is also necessary to examine social modes of validation. The question is therefore to reflect upon construction practises as part of UI, tested through the multiplicity of effective achievements and institutionalization (understood as stabilization rather than explicit recognition by a given institution) of know-how and knowledge.

The **Expanded Model of Praxeology** (Castela, 2008) can be simplified in the following way:

$$\left[\begin{array}{l} T, \tau, \theta^{th}, \Theta \\ \theta^p \end{array} \right] \begin{array}{l} \leftarrow P(K) \\ \leftarrow UI \end{array}$$

Arrows represent social practices of validation in work in the one or other one of the institutions $P(K)$ and IU carrying respectively on the block $[\theta^{th}, \Theta]$ and on θ^p .

Dynamics of mathematical praxeologies

In our work, we focused on mathematical praxeology present in the engineering projects. To account for the way followed by a praxeology from mathematical origin which has to reach the project, we consider different institutions:

Production Institutions

- P(M) Production institution of mathematics
- P(ID) Production of intermediate disciplines

Institutions inside at Vocational Istitute at the University (IUP) (1)

- T(M) Training of mathematics
- T(ID) Training of intermediary disciplines
- Ep Engineering projects

The mathematical praxeologies from production institutions progress to the projects in different ways:

1. P(M)→T(M)→Ep

The first one is from production mathematics to training mathematics until the projects.

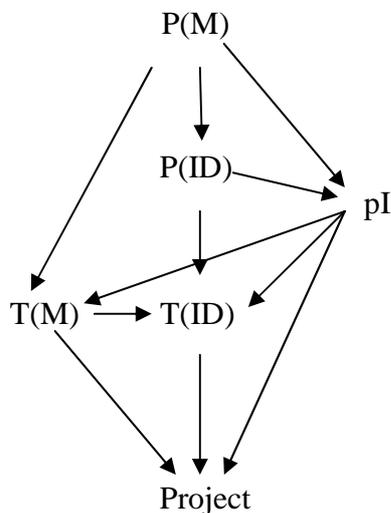
2. P(M)→T(M)→T(ID)→Ep

The second one is from production mathematics to training mathematics through training intermediary disciplines and projects.

3. $P(M) \rightarrow T(ID) \rightarrow T(ID) \rightarrow Ep$

The last one is still production mathematics to intermediary disciplines through training intermediary disciplines until projects.

In our context a vocational training, we shall consider also the profession (professional institution pI). The praxeologies presents in the latter institution are also transposed. These have a specific component θ^p , to promote the use in the professional contexts. We shall take into account the influence from profession to training of mathematics T(M), training of intermediary disciplines T(ID) and Engineering projects Ep. The following schema exhibits the links between the previous components:



Context and methodology of research

In order to realize our study, we have chosen the Vocational Institute at University of Evry (IUP). This Institute uses an educational model of practical education closely related to the industrial world: the university training is combined with training in firm; professional practice takes place during twenty weeks (minimum) over the three years of training. But, the mathematical training is solid, it remains classical at university

The question is: How is the IUP model, which is characterized by a strong nearness with professional middle, inserted in a mathematical training which seems to be designed by this classical model? To answer to this question, our study is focused on an innovative practice, the so-called Projects. These projects intend to connect the official universe of educational disciplines and the professional world of engineers.

The aim of this study is devoted to identify the mathematical praxeology present in the realization of projects and linked with technological tools (TEN).

Therefore, we use this study of praxeology to question the institutional mathematical living in intermediate disciplines or lessons of mathematics.

The projects

The projects are realized by a group of three or four students, very independent, respecting a didactical organization which tries to reflect the real organization in workplaces.

The **engineering projects** are carried out by teams of students in their fourth year of engineering school, over five weeks. The subject of every project is open; there is no previous requirement established by client. The final production and the route towards it have to be built together in the same process. Therefore students have to organize and plan their work, to look for solutions; this generally supposes that they adapt or develop their knowledge.

The projects are realized in two phases. After the first one the students write an intermediary report; in this report they describe the pre-project which is in general justified by a study of the subject. They present the technological solution they have chosen among those they have found during their exploratory work. In the second phase the pre-project must lead to a concrete product.

In this kind of projects, the manager is a college teacher, who plays the role of a client who requests a product from a student's group. All the terms and conditions of the project are described in the schedule of conditions (cahier des charges) which is negotiated between the client (teacher) and the distributor (students). The students are supposed to work on their own to come up to the client's request. The project is assessed from on a double point of view, combining workplace and engineering school requirements. The client must be convinced that the technological solution is the best. But this evaluation is also academic; the students present their work to a jury composed of college teachers. The jury evaluates the use of tools in relation whit knowledge taught in the engineering college. Moreover the students are often asked to justify some of their claims.

Projets Observation methodology

We have realized two observations of the projects. To realize the observation of projects, we used Dumping methodology. In the first phase of project (two weeks) we carried out questionnaires and semi-structured interviews with the students and the clients – tutors. After this phase, we collected institutional data, specifications (document), intermediary reports and documents used for the development of projects. This allowed us to get familiar with projects.

For the second phase we chose only three projects, our aim to be able to realize a deeper and precise observation. To select these projects, we based on the intermediate reports following two criteria: 1) the presence of explicit

mathematical knowledge and 2) the project domain such as aeronautics, mechanics, electronics, etc.

In the third week of the project, we met with the students' teams (three teams for three projects) for an interview about the intermediary report; the aim of this interview was to understand the project and to investigate on the role of the identified mathematical contents. We asked the students to do a brief exposition of their project. The aim of this exposition was to identify the role that they were giving to the mathematical content expressed in their intermediary report. From this, we identified the work division inside the team, and we realized that only one student has the responsibility to develop the mathematical activity. After these meetings, interviews were realized with each student individually.

Praxeological analysis of projects

We carried out a praxeological analysis of the projects. In this paper, we present the analysis of one task accomplished in one of the projects: the Development of a conveyor belt for the aerodynamic study of a light ultra vehicle. The aim of this project was to build a conveyor belt to reproduce the velocity floor. For this, it was necessary to model functioning of the motor and simulates it in Matlab (software).

Task: Modelling of motor

The task is to build a model of the motor through the block diagram. This diagram will allow us to simulate this motor in the Matlab software.

Technique: The modelling of the motor pass by two steps.

1) Mathematical model. The differential equations modelling the electrics and mechanics functioning.

Electrics functioning $u(t) = e(t) + Ri(t) + L \frac{di(t)}{dt}$

Mechanics functioning $C_m(t) - C_r(t) = J \frac{d\omega(t)}{dt} + f\omega(t)$

The electrics and mechanics functioning are linked by two equations. Every single equation contains a flow and couple constant k : $e(t) = k\omega(t)$ and $C_m(t) = ki(t)$

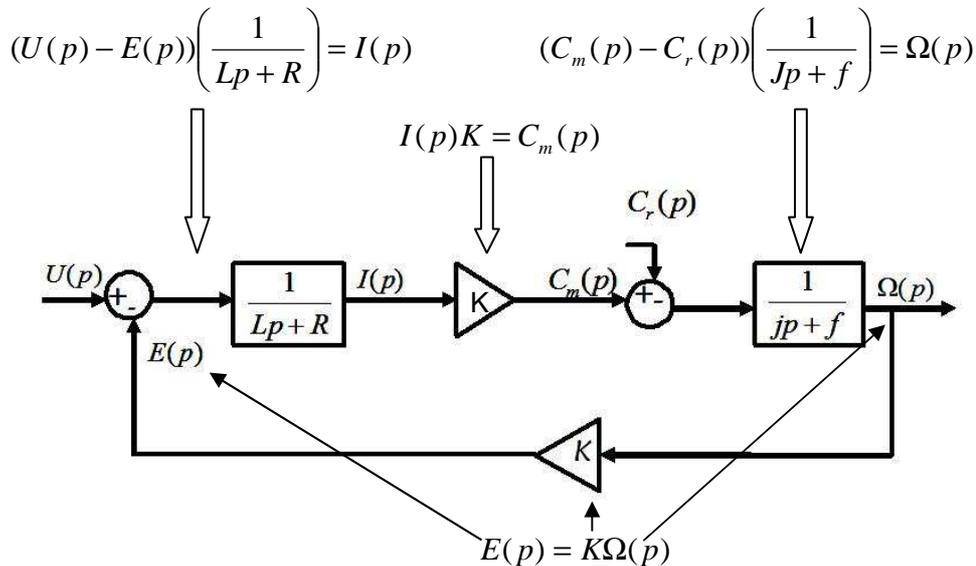
Next, we apply the Laplace transform to every equation:

$$I(p) = \frac{U(p) - E(p)}{R + Lp} \quad (1) \quad \Omega(p) = \frac{C_m(p) - C_r(p)}{Jp + f} \quad (2)$$

$$E(p) = K\Omega(p) \quad (3) \quad I(p)K = C_m(p) \quad (4)$$

2) Construction of block diagrams

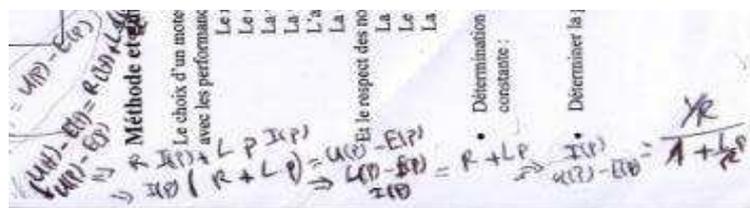
These equations allow us to construct the following block diagrams. Every one element of the equation is represented in the block diagram.



Student techniques:

The student describes the technique utilized to accomplish this task.

“For example if we take this equation (showing $u(t) - e(t) = Ri(t) + L \frac{di}{dt}$) [...] and if we apply Laplace transform we shall have $U(p) - E(p) = RI(p) + LpI(p)$, if we make this (factorize $I(p)$) we shall have this $I(p)(R + Lp) = U(p) - E(p)$, this means that $\frac{U(p) - E(p)}{I(p)} = R + Lp$ and if we make the inverse we shall have $= \frac{I(p)}{U(p) - E(p)} = \frac{1}{R + Lp}$ [...] this equation is modelled by this part” (oral explanation)



Written traces accompanying oral explanation

Technology:

In the description of technique, the student shows the aim of task is to express the “transfer function” of the system. The Laplace transform is for the student a tool which allows to treat an electrical equation as a transfer function. At the

same time, Laplace transform allows to pass from temporary domain (algebraic) to a non temporary domain (differential equation).

“[...] we have $U(p) = E(p) + I(p)R + LpI(p)$ and if we transform $pI(p)$, we apply the inverse Laplace transform, then we obtain the derivative of a temporary function” (Oral explanation)

We see here that motivation appears (function $2 \theta^p$) by the utilization of the Laplace transform. The student focuses in the derivate term $LpI(p)$, showing interest in using the Laplace transform to pass from differential equation (temporary domain) to transfer function (algebraic domain) or the block diagrams.

From the mathematical point of view, there is a notion justifying the block diagram: the transfer function. This notion considers that the physics systems are described by the differential equation:

$$b_n \frac{d^n y}{dt^n} + b_{n-1} \frac{d^{n-1} y}{dt^{n-1}} \dots + b_1 \frac{dy}{dt} + b_0 y = a_m \frac{d^m u}{dt^m} + \dots + a_1 \frac{du}{dt} + a_0 u$$

“If we apply Laplace transform to the differential equation and assume the initial conditions to be null, then the rational fraction which links the output $Y(p)$ to the input $U(p)$ is the transfer function of the system.

$$L\left(\frac{dy}{dt}\right) = p.Y(p) \Rightarrow L\left(\frac{d^2 y}{dt^2}\right) = p^2.Y(p) \Rightarrow \dots \Rightarrow L\left(\frac{d^n y}{dt^n}\right) = p^n.Y(p)$$

$$\Rightarrow b_n p^n Y(p) + \dots + b_1 p Y(p) + b_0 Y(p) = a_m p^m U(p) + \dots + a_1 p U(p) + a_0 U(p)$$

$$Y(p) = H(p).U(p) = \frac{a_m \cdot p^m + \dots + a_1 p + a_0}{b_n \cdot p^n + \dots + b_1 p + b_0} .U(p) \text{ ” (Automatics course: Introduction à$$

l’Automatique des systèmes linéaires, pp.7 -8)

This notion is part of the Automatics course (intermediary discipline).

Conclusion

The task modelling of the motor is the reproduction the existent model. The students are not created a new model. They adapted a type models a specific situation. This adaptation need mobilize the technological elements. These elements are from different institutions: teaching institution of intermediary disciplines T(ID), teaching mathematics T(M) and practical institution pI. We see the processes of transposition of the praxeologies, which pass from one institution to other institution and are transposed. The functions of the practical component θ^p , allows us to analyze the praxeologies used in the projects. To

understand the technologies linked to the students techniques, it is necessary to take in account the intermediary disciplines. These disciplines are intermediary between mathematics teaching and mathematics used in practise.

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FITTING MODELS TO DATA: THE MATHEMATISING STEP IN THE MODELLING PROCESS

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This paper presents a mathematical modelling activity experienced with students of first year university level centred on a problem of forecasting sales using one-variable functions. It then focuses on the back and forth movements between the initial system – a time-series of the term sales of a firm – and the different models proposed to make the forecasting. The analysis of these movements, that are at the core of the ‘mathematising step’ of the modelling cycle, shows how the initial empirical system is being enlarged and progressively enriched with new variables and mathematical objects. Thus the development of a modelling activity initiated with a real-situation may soon lead to a process where the mathematising affects both the system and the model.

1. THE MATHEMATISING STEP IN THE MODELLING PROCESS

In current didactic contracts, the validity of the mathematical knowledge students have to learn usually has its last guarantee in an external source of the activity: the teacher. It is the teacher who, as a last resort, decides if a result is correct or wrong, if the used tool or technique was the best possible choice, etc. Because of this dominant epistemology underlying current didactic contracts of our teaching institutions, research in mathematics education puts forward an ‘experimental epistemology’ more in accordance with the Galilean’s spirit of modern science. According to this epistemology, scientific knowledge (and mathematics in particular) is building up in permanent contrast with ‘empirical facts’ that, added to the principles of theoretical coherence, represent the main elements of proof. The reproduction of this ‘experimental epistemology’ in mathematics underlies the Theory of Didactic Situations (Brousseau, 1997), especially through the notion of didactic situation and the principle of knowledge construction in contrast with a *milieu*. The recent developments of the Anthropological Theory of the Didactic (Chevallard 2004 and 2006) have introduced the notion of ‘media and milieu dialectics’ as an analysis tool of the necessary interaction between a *milieu*, i.e. any system devoid of any didactic intention, and the *media* (in the sense of ‘mass media’) as any source of information or pre-existent knowledge. The aim of this paper is to

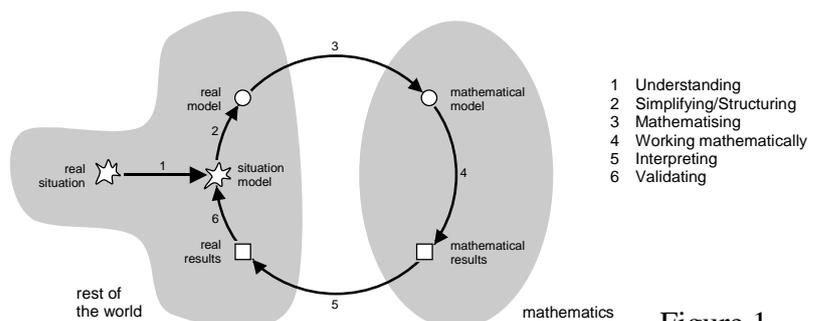


Figure 1

consider how these notions can help analyse a concrete step of the modelling process as it is considered in many research works in the ‘modelling and applications’ domain using the modelling cycle (Blum & Leiß 2006), namely the ‘mathematising’ step (see figure 1).

This paper considers a special modelling activity that has been experimented with first-year students of a mathematics course for economy and business at university level. The real situation that is modelled is a problem of forecasting sales given the historical data or previous term sales. The concrete ‘mathematising’ of this situation consists in choosing an appropriate mathematical model (a one-variable function) fitting the empirical given data. The possibility of choosing different models and the need for a criterion to select one starts a process of contrast between the models and the empirical system acting as a ‘milieu’. The next section presents the conditions of the teaching experience and outlines the work of the students when approaching the sales forecast problem. The analysis of the experience in terms of the ‘media and milieu dialectics’ is detailed in the third section, before concluding about the importance of considering the ‘mathematisation’ of a mathematical system – that is, ‘intra-mathematical modelling’ – as a step of the modelling process analogue to those included in the modelling cycle.

2. A MODELLING WORKSHOP ON ‘FORECASTING SALES’

2.1. Conditions of the experience

The didactical experimentation we present here corresponds to a first course of mathematics in Economics Studies during the academic year 2006/07. It is important to underline that the teaching conditions of this course do not correspond to a traditional one. First, the university we refer to is a private university that organizes teaching in not very large groups (between 30 and 60 students) where every student has a personal laptop computer. Second, the course has been designed by a researcher in mathematics education and the experimentation was carried out by four teachers, three of whom are also researchers in didactics.

The course was designed drawing special attention to modelling activities. Its main goal, as it explicitly appears in the syllabus, is ‘to get students learn to elaborate and use mathematical models for the description, analysis and resolution of problematic situations that can be found in business, economy, finance or daily life. [...] Students should be able to analyze problematic situation in terms of dependence between variables, pointing out the relevant information needed to construct a mathematical model of this situation. And they should know how to use the mathematical model proposed and how to synthesize the results obtained with these models in order to generate new knowledge and new questions about problematic situations considered.’

The programme is divided into three blocks that correspond to the three term periods of an academic year: linear algebra, calculus in one variable, and calculus in several variables/optimization. The course is structured in two weekly sessions of two hours:

the first one is a lecture (teachers' explanations and problem resolutions on the blackboard) and the second one is used to carry out a 'mathematical modelling workshop', centred on the study of a problematic question connected to the field of economy or business. The work here presented corresponds to the workshop experimented during the second term, within the domain of 'one variable calculus', which lasted 5 sessions.

The work at the workshop was organised in the following way: The students work in groups of 3 or 4 and have to write and present a weekly report about the partial results obtained at each session. At the end of the term, an individual final report has to be presented at the moment of the evaluation (a written exam which includes two different problems and a question related to the workshop). This exam represents 50% of the qualification; the written reports 40%, and the remaining 10% corresponds to the individual resolution of problems during some of the lectures.

2.2. The question of 'forecasting sales': analysis of its generative power

The initial question of the workshop was formulated as follows:

A firm registers the term sales of its 7 main products during 3 years. They ask us the following questions:
 → What amount of sales can be forecasted for the next terms? Can we get a formula to estimate the forecasts? Which are its limitations and guarantees? How to explain them?
 → What products sales are increasing more than 10% a term? Less than 12% a term?

The data were 'prepared' by the teachers so that they correspond to seven elementary functions of different types (quadratic, cubic, rational, exponential) with an error term added.¹ The values of each function were slightly changed with the aim of distorting them, but without losing the general "tendency" of the original function.

The workshop's aim was to give students a problem close to a real situation where functions appear as a suitable model. Both the use of Excel in the first term of the course and the students' familiarity with elementary functions (it was the theme of the sessions just preceding the workshop) allowed them to initially detect a tendency in the sales (for example from a graphic representation of the data) and look for a function that fitted this tendency. The firm question proposed also included the idea of percent variation, which we expected would make the study of function variations appear during the workshop. Given that the workshop was run in parallel with the lectures on function derivatives, it was also expected that, at any time, the study of the sales' variation could be connected with them.

¹ The concrete functions were: $0,5(x - 6)^3 + 2000$; $2,5(x + 5)^2 + 100$; $5500/(x + 4)$; $1300 \cdot 085^x$; $1500 - 1200/(x + 1)$; $2,5(x + 5)^2 + 100$; $1300 \cdot 085^x$. The second experimentation in 2007/08 was carried out with 'real' data taken from some macroeconomic magnitudes of different countries: population, oil production, traffic crashes, unemployment rate, etc. The main difference between the two workshops appears in the study of the variations, because the real data have stronger fluctuations and do not always present a clear tendency.

The election of a sales forecast situation was mainly motivated by the fact that it enables to clearly distinguish between the economic system (sales) and the models used (functions). Moreover, working with different products needs to consider different models, raising the problem of the fitting between the model and the modelled system. In other words, the aim of the workshop was to make students use functions as a model of a simple economic system and quickly raise the question of the election of the model and its validation.

2.3. General organisation of the modelling workshop

We here report the four workshops experienced, corresponding to four classes of a (the) first-year course of mathematics for economics and business led by four different teachers working in team. Each group has a teacher, the same one for the lectures and the workshop sessions. All classes were prepared by the team and all sessions were discussed personally or by mail before and after being carried out. Each teacher, at the end of each workshop session, wrote a report in which he/she explained the development of the session, and sent it by mail to the other teachers.

Before the workshop started, the students had four lectures dedicated to introducing the elementary families of functions, from straight lines to exponential functions. The students learned how to use the general expression of every family of functions and to associate them with different graphics. In other words, the students were taught how to assign an algebraic expression to the graphic of a function, among a set of given families. They saw how to deduce the graphic of $y = af(x - b) + c$, from the 'basic function' $y = f(x)$ and, reciprocally, how to deduce the expression of any function $y = af(x - b) + c$ given its graphic and knowing the original 'basic function' $y = f(x)$. The lectures given in parallel with the workshop introduced the notion of absolute and relative variation of a function between two points, the notion of the derivative's function, the notion of straight line tangent, etc. within the general problem of the study of variations. The functions considered were always related to economical situations, such as the *incomes* depending on the *sales*, the *cost* depending on the *production*, the *demand* depending on the *price*, etc.

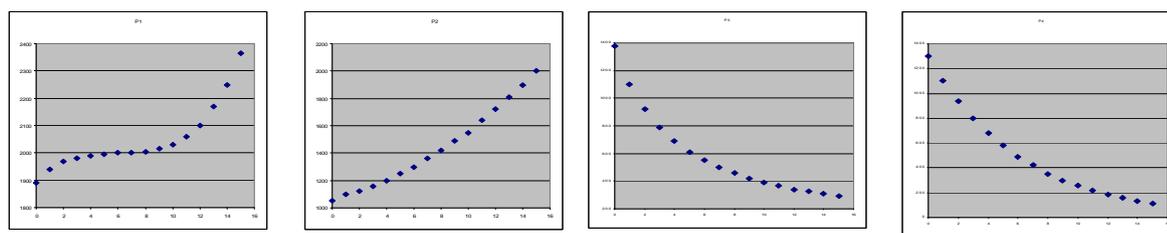
2.4. Description of the workshop sessions

We are now presenting a brief summary of the workshop sessions based on the teachers' reports, the students' weekly summaries of the workshop and the students' individual summaries at the end of the term.

Session 1: Considering the initial question and first exploration of data

The first session is dedicated to present the generative question and the data. Each group is assigned two products from the list. During some time, the students can explore the question and propose a first forecast for the next three-month period. Most of the groups decided to introduce the data in an Excel sheet so as to represent them graphically. Most groups were able to associate the graphic representation with

some of the families of functions previously studied. Some of the graphs obtained were:



Depending on the product considered, different types of functions can be associated with the graphic. The case of product 1 is different because the form of the data clearly suggests a cubic function. In this case, the students easily found an analytic expression $y = a(x - b)^3 + c$ fitting the data, first detecting the inflexion point $(b;c)$ and then testing different values for parameter a . At the end of the session, the teacher asked some of the groups to present their procedure used and results to the whole group. A structure for the Excel sheet was agreed upon and the teams were asked to bring in a possible model with its corresponding forecasts for the next session.

Session 2: Finding different models and comparing them

Each group presented the analytic expression obtained for the products assigned. As each product was assigned to different groups, different possible models appeared for the same set of data. Hence the problem of deciding which forecast was “better” quickly appeared. As it was impossible to decide on at first sight, the teacher introduced a possible criterion to ‘measure how different each model was from the data’. It consists in computing the difference (in absolute value) between the values of the function and the data of the product. A new column was added to the Excel sheet (with) which, at the end, mentioned the arithmetic average of the differences. It was called the ‘average error’.

Then the session work consisted in finding, for each product and within a given family of functions, the model that gives the minimum average error. The first procedure was to modify the parameters of each function to find the best model by trial and error. In the middle of the session, the teacher introduced the Excel tool ‘SOLVER’ that gives the parameter combination that minimizes the average error, when initial values are close to the solution. The Solver function allows finding the best approximation to data when models are considered within the same family of functions, but it is not an effective tool to decide between two models belonging to different families (a parabola and an exponential function, for instance). Besides given two sales forecasts done with functions of a different type, the fact that one of them gave a lower average error than the other, did not always seem a good criterion to determine that it was a better forecast (it is not always so clear graphically, for example). The session concluded by asking the students to bring in ‘the best model’

for each product and the corresponding forecast. In a sense, the first question of the initial problem was almost answered.

Session 3: Study of the average term variations

The session started by sharing the expressions provided by each group. The problem of finding a criterion to select the best model was raised in the case of different models for the same product with a similar average error. At this moment, in one of the four groups, the teacher took advantage of the work done by a team that initially, during the first session, used the term variation of the sales. They found out the rate of the previous terms' variation and then took an average to do the forecast. This idea was introduced to the rest of the teams and also to the other class groups.

Therefore, besides the data of term sales and its possible models, appear a new set of data, the term variations of the sales, which can be modelled in turn. The students were thus asked to proceed with this new data in the same way they did before: doing the graphic representation, deciding which family of functions seems to correspond to the visual tendency, finding the concrete function that gives the lower average error.

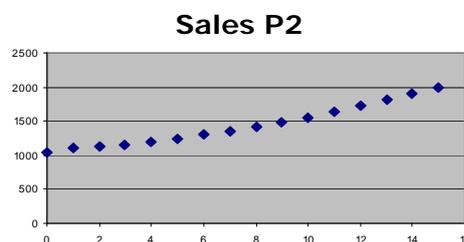
In the case of product 1 (cubic function), the new data appeared as having a quadratic tendency. In the case of products given by a quadratic function, the term variations seemed to correspond to a straight line, in the case of a rational or an exponential function, to another rational or exponential function respectively.

Sessions 4 & 5: Comparing the model of the variations to the variation of the model

When the different groups presented their models for the sales forecast and for the sales variation forecast, the teacher asked for a possible relation between the two models corresponding to the same product. In the case of the products with only one 'good model' (such as product 1 with a 'cubic tendency') the conclusion was quite complicated. With those products accepting more than one model, the variation study led to a better conclusion: the graphic that best fitted the term variations of sales was similar to the graphic of the derivative of the function that best modelled the product.

For example, if we consider product 2, we find:

Term	t	Sales
March-03	0	1050
June-03	1	1100
Sept- 03	2	1120
Dec-03	3	1160
.....



The graphic representation shows a tendency that can be modelled by a linear, a quadratic or an exponential function. The corresponding average errors are:

$a =$	60,92
$b =$	973,03

$a =$	326,96
$b =$	1,10
$c =$	732,97

$a =$	2,46
$b =$	-5,18
$c =$	995,01

t	P2	$y = ax + b$	ERROR
0	1050	973,03	76,98
1	1100	1033,95	66,05
...
AVERAGE			40,625

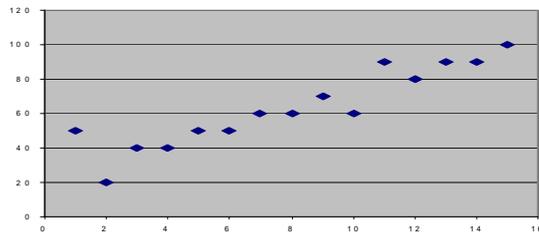
t	P2	$y = ab^x + c$	ERROR
0	1050	1059,92	9,92
1	1100	1091,30	8,70
...
AVERAGE			7,16

t	P2	$y = a(x - b)^2 + c$	ERROR
0	1050	1061,00	11,00
1	1100	1088,95	11,05
...
AVERAGE			3,63

The study of the average error rules the linear model out, but does not provide a good criterion to exclude the exponential function or the parabola. If we consider the term variation of sales and model the new data, we obtain the following:

Sales term average

Time	P2	Term
0	1050	Var.
1	1100	50



$a =$	5,0000
$b =$	24,9999

$a =$	-0,9915
$b =$	-1,2772
$c =$	61,0730

t	P2	T.V.	$y = ax + b$	Error Abs
0	1050	50	30,00	20
1	1100	20	35,00	15
2	1120	40	40,00	0
3	1160	40	45,00	5
4	1200	40
...
AVERAGE				5,67

t	P2	T.V.	$y = ab^x + c$	Error Abs
0	1050	50	62,34	12,34
1	1100	20	59,46	39,46
2	1120	40	63,14	23,14
3	1160	40	58,43	18,43
4	1200	40
...
AVERAGE				16,93

Looking at the two corresponding term variation models, it clearly appears that the linear model has a lower average error than the exponential one. To summarize, we have found two models that fit the initial data in a similar way. Their analytic expressions, using the Excel tool 'Solver', are:

OPTION 1: $y = 326,96 (1,09)^x + 732,96 \rightarrow$ average error: 7,16

OPTION 2: $y = 2,46 (x + 5,18)^2 + 995,01 \rightarrow$ average error: 3,63

The lower error corresponds to the parabola, but both are similar (comparing to other considered possible models). When considering the term average of the sales, the model that fits better is: $y = 5x + 25$. Finally, if we take the first model expression $y = 2,46 (x + 5,18)^2 + 995,01$ and derivate it, we get an expression very similar to the model found: $y' = 2,46 \cdot 2 \cdot (x + 5,18) = 5,2x + 26,936 \approx 5x + 25$

Therefore, we have a new criterion to decide between two models: studying both the tendency of the sales and of their term variation, and choosing as 'best model' the

function that has a derivative that fits the model of the term variation. At this moment, further work on the mathematical model can follow, looking at the derivative as a model of the term variation $\Delta f(x) = f(x) - f(x - 1)$. The use of a symbolic calculator was an important tool for this final step of the modelling process, which was left to the students as a complementary theoretical analysis of the whole work done in the workshop. After these five sessions, students were able to use all the information to present a forecast for the sales and report a complete answer to the initial question.

3. THE ‘MATHEMATISING STEP’: CONTRASTING MODELS TO DATA

3.1. First part of the workshop: the problem of choosing the best model

The process of mathematising or assigning an appropriate mathematical model to a given system can be done in a simple way by directly choosing a previously available model given by an external source (a ‘media’). However, the productivity of the model, that is, the fact that it produces new knowledge about the system, requires a certain ‘fit’ or ‘adaptation’ to the system. This process is rarely done once and for all. It requires a forth and back movement between the model and the system, in a sort of questions-answers or trial-error dynamics. We will now see the details of this process in the concrete modelling process of the workshop presented below.

In the first part of workshop, the aim is to look for a function that accurately reproduces the sales dynamic. The first decision to take is to fix the family function that seems to reproduce the observed dynamic in the data. The students’ first gesture was to represent the data in a calculus sheet and determine a priori which type of function would be chosen². In terms of the ‘media and milieu dialectics’, we can consider that the Excel graphic works as a *milieu*: when representing the chosen function, it allows to visually contrast the ‘proximity’ between the model and the data.

The problem about how to construct a criterion to determine the best fit is the crucial question that drives the study process. Except in one or two cases, the only visual comparison between different sales models becomes an early limited *milieu*. The necessity of establishing a ‘measure of the fit’ comes up, and enriches the initial *milieu* given by the numeric data series and its graphic representation. The option chosen –a new message (*media*) given by teacher– is to calculate the average of the differences (in absolute value) between the data and the values of the considered function. The incorporation of the Solver function –that works as a black box for the students– provides another *milieu* that makes the search of the function that minimizes the error more dynamic. However, this new enriched milieu can also show its limitations when the errors between different ‘competitive’ models are similar.

² The fact that students work with a small group of a pre-established family of functions does not have to be considered as a didactic limitation. It reproduces the usual situation of the genuine modelling work.

3.2. Second part: the model of the variations and the variations of the model

In the case of having different models with similar errors, the *milieu* made up of numeric values and the graphics of both sales and models is newly enriched by the introduction of a new variable: the sales variation. A new modelling process starts, similar to the previous one. The derivative function, as an approximation of the variation, soon becomes a new element of the *milieu* brought by the teacher acting as a *media*. It will contribute as a new criterion of validation: if a model fits the sales, the derivative of the model should be a good fit of the sales variation. For example, if sales seem to follow a parabolic growth, it is expected that the sales variation will follow a straight line growth. In this case, the *milieu* is all the work done during the first part of the workshop, that is, the construction of different models to each data series.

The teacher is who introduces the relation between the term variations and the derivative of the pre-established model (*media*). Besides, as students had a symbolic calculator that allowed them to easily calculate the algebraic expression of the average value $f(x + 1) - f(x)$ of any function, it was also possible to compare the derivative value of the model with this average value and confirm the approximation. It is important to underline that the increase of the '*milieu*'s complexity' made the development of this second part of the workshop more difficult, the 'system' that was to be modelled being less known and 'unstable' for the students. However, the work done represents an exemplary case of the functionality of the derivative as a simple way to calculate the average variation of a function between two points.

4. CONCLUSIONS

Using the modelling cycle proposed by Blum & Leiß (2006), the whole process can be described in the following way. The problem of forecasting sales given a time-series of data constitutes the initial extra-mathematical situation, that we will call the 'system' (as opposed to the 'model'). At this stage, the system considered was a 'real one' (extra-mathematical). The first step of the modelling process consists in representing the data graphically to make a first hypothesis about the tendency of the time series. This first graphical model helps to decide on the type of functional model that best fits the data, giving rise to a mathematising process aimed to decide on the parameters of the chosen concrete function by a trial and error procedure using Excel, going forth and back from the model to the system. A new question arises when different types of functions are used to fit the data and one has to decide which model is best. The search for a criterion needs to consider a new 'real system' formed by the data and the possible models, with the problematic question of how to determine the 'best fit', that is, how to mathematically model the 'fitness' of a model. This new system is in turn mathematised by the average error of the fit. Again, the insufficiencies of this new model lead to the consideration of a new enriched 'system': the one formed by the original data and the term variation of the sales. A

possible criterion is set up by considering the double modelling of the sales and the term variation of the sales. Finally, considering the derivative as a model of the term variation constitutes the last mathematisation step that leads to a final conclusion for the forecast problem.

It is important to note that, in this entire process, the successive ‘systems’ that are modelled are more and more mathematised, and that the successive ‘models’ constructed progressively integrate the previous systems, creating new problems and, thus, generating the need to go on with the modelling process. We have interpreted these successive mathematising processes using the ‘milieu and media dialectics’ introduced by Chevallard (2004), which has helped us provide a detailed analysis of the mathematising step of the modelling process, showing how being a ‘system’ to be modelled or a ‘model’ of the system is more related to the *function* assigned to a given object during the modelling process than to its very ‘nature’ (it being mathematical or extra-mathematical). The example here described shows how the development of a modelling activity, even if initiated with an extra-mathematical situation, leads to consider, not only a sequence of new models, but also new and enriched systems more and more mathematised. Hence, extra-mathematical and intra-mathematical modellings appear as strongly intertwined.

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WHAT ROLES CAN MODELLING PLAY IN MULTIDISCIPLINARY TEACHING

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This paper presents a research- and development project about mathematics in multidisciplinary teaching, running as a pilot in 2008-2009 and planned to run in full scale in 2009-2010. Its aim is to inquire how learning potentials in mathematics are realised in a number of cases of good practice and, besides, to prepare materials for such teaching. The issue of this paper is to report on potentials and drawbacks experienced so far in the project and to discuss how to avoid the major drawbacks. The discussion takes as its starting point one example of modelling from the project, which invites critical discussions in the classroom about the use of mathematical models in statistics.

NEW CHALLENGES TO THE SCHOOL SUBJECT MATHEMATICS

As one consequence of a reform in 2006 of upper secondary school in Denmark, there is a need for examples of good teaching throwing light on and demonstrating what works for the learning of mathematics in multidisciplinary contexts. Furthermore, the reform's introduction of multi disciplinarity draws attention to the role of mathematics in different types of collaborations: It is not uncommon that multidisciplinary projects involve cultural, historical or philosophical aspects which are important but not at the heart of mathematics taught in schools. To balance this tendency, there is a need for advice and ideas about how to empower the learning of what one might call 'core mathematics' within a multidisciplinary teaching context.

THE DASG – NAVIMAT COLLABORATION PROJECT

This paper presents a research- and development project, which is running as a pilot (15 teachers in 4 schools) in 2008-2009 and planned to run in full scale (about 20 classes) in 2009-2010. The project deals with mathematics in multidisciplinary teaching projects. Its aim is to inquire how learning potentials in mathematics are realised in a number of cases of good practice and, besides, to prepare materials for such teaching.

The project is conducted in collaboration between Danish Science Gymnasiums (DASG)¹ and Nat. Knowledge Centre for Math. Ed. (NAVIMAT)². DASG is a network³, incorporating about 36 Danish Upper Secondary Schools (out of 200⁴). Membership implies an obligation for the school to spend resources, in the form of teachers' working hours, on participation in at least one of the 5 – 8 sub-projects, which are formulated and announced every year. The sub-projects run for two or three years and each one involves about 25 teachers. The collaboration between DASG and NAVIMAT encompasses a two-stage project. During the first year, three different types of teaching materials will be produced and tried out in a pilot; each of

these materials represents the interplay between mathematics and one of the three participating faculties Human Sciences, Social Sciences and Natural Sciences. During the next stage, the following year, trials and evaluations of revised teaching materials from the pilot will be offered to the DASG schools as sub-projects. The revised versions of these materials hopefully will be published by NAVIMAT to provide inspiration for teachers at the conclusion of the trials. Teams of two to four teachers, a researcher in mathematics education and a professional specialist prepare the materials. The teams autonomously plan and make arrangements for their work during the first year of the project. DASG organises joint seminars for all the teams during this stage, for the exchange of ideas and experiences so far.

The professional specialist's are picked out depending of the mathematics teachers' choice of subject. The professional specialist's role in the team is to provide inspiration and expertise with regard to the content of the teaching materials. The mathematics education researcher provides inspiration and expertise with regard to the design of the materials and observes and evaluates the teaching experiment. The researcher is responsible for development and formulation of guidelines for good practice in multidisciplinary teaching. The team's teachers design and produce the teaching materials and carry out the teaching sequences. The teachers participate in the evaluation and discuss the results with the researchers.

THE POTENTIAL OF MULTI DISCIPLINARITY

Some of the potentials of multidisciplinary mathematics teaching were discussed in (Andresen and Lindenskov 2008). We see potentials achieving a number of different goals.

- i) Students' motivation and interest. Multi-disciplinary projects can stimulate the students' interest and engagement in mathematics because the usefulness of the mathematics taught, and its links with the students' own, experienced world are in constant request in Danish school. Multi-disciplinary projects set the stage for the teaching of useful applications of mathematics in authentic, daily life settings. Hence, such projects can serve to meet the students' requests and to improve their desired understanding of connections between subjects and the world outside school. This is in accordance with Michelsen, Glargaard and Dejgaard (2005 p 33) who point to an alternative approach that stresses the importance of modelling activities in an inter-disciplinary context between the two school subjects physics and mathematics. Similarly, R. Filo and M. Yarkoni (2005) reported on a project, which integrated geometry and art, aiming at inter-disciplinary learning of parallel concepts. Filo and Yarkoni's assumption in this case was that an enriched concept formation was supplied by an advanced status of both subjects in the students' minds.
- ii) Transfer. The authors report on their observations of the classroom that indicated
 - o Students' awareness of the possibilities to transfer concepts and results between subjects

- Students' consciousness about benefits, traps and misunderstandings caused by such transfer
- Students' reflections upon the relations between the project's subjects

The observations were interpreted in accordance with an interactionist's perspective like Heinrich Bauersfeld presents it in (Bauersfeld 1994 p 137-139). Hence, we looked for indications of a classroom culture where, for example, arguments from one subject (mathematics) were used and accepted in discussions within another subject (chemistry or physics) or used to convince other members of the group in discussions of problem-solving strategies etc. Besides, we evaluated signs of the students' formation of conceptions. The students seemed to build relations between the subjects in parallel with their formation of concepts and new skills belonging to the single subject.

- iii) Implementation. Multi-disciplinarity can be seen as a means to revise the role of school mathematics and, thereby, to embed students' mathematical competence into a broad and reflected view of math and science. Compared to cross (or inter)-disciplinarity or to trans-disciplinarity, multi-disciplinarity has better odds for successful implementation because it resonates with the following four main elements of Fullan and Hargreaves's (1992 p 5) framework for understanding teacher development; 1) the teachers' purpose, 2) the teachers as a person, 3) the real world context for the teacher's work and 4) the culture of teaching

Hence, we argue (Andresen and Lindenskov 2008) that multidisciplinary teaching has important potentials for improving students' motivation and interest and for an enhanced transfer between subjects. We expect multidisciplinary to be successfully implemented, and we expect it to serve as a means in the future to support the embedding of the students' competencies into broad and reflected view on mathematics.

MODELLING FOR CONCEPT FORMATION

In addition to this, the didactical potentials of a multi-disciplinary project rest on the role of mathematical modelling and reflections for concept formation. Mathematical models in multidisciplinary projects play a double role: on the one hand, the model can serve as the link between subjects and daily life, authentic problems etc., dealt with above. On the other hand, modelling plays an important role for concept formation. The role of modelling for concept formation in learning mathematics is described in the domain-specific instruction theory for realistic mathematics education, RME. (Gravemeijer and Stephan 2002 p 147ff). From this point of view, all mathematical activity concerns modelling, and it gives little meaning to try to discern theoretically between to learn, to apply or to develop new mathematics. Strict borderlines between the three are not to be drawn. In general, the use of the term 'modelling', therefore, has to be specified, since it depends on the context. (In this

paper, though, we still use the terms in the ‘common way’ sense unless stated otherwise.)

POTENTIAL DRAWBACKS AND HOW TO AVOID THEM

Teaching multi-disciplinary projects in accordance with the Danish 2006 reform, hence, is a promising prospect. We also see some potential drawbacks. In some aspects, the impact of multi-disciplinarity on the students’ view on mathematics is comparable to the impact of use of computers. The 2006-reform also imposed the introduction of compulsory use of computer algebra systems (CAS) in mathematics. Obviously, CAS has the potential for a huge extension and development of the teaching of models and technical modelling in the sense of comparing a number of models and fitting them with a set of data (Andresen 2007a p5). It also has potentials to support students’ model recognition and capability to understand and criticize authentic use of ready-made models in different contexts.

Results from our previous research, however, show that in general, the use of CAS tends to change focus of attention to the technical and practical aspects of upper secondary school mathematics. In general, teaching with a computer is centred upon solving tasks, whereas the reading of proofs and theoretical treatments in general are carried out without use of computer (Andresen 2006 p 28).

There is a potential danger that the same trend might direct the multi-disciplinary teaching into a skills based view of mathematics by the students, at the expense of giving the students a more profound insight into mathematical activities, theory and knowledge. To avoid this, I suggest that the students’ more technical and practical view on models and modelling, should be balanced by explicit reflections upon the use of models and upon the modelling process, that is, upon horizontal and vertical mathematizing.

MODELLING AND MATHEMATICAL REFLECTIONS

Reflection is the driving force for the process of mathematical modelling in the sense of progressive mathematizing (Gravemeijer and Stephan 2002 p 147 ff). Hence, Andresen and Froelund (2008) discuss how to make the students’ philosophical reflections explicit, as a tool for mathematical reasoning and, thereby, to strengthening the students’ consciousness of the art of reflection and of the relationship between reflection and learning. In line with the idea that awareness and consciousness about one’s own learning support learning outcome, Andresen and Froelund suggest the explication of mathematical reflections as a tool for learning. The use of philosophical reflections as a tool for mathematical reasoning was recently discussed (Prediger 2007). Prediger’s discussion was based on the stratification (Neubrand, 2000) of reflective practice in mathematics into four levels:

1. Questions at the level of the mathematician concern isolated, mathematical details. The questions are meant to deepen the students’ understanding of the rise from a situational to a referential model which means that a preliminary or

emergent model is to be constructed. At this level objects still are marked by the context and, for example, referred to as ‘people’, ‘number of heart attacks’ etc.

2. Questions at the level of the deliberately working mathematician concern conscious use of mathematical objects and processes. The questions set focus of attention on generalisation of entities and their relations and, thereby, on the construction of a model for the case based on the model of the contextualised problem. The same type of questions could start discussion after the rise from referential level to general level; in the actual case by introduction of several distributions etc. The later discussion could lead to the next level of questions:
3. Questions at the level of the philosopher of mathematics concern mathematical methods and applications. Rise from general to formal level tends to happen over time, sometimes in a somehow subtle way. In the actual case discussions about the range of applicability and validity of hypothesis-test methods etc. serves to support the rise and make it more explicit to the students.
4. Questions at the level of the epistemologist concern the characteristics of mathematics compared to and delineated from other sciences. These questions relate to activities at the formal level which may be widened by further reflections. In the actual case, the multidisciplinary setting itself may lead to questions and discussions of the intended type.

Andresen and Froelund (2008) argue for the teaching of mathematics based on the use of a reflection guide containing thought-provoking questions at these four levels. A short analysis of the modelling process is prerequisite for the design of a reflection guide. The aim of this analysis is to identify potential levels of mathematical activity, referring to Gravemeijer’s model which includes four levels: situational, referential, general and formal. (Gravemeijer, K. & Stephan, M. (2002). p 159)

Teaching in a multi-disciplinary setting like in the example, provides a design that particularly favours explication of mathematical reflections. The didactical potential of such multi-disciplinary teaching, though, depends on its design: the design has to ensure that the project’s modelling processes are visible to the students as well as providing the opportunity to make students’ mathematical reflections explicit during classroom discussion etc.

ONE EXAMPLE OF THE PILOT’S TOPICS: THE VIOXX CASE

The materials presented in the following example takes the Vioxx case, described below, as its starting point and concentrate on probability theory and statistics in mathematics. Preparation of the materials is still ongoing (autumn 2008), based on experiences and notes from a pre-pilot teaching experiment carried out in 2007-2008. In the pre-pilot, all the project’s lessons were spent in mathematics, although the envisioned partner subject was the school subject social science. Philosophical ethics or chemistry might also have been appropriate. The teacher with his teaching experiences referred to below are from this pre-pilot.

The VIOXX case

Vioxx was a pain-reducing drug produced by Merck, and the case was about the statistical estimation of its long-term effects. In such cases it is impossible to carry out large-scale trials to determine the serious or long-term effects of drugs such as Vioxx. Therefore, when the drug is approved, such trials may be substituted by statistical inquiry of the population of users. For such inquiries, though, statistical models suitable for large-scale trials have to be modified and in particular, the criteria for the acceptance or rejection of hypotheses must be changed. Hence, the Vioxx case served as a context for the students in mathematics to study probability value (p-value), statistical significance and confidence intervals.

Vioxx, which was withdrawn from the U.S. market in 2004, is part of the class of drugs known as nonsteroidal anti-inflammatory drugs (NSAIDs). Vioxx was used to reduce pain, inflammation and stiffness caused by osteoarthritis; to manage acute pain in adults; to treat migraines and to treat menstrual pain. Merck, the manufacturer of Vioxx, announced a voluntary withdrawal of the drug from the U. S. and worldwide market, due to safety concerns of an increased risk of cardiovascular events (including heart attack and stroke) in patients taking Vioxx.

According to the U. S. Food and Drug Administration (FDA)'s website⁵, FDA originally approved Vioxx in May 1999. The original safety database included approximately 5000 patients on Vioxx and did not show an increased risk of heart attack or stroke. A second study was primarily designed to look at the side effects of Vioxx such as stomach ulcers and bleeding and was submitted to the FDA in June 2000. The second study showed that patients taking Vioxx had fewer stomach ulcers and bleeding than patients taking naproxen, another NSAID, however, the study also showed a greater number of heart attacks in patients taking Vioxx. This second study was discussed at a February 2001 Arthritis Advisory Committee and the new safety information from this study were added to the labelling for Vioxx in April 2002. Merck then began to conduct longer-term trials to obtain more data on the risk of heart attack and stroke with long term users of Vioxx.

Merck's decision to withdraw Vioxx from the market was based on new data from this, later, trial in which Vioxx was compared to placebo (sugar-pill). The purpose of the trial was to see if Vioxx 25 mg was effective in preventing the recurrence of colon polyps. This trial was stopped early because there was an increased risk for serious cardiovascular events, such as heart attacks and strokes, first observed after 18 months of continuous treatment with Vioxx compared with placebo.

The Vioxx case attracted public attention since a large number of people had been taking Vioxx and amongst them, some had heart attacks. Heart attack victims and surviving relatives had taken legal action and were, in a number of cases, rewarded. For example, John McDarby, 77, and his wife were rewarded a \$4.5 million dollar verdict and \$9 million in punitive damages from a New Jersey jury in one of the first Vioxx trial cases against Merck⁶. The controversial question for judgement about

Merck's responsibility was to determine, whether data were sufficient to validate any hypothesis about correlation between Vioxx and the heart attacks.

Role and content of mathematics lessons

From the viewpoint of mathematics, Binomial distribution, Poisson distribution and Normal distribution were sufficiently strong tools to deal with these issues. Data from the original and from the later trials are available on Merck's website and then, the determination rests on decisions about level of significance and the confidence intervals. More profound model discussions may concern standards for comparison, compatibility and transfer of results etc.

In the pre-pilot, the teacher designed a sequence of about twenty lessons. This teacher had economy as his minor, so he agreed to spend some time and efforts on the inclusion of societal economics aspects in his teaching. The design was based on preceding discussions at a two-day seminar on authentic mathematics in upper secondary school and, subsequently, in a team with another mathematics teacher; a bio statistician and a researcher in mathematics education. This small group gathered twice during the semester where the experiment took place, for inspiration, exchange of ideas and evaluation.

The students had no prior experiences with probability or statistics. Consequently, the major part of the time was spent on the introduction and training of basic terms and relations within these branches. This introduction and training was based on the textbook with additional tasks collected from the web. In addition, the team prepared a spreadsheet for the students to experiment with distribution, confidence intervals and correlation coefficients.

In parallel, the students learned about the Vioxx case. Different aspects of the case were discussed in the class; in particular, the weighting between ethical and economical aspects and the role of mathematising in such cases were examined and debated. This part of the teaching might have taken place in the lessons on social science as well.

The challenge for the teacher was to combine the following three elements:

- i) The mathematical content: introduction and basic training of terms and relations in probability and statistics. The content was taught in line with common practice in this class, based on the same textbook.
- ii) The role of mathematics in the Vioxx case. In the Vioxx case the process of mathematising, obviously, was an issue of debate because of its implications for clients, the Merck Company etc. Thus, the case did not serve as a bare illustration of a 'neutral application' of mathematics. On the contrary, the case intended to draw attention to the modelling process itself.
- iii) A look from outside at the societal role of mathematics. Development, test and application of medical treatments are based on the use of bio statistics

and play an important role for healthcare at individual and societal levels. Though, it implies ethical and economical perspectives. Besides, public discussion of these issues may be seen as one element of democracy.

RESULTS AND OUTCOMES OF THE PRE - PILOT

The design of revised teaching materials and plans in the pilot will be based on the following summary of resulting outcomes related to the bullets i) – iii) above:

Mathematical Content: During the teaching experiment, the students showed large interest in the subject and in the Vioxx case. According to the teacher, the students were so eager to understand and to feel comfortable with the mathematical terms and relations and as a consequence, the class had to spend more time than expected on the technical-mathematical part of the course. For example, they spent six lessons just on working with level of significance. The teacher noted that this part of the sequence worked very well for the students.

Mathematical Modelling: The teacher indicated that the discussions stayed at the level of ready-made models. No attempts were made to modify the binomial distribution or to critically sort out the website's data. Modelling as such appeared not to be self-explanatory; on the contrary, every step had to be pointed out explicitly if the students were expected to be aware of it. For example, it was complicated for the students to make mathematical decisions, such as stating the level of significance. The teaching experiment, evidently, intended to demonstrate exactly that point to the students; so the stage was set to go deeper into the – complex - questions. The class spent time on changing the levels of significance and studied the consequences and effects. But they did not have time to follow these studies up.

Societal Role of mathematics: The teacher had the impression that the envisioned 'look from outside' on mathematics and its role could give input to very interesting and fruitful lessons on societal economy, law and on issues about democracy, public opinion and politics. It could be great, according to the teacher, to arrange a replay of one of the big hearings as a game with students arguing for and against. To complete this, teachers from both subjects should collaborate. The instructional materials for such a replay could be found on the various web sites but it should, preferably, be prepared in the – enlarged – team, including a teacher of social sciences.

GUIDELINES FOR THE PILOT

The big challenge for the pilot project will be to make the three elements, listed above as a coherent and convincing whole. In the pre- pilot part i), the mathematics content, was marked by its status at the introduction. In the pilot the sequence, consequently, will start only after the students' introduction to basic probability and statistics. They will then be able to concentrate on the role of mathematics and to work deliberately with the involved models and modelling. Connections, then, should be established more easily between the mathematical activities and the other elements of the Vioxx case. Such connections will be established, based on the teacher's guiding questions

and sub tasks, focusing on specific aspects of the distribution, the level of significance or the parameters influencing the basic probabilities etc. combined with special points of interest from a social science point of view.

To sum up, the teacher recommended that part i) precede the proper multidisciplinary part which should then combine parts ii) and iii). The complete project should build up to a student role play of one of the big hearings, with arguments and a final verdict in the form of a verdict.

Further, preparation of a reflection guide should be included in the design of the pilot. The reflection guide should contain thought-provoking questions, which aim to stimulate the students' mathematical reflections and put them in focus of attention. The guide should be tailored to fit with the teaching materials, not vice versa. Preparation of one example of a guide is outlined in the following – a more detailed example may be found in (Andresen 2008).

CONCLUSION

In this actual case, the reflection guide's questions to the students can give rise to reflections upon the modelling process as a whole, as well as reflections upon the single parameters and how they are related, what they stand for etc. (level one and two) and, besides, to reflections upon smaller parts of the modelling process picked out to be studied separately. So, the scene is successfully set for reflections at all four levels in Prediger/Neubrand's model. Hence, it may be concluded that even if the teacher chooses a design where the technical mathematical part of the sequence precedes the other parts, and even if the VIOXX case in itself attracts the students' attention, it is still possible to choose a design that 1) makes the mathematical content, the role of mathematics in the Vioxx case and the societal role of mathematics as a coherent and convincing whole and 2) gives the students a profound insight into mathematical activities, theory and knowledge.

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1 <http://www.emu.dk/gym/tvaers/sciencegym/english.html> (loc. 22 May 2008)

2 www.navimat.dk (in Danish, loc. 22 May 2008)

³ The Lundbeck Foundation in 2006 provided a five year sponsorship for the DASG networks activities³ and each gymnasium in DASG has applied for its membership on a voluntary basis.

4 155 general and 42 technical upper secondary schools (gymnasiums)

5 <http://www.fda.gov/cder/drug/infopage/vioxx/vioxxQA.htm> (loc. June 2008)

6 http://www.weitzlux.com/vioxx/trial_402790.html (loc. June 2008)

MODELLING IN ENVIRONMENTS WITHOUT NUMBERS – A CASE STUDY

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In order to study how students are mathematising in modelling situations, students' work on problems having no obvious mathematical character is investigated. The task design aims at preventing students from concentrating on calculations, but challenges them to get involved in social interactions, where they argue and defend their ideas. The students' approaches to these mathematisation tasks are analysed; in particular it is discussed to what extent the students work mathematically. The concept of fundamental mathematical ideas is used in order to structure the way mathematics occurs in the students' works.

Keywords: modelling, mathematising, fundamental ideas, approximation, measuring

INTRODUCTION

In mathematics education, word problems are regarded as that type of mathematical exercises where information is provided in narrative, descriptive form, rather than in terms of numbers, variables, and so on. In extension, modelling problems are word problem solving activities, which involve not only handling data or calculating, but also observing patterns, testing conjectures and estimations of results (Schoenfeld, 1992). Tightly connected with modelling is the process of mathematising, i.e. the structuring of reality by mathematical means (Freudenthal, 1991). The aim of this case study is to understand and identify how mathematising emerges while students work on certain tasks of non-obvious mathematical nature.

THEORETICAL FRAMEWORK

Mathematising and modelling

Modelling can be viewed as linking the two sides of mathematics, namely its grounding in aspects of reality - and the development of abstract formal structures (Greer, 1997). In the modelling cycle described by Maaß (2006) (originating from Blum) reality and mathematics are regarded as distinct environments, and the process of modelling includes a number of phases between and within these 'worlds'. The 'step' in which the real-world model is translated into mathematics, leading to a mathematical model of the original situation is regarded as mathematising (Kaiser, 2006).

As working definition, mathematising is denoted here as the activity or process of representing and structuring real world artefacts and/or situations by mathematical

means. The overall aim is to enable a logical, traceable and rational treatment of the given artefacts and situations with the help of mathematical knowledge and tools.

Modelling asks for certain cognitive demands, being determined by competencies like designing and applying problem solving strategies, arguing or representing, but it involves also communication skills, as well as real life knowledge (Blum and Borromeo-Ferri, 2007, Kaiser, 2006). Unlike the majority of problem situations,

modeling activities are inherently social experiences, where students work in small teams to develop a product that is explicitly shareable. Numerous questions, issues, conflicts resolutions, and revisions arise as students develop, assess, and prepare to communicate their products. (English and Doerr, 2004, p. 3)

At the same time, mathematising is part of the modelling process and it is surely not possible to define neatly a border between mathematics and reality. They are interfering and depend on the contextual situation.

The role of context is very important in mathematical modeling, since modeling requires a context in which to 'frame' the problem and 'develop' the mathematics. (Mousoulides, Sriraman and Christou 2007, p. 29)

According to Freudenthal, mathematising is the human activity consisting in organising matters from reality or mathematical matters, and “there is no mathematics without mathematising”. Later on, Treffers (1987) treated, in an educational context, the idea of two ways of mathematising, which led to a reformulation by Freudenthal in terms of 'horizontal' and 'vertical' mathematisation. In the horizontal mathematisation, mathematical tools are promoted and used to structure and solve a real-life problem, whereas vertical mathematisation supposes reorganisations and operations executed by students within mathematics. Adopting Freudenthal's (1991) formulation, mathematising horizontally means to go from the real world to the world of symbols, while mathematising vertically means to move within the symbols' world.

Maria van den Heuvel-Panhuizen studied the didactical use of models, which in Realistic Mathematics Education (RME) are

seen as representations of problem situations, which necessarily reflect essential aspects of mathematical concepts and structures that are relevant for the problem situation, but that can have various manifestations. (Maria van den Heuvel-Panhuizen, 2003, p. 13)

Modelling always involves mathematising, which is regarded as the activity of observing, structuring and interpreting the world by means of mathematical models. Since the promotion of critical thinking by students represents one of the main pedagogical aims, “reflexive discussions amongst the students within the modelling process are seen as an indispensable part of the modelling process” (Kaiser and Sriraman, 2006).

Fundamental mathematical ideas

Often, it is no question what “mathematisation” is. If students work in a modelling framework, it cannot be expected that they develop a mathematical idea themselves if they are novices. Since they do not have formed a clear picture of mathematics, it is likely to see elements of different mathematical cultures in their modelling framework: mathematics in every day life or social practice, mathematics as a toolbox for applications, mathematics in school, and mathematics as a science.

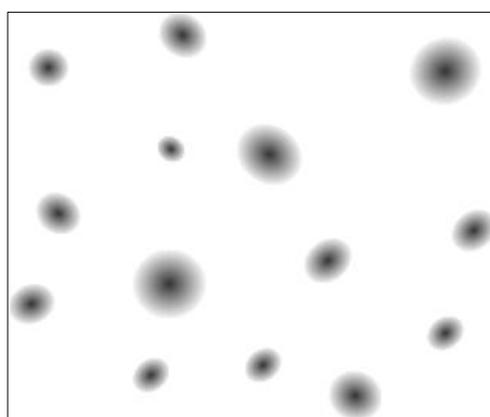
Fundamental ideas in mathematics may serve as a framework in this setting because they connect different mathematical cultures (Schweiger, 2006). Fundamental ideas recur in four dimensions: the historical development of mathematics (time dimension), in different areas of mathematics (horizontal dimension), at different levels (vertical dimension), in everyday activities (human dimension). Schweiger lists a synopsis of fundamental mathematical ideas from different sources: *algorithm, characterisation, combining, designing, exhaustion/approximation, explaining, function, geometrisation, infinity, invariance, linearisation, locating, measuring, modelling, number/counting, optimality, playing, probability, shaping*. Since modelling is discussed in detail and consists of the worked out tasks, it will not be considered in the sequel.

The main aim of this investigation is to see to what extent these fundamental ideas can be recognised in the answers to the rather open-ended Mars task (see next section). The overall pedagogical aim is to design such tasks that students are led to the consideration of fundamental mathematical ideas in a natural way.

EMPIRICAL SETTING

The task of non-obvious mathematical character that students have been given to work out is as follows:

“Imagine you are a scientist at NASA and you have a picture of the planet Mars. This picture shows different spots which indicate craters. These craters were obviously generated by impacts of several meteorites. It is possible that such an impact



generates more craters.

Fig. 1: Picture of the planet Mars depicting a crater

1. Write to a colleague a half-page report about the spots in Figure 1.
2. Describe, respectively label the spots.
3. Find out how the position of each spot could be described.
4. How would you specify the relationships between spots?
5. Could you order the spots by means of mathematical criteria? How?"

The task was given to 13-14 - aged students - in group-work in the classroom environment. Teams of three students were video-taped while working. The present study focuses only on one working group. No intervention from the teacher's side took place, unless students wanted to clarify the formulation of the task.

Data analysis

In the following excerpt, one can see a typical mathematical debate (see Figure 3).

- | | |
|------|--|
| 32 J | So, which points are farthest away from each other?... K13 and K2... |
| 33 A | K10 and K11... come, we measure them! |
| 34 J | K13 and K2 are farther away from each other... |
| 35 A | We take the middle point of the crater. |
| 36 F | This is 8... |
| 37 A | 7.5... |
| 38 J | They are both 8. |
| 39 F | Where from, do you mean? |
| 40 A | We start from the middle point. |
| 41 F | Yes, I mean... which one do you mean? |
| 42 A | K10 and K11. |
| 43 J | K2 and K13 are a bit more... |
| 44 F | Yes, 0.6cm |

The students formulated themselves a small task, generated by the idea of finding 'extreme' points. This yielded the need to measure (line 33, as verifying action), which was not really unproblematic, since the 'spots' are of irregular form. J raised then the idea of comparing, which brought student A to the decision of taking the middle point. That means implicitly that the spots were seen as circles (or even ellipse, though they most probably did not meet it so far as subject in school). The idea of considering the middle point was proposed (line 35), but apparently no attitude was taken by the other two team-colleagues. Nevertheless, the idea was somehow tacitly adopted and they measured (lines 36, 37, 38) distances between points, which involves the assumption that the middle point was taken. In line 40, student A reminded of the middle point, but again no certain remark in this sense was

made by his colleagues. However, the idea was carried out, and after some approximation trials (lines 36, 37), they obtained a very small result, namely that K2 and K13 were the searched points; thus, their initial claim was checked.

Another mathematical idea arose when mentioning 'coordinate system' in line 69.

- 67 J This is a brilliant idea!
 68 A What?
 69 J This with the coordinate system... It came from me...
 70 A It came from me!!!
 71 J So, if all the points have now to be mapped there... We can write, yes, Z-point, G-point... and then somehow one-two maps... or so
 72 A Do we now want to mark all the points?
 73 F&J No!
 74 J We do just an example. K11 is simple..
 75 F No, also K10... K10 is also simple... that is 1... ehh... 8... 1-8

The students became quite enthusiastic about the idea. They appreciated it as being 'brilliant' and two of the students were almost simultaneously claiming it. They saw this as a mathematical criterion for describing the position of the spots, but the idea offered them an expanded perspective and view, which six minutes later (see next excerpt) brought A (who had the idea, in fact) to the vision of a virtual map.

Further on students wanted to perform measurements, but it was not really clear for them how, probably because the exact corresponding geometrical figure associated to the 'spots' was not explicitly debated and agreed on.

- 93 A ... Measure the dimension...
 94 J The dimension...
 95 F The dimension...
 96 A We cannot measure that, a crater goes also up... doesn't it? ... and also down...
 97 J We cannot do that, because we have no photo.
 98 F You now want to position this somehow like this (placing the set square perpendicularly on a crater) and measure the dimension?... Now tell me shortly how should we get this?
 99 A ... in order to build a virtual map!... Measure the volume and building a virtual map.



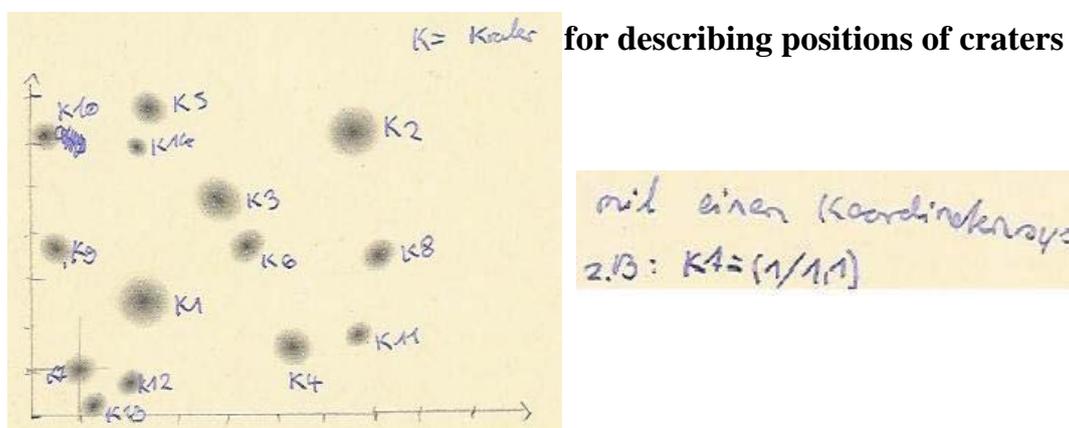
Fig. 2: Student's transfer from 2-D to 3-D

In fact, A had something indefinite in mind, following from the idea of coordinate system, an idea to construct a virtual map, which would have also allowed calculations for the volume of the crater. His colleagues showed themselves quite sceptical with respect to the idea, they faced a lack of understanding of A's mental representation, then the idea vanished, and A did not bring further arguments for sustaining it. The basis of A's idea of coordinate system seemed to be the experience he probably had with the geographical atlas or the moving of chess pieces, which is expressed in the form of horizontal coordinate and its corresponding vertical one. It might also be that A's idea of virtual map is inspired by computer games. Student A realised a mental transfer from 2-D to 3-D, unfortunately without a further development.

Findings on used mathematical ideas

As main fundamental ideas, approximation, geometrisation, locating, measuring, number/counting, and optimality were observed several times. The students *approximated* the craters with a circle. They did not state it explicitly, but this assumption was carried out during their work. The middle point was only roughly marked. *Geometric* shapes were developed, which helped students to describe the situation. These were not named explicitly, but the concepts of distance, area and volume were used by students. A coordinate system was used to *locate* the craters. Some groups described the location of the craters absolutely, some relative to a coordinate system or somehow relative to other craters in a fancy way. *Measuring* appeared several times and was discussed intensively. The idea was adopted for measuring distances between the craters, as well as the size of the craters. One of the very first actions of the groups was to *count/order* the craters. In the "relations" (see task), *optimality* occurs, e.g. as the closest/furthest distances between craters.

Some students chose to label the spots in a rather mathematical way, as seen in Figure 3. They also had the idea to give the name 'N250i' to a probe, as being the instrument used to observe the given task phenomena, i.e. the generation of the craters.



The task was formulated in such a way that it was not clear to the students what the actual aim of the task was. In fact, one might investigate the given data with different goals in mind. For example, one might want to have an estimate of the number of meteorites creating the craters (on falling apart), which leads to a clustering problem, or one might just want to have a precise map of the craters, yielding a position measuring mainly. It seems that the students interpreted the task as having this latter aim. Basically, there is no 'ideal' solution. The students had to come up with their own interpretation of the goal. The quality of their answers could be judged by the 'depth' of their analysis. The task allows an analysis on different levels of sophistication.

The theoretical framework of Freudenthal, in particular horizontal mathematisation, could be recognised in students' answers, as is apparent from the coordinates they introduced. However, vertical mathematisation, relating these coordinates, for example in a clustering procedure, did not take place. This is probably due to the fact that no specific goal was mentioned, and therefore students were not guided to mathematise in a vertical direction. Their considerations stayed on the level of description.

As characteristic of the modelling processes, the frequent moving between environments, see also (Grigoraş and Halverscheid, 2008), seemed to happen not randomly, but generated by certain 'needs' (e.g. additional data demands). During the discussions towards finding a solution for the problem in a systematic way, students posed questions and set themselves small tasks.

Besides the initial idea of naming the spots, which might not necessarily be a mathematical act, but rather seen as usual labelling (see Figure 3), some students proposed a coordinate system as idea of describing the position of the 'spots', which is a mathematising action. Further on, they started to calculate positions of several 'spots', but finally they decided to give just examples, e.g. 'spot' K10 having 1 as horizontal coordinate and 8 as vertical one (see second excerpt of the previous section). The measuring idea was also found in their talks, and students debated on it for some time, while trying to find out which spots are in extremal position. These acts count as at least two mathematisation achievements.

In this case, but also in several previous surveys carried out on tasks without numbers, it was seen that many fundamental ideas occurred as mathematisation acts. However, not all of these ideas lead to intensive mathematical modelling activities. As for the task discussed here, deeper mathematical activities were started concerning measuring and optimality. The students proceeded by taking the set square or ruler and measured the distance between the 'spots', whereas for the other mentioned fundamental ideas no mathematical activities were performed. Students also handled the approximation of spots by circles in a mathematical manner, and measured

distances between them by taking the middle points of the 'spots'. Ideas like radius, circumference, volume (even unclear whether applicable in this case) completed the mathematical 'picture' of what students built up around the 'spots'.

A finer look at the last excerpt

The situation may seem at a first sight somehow simplistic, such as students using mathematics when working out tasks without numbers. But it is interesting to examine the subtle aspects and reasons behind the usage of mathematics. This is done here with respect to the question whether and how the need of mathematising occurs.

It should be remarked that simplification in various forms (schema, drawings, etc.) is a characteristic of modelling itself. For students in their age, modelling rests on the principle of representing a situation type in a simplified, general manner which allows extended applications.

While tackling the task of finding mathematical criteria, the students approached ideas one by one like finding distances (they coped well with planning and measuring distances between extremal 'spots'), coordinate system, then finally the volume, then they stopped doing further things, since their 'tools' for calculating things were not sufficient. Mathematical concepts used by students - *distance*, *area* (implicit, through the coordinate system and middle point of a plane figure), and *volume* originated from a need of simplification of the initially given problem. Therefore the approximation of spots by circles was done, though never explicitly stated.

The idea of finding distances between extremal 'spots' was conducted through measuring, since students found something they could do. Somewhat further, they came to the idea of coordinate system, by which they were quite absorbed and dealt successfully with in the 2-D situation. Then the *dimension* was mentioned, but the students faced up to some problems with the data, that seemed not to suffice (line 97 in the transcript). Once they met this data demand, students were confronted with an unclear situation of the model, since they did not know which mathematical object would fit to that stage, where a virtual map was proposed. At that point, their debate stopped, hence no simplification was achieved. Therefore the 3-D situation failed, because of a lack of tools and/or data.

DISCUSSION

It is intriguing in this case to study how fundamental mathematical ideas occurred through mathematising. There were fundamental ideas leading to the model (biggest, smallest, extreme, measure). But how did students build a modelling idea? It seems they looked in mathematics for 'tools' which would allow them to work out a model.

When discussing the real situation, students emitted sometimes mathematical ideas, e.g. the idea of measuring, which means that a transfer to mathematics took place. There then were two possibilities: either they remained within mathematics and

performed further on, or they turned back to the real situation. Such frequent forth-back transitions are analysed in (Borromeo-Ferri, 2007). The decision (often unconsciously taken) whether to stay or not within mathematics seemed to be influenced and caused by a number of factors, as described in the following. We refer to the real situation as being *data situation*, as the existing task formulation students have at their disposal. When being situated in mathematics, there could be *tools, knowledge, experience, motivation*, among others, all of these determining whether students stayed and worked with and within mathematics, or they turned again to data and tried to handle them and searched for next steps. If one or more of these items were missing and students were facing a dilemma in performing further on with and within mathematics, then they came back to the data.

The analysis of the present study showed that while acquiring a real modelling experience, students produced many fundamental mathematical ideas, but when confronted with a lack of tools, knowledge, or even experience, their activities stopped at the level of ideas' supply.

As for a future analysis, a first hypothesis is that a modelling task develops through fundamental ideas. A second hypothesis is that mathematics is reached by means of assumptions, which students proposed and agreed on while taking decisions during solving a modelling problem. It would be interesting to examine how mathematising differs according to the mathematical nature of the task formulation.

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MODELLING ACTIVITIES WHILE DOING EXPERIMENTS TO DISCOVER THE CONCEPT OF VARIABLE

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Physical experiments have a great potential in math lessons. Students discover the aspects of the concept of variable and while doing that run through the whole modelling cycle. In this paper we show how physical experiments can contribute to the modelling activities and the concept of variable and how scientific issues influence the students' conceptions based on interviews with them.

MODELLING AND PHYSICAL EXPERIMENTS

In the PISA framework the authors emphasize the functional use of mathematics. Students should discover problems, formulate them and should then be able to solve and interpret them. While doing that different mathematical contents and competencies are activated. One of these competencies is modelling, which has a central place within the framework:

This involves structuring the field or situation to be modelled; translating reality into mathematical structures; interpreting mathematical models in terms of reality; working with a mathematical model; validating the model; reflecting, analysing and offering a critique of a model and its results; communicating about the model and its results (including the limitations of such results); and monitoring and controlling the modelling process.

Mathematics is a tool often used in real world and in Science. The role of mathematics is predominantly brought through the building, employment and assessment of mathematical models (Michelsen, 2006).

How can physical experiments contribute to modelling activities in math lessons? If you look at the different steps mentioned above, physical experiments have a great potential. The experiments are derived from a phenomenon of everyday life and represent an idealized setting, considering certain factors only. Students doing physical experiments work with concrete terms. These terms are in a functional relationship with each other. If you want to describe them in a quantitative way you have to translate that relationship into mathematical structures. All kind of representations (graph, tables, etc.) can be applied. Students have to communicate about the phenomenon and the correspondent formula. A modelled formula can be checked directly through the measuring values and by new measurements. Because of measurement errors the formula is never correct. So it is natural to talk about the correctness and the limitations of the model and its results. If one slightly changes the setting of the experiment, the formula might change. Hence there is a strong emphasis on the validation process which often plays a minor role in the modelling process.

THE CONCEPT OF VARIABLE – ASPECTS

Malle (1986) differentiates three different aspects of variables:

- Variable as an object (Gegenstandsaspekt)
A variable stands for an unknown item or an unknown object.
- Placeholder aspect (Einsetzaspekt)
A variable means a placeholder, which you can substitute through a number.
- Calculational aspect (Kalkülaspekt)
A variable stands for a meaningless symbol, with which you can apply certain rules.

He differentiates variable as an object into single number aspect and interval aspect. Single number aspect means an arbitrary but fixed number within a given domain. Variables which match to the interval aspect represent the whole domain. Within that interval aspect it can be differentiated between simultaneous aspect (representation at the same time) or changing aspect (representation in succession).

On the other hand variable as an object can be classified by a dynamic and a static component. Dynamic component means a changing number and static component means a specific unknown, i.e. it might change in another context.

If you compare the decomposition of the concept of variable according to Trigueros et. al (1996) into generalized number (representing a general entity, which can assume any value), specific constant (representing a constant value, which might change in another situation) and variable in a functional relationship, generalized number can be attributed to variable as an object, which can be represented at the same time or in succession. Specific constant is equivalent to the static component. To conceptualize variables in a functional relationship, knowledge of dynamic and static components is needed.

Malle demands among other things that in the beginning emphasis should be put to variable as an object and to the conception and interpretation of formulas.

THE CONCEPT OF VARIABLE AND PHYSICAL EXPERIMENTS

Michelsen (2006) proposes that by expanding the domain, mathematical concepts can be developed in a more practical and coherent structure, since

the student's conceptions of a mathematical concept is determined by the set of specific domains in which that concept has been introduced for the student.

If students do physical experiments they can identify variables with concrete terms. That's why these variables can be classified to the aspect variable as an object. Students' major problem seeing variables as symbols to be manipulated (Schoenfeld & Arcavi, 1988) can therefore be diminished. Both dynamic and static components

are touched since the values of the measurands change with each new measurement and the (anti-)proportional constant is constant in the same context. The possible values of the measurands determine the domain of the corresponding variable. The (anti-)proportional constants mostly are representatives of a discrete set.

By experimenting, students can discover the aspects of the concept of variable *before* they are properly defined in class. This is in accordance to Freudenthal's philosophy that context problems and real life problems are used to constitute and apply mathematical concepts. The aspects don't have to be touched in the abstract level at once. If they are touched on a descriptive level that can be enough. One example is the functional relationship of two measurands. While doing the experiment, students actively discover that change of one measurand causes a change of the other measurand. Especially, weaker students have problems to interpret this into a formula. But if you present a formula and give further explanations after the experiments the formula will not seem that abstract anymore because they can identify the formula with their experiences made while doing the experiment.

PHYSICAL EXPERIMENTS IN MATHEMATICS LESSON

In the above sections a few advantages and commonalities have been shown. But there are also subject specific characteristics, which have to be taken into consideration. In physics math is mostly seen as a tool for describing phenomena in a quantitative way. On the other hand mathematicians don't care how data was gained in detail; their only interest is the correctness of that data. Algebra is a correct theory. Experiments are never exact, because measurement errors always occur, even if they are very small. If you want to find relationships those measurement errors have to be kept in mind. School physics shows that all the time; school math only in a few fields. Therefore students have to be prepared to handle measurement errors.

If one wants to use experiments for mathematical concepts, emphasis should be given to the common and mathematical aspects. That means

- Experiments should have an easy setting

Mathematics' interest is data and not how to get data. Therefore the experiment should be done with few materials and measured quickly allowing students to concentrate more on the math.

- Intervals of measurement errors should be small

To find the relationship between the measurands quickly, there should be (if possible) no chances for systematic measurement errors and small intervals for random ones.

- The physical terms should be familiar to the students

This doesn't mean that physical terms not covered in physics class are forbidden. It is legitimate to use terms which are familiar in every-day life, like pressure, volume, temperature and so on.

- The interval of the measurands should be suitable

Especially in experiments which contain an antiproportional relationship intervals should be chosen where the constant product stands out. Otherwise students may see (with consideration of measurement errors) a proportional relationship.

Physical experiments can be used within interdisciplinary lessons. This can be in separate classes, i.e. each class covers subject specific aspects; or for a short period in a common class in which all aspects are covered. On overview of different forms of cooperation can be found in Beckmann (2003, p. 9ff).

CONCEPTUALISATION IN SCHOOL

The use of experiments to introduce the concept of variable has been tested on 90 students of 7th grade in three different schools. They were required to do three out of five physical experiments. After the experiments, the concepts of variable and term were introduced formally. This was done to see if physical experiments can be applied in class and which experiments are appropriate. To get a deeper insight of the concept of variable and of reflection and validation of their modelled formula, another examination was done in spring 2008. 18 Students of 6th grade attending a German Gymnasium were required to do one experiment out of three working in groups of two. These 18 students knew the placeholder aspect, i.e. variables can be substituted by numbers, and that variables stand for a number which is unknown and changes continuously. Theoretical knowledge of the concept of variable concerning the object aspect hasn't taught yet. While doing the experiments, they were observed by students of the University of Education Schwaebisch Gmuend. After the experiments the 6th graders were interviewed by the students. The main research questions covered the aspects of the concept of variable touched by the experiment and of how convinced the students were of the formula found. The second question is to determine students' abilities to reflect and validate their results. Problem oriented interviews were chosen, so students could talk freely and were only slightly guided by the interviewers through open questions. The interviews were transcribed. Emotional factors like emphasizing words etc. were not considered during the process of transcription. Students' answers were categorized in the different aspects of variable and how they reflected the validity of their modelled formula.

The following experiments were done by the students:

- Buoyancy

The students measure the force of different masses in air and in water and conceptualize a formula which describes a proportional relationship between the forces in air and in water.

- Thermal expansion of a liquid

The students measure the heights of an uncalibrated thermometer at different temperatures. Then they conceptualize a formula which describes a proportional relationship between difference of heights and difference of temperatures.

- Law of Boyle-Mariotte

The students measure the pressure as well as position or volume of a piston. Then they conceptualize a formula which describes an antiproportional relationship between pressure respective to position or volume.

The design of the instructional sheets allows students to work by themselves. Assistance is only given, if students are at a loss and if tasks are essential for the following tasks. In that case hints were given and written down for consideration of students' results. No solution of tasks was given to the students.

The instructional sheets start with an impulse from real life. It shall motivate the students towards the experiment and shall put the experiment in a real setting. Through measuring different measurands students shall qualitatively experience the functional relationship of the two measurands. After measuring at least six different values, students are asked to describe the relationship first in their own words and then through a formula. This formula shall then be used to calculate measurands. These values shall be checked by looking at the values they measured before. This is to reflect their formula found. After that there follow questions concerning the domain of the variables and their properties. To touch the specific constant and change of formula in different contexts they were asked how the formula changes if one alters the setting of the experiment followed by a question for a more general formula. In the three classes students had to write a protocol containing the most important aspects. The 6th graders didn't have to write a protocol since they were interviewed after the experiment.

RESULTS

Concept of variable

Variable as an object according to Malle is touched. Students can identify the measurands with their chosen variables. A few examples:

Buoyancy experiment:

I2: Can you tell how you recognize (the experiment in your formula)?

S6: yes, you see the statement for air and for water. And yes the result, yes...

Here the group chose word variables. If they didn't choose words they chose the units of the measurands.

Boyle-Mariotte experiment

I1: what are those cm? What do they stand for?

S1: mmh here at that strip for example 6cm

I1: mmhmm

S1: so for the respective number

I1: and the x?

S1: for the respective pressure

Here the student chose the units of the physical terms as the name of his variables. Since he didn't know the unit of pressure, he chose x.

The functional relationship between the two variables has been recognized by the students both statically and dynamically.

Buoyancy experiment

S4: Then we agreed that if you divide air by water, the result is always the same. It doesn't matter, if there are 1, 2,3,4,5 cylinders. The result is always 1.2.

Boyle-Mariotte experiment

I1: What have you found out?

S1: yes, that device. If you turn further that thing moves forward and the further it moves the measuring number gets smaller and the pressure gets higher.

Thermal expansion experiment

S17: We had to find formulas. These were height times x is difference of temperature and difference of temperature divided by x is then height and difference of temperature divided by height is then x.

[...]

I8: and what changes in general in your formula?

S17: temperature and the head of liquid there, both get higher the more water you add.

Modelling process

Students went through the first part of the modelling cycle by examining the phenomenon and structuring it in a formula. That has been done on different levels. Weaker students could only explain in their own words and the strongest students have even presented three equivalent formulas.

To check how they reflected their formula students were asked if their findings are valid, since their measured values and the corresponding quotients/products weren't constant. They differed in small intervals due to random measurement errors. Before the experiments began, the instructor told the students that one could never measure exactly and that they had to keep that in mind. That is not easy as the following example shows:

Buoyancy experiment

S6: In the beginning I thought that I had to take the numbers which we had measured and then I thought for a longer time, if that was right.

After they accepted the influence of measurement errors, they rounded the quotients and then all but one were constant. Then they were convinced about the constant quotient.

I2: Did you notice anything about your result? In the case of normal water and air?

S5: Yes, the result was always 1.1; always the same.

They are convinced of the correctness of their formula because they have actively experienced that their results weren't always correct, but close to the "correct" answer.

Boyle-Mariotte experiment

S2: they aren't that correct.

I1: But the formula, that you have written down, is exact, isn't it?

S2: Well not that exact. It is... It could be also 7.1 instead of seven.

I1: Would you say your relationship is valid or your relationship is wrong?

S2: I would say, the relationship is valid, because with this device you can't determine that number that exactly. And the numbers I have written down, are actually as exact as possibly can be done with this device.

The use of experiments stimulates one to critically review the results and actively discuss the validity of the formulas found. Some students tend to extrapolate their formula after measuring a few values.

Buoyancy experiment

S3: (constant quotient) It is actually with all numbers! With six it is the same.

S4: That I don't know. You can't say ... You don't know, what is with six. We haven't done that.

S3: Yes, but with 1 and 2 it is same, too.

But student 3 would not be convinced anymore, when one measures other values.

S3: and if we were to repeat that and would get other results, then we would be in a fix and wouldn't know what would be right.

But the more students measure the more convinced they are about their formula.

Thermal expansion experiment:

I8: You have found a formula, if a teacher comes to you [...] and says your formula is wrong, would you say your formula is wrong or your formula is right?

S17: Yes, I think it is right, because of the different experiments we have done. Well with the different degrees and with the table at the beginning. We have measured the head of liquid and the difference of temperature six times and that was true all the time.

Hence physical experiments stimulate discussion about the formula. Reflection and validation of their formula is promoted.

Static component of variable & limit of modelled formula

If you ask students about the specific constant, you implicitly ask about the limit of their formula found. The (anti-)proportional constant is only constant in the same context. Changes of the context might cause a change of those constants. In the experiments, students were asked if the formula changes when you change the setting. In the buoyancy experiment, they were shown a man reading newspaper in the Dead Sea and asked if and how their formula would change, if they did the same experiment with salt water. Students doing the thermal expansion experiment were asked, if they were to change the thermometer, would that cause a change of their formula. In the Boyle-Mariotte experiment, students were asked if changes in the environment would cause changes of the formula.

Most of the students say that the formula changes and explain it on a descriptive level. Stronger students can tell which part changes while the strongest students set up a general formula.

Thermal expansion experiment

S10: We have found out that, if the glass tube is thicker, then it raises slower and if the glass tube is thinner, the liquid raises faster.

Buoyancy experiment

I3: Good. Is there a term, which doesn't change? Or changes everything?

S3: I think, if you stay in normal water, then it is always 1.2.

S4: Well, once you add a liquid, it will be heavier

S3: Yes, salt water or – then

S4: is, I think, heavier.

S3: Then 1.2 will be

S4: bigger.

Boyle-Mariotte experiment

I1: If I change anything on this device, how would your position and pressure change?

S1: mmhmm. Well, I think. Well, position will be the same but pressure will change.

Thermal expansion experiment

S17: If you change the glass tube, that means making them wider or yes thicker or thinner, then the constant changes. Otherwise it stays constant with the same glass tube.

I8: How does it change if you have a wider or thinner ...?

S17: There it changes, well with a thinner, when it gets thinner, then the constant will get higher and when it gets thicker, then it will get lower. [...]

That x is the constant, well, in our experiment it was six and it can change when the glass tube gets thicker or thinner.

As you have seen, questions about the specific constant have a great potential for discussion about the limits of a given model.

CONCLUSION

The use of physical experiments to introduce the concept of variable is as well a good way to promote the modelling process.

All of the aspects of variable as an object according to Malle are touched, especially within the functional relationship of two measurands. Formulas make sense to them, because they can identify variables with concrete objects. Not everybody touches the aspects on an abstract level but most do on a descriptive level. In the lessons afterwards those students will have fewer problems to understand abstract formulas because they can make connections to those experiments.

Like Maass (2006) found out, that students of lower secondary level were able to develop modelling competencies. Physical experiments can contribute to those competencies since the complete modelling cycle is covered. Especially “reflecting, analysing and offering a critique of a model and its results” has a main role in that concept. That is mainly through the appearance of measurement errors. Students learn that the modelled formula is an idealization, but still a good representation of the

phenomenon. They can discover the limit of the formula found by scenarios they can imagine. Experimenting in groups stimulates the discussion about the model.

Students are motivated to do experiments, but finding a formula is a cognitive challenge. That's why students might get frustrated. A working sheet covering all aspects on the concept of variable and modelling on a descriptive level, i.e. without students coming up with a formula by themselves, would be better in cognitively weaker classes. If one stays on the descriptive level major phases of the modelling process are still touched. Then emphasis goes even more to analysing and criticizing the model.

This sequence is also a good basis for interdisciplinary teaching to see the same phenomenon with "subject driven" eyes. An overview gives the framework "Math and Science under one roof" which can be found on the homepage of the EU ScienceMath Project <http://www.sciencemath.ph-gmuend.de>.

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MODELING WITH TECHNOLOGY IN ELEMENTARY CLASSROOMS

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In this study we report on an analysis of the mathematical developments of twenty two 11 year old students as they worked on a complex environmental modeling problem. The activity required students to analyse a real-world situation based on the water shortage problem in Cyprus using Google Earth and spreadsheet software, to pose and test conjectures, to compare alternatives, and to construct models that are generalizable and re-usable. Results provide evidence that students successfully used the available tools in constructing models for solving the environmental problem. Students' mathematical developments included creating models for selecting the best place to supply Cyprus with water, finding and relating variant and invariant measures such as tanker capacity, oil consumption, and water price. Finally, implications for further research are discussed.

Keywords: Modeling, technological tools, environmental modeling problem.

INTRODUCTION

The importance of modeling and applications has been well documented and a significant number of researchers discussed the impact of modeling in the teaching and learning of mathematics (Pollak, 1970; Blum & Niss, 1991; Lesh & Doerr, 2003). Additionally, professional organizations, like the National Council of Teachers Mathematics (NCTM, 2000), recommended that the inclusion of real world based problems in the curriculum can capture students' interest and students will gain mathematical problem solving skills, as well as an appreciation of the power of mathematics and some essential mathematical concepts and skills (NCTM, 2000).

Students, even at the elementary school level, need to be able to successfully work with complex systems that daily appear to the mass media (English, 2006). More than ever before, the nature of the mathematical problem-solving experiences has to be changed, if we want to prepare students to adequately deal with the complexity of the rapidly changing world (English, 2006; Lesh & Zawojewski, 2007). Traditional forms of problem solving constrain opportunities for students to explore complex, messy, real-world data and to generate their own constructs and processes for solving authentic problems (Kaiser & Sriraman, 2006). In contrast, mathematical modeling provides rich opportunities for students to experience complex data within challenging, yet meaningful contexts. Students' interactions within these experiences can assist them in building mathematical understandings and in developing their problem solving skills (Mousoulides & English, 2008).

In this attempt, given the potential value of technology for enhancing learning, it is imperative that students undertake realistic modeling problems and appropriately use technological tools for developing their ideas about and their understandings of related mathematical concepts (Mousoulides, Sriraman, & Lesh, 2008; Mousoulides, 2007). Although the increased interest on modeling and applications, even at the elementary school level, only a limited number of researchers focused their agendas on investigating the role of technology in mathematical modeling, on exploring how spreadsheets are used in constructing models (Blomhøj, 1993; Mousoulides et al., 2008), and on identifying how dynamic geometry software features might influence the modeling process (Christou et al., 2005).

This paper reports on the mathematical developments of one class of eleven year old students, as they worked on an environmental modeling problem that involved interpreting a real world situation and dealing with digital maps, tracing ship routes, working with tables of data, exploring relationships among data, and representing findings in visual and written forms. We were particularly interested in exploring the ways in which the students used the available tools (Google Earth and spreadsheets) in constructing the necessary mathematical developments for solving the problem.

MATHEMATICAL MODELING AND TECHNOLOGY IN THE ELEMENTARY SCHOOL

Mathematical models and modeling have been defined variously in the literature (e.g., Greer, 1997; Lesh & Doerr, 2003). In this paper, models are defined as “systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behavior of some other familiar system” (Doerr & English, 2003, p.112). A definition of modeling, as a problem solving approach, is presented in Lesh and Zawojewski (2007): “A task, or goal-directed activity, becomes a problem (or problematic) when the “problem solver” (which may be a collaborating group of specialists) needs to develop a more productive way of thinking about the given situation” (p. 782).

Research studies have shown that mathematical modeling can be considered as an effective medium to improve students’ problem solving abilities in working with unfamiliar complex real world situations by thinking flexibly and creatively (Haines, Galbraith, Blum, & Khan, 2007; English, 2006). One approach to having students solve complex problems is through team oriented activities, called model eliciting activities (MEAs). These activities are based upon the models and modeling perspective (Lesh & Doerr, 2003), and they are designed to document students’ thinking. MEAs, therefore, provide an ideal setting to assess the knowledge and the abilities that students express during the modeling process (Lesh & Doerr, 2003). MEAs usually consist of three sessions. The first session provides the problem statement and introduces students to the modeling activity. Students define for themselves the problem, assess the problem situation and create a plan of action to

successfully solve the problem. During the problem solving session of the modeling problem students work in small groups and go through multiple iterations of testing and revising their solution(s) to ensure that their solution(s) is the best possible for the problem situation. In the third session of the modeling activity each group of students present their solution(s) to the rest of the class for constructive feedback and discussion of the mathematical ideas presented in the modeling activity (Mousoulides, 2007; Lesh & Doerr, 2003).

Modeling activities, set within authentic contexts, engage students in mathematical thinking that extends beyond the traditional curriculum, as they embed the important mathematical processes within the problem context and students elicit these as they work the problem (English, 2006). Problems presented in modeling activities are not carefully mathematized for the students, and therefore students have to unmask the mathematics by mapping the problem information in such a way as to produce an answer using familiar quantities and basic operations (English, 2006). The problems necessitate the use of important, yet underrepresented in traditional mathematical curriculum, mathematical processes such as constructing, describing, explaining, predicting, and representing, together with quantifying, coordinating, and organizing data (Mousoulides, 2007). Key mathematical ideas that appear in the modeling problems can be accessed at different levels of sophistication and therefore all students through questions, revisions and communication can have access to the important modeling and mathematical content. This can result in improving competencies in using mathematics to solve problems beyond the classroom (English, 2006; Kaiser & Sriraman, 2006; Mousoulides et al., 2008).

Recent research studies focusing on mathematical modeling at the elementary school level indicated that students can build on their existing knowledge and develop their mathematical ideas and modeling competencies that they would not meet in the traditional school curriculum (English, 2006; Mousoulides & English, 2008). Students' informal knowledge and ideas assist students in understanding the problem presented in the modeling activity, in identifying variables and constraints, and in building mathematical models for solving the modeling problem (Mousoulides, 2007). The framework of modeling activities does not narrow students' work in only performing calculations or working with ready made models; on the contrary, students need to construct models in a meaningful way for solving a real problem and this approach can lead to conceptual understanding and mathematization (Greer, 1997; Mousoulides et al., 2008; Mousoulides & English, 2008). Conceptual understanding was also reported as students worked in modeling activities in exploring quantitative relationships and in comparing varying rates of change (Doerr & English, 2003), in probabilistic reasoning (English, 2006), and in geometric reasoning and spatial abilities (Mousoulides et al., 2006).

The availability of technological tools is one factor that might influence students' work and outcomes in working with modeling activities (Mousoulides et al., 2006). Recent research studies indicate that appropriate use of technological tools can

enhance students' work and therefore result in better models and solutions. In Blomhøj's (1993) research, students successfully used a specially designed spreadsheet for setting models and for expressing relations between variables in spreadsheet notation. More recently, Mousoulides (2007) reported that school and undergraduate students successfully used spreadsheets in developing simple and more complex models for connecting the real world problem with the mathematical world. The contribution of technological tools in modeling problems was also examined in the areas of geometry and spatial geometry. Christou and colleagues (2005) reported that students, using a dynamic geometry package, modelled and mathematized a real world problem, and utilized the dragging features of the software for verifying and documenting their results. In line with previous findings, Mousoulides and colleagues (2007) reported that students' work with a spatial geometry software broadened students' explorations and visualization skills through the process of constructing visual images and these explorations assisted students in reaching models and solutions that they could not probably do without using the software. As a concluding point, it is important to underline that the inclusion of appropriate software in modeling activities can provide a pathway in better understanding how students approach a real world problem and how they might develop technology-based solutions for these problems.

THE PRESENT STUDY

Participants and Procedures

One class of 22 eleven year olds and their teacher worked on an environmental modeling problem as part of a longitudinal study, which focuses on exploring students' development of models and processes in working with modeling problems. The students are from a public K-6 elementary school in the urban area of a major city in Cyprus. The students only met such modeling problems before during their participation in the current project, as the mathematics curriculum in Cyprus rarely includes any modelling activities. Students were quite familiar in working in groups for solving more complex problems than those appear in their mathematics textbooks. However, this was the first time students had the opportunity to work with spreadsheets and Google Earth for solving a real world modeling problem.

The data reported here are drawn from the problem activities the students completed during the first year of the project. The *Water Shortage* modeling problem (appears in the appendix) entails: (a) a warm-up task comprising a mathematically rich "newspaper article" designed to familiarize the students with the context of the modeling activity, (b) "readiness" questions to be answered about the article, and (c) the problem to be solved, including the tables of data (see Table 1). This environmental modeling problem presented in the activity asked from students to help the local authorities in finding the best country for supplying Cyprus with water. Water shortage is one of the biggest problems Cyprus face these days. As a result,

students were very familiar with the problem, since almost everyday there are discussions on TV about the possible solutions to the problem.

Table 1: The Water Shortage Problem Data

Country	Water Supply per week (metric tons)	Water Price (metric ton)	Tanker Capacity	Oil cost per 100 km	Port Facilities for Tankers
Egypt	3 000 000	€3.50	30 000	€20 000	Good
Greece	4 000 000	€2.00	50 000	€25 000	Very Good
Lebanon	2 000 000	€4.00	50 000	€25 000	Average

The problem was implemented by the authors and the classroom teacher. Working in groups of three to four, the children spent five 40-minute sessions on the activity. During the first two sessions the children worked on the newspaper article and the readiness questions and familiarize themselves with the Google Earth and spreadsheet software. Introduction to Google Earth focused on the following commands: “Fly to” for visiting a place, “Add Placemark” and “Ruler” for calculating the distance between two points, and “Path” for drawing a path between two points. In contrast to regular maps, Google Earth can help students in making accurate calculations, being more precise in drawing the tanker routes, in “visiting” the different countries for exploring their major ports and finally in observing country’s landscape. In the next three sessions the children developed their models, wrote letters to local authorities, explaining and documenting their models/solutions, and presented their work to the class for questioning and constructive feedback. A class discussion followed that focused on the key mathematical ideas and relationships students had generated.

Data Sources and Analysis

The data sources were collected through audio- and video-tapes of the students’ responses to the modeling activity, together with the Google Earth and spreadsheet files, student worksheets and researchers’ field notes. Data were analysed using interpretative techniques (Miles & Huberman, 1994) to identify developments in the model creations with respect to the ways in which the students: (a) interpreted and understood the problem, (b) used and interacted with the software capabilities and features in solving the environmental problem, and (c) selected and categorized the data sets, used digital maps and applied mathematical operations in transforming data. In the next section we summarize the model creations of the student groups in solving the *Water Shortage* activity.

RESULTS AND DISCUSSION

Group A Model Creations

Group A started their exploration by visiting Lebanon, a nearby country, using the “Fly to” command. This approach helped students in identifying that there were many mountains and therefore Lebanon could supply Cyprus with water. In their final report, students documented that: “Lebanon has a high percentage of precipitation, because there are many mountains there. So, they will probably sell water to Cyprus”. They then “zoom in” for finding a port. They decided that Tripoli was a major port and their next step was to add a placemark to Tripoli. Students then “zoom out” from Lebanon and gradually moved to the west for finding Cyprus. Students in group A directly focused on Limassol, the major port in Cyprus and added a second placemark. Group A then used the “ruler” feature of the software for calculating the distance between Tripoli and Limassol.

Students followed the same approach for placing placemarks in Pireus (in Greece) and Cairo (Egypt), and for finding the distances between Cyprus and the other three countries. Since the data table (see Appendix) was supplied in spreadsheet software, students added one column presenting the distances between the three different countries and Cyprus. Students explicitly discussed about oil price, and they reached the conclusion that buying water from Greece would be more expensive than buying water from Lebanon or Egypt due to the greater distance between Greece and Cyprus. Students, however, failed to successfully use the provided data and they finally based their choice (Lebanon) partly on the provided data and on their calculations, without providing a coherent model.

Group B Model Creations

Similar to the work of Group A, students in this group quite easily visited the three countries and added placemarks in their major ports. They drew precise paths between each country’s port and Limassol and used ruler to calculate the distances (see Figure 1). They reported that: “It is not easy to decide from which country Cyprus should buy water. Lebanon for example is closer than Greece, but water from Greece is much cheaper than water from Lebanon. After calculating the distances between the countries using Google Earth, they moved into the spreadsheet software and added one column in the provided table, presenting the distances. They, however, failed to incorporate into their model the provided data about oil cost, tanker capacity and water price.

Group C Model Creations

This group commenced the problem by finding a major port in each one of the three countries and by drawing paths from these ports to Limassol. Students in this group then calculated the distances between the ports and continued in calculating oil and

water cost for *each* tanker trip. In contrast to Groups A and B, students in this group incorporated within their model one more factor; instead of calculating the total cost for each trip and then ranking the three countries, they decided to calculate the cost

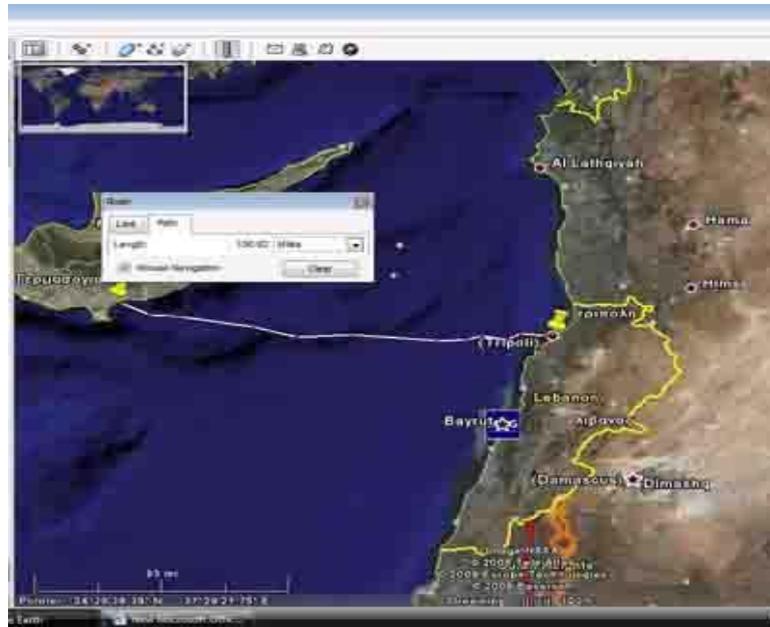


Figure 1: Finding the distance between Tripoli and Limassol.

per water metric ton and based their ranking on this factor. As a result, this model ranked Lebanon as the best possible choice, since the average cost per water ton was only €4.20. On the contrary, the average costs for Egypt and Greece were €6.70 and €7.00 respectively. Student calculations and final selection are presented in Table 2.

Table 2: Group C calculations and final model

Country	Distance	Oil cost	Water cost per tanker	Total cost	Average water cost per ton
Egypt	480	€96000	€105000	€201000	€6.70
Greece	1100	€275000	€75000	€350000	€7.00
Lebanon	240	€60000	€150000	€210000	€4.20

Although this group differed from other groups in that they used a more refined model, they also failed to apply in their model factors such as port facilities for tankers and each country's resources for supplying water to Cyprus. Students in this group, similar to group A and B did not use in their calculations round trips but they rather based their calculations on single trips.

Remaining Groups' Model Creations

Students in the remaining four groups faced a number of difficulties in ranking the different countries. In the first component of the problem, using Google Earth for finding appropriate ports and calculating the distances between Cyprus and the three countries, two groups focused their efforts only on Greece, by finding the distance between Pireus and Limassol. Some other groups faced a number of difficulties in using the software itself.

In the second component of the problem, transferring the distance measurements in the spreadsheet software and calculating the different costs, the students faced more difficulties. Most of their approaches to problem solution were not successful. Many students, for example, just made random calculations, using partially the provided data, and finally making a number of data misinterpretations. One group, for example reported that buying water from Greece is the best solution, since the water price per ton from Greece was only €2.00 (see Table 1).

CONCLUDING POINTS

There are a number of aspects of this study that have particular significance for the use of modeling in mathematical problem solving in elementary school mathematics. First, although a number of students in the present study experienced some difficulties in solving the problem, elementary school students can successfully participate and satisfactorily solve complex environmental modeling problems when presented as meaningful, real-world case studies. Second, our findings show that the available software broadened students' explorations and visualization skills through the process of constructing visual images to analyze the problem, and by using appropriately the spreadsheet's formulas they performed quite complex calculations.

The students' models varied in the number of problem factors they took into consideration. Interestingly, at least three groups succeeded in identifying dependent and independent variables for inclusion in an algebraic model and in representing elements mathematically so formulae can be applied. A number of groups of students made the relevant assumptions for simplifying the problem and ranking the three countries. Finally, the first three groups (as presented in the results session) successfully chose the technological tools/mathematical tables to make precise graphical models in Google Earth and to enable calculations in spreadsheets.

Substantial more research is clearly needed in the design and implementation of technology-based modeling problems and in studying the learning generated. Of interest are, for example, the developments in elementary school students' learning in solving technology-based modeling problems, the ways in which the features of the technological tools can assist students in broadening their explorations and in constructing better models for solving modeling problems, and the teacher professional development training programs that are needed to facilitate mathematical

modeling as a problem solving. In concluding, using computer based learning environments for mathematical modeling, at the school level, are a seductive notion in mathematics education. However, further research towards the investigation of their role is needed, to promote both students' conceptual understandings and mathematical developments.

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APPENDIX

Water Shortage Problem: Cyprus will buy Water from Nearby Countries

Background Information: One of the biggest problems that Cyprus face nowadays is the water shortage problem. Instead of constructing new desalination plants, local authorities decided to use oil tankers for importing water from other countries. Lebanon, Greece and Egypt expressed their willingness to supply Cyprus with water. Local authorities have received information about the water price, how much water they can supply Cyprus with during summer, tanker oil cost and the port facilities.

Problem: The local authorities need to decide from which country Cyprus will import water for the next summer period. Using the information provided, assist the local authorities in making the best possible choice. Write a letter explaining the method you used to make your decision so that they can use your method for selecting the best available option (The following **table** was supplied).

Country	Water Supply per week (metric tons)	Water Price (metric ton)	Tanker Capacity	Oil cost per 100 km	Port Facilities for Tankers
Egypt	3 000 000	€3.50	30 000	€20000	Good
Greece	4 000 000	€2.00	50 000	€25000	Very Good
Lebanon	2 000 000	€4.00	50 000	€25000	Average

CERME 6 – WORKING GROUP 12

ADVANCED MATHEMATICAL THINKING

TABLE OF CONTENTS

Introduction.....	2238
<i>Roza Leikin, Claire Cazes, Joanna Mamona-Dawns, Paul Vanderlind</i>	
A theoretical model for visual-spatial thinking.....	2246
<i>Conceição Costa, José Manuel Matos, Jaime Carvalho e Silva</i>	
Secondary-tertiary transition and students' difficulties: the example of duality	2256
<i>Martine De Vleeschouwer</i>	
Learning advanced mathematical concepts: the concept of limit	2266
<i>António Domingos</i>	
Conceptual change and connections in analysis	2276
<i>Kristina Juter</i>	
Using the onto-semiotic approach to identify and analyze mathematical meaning in a multivariate context.....	2286
<i>Mariana Montiel, Miguel R. Wilhelmi, Draga Vidakovic, Iwan Elstak</i>	
Derivatives and applications; development of one student's understanding	2296
<i>Gerrit Roorda, Pauline Vos, Martin Goedhart</i>	
Finding the shortest path on a spherical surface: "academics" and "reactors" in a mathematics dialogue.....	2306
<i>Maria Kaisari, Tasos Patronis</i>	
Number theory in the national compulsory examination at the end of the French secondary level: between organising and operative dimensions.....	2316
<i>Véronique Battie</i>	
Defining, proving and modelling: a background for the advanced mathematical thinking.....	2326
<i>Mercedes García, Victoria Sánchez, Isabel Escudero</i>	
Necessary realignments from mental argumentation to proof presentation	2336
<i>Joanna Mamona-Downs, Martin Downs</i>	
An introduction to defining processes	2346
<i>Cécile Ouvrier-Bufferet</i>	

Problem posing by novice and experts: comparison between students and teachers	2356
<i>Cristian Voica, Ildikó Pelczer</i>	
Advanced mathematical knowledge: how is it used in teaching?.....	2366
<i>Rina Zazkis, Roza Leikin</i>	
Urging calculus students to be active learners: what works and what doesn't.....	2376
<i>Buma Abramovitz, Miryam Berezina, Boris Koichu, Ludmila Shvartsman</i>	
From numbers to limits: situations as a way to a process of abstraction	2386
<i>Isabelle Bloch, Imène Ghedamsi</i>	
From historical analysis to classroom work: function variation and long-term development of functional thinking	2396
<i>Renaud Chorlay</i>	
Experimental and mathematical control in mathematics	2406
<i>Nicolas Giroud</i>	
Introduction of the notions of limit and derivative of a function at a point.....	2416
<i>Ján Gunčaga</i>	
Factors influencing teacher's design of assessment material at tertiary level	2426
<i>Marie-Pierre Lebaud</i>	
Design of a system of teaching elements of group theory	2436
<i>Ildar Safuanov</i>	

INTRODUCTION

ADVANCED MATHEMATICAL THINKING

Reflection on the work at the conference

Roza Leikin, *Israel*, Claire Cazes, *France*,
Joanna Mamona-Dawns, *Greece*, Paul Vanderlind, *Sweden*

AGENDA

In 1988 D. Tall argued that "Advanced Mathematical Thinking" (AMT) can be interpreted in at least two distinct ways as *thinking related to advanced mathematics*, or as *advanced forms of mathematical thinking*. Following this distinction, we suggested to the participants to take part in the discussion in two interrelated perspectives:

According to *mathematically-centered perspective* we planned to consider AM-T as being related to mathematical content and concepts at the following levels: upper secondary level, tertiary educational level, the transition stages between and within the two secondary and tertiary levels. The research presented in this category included (but was not bounded to) conceptual attainment, proof techniques, problem-solving, instructional techniques and processes of abstraction.

According to *thinking-centered perspective* we suggest to address A-MT through focusing on students with high intellectual potential in mathematics (e.g., mathematically gifted students). The research in this perspective can, for example, ask how these students differ in their actions from other students of the same age group. In this perspective we can address such characteristics of mathematical thinking as creativity, reasoning in a critical mode, persistence and motivation.

In this perspective, we planned to encourage participants to attain their attention on individual and group differences related to advanced mathematical contents. We shall note that thinking-related perspective was less enlightened in the contributions and during the work at the conference.

The group was focused on original research mainly of the first perspective. Contributors adopted different the research paradigms, theoretical frameworks and research methodologies. Contributors addressed a variety of issues in the field of AMT, amongst the following themes:

- A. Learning processes associated with development of AMT
- B. Problem-solving, conjecturing, defining, proving and exemplifying at the advanced level
- C. Effective instructional settings, teaching approaches and curriculum design at the advanced level.

Setting

All the participants of WG-12 were divided in three small groups according to the abovementioned themes (Groups A, B and C). Participants of Groups A, B and C prepared main questions for the discussion in Groups B, C, and A correspondingly. Of these questions, participants in each small group chose questions that they considered as most important and interesting for the discussion. Below we present our reflection on the outcomes of our work at the conference.

FOCAL TOPICS

Learning processes associated with development of AMT

Discussion on this topic was coordinated by Claire Cazes. The participants of the small group focused their discussion on Learning processes associated with development of AMT, students' difficulties, concept image-concept definition on advanced level. This group included the following contributions: Theoretical model for visual-spatial thinking (by Conceição Costa and her colleagues), Secondary-tertiary transition and students' difficulties: the example of duality (presented by Martine De Vleeschouwer), Learning advanced mathematical concepts: the concept of limit (António Domingos), Conceptual change and connections in analysis (Kristina Juter), Using the onto-semiotic approach to identify and analyze mathematical meaning in a multivariate context (presented by Miguel R. Wilhelmi et al.), Derivatives and applications: Development of ONE student's understanding (Gerrit Roorda et al.), and Finding the shortest path on a spherical surface: "Academics" and "Reactors" in a mathematics dialogue (Maria Kaisari and Tasos Patronis).

The most intriguing distinction between the papers in this group was connected to the conceptual frameworks chosen by the authors for their studies. These frameworks related to AMT include different basic concepts. Thus, among other questions, formulated by group C, members of group A chose to focus on the following questions:

- How could you compare the meanings of the basic concepts in the theoretical frameworks addressed in different papers? How are they different? How are they similar or interchangeable?"

Group A found that the complexity of the topic that concerning in the diversity of the approaches and diversity of the frameworks that were raised. Figure 1 demonstrated main points addressed in this discussion:

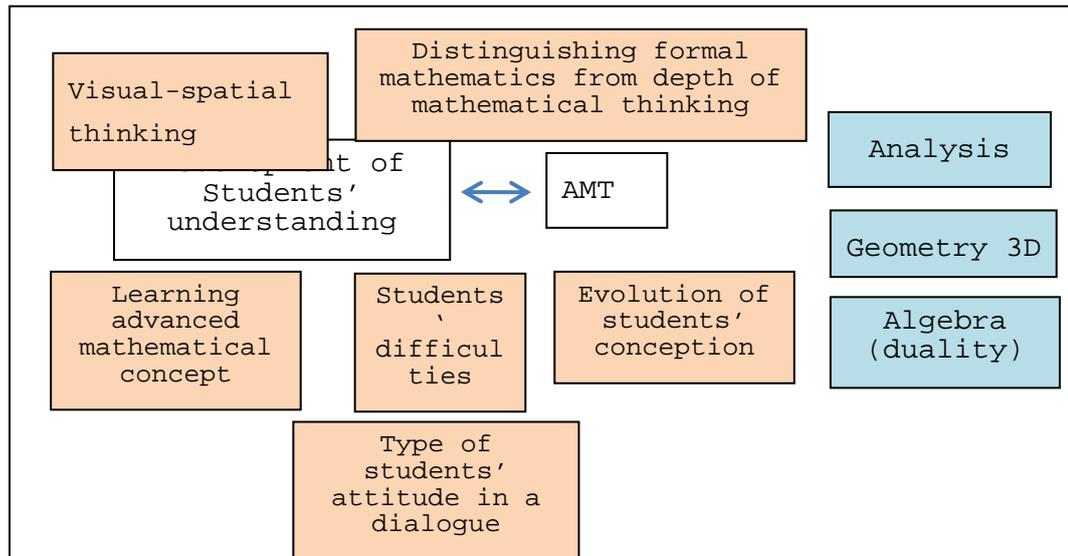


Figure 1: Complexity of the topics

Based on the papers of the participants of group A, the members presented the following theoretical frameworks: Antonio Domingos discussed Tall and Vinner (1981) concept-image, concept definition framework as the central framework for research on AMT. Additionally he presented Tall's view on the development of mathematical understanding through embodied, symbolic and axiomatic worlds (Tall, 2006a, b).

Gerrit Roorda stressed the better mathematical understanding might be reflected by more and better connections between representations, within representations, between applications and mathematics (for elaboration see Roorda, et al. in the proceedings of CERME-6). Conceição Costa framed her framework based on the views on cognitive processes, embodiment, sociocultural perspectives, and theoretical perspectives on teaching and learning geometry. She presented her own framework developed through studying visual reasoning (see figure 2, for elaboration see Conceição et al. in the proceedings of CERME-6).

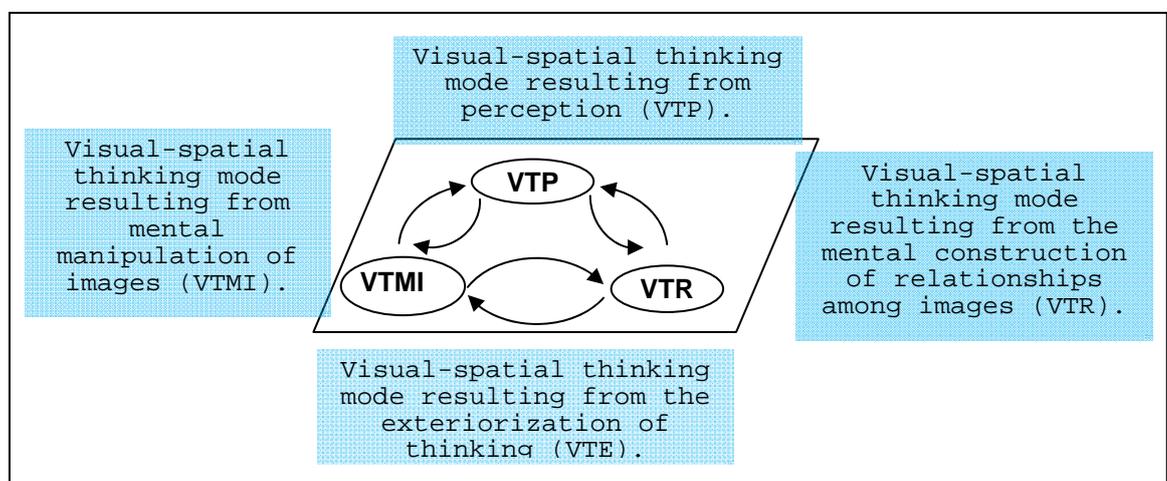


Figure 2: Costa (2008) –AMT and visual reasoning

Martine De Vleeschouwer presented Chevallard's *Institutional point of view* as the main theoretical framework that allows exploring advances mathematical thinking. This framework focuses on four main components: Type of tasks, Technique, Technology, and Theory. Milguel R. Wilhelmi presented Epistemic Configuration that they developed for the development of didactical situations of different kinds and the analysis of AMT developed in these situations. Definitions, procedures and propositions in this framework are the "the rules of the game", argumentation and justification are integral characteristics of the situations associated with AMT (see Figure 3).

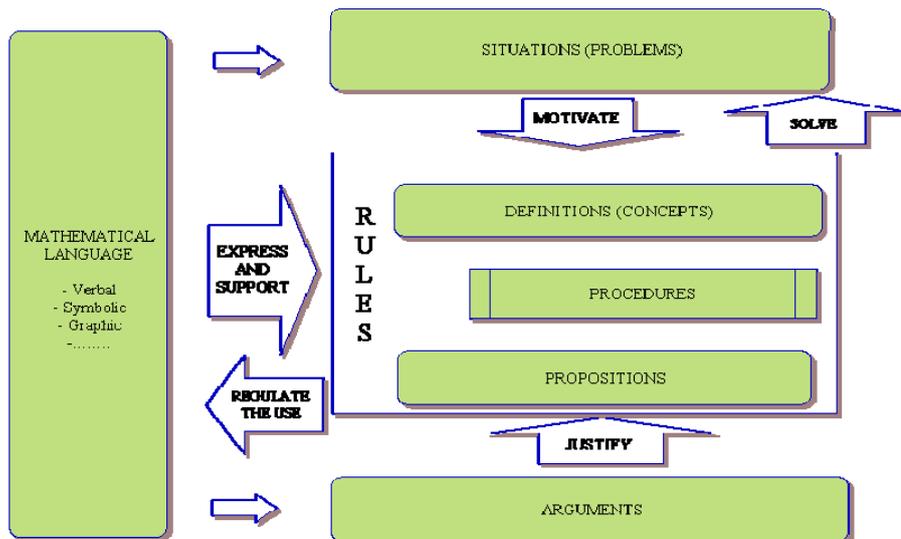


Figure 3: Epistemic Configuration

Claire Cazes summarized this discussion and outlined further directions to be addressed in future research. She stressed the need in *finding connections between five theoretical frameworks used* in different studies (see Figure 4). She also pointed out the need (a) to specify *why each approach is useful* for study AMT, (b) to make "cross analysis" by working by pairs and analyse the same data with two different frameworks. Then the following questions are important and interesting for the future exploration: *Do we focus on the same points? Are the results: opposite, additional, identical?*

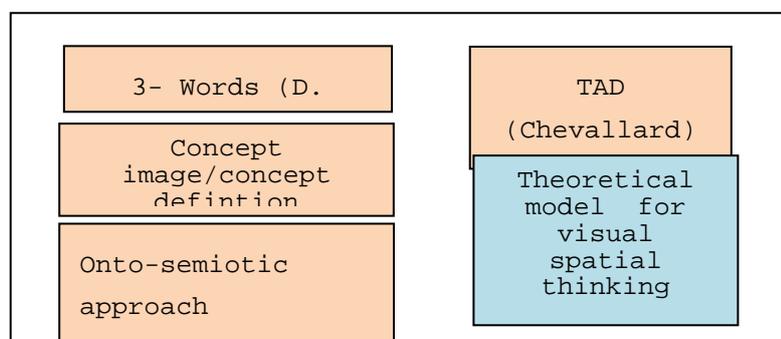


Figure 4: Theoretical frameworks observed in the Group.

Problem-solving, conjecturing, defining, proving and exemplifying at the advanced level.

This theme was coordinated by Joanna Mamona-Downs. The group participants based their discussion on the following contributions: Number theory in the national compulsory examination at the end of the French secondary level: between organising and operative dimensions (Véronique Battie), Defining, proving and modelling: a background for the advanced mathematical thinking (García M., V. Sánchez, and I. Escudero), Necessary realignments from mental argumentation to proof presentation (Joanna Mamona-Downs and Martin Downs), An introduction to defining processes (Cécile Ouvrier-Bufferet), Problem posing by novice and experts: Comparison between students and teachers (Cristian Voica and Ildikó Pelczer), and Advanced Mathematical Knowledge: How is it used in teaching? (Rina Zazkis, Roza Leikin).

The group chose to focus on the questions:

- What are the relationships between problem solving, conjecturing, defining and proving?
- What is the effective use of problem solving?
- How to help students in justifying formal proof?

The group decided that *features of Problem Solving depend* on the level of problem solver, the place in a course, the context and other factors. Problem Solving Features depend on the problem solving aspects the solver is engaged in: (a) formulating questions (b) engaging in a proof process or in a modeling process, (c) making mistakes, (d) expecting posing more questions, (e) communicating with other persons while solving or redefining the problem, (f) communicating about results.

Veronique Battie performed her research in the number theory. She focused on two following dimensions and the relationships between them: The *Organizing dimension* concerns the mathematician's "aim" (i.e., his or her "program", explicit or not); induction, reduction ad absurdum (minimality condition); Reduction to the study of a finite number of cases; Factorial ring's method; Local-global principle. The *Operative dimension* relates to those treatments operated on objects and developed for implementing the different steps of the aim, forms of representation of objects, algebraic manipulations, using key theorems, distinguishing divisibility order and standard order.

Cristian Voica presented distinctions in problem posing activities for teachers and students. He argued that teachers' views on problem posing are influenced by the curricula and the exams subjects, guided by pedagogical goals and by attention to the formulation of the problem. Students are interested in extra-curricular contexts and solution techniques, see problem posing as a self-referenced activity, and (many of

them) generate problems with an unclear statement, or does not choose a good question.

Cecile Ouvrier-Bufferet explored defining processes. Her design of a didactical situation is aimed to make students acquire the fundamental skills involved in defining, modelling and proving, at various levels of knowledge; to work in discrete mathematics but also in linear algebra because similar concepts are involved in this situation; and to have a mathematical experience and to raise mathematical questionings. While she chooses an epistemological approach to data analysis, she considers defining processes as a tool for characterizing mathematical concept.

All the participants shared concerns regarding connections between school and University mathematics. They observed the gap between the teaching approaches, the requirement for rigor mathematics and the role of defining and proving in learning process in these two contexts. Zazkis and Leikin pointed out that school teachers' conceptions of advanced mathematics and its' role in school mathematical curriculum reflect this gap. They argued that mathematics teacher preparation should explicitly introduce connections between school and tertiary mathematics.

Effective instructional settings, teaching approaches and curriculum design at the advanced level

Group C, coordinated by Isabelle Bloch, discussed Effective instructional settings, teaching approaches and curriculum design at the advanced level Urging calculus students to be active learners: what works and what doesn't (Buma Abramovitz, Miryam Berezina, Boris Koichu, and Ludmila Shvartsman), From numbers to limits: situations as a way to a process of abstraction (Isabelle Bloch and Imène Ghedamsi), From historical analysis to classroom work: function variation and long-term development of functional thinking (Renaud Chorlay), Experimental and mathematical control in mathematics (Nicolas Giroud), Introduction of the notions of limit and derivative of a function at a point (Ján Gunčaga), Advanced mathematical thinking and the learning of numerical analysis in a context of investigation activities (poster presented by Ana Henriques), Factors influencing teacher's design of assessment material at tertiary level (Marie-Pierre Lebaud), Design of a system of teaching elements of group theory (Ildar Safuanov).

This group chose to focus on the following points

- Importance for the students to be active learners when they study AM.
- Making abstraction accessible (“Abstract” and “formal” are not the same).
- Minding the secondary – tertiary gap.

The group argued that generally speaking they look for more opportunities for high school students to be engaged in high-level abstracting and proving, and for university students to be engaged in activities elaborating the meaning of (abstract)

concepts they study. It implies the necessity for gradual change in didactical contracts, both in secondary and university education

Buma Abramovich with colleagues reported an on-going design experiment in the context of a compulsory calculus course for engineering students. The purpose of the experiment was to explore the feasibility of incorporating ideas of active learning in the course and evaluate its effects on the students' knowledge and attitudes. Two one-semester long iterations of the experiment involved comparison between the experimental group and two control groups. The (preliminary) results showed that active learning can have a positive effect on the students' grades on condition that the students are urged to invest considerable time in independent study. They presented two episodes from different settings and concluded that the answer to their research question appears to be more complex than expected (see for elaboration Abramovich et al.).

Isabelle Bloch discussed ways of designing a milieu that helps students constructing mathematical meaning. She argued that when they enter the University, students have a weak conception of real numbers; they do not assign an appropriate meaning to $\sqrt{2}$, or π , or to variables and parameters. This prevents them to have a control about formal proofs in the field of calculus. She presents some situations to improve students' real numbers understanding, situations that must lead them to experiment with approximations and to seize the link between real numbers and limits. They can revisit the theorems they were taught and experience their necessity to work about unknown mathematical objects (see Bloch in this proceedings).

Nicolas Giroud focused on mathematical games as an effective didactical tool for development AMT. He presented a problem which can put students in the role of a mathematical researcher and so, let them work on mathematical thinking and problem solving. Especially, in this problem students have to validate by themselves their results and monitor their actions. His purpose was centered on how students validate their mathematical results. His paper is related to learning processes associated with the development of advanced mathematical thinking and problem-solving, conjecturing, defining, proving and exemplifying.

Renaud Chorlay presented work on mathematical understanding in function theory. Based on a historical study of the differentiation of viewpoints on functions in 19th century involving both elementary and non-elementary mathematics he formulated a series of hypotheses as to the long-term development of functional thinking, throughout upper-secondary and tertiary education. The research started testing empirically three main aspects, focusing on the notion of functional variation: (1) "ghost curriculum" hypothesis; (2) didactical engineering for the formal introduction of the definition; (3) assessment of long-term development of cognitive versatility.

CONCLUDING REMARK

Very naturally all the three groups admitted the gap between school and tertiary mathematics. Rina Zazkis managed a special discussion on the way of bridging school and university mathematics. Most of the examples provided by the participants were extracurricular tasks from the university courses that in the presenter's opinion may be used in school as well. However the question of the integration of AM-T in school teaching and learning remains open.

A-MT is another issue that needs further attention of the educational community. This perspective was less addressed and requires investigations associated with AMT. It may be suggested as one of the topics for the discussion at the future meetings of AMT group.

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A THEORETICAL MODEL FOR VISUAL-SPATIAL THINKING

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This paper presents part of a study (Costa, 2005) intending to create, explore and refine a theoretical model for visual-spatial thinking that includes three visual-spatial thinking modes along with the thinking processes associated to them. This paper will focus on the final theoretical model.

Many researchers have emphasized the value of the visualization and the visual reasoning in the mathematics learning (Bishop, 1989; Presmeg, 1989, Zimmerman & Cunningham, 1991). In the literature we find terms such as visualization, visual thinking, visual reasoning, spatial reasoning, spatial thinking to name mental acts combining visual, spatial, and visual-spatial thinking. The visual reasoning often parallels visualization (Hershkowitz, Parzysz & Dormolen, 1996) and visualization itself has different definitions according to the context of mathematics education, mathematics, or psychology. The terms, spatial thinking or spatial reasoning appear frequently tied to spatial abilities (Clausen-May e Smith, 1998). Dreyfus (1991) included visualization as a component of representation crucial in AMT.

This paper presents part of a research (Costa, 2005) intending to create, explore and refine a theoretical model for visual-spatial thinking, thus deepening meaning of a thinking-centered perspective on AMT. This research was developed through a three-stage process. Firstly, an initial model for visual-spatial thinking, condensed from relevant literature, was developed; secondly, this initial model was confronted with data from an empirical study; finally, the initial model was refined. The methodology for the empirical study was qualitative, integrating video registrations of individual answers and tasks performed in classroom activity. These episodes were analyzed and a constant comparison approach was used to fine-tune the initial model. The refined version of the model was elaborated and evaluated according to the standards for judging theories, models and results proposed by Schoenfeld (2002).

This paper will focus on the final theoretical model. The theoretical framework took into account research in the areas of cognitive processes in mathematics education, embodiment in mathematics, a perspective on learning with emphasis on the social construction of knowledge and on semiotic mediation, theoretical perspectives on the teaching and learning of geometric concept.

A THEORETICAL VISUAL-SPATIAL THINKING MODEL

The final model for understanding the visual-spatial thinking differentiates four distinct modes of thinking: the visual-spatial thinking resulting from perception (VTP) — intellectual operations on sensory, perceptual and memory material —; the visual-spatial thinking resulting from mental manipulation of images (VTMI) —

intellectual operations related to the manipulation and the transformation of images —; the visual-spatial thinking resulting from the mental construction of relationships among images (VTR) — intellectual operations related to the mental construction of relationships among images, the comparison of ideas, concepts and model—; the visual-spatial thinking connected with transmission-communication and representation, that is to say, connected with the exteriorization of the thinking (VTE) — intellectual operations related to the representation, translation and communication of ideas, concepts and methods.

Visual-spatial thinking modes	Definition
Visual-spatial thinking resulting from perception (VTP).	Intellectual operations on sensory, perceptual and memory material.
Visual-spatial thinking resulting from mental manipulation of images (VTMI).	Intellectual operations related to the manipulation and the transformation of images.
Visual-spatial thinking resulting from the mental construction of relationships among images (VTR).	Intellectual operations related to the mental construction of relationships among images, the comparison of ideas, concepts and models.
Visual-spatial thinking resulting from the exteriorization of thinking (VTE).	Intellectual operations related to the representation, translation and communication of ideas, concepts and methods.

TABLE I. Visual-spatial thinking modes and respective definitions.

In the next sections, we will discuss each mode and characterize the associated mental processes.

VISUAL-SPATIAL THINKING RESULTING FROM PERCEPTION

The visual-spatial thinking mode resulting from perception (VTP) is the nearest to sensations, that is to say, to the electric impulses that arrive at the brain. Its intellectual operations occur on sensory, perceptual and memory material. It is constructed from sensory stimulus and takes advantage of information gained from experience. This thinking mode involves experiences of mental concentration, of control, and observation. The observation experiences involve perception and interpretation, depend on past experience, memory, motivation, emotions, attention, the individual neuronal mechanisms, previous knowledge, verbalizations, and cultural aspects and so, what we saw depends on our relationship to the situation. The sociocultural factors, from which the perception depends on, are not less importance and they regulate how the members of a culture see.

This mode uses concrete images and memory images (Brown & Presmeg, 1993). *Concrete images* may be thought of as “a picture in the mind”, and are not the same for all persons; *memory images* are produced when images of experience are brought up again. These are representations of visual information connected to the perception of movement, for example, the images remaining immediately after we visually check for in-coming vehicles, before crossing the street.

Mental processes of this mode

Thinking processes involved in this visual-spatial thinking mode are: primary intuitions; intuitive inference; visual construction; representation again and image evaluation; visual recognition; objects and models identifications, formation of a “gestalt”, global apprehension of a geometrical configuration; perceptual abstraction and abstraction connected with recognition; and generation of concepts.

The first mental processes associated with the VTP mode are intuitions. Using the terminology of Fischbein (1987), we include in this mode the primary intuitions, — cognitive acquisitions that develop in individuals independently of any systematic instruction as an effect of personal experience. The primary intuitions are connected, for instance, with space representation related to body movement, and to images as models. Images may inject properties and relationships in the process of concepts construction that do not belong to the conceptual structure (points as spots, lines as bands). It also includes intuitive inferences, which are shown, for example, when a child sees a ball, runs after it according to the ball’s position and adapts his reactions to the ball’s movements. The child not only sees the ball moving, but also expects that it goes on moving, existing and preserving its shape and properties.

Visual construction is a mental process, which is present in this mode and may be illustrated, for instance, when alterations of distance or size “are seen” in optic illusions (even though the mind knows the perception is illusory), or when we perceive the fluctuations of the figure-ground in ambiguous designs.

The mental process of evaluating an image consists in representing again the image and this act of re-representation is complex and subtle (Wheatley, 1998). These re-presented images are not immutable, because they may undergo change over time. In many cases the re-presented image may have been modified or it might be a prototype, which is then transformed, based on the demands of the context. The nature of the re-representation is greatly influenced by the intentions of the individual and in many cases the re-presented image may come again more elaborated.

The information that comes through our eyes is involved in visual perception containing two phases (Gal & Linchevski, 2002), the visual information processing phase which consists in registering the sensory information, and the visual pattern recognition phase, which involves the interpretation of the identified shapes and objects. In the first stage of visual perception, shapes and objects are extracted from the visual scene. To form the object we need to know “what goes with what” and they

are organized into groups similar to the gestalt principles. In the second phase of visual perception, shapes and objects are recognised. Recognition is the result of feature analysis, in which the object is segmented into a set of sub-objects, as the output of early visual processing of the first phase. Each sub-object is classified, and when the pieces out of which the object is composed and their configuration are determined, the object is recognized as a pattern composed of these pieces. The cognitive processes designated by visual recognitions, objects and models identifications, formation of a *gestalt*, global apprehension of a geometrical configuration belong to the second phase of visual perception while the remainder are included in the first phase of visual perception.

Although abstraction is more developed in the others thinking modes, it shows in VTP as a basic perceptual procedure — when we isolate (identify) something from the visual scene —, or in the recognition of a familiar structure in a given situation. Generation of concepts is done when the recognition of relations and idea emerge.

VISUAL-SPATIAL THINKING RESULTING FROM MENTAL MANIPULATION OF IMAGES

Visual-spatial thinking mode resulting from mental manipulation of images (VTMI) embraces different levels of imagery processing, mainly to foresee the result of transforming an image and envision the trajectory of that same transformation. We will include in this thinking mode the dynamic imagery and the pattern imagery proposed by Brown and Presmeg (1993). Dynamic imagery involves the ability to move or to transform a concrete visual image and pattern imagery is a highly abstract form of imagery where concrete details are rejected and pure relationships are depicted in a visual-spatial scheme. Owens (without date) using the conceptual frame of Presmeg, showed a kindergarten child extending a square using pieces of bread to make a “skinny” rectangle. This child also used dynamic imagery foreseeing (mentally) the result of the transformation a square into a rectangle before executing (physically) this same transformation. According to Owens (1994) the dynamic imagery was the means by which the child was linking her images for the concepts of squares and rectangles. Another child, for instance, makes the medium triangle with the small triangles in the tangram puzzle (Owens, without date). This child also used a patterned imagery because she can see a certain configuration, structure (triangle) as a composition of other structures.

The VTMI mode incorporates the transformational reasoning referring to the foresight and mental transformations of objects, postulated by Simon (1996). Simon assumes, more than the inductive and deductive reasoning used in the comprehension and validation of mathematics ideas, a third type of reasoning, transformational reasoning, is defined as

“The mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and

the sets of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated” (p. 201).

This transformational reasoning is supported by transformational reproductive images or by anticipatory images. Reproductive images evoke objects or events already known and anticipatory images represent, through figural imagination, events (movements or transformations, for example) that have not previously been perceived. In either case, someone is able to visualize the transformation resulting from an operation; however, transformational reasoning is not restricted to mental imaging of transformations. A physical enactment may be used to examine the results of a transformation. For example, a student who is exploring the validity of the statement, “If you know the perimeter of a rectangle, you know its area”, might work with a loop of string observing what happens to the area as she makes the rectangle longer and thinner. But in order for the student to model this problem it is required a mental anticipation, that is, he must know, before handling the string, how to model the rectangles and use the string to observe the results of the operation (Simon, 1996). In both transformational reasoning and VTMI mode, mental operations or transformations on objects may be made and mentally envisioned as well their results.

Mental processes of this mode

The following mental processes are associated with this visual-spatial thinking mode: secondary intuitions and anticipatory intuitions; unitizing; mental transformations; reflective abstraction, constructive generalization; synthesizing; spatial structure; coordination; and visual construction.

The intuitions associated to VTMI, following the Fischbein’s terminology, are of two types: secondary intuitions and anticipatory intuitions. The secondary intuitions are affirmative intuitions that represent a stable cognitive attitude with regard to a more general, common, situation. The secondary intuitions are developed as the result of a systematic intellectual formation and they are interpretations of various facts taken as assured. Integration into dynamic and perceptively rich situations, as for instance, the use of a microworld, seems to enrich the acquisition of intuitions. Particularly secondary intuitions may be acquired (Fischbein, 1987).

Anticipatory intuitions also characterize this visual-spatial thinking mode. These intuitions do not simply establish a (apparently) given fact. They appear as a discovery, a preliminary solution to a problem, and the sudden resolution of a previous endeavour. Moreover, one may assume that anticipatory intuitions are inspired, directed, stimulated or blocked by existing affirmative intuitions. The anticipatory intuitions may be the effect of a creative activity in mathematics, of a constructive process in which inductive procedures, analogies and plausible guesses play a fundamental role (Fischbein, 1987).

Unitizing, which consists in the mental operation of constructing, creating and coordinating abstracts mathematical units, identified as a base for much mathematical activity in both geometric and numeral settings, are present in VTMI.

The term mental transformation is used to refer a type of process which involve the change of a mental representation in one of two aspects or in a composition of the two: to dislocate, that is to say, to change the position and to transform, where there is only a change of shape. These two aspects are related to each other and there is only a difference of complexity between displacements and transformations. In particular, to change the shape of an object may consist in dislocating the parts. Reciprocally, when we dislocate an object without changing its shape, this may dislocate en reference to another and changing the configuration of the whole.

Gusev and Safuanov revealed three types of operating with images (in order of their increasing complexity): transformations resulting in the change of a spatial position of an image (1st type); transformations changing the structure of an image (2nd type); long and repeated performance of transformations of first two types (3rd type).

This thinking mode is characterized by a particular type of abstraction, the reflective abstraction — essentially the construction by the subject of mental objects and of mental actions on these objects. The subject, in order to understand, deal with, organize, or make sense out of a perceived problem situation or to know a mathematic concept, uses schemes that invoke a more or less coherent collection of objects and processes. Understanding the trajectory as a coordination of successive displacements to form a continuous whole is an example of reflective abstraction in children thinking (Dubinsky, 1991). The pseudo-empirical abstraction (in the Piaget sense) as a sub-variety of the reflective abstraction is present in this visual-spatial thinking mode, focused on children actions and the properties of the actions and it appears from their successive coordinations.

Constructive generalization creates new forms, new contents, that is to say, a new structural organization. The mental process synthesizing that means to combine or compose parts in such way that they form a whole, an entity, is a basic prerequisite to the abstraction. The spatial structuring is the mental act of constructing an organization or form for an object or set of objects. It determines an object's nature or shape by identifying its spatial components, combining components into spatial composites and establishing interrelationships between and among components and composites (Battista, 2003).

A fundamental cognitive process to the understanding of the reasoning in this thinking mode VTMI is the coordination which involves diverse aspects, one of them is that indicated by Battista (2003, p. 79) “it arranges abstracted items in proper position relative to each other and relative to the wholes to which they belong”. Another aspect of the coordination is related with the ability of using structures (references systems) as a way to organize the thinking. So, for instance, a student adopts structures of references to codify the spatial positions of the objects that may

come to be defined: references systems centred in himself, references systems centred in the objects or in external structures which are or provided by the spatial structure or they are imposed mentally by the space (environment).

The visual construction process included in this visual-spatial thinking mode is related with making or modifying a spatial structure in such way that it meets certain predetermined geometric criteria. The visual construction comprises abilities such as the anticipation and the logic organization.

VISUAL-SPATIAL THINKING RESULTING FROM THE MENTAL CONSTRUCTION OF RELATIONSHIPS BETWEEN IMAGES

The intellectual operations of the visual-spatial thinking mode resulting from the mental construction of relationships between images (VTR) are related to the mental construction of relationships between images, the comparison of ideas, concepts and models.

Mental processes of this mode

We consider that the visual-spatial thinking resulting from the mental construction of relationships between images, mode VTR, may be associated to the following thinking processes: anticipatory intuitions; discovery of relationships between images, properties and facts; comparisons; synthesis; reflective abstraction; metacognition. The metacognition process is fundamentally understood as a regulation of cognition which includes the planning before beginning to solve the problem and the continuous evaluation while solving the problem.

VISUAL-SPATIAL THINKING RESULTING FROM THE EXTERIORIZATION OF THINKING

The visual-spatial thinking mode resulting from the exteriorization of thinking (VTE) is connected to the process by which mental representations are materialized, to the communication and the dissemination of ideas, to the construction of argumentation, to the description of the mental dynamics and to the support of conceptualizing abstract entities. The VTE mode has a nature different from the other thinking modes because is like the conveyor of those thinking modes. The VTE mode is a cognitive space of action, representation, construction and communication and as a whole may integrate components such the body, the physic world and the culture. This mode allows us to infer the imagery and the mental dynamics of students and to understand how they perform mathematical tasks.

For communicating their mental representations, the students may construct patterns, drawings, figures, and graphics, musical and rhythmic productions, to use gestures (corporal language, facial expression), actions, verbal descriptions (spoken or written), mathematic representations, etc. The VTE thinking mode relies fundamentally on verbal and gestured, visual language and it requires the use of concrete, memory, dynamic, pattern images and also kinaesthetic images (Brown &

Presmeg, 1993) which involve muscular activity of some type (the muscular activity may be limited to the use of hands and fingers).

Mental processes of this mode

The mental processes associated to the visual-spatial thinking mode resulting from the exteriorization of thinking are: representations; translation; description of the mental dynamics through verbalization and gestures; construction of argumentation, of conjectures; and the use of analogies. The concept of representation is essential to understanding constructive processes in learning and doing of mathematics and, roughly speaking, an external representation is a configuration of some kind that represents something in a special manner. For instance a word may represent an object of the real life, a numeral may represent the cardinal of a set, or even the same numeral may represent a position in a numeric line. The representations do not occur in isolation and usually they belong to highly structured systems, either personal and idiosyncratic or cultural and conventional (Goldin & Kaput, 1996). Among the external representations we find external physic embodiments, structured external physical situations or a set of situations which may be mathematically described or seen as embodying a mathematic idea; linguistic expressions, verbal or syntactic and formal mathematic constructions.

The representation of visual-spatial information used by the student is going to depend on the context where the problem is posed. The same task may require from the student different spatial abilities or different levels of abstraction. This representation may be a concrete image or a diagram or a concept representation: the reflection around a line, or the pattern construction or a tessellation.

Translation is a process that is intimately related to the conversion among representations. For example, the conversion of what is given of symbolic form in information given by figures or passing a problem from natural language or graphic form to some other form.

The description of the mental dynamic designates mental images evidenced in oral language, actions or gestures and in metaphoric expressions. Gesture is used to refer to any of a variety of movements, we want to identify mainly movements of hands, non-conventional gestures (gesticulations and language-like gestures) that accompany the speech with which they form an integrated whole. The description of the mental dynamic is going to be designated by factual if the objects of description are geometric objects and by analytical if the objects of description are geometric properties.

Analogies or metaphoric expressions are appealing modes of externalizing visual-spatial thinking, particularly ways of mathematic communication and of building of meaning. Two objects, two systems are said to be analogical if, on the basis of a certain partial similarity, one feels entitled to assume that the respective entities are similar in other respects as well. The difference between analogy and trivial similarity

is that analogy justifies plausible inferences. So analogies imply similarity of structure (Fischbein, 1987). The visual-spatial thinking mode VTR may involve the use of analogies, which may conduct to new images, to new models or to draw comparisons, transformations and discoveries of relationships between images. Gusev and Safuanov (2003) say that the new images processed under the influence of some associations and analogies emerge frequently with unexpected qualities, creative imagination and they are the result basically operating the second and third type of transformations (behind explained). The visual-spatial thinking mode VTE is the conductor of those analogies, is linked to the externalization through the language, actions and gestures or through a distributed blend of perceptual sources coming from the screen and the gestures, if the student has not yet a language to describe and to theorize the events, appropriately.

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SECONDARY-TERTIARY TRANSITION AND STUDENTS' DIFFICULTIES: THE EXAMPLE OF DUALITY

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Abstract : *We are presenting a study about duality and its learning in linear algebra. We have elaborated a device of follow-up of knowledge and difficulties of students enrolled in first-year university mathematics or physics programs, concerning this theme. We are presenting the results of this device categorizing students' difficulties. We present moreover a perspective on transition allowing us to interpret students' difficulties in duality in terms of transition.*

Key-words : *linear algebra, duality, tertiary level, institutional transition*

1. INTRODUCTION AND THEORETICAL FRAMEWORKS

The study presented here focuses on the teaching of duality at university. This work is thus naturally related with WG12 theme “Advanced mathematical thinking (AMT)” of CERME6, and is more precisely connected with the sub-theme “Effective instructional settings, teaching approaches and curriculum design at the advanced level”.

Duality is taught in most countries only at tertiary level, and is even more ‘advanced’ than elementary linear algebra. One aspect of our contribution is to precise possible meanings of ‘advanced’, in order to enlighten students’ difficulties, a necessary step before proposing a teaching design.

From an epistemological point of view, duality takes a central place in linear algebra. Indeed, the notion of rank, essential in linear algebra, has first emerged in what Dorier terms the *dual aspect*, meaning the smallest number of linearly independent equations (Dorier 1993, p. 159).

Even if since the mid-eighties didactical works are interested in linear algebra, they mostly concern elementary notions of this part of mathematics (Dorier 2000, Trigueros & Oktac 2005,...).

However, when the duality is studied as an object (Douady 1987) in a course of linear algebra in first year of university, we notice that the students are confronted with numerous difficulties. Our main objective is to understand the origin of these difficulties, and to be able, in a later work, to propose adapted teaching devices.

In our work, we try, in a first step, to identify different kinds of difficulties, according to mathematical content that can be problematic, and after to interpret these

difficulties from an institutional point of view. So we try to answer the following questions :

- What are the difficulties tied to duality itself, those that are linked more generally to linear algebra, or also to other connected contents?
- How can we interpret these difficulties, which hypotheses can we do about their causes ?

Our work, beyond duality, also has for objective to enlighten the specific difficulties of novice university students. These difficulties have already been the object of numerous works (Artigue 2004, Gueudet 2008). Here we adopt an institutional point of view (Chevallard 2005). The difficulties don't only result from the fact that new knowledge is met. They can be caused by the fact that the same knowledge will be differently approached in the secondary school institution and in the undergraduate institution. So a same *type of tasks* can be associated with a new *technique, to solve the corresponding exercises* ; a same technique will be differently justified... So, in our research, we use the « praxeology » notion, also named « mathematical organization », introduced by Chevallard (2002). He defines a punctual mathematical organization as an union of two blocks $[\Pi / \Lambda]$, each one containing two parts. The first block, $\Pi = [T / \tau]$, named « practico-technical » block, is made of a type of tasks T and a technique τ allowed to realize tasks related to type T . The second block, $\Lambda = [\theta / \Theta]$, named « technologico-theoretical », is made of a *technology* θ , which is a discourse justifying the technique τ , and a *theory* Θ justifying the technology θ . A complete mathematical organization is then an organization that we can note $[\Pi / \Lambda]$ or $[T / \tau / \theta / \Theta]$.

Let us illustrate these concepts by an example. Suppose we propose to a student to solve the following *exercice*: « Compute the dual basis of the canonical basis of \square^4 ». We can say that this exercise is related with the *type of task* T « given a n -(sub-)vector space E and one of its bases, to determine the dual basis of the given basis ». A *technic* τ associated with this type of tasks T consists in solving n systems ($i =$

$1, \dots, n$) of n equations in n unknowns (α_{ip}):

$$\begin{cases} \sum_{p=1}^n \alpha_{ip} x_{1p} = \delta_{i1} \\ \dots \\ \sum_{p=1}^n \alpha_{ip} x_{np} = \delta_{in} \end{cases} \quad \text{where } x_{jp} \text{ are the coordinates}$$

of the j^{th} vector of the given basis. This technic τ is justified by a discourse, called *technology* θ : « To find the dual basis, firstly define the general expression of any linear form y in the given space : $\forall x \in E, y(x) = \sum_{p=1}^n \alpha_p x_p$ where x_p are the coordinates of a vector x in E . Then solve n systems of n equations in n unknowns : $\forall i, j = 1, \dots, n : y_i(x_j) = \delta_{ij}$ where x_j are the vectors of the basis given in the type of task ». This technology θ is justified by the *theory* 1 : « Given E an n -vector space, and $\{x_i\}_{i=1}^n$ a basis of E . Then there is a basis $\{y_i\}_{i=1}^n$ of the dual space E' so that

$\forall i, j = 1, \dots, n : y_i(x_j) = \delta_{ij}$. The defined basis $\{y_i\}_{i=1}^n$ is also called the dual basis associated with a basis of the primal space E ».

We also use a framework proposed by Winsløw (2008), especially focused on “concrete-abstract” transition issues, and drawing on praxeologies. Winsløw considers that when a student arrives in an undergraduate institution, he/she is confronted with two types of transition. The first type of transition originates in the secondary school’s teaching, where almost only the block « practico-technic » intervenes. The first transition that a student meets changing institution, is that at university, the « technologico-theoric » block is also present, completing the mathematical organizations. But a second transition appears when the recently introduced elements of « technologico-theoretical » block also become objects that the students have to manipulate, constituting then the « practico-technic » block of new mathematical organizations. We will explain why the learning of duality in linear algebra at university depends of this second type of transition.

In this article we present the analysis of responses to a survey that has been proposed to students enrolled in first year university mathematics or physics programs in the University of Namur (Belgium) concerning duality. In a first step (part 2), we describe the survey. Then in part 3 we present the analysis of the survey’s results.

2. DESCRIPTION OF THE SURVEY

In (DeVleeschouwer 2008), we describe how the teaching of the duality in linear algebra is structured, focusing on the concepts of dual (as vector space), linear form, dual basis, annihilator and transposed transformation. Through the analysis of various textbooks (books and course notes), we have analysed the duality as an object (Douady 1987) of teaching in the university institution. We also studied the different aims of the tool function of the duality : we distinguished the analogy-tool, the resolution-tool, the illustration-tool, the definition-tool and the demonstration-tool for duality.

Thanks to these analyses we have designed a survey addressed to students enrolled in first year of university, meeting the teaching of duality in linear algebra. This survey, which focuses on the duality in its ‘object’ aspect (Douady 1987), is based on the elements identified in the analysis of textbooks, and will enable us to precise the difficulties faced by the students.

This survey contains two parts :

- The first one is constituted of a *questionnaire*. 37 students enrolled in the first year of mathematics or physics programs at the University of Namur answered to this (written) questionnaire (February 2008). The students had two hours to answer it. Some interviews allowed to highlight the answers brought to the questionnaire for 16 of these students (May 2008).

- The second part of the survey is a *group work*. 23 students enrolled in the first year of mathematics programs took part of this group work. The students, divided in four groups of 5 or 6, had 5 weeks to return a written report about the asked work. It was recommended then to consult an assistant during the two first weeks of their work ; and an interview (varying from 30 to 90 minutes) was mandatory when giving the written report (March 2008).

Before the survey, the students have already seen, in the theoretical course and in the exercises, the vector spaces (algebraic structures, linear dependence and dimension, sub-vector spaces) ; the linear applications, the associated matrices; the linear forms, and also the dual space (and bases) and the reflexivity; the linear and transpose transformations. The theoretical course had already approached determinants (without exercises).

We have to precise that in the secondary school Belgian pupils have only approached the vector's notion at the *geometric level* (Hillel 2000, p.193). The notion of transpose was only presented to the pupils of the secondary school who specialize in mathematics, principally when approaching the definition of the inverse matrix (using the transpose of the cofactors *matrix*).

2.1. THE QUESTIONNAIRE

The questionnaire (appendix 1) comprises two parts, each one composed by the same questions but contextualized in different frames. The two chosen frames are the vector space \mathbb{R}^4 ; and the frame of matrices with real coefficients, particularized to 2 by 2 matrices ($\mathcal{M}_{2 \times 2}$).

The different types of tasks (Chevallard 2005) associated with the exercises proposed in the questionnaire are described in (De Vleeschouwer 2008). We only propose here a short description of types of tasks present in the questionnaire :

- « Example of linear form », noted **T_Exemp_FL** : given a (sub-)vector space, give an example or counter-example of a linear form.
- « General expression of a linear form », noted **T_ExpGen_FL** : given a (sub-)vector space, describe a general expression of a linear form defined on the studied space.
- « Primal and dual basis », noted **T_Base_P&D** : given a n -(sub-)vector space and a set of n vectors of the considered vector space, determine if this set is a basis of the vector space and if it is, to find the dual basis.

For the rest of or study, we had to subdivide the type of tasks **T_Base_P&D** into sub-types of tasks :

- « Primal basis », noted **ST_Base_P** : given a n -(sub-)vector space and a set of n vectors of the considered vector space, determine if this set is a basis of the vector space .
- « Dual basis », noted **ST_Base_D** : given a n -(sub-)vector space and a set of n vectors of the considered vector space, determine its dual basis.

- « Coordinates functions », noted **T_FctCoor** : given a basis and its dual basis, determining the coordinates of a vector from the primal vector space.
- « Definition of the transpose transformation », noted **T_Def_TTransp** : given a linear transformation defined on a (sub-)vector space, to define its transpose transformation.

2.2. THE GROUP WORK

The group work (GW) is composed of several parts, that we will not present in details in this article. The two first parts of the GW are corresponding to the questionnaire. What follows complements then the questionnaire, notably :

- asking for the relation between the two parts of the questionnaire ;
- taking the same plan that the two parts of the questionnaire, but in the algebraic theoretical frame because « il s'agit de proposer des apprentissages qui portent sur divers cadres à propos d'une même connaissance »¹ [Robert 1998, p.155]. Knowing that « ce n'est pas toujours le travail dans un cadre général, formel, qui est le plus difficile »² [Robert 1998, p.151], we adapt the common plan of the two parts of the questionnaire notably with bringing new types of tasks for the algebraic theoretical frame. For example, concerning the transpose :
 - « Representation of the transpose », noted **T_Repr_TTransp** : explain, choosing one or several semiotic representation registers, what represents the transpose transformation. We want to know if the students think that the transpose transformation is defined on the dual space, or if they feel that the transpose transformation applied to a linear form is in fact the compound of the linear form and the initial transformation.
 - « Properties of the transpose », noted **T_Prop_TTrans** : establish or prove transpose's properties. Especially, we ask the students if it is possible to claim that $(f')' = f$. They have then to justify their answer. That question allows us to investigate the students' perception about the relation between the bidual and the primal and more especially about the canonic isomorphism between these two finite-dimensional spaces.

3. RESULTS OF THE SURVEY

The first observations of the analysis of the student's answers to the survey lead us to perceive different natures of students' difficulties when learning duality. Drawing on this analysis, and on our analysis of the way duality is structured in textbooks, and articulated with linear algebra (DeVleeschouwer 2008), we have chosen to classify the appeared difficulties in three main categories: difficulties tied to an insufficient mastery of elementary concepts of linear algebra, difficulties common to the

¹ « We have to propose learnings which concern diverse frames about the same knowledge. »

² « It is not still the work in the general, formal frame, that is most difficult. »

elementary linear algebra and duality, and finally difficulties specific to duality. Naturally, intersections between these categories are possible.

Some difficulties, obviously, are even more general : for example, we observed a confusion between a function f and the value of the function in an element of the departure's space : $f(x,y,z,t)$. Another well-known fact is that mathematical writing is not mastered by the students yet (obstacle of formalism, Dorier 2000). We don't detail here these types of difficulties, preferring to focus on linear algebra.

All the listed difficulties can be analyzed from an institutional point of view (the same object is differently considered in different institutions). In particular, we shall show (section 3.2) that the difficulties listed in the third category can be interpreted in term of second type of transition (Winsløw 2008).

3.1. OBSERVATION AND CLASSIFICATION OF DIFFICULTIES IN DUALITY

3.1.1. Insufficient mastery of elementary concepts of linear algebra

By elementary concepts of linear algebra, we mean concepts considered as elementary *with regard to* the notion of duality which we study.

Let us consider for example the notion of linear application or linear form. Indeed, only 62% of the students who answered to the questionnaire give a correct example of linear form within the frame of \mathbb{R}^4 . This rate decreases to 27% in the matrix frame.

The students also have difficulties to build examples of vector spaces. They propose for example the set of polynomials of degree 3; or still the set of polynomials of degree superior or equal to 3. Asking the students to design for examples, is frequent at the university, and hardly present at secondary school; it is thus difficult for novice students (Praslon 2000).

We can also notice that generally speaking, the students prefer to work within the frame of \mathbb{R}^4 rather than within the frame of matrices. The exercises corresponding to the various types of tasks are also better solved there. The vector space of the 2×2 matrices is not familiar to the students. In the University institution, it is necessary to consider objects recently defined in linear algebra as familiar objects on which and from which we are going to work. For example the fact that the object *matrix* can be considered as an element of a vector space, that's to say a vector. We can thus consider the coordinates of a matrix, or define linear applications acting on matrices.

Being able to change frames is important for the learning of a notion. In the case of duality this requires in particular the knowledge of several vector spaces.

3.1.2. Difficulties common to linear algebra and duality

We also observe difficulties common to elementary linear algebra and to duality, for example the confusion between a vector and its coordinates. This confusion, well known in linear algebra (Dorier 1997), becomes crucial when learning duality.

Within the framework of 4-tuples, we could say that the confusion between vectors and coordinates is natural or unnoticed. We can think that it is one of the reasons for which the students privilege this frame in the questionnaire. We notice that the students tend to work with the coordinates of objects (vectors, matrices, linear forms) and not with objects in themselves. So, it is frequent to see appearing in the answers the equality between the i^{th} linear form of dual basis (often noted y_i by the students) and the 4-tuple taking back its coordinates in the canonic base (that the students nevertheless learnt to note $[y_i]^{e'}$).

Another problem that we identified is the fact that the students prefer to present the solution of an exercise as an element of the vector space being of use as frame to the task (\mathbb{R}^4 or $\mathcal{M}_{2 \times 2}$). So, during the resolution of exercises corresponding to the type of task T_FctCoor, concerning the computation of the coordinates of an element (quadruplet or matrix) of the considered vector space, it is frequent to see students presenting calculated or deducted coordinates (in the second part of the questionnaire by analogy with regard to the first part) as a 4-tuple or as a matrix.

So, the only student having correctly solved the exercise corresponding to the type of task T_TTransp within the framework of 4-tuples ends then his answer by identifying $f'(y)$ with a 4-uplet containing his coordinates in the canonic dual basis, without mentioning however these are coordinates in this basis. In the matrix frame, this student presents the transpose in the form of matrix.

3.1.3. Difficulties directly related with duality

We can also classify difficulties directly related with duality, often connected with the very abstract character of the involved objects. It will lead us naturally to the following section dealing with the “concrete-abstract” transition (Winsløw 2008).

The definition of the transpose transformation can illustrate our comments because it is about a transformation defined on a vector space which elements are linear forms.

So, during the resolution of an exercise corresponding to the type of tasks T_Def_TTransp, within the frame of 4-uplets, three students mix up the transpose transformation with the inverse transformation. They have a general idea of a “reverse” process, associated both with inverse and with transpose. We also can notice, within the frame of \mathbb{R}^4 , that some students don't even try to work with the given transformation : they only give the theoretical definition of the transpose or another explanation onto what they think the transpose should be, without trying however to resolve effectively the proposed task. For these students, the transpose is only a part of the abstract world, and they don't manage to mobilize it in a contextualised frame.

Within the frame of the 2x2 matrices, we find almost the same proportion of students working with the given transformation among the students trying to solve the question corresponding to the type of tasks T_Def_TTransp. But in this frame, the answers are more varied because the students associate the proposed type of task with

a notion approached on the institution secondary school in Belgium : the transpose matrix. For example, to resolve an exercise depending from the type of tasks $T_Def_TTransp$ in the matrix frame, some students simply take back the matrix which is given to them in the statement and transpose it. The notion of matrix dominates on the notion of application when the term “transpose” is used.

3.2. « CONCRETE-ABSTRACT » TRANSITION

The difficulties directly related to duality presented in the previous section can be interpreted in terms of "concrete-abstract" transition (Winsløw 2008), which corresponds to the second type of transition described in the section 1. According to Winsløw, in the secondary school institution, it is essentially the "practico-technical" block of the mathematical organizations that is worked. This coincides with what we can notice when we analyze the answers of the students who were asked to say, in the work group, if there is, according to them, a link between the first two parts (\mathbb{R}^4 frame and matrix frame). The students concentrate themselves on the practico-technical part of mathematical organizations described by Chevallard (2005), and generally let down the technologico-theoretical block. Indeed, students answer that “both exercises represent the same transformations in two very similar vector spaces” and that “the question 2 is exactly the same than the question 1, there is only their representation which changes”. By using the term “similar”, the students do not identify the vector spaces, but indeed elements constituting the vectors of each of these two spaces. The students notice that only the “representation changes”. We can suppose that by writing it, the students think of applying identical techniques (computation of dual basis,...) to the various proposed statements. Always concerning the link between both parts of the questionnaire, the other students say, in the end, that "we find the same solutions". They fall again into the practico-technical block : according to them, the numerical values appearing in the solution are the most important. They do not mention the isomorphism used to justify this practice.

In the University institution, the technologico-theoretical block takes more importance. It is a first transition. Some students already adapted to this evolution. To illustrate our comments, let us turn to the exercises corresponding to the types of tasks T_Exemp_FL and T_ExpGen_FL . Even if these exercises did not a priori require any justification, a student justifies explicitly the fact that the supplied example is a form and also that the linearity is verified.

A second transition appears when elements constituting the technologico-theoretical block of a mathematical organization become elements on which calculations will be made and in which techniques are going to be applied. These elements constitute then the practico-technical block of new mathematical organizations. It is what happens when we work with the duality as an object : linear forms are considered as vectors because the set of linear forms is a vector space. The theories developed on the dual justify techniques applied to the linear forms. But when we consider the transpose transformation, the dual shifts from the technologico-theoretical block of a previous

mathematical organization to integrate the practico-technical block of a new mathematical organization, because the dual is then considered as the departure space of the transpose transformation. According to Winsløw, this second transition is even more difficult than the first one. Indeed, concerning the type of tasks T_Def_TTransp for example, we observe that the students have difficulties to define correctly the departure space of the transpose transformation.

However, when we ask the students, in the group work, if we can assert that $(f')' = f$, we notice that the question is very well answered by all groups. To solve a task of the type T_Prop_TTrans presented in an algebraic theoretical frame, the students choose, rightly, the technologico-theoretical block. For the transpose of the transpose, the students agree spontaneously to look for the solution in the theory. Sometimes, to make the link between the theory and the examples is more difficult than to stay in the theory.

4. CONCLUSIONS, DISCUSSION AND PERSPECTIVES

We classified the difficulties observed in the students' answers in three principal categories: the difficulties tied to an insufficient mastery of elementary concepts of linear algebra, those common to the elementary linear algebra and duality, and finally those specific to duality. We have seen, particularly, that the movement from elementary linear algebra to duality can be interpreted as a transition, according to Winsløw's meaning (2008). This confirms that transitions exist beyond the precise moment of the university's entry.

So, proposing a teaching device which searches to improve the learning of duality, asks to sit solid bases of linear algebra, and to devote specific attention to very abstract concepts as the transpose; but also to think about transition between elementary linear algebra and duality.

We will use these facts to propose an experimental teaching of duality in first year of university, in a further stage of our work.

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APPENDIX 1 : Questionnaire

To answer the questions below, you may use as you prefer, the formal mathematical language, the French language, graphics or drawings,...

- Consider the vector space, built on the field of reals.
 - Give an example on a linear form defined on \mathbb{R}^4 .
 - Give the general expression of a linear form defined on \mathbb{R}^4 .
 - Given $x_1 = (1, 2, 0, 4)$, $x_2 = (2, 0, -1, 2)$, $x_3 = (1, 0, 0, -1)$, $x_4 = (2, 0, 0, 3)$; given $X = \{x_1, x_2, x_3, x_4\}$. Is the set X a base of \mathbb{R}^4 ?
If yes, determine its dual basis.
 - If the set $X = \{x_1, x_2, x_3, x_4\}$ defined above is a basis and if you were able to compute its dual basis, what could be the coordinates of the vector $(15, 8, 10, 5)$ in the basis X ? Please explain your solution.
 - Given the linear transformation $f: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ so that $f(x, y, z, t) = (2x - t, 2y - z, x - y - t, -3z)$.
How will you define the transpose transformation?
- Consider the vector space $\mathcal{M}_{2 \times 2}$, the vector space of 2 lines, 2 columns matrices, with real coefficients, built on the field of reals.
 - Give an example of linear form defined on $\mathcal{M}_{2 \times 2}$.
 - Give the general expression of a linear form defined on $\mathcal{M}_{2 \times 2}$.
 - Given $M_1 = \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}$, $M_2 = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$, $M_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $M_4 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$; given $X = \{M_1, M_2, M_3, M_4\}$.
Is the set X a basis of $\mathcal{M}_{2 \times 2}$? If yes, determine its dual basis.
 - If the set $X = \{M_1, M_2, M_3, M_4\}$ defined above is a base and you had computed the dual base, what could be the coordinates of the matrix $\begin{pmatrix} 30 & 20 \\ 16 & 10 \end{pmatrix}$ into the base X ? Please explain your solution.
 - Given the linear transformation $f: \mathcal{M}_{2 \times 2} \rightarrow \mathcal{M}_{2 \times 2}$ so that $f \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 2a - d & a - b - d \\ 2b - c & -3c \end{pmatrix}$.
How will you define the transpose transformation?

LEARNING ADVANCED MATHEMATICAL CONCEPTS: THE CONCEPT OF LIMIT

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This paper looks for the difficulties of the students of tertiary educational level in the understanding of the mathematical concepts. Based on the Advanced Mathematical Thinking (AMT) notion and some cognitive theories about the construction of the concepts, it is intended to characterize the understanding of the concept of limit revealed by students in the beginning of tertiary educational level. Using the notion of concept definition and concept image, the theory of the reification and the proceptual nature of the concepts we try to identify these difficulties in students at a course of first year in Calculus. More specifically the main research question is to characterize understandings of advanced mathematical concepts at the beginning of tertiary education. A discussion of a mathematical-centred perspective of AMT is undertaken. The methodology used is of qualitative nature involving a teaching experiment. We conclude that it is possible to define three levels of concept image, incipient concept image, instrumental concept image and relational concept image that represent a progression in the level of understanding of the concept in study. These levels are based on objects, processes, properties, translation between representations and proceptual thinking that these students use when they intend to explain the concept.

CHARACTERISTICS OF ADVANCED MATHEMATICAL THINKING

The development of the mathematical thinking of students since the elementary level until the tertiary level or has been considered an important theme of study. David Tall and Tommy Dreyfus have written about these problems showing some of their essential characteristics in concrete situations. Tall (1995, 2004, 2007) characterizes the evolution of three worlds of mathematics under a perspective that shows the cognitive growth of the mathematical thinking. The conceptual-embodied world, based on perception of and reflection on properties of objects, the proceptual-symbolic world that grows out of the embodied world through action and symbolization into thinkable concepts, developing symbols that may be used as *procepts*, and the axiomatic-formal world that is based on formal definitions and proof.

The perceived objects are first seen like visual-spatial structures. When these structures are analyzed and their properties tested, these objects are described verbally and submitted to a classification (first in collections, later in hierarchies).

his corresponds to the beginning of a verbal deduction related to the properties and to a systematic development of a verbal demonstration.

Actions on the objects, for example, to count, lead to a type of different development. The process of counting is developed using numerical words and symbols that will be conceptualized as number concepts. These actions become symbolized as processes that later are encapsulated in *procepts*. This type of development that begins with Arithmetic, develops into Algebra and then in Advanced Algebra. In this approach, Tall (1995) makes a distinction between elementary and advanced mathematics, considering that the transition for the advanced mathematics occurs on the level of Euclidean demonstration and Advanced Algebra. This characterization, that places advanced mathematical thinking on the level of formal geometry, of the formal analysis and formal algebra supported by the formal definitions and logic supports the development of a creative thought and the investigation.

The distinction between the two ways of thinking is blurred in Dreyfus (1991) when he considers that it is possible to think on topics of advanced mathematics using an elementary form. He distinguishes between these two types of thinking by performing on the complexity which. He considers that there is not a prefunded distinction between many of the processes that are used in the elementary and advanced mathematical thinking. However advanced mathematics is essentially based in the abstractions of definition and deduction.

The processes that Dreyfus considers in the two types of thought are the processes of *abstraction* and *representation*, and the main difference is marked by the complexity that is demanded in each one. The processes involved in the representation are the process of representation beyond itself, the change of representations and the translation between them and modelling. The processes involved in the abstraction are generalization and synthesis. Dreyfus (1991) considers that, through representation and abstraction, we can move between one level of detail to another one and based on this movement we can manage the increasing complexity in the passage from a way of thought to the other. This vision of the Advanced Mathematical Thinking seems to be more useful for the study of the mathematical concepts because it places the emphasis in the complexity of these concepts and not in the level of formalization needed to develop understanding.

COGNITIVE THEORIES ON THE CONSTRUCTION OF THE MATHEMATICAL CONCEPTS

This study intends to identify the difficulties felt by the students in the understanding of complex mathematical concepts. We will briefly discuss the theories about concept definition and concept image, theory of reification and the proceptual thinking, where the symbols have an essential role.

Concept definition and concept image

The formation of the concepts is the one of the topics of main importance in the psychology of the learning. According to Vinner (1983) there were two main difficulties to deal with this question: one is linked with the notion of the concept itself and another with the determination of when the concept is correctly formed in the mind of somebody. A model of this cognitive process was based on the notions of *concept image* and *concept definition*. The concept image is something not verbal associated in our mind to the name of the concept. It can be used to describe the cognitive structure associated to the concept, that includes all *mental images*, all properties and all processes that may be associated to him. For *concept definition* it was understood the verbal definition that explains the concept in an exact mode and in a not circular manner (Tall and Vinner, 1981; Vinner, 1983, 1991). This vision of the concept definition seems to be based on the teaching of the mathematical concepts at the end of secondary education and in tertiary education, where is possible to present a formal mathematical definition for the concept. It is this definition that is reported by Vinner as being part of the concept definition, being all the other representations associated to the concept included in the concept image. This form of thinking seems to induce that the mind and the brain can be separate. However for Tall (2008) the mind is thought as the way in which the brain works and consequently it is an indivisible part of the structure of the brain. Thus, instead of a separation between concept definition and concept image, Tall considers that the concept definition is no more than one part of the total concept image that exists in our mind. For him, the concept image describes the total cognitive structure that is associated with the concept, this formularization is very close to that detailed above, while the concept definition acquires a statute that is not only linked to the formal definition such as it is conceived by the mathematicians. It is this conception that is followed in the development of the present study.

Theory of reification

Making the analysis of different representations and mathematical definitions we can conclude that the abstract concepts can be conceived of two different forms: *structurally*, as objects, and *operationally*, as processes (Sfard, 1987, 1991, 1992; Sfard and Linchevki, 1994). These two views seem to be incompatible, but they are complementary. It is possible to show that learning processes can be explained based in an interrelation between operational and structural conceptions of the same concepts. Based on historical examples and in light of some cognitive theories Sfard shows that the operational conception is usually the first step in the acquisition of new mathematical concepts. Through the analysis of stages of the formation of the concepts, she concludes that the transition from the operational mode to the abstract objects is a long and difficult process composed by the phases of *interiorization*, *condensation*, and *reification*. In *the interiorization* phase the individual makes familiar itself to the processes that eventually give origin to a new concept. The phase of *condensation* is a period of compression of long sequences of operations in more

easy manipulated. This phase is real while the new entity remains firmly linked to the process. But when the person will be able to conceive the notion as a finished object we can say that the concept was reified. *The reification* refers to the sudden capacity to see something familiar in a totally new form. The individual suddenly sees a new mathematical entity as a complete and autonomous object endowing with meaning. Thus, while interiorization and condensation are gradual and involve quantitative changes, the reification is an instantaneous jump: the process solidifies in one object, in a static structure. The new entity is quickly disconnected from the process that gave origin to it and starts to acquire its meaning by the fact it belongs to one definitive category. This state is also the point where the interiorization of concepts of higher level starts.

Proceptual thinking

Another perspective on the construction of the mathematical knowledge is proposed by David Tall (1995) and is based on the form as the human being, based in activities that interact with the environment, develop sufficiently subtle abstract concepts. The appearance of the Symbolic Mathematics has special relevance here. Given the nature of this type of conceptual development, symbols have an essential role, joining thinking the symbol as a concept or as a process. This allows us to think about symbols as manipulable entities to make mathematics. Gray and Tall (1994) consider thus that the ambiguity of the symbolism expressed in the flexible duality between process and concept is not completely used if the distinction between both remains in the mind. It is necessary a cognitive combination of process and concept with its own terminology. Consequently, the authors appeal to the term *procept* to mention the set of concept and process represented by the same symbol. An elementary *procept* will therefore be an amalgam of three components: *a process* that produces an mathematical *object* and *a symbol* that represents at the same time the process and object. To explain the performance in the mathematical processes Gray and Tall (1994) leave of the nature of the mathematical activities where the terms procedure, process and procept represent a sequence in the development of the concepts more and more sophisticated.

The proceptual thinking can be characterized by the ability to compress phases in the manipulation of the symbols, where they are seen as objects that can be decomposed and be recombined in a flexible way. This kind of thinking plays an essential role in the understanding of the mathematical concepts being the symbolism and its ambiguity the privileged vehicle for the development of this thought.

METHODOLOGY OF THE STUDY

This study is based on a qualitative methodology supported by observation of lessons. A design akin to a teaching experiment, involving semi-structured interviews, where students are invited to solve mathematical problems related to the tasks developed in classes followed-up by a discussion of their procedures, was used.

The study was performed at an institution of tertiary education of the region of Lisbon, where engineering courses are taught. The participants belonged to the course of Mathematics, Engineering Electrotechnic and Computers and Teaching of Natural Sciences. All the students attend during a semester the discipline of Mathematical Analysis I. The education process was developed around theoretical and practical lessons, where the concepts were essentially introduced based on their formal definition, which was later worked in the practical lessons based on the resolution of exercises. The lessons were observed by the investigator, having in the end of the semester lead interviews semi-structured to some of the students. Based on the interviews, in the comments of the lessons and documents produced by the students, we made an analysis of content and three levels of concept image of the students were identified: *incipient concept image*, *instrumental concept image* and *relational concept image*. The establishment of these levels is elaborated on the basis of the objects, processes, translation between representations, properties and proceptual thinking that the pupils reveal when answers to the cognitive tasks that are placed to it. The case of the limit concept and examples of each one of the levels of the concept image are now presented.

IMAGES OF THE CONCEPT OF LIMIT

During the teaching process, the concept of limit was introduced on the basis of the following definition:

"Let's $f: D \subset \mathbf{R} \rightarrow \mathbf{R}$ and a an adherent point to the domain of f . One says that b is limit of f in the point a (or when x tends for a), and it is written $\lim_{x \rightarrow a} f(x) = b$, if $\forall \delta > 0 \exists \varepsilon > 0: x \in D \wedge |x - a| < \varepsilon \Rightarrow |f(x) - b| < \delta$.

The data presented below was part of a more general study (Domingos, 2003).

In the task placed to the students in the interview situation we made an approach that we can consider with characteristics of an teaching experiment. We started with an concrete example, the expression

$\lim_{x \rightarrow 1} \left(\frac{x^2 - 1}{x - 1} \right) = 2$ and graphical representation

of the function $\left(\frac{x^2 - 1}{x - 1} \right)$ (figure 1), so that the

students could give a geometric interpretation that allowed them to support the symbolic translation of this concept. It is presented below a detailed characterization of each one of the concept images founded.

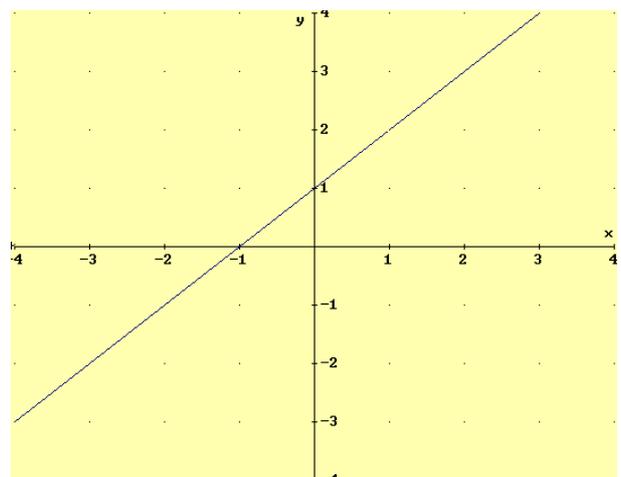


Figure1. Graph of the function $\frac{x^2 - 1}{x - 1}$ presented to the students (it has a "hole" in the graph in the point of absciss 1)

Incipient concept image

When Mariana is asked to explain the meaning of the expression $\lim_{x \rightarrow 1} \left(\frac{x^2 - 1}{x - 1} \right) = 2$, she says:

Mariana – Then, aaa... When *the* x tends... when *the* x tends to 1... the function comes closer to the image, of its image that is two... It is approaching 2...

She considers that the value of the limit is the image of 1. For such she relates the proximity of the images of the point 2 when x approaches 1. When the graph of the function (figure 1) is showed and she is asked for to explain the same situation based on it, she use the notion of proximity cited previously in terms of intervals:

Mariana – Then, aaa... In a small interval near of the 1, to the left [points to the graph] comes close to the 2. And on the right also it comes close to the 2.

Inv. – Therefore, you consider an interval here [indicated a neighbourhood of the 1, in the horizontal axis] and what happens here? [indicated a neighbourhood of the 2, in the vertical axis]... It has that to be always very close...

Mariana – Of the 2. In a neighbourhood ε .

Inv. – (...) Therefore, what you says is: when the x is in the neighbourhood of the 1... the images ...

Mariana – Are in the neighbourhood of the 2.

She makes use of to the lateral limits to explain her notion of limit considering separately a neighbourhood to the left of 1 and another one to the right of 1, but without having the concern to define also a neighbourhood in terms of the images. When the interviewer points to a singular interval at the neighbourhood of 1, she mentions the existence of a neighbourhood of 2 with ray ε . Using only the resources of the language of the neighbourhoods she does not provide the symbolic translation of any part of the definition. Then the interviewer supplied the formal description of this example as it might have occurred class (figure 2).

$$\forall \delta > 0 \exists \varepsilon > 0 : x \in D \wedge |x - 1| < \varepsilon \Rightarrow |f(x) - 2| < \delta$$

Figure 2. Symbolic representation of the expression $\lim_{x \rightarrow 1} \left(\frac{x^2 - 1}{x - 1} \right) = 2$ presented to students.

When she was asked to explain the meaning of $|x - 1| < \varepsilon$ in terms of neighbourhoods, Mariana did not provide any intended translation between the two representations:

Mariana – This $[|x - 1| < \varepsilon]$ is the neighborhood of the 1... Of ray 1. Not? ...

Her conception of neighbourhood seems to be based essentially on a relation of proximity in geometric terms but for which she does not provide a symbolic representation. She does not provide the translation between the different representations that are presented to her, showing some difficulty in following the suggestions made by the interviewer.

Mariana presents thus a concept image of limit essentially based on a geometric interpretation from which she retains a dynamic relation between objects and images. This does not allow her to attribute meaning to the symbolic definition where some of the most elementary procedures are translated by symbols.

Instrumental concept image

For José the explanation of the expression $\lim_{x \rightarrow 1} \left(\frac{x^2 - 1}{x - 1} \right) = 2$ his based on a graphical representation, even when such representations are not present. When mentioning the previous expression he detaches what happens in the vertical axis "is that the function is come close to the 2... of the *YYs*". He relates what happens with the images in the vertical axis and when confronted with the graph of figure 1, finishes by saying:

José – When we approach here in the axis of the *XXs* for 1, of the two sides... It is going to tend for 2, in the axis of the *YYs*. It's approaching the 2.

José shows the processes that underlie the relation between the objects and the images. He also shows that he sees as a dynamic relationship.

When asked to establish the symbolic representation of limit he says he cannot do it, but provides the translation of some of the processes that he described previously. Thus when he refers to the fact that the x approaches 1 he suggests that it can be represented by "1 minus x less than anything" and as the x approaches the right and the left he considers that it can use the module and writes $|1-x|$. He even considers that this module must be smaller than a very small value, he does not use any symbol to represent it and when the investigator suggests that he can be ε , he does not know how to write this symbol. In the same way he establishes what happens in the neighbourhood of the limit. Using the module symbol he writes $|2 - f(x)|$ considering that also it can be minor that ε . He uses the same symbol ε in both cases, not because he is convinced that both must be equal, but because he does not remember of another different symbol. When the investigator tries to explain that this parameter cannot be the same, he uses α , and writes $|2 - f(x)| < \alpha$. When asked to describe the role of quantifiers José imagines that the universal quantifier is applied to ε . It seems that he considers that any object has an image and therefore the universal quantifier would be related to the objects. Finally, he writes a symbolic definition (figure 3), showing some difficulty in drawing the symbols of the quantifiers, and was not able to explain their role in the definition.

$$\forall \alpha > 0 \exists \varepsilon > 0 : \omega \in A \wedge |1 - \omega| < \varepsilon \Rightarrow |2 - f(\omega)| < \alpha$$

Figure 3. José's symbolic representation of $\lim_{x \rightarrow 1} \left(\frac{x^2 - 1}{x - 1} \right) = 2$.

José's concept image of limit it can thus be characterized by incorporating a complete graphical component that allows him to relate the objects and the images

dynamically. Based in this component he symbolically translates some parts of the concept, namely what happens in the neighbourhood of the point for which the function tends and on the limit point. However he is not able to give meaning to the quantifiers as well as identifying the symbols that represent them.

Relational concept image

To Sofia the explanation of the expression $\lim_{x \rightarrow 1} \left(\frac{x^2 - 1}{x - 1} \right) = 2$ is based in a graphical sketch (figure 4):

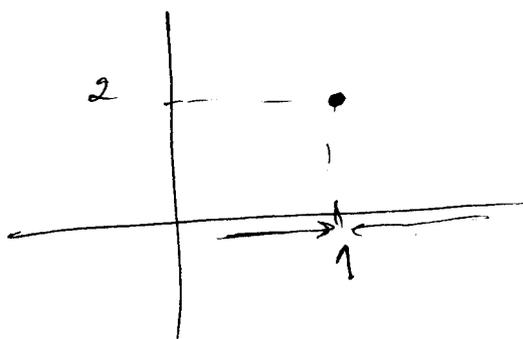


Figure 4. Graphical sketch that translate the notion of limit of Sofia.

Sofia – Then we are saying that when the x , that is... If here we will have the 1. We are to say here in this in case that, when the x is to tend for 1.

Inv. – Hum, hum.

Sofia – For different values of 1, I think that is different, yes because this never can... the images is to approach it... (...) of 2. Therefore the function, here is the point of the function or ...

Sofia starts to explain her notion of limit using a system of axis, without representing the function graphically. She uses it to describe the fact that x tends to 1 and the images tends to the value of the limit, 2. This representation caused some apprehension to her because she needs to materialize the image of the 1 in the sketch. She finishes her concluding that this point does not belong to the domain, and then she needs to consider that it should tend for different values of point itself. Based on this boarding she establishes the symbolic definition:

Sofia – I think that it is thus. For all the positive delta, exists one epsilon positive, such that the x belongs to \mathbb{R} except the 1... And... x aaa... $x-1$ has that to be minor that epsilon, and there that is ... $f(x)$ minus 2, module, minor that delta.

[She writes the expression of figure 5]

$$\forall \delta > 0 \exists \varepsilon > 0 : x \in \mathbb{R} \wedge |x-1| < \varepsilon \Rightarrow |f(x)-2| < \delta$$

Figure 5. Sofia's symbolic writing of $\lim_{x \rightarrow 1} \left(\frac{x^2-1}{x-1} \right) = 2$.

In this way Sofia translates symbolically the limit under study. It seems that she did not memorize the definition, because when she establishes the role of the parameters ε and δ , she draws them in the graph of figure 1, representing the ray of the neighbourhoods centred in the points of abscissa 1 and ordinate 2 respectively. It is in the role of the quantifiers that inhabits the main difficulty, over all when she intends to explain how they influence the reach of the definition.

Sofia's concept image of limit seems to be the result of the coordination of the some underlying processes, through which she relates the different representations of the concept, conferring to them some generality, with exception to the role played for the quantifiers.

CONCLUSIONS

Based on cognitive theories of the learning and in the notion of advanced mathematical thinking it is possible to identify the complexity involved in the understanding of these concepts. In the cases studied, the analysis of the answers of students allowed us to verify a satisfactory verbal performance of the concept. However, when translating this verbal ability into a symbolic representation, performance decreases significantly as anticipated. The key findings of this study, however, lie on the distinction among three levels of concept image, namely: a) an *incipient concept image*, translating verbally only some parts of the symbolic definition; b) an *instrumental concept image*, making the symbolic translation of some parts of the concept; and c) a *relational concept image* that is translated into the capacity to represent the concept symbolically. These findings are relevant to AMT in the sense that they characterize complex concept images with greater accuracy. Further studies must deepen these distinctions.

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CONCEPTUAL CHANGE AND CONNECTIONS IN ANALYSIS

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The paper presents a work in progress which is part of a larger study. Students learning analysis was investigated with the aim to find out how their concept images changed from the beginning of an analysis course to a year after the course. Their links between concepts were studied after the year had passed. The influence of the students' pre-knowledge was durable and sometimes prevented students from making connections or abstractions.

Key-words: Mathematics, analysis, university students, concept development, concept image.

INTRODUCTION

Mathematical analysis comprises several challenging concepts to link together. Conceptions change as they are evoked. The changes may be irrelevant to the over all conception, for example just another experience of a routine operation, or they can have an important impact on related concepts if, for example, a misconception is revealed and rectified. Conceptions that are not evoked may also change over time. The changes, if not sturdily enough integrated to prior knowledge or used, sometimes revert to former constellations as if they never occurred (Smith, diSessa & Rochelle, 1993). The present study deals with changes over time as three students were asked to explain their conceptions of functions, limits, derivatives, integrals and continuity before a course and then again a year after.

The research questions posed are: What relevant pre-knowledge do students have at the start of a basic analysis course? How have the conceptions changed a year after the analysis course? How do the students connect different concepts in analysis a year after the analysis course?

DEVELOPMENT OF CONCEPT IMAGES

A concept image (Tall & Vinner, 1981) encompasses representations of concepts and processes learned or just briefly perceived arranged in mental networks. Impressions from instructions, discussions, solving tasks and reading, which all lead to mathematical thinking, have an impact on the development of the concept image. Tall's (2004) three worlds of mathematics depict a development from just perceiving a concept through actions to formal comprehension of the concept. The first world is called *the embodied world* and here individuals use their physical perceptions of the real world to perform mental experiments to create conceptions of mathematical concepts. Intuitive representations naturally develop here from the lack of stringency. The second world is called *the proceptual world*. Here individuals start with procedural actions on mental conceptions from the first world, as counting, which by

using symbols become encapsulated as concepts. The third world is called *the formal world* and here properties are expressed with formal definitions and axiomatic theories comprising formal proofs and deductions. Individuals go between the worlds as their needs and experiences change and mental representations of concepts are formed and altered in the concept images.

Understanding a concept means that an individual is able to connect that certain concept to his or her concept image in a significant way (Hiebert & Lefevre, 1986) which is different from just being able to perform a particular operation. Pinto and Tall (2001) described two ways of understanding a concept, through formal or natural learning. A formal learner uses definitions and symbols as a ground, whereas natural learners logically construct new knowledge from their concept images. The former has, if successful, a neat structure to build on, but, if not, a meaningless mass of symbols. The latter may have problems to formalise the knowledge from their concept images as there is a risk of problems to separate formal representations from their own, perhaps intuitive or naive, images. One benefit from natural learning is the logical understanding of concepts' relatedness that comes from reconstruction.

New concepts are sometimes introduced intuitively, perhaps with an image, which lays the ground for more strict representations later on as the learner is able to link the intuitive representation to a stricter one or a complete one. Images of concepts can however work in a way opposed to the intended as Aspinwall, Shaw and Presmeg (1997) found in their case study on mental imagery. A person's concept image can confuse, rather than ease making sense of concepts and links between them, if it does not cohere with formal concept definitions, i.e. definitions of mathematical concepts generally used in the mathematics society.

Research expose students' struggle to link intuitive representations to formal representations (e.g. Cornu, 1991; Juter, 2006; Sirotic & Zazkis, 2007; Williams, 1991). Sirotic and Zazkis claimed that underdeveloped intuitions often are due to flaws in formal knowledge and an absence of algorithmic experience. Links between intuitions, formal knowledge and algorithms are necessary for anyone to understand the topic at hand. Functions, limits, derivatives, integrals and continuity are tightly linked together in an analysis course. All topics comprise studies of functions. Derivatives and integrals are defined by limits of different kinds (limits of difference quotients and sums of infinitely thin rods respectively). Derivatives and integrals have a quality of being each others inverses with the possible exception of constants. Continuity is closely linked to limits by their definitions, and also to derivatives since differentiability is a stricter condition than continuity of the function's smoothness. Merenluoto and Lehtinen (2004) studied students' conceptual changes at upper secondary school. The concepts density, limit and continuity were studied in connection to number. The students showed almost no links relating the different concepts. The endurance of prior knowledge was one reason for the students' disjoint concept images. Hähkiöniemi (2006) investigated students learning the derivative and

concluded that students had difficulties to link their procedural conceptions to formal mathematics. A similar result was drawn from a study on students learning limits of functions (Juter, 2006) where students' intuitive perceptions often were incompatible with the formal concept image leaving the students with two incoherent representations, one for theory and one for problem solving. Students' struggle with separated concept images from disability to formalise the intuitive representations and the lack of links to other concepts causes the feeling of a threshold for the students to surmount. Viholainen (2006) has also presented results of students' difficulties to use concepts in the embodied world in a constructive correct manner when they worked with continuity and differentiability. This means that some students have an intuitive sometimes procedural conception of the concepts and need guidance to take the next step to formalise their knowledge.

THREE STUDENTS' CONCEPTIONS

The students investigated were enrolled in an analysis course. The part presented here is part of a larger study of students' pre-knowledge and their knowledge at times after analysis courses in mathematics teacher education (Juter, in press). The students were aged 19 years or older. Three students were selected in a group of 15 for further investigations, based on their results on the exam and on their responses to initial queries, so that there was an average achieving student, one higher achieving and one lower achieving student. The course was part of their teacher education programme, but it was also given outside the program. All students had, at least, had an introduction to the concepts studied in this article at upper secondary school.

The course was given fulltime over ten weeks. The students had two lectures (40 minutes each) and two sessions for problem solving (40 minutes each) twice every week which gives a total of 80 lessons and problem solving sessions. The syllabus of the course included limits of functions, continuity, derivatives, and integrals (i.e. the topics studied in this paper) with derivatives and integrals as main parts of the course. Differential equations, parametric equations, polar coordinates and infinite sequences and series (Taylor and Maclaurin series) were also taught. The students worked in groups with tasks between the scheduled sessions. The tasks were designed to help the students understand definitions and theorems, e.g. the intermediate value theorem and the limit definition: A is called the limit of $f(x)$ as $x \rightarrow a$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - A| < \varepsilon$ for every x in the domain with $0 < |x - a| < \delta$.

On their first session of the course the students filled out a questionnaire where they were asked to describe the concepts and also to write what the concepts are used for. The concepts in the tasks were not specified other than functions, limits, derivatives, integrals and continuity. The reason for this open approach was to prevent the students from becoming restrained with other formulations than their own. The aim was to keep the students from writing what they thought was expected of them and in stead let them explain in their own words.

One year after the course, the three selected students were interviewed. They got the same questionnaire about functions, limits, derivatives, integrals and continuity as before the course. In addition, four graphs were presented for the students to determine differentiability, integrability, limits and continuity at all points. At the end of the interview they got a table with words or phrases listed in connection to the concepts studied. The words were selected from the students' prior descriptions in the questionnaire and from formulations in the textbooks used in the course and lectures. The aim was to evoke different characteristics in the students' concept images of the different concepts and from that see how they linked them together.

The design with only a questionnaire at the beginning of the course and interviews after means that there is much more information about the students' concept images after the course leaving the results somewhat unbalanced. The questionnaire was used for selecting students to interviews as well as revealing their conceptions of the concepts and it was not possible to conduct interviews with all students to make such a selection.

Pseudonym names, Alex, Ian and Kitty, are used to retain anonymity for the students. The sample selection was done based on their questionnaire responses to become as representative as possible of the group. Kitty was achieving a bit higher than average students scoring the highest mark, VG (passed with honours), on the exam, Ian was a typical average student awarded the mark G and Alex achieved somewhat lower as he did not pass on the first try, but got a G (passed) on the second.

Students' conceptual change over a year

The results are presented in tables 1 to 3 which show the students' individual responses, before and a year after the course, to the five tasks: Describe the concept of function/limit/derivative/integral/continuity in your own words.

Table 1. Alex's responses to the five tasks before and one year after the course

Alex	Before the course	A year after the course
Function	A function is an approximation like an equation with the difference that you can picture a function on a graph.	$y = kx + m$ is a function for me, you use x and y . You can draw a graph on it.
Limit	A limit is what the word "says", a limitation so you know for example within what values to stay.	When you press these [the end points of an interval on the y -axis close to the function] together as much as possible

		you get a limit.
Derivative	You can describe a derivative as a means to “simplify” equations. It is something you do to get other functions in a graph.	You change the function [...] you can get more information from the function, you see the function differently.
Integral	The opposite to derivative. Is used as derivative but in reversed meaning.	You change a function, get different information.
Continuity	It [the function] behaves the same way all the time. There are no “surprises” in the graph.	A continuous function [...] changes in a re-occurring pattern all the time. [Linear and sine functions are given as examples]

Alex’s perceptions from before the course endured the course and a year after for the concepts function, derivative and integral. A severe misconception is clear from his descriptions of derivative and integral as he saw them as means to simplify or change functions. He was unable to explain the concepts in more detail. The changes he made on limits remained for the year with an emphasis on the limit definition and the illustration used in the course literature and in the lectures. Illustrations worked in a fruitful manner as the image had become a constructive part of his concept image. He was not able to present a formal definition of any of the concepts.

Table 2. Ian’s responses to the five tasks before and one year after the course

Ian	Before the course	A year after the course
Function	A sequence of events presented by a formula or a coordinate system.	A sequence of events but on paper in a graph so to say [...] or a system, a coordination [changed later to coordinate] system.
Limit	Limits are either maximum or minimum values in the function	There are several kinds of limits [...] maximum and minimum values [...] average value of the curve.
Derivative	The derivative of a function is used to show what values are maximum and minimum.	If you take the derivative of something, you get for example velocity and acceleration and so, but I do not remember.

Integral	-	You go in the opposite direction [to derivative]. In stead of acceleration to velocity you take velocity to acceleration.
Continuity	It [the function] moves the same way all the time, for example the sine curve.	It was this funny thing [...] it did not have an infinite value. The curve may not shoot off upwards or downwards [...] it often becomes a gap in the curve but then it may shoot straight up or something. [...] If it is continuous then it is whole.

Ian used similar descriptions before and after the year on the concepts of function and limit. He perceived a function both as a process, a sequence, and an object, the coordinate system, at both times. Limits, integrals (after the year) and derivatives were process oriented in their descriptions with an emphasis on applications. Continuity was first seen as a process, i.e. as a function that moves the same way. A year later, his description focused the graph as an entity with the feature of being whole. Before the course, he had no description of integral despite his experiences from upper secondary school.

He was unable to give any formal definition for the concepts.

Table 3. Kitty's responses to the five tasks before and one year after the course

Kitty	Before the course	A year after the course
Function	A function is a constructed series of events.	Numbers and an x to determine. A graph.
Limit	A limit is something you calculate as something tends to for example zero or infinity.	A graph [...] closing in on a value but it never gets there.
Derivative	You derive a function and get for example zero values.	Area under a graph. [first but after some thought about integrals changed to:] A measure on how fast something accelerates.

Integral	Reversed derivative where you calculate the area under a function on a certain interval.	Area under a graph divided in small rectangles depending on how accurate you are.
Continuity	When there are no gaps in the graph and there is only one x -value per y -value.	If you go from one value to another there can not be any gaps in it.

Kitty had a conception of functions similar to Ian's before the course as a series of events, but she changed it to a view of the objects used when working with functions. On limits, she went from calculating to the limiting process, with the not so unusual misinterpretation that limits are unattainable (e.g. Cornu, 1991; Juter, 2006; Williams, 1991). There was obvious development in Kitty's concept image that remained for the year on derivative and integral. She presented no formal definition though.

Kitty had some confusion of her conceptions during the interview but she was often able to alter her concept image when needed. One example is concerning continuity and derivatives when she had answered the question about continuity in table 3:

Kitty: And then there was something about not having any edges.

Interviewer: Peaks and so you mean?

Kitty: Yes ... or perhaps it was continuous then too, but there were something about those peaks anyway.

Interviewer: Yes.

Kitty: Maybe that you could not take the derivative on those peaks or something like that ... no I might be thinking incorrectly.

[The interview goes on and four graphs are presented where Kitty shall determine differentiability, integrability, continuity and limits. One has a peak.]

Kitty: If you derive, to determine how the other curve [the derivative] shall look, are you not supposed to draw those lines to see? [She shows a tangent line with her finger]

Interviewer: Mm.

Kitty: And that is impossible at the peak there because then you do not know if it, because it is pointy, you do not know what slope it has.

Kitty worked with her existing knowledge and found out the logical and correct properties. This way of reasoning was typical for her during the interview.

Networks of concepts

The students connected different concepts and processes together with relevant links, i.e. links that are correctly justified and true as well as irrelevant or untrue links.

Typical examples of relevant and true links were, for example, Alex's link between difference and limit in the sense that the difference is between the borders in the interval $|f(x) - A| < \varepsilon$ from the limit definition. Kitty connected change to derivative as she said: "Derivative [...] is a measure of change [...] how the velocity change, kind of, and then you draw it". Ian, slightly vaguely, linked sums and integrals and explained: "If you calculate the area under the curve you get a sum". Ian had a revelation when he tried to explain the connection between limits and continuity:

Ian: A graph can be continuous, that is what you mean?

Interviewer: Mm.

Ian: But it can at the same time be a straight line or go straight up.

Interviewer: Mm.

Ian: And then there is no limit on it so ... yes there is an outer limit ... but then there is a limit. Yes, then we take continuity on it [marks the box linking limits and continuity at the paper].

Ian managed to reason with himself to make sense of the relation between the concepts, similarly to Kitty's strategy.

The patterns of links were different for the students. Alex had, by far, the highest number of links between concepts but if the selection was restricted to relevant links Kitty had the most links. She also had irrelevant or wrong links, but only few. Alex had several links to continuity, none of them relevant whereas Ian and Kitty only had a few each where Kitty had one and Ian three relevant links. Derivatives and integrals were the two topics with the highest rate of links as could be expected from the syllabus.

CONCLUSIONS

The students had pre-knowledge of various characters when they came to the course as tables 1 to 3 show. Some pre-conceptions endured the course and a year, for example Alex's unfortunate perceptions of derivative and integral as means to change functions and Ian's more practical view of functions. Kitty's concept image of integrals was partly the same but a development of further understanding had occurred (table 3). Building up concepts this way is stable since no changes of prior knowledge are required, there is only a phase of adding new knowledge strongly linked to the former.

A drawback of pre-conceptions is when they are wrong and remain, despite teaching and own work within a course stating the opposite of the pre-conceptions (Smith, diSessa & Rochelle, 1993). Alex's interpretations of derivatives and integrals are obvious examples of such wrongly established conceptions. A conception that has been there for some time is not easily changed since it also demands changes in the

nearby parts of the concept image. Another reason to retain familiar conceptions is the comfort and security of the known that may not be readily surrendered.

Mental representations naturally connect to pictures, self constructed or otherwise, supporting understanding. All three students mentioned graphs. Ian, for example, described continuity as from a picture at the latter data collection. Kitty mixed up derivatives with integrals as she stated that the derivative is the area under the graph. When she, shortly after, was describing integrals she was able to make sense of her pictures of ‘areas under graphs’ and she went back to rethink derivatives. In Alex’s case of limits after the year the picture is easily recognised from lectures in the course. He had used the picture to strengthen his concept image in a, for him, useful manner. Pictures can however, as afore mentioned (Aspinwall, Shaw & Presmeg, 1997), cause confusion rather than insight. The same picture as Alex used give many students the impression that limits actually are the limits of the intervals from the absolute values in the limit definition mentioned before (Juter, 2006).

The lack of connections between limits and continuity and other concepts is clear and consistent with Merenluoto and Lehtinen’s (2004) results. The present study explicitly investigates the links between further concepts which gives a fuller image of the scarcity of appropriate links. The students’ naive or wrong pre-knowledge was not easily changed with the effect that they were held back from reaching Tall’s formal world (2004). Understanding these concepts is not the same as being able to formally express them. Students also need to have a strong and rich foundation tightly linked to the formal expressions which has been proven to be difficult (Hähkiöniemi, 2006; Juter, 2006; Viholainen, 2006). Kitty had a functional foundation to formalise and she showed evidence to be on her way to reach the formal world. Ian had less such evidence and Alex essentially none. The students in the study are future upper secondary school teachers in mathematics and their mathematical understanding need to be rich and well connected in order for them to be able to perform their profession satisfactorily.

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USING THE ONTO-SEMIOTIC APPROACH TO IDENTIFY AND ANALYZE MATHEMATICAL MEANING IN A MULTIVARIATE CONTEXT

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The main objective of this paper was to apply the onto-semiotic approach to analyze the mathematical concept of different coordinate systems, as well as some situations and university students' actions related to these coordinate systems. The identification of mathematical objects that emerge from the operative and discursive systems of practices, and a first intent to describe an epistemic network that relates these operative and discursive systems was carried out. Multivariate calculus students' responses to questions involving single and multivariate functions in polar, cylindrical and spherical coordinates were used to classify semiotic functions that relate the different mathematical objects.

Introduction

This study, in particular, embraces the aspect of thinking related to advanced mathematics. Mathematics education literature concerning university level mathematics, such as multivariable calculus, is relatively sparse. Yet it cannot be taken for granted that mathematical understanding at this level is unproblematic: the data from research such as that represented in this paper makes this clear.

The subject of curvilinear coordinates in the context of advanced mathematics requires transiting between the different coordinate systems (change of basis in the language of linear algebra) within a framework of flexible mathematical thinking. The achievement of conceptual clarity, while important in itself, is required in the context of applications in different areas (physics, geography, engineering) where a total lack of homogeneity in terms of notation, especially notorious when comparing calculus textbooks with those of other sciences, is presented (Dray & Manoge, 2002).

The issue of transiting between different coordinate systems, as well as the notion of dimension in its algebraic and geometric representations, are significant within undergraduate mathematics. Deep demands are made in both conceptual and application fields with respect to understanding and competence.

“The move into more advanced algebra (such as vectors in three and higher dimensions) involves such things as the vector product which violates the commutative law of multiplication, or the idea of four or more dimensions, which overstretches and even severs the visual link between equations and imaginable geometry.” (Tall, 1995).

On the other hand, argument is made for the onto-semiotic approach as representing a distinct difference from approaches seen as situated within paradigms of mathematical theories represented by set theory and classical logic. This opens the door to a possible modelling of the communication of advanced mathematics as a

semiotic system. The concept of semiotic function is addressed and related substantively to linguistic, symbolic and gestural expressions documented in situations that involve demanding mathematical connections.

Different Coordinate Systems

The mathematical notion of *different coordinate systems* is introduced formally at a precalculus level, with the polar system as the first topological and algebraic example. The emphasis is placed on the geometrical (topological) representation, and transformations between systems are introduced as formulas, under the notion of equality ($x = r \cos \theta, r = \sqrt{x^2 + y^2}$, etc.). The polar system is usually revisited as part of the calculus sequence; in single variable calculus, the formula for integration in the polar context is covered, as a means to calculate area. In multivariate calculus, work with polar coordinates, and transformations in general, is performed in the context of multivariable functions. It is in calculus applications that the different systems become more than geometrical representations of curves.

The different systems, which are related to each other by transformations, are meant to be dealt with through the algebraic and analytic theory of functions, although the geometric representation will still play a large role in the didactic process. As has been established (Montiel, Vidakovic & Kabaël, 2008), the geometric representations need to be dealt with very carefully. For example, it was reported that techniques such as the vertical line test, used to determine if a relation is a function in the rectangular context, were transferred automatically to the polar context. Hence the circle in the single variable polar context, whose algebraic formula $r = a$ certainly represents a function of the angle θ (the constant function), when θ is defined as the independent variable and r as the dependent variable, was often not identified as a function because, in the Cartesian system, it doesn't pass the vertical line test.

The graphs are symbolic representations of the process with their own grammar and their own semantics. It is for this reason that their interpretation is not unproblematic (Noss, Bakker, Hoyles & Kent, 2007, 381).

When multivariate functions are introduced in the rectangular context, in particular functions with domain some subset of \mathbf{R}^2 or \mathbf{R}^3 and range some subset of \mathbf{R} , the *institutional expectation* is that the student will “generalize” the definition of function. The assumption is that students have *flexible mathematical thinking*, that is, that they are capable of transiting in a routine manner between the different meaning of a mathematical notion, accepting the restrictions and possibilities in different contexts (Wilhelmi, Godino & Lacasta, 2007a, 2007b).

Research on the epistemology and didactics in general of multivariate calculus is virtually non-existent, and it is for this reason that no real literature review is given on the subject. It is a “new territory” that is being charted in this respect. Nonetheless, it is in the multivariate calculus course where students, many for the first time, are expected to deal with space on a geometric and algebraic level after years of single variable functions and the Cartesian plane. They must define multivariable and vector

functions, deal with hyperspace (triple integrals), find that certain geometrical axioms for the plane do not hold over (lines cannot only intersect or be parallel, they can also be skew), and work with functions in different coordinate systems. Students must learn operations that are dimension-specific (such as the cross product) and make generalizations which require *flexible mathematical thinking*. These are just some of the aspects which make multivariate calculus a rich subject for many of the research questions that arise when trying to analyze the epistemology, as well as the didactical processes, in the transition to higher mathematics.

On the other hand, multivariate calculus in itself, with its applications, is an important subject for science (physics, chemistry and biology), engineering, computer science, actuarial sciences, and economics students. For this reason, it is important to analyze the contexts and metaphors used in its introduction and development, because generally there aren't evident translations between college and workplace mathematics (Williams & Wake, 2007).

Conceptual Framework

Clarifying the meaning of mathematical objects is a priority area for research in Mathematics Education (Godino & Batanero, 1997). In this paper, a mathematical object is: "anything that can be used, suggested or pointed to when doing, communicating or learning mathematics." The onto-semiotic approach considers six primary entities which are (Godino, Batanero & Roa, 2005, 5): (1) *language* (terms, expressions, notations, graphics); (2) *situations* (problems, extra or intra-mathematical applications, etc.); (3) *subjects' actions* when solving mathematical tasks (operations, algorithms, techniques); (4) *concepts*, given by their definitions or descriptions (number, point, straight line, mean, function, etc.); (5) *properties* or attributes, which usually are given as statements or propositions; and, finally, (6) *arguments* used to validate and explain the propositions (deductive, inductive, etc.).

The following dual dimensions are considered when analyzing mathematical objects (Godino et al., 2005, 5): (1) personal / institutional; (2) ostensive / non-ostensive. (3) example / type; (4) elemental / systemic; and (5) expression / content.

The present study carries out analysis with this classification, and relies on the reader's intuition and previous knowledge to understand how they are used in the context. The emphasis on mathematical objects in the present study is represented by the words of Harel (2006) when referring to Schoenfeld:

A key term in Schoenfeld's statement is *mathematics*. It is the *mathematics, its unique constructs, its history, and its epistemology* that makes *mathematics education* a discipline in its own right. (p. 61)

The situating of onto-semiotic approach within the domain of theories such as category theory, and non-bivalent logic is much more than a mere academic exercise. In the ICMI study *Mathematics Education as a Research Domain: A Search for Identity*, Sfard (1997) stated that:

Our ultimate objective is the enhancement of learning mathematics...Therefore we are faced with the crucial question what is knowledge and, in particular, what is mathematical knowledge for us? Here we find ourselves caught between two incompatible paradigms: the paradigm of human sciences... and the paradigm of mathematics. These two are completely different: whereas mathematics is a bastion of objectivity, of clear distinction between TRUE and FALSE... there is nothing like that for us. (p. 14)

It is clear that the possibility of situating research in mathematics education within the paradigm of mathematical theories other than set theory and classical logic was not contemplated in the previous quote.

The onto-semiotic approach to knowledge proposes five levels of analysis for instruction processes (Font & Contreras, 2008; Font, Godino, & Contreras, 2008; Font, Godino & D'Amore 2007; Godino, Bencomo, Font & Wilhelmi, 2006; Godino, Contreras & Font, 2006; Godino, Font & Wilhemi, 2006):

- 1) Analysis of types of problems and systems of practices;
- 2) Elaboration of configurations of mathematical objects and processes;
- 3) Analysis of didactical trajectories and interactions;
- 4) Identification of systems of norms and metanorms;
- 5) Evaluation of the didactical suitability of study processes.

The present study concentrates on the first level, while touching on the second as well. The same empirical basis, with the same notions, processes and mathematical meanings will be used in future studies to develop the second and third aspects.

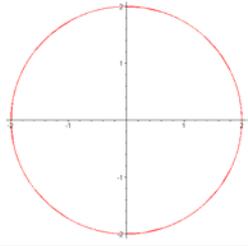
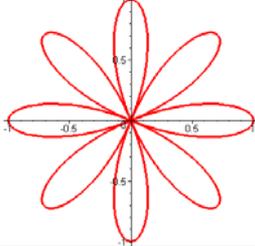
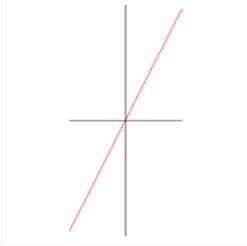
Context, Methodology and Instrument

The context of the present study is multivariate calculus as the final course of a three course calculus sequence, taught at a large public research university in the southern United States. Six students were interviewed, in groups of three, and the interviews were video-recorded. The students were first given four questions in a questionnaire (figure 1), on which they wrote down their responses, and they were then asked to explain them. In this paper, we analyze exclusively the first question because of limited space. In the figure 2, a semblance of the answers that were expected from the students by the researchers is given, as well as selected student work.

For each question, the students were chosen in a different order, but it was inevitable that who spoke first would influence, in some way, the other two. They were asked to explain verbally on an individual basis, but group discussion was encouraged when it presented itself. It should be noted that these students participated after taking their final exam, so they had completed the course. The students were assured that their professor would not have access to the video-recordings until after the final grades had been submitted.

Question 1. Are the given graphs functions in the single variable set up of polar coordinates, when r is considered a function of θ ($r = \rho(\theta)$)?

Circle your choice and explain the reason.

Function	$r = 2$	$r = \cos(4\theta)$	$\theta = \pi/3$
Graphs			
Answer	YES NO Explanation: ...	YES NO Explanation: ...	YES NO Explanation: ...

Question 2. Shade the region and set up how would you calculate the area enclosed by: outside $r = 2$, but inside $r = 4 \sin(\theta)$; Use DOUBLE integration. [DO NOT CALCULATE THE INTEGRAL.]

Question 3. In rectangular coordinates the coordinate surfaces: $x = x_0$, $y = y_0$, $z = z_0$ are three planes.

- (a) In cylindrical coordinates, what are the three surfaces described by the equations: $r = r_0$, $\theta = \theta_0$, $z = z_0$? Sketch.
- (b) In spherical coordinates, what are the three surfaces described by the equations: $\rho = \rho_0$, $\theta = \theta_0$, $z = z_0$? Sketch.

Question 4. What are the names of the following surfaces that are expressed as the polar functions:

- (a) $z = f(r, \theta) = r$. Sketch the surface. Find the volume of the solid by triple integration (use cylindrical coordinates) when $0 \leq r \leq 2$. Does your answer coincide with the formula for the volume of this solid (if you happen to remember)?
- (b) $z = f(r, \theta) = r^2$. Sketch the surface. Find the volume of the solid by triple integration.

Figure 1. Questionnaire

The nature of this study does not require the reader to have detailed information on each of the students, as the focus is upon the mathematical objects and not on the cognitive processes of the participants. Another article, with a more cognitive focus, will be developed with this same data, as the onto-semiotic approach can be used as a framework in theories of learning and teaching mathematics (communication), as well as the epistemology and nature of mathematical objects.

The first question was in three parts, and was identical to the question presented to second course calculus students (calculus of a single variable) and reported upon in Montiel, Vidakovic and Kabael (2008). The objective was to determine if the students could distinguish when a relation between r and θ was a function or not, taking θ as the independent variable and r as the dependent variable. This is not a trivial question, as the geometric representation of the constant function in polar coordinates, $r = a$, is a circle, which is not a function in rectangular coordinates, as was reported in the previous study.

The generic definition of function, which we can paraphrase as ‘a transformation in which to every input there corresponds only one output’, seems to often be lost amongst the different representations students are exposed to, without recognizing any implicit hierarchy. (p. 18)

For this reason, in the previous study the vertical line test, valid for the rectangular system but not for the polar coordinate system, was used as a criterion to say, mistakenly, that $r = a$ was not a function. This same question was now asked to students who had completed a multivariate calculus course, and who were expected to know how to identify and “do calculus” with not only single variable functions, but multivariable functions as well, in rectangular, cylindrical and spherical systems. It was of interest to analyze the answers and explanations to question 1 with this new student sample.

Analysis Using the Onto-Semiotic Approach

The plan will be to go through the question; as there are six subjects and two groups, S1, S2 and S3 will represent the participants in the first group, and S4, S5 and S6 the participants in the second interview session. Usually the two sessions will not be differentiated as emphasis will be placed on the questions themselves and the mathematical content. There are also written answers which will be referred to at times.

The essence of the first question is the fact that the exact same geometrical representation, a circle, which is not considered a function in rectangular coordinates, is in fact a function in the polar coordinate system. Language seen as a mathematical object, one of the primary entities, and understood as terms, expressions, notations and graphics, and semiotic functions that map language (expression) to content (meaning), play an important role here. For example, S2 specifically mentioned that the vertical line test could not be used, making it understood that the “definition of a function by the vertical line test” was not valid in polar coordinates, because in polar coordinates “anything goes”. What is inside the quotations, of course, are personal objects in a very colloquial language, although from the institutional point of view the answer is correct, given that she circled “yes” for “a” and “b”, and “no” for “c”. However, as can be seen in Appendix, her explanation differs from the usual institutional expression.

In figure 2, it can be appreciated that S3 gave as his explanation “for every θ there is only one r ”, using the concept (definition) and properties of function in its underlying, structural meaning, which does not rely on a particular coordinate system, as well as employing impeccable institutional expression. S4 related the two systems by saying that “in the rectangular system there is one y for each x , so here there is one r for each θ ”, while S1 used the radial line test to justify the equation as representing a function; the radial line test had been briefly mentioned in class.

The concept (definition) of function, as seen from the onto-semiotic approach (Wilhelmi et al, 2007a), can be understood in different mathematical contexts, such

as topological, algebraic or analytical. Furthermore, when the concept of function is first introduced, usually at the secondary algebra level, it is not possible to embrace all the systems of practices, so even when the underlying structural definition is given (“for every element in the domain, there corresponds one and only one element in the codomain”, or, “for every input there is only one output”), what often remains in students’ minds (Montiel et al., 2008) is the geometric language with the vertical line test, as different coordinate systems are not included. Even though polar coordinates are introduced at the precalculus level, their geometric representations are usually presented in textbooks as exotic curves (lemniscate, etc.), not as functions.

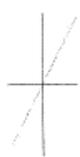
<p>Expected answer.</p> <p>(a) Answer: <u>YES</u> NO. Explanation: For every element θ in the domain, there corresponds one, and only one, element in the codomain. For every input θ, there corresponds one, and only one, output.</p> <p>(b) Answer: <u>YES</u> NO. Explanation: Same as in part (a).</p> <p>(c) Answer: YES <u>NO</u>. Explanation: For $\pi/3$ there are infinite values (more than one) of r.</p>	
<p>Answer from S2.</p> <p>(a) Answer: YES NO Explanation: Even though r is constant, θ could be anything.</p> <p>(b) Answer: YES NO Explanation: because \cos is a function and H_0 represents the number petals you have.</p>	<p>Answer from S3.</p> <p>(a) Answer: YES NO Explanation: For every θ, there is only one r.</p> <p>(b) Answer: YES NO Explanation: same</p>
<p>(c) Answer: YES NO Explanation: the graph would need a radius</p>	<p>(c) $r = \theta$ $y = x$ c) $\theta = \pi/3$  Answer: YES NO</p>

Figure 2. Expected answers and actual student answers

The elementary-systemic dichotomy also is applicable here, because all the different coordinate systems, including the general “curvilinear” coordinates, and the transformations between them together with the determinant of the Jacobian matrix, form a compound object, that is, a system. The actual curve in a particular system, as graphical language, would be an example of an elementary - or unitary- object. At the same time, the ostensive/non-ostensive duality is also relevant, as the graphical representations and the set up of double and triple integrals in different systems lead up to the mathematical concept of *changing variables in multiple integration*.

On the other hand, it is interesting to observe that in this study the students had no problem with realizing that θ was changing, although the point on the graph appeared to be in the same place. That is, that a point with polar coordinates, say, $(4, \pi/2)$ was different from the points $(4, 5\pi/2), (4, -3\pi/2)$ and so on. They also recognized θ and r as independent and dependent variables, even though the pairing (r, θ) often creates confusion, as it is reversed when compared to the convention in the rectangular system, where the independent variable is the first component and the dependent variable is the second component $((x, y))$. In these cases students portrayed much more adhesion to the following mathematical norm: “the determination of an ordered pair consists of knowledge about the elements, the order in which they should be expressed and the meaning of each component”, as compared to the single variable calculus students faced with the same problem (Montiel et al, 2008).

Many standard calculus textbooks do not help in clarifying the concept of function in polar coordinates. Varberg and Purcell (2006) state that:

...There is a phenomenon in the polar system that did not occur in the Cartesian system. Each point has many sets of polar coordinates due to the fact that the angles $\theta + 2\pi n$, $n = 0, \pm 1, \pm 2, \dots$, have the same terminal sides. For example, the point with polar coordinates $(4, \pi/2)$ also has coordinates $(4, 5\pi/2), (4, 9\pi/2), (4, -3\pi/2)$, and so on (p. 572).

However, we ask, if there is a switch from Cartesian to polar coordinates, is the element $(4, \pi/2)$ really the same as $(4, 9\pi/2)$?

It should be pointed out that, this “phenomenon” comes about because a point in polar coordinates is being identified with an equivalence class. That is, a point (r, θ) is equivalent to another point (r, θ') if $\theta' = \theta \pm 2\pi$. In other words, it is presupposed that the dual dimensions example/type and expression/content should be avoided, as they constitute an unnecessary difficulty. However, this “simplification” can limit students’ access to the overall institutional meaning.

In Salas, Hille and Etgen (2007, 479), it is also stated “Polar coordinates are not unique. Many pairs (r, θ) can represent the same point”. On page 492, the problem is avoided by strictly stating the domain of the variable θ as limited to $(0, 2\pi)$. There is no mention of the radial line test in any of these texts.

When the geometric language, and the system of practices developed around it, are not taken specifically into account, the elementary algebraic entity, in the example

above, is a perfectly defined function $r(\theta) = 4$, with no restriction on the domain. If the formal structure of the object “function” must be coherent in all coordinate systems, then the fact that the point is “apparently” the same does not make for sound mathematics. If “for every input there is only one output” captures this underlying structure, then the textbooks might need to take this into account.

Conclusion

Different coordinate systems, apart from their intrinsic mathematical interest, are used in many types of applications in science and engineering. The main objective of this paper was to apply aspects of the onto-semiotic approach, especially those related to the notion of meaning and mathematical objects to *different coordinate systems*. In the process, the systems of operative and discursive practices associated with this mathematical concept were identified. As previous research, within any framework, on this mathematical concept, and on multivariate functions in analysis in general, is practically non-existent, a much more sophisticated description of an epistemic network for this subject is a goal that we hope to reach in the near future. The transformation of expressions to content through semiotic functions, and the identification of chains of signifiers and meanings, could be accomplished because of the rich layering and complexity of the mathematical concept at hand.

“The notion of meaning, in spite of its complexity, is essential in the foundation and orientation of mathematics education research” (Godino et al., 2005).

It is essential to organize what must be known in order to do mathematics. This knowledge includes, and even privileges, mathematical concepts, and it is the search for meaning and knowledge representation that has stimulated the development of the mathematical ontology. However, the onto-semiotic approach gives us a framework to analyze, as mathematical objects, all that is involved in the communication of mathematical ideas as well, drawing on a wealth of instruments developed in the study of semiotics. It is hoped that this attempt to apply this ontology and these instruments to a mathematical concept that involves so many subsystems, provides an example of the kinds of studies that can and should be undertaken. Further studies on this particular mathematical concept can only clarify aspects of the knowledge needed in the communication and understanding of it.

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DERIVATIVES AND APPLICATIONS; DEVELOPMENT OF ONE STUDENT'S UNDERSTANDING

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This paper reports on a longitudinal observation study characterising student's development in their understanding of derivatives. Through the Dutch context-based curriculum, students learn this concept in relation to applications. In our study, we assess student's understanding. We used a framework for data analysis, which focuses on representations and their connections as part of understanding derivatives, and it includes applications as well. We followed students from grade 10 to grade 12, and in these years we administered four task-based interviews. In this paper we report on the development of one 'average' student Otto. His growth consists of an increasing variety of relations, both between and within representations and also between a physical application and mathematical representations. We also find continuity in his preferences for and avoidances of certain relations.

Keywords: Derivative, applications, procedural and conceptual knowledge, process-object pairs, case study.

INTRODUCTION

In the Dutch mathematics curriculum for secondary schools, the role of applications increased over the past 15 years. When the concept of the derivative is taught in grades 10-12, most textbooks provide students with opportunities to learn the concept in different contexts. Often an introduction in grade 10 starts with contexts related to velocity, steepness of graphs and, for example, increasing or decreasing temperatures. Textbooks provide tasks on the average rate of change, average velocity and the slope of a secant. The step towards instantaneous rate of change is kept intuitive, as most textbooks avoid the use of the formal limit definition, or only mention it on one page without using the notation with a 'limit'. Also in the conceptual extension of the derivative in grades 11 and 12, most chapters contain applications.

During their school time, students construct their knowledge of different concepts. One of these concepts is the derivative, which is not only a multifaceted mathematical concept, it also has relations to other school subjects. Knowledge of the derivative may support the learning of physics and economics, but physics teachers complain that students cannot apply what they have learned in their mathematics classes (e.g. Basson, 2002). In our research, we investigate which aspects of the concept derivative are becoming available to students, and whether and how students can relate the concept between different subjects such as mathematics, physics and

economics. Our aim is to describe and analyse the development of students' understanding of derivatives, not just as a mathematical concept in itself, but as a mathematical concept in relation to applications.

THEORETICAL BACKGROUND

Understanding the concept of the derivative

It is complex to determine to what extent a student understands the concept of derivative. Many publications on understanding concepts use words such as scheme, structure, connections and relations. Anderson and Krathwohl (2001) define conceptual knowledge as: the interrelationship between the basic elements within a larger structure that enables them to function together. Thus, they perceive it as more complex and organized forms of knowledge. Procedural knowledge is defined as: methods of inquiry and criteria for using skills, algorithms, techniques and methods. Hiebert and Carpenter (1992) describe understanding in terms of the way, in which information is represented and structured. The degree of understanding depends on the number and strengths of connections between facts, representations, procedures or ideas. Connections can have different characteristics. In our analysis of students' connections, we identify procedural and conceptual knowledge. To describe a student's understanding of the derivative in relation to applications, we describe the connections made by a student (Roorda, Vos & Goedhart, 2007), distinguishing:

- (i) Connections between mathematical representations,
- (ii) Connections within mathematical representations and
- (iii) Connections between an application and mathematical representations.

We will explore these three types of connections further.

Connections between representations

Hähkiöniemi (2006) discusses different viewpoints on representations. According to him, the traditional view on representations is that a representation is conceived as something that stands for something else, and representations are divided into external and internal ones (cf. Janvier, 1987). In his study Hähkiöniemi defines a representation broader as:

“.. a tool to think of something, which is constructed through the use of the tool; a representation had the potential to stand for something else but this is not necessary. A representation consists of external and internal sides which are equally important and do not necessarily stand for each other but are inseparable.” (p. 39)

As such, a gesture by a hand in the air can be a representation of a tangent. Without ignoring the existence of internal representations, we will follow the more traditional view, because external representations can be observed and they can be considered as external indicators of someone's internal representations. In different research the following representations are distinguished: formula, graph, table, words, physical background, gestures (Asiala, Cotrill, Dubinsky & Swingendorf, 1997; Hähkiöniemi,

2006; Kendal & Stacey, 2003; Kindt, 1979; Zandieh, 2000). Kendal and Stacey (2003) look especially at three mathematical representations: formula, graph and table. Students can talk about derivatives from a formulae viewpoint (such as rate of change), from a graphical viewpoint (slope), or from a numerical viewpoint (such as average increase).

Connections between representations and the ability to switch between these are important features for solving tasks (Dreyfus, 1991; Hiebert & Carpenter, 1992). Hähkiöniemi (2006) states that conceptual knowledge often refers to the making of connections from one representation to another. However, we will show in this paper that a connection between two representations can also have a more procedural character.

Connections within representations

As mentioned above, not only connections between representations but also within one representation are important (Dreyfus, 1991). For the derivative, Kindt (1979) distinguishes four levels within each representation. For example, in the formulae representation the four levels are: function, difference quotient, differential quotient and derivative, in the graphical representation: graph, slope of a chord, slope of the tangent and graph of the derivative. Zandieh (2000) indicates the steps between these four levels as process-object pairs, since each level can be viewed both as dynamic process and as static object. To illustrate the idea of process-object pairs we look at the second level of the formulae representation, the difference quotient. A difference quotient $\Delta y : \Delta x$ is a division, which can be viewed as a process: divide a difference in y by a difference in x . The outcome of this division, denoted by $\frac{\Delta y}{\Delta x}$, is a value

which can be seen as an object. Likewise, in the graphical representation: the division of two lengths is the process, which results in an object, the slope of a chord.

Zandieh (2000) explains why the differential quotient and the derivative function both also can be viewed as process-object pairs. In the difference quotient a limiting process is involved, and ‘the derivative acts as a process of passing through (possibly) infinitely many input values and for each determining an output value given by the limit of the difference quotient at a point.’

When a student makes connections between levels within a representation, Hähkiöniemi claims this to be mostly procedural. However, these connections can also be conceptual, for example in a graphical explanation of the limiting process.

Connections between applications and mathematics

The mathematical concept ‘derivative’ has relations with different applications. Thurston (1994) describes different ways of understanding derivatives. One way is to understand derivatives in terms of the instantaneous speed of $f(t)$ when t is time. Also, derivatives are used in physics lessons for concepts such as velocity, acceleration or radioactive decay, and in economics lessons for calculating maximum profits of

marginal costs and revenues. Zandieh (2000) included a column *physical* into her framework. She argued that the context of motion serves as a model for the derivative. This extension can be made to other applications of the derivative as well.

Our research question in terms of the described framework is: what are characteristics of a student's development with respect to connections made between and within representations, and between applications and mathematical representations?

METHODOLOGICAL DESIGN

To study the development of students' understanding, we designed a longitudinal multiple case study with twelve students. Between April 2006 and December 2007, approximately every six months a task-based interview was conducted, yielding four interviews of 75 minutes with each student. In the interviews, we used think-aloud and stimulated recall techniques. The interviews were videotaped and transcribed.

The first interview was held before students were introduced to the theory of derivatives. Between the second and the last interviews, derivatives were a re-occurring topic in mathematics lessons. For this paper, we report on interview 2 (I-2) in November 2006 and interview 4 (I-4) in November 2007, because these contained the same five tasks, enabling us to compare in time. We will report on the work of one student, Otto. By zooming in on the work of one student, we can look more precisely at the solution strategies and statements of this student. We selected an average student with a positive attitude.

All tasks in the test dealt with the concept of derivative, but this was not explicitly mentioned. The tasks were designed to give students many opportunities to show their understanding of derivatives in different representations and applications. We describe three exemplary tasks, named *Emptying a Barrel*, *Petrol* and *Ball*.

Barrel: A barrel is emptied through a hole in the bottom (Figure 1). For the volume of the liquid in the barrel, the formula $V = 10(2 - \frac{1}{60}t)^2$ and its graph are presented. The question is to calculate the out-flow velocity at $t = 40$.

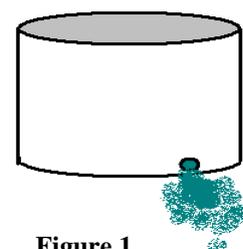


Figure 1

Petrol (Kaiser-Messmer, 1986): In a car an installation measures the petrol consumption related to the distance driven. The amount of petrol, used by a car, depends on the travelled distance. The task includes a graph and a table. $V(a)$ is the petrol consumption after a km. The question is to interpret

$$\frac{V(a+h) - V(a)}{h} \quad (\text{h is a value, which you can choose}).$$

Ball: A ball falls from a height of 90 cm. A table, a graph and the formula for the height $h(t) = 0,9 - 4,9t^2$ are presented. The question is to calculate the velocity at a certain point.

Our analytic framework (presented in Roorda et al. 2007) contains elements of earlier frameworks of Zandieh (2000), Kindt (1979) and Kendal & Stacey (2003) In one dimension we have three mathematical representations: (a) formulae, (b) graphical; (c) numerical. In the other dimension we have the three object-process layers as connections between the four levels. See Table 1.

Table 1: Representations and levels of the concept derivative

	Formulae	Graphical	Numerical
Level 1	F1: f : function	G1: graph	N1: table
Level 2	F2: $\frac{\Delta f}{\Delta x}$ difference quotient	G2: average slope	N2: average increase
Level 3	F3: $\frac{df}{dx}$ differential quotient	G3: slope of a tangent	N3: instantaneous rate of change
Level 4	F4 : f' derivative	G4: graph of derivative	N4: table with rates of change

To solve an application problem, students can choose which mathematical representation can be helpful. In this way, they make a connection between an application and a mathematical representation. In the table below, different non-mathematical representations are displayed, matching the format of the table above.

Table 2: Different applications

	General application	Economics	Physics: velocity	Physics: acceleration
Level 1	S1: $A(p)$: A depends on p	E1: TK total costs	Pa1: $s(t)$ displacement	Pb1: $v(t)$ velocity
Level 2	S2: $\frac{\Delta A}{\Delta p}$ average change of A	E2: $\frac{\Delta [TC]}{\Delta q}$ average increase of costs	Pa2: $\frac{\Delta s}{\Delta t}$ average velocity	Pb2: $\frac{\Delta v}{\Delta t}$ average acceleration
Level 3	S3: $\frac{dA}{dp}$ instantaneous rate of change	E3: $\frac{d[TC]}{dq}$ marginal costs	Pa3: $\frac{ds}{dt}$ instantaneous velocity	Pb3: $\frac{dv}{dt}$ for $t = a$ instantaneous acc.
Level 4	S4: $A'(p)$ derivative	E4: MC marginal costs	Pa4: $v(t)$ velocity	Pb4: $a(t)$ acceleration

The difference with earlier frameworks is that we operationalise understanding of the concept of the derivative through the *connections* between representations, within representations and between representations and applications. In our analysis, we use arrows (as connectors) to visualize the connections in the scheme above. During the problem solving process a student may switch, for example, from a function (F1) to the derivative function (F4), yielding the code F1→F4. Another difference is the role of applications: these are not only viewed of as a support for understanding mathematics, but also as a part of other school subjects. When, for example in an

economic problem, a student focused on the graph, drew a tangent line, and calculated the slope, without economic interpretation, we will denote this as: $E1 \rightarrow G1 \rightarrow G3$. However, when a student solves a problem by calculating marginal costs, without mentioning relations with functions, graphs or table, we will denote this as $E1 \rightarrow E4 \rightarrow E3$.

RESULTS

In this section, the analysis and coding of students' strategies in terms of our framework is illustrated by looking at the task *Barrel*. In Table 3 we summarise Otto's work on this task during I-2 and I-4.

Table 3: Otto's typical statements and activities; Associated codes for Otto's connections; task *Barrel*

Interview 2 (I-2)	Interview 4 (I-4)
<p>Otto: <i>I have to calculate the velocity at that point</i> [plots the graph and uses the option 'Tangent' of his graphing calculator. In the window of the calculator the tangent appears and the formula $y = -0,4428191485x + 35,49..$]</p> <p>Otto goes on to say: <i>I think I have to differentiate, I get the formula of the tangent by differentiating.</i> He calculates the derivative, without using the chain rule, fills in $t = 40$, makes a calculation error, writes down $V'(40) = -493,333$.</p> <p>To check his answer, Otto tries to calculate the average out-flow velocity of the tank over the whole period, by a self-made rule: $\frac{\text{begin} + \text{end}}{2}$</p>	<p>Otto calculates the derivative with some errors: $V'(40) = 59,8$. He discovers a miscalculation, corrects his answer into $-555,56$ litre per minute. To check his answer, Otto draws a tangent into the graph of the task and calculates $\frac{\Delta y}{\Delta x} = \frac{35-0}{80} = 437,5$ l/m. He says: <i>This is a bit imprecise. I think it is possible. [...] you can check with a graphical calculator by drawing a tangent.</i></p> <p>Otto plots the graph and the tangent: [O writes down: GR \rightarrow tangent(40) \rightarrow $-0,444x + 35,56$]</p> <p>He writes down 444,4 l/min. He thinks he made a miscalculation in the derivative.</p>
<p>Connections interview 2</p> <p>S1 \rightarrow F1 \rightarrow G1 \rightarrow G3: use of formula; plots the graph; plots tangent</p> <p>S1 \rightarrow F1 \rightarrow F4 \rightarrow F3 \rightarrow S3: derivative (with error); derivative at $t = 40$; back to application</p>	<p>Connections interview 4</p> <p>S1 \rightarrow F1 \rightarrow F4 \rightarrow F3 \rightarrow S3: formula; derivative (with error); fills in $t = 40$; back to application</p> <p>S1 \rightarrow G1 \rightarrow G3 \rightarrow S3: graph; tangent; application</p> <p>F2 \rightarrow G2 slope of tangent with $\frac{\Delta y}{\Delta x}$</p> <p>F1 \rightarrow G1 \rightarrow G3 \rightarrow S3 graph, tangent; application</p>

Some observations: Otto used in I-2 and I-4 similar solution methods, such as differentiating the formula and plotting the tangent. Differences are also visible, for example in I-4 Otto checked his solution additionally by drawing the tangent on paper. Also, the connection between applications and mathematics $G3 \rightarrow S3$ was added, because Otto interpreted the slope of the tangent in terms of the application. In table 4 the same overview is given for the tasks *Ball* and *Petrol*. We will analyse

the data of these three tasks by examining the connections between representations, within representations and between application and mathematical representations.

Connections between representations

In I-2 the connection F1→G1 is frequently observed. In the tasks, Otto used the given formula as a starting point to plot a graph on his graphical calculator. Only one time we saw Otto make a table with his graphing calculator. Throughout I-2, Otto made a connection between derivative and tangent (F3/F4 →G3), but he could not explain this relation precisely. He said, for example: *When you differentiate you get the formula of the tangent* (see Table 3) and: *to approximate the tangent, you use the formula $\frac{V(a+h) - V(a)}{h}$* (see Table 4).

Table 4: Otto's typical statements and activities; Associated codes; tasks *Ball* and *Petrol*

Interview 2	Interview 4
<p>Otto reads the task <i>Ball</i> and says: <i>I think I have to use a derivative</i>. He calculates the derivative but he fills in $t = 2,4$ instead of $t = 0,24$. Then he says: <i>When you differentiate you get the formula for the tangent, and that corresponds to the velocity, I think</i>. On his graphing calculator he plots a graph and a tangent but after a long silence he states: <i>I don't get any wiser from this</i>.</p> <p>Connections: Pa1→F1→F4→F3 formula; derivative; fills in a wrong value for t. F1→G1 →G3 graph; tangent</p>	<p>Otto thinks he can calculate the velocity of the <i>ball</i> by the formula $v = \Delta x/t$. He calculates the average velocity over the first 0,24 seconds. This is followed by some confusion because Otto thinks the ball also moves horizontally. When the interviewer asks him to check his answer, Otto calculates the derivative. This answer is better, according to him, because in it he recognizes the derivative 9,8 as the gravity acceleration. He also says: <i>I could draw a tangent and calculate the slope of it</i>. At last Otto mentions a method with kinetic energy, but for that he needs the mass of the ball.</p> <p>Connections: Pa1→F1→F4→F3→Pa3: formula; derivative; fills in a value for t; velocity G1→G3 slope of tangent</p>
<p>Statements of Otto in the task <i>Petrol</i> <i>It's the oil consumption at that point.</i> <i>On a small interval it becomes precise.</i> <i>On a small part you can approximate the tangent.</i> <i>Differentiating is for the formula of the tangent.</i> <i>It is a specific value for the tangent</i> <i>How many liters per kilometer he uses (F2→S2)</i></p>	<p>Statements of Otto in the task <i>Petrol</i> <i>It is the approximation on a certain point;</i> <i>It is a certain slope, when you take a small h you calculate exactly the slope at a certain point (F3→G3; F2→F3);</i> <i>You get the consumption very precisely;</i> <i>When h is larger it is the average consumption over a certain distance. (F2→S2);</i> <i>It is a formula to calculate the consumption over a certain period of time.</i></p>

Compared to I-2, in I-4 we observed more relations between representations, also at different levels of the concept. Otto more often used the given graph to solve the task. In I-4 Otto stated that the value of the derivative equals the *slope* of the tangent. He

also made a connection between the formula of the difference quotient and the slope of a secant. He never used the numerical representation.

Connections within representations

Both in I-2 and I-4, we often coded the connection between levels $F1 \rightarrow F4 \rightarrow F3$ and $G1 \rightarrow G3$. These two connection strings (calculating a derivative and plotting a tangent) were standard procedures for Otto, displaying a strong procedural understanding, but in I-2 Otto cannot yet explain this relation accurately.

In the tasks *Barrel* and *Ball*, Otto never mentioned the difference quotient at a small interval or slope of a secant; the tasks obviously did not activate his potential knowledge of the limiting process of the derivative (connections within level 2 and 3) although the task *Petrol* gave ample opportunities to reason about the impact of a larger or smaller h . In both interviews, Otto was unable to explain the formula precisely, but in I-4 Otto made more correct statements than in I-2 (see table 4). As we see in I-4, Otto tried to explain the limiting process, but even in I-4 his formulations are not very accurate.

Connections between applications and representations

In I-2 Otto connected derivative, tangent and velocity, when saying: “*When you differentiate you get the formula of the tangent, and that corresponds to the velocity, I think.*” Nevertheless, Otto did not accurately put these concepts together. In I-4 Otto mentioned and used more relations between formula/graph and applications. He interpreted the tangent-formula correctly to find the velocity of the ball, and in the *Petrol*-task the link between the mathematical notation and the application is correctly described by Otto.

In I-2 Otto did not connect mathematical and physical methods (such as using the formula $v = a \cdot t$). A year later, in I-4 Otto made a few remarks, in which he connected mathematics and physics. For example, Otto noticed that in the derivative $h'(t) = -9,8t$ the value 9,8 is the acceleration of gravity, and he mentioned a calculation method using kinetic energy. In I-4 Otto stated (in another task): “*the derivative is the formula for the velocity, and the second derivative is for **distance moved** [...] Once, my math teacher gave this as notes.*” This is an incorrect formulation, because Otto meant ‘acceleration’ instead of ‘distance moved’.

CONCLUSIONS AND DISCUSSION

This study uses a case study methodology, the focus of the data analysis is on the student as an individual. From individual results we can not prove any generalizations, which is clearly a limitation of this paper, but we can find counterexamples and existence proofs.

In this paper, we reported on Otto’s development in understanding the derivative. Compared with I-2, we measured in I-4 an increased number of connections, both

between and within representations. Connections made in I-2 reoccurred in I-4. Otto's preference for the graphical and the formulae representation was continued in I-4 and also his avoidances of the numerical representation. The preference for graphical representation corresponds to research by Zandieh (2000), who observed that six out of nine students prefer the graphical representation in tasks and explanations about derivatives. In the case of Otto, we saw that this preference prevailed throughout the learning process.

In I-2 at several occasions, Otto equalled the derivative to the tangent, instead of 'the slope of the tangent'. This was not a slip of the tongue, because Otto repeatedly displayed an incorrect idea about the connection between 'tangent' and 'derivative'. This phenomenon is also reported by Asiala et al.(1997) and Zandieh (2006). In addition to the research of Zandieh, we see that Otto's misstatements hinder him during problemsolving. A year later in I-4, Otto knows that the derivative yields the *slope* of the tangent, so his understanding of the formula of a tangent is corrected.

Basson (2002) reported that physics teachers frequently complain that students cannot use what they have learned in their mathematics classes. In the case of an average student such as Otto, we observe indeed difficulties to connect mathematics and physics correctly. Although there is some progress in the accuracy of statements, for example in recognizing the gravity acceleration, the use of the rule 'derivative is velocity', his understanding of these connections stays weak.

Otto improved his procedural knowledge. Although he often uses the same procedures, especially plotting the graph ($F1 \rightarrow G1$), plotting a tangent ($G1 \rightarrow G3$), or calculating a derivative at a point ($F1 \rightarrow F4 \rightarrow F3$), he seems to be more certain of his work and he is more sure about the connections between the different procedures. On the other hand, a recurring feature with Otto was that he sometimes chose an incorrect method, for example in the task *Ball*, in which he calculates in I-4 an average velocity instead of an instantaneous velocity, without any corrections on his work.

Between I-2 and I-4, his conceptual knowledge increased. In I-4 Otto could explain relations between mathematics and physics to a certain extent, the connection between tangents and the derivative function improved and he connected more frequently to the levels 2 and 3 of the derivative. On the other hand, the connections made were not verbally well explained and some possible connections were not mentioned. So his conceptual knowledge increased, but nevertheless remained weak.

We have used a framework for analysing students' understanding of the derivative in application problems. The resulting arrow-schemes describe students' strategies in a structured way by indicating patterns between cells of the table (see table 1). This facilitates the interpretation of students' statements and operations. Our framework also gives a clear description of transitions between applications and mathematical representations, which students make during problem solving. We added notes on procedural and conceptual knowledge displayed by the students. A challenge remains

to use students' misstatements, which are presently not described although these can be indicators of students' understanding.

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FINDING THE SHORTEST PATH ON A SPHERICAL SURFACE: “ACADEMICS” AND “REACTORS” IN A MATHEMATICS DIALOGUE

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The geometry of the surface of the Earth (considered as spherical) can serve as a thematic approach to Non-Euclidean Geometries. A group of mathematics students at the University of Patras, Greece, was asked to find the shortest path on a spherical surface. Advanced Mathematics provides different aspects of students' mathematical thinking. In this paper we focus on a dialectic of two types of students' attitude, which we call “academics” and “reactors”, and we analyze students' dialogue according to a theoretical framework consisting in three main frames of understanding mathematical meaning.

Keywords: Thematic approach, project method, academics, reactors.

INTRODUCTION AND THEORETICAL FRAMEWORK

As a well-known research team at the Freudenthal Institute has shown, Spherical Geometry can give opportunities to students for exciting “mathematical adventures” (van den Brink 1993; 1994; 1995). Van den Brink's descriptions of designing and carrying out a series of lessons on spherical geometry for high school students are convincing enough (however see Patronis, 1994, for students' difficulty to accept the ideas of non-Euclidean Geometry). In particular, an intuitive, non-analytical mode of presentation and discussion in the classroom seems to be very satisfactory at this level: perhaps this is the most natural way to link this geometry with everyday problems of location, orientation and related cultural practices.

Project method, discussed in the context of Critical Mathematical Education (see Skovsmose, 1994a; Nielsen, Patronis, & Skovmose, 1999), involves the selection of *themes* of general or special interest. For us, a *thematic approach* to non-Euclidean Geometry involves a choice of a *main theme* according to the following criteria. First, this theme should be formulated in a language familiar to students and create a link between Elementary and Higher Geometry. On the other hand, the same theme might represent some critical conflicts in the History of Mathematics and function as an epistemological “dialogue” between different conceptions and views. The geometry of the Surface of the Earth (taken as spherical) was taken as such a theme of more general interest, which was used as a starting point in our project and provided opportunities for the formulation of more special tasks.

One of the most significant tasks in the Freudenthal Institute experience mentioned above was to determine the path of shortest length between two places on the surface of the Earth. The present paper describes and analyses a mathematics dialogue

between university students on the same task. This dialogue is part of a long-term project in the Mathematics Department of Patras University, during two academic semesters, with a group of students of 3rd or 4th year. The paper focus on a dialectic of two types of participants' attitudes in this experience. The first type of attitude corresponds to the role of an «academic» and consists in students' tendency to choose coherent theoretical models or methods for solving the given tasks. The second type of attitude corresponds to the role of a «reactor» and amounts to exercise control, or “improve” academics' proposals. The first type corresponds more or less, to a *formalist's* view and the second may include various reactions to formalism (Davis& Hersh 1981 ch.1, Tall 1991 p.5). Thus we decided to focus on these two attitudes, as the analogues of formalist and non-formalist views of mathematics in students. We shall describe the dialectic of the attitudes of academics and reactors in terms of a framework of understanding mathematical meaning, which follows.

According to Sierpinska (1994, p.22-24) *meaning* and *understanding* are related in several ways. One of these, which we follow here, is typical in Philosophical Hermeneutics: *understanding* is an interpretation (of a text, or an action) according to a network of already existing “*horizons*” of *sense* or *meaning* (see also Pietersma 1973 for “horizon” as implicit context in phenomenology). Thus we are going to analyze our empirical data according to a theoretical framework involving three main frames (or “horizons”) of understanding mathematical meaning namely: i) mathematical meaning as related to students' common background, ii) mathematical meaning as specialized theoretical knowledge, and iii) mathematical meaning as pragmatic meaning.

I. Mathematical meaning as related to students' common background

The first main frame of understanding mathematical meaning in our framework consists, roughly speaking, in what almost all students «carry with them» from school mathematics or first year calculus and analytic geometry. Mathematical terms in this frame may have an intuitive as well as a formal meaning. The mathematical language used is mixed and some times ambiguous (as e.g. it is the case with the word “curve” in school mathematics). The influence of this frame of understanding meaning is very strong may become an «obstacle» in the construction of new mathematical knowledge (Brown et al 2005).

II. Mathematical meaning as specialized theoretical knowledge

The second main frame of understanding mathematical meaning is typical in specialized university programs in Mathematics, at an advanced undergraduate or a postgraduate level. Examples of this frame of understanding mathematical meaning are offered by advanced courses of Algebra, Topology, and Differential Geometry (or Geometry of Manifolds). Mathematical terms in this frame are coherently and formally defined (usually by means an axiomatic system) and proofs are given independently of common sense (Tall 1991).

III. Mathematical meaning as (socially negotiated) pragmatic meaning

As the third main frame we consider pragmatic meaning: the meaning of a sentence or a word is determined by its use in real life situations or in given practices. An important example in this frame of understanding mathematical meaning is offered by the case of practitioners in the field of navigation and cartography during 16th century (Schemmel 2008 p.15-23). In some classroom situations we can also consider this kind of meaning as socially negotiated meaning. It has been observed that in interactive situations negotiation of meaning involves attempts of the participants to develop, not only their mathematical understanding, but also their understanding of each other (Cobb, 1986, p.7).

PARTICIPANTS AND COLLECTION OF DATA

During the first semester of the year 2003-2004, all mathematics students at Patras University, attending a course titled “Contemporary view of Elementary Mathematics”¹, were informed about the project «Geometry of the Spherical Surface» and were invited to participate. Eleven students responded. Five of them, who were particularly involved in the project, formed the final group of participants. Only one of the participants was a girl (Electra²), who worked together with one of the boys (Orestes), while the rest worked alone. Orestes, Electra and Paris were students of the third year and Achilles was at the last (fourth) academic year. An exceptional case is Agamemnon, who was not normally attending this course but participated by pure interest.

A narrative text was given to the participant students adapting Jules Verne’s novel “Un capitaine de 15 ans” (in Greek translation). After reading this text we had a discussion with the students in the classroom, which led to the formulation of the task examined in the present paper:

Which is the shortest path between two points on the surface of the Earth (considered as spherical) and why?

During of the project we collected data by personal interviews (formal or informal), by recording classroom meetings and by gathering students’ essays or intermediate writings in incomplete form.

ANALYSIS

As we already announced, we are going to analyze students’ dialogue and some of their essays by using the crucial distinction between *academics* and *reactors*.

Academics

As we already said, this type of attitude characterizes the students who use conventional and/or coherent methods or higher mathematics to solve a problem.

Mathematical knowledge used may have different origins, but usually *academics* use school or first year university mathematics. This choice corresponds to the first frame of understanding mathematical meaning. More specifically, academics may try to use elementary mathematics in order to solve an advanced mathematical problem. On the other hand, students of the same type of attitude may follow the second frame of understanding mathematical meaning. According to this frame students use advanced mathematical knowledge from university courses in order to solve (advanced) mathematical problems. They may also use knowledge even from postgraduate courses, producing formal proofs without originality and intuitive understanding. A general characteristic of academics is that they can only act in a single frame (first or second) and not in many frames at the same time. They seem to have a difficulty to change frames of meaning.

Our first case, representing academics following the first frame of understanding mathematical meaning, is *Agamemnon*. On the other hand *Achilles* represents academics at the second frame of understanding meaning. As we shall see, Achilles uses advanced mathematical tools from differential geometry in order to prove that great circles are geodesic lines on a spherical surface. Here are some extracts from his presentation in the classroom.

Achilles: We are going to define a very important concept, the concept of geodesic curvature. The definition is $k_g = k \sin \theta$ (Where k is the curvature of a space curve). According to Darboux formulas we have

$$\frac{d\vec{t}}{ds} = k_g \vec{n}_g + k_n \vec{N} \quad (1)$$

$$\frac{d\vec{N}}{ds} = -k_n \vec{t} - \tau_g \vec{n}_g \quad (2)$$

$$\frac{d\vec{n}_g}{ds} = -k_g \vec{t} + \tau_g \vec{N} \quad (3)$$

Forming the scalar product of the first member of (3) with \vec{t} we have

$$k_g = - \left\langle \vec{t}, \frac{d\vec{n}_g}{ds} \right\rangle \quad (4)$$

...I suppose we don't need this formula but the equivalent one:

$$k_g = \left\langle \frac{d\vec{t}}{ds}, \vec{n}_g \right\rangle \quad (5).$$

The participant observer intervenes and asks why (4) and (5) are equivalent. After some thought, Achilles says that formula (5) results from (1) by scalar multiplication with \vec{n}_g .

Meanwhile, Agamemnon writes his own answer to the participant observer's question:

$$\langle a, b \rangle = 0 \Rightarrow \langle a', b \rangle + \langle a, b' \rangle = 0 \Rightarrow \langle a', b \rangle = -\langle a, b' \rangle$$

(Agamemnon means that a, b can be any vector functions $\vec{a}(t), \vec{b}(t)$.)

Achilles continues by proving that a curve γ is a geodesic on a surface if and only if $\vec{n}_0 = \pm \vec{N}_0$. He concludes that great circles are geodesic for the surface of the sphere.

This proof involves concepts from the postgraduate course “Geometry I”, taught at the first year of the postgraduate program of the department of Mathematics. Achilles ignores the formulation in the given context (as we described in section 2) and focuses at the mathematical task. This choice to use differential geometry is not accidental. At the end of his presentation he said that this solution is the better and the prettier one because, given a curve on a surface we must use Curve Theory and Surface Theory. It is also interest to compare the reactions of Achilles and Agamemnon to the participant’s observer question: Achilles acts in the second frame of understanding and gives an answer by using again advanced mathematical tools. On the other hand Agamemnon acts in the first frame of understanding meaning and using elementary mathematics gives an answer that is in fact a new proposition (a lemma).

Agamemnon’s project is quite different and uses a notation of his own.

Agamemnon: We define a function

$$\begin{aligned} \mu_R : (0, 2R] &\rightarrow (0, \pi R] \\ x &\rightarrow \mu_R(x) \end{aligned}$$

where $\mu_R(x)$ is the length of the smaller arc corresponding to the spherical chord x .

Let $A_1, A_2, \dots, A_n \in \Sigma_\varepsilon$ be $n \geq 3$ points on the spherical surface. We can prove that...I will first write and then explain:

$$\sum \mu_R(|A_i A_{i+1}|) \geq \mu_R(|A_0 A_k|) \quad (1)$$

Agamemnon proves inequality (1) (a generalization of the well known Triangle Inequality for Spherical Triangles) using mathematical induction.

Let a curve in three dimensional space, with ends A, B . We try to approximate the length of this curve with polygonal lines.

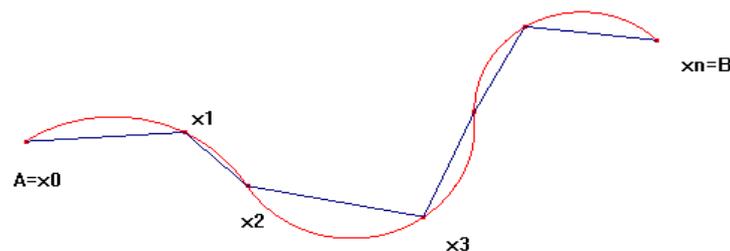


Fig. 1

For every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$

such that $A = x_0 < x_1 < x_2 < \dots < x_n = B$ and $|x_i x_{i+1}| < \delta(\varepsilon)$

Then $\left| \mu_\gamma - \sum_{i=0}^n |x_i x_{i+1}| \right| < \varepsilon$

Agamemnon tries to approximate a curve on a spherical surface by arcs of great circles:

Let now be $\mu_{\overline{AB}}$ the length of the great circle that passes through A, B and μ_γ the length of an arbitrary line connecting A, B. We are going to prove that $\mu_{\overline{AB}} \leq \mu_\gamma$. We approach μ_γ with spherical broken lines... If we assume that $\mu_{\overline{AB}} > \mu_\gamma$ then, by using (2) for a suitable choice of points x_i on the spherical surface we have:

$$\left| \mu_\gamma - \sum \mu_R(|A_i A_{i+1}|) \right| < \varepsilon, \text{ a contradiction with (1).}$$

Although Agamemnon promises that he will explain his choices, in fact he is not in a position to do this, and his peers cannot follow his thought.

As we already said, Agamemnon acts in the first frame of understanding mathematical meaning. His proof is characteristic of this frame following a similar idea with that of the proof concerning plane curves. We find essentially the same proof in Lyusternik (1976) but in a more intuitive formulation, without using formal mathematical notation. Agamemnon was not aware of this proof since he used school and first year geometry textbooks in Greek. The notation he used is a creation of his own, expressing his formal kind of thinking. Contrary to Achilles he is interested in creating a new proof, and despite his difficulties he never consults the University Library.

Reactors

The second type of students' attitude expresses itself in the form of, either a disagreement, or a proposal of "simplification" or "improvement". Students of this type of attitude can act in at least two frames of understanding mathematical meaning at the same time. Moreover, a frame of meaning particularly use by reactors it is the third one. Pragmatic meaning is provided by the *scene* of action and transforms the first frame of mathematical meaning in a non-conventional way. Some of these students act within the given social context and are mainly inspired by it. Thus not only they react to academics' proposals, but they also try to introduce a different way of thinking.

Before their final presentation, students interchanged opinions. Agamemnon tries to communicate with others students by expounding his thought. In this phase Orestes

reacts to him by proposing a “simpler” solution by using orthogonal projection and Orestes himself interacts with Paris.

Agamemnon: Consider a curve on the spherical surface and a sequence of points on this curve. For any two points we consider the smaller arc of a great circle... I think we can call these lines spherical broken lines.

Agamemnon draws Figure 1 and Orestes reacts as follows:

Orestes: Let us draw the perpendiculars from the end points of these arcs to the chord AB, and compare, for example, chord AM with segment AH. Since AM is the hypotenuse of the triangle AHM, it is greater than AH. Similarly MN is greater than ME=HZ. Continuing in the same way we find that the sum of all those chords is greater than the chord AB. Now we wish to find a relation between chords and arcs.

At this point the participant observer asks Orestes where all those chords (arcs and perpendiculars) lie on. Orestes knows that they lie on different planes. Paris shows with his hands a warped triangle. Orestes makes Fig.2 and continues:

Orestes: The only thing that matters is the length. That the hypotenuse is greater than perpendicular...

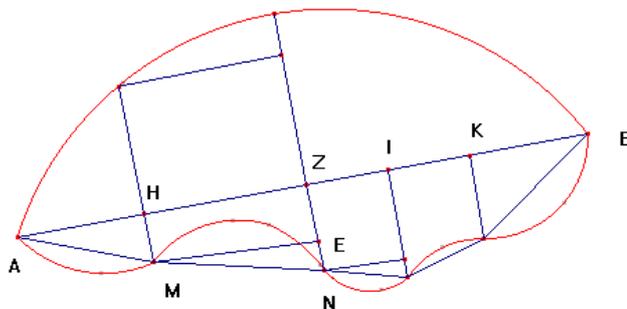


Fig.2

Paris has a difficulty to imagine the figure in 3D-space:

Paris: From what Orestes said, I thought that we could project the figure in the plane... like Mercator projection. Then we could work in the plane...that will be easier.

Achilles: This projection must be isometric and Mercator's projection I do not think is going to help.

Paris: If we project small areas from a part of the Earth.

Achilles: For large areas France will be came equal to North America.

Paris: We can make divisions as we do in integrals ...I' ill thing about that.

As we see here, both *academics* and *reactors* act and react to each other. Agamemnon tries to expose his thought and Orestes responses by trying to “simplify” his attempt. It is difficult, however, to communicate their ideas each other in a way to understand each other. Although Orestes responses to Agamemnon, it is obvious that he cannot follow his thought. Moreover Orestes is not concerned about the context when he says that the only thing that matters is the “length” and seems to ignore that he is working on a spherical surface. Paris reacts to Orestes and proposes a projection on the plane. Achilles reacts to Paris by disputing the suitability of this proposal.

In a later essay Paris presented three different plans of proof, neither of which was complete. In one of these plans he formulated the following lemma, which is typical of the first frame of understanding mathematical meaning:

Let (K,R) be a great circle on a spherical surface and (K', R') a small circle so that the chords AB and $A'B'$ are equals (Fig.3). Then the arc of the small circle is longer than the arc of the great circle with the same chord because the small circle has a greater curvature.

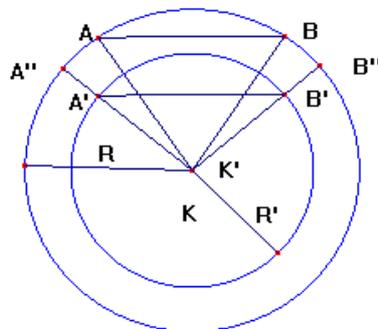


Fig.3

In another plan, Paris introduces a system of parallel circles (similar to that used for the Globe) and tries to combine the first and second frame, by using chords instead of corresponding circular arcs.

We could say that Paris acts in first but also in the third frame of understanding mathematical meaning since the globe but also the planar projections have central position in his attempts.

Finally, some of the *reactors* act in the third frame by “transferring” knowledge from navigation practices to the given problem, without any further elaboration. For example Orestes (in his final essay) uses the globe in order to describe the concepts of loxodrome and orthodrome.

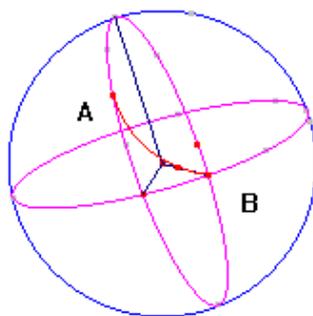


Fig. 4

Orestes finally chooses the method of “logistic orthodrome”, in which middle points must be found between A and B (Fig.4). He describes this method without using any projection, working this time on the spherical surface of the Earth.

FURTHER DISCUSSION AND PERSPECTIVES

The three frames of understanding mathematical meaning, which we used in our analysis, may be helpful into some more general perspectives, which perhaps are already present in our experience but are not yet thoroughly studied in this context. One of these perspectives comprises argumentation and proving processes at the tertiary level of geometry teaching. In this direction the frames introduced here may be seen as different *frames of arguing and proving* or of *understanding proofs*. As an example of a *proof in the first frame* we may consider the elementary mathematical proof of the fact that great circles are geodesic lines on a spherical surface, which we find in Lyusternik (1976; p.30-35). An example of a *proof in the second frame* is the proof of the same fact in the context of Differential Geometry (followed by Achilles in our experience - for a complete proof see Spivac 1979). Again Lyusternik (1976) offers us an example of (pragmatic) *argumentation in the third frame* in p.49-51, of his book by which he establishes Bernoulli's theorem: *For an elastic thread q stretched on surface S to be in a state of equilibrium it is necessary that at any point of q , the principal normal of q coincides with the normal to the surface S (i.e. q is stretched along a geodesic of S).*

It seems difficult, in general, to combine any two of the above three frames of understanding mathematical meaning (and proof). As we have already said, *academics* act either in the first or in the second frame, being almost unable to combine frames. This combination provides a link between Elementary and Advanced Mathematics that is essential in Tertiary Mathematics Education. On the other hand, *reactors* can combine the two first frames (students' common background and pragmatic meaning), while there is no combination of the second with the third frame, which shows a need for enrichment of the scheme *academic/reactor* with more special categories of attitudes. Here a question arises for further theoretical and empirical study, namely how can old textbooks of mathematics or other related

historical sources be used in teaching to provide a “dialogue” between various epistemological perspectives.

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NOTES

¹This course is addressed to students in the third year of study. The subject matter of this course is not fixed for all academic years, so students have the opportunity to study new issues.

² These names are not students’ real names.

NUMBER THEORY IN THE NATIONAL COMPULSORY EXAMINATION AT THE END OF THE FRENCH SECONDARY LEVEL: BETWEEN ORGANISING AND OPERATIVE DIMENSIONS

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In our researches in didactic of number theory, we are especially interested in proving in the secondary-tertiary transition. In this paper, we focus on the “baccalauréat”, the national examination that pupils have to take at the end of French secondary level. In reasoning in number theory, we distinguish two complementary dimensions, namely the organising one and the operative one, and this distinction permits to situate the autonomy devolved to learners in number theory problems such as baccalauréat’s exercises. We have analysed 38 exercises, from 1999 to 2008, and we present the results obtained giving emblematic examples.

INTRODUCTION

At the end of French secondary level (Grade 12), there is a national compulsory examination called *baccalauréat* and the mathematics test includes three to five exercises (each one out of 3 to 10 points). In French Grade 12, there is an optional mathematics course in geometry and number theory and the test for candidates who have attended this optional course differs from that for others candidates by one exercise (out of 5 points); this exercise includes or not number theory. In our researches in didactic of number theory, we are especially interested in the secondary-tertiary transition¹, so especially interested in the *baccalauréat* which plays a crucial role in this transition. Within didactic researches related to secondary-tertiary transition (Gueudet, 2008), we propose to study some of the ruptures at stake in terms of autonomy devolved to Grade 12-pupils and students. In this paper, we focus on characterizing this autonomy in *baccalauréat*’s exercises using the distinction that we make in the reasoning in number theory between the organising dimension and the operative dimension (Battie, 2007).

We distinguish two complementary dimensions. The organising dimension concerns the mathematician’s « aim » (i.e. his or her « program », explicit or not). For example, besides usual figures of mathematical reasoning, especially *reductio ad absurdum*, we identify in organising dimension induction (and other forms of exploitation in reasoning of the well-ordering \leq of the natural numbers), reduction to

¹ In secondary-tertiary transition, number theory is primarily concerned with structures and properties of the integers (i.e. Elementary number theory). For a detailed consideration of various facets falling under the rubric of number theory, see Campbell and Zazkis, 2002.

the study of finite number of cases (separating cases and exhaustive search²), factorial ring's method and local-global principle³. The operative dimension relates to those treatments operated on objects and developed for implementing the different steps of the program. For instance, we identify forms of representation chosen for the objects, the use of key theorems, algebraic manipulations and all treatments related to the articulation between divisibility order (the ring \mathbb{Z}) and standard order \leq (the well-ordered set \mathbb{N}). Among the numerous didactic researches on mathematical reasoning and proving (International Newsletter on the Teaching and Learning of Mathematical Proof and, especially for Number theory, see (Zazkis & Campbell, 2002 & 2006)), we can put into perspective our distinction between organising dimension and operative dimension (in the reasoning in number theory) with the “structuring mathematical proofs” of Leron (1983). As we showed (Battie, 2007), an analogy is a priori possible, but only on certain types of proofs. According to us, the theoretical approach of Leron is primarily a hierarchical organization of mathematical sub-results necessary to demonstrate the main result, independently of the specificity of mathematical domains at stake. As far as we know, Leron's point of view does not permit access that gives our analysis in terms of organising and operative dimensions, namely the different nature of mathematical work according to whether a dimension or another and, so essential, interactions that take place between this two dimensions.

In this paper, we present the results obtained analyzing 38 *baccalauréat*'s exercises, from 1999 to 2008, in terms of organising and operative dimensions. In the first part, we study the period from the reintroduction of number theory in French secondary level (1998) to the change of the curriculum in 2002 (addition of congruences). In the second part, we focus on the next period, from 2002 to 2008.

NUMBER THEORY IN BACCALAUREAT'S EXERCISES FROM 1999 TO 2002

After 15 years of absence, number theory reappeared in 1998 in French secondary level, first in Grade 12 as an optional course (with geometry). From 1998 to 2002,

² For example, an exhaustive search to find the divisors of a natural number n is to enumerate all integers from 1 to n , and check whether each of them divides n without remainder. We talk about *strict exhaustive search* when there is not a limitation phase of possible candidates (for the solution) before checking whether each candidate satisfies the problem's statement.

³ An elementary example is given in (Harary, 2006): *Proposition*. Let m be an integer checking $m = 4^r(8s + 7)$, r and s integers > 0 . Then the equation $x^2 + y^2 + z^2 = m$ has no rational solution. *Demonstration*. If there was a rational solution, there would be a non-trivial integer solution (in “hunting” denominators) for the equation $(8s+7)t^2 = x^2 + y^2 + z^2$. Even if it means to divide x , y , z , t by the same number, then we can assume they are relatively prime. Then we look at the equation modulo 4: in $\mathbb{Z}/4\mathbb{Z}$, the squares are 0 and 1; and t can not be even otherwise $x^2 + y^2 + z^2$ would be divisible by 4 implying that x , y , z are all even, contradicts the hypothesis. But if t is odd, then $(8s+7)t^2$ is congruent to -1 modulo 8 and $x^2 + y^2 + z^2$ too, which is impossible because the squares of $\mathbb{Z}/8\mathbb{Z}$ are 0, 1, 4.

Number theory curriculum as an option comprised: divisibility, Euclidian division, Euclid's algorithm, integers relatively prime, prime numbers, existence and uniqueness of prime factorization, least common multiple (LCM), Bézout's identity and Gauss' theorem⁴. In one of our researches (Battie, 2003), we tried to find all *baccalauréat*'s exercises related to the optional course in number theory (and geometry) in French education centers in the world. From 1999 (in 1998 there was only geometry exercises) to 2002, within the 40 exercises we found, 20 concern exclusively number theory, 10 are mixed (number theory and geometry) and 10 concern exclusively geometry. We analysed therefore 30 *baccalauréat*'s exercises⁵. In this ecological study, after grouping together exercises related to the same mathematical problems, the objectif is to assess the richness of what is "alive" in these exercises and to situate the autonomy devolved to pupils in terms of organising and operative dimensions. What are the results of this study?

The identification of mathematical problems involved in these 30 *baccalauréat*'s exercises highlights a real diversity through the existence of three possible groups⁶ : a first one defined by solving Diophantine equations (18 exercises), a second group defined by divisibility (21 exercises) and a third one characterized by exogenous questions compared to the first two groups (3 exercises associated with at least one of the first two groups). However, refining the analysis, we observe that all exercises are constructed from a relatively small number of types of tasks. This is primarily solving in \mathbb{Z} Diophantine equations $ax + by = c$ ($\text{gcd}(a,b)$ divide c) in the first group of exercises (we'll note T afterwards) and, for the second group, proving that a number is divisible by another one or determining gcd of two numbers.

The analysis of first group's exercises confirms the emblematic character of T : we identify T in 16 of the 30 exercises. There is three cases related to its role in each exercise: T , as an object, is essential in the exercise and comes with direct applications (8 exercises), T occupies a central place and comes others problems (3 exercises), T is an essential tool to solve a problem outside number theory (5 exercises). The autonomy devolved to pupils to realize T is almost complete, at the organising dimension and at the operative dimension, undoubtedly because of routine characteristic. Indications for the organising dimension, according to the technique taught in Grade 12, appear through cutting the resolution in two questions: a first

⁴ If an integer divides the product of two other integers, and the first and second integers are coprime, then the first integer divides the third integer.

⁵ France (june 2002, 2001, 1999, september 2002, 2001), Asia (june 2002, 2000, 1999), North America (june 2002, 2001, 1999), South America (november 2001), Foreign centers group 1 (june 2002, 2001, 1999), Pondicherry (may 2001, 1999, june 2002, 2000), La Réunion (june 2000), Guadeloupe – Guyana – Martinique (june 2001, 2000, 1999, september 2001), Polynesia (june 2002, 2001, 2000, 1999), New Caledonia (november 2001, march 2001).

⁶ It's not a classification: an exercise can be associated to several groups.

question about existence of a solution and another one about obtaining all solutions from this solution (linearity phenomena); the set of solutions is given only in one exercise. The treatment of the logical equivalence at stake is under the responsibility of pupils in almost all exercises. At the operative dimension, Bézout's identity and Gauss' theorem, both emblematic of Grade 12 curriculum, are respectively the operative key for finding a particular solution and to obtain all solutions from this particular solution. We identify four types of exercises for the first step (finding a particular solution): 4 exercises with only checking whether a given candidate satisfies the equation, one exercise where an obvious solution is requested, 5 exercises where using Euclid's algorithm is recommended more or less directly and 5 exercises without indication. Note that a justification for such a solution is at stake in a third of exercises; Bézout's identity is expected. For the second step (obtaining all solutions from the particular solution), the operative dimension is entirely under responsibility of pupils (except for one exercise). Despite the important role of T , both qualitatively and quantitatively, this type of tasks is not completely standardized: we highlight levers chosen by *baccalauréat*'s authors to go beyond its routine. Generally, such an extension is achieved by reducing the resolution to \mathbb{N} or to a finite \mathbb{Z} -subset (12 exercises on the 16 at stake) and is often "dressing" the problem which naturally leads to this reduction (geometry (9 exercises), astronomy (2 exercises), context of "life" (1 exercise)). The organising dimension favoured by the authors is one whose aim is using \mathbb{Z} -resolution. This dimension is clarified in 5 exercises (through the phrase "Deduce" or "application"); these include especially those where the set of solutions is infinite. When the set of solutions is finite and when the resolution is in a finite \mathbb{Z} -subset, there is no explicit indication and we identify an opening in terms of autonomy devolved to pupils at the organising dimension; this is the example of [Polynesia, June 2001]:

1. Let x and y be integers and (E) be the equation $91x + 10y = 1$.
 - a) Give the statement of a theorem to justify the existence of a solution of the equation (E) .
 - b) Determine a particular solution of (E) and deduce a particular solution of the equation (E') $91x + 10y = 412$.
 - c) Solve (E') .
2. Prove that the integers $A_n = 3^{2n} - 1$, with n a non-zero natural number, are divisible by 8 (one of the possible methods is an induction).
3. Let (E'') be the equation $A_3x + A_2y = 3\,296$.
 - a) Determine the ordered pairs of integers (x, y) solutions of the equation (E'') .
 - b) Prove that an ordered pair of natural numbers is a solution of (E'') . Determine it.

We can analyze the issue 3. by identifying Z-resolution and N-resolution as two separate problems, i.e. without giving to Z-resolution the status of under problem in issue 3.b. This is a N-resolution of (E'') according to this aim:

$$91x + 10y = 412$$

$$91x = 2(206 - 5y)$$

Necessarily 2 divide x by using Gauss' theorem. x and y are natural numbers so

$91x \leq 412$ and then $x \in \{2; 4\}$. Only $x = 2$ is ok ($y = 23$).

The specificity of possible solutions is exploited in operative work to reduce the research by containing the set of solutions: the organising dimension is an exhaustive search with limitation phase. The uniqueness of the solution announced, we can also choose a strict exhaustive search. However, it seems unlikely that a student does not use the Z-resolution, in particular because of the didactic contract. We have an exception, [France, June 2002], related to levers chosen by *baccalauréat*'s authors to go beyond the routine characteristic of T :

1. Let (E) be the equation $6x + 7y = 57$ in unknown x and y integers.

a) Determine an ordered pair (u, v) of integers checking $6u + 7v = 1$. Deduce a particular solution (x_0, y_0) of the equation (E).

b) Determine the ordered pairs of integers, solutions of the equation (E).

2. Let $(O, \vec{i}, \vec{j}, \vec{k})$ be an orthonormal space's basis and let's call (P) the plane defined by the equation $6x + 7y + 8z = 57$.

Prove that only one of the points of (P) contained in the plane (O, \vec{i}, \vec{j}) has got coordinates in N , the set of natural numbers.

3. Let $M(x, y, z)$ be a point of the plane (P), x , y and z natural numbers.

a) Prove that y is an odd number.

b) $y = 2p + 1$ with p a natural number. Prove that the remainder of the Euclidian division of $p + z$ by 3 is 1.

c) $p + z = 3q + 1$ with q a natural number. Prove that x , p and q check $x + p + 4q = 7$. By deduction, prove that q is equal to 0 or equal to 1.

d) Deduce the coordinates of all points of (P) whose coordinates are natural numbers.

In this exercise, the routine characteristic of T is broken by its extension through an original (related to Grade 12 teaching culture) type of problems: the N-resolution of Diophantine equations $ax + by + cz = d$ (a , b and c relatively prime). A characteristic of the organising dimension behind the exercise's statement is that it does not use the Z-resolution, breaking with the conception of other exercises. The organising dimension is an exhaustive search with limitation phase, and in this case, autonomy devolved to pupils is very small (throughout the limitation phase). However,

confirming the analysis of other exercises, the (phase of) strict exhaustive search and the logical equivalence at stake is under responsibility of pupils.

In the second group of exercises around the concept of divisibility, we find all main operative dimensions used in our epistemological analysis: forms of representation chosen for the objects, the use of key theorems, algebraic manipulations and all treatments related to the articulation between divisibility order (the ring *Proceedings of the 28th International Conference for the Psychology of Mathematics Education.*) and standard order \leq (the well-ordered set N). The autonomy devolved to pupils at operative dimension is very variable, unlike T which it is almost complete. This variability is a function of the complexity of operative treatments to be developed. For example, we find the extreme case where nothing is provided to pupils when he can use Bézout's identity to show that two numbers are relatively prime and, conversely, we have 2 exercises where an algebraic identity, operative key expected, is given to show a divisibility relation. Regarding the organising dimension, the algorithmic approach of strict exhaustive search is most relevant to resolve many issues of divisibility. Using induction is explicitly expected 5 times in 3 exercises (this organising dimension is also explicit in one of the first group but in a geometry issue). We identify several times reasoning by separating cases. The autonomy devolved to pupils is defined as follows: for reasoning by separating cases there are the two extreme positions (autonomy empty or not) and, for the strict exhaustive search and induction, autonomy is complete. We suppose that the existence of substantial autonomy devolved to pupils demonstrates that organising dimensions at stake are not considered as problematic by the educational institution, as the case of logical equivalence.

According to us, exploitation of the potentialities highlighted in *baccalauréat's* exercises is poor because the conception of this examination is strongly governed by the will assess pupils on emblematic and routine Grade-12 tasks. In addition, we believe that the authors seek a compromise between assess pupils on different things to "cover" maximum the curriculum (one of the recommendations for authors) and build up a coherent mathematical point of view. It seems that the aspect "patchwork" of certain exercises, especially those attached to the third group, reflects this institutional constraint.

Now, we're going to study the 2002 change of curriculum limiting us to national *baccalauréat's* exercises: how the new curriculum alter the conception of the this examination? Especially for the autonomy devolved to pupils: is it situate as the same way than before 2002 (2002 exercises included)?

NUMBER THEORY IN BACCALAUREAT'S EXERCISES FROM 2003 TO 2008

At the start of the 2002 academic year, Grade 12 number theory curriculum has been modified with the addition of congruences (without the algebraic structures are

clarified). We are interested here in *baccalauréat*'s exercises given in France since the curriculum's change so from June 2003 to June 2008. Within the 11 exercises at stake, 5 concern exclusively number theory, 3 are mixed (number theory and geometry) and 3 concern exclusively geometry; we find significantly same proportions than in the 40 exercises mentioned in the first part of this paper. We now focus on the 8 exercises with number theory issues (note that exercise of September 2005 is a QCM, a new form of assessment for this examination).

Resuming the three groups of exercises defined in the first part: 3 exercises (June 2008, September 2005 and 2006) can be associated to the T group and only one exercise (June 2004) in the second group (concept of divisibility), without congruences are mentioned, and the two types of tasks that we have identified are represented in this exercise. For these 4 exercises, conclusions of an analysis in terms of organising and operative dimensions are the same as before 2003 (except in the case of QCM where no indication is given, except from the data sets of potential solutions). Closely associated with the second group, a third one is possible from congruences and 5 exercises can be linked (June 2006, 2003, September 2007, 2005, 2003). Now, we focus on this third new group.

The main types of tasks encountered in this third group are calculating in $\mathbb{Z}/n\mathbb{Z}$ and solving congruences equations, particularly in relation to the field structure of $\mathbb{Z}/p\mathbb{Z}$ (p prime), both without the algebraic structure is clarified. With one exception (June 2003), congruences have only the status of object (not a tool) in exercises. The introduction of congruences enriches potentialities of the curriculum in terms of operative dimension and specifically in terms of forms of representation chosen for the objects. In an interactive way, this enrichment could be extended in terms of organising dimension with the local-global principle announced in the introduction, but we only identify the strict exhaustive search associated with the direct work in $\mathbb{Z}/n\mathbb{Z}$. As in the first part, we find that this organising dimension is under the responsibility of pupils in *baccalauréat*'s exercises. We have the example of the issue 3.a. of the exercise of June 2003:

[...]

3. a) Prove that the equation $x^2 \equiv 3[7]$, in unknown x an integer, has no solution.

b) Prove the following property:

for all integers a and b , if 7 divides a^2+b^2 , then 7 divides a and 7 divides b .

4. a) Let a , b and c non-zero integers. Prove the following property:

If the point $A(a, b, c)$ is a point of the cone Γ [equation $y^2+z^2=7x^2$], then a , b and c are divisible by 7.

b) Deduce that the only point of Γ whose coordinates are integers is the vertex of this cone.

Emphasize the unusual nature of this issue in a exercise in all issues, except this one, are unified by a unique mathematical problem (research of points of a cone with N -coordinates). According to us, this unusual characteristic refers to the institutional constraint mentioned in the first part, so to emblematic characteristic of this type of tasks entirely under the responsibility of pupils. Beyond the desire to assess pupils in relation to a emblematic type of tasks, we are assuming that this issue 3.a, by the effect of didactic contract, is an operative indication for the issue 3.b, namely using congruences (modulo 7) to study divisibility by 7.

Finally, we zoom on the June 2006 exercise:

Part A

- 1) Enunciate Bézout's identity and Gauss' theorem.
- 2) Demonstrate Gauss' theorem using Bézout's identity.

Part B

The purpose is to solve in \mathbb{Z} the system (S)
$$\begin{cases} n \equiv 13 \pmod{19} \\ n \equiv 6 \pmod{12} \end{cases}$$

1) Prove that exists an ordered pair of integers (u, v) such that $19u + 12v = 1$ (in this question it's not required to give an example of such an ordered pair). Check that for such an ordered pair $N = 13 \times 12v + 6 \times 9u$ is a solution of (S).

2) a) Let n_0 be a solution of (S). Check that the system (S) is equivalent to
$$\begin{cases} n \equiv n_0 \pmod{19} \\ n \equiv n_0 \pmod{12} \end{cases}$$

b) Prove that the system
$$\begin{cases} n \equiv n_0 \pmod{19} \\ n \equiv n_0 \pmod{12} \end{cases}$$
 is equivalent to $n \equiv n_0 \pmod{12 \times 19}$.

3) a) Find a ordered pair (u, v) solution of the equation $19u + 12v = 1$ and calculate the corresponding value of N .

b) Determine the set of solutions of (S) (it's possible using question 2)b).

This problem is a particular case of Chinese remainder theorem. To prove this theorem, the main organising dimension refers to an equivalence that can be interpreted in terms of existence and uniqueness of a solution of the system or in terms of surjective and injective function which is, in this case, a ring's isomorphism (let m_1, m_2 be coprime integers, for all x , element of \mathbb{Z} , the application at stake, from \mathbb{Z}/m_1m_2 to $\mathbb{Z}/m_1 \times \mathbb{Z}/m_2$, associates to each element $x \pmod{m_1m_2}$ the sequence of $x \pmod{m_1}$ and $x \pmod{m_2}$). For the operative dimension, the key to prove the existence of a solution is Bézout's identity (m_1 and m_2 are relatively prime); this is precisely the subject of Question 1. To prove the uniqueness of such a solution, the essential operative element is the result stating that if an integer is divisible by m_1 and m_2 then it is divisible by the product m_1m_2 and this can be achieved here as a consequence of Gauss' theorem (but also via the concept of LCM); this is the subject of Question 2b. In this exercise, we find again the importance of Bézout's identity and Gauss' theorem in the operative dimension underlying *baccalauréat*'s exercises; both are in Part A, a course issue, and using them in the resolution of the problem (Part B) is under the responsibility of pupils. For the organising dimension, many indications are

given; it is not a problem associated with a routine type of tasks of Grade 12. Indeed, breaking with what is proposed in this exercise, a change of objects in the operative dimension (equivalent transformation of the system (S) into the equation $12v-19u = 7$) offers the possibility of a new organising dimension via the emergence of the type of tasks T .

CONCLUSION

An analysis in terms of organising and operative dimensions permits to situate the autonomy devolved to pupils in number theory *baccalauréat*'s exercises. This autonomy is mainly located at the operative dimension. The organising dimension is under pupils' responsibility only for routine tasks as resolution of Diophantine equations $ax+by=c$ ($\gcd(a,b)$ divide c), and when it considered as non-problematic by the institution, such as the treatment of logical equivalences, or strict exhaustive search much more important since the introduction of congruences in 2002 in Grade 12 number theory curriculum. In Grade 12-University transition, we observe a transfer of the autonomy devolved to learners in proving tasks (proposal contribution for the ICMI Study 19 "Proof and proving in mathematics education"⁷): breaking with the culture of Grade 12-teaching, the skills related to organising dimension become important at the University. According to us, this transfer is one of the sources of difficulties encountered by students arriving at University to prove in number theory: except for routine tasks, their control of organising level is very too low.

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⁷ <http://jps.library.utoronto.ca/ocs-2.0.0-1/index.php/icmi/>

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DEFINING, PROVING AND MODELLING: A BACKGROUND FOR THE ADVANCED MATHEMATICAL THINKING

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This paper is a part of the large study that explores what 16-18 year old students have learnt with respect to defining, proving and modelling, considered as metaconcepts that constitute a background to the advanced mathematical thinking. In particular, we focus on the characterization of students' justifications and its persistence (or not) when making decisions related to tasks that involve those metaconcepts. Through the study, we have identified different types of considerations that underlie students' justifications. Our results have shown how students that maintain different types of considerations do not react in the same way to the same mathematical situations.

Key words: students' understanding, students' justifications, defining, proving, modelling

INTRODUCTION

The mathematical background of first year university students is an issue of concern and debate in our country. Throughout the last years, university mathematics teachers have been observing in the first year students a lack of understanding of basic mathematical ideas, which affects in a significant way the access to the mathematical advanced thinking. In order to improve this situation, some Spanish universities are offering courses of basic mathematics to students who want to access scientific and technological degrees. In this context, the highest grade (16-18 year-old students) of Secondary Education in Spain requires special interest. This grade is a non-compulsory level and its duration is two academic years. Among their aims is its importance as preparatory stage, which should guarantee the bases for tertiary studies.

Our study seeks to explore the understanding of students of the 16-18 level with respect to three metaconcepts that we consider fundamental in mathematics and didactics of mathematics: defining, proving and modelling. We consider them metaconcepts, due to their complex, multidimensional and universal configuration, admitting that each of them includes several aspects of very different complexity. In addition, we assume that they are key elements in the construction of the mathematical knowledge, and we decide to approach them jointly, since they contribute in different and interrelated ways to the above mentioned construction, and therefore to the students' learning process.

We want to emphasize that, at least in Spain, those metaconcepts are not explicitly mentioned in the school curriculum, but students approach them in an indirect way, through other mathematics curricular topics.

CONCEPTUAL FRAMEWORK

We think that the acquisition of intellectual skills is closely linked to sociocultural context (Brown, Collins & Duguid, 1989; Lave & Wenger, 1991). From this basic assumption, we approach students' understanding related to metaconcepts through:

- the use they make of the metaconcepts when they solve tasks in which the mathematical objects are those metaconcepts (metaconcepts are involved), and
- the justifications that they provide about their decision-making.

From a theoretical point of view, we needed to select some elements that allowed us accessing to that 'use' and those justifications.

With respect to the use, in an initial phase of our research we selected some elements that were considered the 'variables' of our study:

- identification variables, considered the *characteristics* that allow for a clear identification of metaconcept, and
- differentiation variables: *role*, representing different facets of the metaconcepts, and *type*, establishing differences inside them, including different systems of representation.

We think these variables are 'aspects' that can represent or describe in some way the metaconcepts and, furthermore, the relationship between the student and those aspects can inform us about his/her understanding of those metaconcepts.

These variables were specified for each metaconcept.

The variables in the case of defining. We considered "defining", among other characteristics, as prescribing the meaning of a word or phrase in a very specific form in terms of a list of properties that have to be all real ones. This prescription had characteristics that could be imperative (not contradictory, not ambiguous, and invariant under the change of representation, hierarchic nature) or optional (for example, minimality) (van Dormolen & Zaslavsky, 2003; Zaslavsky & Shir, 2005).

With respect to the differentiation variables, we selected the four roles mentioned by Zaslavsky & Shir (2005), which included: introducing the objects of a theory and capturing the essence of a concept by conveying its characterizing properties, constituting fundamental components for concept formation, establishing the foundation for proofs and creating uniformity in the meaning of concepts. In addition, we contemplated two types of definitions. Procedural type refers to what different authors consider definitions for genesis (Borasi, 1991; Pimm, 1993), which included what has to be done to obtain the mathematical defined object. Structural type referred to a common property of the object that is defined, or of the elements that constitute the object.

The variables in the case of proving. The contributions of different authors (Balacheff, 1987; Moore 1994; Hanna, 2000; Healy & Hoyles, 2000; Knuth, 2002; Weber, 2002) led us to include among the characteristics of proving the existence of both a premise / terms of reference / proposition and a sequence of logical inferences, which are accepted as valid characteristics by the mathematical community in the sense of ‘not erroneous’.

Moreover, we took into account the five roles proposed by Knuth (2002). This author, on the basis of several roles identified by previous authors and proposed in terms of the discipline of mathematics, which he considered to be useful for thinking about proof in school mathematics, suggested the following roles:

“to verify that a statement is true, to explain why a statement is true, to communicate mathematical knowledge, to discover or create new mathematics, or to systematize statements into an axiomatic system” (Knuth, 2002, p.63).

In addition, we identified three types: pragmatic proof, intellectual proof and formal proof. Pragmatic proof is restricted by the singularity of the event. That is, it fails in accepting the generic character and, in occasions, it depends on a contingent material that can be imprecise or depending on local particularities. Intellectual proof requires the linguistic expression of mathematical objects that intervene and of their mutual relationships. Lastly, formal proof makes use of some rules and conventions, universally accepted as valid by the mathematical community (Balacheff, 1987; García & Llinares, 2001).

The variables in the case of modelling. Mathematical modelling was characterized as a translation of a real-world problem into mathematics, working the mathematics, and translating the results back into the real-world context (Gravemeijer, 2004). Among the different roles, we included solving word problems and engaging in applied problem solving, posing and solving open-ended questions, creating refining and validating models, designing and conducting simulations, and mathematising situations. We selected two types: ‘model of’ and ‘model for’. ‘Model of’ deals with a model of specific situations. ‘Model for’, deals with a model for situations of the same type (Cobb, 2002; Lesh & Doerr, 2003; Lesh & Harel, 2003).

With respect to the students’ justifications, they have been considered in mathematics education from very different context and points of view (Yackel, 2001; Harel & Sowder, 1998). In particular, in our case they were analyzed according to the two main types of considerations identified by Zaslavsky and colleagues (Shir & Zaslavsky, 2002; Zaslavsky & Shir, 2005). Mathematical considerations included principally arguments in which mathematical concepts and relationships are involved. Communicative considerations were mainly based on ideas as clarity and comprehensibility, among others.

The part of the large study reported here focuses on the characterization of students’ justifications and its persistence (or not) when making decisions related to tasks that involve the different metaconcepts.

METHOD

Participants

Ninety-eight students (aged 16-18 years) participated in this part of the study. They belonged to three different Secondary schools (A, T and C in the text) of three different towns, with no special characteristics in relation to their socio-cultural context. The role of teachers and schools was not considered in the part of research reported here.

Data collection

Our data source included questionnaires and semi-structured interviews for teachers and students. Considering the aims of this part of research, we focus on the results of students' questionnaire, we will detail only this research instrument.

The questionnaire consisted of an initial presentation followed by three parts (one for each metaconcept). These parts had in general lines the same structure. They included two types of statements to access to different aspects related to the way in which the students had constructed the different metaconcepts, so that they allowed gathering a variety of points of view (Healy & Hoyles, 2000).

In the first type of statements, students were asked to provide descriptions on every metaconcept, expressing in their own words the associated meaning, and including an example that they were considering more suitable.

The second type of statements presented different possibilities for each metaconcept according to the type and role (differentiation variables). These statements were related to two mathematical topics. They included three correct/incorrect expressions for each topic. The mathematical topics belonged to different mathematical domains (Algebra, Analysis and Geometry), and were practically extracted from the textbooks used at school. For example, with respect to the metaconcept defining, we selected three definitions of perpendicular bisector (mediatrix) and three of the greatest common divisor (they are not included due to the limitation in extension of this paper). The students had to indicate whether or not these definitions were correct, which one they preferred and which one they thought their teacher would prefer, giving reasons for each of their answers.

The initial version of the questionnaire thus obtained was then sent to five expert secondary teachers, who were asked to comment on the general structure of the set of statements, and to give comments and suggestions about specific items. Their comments were used to modify the formulation of almost every statement.

Next, the revised version of the questionnaire was piloted. For this purpose, a sample of 26 secondary students was chosen. These students belonged to one of the secondary schools that participated in our study, but they were not included in the final sample. According to the analysis of their answers, some items were subsequently deleted from the questionnaire, because the original formulation was

ambiguous or unclear, or not provided important information. The final version of the questionnaire was administered to the 98 students.

Data analysis

The data in this part of the study consisted of individual students' written responses to the different items of the questionnaire. From a qualitative / interpretive approach, in a first step we followed an inductive and iterative process in which every response was divided in units of analysis. In a second step, these units were categorized depending on the type of considerations (mathematical or communicative) identified in the justifications. We exclusively considered the questionnaires belonging to students that had answered all the items. Because of that, only 67 were selected.

RESULTS

This section reports and discusses the results of the study and is organized around the two aforementioned research questions: the characterization of students' justifications and its persistence (or not) when they make decisions related to tasks that involve the different metaconcepts.

In the *justifications* provided by our students, we have found the two main types of considerations identified for Zaslavsky and colleagues (Shir & Zaslavsky, 2002; Zaslavsky & Shir, 2005). In addition, we have found some considerations on the basis on institutional-cultural aspects. This type of considerations was based in the context provided by schools that includes teachers, curriculum, principals and so on. The students identified as A217 and T17 (the first letter identifies the school, the following number the course (1 or 2) and, finally, the last numbers indicate the student) were representatives of this type of considerations:

Student A217: [I chose this...] because teachers explained it this way and this is how they taught me this topic

Student T17: Because that is how we were taught this topic at primary school and I have got used to it

With respect to the *persistence* of the students' justifications through the different metaconcepts, we have been able to identify:

- seventeen students that always followed considerations communicative or mathematical, independently of the considered metaconcept;
- six students that always combined mathematical and communicative (mathematical/commnicative) considerations, independently of the considered metaconcept;
- thirty-one students varied their considerations depending on the metaconcept. These considerations could be mathematical, communicative, institutional/cultural or they combined these types the considerations;

and

- thirteen students that used different considerations depending on the different statements in each metaconcept; in this case, we were not able to identify the type of consideration and they were not considered here.

In relation to the 17 students that maintained a common consideration, we show in the Table 1 the types of considerations identified and the corresponding students:

<i>Types of considerations</i>	<i>Students</i>
Communicative	A15,A16,A28,A213,A216,C16,C19,C120,C135
Mathematical	A25,T13,T14,T113,T114,T21,T25,C127

Table 1: Students that maintained communicative or mathematical considerations

The nine students situated in a communicative perspective considered their own person as the ‘centre’ of the considerations. The following excerpt is representative of this:

Student A16: I like statement 1 because it seems to be the easiest one for me

In general, communicative students’ decisions were related with ideas as clarity, comprehensibility and so on. They saw *mathematics* and *teacher* (considered as a vehicle of communication between student/mathematics) from a very personal point of view.

In the case of the eight students situated in a mathematical perspective, their considerations were related to the use of mathematical expressions, lack of accuracy and so on. The following excerpt exemplifies this aspect:

Student A25: Statement 1 is not correct because it tells you what normally happens ... in the majority of cases is the greatest number... but it doesn’t not always have to be this way ... it is incomplete

These students were able to consider separately the mathematical aspects from the personal aspects.

In addition, communicative students made a weak distinction of the identification variables (characteristics that allow the identification of a metaconcept). In relation to students situated in mathematical considerations, we can say that the majority of these students identified the incorrect expressions of the three metaconcepts, although they showed different degrees of accuracy in their mathematical arguments for justifying their decisions. The percentage of communicative students that were able to decide whether or not a statement on the different metaconcepts was correct was less than 40% in all cases. This percentage increased up to a 90% in the case of students that adopt mathematical considerations.

In particular, in the case of defining, 7 out of 9 communicative students chose both for teacher and students the same definition of mediatrix and the greatest common divisor, independently of characteristics, role and type and representation system. The

communicative students did not see these characteristics as relevant because the centre was his/her own person. This result was also found in proving, with a slight difference between topics (7 of 9 and 6 of 9 in each case), and in modelling. This result differed in the case of mathematical students, who did not show a clear coincidence.

With respect to the thirty-one students who adopted different justifications depending on the metaconcept, the three main types of considerations (communicative, mathematical and institutional-cultural) were combined in some cases. We were able to identify several types of mixed considerations (communicative/ institutional-cultural, communicative /mathematical, mathematical/ institutional-cultural). We show in the Table 2 the students that were situated in each consideration.

	<i>Proving</i>	<i>Defining</i>	<i>Modelling</i>
<i>Communicative considerations/mathematical considerations in each metaconcept</i>			
Communicative	A210	A215, A217	A210, A215
	T11, T15	T112	T17, T112
	C12, C116, C119 C122, C123, C132 C138, C139		C119, C122, C123 C134, C138
Mathematical		A211	A29, A211
	T18, T19, T112 T22, T23, T29, T210	T12, T28, T119	T11, T18, T19 T115, T119 T22, T23, T28, T29
	C137	C116, C139	C129
<i>Mixed considerations in each metaconcept</i>			
Communicative /Institutional- cultural	A217		A217
	T17		
	C129	C119, C123, C134	
Communicative /Mathematical	A29, A211, A215	A29, A210	
	T12, T115, T119 T28	T11, T17, T18, T19, T115 T22, T29, T210	T12 T210
	C134	C12, C122, C129, C132, C137, C138	C12, C116, C132, C137, C139
Mathematical/ Institutional- cultural	T118	T15, T23, T118	T15, T118

Table 2: Students that varied their considerations depending on the metaconcept

As we can see in the Table 2, globally considered there were not significant differences between the number of communicative or mathematical considerations (23 and 26 respectively). The communicative/mathematical considerations (C/M) prevailed, being the most common in the three metaconcepts. Communicative considerations had a significant presence in proving and modelling with respect to defining.

In addition, 6 students (A14, A19, A21, C110, C126, and C130) maintained communicative/mathematical considerations in all metaconcepts. These students used communicative considerations when the focus of their justification was the relationship between the metaconcept and themselves; when the relationship was between metaconcepts and the teacher, the type of consideration was mathematical. We can say that in these cases those considerations were associated with the 'character' (student or teacher).

It is worth to point out to the great number of students that belong to the Secondary School T and who were situated in mathematical considerations. Although the reasons provided by the teachers in the large research have been very useful in explaining, from their point of view, some of the differences between the different Secondary Schools, as we mentioned above this is not the aim of the part the research reported here.

CONCLUSIONS

Our study examines three metaconcepts that we consider basic in the construction of students' mathematical knowledge. The findings suggest that the type of research instrument we designed has proven to be a valuable research tool in the identification of students' justifications.

Students' communicative and mathematical considerations proposed by authors as Shir & Zaslavsky (2002) for defining have been enlarged in the case of other metaconcepts as proving and modelling. In addition, the presence of institutional-cultural considerations showed in the other kind of justifications, which indicate the importance of the aspects linked to school context, that are considered as a 'source' for the justifications. Moreover, we were able to see the presence of mixed considerations (Communicative/institutional-cultural, communicative/ mathematical, and so on).

Our results have shown the students that justify their decisions on the basis of mathematical or communicative considerations do not react in the same way to the same mathematical situations. In particular, we have been able to see the difficulties communicative students have in making decisions both on distinguishing the characteristics of metaconcepts and on differentiating between the teacher and themselves, showing that their decisions are related to personal aspects. For mathematics teachers this fact implies the importance of considering the existence of students whose analytical tools are based on communicative aspects and the difficulties that means in helping them to construct other types of reasoning.

With respect to the findings related to the students that varied their type of considerations depending on the metaconcepts, they inform us about the necessity of going deep into the relationships among the motives that students have to link a specific type of considerations to a specific metaconcept. In some way, these relationships could inform us about some characteristics of students' understanding.

Finally, although it has not been considered in this paper, the differences among secondary schools that we have identified in our findings lead us to the need to incorporate in the design of future research some instruments that allow us to answer the following question: up to which point is the adoption of any determined consideration influenced by the specific education (training) of a secondary school and particularly by secondary school teachers? As researchers, we need to deepen the characteristics of the relationships between students and teachers in a specific secondary school that might encourage a determinate type of considerations.

NOTES

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NECESSARY REALIGNMENTS FROM MENTAL ARGUMENTATION TO PROOF PRESENTATION

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This paper deals with students' difficulties in transforming mental argumentation into proof presentation. A teaching / research tool is put forward, where the statement of a task is accompanied by a given written piece of argumentation suggesting a way to resolve the task intuitively. The student must convert this into an acceptable mathematical form. Three illustrative examples are given.

Key words: mental argumentation; proof presentation; mathematical language; refinement of expression; transparency.

INTRODUCTION

It has been noted in several papers (eg. Gusman, 2002; Moore, 1994) that in certain circumstances students can 'see' a proof but they cannot express their intuitive ideas in terms of mathematical language. The students use representations that are or have become over time divorced from the mathematical frameworks that allow explicit tools of exact analysis. Thus an impasse occurs.

On the other hand, the usual style of presentation of proof can seem 'monolithic'. It denies in most cases not only a history of aborted attempts, but also it does not communicate essential conceptual and cognitive input that supported the initial formation of the proof. In this respect, reading a proof has a facet that has to be deciphered. When assessing proofs we should not be only concerned in investigating the 'mechanics' that explain how a given proof succeeds in what it was meant to achieve. We also should be concerned with the creative processes involved in producing the 'mechanics' in the first place.

Hence, the circumstance where a student can discern an argument informally but cannot express it in a ratified mathematical format is exacerbated by the fact that past exposure to proof presentation hardly is supportive. A possible remedial measure might be to seek for a radical change in how proofs are written, to better reflect the cognitive input that otherwise would be repressed. However in the next section we will argue that there are compelling reasons to retain the traditional styling of proof presentation. Taking this in mind, if students are to develop the skills to convert mental argumentation into mathematical frameworks allowing deductive reasoning, channels have to be found to help the students to achieve this. In this paper, we put forward such a channel.

In particular, we consider the situation where a student is given not only a task, but also has an informal description how to deal with the task. The description can be self, peer, or teacher generated. The job of the teacher is to guide the student to

transform the information that is provided into a strict proof. This is envisaged as a sustained teaching practice, which hopefully would encourage student emulation in their independent work. The education researcher also has a role. Beyond investigating which kinds of guidance given by the teacher will be the most effective, the researcher would be interested in identifying specific types of discrepancies that can occur between informal and formal reasoning, and their effect in cognitive terms.

The main body of this largely theoretical paper will comprise a discussion of three worked examples. These worked examples follow a certain format of design. We envisage that this format could be consistently adopted as a research tool for an educational program of a larger scale. For each example, its content will be carefully separated between the 'givens' and the 'material to be produced'. The 'givens' have two components; the first is a task or a proposition, the second is a mental argument that addresses it informally. The material to be produced will include a 'rigorous' solution or proof influenced by the given mental argument. In addition, in order to ease the transition to the proof, the material to be produced may further involve the formation of an enhanced version of the initial informal argument.

The examples are chosen to illustrate how the identification of structural properties in the informal argumentation can lead to an entry point into a mathematical framework, and ways that proof presentation may seem not to respect the informal line of thought. The approach taken here would be most pertinent to the upper-secondary and tertiary levels, as it is at these levels that the insistence of proof production becomes more poignant.

We acknowledge some points in our undertaking might deny some important aspects in combining intuitive and formal sources in the doing of mathematics. For example, ideally the students themselves could be constructing their own representations and mental argumentation. Representations and mental argumentation made by peers or the teacher may not be comprehended by the students. Further, often it is the case that mental argumentation and the thinking consonant to mathematical frameworks might evolve mutually. These points might suggest that what we are endeavouring to do in this paper has its limitations. However, we do believe that the direction we take constitutes an important device for analysing the learning and teaching of mathematical modelling, and the potential difficulties that are involved.

MENTAL ARGUMENTATION AND PROOF; HOW DO THEY DIFFER?

It has often been observed both by mathematicians and educators that the proofs published in mathematical journals are far from being completely rigorous (e.g., Thurston, 1995; Hanna & Jahnke, 1996). This has prompted some educators to view proof mostly in terms of conviction. However, in certain circumstances even a highly naive argument can be so compelling that any reasonable person would be 'convinced' of the proposed conclusion. The problem is that however 'obvious' or 'transparent' an intuitive argument is, there might not be a way to directly reduce it to

fundamental principles. The point is not so much about conviction, but how we can clarify the bases of the reasoning employed. The notion of a 'mathematical warrant' (Rodd, 2000) addresses the issue of justifying the grounds that support students' belief in the truth of a mathematical proposition. Still, in how this term is employed suggests a certain primacy to 'embodied processes' over any mathematical setting demanding deductive argumentation.

This primacy might be challenged by some. For example, the construction of a proof can be regarded as an activity to make argumentation more precise. From this viewpoint, proof refines any intuitively based argument. Perhaps a more balanced stance to take is that it is artificial to try to distinguish informal thinking from formal thinking. Thurston talks about a mathematical language (replacing the 'myth' of complete rigour). As in any language, there is ample space to express ideas in casual, incomplete, or inexact formulations. However mathematical language is strongly rooted to a vocabulary referring directly to defined mathematical entities, and its expression is conditioned by respecting previously ascertained properties. Drawing a sharp characterisation of this language might be a difficult undertaking, though preliminary remarks are made in Downs & Mamona -Downs (2005). Assertions made by Thurston are that it is very difficult for students to become fluent in the mathematical language, but ultimately it is in this medium that mathematical thought evolves.

In the introduction we employed the term 'mental argumentation'. What place does this have in our discussion above? From our perspective, mental argumentation rests on collating sources of intuitive knowledge. One character of intuitive knowledge is that, cognitively, it deals with self-evident statements. Unlike perception, intuitive knowledge exceeds the given facts (see Fischbein, 1987). Also, it is accumulative; it depends on past assimilation of conceptual matter. The collation involved in mental argumentation can be made either at the level of instinct or at the level of insight. Both rely on a certain degree of vagueness (see Rowland, 2000, for the importance of vagueness in the doing of mathematics). Mental argumentation should convince the practitioner but not necessarily others; the practitioner would be aware that someone else might demand a warrant. Mental argumentation can lie either inside or outside the mathematical language. Which of the two depends on whether the collation of intuitive knowledge is guided by mathematical insight rather than instinct. Indeed if the argument is based on instinct, there is a lack of self-awareness of the sources drawn on in making the reasoning, including mathematical backing.

Harel, Selden & Selden (2007) have put forward a framework for the production of proof by distinguishing a 'problem - oriented' part and a 'formal - rhetorical' part. (The word rhetorical here serves to point out that what is accepted as formal proof can include some standard linguistic devices beyond strict logic). We suggest that mental argumentation stresses the 'problem - oriented' part; the 'formal - rhetorical' part is as yet opaque, and it is drawn on only when it is required to bolster the intuitive line of thought. A 'naturalistic' proof is obtained by respecting the original

problem solving aspects, but fills the 'gaps' in the reasoning by explicitly bringing in mathematical sources permitting tight deduction. A 'naturalistic' proof should be explanatory; Hanna & Jahnke (1996) suggest that proof that explains is preferable to proof that does not. However 'naturalistic' proofs are not always feasible; in the process of converting the original mental argumentation into a framework allowing deductive argument, certain mathematical constructs have to be made to accommodate the intuition, but in doing this there might well be clashes in cognition that cannot be side-stepped. Because of this, formal proof presentation often does not seem to communicate the thinking processes that first motivated its formulation. However, the formal presentation is not simply a contrived imposition, stipulating that your argument has to be validated by a vague standard of rigour. It is something that is encompassed in the mathematical language. In that context, the original thinking processes should be retrievable. Hence, we have a duality between the problem-solving element needed in forming a proof and that needed in reading a proof (see Mamona-Downs and Downs, 2005).

A teaching/research practise similar to that proposed in the introduction is forwarded by Zazkis (2000). It deals with relatively simple examples that only involve translation from mental argumentation to naturalistic proof.

THREE ILLUSTRATING EXAMPLES

In this section we write down and discuss three tasks and proposed solutions. The purpose is to illustrate some cognitive issues concerning the conversion of mental argumentation into proof presentation. In considering just three tasks, our exposition will bring up only a sample of the points that potentially can be made; we believe that many other points and elaborations can be drawn in the future.

Each example will be divided into three parts. The 'givens' is the material that would be given to the student if a fieldwork were undertaken. The 'material to be produced' always includes a form of a suitable proof presentation, but might also involve a middle step enhancing the original mental argumentation. The 'material to be produced' is made in a putative spirit rather than regarding it as a 'model solution'. Finally, the 'comments' relate the cognitive factors extracted from the examples.

Example 1

Givens

Task: Two persons, A and B, start a walk at the same time and place along a particular path of length d . Person A walks at speed v_1 for half of the time that A takes to complete the walk; after he walks at speed v_2 , where $v_2 < v_1$. Person B walks at v_1 for half of the distance, and after walks at v_2 . Who finishes the walk first?

Mental argumentation: Person A covers more distance in the first half of the time when walking at v_1 than the distance achieved in the second half of the time walking at v_2 (as $v_1 > v_2$). Thus A walks further than the half point in distance, i.e. $d/2$, at the faster speed v_1 , whereas person B walks only the half- distance at v_1 ; hence A arrives first.

Material to produce

Proof presentation: Let d_1 be the distance at which A changes speed. Let t_1, t_2 be the time for A, B to complete the walk respectively. Then

$$d_1 = \frac{1}{2}t_1 v_1 \Rightarrow d - d_1 < d_1 \quad (\text{as } v_1 > v_2) \Rightarrow d_1 > \frac{d}{2}$$

$$d - d_1 = \frac{1}{2}t_1 v_2$$

$$t_2 = \frac{\frac{d}{2}}{v_1} + \frac{\frac{d}{2}}{v_2} = \frac{d}{2} \left(\frac{1}{v_1} + \frac{1}{v_2} \right) + \frac{(d_1 - \frac{d}{2})}{v_2} + \frac{(d - d_1)}{v_2} > \frac{d}{2} \left(\frac{1}{v_1} + \frac{1}{v_2} \right) + \frac{(d - d_1)}{v_2} \quad (\text{as } v_1 > v_2)$$

$$= \frac{1}{2}t_1 + \frac{1}{2}t_1 = t_1$$

Comments

This example constitutes a relatively smooth transition from the mental argumentation to the proof presentation. Even so, we envisage that many students might have problems in executing it. Even the required assignation of symbols (d_1, t_1, t_2) has a modest constructive element that should not be assumed easy for the students to adopt. The thrust of the proof lies in the transformation of $d/2$ into $(d_1 - d/2) + (d - d_1)$. The motivation in doing this is $(d_1 - d/2)$ represents the distance that A walks at the highest speed v_1 beyond B does; $(d - d_1)$ represents the distance for which both A and B walk at the lower speed v_2 . Hence one term pinpoints where the behaviour of A and B is different, the other where their behaviour is the same. This 'move' might be difficult to make unless you have the support of the mental argumentation, so the student would have to have a tight grasp of how the intuitive reasoning is guiding the algebra.

This task appears in Leikin & Levav-Waynberg (2007) in the context of connecting tasks. Another approach different to the one above would be to take the strategy: explicitly determine the time that A and B take separately and then argue which time is the shorter. However, there is not a sense here that a mental argumentation is playing a role; the task is immediately modelled into an algebraic context, and the argumentation is accomplished completely at this level. This latter approach certainly provides more explicit information (beyond what was demanded), but lacks the transparency that the first provides.

Example 2

Givens

Task: Suppose that the real sequence (a_n) is convergent, and there is an infinite subset M of the set of natural numbers \mathbb{N} and a real number t such that $a_n=t$ whenever $n \in M$. Prove that the limit of (a_n) is t .

Mental Argumentation: There is an 'infinite number of terms' that take the value t , so however far the sequence has progressed there must still be a term having the value t not reached as yet. At the limit, the terms must be tending to the limiting value, but as far progressed the sequence is, t 'occurs', so the limiting value must be t .

Material to produce

Enhanced mental argumentation: Suppose that in fact it is not true that the limiting value is t . Then the value must be a number $l \neq t$. There is an explicit number expressing the distance between l and t . However progressed is the sequence, the value t 'occurs' and so there will always be terms that have a certain fixed distance from the limiting value. This contradicts the idea that the sequence is tending to the limiting value. Thus it cannot be true that l and t are different.

Proof Presentation: Suppose that $\lim a_n=l$ and $l \neq t$. Let $\varepsilon = (|l-t|)/2$. Then there is a natural number N such that for all $n > N$, $a_n \in (l-\varepsilon, l+\varepsilon)$ and we have chosen ε such that $t \notin (l-\varepsilon, l+\varepsilon)$. Now there are only a finite number of $n \in \mathbb{N}$ such that $a_n \notin (l-\varepsilon, l+\varepsilon)$. This means that only a finite number of $n \in \mathbb{N}$ satisfy $a_n=t$. This is a contradiction.

Comments

The first mental argument could persuade some students on reading it, but the basis of its acceptance rests on a degree of personal instinct that likely would not be shared by others. An enhanced mental argument might arise as an attempt to remedy some of the shortcomings of the first; if the argument lacks concreteness when it is used to justify a proposal, you might be forced to consider the consequences if the proposal was not true. These consequences might run contrary to the specifications of the task environment. In this way, we believe that logical devices such as proof by contradiction can, up to a point, be naturally handled in the confines of mental argumentation.

There remains a point of vagueness shared by both mental arguments, i.e. the claim 'however the sequence has progressed there must still be a term having the value t not reached as yet'. Likely the acceptance of this would depend much on the student having a suitable mental image of what an infinite sequence is. Without this, a student might be doubtful about how the claim could be justified.

For a justification, one has to refer to the mathematical definitions providing the means to decide on issues dealing with limits. Much research has reported clashes of intuitive images with the dictates of the definition of the limit. With this in mind, it is not surprising that some switches of focus have to be made to transform the mental

argumentation into a proof presentation, Mamona-Downs (2001). What the definition provides is an ' ε -strip' around l that stipulates that however small ε is, there is a 'stage' of the sequence beyond which the values taken must be trapped in the strip. (This makes use of imagery that is usually made available in the teaching process.) By choosing ε small enough, we can arrange the ε -strip to 'avoid' the value of t if $t \neq l$. Then there are only a finite number of terms 'at the start of the sequence' that can possibly take the value of t , and we reach a contradiction.

The switch then is that instead of employing the fact that there are infinitely many terms taking the value of t as a basis for argument, one employs the definition of the limit of a sequence as a basis for finding contrary evidence. The character of the contradiction here is somehow different from the one found in the enhanced mental argument. The difference could be expressed by comparing "if the result was not correct, then a condition is transgressed" with "a perceived property (tending to the limit) is contravened".

Note that the negotiation of what direction the proof should follow is itself couched in casual terms. This illustrates how mental argument can be a part of the mathematical language. Even though the supporting mental argument guides the structure of the proof, the proof presentation does not acknowledge its role. Particularly stark is the setting, almost as a fiat, of the value of ε . However, from our strategy making, the choice of ε is pre-motivated, and it could take any value in the interval $(0, |1-t|)$. A reader of the proof might not appreciate this. Another feature of the proof presentation is the compression involved in the statement 'we have chosen ε such that $t \notin (l-\varepsilon, l+\varepsilon)$ '. Set theoretically, a justification of it would take several lines. But because the value of ε was picked especially to satisfy the property involved, these details can be safely suppressed. In general, the transition from one line to another in a proof presentation often goes beyond deductive implication; it often 'hides' input from mental argumentation. The skeletal form of the proof presentation has an advantage in that the 'gaps' that appear can be filled through insight, but if this fails one can always resort to the mathematical tools available to complete the minutiae synthetically. This discussion throws a light on the respective roles of mental argumentation and proof presentation in the mathematical language.

Example 3

Givens

Task: Let n be a natural number. Suppose that r_n is the highest power of two dividing the factorial of 2^n . Find r_n .

Mental argumentation: (Student produced)

"We know that from the numbers $1, 2, 3, \dots, 2^n$, there are 2^{n-1} numbers which are divisible by 2. We note that from the numbers

1, 2, 3, ..., 2^{n-1} , there are 2^{n-2} numbers that are divisible by 2. We note that from the numbers 1, 2, 3, ..., 2^{n-2} , there are 2^{n-3} numbers that are divisible by 2. Continuing to the end we have that $2^n! = 1.2.3...2^n$ is divisible by 2 raised to the power

$$2^{n-1} + 2^{n-2} + 2^{n-3} + \dots + 2^2 + 2 + 1.$$

This means that r_n equals $2^n - 1$."

Material to produce

Proof production: Here there is a choice. One tack that can be taken is to conjecture that the result obtained is correct and then use induction. This is fairly easy to do, and it will be left to the reader. The other tack is to produce a proof not assuming the result. Such a proof might follow the lines as below:

For each $i = 1, \dots, n$, let

$$A_i = \{s \in \mathbf{N}: s \leq 2^n \text{ and is a multiple of } 2^i\}$$

$$B_i = \{t \in \mathbf{N}: 2^i \text{ divides } t \text{ and } t/2^i \text{ is odd}\}$$

$$a_i := |A_i|, \quad b_i := |B_i|$$

By construction,

$$r_n = \sum_{i=1}^n i b_i, \quad a_i = b_i + b_{i+1} + \dots + b_n \quad \text{and} \quad a_i = 2^{n-i}$$

Hence, for $i \neq n$

$$a_i = b_i + a_{i+1} \Rightarrow b_i = a_i - a_{i+1}$$

$$\Rightarrow r_n = n + \sum_{i=1}^{n-1} i(a_i - a_{i+1}) = n - (n-1) + \left(\sum_{i=2}^{n-1} (i - (i-1))a_i \right) + a_1$$

$$= 1 + \sum_{i=2}^{n-1} a_i + 2^{n-1} = \sum_{i=1}^{n-1} 2^i = 2^n - 1$$

Comments

In this example, contrary to the previous two, the mental argumentation was produced by two students (working together) whilst doing project work, and this constituted their final answer. In a subsequent interview, it became clear that they did not consider their response to constitute a proof, however the terse manner of their exposition seems to be influenced by an image of a proof being minimally expressed. In the interview the students were able to explain the origin of the stated lists of numbers, but only in informal terms. It is significant that the students did not spot the induction option, as in other work they showed themselves adept in identifying and applying this general proof technique. The impression was that they wanted a proof that reflects and respects the procedure for which they invested a lot to obtain the answer, rather than building up an argument employing the answer as a working conjecture. Quite likely, if their presentation were shown to other students

to refine, those students would be more inclined to take the induction method. This proposition illustrates that we should expect some differences in student behaviour when they are reacting to their own mental argumentation rather than that provided by others.

The proof stated was achieved by the students with guidance of one of the authors during the follow-up interview. The degree of guidance will not be described here; in accordance with the other two examples, the proof will be discussed hypothetically in terms of cognitive demands in producing it from the existing mental argumentation. First, notice that the proof involves the construction of families of sets. Although the importance of sets (and functions) to the foundations of mathematics is usually emphasized in teaching at the tertiary level, generally students tend to be poorly equipped to design sets for specific purposes. Returning to the example, the family of sets A_i reflects the process that is implied in the mental argumentation; had the two students based their argumentation on these sets, the exposition of the solving algorithm would have been clarified. The family of sets B_i had the role to model the situation given by the task environment. The B_i 's give the grounding, the A_i 's the calculating power. Thus the B_i 's appear from theoretical considerations, and are related (in the form of their orders) to the A_i 's to realize the numeric expression sought. In this way, the translation from the mental argumentation to a proof presentation needed the construction of sets together with a strategic understanding how these sets would avail what was desired. We see then that proof production can involve significant problem solving aspects, as noted before.

CONCLUSIONS

There is plenty of evidence that students experience severe difficulties in the production of mathematical proofs. A particularly frustrating circumstance for a student is when he/she can 'see' a reason why a mathematical proposition is true, but lacks the means to express it as an explicit argument in one form or another. One problem is that students feel that the 'reason' has to be immediately couched in 'rigorous' mathematical terms. In fact, there is no harm in trying to write informal descriptions, which can be a first step in developing mental argumentation ultimately giving access to 'mathematization'. The paper proposes a teaching / research tool designed to give students support in this process. This tool provides, beyond the stated aim of the task, an informal account how the aim might be achieved. This format has several advantages. One is that it should help students to regard mental argumentation as being legitimate. Second, mental argument comprises an environment that allows refinement of expression. Third, mental argumentation is not just a way of negotiating an entry into established mathematical systems, but even the writing of proof presentation is highly dependent on it, though its influence is usually left implicit.

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AN INTRODUCTION TO DEFINING PROCESSES

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Abstract. The aim of this paper is to bring some theoretical elements useful for the characterization of defining processes. A focus is made on a situation which engages students in the construction and the definition of concepts used in linear algebra (such as generator, independence). Such concepts have a reputation of being difficult to learn and to teach. The specificity of such a situation is that it comes from discrete mathematics and it allows a mathematical questioning and a mathematical experience.

Key words: defining processes, concept image, discrete mathematics, linear algebra, (in)dependence, minimality, generator.

INTRODUCTION

The defining process represents a specific constant of the language and of the human thought. In mathematics, as well as in all the scientific fields, to define is intrinsically linked to the objects: the action of “defining” attests to the existence of new objects and gives them the status of “scientific objects”. In a formal theory, definitions seem to be undeniable, immutable and appear like definitive statements. Nevertheless, the forms, the status and the roles of definitions change notably, throughout the centuries (history of mathematics teaches us a lot), but also through teaching and learning processes. From one point of view among others, a definition can be a statement given in order to know what one talks about (such as Euclidian definitions which are declarative statements: everybody already knows what it refers to). A definition can also be the only way one can grasp a concept, at the beginning of a presentation. From another perspective, a search of a proof can make room for a new concept: that is the notion of *proof-generated definitions* introduced by Lakatos (1961, 1976). All these elements underline the gap between defining processes in real live mathematics where definitions come at the end of a research process and are generally intrinsically linked to a proof perspective and formal theories where definitions come at the beginning of a presentation. In fact, the way one considers definitions depends on the view one has about the mathematical experience, and then the view one has about “proof”. Formal and axiomatic mathematical presentations hide scientific concepts, their pertinence and their usefulness. That obviously explains why students have difficulty learning and understanding new concepts. Indeed, students must construct concepts from the definitions given at the start of a chapter where all concepts appear as divided into compartments. Moreover a formal definition is generally a minimal one because axiomatic theories should be nice with a small number of axioms and non-redundant definitions. Then, with a formal minimal definition, a student only has a view of a concept. But, when grasping a new concept, a student needs to have several properties of this concept, several representations, links with other concepts

and equivalences between different kinds of properties. Furthermore, a definition can become a proposition used in a proof in order to make an inference. That prevents students to distinguish clearly among axioms, theorems and definitions.

In my opinion, the question of mathematical definitions is a crucial one in an advanced mathematical perspective. The existence of formal definitions and formal proofs marks Advanced Mathematical Thinking. It is taken into account by Tall and Vinner with the notions of *concept definition* and *concept images*. Students construct concept images to give meaning to formal mathematical concepts. Therefore, studying concept images represents one way of characterizing concept formation and a part of the students' understanding of a concept, even if the students' concept images are not always easily accessible. I suggest focusing my paper in a mathematically-centered perspective as proposed in this working group, studying more specifically definitions in the general background of "Problem-solving, conjecturing, defining, proving and exemplifying at the advanced level". Questioning the defining processes at stake in the work of real live mathematicians can bring answers to didactical research about concept formation. My approach is an epistemological one and tends to question the practice of mathematicians concerning definition construction processes. I intend to explore what a mathematical experience can be, focusing on defining processes, which are difficult to characterize meta-processes. I will also propose the broad outlines of a framework useful for analyzing a situation for the transition stage between upper secondary level and university.

KEY CONCEPTS FOR THINKING MATHEMATICAL GROWTH THROUGH DEFINING PROCESSES

The work of Tall (2004) is ambitious and paramount. I have commented it (Ouvrier-Buffer, 2006), taking into account the specific perspective of definitions, in the following way. Tall (2004, p. 287) gains "an overview of the full range of mathematical cognitive development" by scanning a whole range of theories. A global vision of mathematical growth then emerges, making room for three worlds of thinking: the "embodied world", the "proceptual world" and the "formal world". In this way, a more coherent view of cognitive development may be obtained. Endorsing this point of view, I will question the place of definitions in such a theory. "Formal definitions" admittedly belong to Tall's "formal world". What happened before the "smooth" definitions were arrived at? What were the heuristic processes involved? Although the apprehension of new mathematical concepts began in the "embodied world" through perception, I still assume that the "proceptual world" is not always adequate to characterize a concept which is being constructed. So how are we going to grasp the dialectic between concept formation and definition construction within this theoretical range? I think we can safely assume that there is another world, different from the "embodied", "proceptual" and "formal" worlds, which is both transversal and complementary, fostering the characterisation of mathematical growth through definition construction processes in particular. I will not characterize such a

fourth world (because it is a transversal one to the previous three), but I will try to give key concepts for thinking mathematical growth (i.e. concept formation in my perspective) through defining processes.

What does it “defining processes” mean? This wide question cannot be entirely dealt with in such a paper. Let me give some elements of my research perspective.

The concept of “definition” can actually be approached in several ways because it is at the intersection of different fields. Studying “definitions” inevitably leads us to philosophical questions, joining the famous nominalism/essentialism debate, the problem of the existence of the objects one defines, and logic and linguistic considerations. Because a definition is a part of a theoretical system, the field of logic and meta-mathematics (how to build formal and axiomatized theories) should be explored but is not the purpose of this paper.

The heuristic approach as proposed by Lakatos (1961, 1976), where a definition is an answer to a problem, and the concept formation approach, as proposed in different directions by Vygotsky and by Vinner for instance, represent my research interests. Vygotsky (1962), in the famous Chapter 6, underlines the structure of scientific concepts organized in systems (interdependence of concepts within networks) and the distance between the growth of scientific concepts and the growth of everyday and spontaneous concepts. But Vygotsky does not take into account the nature of the concepts. Vinner does. To map the concept formation implies to grasp students’ concepts images and the links which they are able to do with other knowledge.

Let me now summarize two fundamental notions about definitions. Tall and Vinner made a distinction between the individual way of thinking of a concept and its formal definition, introducing the notions of *concept image* and *concept definition*. It allows to take into account mathematics as a mental activity and mathematics as a formal system. Then, practice of mathematicians and students’ cognitive products can be studied from that perspective. Moreover, I retain that Vinner emphasizes the importance of constructing definitions: “the ability to construct a formal definition is for us a possible indication of deep understanding” (Vinner, 1991, p. 79) and explains the “scaffolding metaphor” which presents the role of a definition as a moment of concept formation. Within his theoretical framework, Vinner suggests to expose a flaw in the students’ concept image of a mathematical concept, in order to induce students to enter into a process of reconstruction of the concept definition and proposes some interplay between definition and image. Vinner assumes that “to acquire a concept means to form a concept image for it (...) but the moment the image is formed, the definition becomes dispensable” (p. 69, *ibid*). I underline the first part of this quotation and the main interest of using concept image (and concept definition) as a theoretical tool to analyze dynamical defining processes. From a didactical perspective, the main question is the following: how can one make easier the construction of students’ concept image? And how can one use markers in order

to characterize such a process? The notion of concept image, according to Watson and Mason, is used:

to encompass all the images, definitions, examples and counterexamples, associated links, and their interrelationships that are all held together in a structured way and constitute the learner's complex understanding of the concept (Watson & Mason, 2005, p. 97).

It is time to introduce Vergnaud's idea of invariants which make the students' action operational. Vergnaud (1996) distinguishes *concepts-in-action* and *theorems-in-action*, in reference to the concepts and the theorems of mathematics. In particular, he defined concepts-in-action in the following way:

Concepts-in-action are categories (objects, properties, relationships, transformations, processes, etc.) that enable the subject to cut the real world into distinct elements and aspects, and pick up the most adequate selection of information according to the situation and scheme involved (Vergnaud, 1996, p. 225).

I extend these notions to *definitions-in-action* and *properties-in-action* in order to guide an analysis on the students' invariants.

My research about definitions had led me to also adopt an epistemological point of view, taking into account simultaneously logic, linguistic, axiomatic and heuristic approaches. Let me focus here on the Lakatosian heuristic point of view (and not on the formal aspect of the reconstruction of a theory), where definitions are temporary sentences and also at the dialectic interplay with proofs. Therefore, I use Lakatos' categories of definitions, namely *zero-definitions*, emerging at the start of an investigation, and *proof-generated definitions*, directly linked to problem situations and attempts at proof. In the context of the immersion of a proof in a classification task (Euler's formula and polyedra), Lakatos has showed that a definition is not only a tool for communication, but also a mathematical process taking part in the formation of concepts. In the example at hand, the aim consists in a characterization of markers in order to examine the concept formation process, and, in particular, to identify specific statements in the defining process. Let me underline that the kind of problem proposed by Lakatos can be inscribed in a problem-solving perspective because of the dialectic between the construction of a definition and the validity of a proof (involved in Euler's formula). But the starting point is "only" a classification task. Such a situation can be kept in mind. We now have some cognitive and epistemological elements in order to try to grasp defining processes (namely concept image, definitions-in-action, zero-definitions and proof-generated definitions).

SITUATIONS INVOLVING DEFINING PROCESSES

Can we now imagine several kinds of situations involving defining processes? Of course, there is the case of the construction of a theory, when several theories are in competition (Popper, 1961). However, I will not develop this aspect, even if it plays a leading role in the defining processes (indeed definitions are chosen, reconstructed

etc. during axiomatization), because it is not a beginning from a didactical perspective when one wants students to be engaged in a process of knowledge construction. There are not a lot of propositions for constructing definitions and building new concepts in the relevant literature in mathematics education (I do not take into consideration the situations of reconstruction of definition of a known concept). My research is focused on the design and on the analysis of situations in which students are engaged in defining processes in order to build new concepts. I therefore had to work out a theoretical framework through epistemological, didactical and empirical research in order to characterize definitions construction processes (Ouvrier-Bufferet, 2006). My experiments were conducted in discrete mathematics with the following concepts which are of different natures: trees (a known discrete concept, graspable in several ways), discrete straight lines (a concept which is still at work, for instance in the perspective of the design of a discrete geometry) and a wide study of properties of displacements on a regular grid. I have chosen to develop this last point for two reasons. Firstly, this kind of situation contributes to make students acquire the fundamental skills involved in defining, modelling and proving, at various levels of knowledge. A mathematical work on (“linear”) positive integer combinations of discrete displacements actually mobilizes skills such as defining, proving and building new concepts. Secondly, it leads us to work in discrete mathematics but also in linear algebra because similar concepts are involved in this situation. So we can focus on concepts which are known as difficult, at the university level, namely concepts of linear algebra. These concepts have the specificity of being inscribed in a very formalized theory, and historically, they have a unifying and generalizing power. They are well-known for being difficult to learn... and to teach.

The challenge, from my point of view, is to find a “good” situation i.e.: 1) a situation which allows the construction of some concepts and leads students to explain and to explore a mathematical questioning and then, to have a mathematical experience; 2) a situation which does not generate well-known obstacles in teaching and learning linear algebra (and so which avoids the problems connected to the lack of practice in basic logic and set theory of students for instance and their difficulty connecting new concepts to previous knowledge etc.); 3) a situation which allows the construction of zero-definitions and the catalysis of proof-generated definitions, trying to instil a kind of concept images in particular (the study of Harel (1998) underlines that the students do not build effective concept images for the concepts of linear algebra, in particular for the notion of independence); 4) a situation which brings a kind of useful and dynamic representation of some concepts of linear algebra, avoiding the trap of using 2D or 3D geometry: indeed, the attempts to connect linear algebra to 2D and 3D geometry in order to give an image of some concepts (linear (in)dependence in particular) have showed their limits (Hillel, 2000; Harel, 1990 & 1998 for instance). What a challenge... Is it really sensible?

A CASE STUDY: DISPLACEMENTS ON A REGULAR SQUARED GRID (\mathbb{Z}^2)

A situation in discrete mathematics

Let G be a discrete regular grid. This grid can be squared or triangulated for instance. For the rest of this article, G is a squared regular grid. A “point” of the grid is a point at the intersection of the lines. Let A be a starting point. An elementary displacement is a vector with 4 positive integer coordinates (it can be described with the directions: up, down, left and right, for instance “2 squares right and 3 squares down”). A displacement is a positive integer combination of k elementary displacements, written $a_1d_1 + a_2d_2 + \dots + a_kd_k$ (a_i are natural numbers, $1 \leq i \leq k$).

The general problem is: let E be a set of k vectors with integer coordinates. Starting from a given point, which points of the grid can one reach using positive integer combinations of vectors of E ?

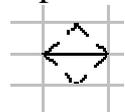
In vector space, the notions of generator and dependence are highly correlated. In a discrete situation, the lack of definitions of these notions may allow an activity of definition-construction. The situation above is decontextualized with regard to classical introduction of concepts in linear algebra. It is an open problem, which the students do not know. The concepts of generator, minimality but also (in)dependence and basis can be studied. I stress the fact that the linear algebra is not the model for the situation of displacements. Linear algebra brings well-known obstacles, in particular with its definitions and a unifying formalism. So this explains the necessity of a “decontextualization” in order to give an access to the mathematical problematic. This decontextualization in discrete mathematics allows a work on properties which are co-dependant in the continuous case.

As seen in the mathematical study below, the situation suggests an activity on the definition of “different” paths, but also the definition of generator, minimality, density and “a little bit everywhere”. The students were induced to define besides being challenged to discover an answer to the “natural” questions: How can we reach each point of the regular grid? What does it mean? Does a minimal set of displacements exist in order to go everywhere? Furthermore, I assume that the notion of generator should come naturally and will lead students to the notions of (in)dependence and minimal generator (basis).

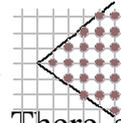
The mathematical study in brief

1) *How to reach all the points of the grid?*

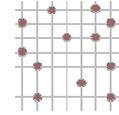
There exists a set of displacements which allows all the points of the grid to be reached. The four elementary displacements represented here obviously form one such set. Now, can we characterize all the sets of displacements which allow us to reach all the points of the grid? We have to work on two different properties simultaneously:



- the “density”: all the points of a zone of the grid are reached.



- and “a little bit everywhere”: let P be a point of the grid. There exists a reached point, called A , “close to P ”, i.e. such that the distance between A and P is bounded (for every P , independently of P). We will call this property “ALBE”.



We can reach all the points of the grid when these two properties (“density” and “ALBE”) are satisfied simultaneously. These properties imply the definition of “generator set”.

2) *Reciprocal problem and minimality*

Let E be a set of elementary displacements. What points can one reach with E ? When the set of reached points is characterized, a new question emerges: is it possible to remove an elementary displacement of E without changing the reached points? This is a question about the minimality of the E set. E is called minimal when removing one of its elementary displacements modifies the set of reached points. With this definition, how do we characterize a minimal set and a generator set of displacements? Furthermore, are the minimal and generator sets of displacements minimum too, i.e. do they have the same cardinality?

3) *Paths and different paths*

Let E be a set of k elementary displacements written as d_1, d_2, \dots, d_k . What can we say about the paths from the fixed point A to the reached point B ? A path from A to B is an integer combination of elementary displacements of E . A path can be described by a k -tuple (a_1, a_2, \dots, a_k) where a_i , for $1 \leq i \leq k$, are the integer coefficients of this combination.

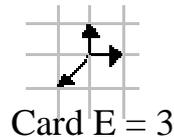
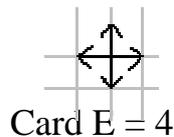
Two paths from A to B are called different if and only if the k -tuples characterizing them are different. Note that the order of the elementary displacements does not interfere because of the commutativity of displacements. Then, we can form a question on the relationship between the number of the paths from A to B and the minimality of E : when there are (at most) two different paths, is it possible to remove an elementary displacement in E ? The answer is ‘No’: the study of that is a difficult one, even if we limit the study to \mathbb{N} . Here is a counter-example on the discrete line. Let E be a “displacement” composed by 2 squares to the right and 3 to the right, i.e. E is composed by the natural numbers 2 and 3, and we look at the numbers which can be generated by 2 and 3. With the displacements of E , we can reach 11 in two different ways: either with $4 \times 2 + 1 \times 3$, or with $1 \times 2 + 3 \times 3$. But we cannot remove 2 or 3 from E otherwise 11 will not be reached. Then, E is generator and minimal for 11. It can lead us to the famous Frobenius problem (Ramirez-Alphonsin, 2002).

We notice that the existence of several paths does not necessarily imply the non-minimality of E . Then we have to consider three kinds of E sets. 1) There is no uniqueness of the path for one point at least i.e. there exists at least one point which

can be reached with at least two different ways. This does not imply that E is non-minimal. 2) Every point of the grid can be reached in at least two different ways. We call this property “redundant everywhere”. Thus, the E set is non-minimal: this is the case when an elementary displacement of E is an integer combination of other elements of E . 3) Every point of the grid can be reached in only one way (uniqueness of the path): we call this property “redundant nowhere”. The E set is clearly minimal.

4) Discussion on the minimal generator sets of \mathbb{Z} and their cardinalities

The minimal generator sets can have different cardinalities. For example, you can see below a minimal generator set with 4 elements and another one with 3 elements: with both of them you can go everywhere on the grid, that is to say “ALBE” and with “density”.



We can succinctly study this specificity of the discrete case with the integers.

In order to build a set of minimal generator elementary displacements on \mathbb{Z} , we have to use coprime numbers (i.e. gcd of them is equal to 1). Thus, the “density” property is true for natural integers (Bezout’s theorem). Some of these coprime numbers should be negative in order to go “a little bit everywhere” (a little bit to the right and a little bit to the left). For example, if we want to generate \mathbb{Z} with 4 integers, we build 4 natural numbers which are coprime as a whole (for instance $2 \times 3 \times 7$, $3 \times 5 \times 7$, $2 \times 3 \times 5$, $2 \times 5 \times 7$ i.e. 42, 105, 30 and 70). Then we can reach 1 (according to Bezout’s theorem) that is to say we can go with density on \mathbb{N} . Now if we take one of these numbers as a negative one, we can go “a little bit everywhere” and we get: $E = \{42; 105; -30; 70\}$ is a generator of \mathbb{Z} . So we can build several sets of minimal generator displacements with different cardinalities. Another example: $E = \{1; -1\}$ and $F = \{2; 3; -6\}$ are generator and minimal, $\text{card}(E)$ is 2 and $\text{card}(F)$ is 3.

Then, we have the following theorem:

Theorem: there exists, in \mathbb{Z} , sets of minimal generator elementary displacements with k elements, k being as big as one wants.

Therefore, the cardinality of sets of minimal generator elementary displacements of \mathbb{Z} is not an invariant feature. However, the study of the generation of integers has showed that this problem is mathematically closed for \mathbb{Z} . The reader can consult the wider and more complex NP-hard Frobenius Problem (Ramirez-Alphonsin, 2002).

We will show that the problem is not mathematically closed in \mathbb{Z}^2 , by proving that we can build minimal generator sets with as many elementary displacements as we want.

5) Construction of sets of minimal generator elementary displacements, in \mathbb{Z}^2 , with k elements

We call E_k the set of all generator displacements with k elementary displacements. We want to generate all the points of the regular grid. A starting point is given. The study of the “generator” and “minimal” properties on a discrete grid is more complex than on \mathbb{Z} : that is the reason why the study of the first cases (homework for the reader) E_k , $k = 2, \dots, 5$, is necessary. It leads us to a theorem of existence.

Theorem: there exist, in \mathbb{Z}^2 , sets of minimal generator elementary displacements with k elements, k being as big as one wants.

Indications for the proof: one constructs a set of horizontal minimal generator elementary displacements with $(k-2)$ elements in order to generate \mathbb{Z} and then add two vertical elementary displacements in order to go everywhere by translation.

But, k being given (as big as one wants), we do not know how to construct all the sets with k minimal generator elementary displacements. The next crucial question is: how to prove that a set of elementary displacements is generator or minimal?

CONCLUSION: PRESENTATION OF SOME EXPERIMENTAL RESULTS

I will present a complete analysis of students’ procedures during the Conference, exploring the concept formation and the perspectives that the situation of displacements offers to other fields of mathematics. But let me briefly outline some experimental results coming from an experiment with freshmen audiotaped recorded.

The situation of displacements allows a work on mathematical objects (displacements, paths) graspable through a basic representation close to that of vectors. The main difficulty lies in the fact that properties have to be defined (generator, independence, redundancy, minimality). Indeed, the objects we work with do not need to be explicitly defined at first: we have to focus on properties, to characterize and to define them. These specificities of the situation of displacements partially explain why the students did not engage in characterizing mathematical properties. Indeed, only some zero-definitions were produced but they did not evolve into operational definitions. Nevertheless, a “natural” definition of “generator” (i.e. “to reach all the points of the grid”) has been produced and has been transformed into an operational property (“to generate four points or elementary displacements”). Furthermore, I have identified two definitions-in-action: one for “generator minimal” and one for “minimal set”. The presence of definitions-in-actions proves that students can not stand back from the manipulated objects: students stayed in the action, in the proposed configurations. Their process did not move to a generalization which would have allowed a mathematical evolution of zero-definitions or definitions-in-action. A plausible hypothesis is that this distance (between manipulation and formalization, formalization merely a first step, not a complete theorization) is too rarely approached in the teaching process. It goes along the lines of previous

epistemological and didactical results which conclude that formalism is a crucial obstacle in the teaching of linear algebra.

The didactical analysis of the productions of the students is very difficult. In fact, the dialectic involving definition construction and concept formation is useful to understand the students' procedures and their ability to define new concepts in order to solve a problem. To understand how concept formation works implies exploring the wide field of mathematical definitions considered as concepts holders. That will be discussed during the Conference.

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PROBLEM POSING BY NOVICE AND EXPERTS: COMPARISON BETWEEN STUDENTS AND TEACHERS

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Lately, problem posing gained terrain in mathematical education research due to its connection with mathematical understanding and thinking. Still, comparisons between novice' and experts' problem posing are still scarce. In this paper we compare students' and teachers' generated problems on three aspects: variety of problem types and of tasks, and quality of questions. We found that teachers use their pedagogical knowledge to constrain problem types and tasks, and that teachers' classroom experience shapes their view on difficulty. In conclusion, teachers are always guided by the audience they have in their mind in contrast with what can be observed at students.

INTRODUCTION

Research on problem posing can be structured along several lines. First, there is a research trend on relating problem posing to instruction: by which means a problem posing approach can be beneficial in the classroom. Studies that can be subscribed to this category look at the relation between problem posing and problem solving (in case of pre-service teachers – Crespo, 2003; in-service teachers – Chang, 2007; both – Silver et al., 1996; students – Imaoka, 2001), international comparisons (Cai, 1997) or problem posing and mathematical understanding, modelling and open ended problems (Lin, 2004; Pirie, 2002). Another line of research focuses on enhancing problem posing skills: in traditional (Yevdokimov, 2005) or by development of computational settings (de Corte *et al.*, 2002). There are also a series of studies that relate problem posing to individual attitudes towards mathematics and affect (Akay & Boz, 2008). A fourth line of research connects problem posing to creativity and evaluates the posing process and results from creativity point of view (Silver, 1997). However, comparisons between novices (from some particular point of view) and experts are scarce and there is no commonly agreed framework that would allow this.

One explanation to such a situation is the fact that mathematical problems need a rich characterization of them. However, such an inquiry leads to questions like: when a situation turns into a problem, what makes it to belong to a particular topic, which of the problems elements (like given, asked for) should be considered and which meta-characteristics are important (like solvability, cognitive resources involved in

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solution, etc.). In conclusion, researchers need to take into account the particular topic, beside general aspects, in order to define their evaluation criteria.

In the present paper we intent to contribute on this line by proposing a framework for the evaluation of problems and apply it to compare problems posed by university students (pre-service teachers, considered as novice from the point of view of classroom teaching) and in-service teachers (considered as experts). The categorization into novice and expert is done on terms of pedagogical, mathematical knowledge and classroom teaching experience.

METHODOLOGY

In the present study, 88 persons from Romania (25 first year or second year mathematics students, 41 middle school teachers, and 22 high school teachers) completed a problem posing task. Students were of 18-20 years old and entered to university after completing an admission exam. Teachers had more that 5 years teaching experience. Participants were selected randomly, without any reference to their professional or scientific performance. None of them has been subject of training in problem posing, however it is possible that some of them would have experience in Olympiads as students or teachers.

The participants had to generate three sequence problems (as home assignment task) such that to have an easy, one of average difficulty and a difficult problem. They had a week at their disposal to finish; at the end, they responded a questionnaire regarding their problem posing process. It was requested to hand in not only the final formulations, but also the scratch work. The questions were about the following aspects of the problem posing process: the existence of an initial idea (for each problem of different difficulty), change of the idea during generation, problem types from which to start the generation process, a theorem or generalization as from where to trigger the problem posing process and difficulty criteria they used.

ANALYSIS OF THE POSED PROBLEMS

It has to be mentioned, before the presentation of results, that we found two situations along with the expected one: first, not all participants posed problems for each difficulty level and, second, some of them, posed more than one problem for a specific difficulty level. The problems were analyzed from three perspectives: variety of problem types and of questions, and problem formulation .

Problem - type analysis

The problem typology for sequences was taken from Pelczer and Gamboa (2006). Theoretical problems are the ones in which there is no quantitatively specified sequence, but rather a generic sequence is specified as the mathematical object under inquiry. The term “contextual” was employed as in Borasi (1986), meaning the situation into which the problem is embedded. The rest of categories refer to the way in which the general term is specified. Table 1 contains the results concerning

problem types, in percent (E – easy, A – average, D - difficult). The total number of problems appears in the last line of this table.

Table 1. Statistical results on problem types. For each problem type we specify, in parenthesis, as a triplet the number of problems posed by students, secondary and high school teachers.

Problem types	Students			Secondary			High school		
	E	A	D	E	A	D	E	A	D
<i>Theoretical</i>	-	-	-	-	-	-	-	-	-
<i>Contextual (8,-,-)</i>	12	12	10	-	-	-	-	-	-
<i>Explicit (13,42,40)</i>	28	4	5	41	38	27	73	67	43
<i>Implicit (15,6,1)</i>	12	36	14	5	2	8	4	-	-
<i>Linear Recurrence (27,4,5)</i>	44	36	33	-	5	5	-	16	10
<i>Non-linear Recurrence (8,3,3)</i>	4	12	18	-	2	5	-	-	14
<i>Enumeration (2,37,5)</i>	-	-	10	40	30	25	19	4	-
<i>Sum, Product (2,26,11)</i>	-	-	10	14	23	30	4	13	33
Total nr. of problems	25	25	21	41	40	36	22	22	21

We can observe from table 1 that at students recurrence problems dominate; at high school teachers prevails the problem in which the general term is expressed explicitly by a formula and at secondary teachers the “enumeration” type (sequence specified by the enumeration of few initial terms) is the most frequent. The dominance of enumeration type at secondary teachers can be explained by the curricula: the accent is on identifying and formalizing the sequence’s patterns and moving between different representations of the sequences (geometric, analytic, formal and recurrence).

The observation holds for high school teachers, too, with the remark that in their case there is an increase also in non-linear recurrence problems. In case of high school teachers, the dominance is one of the explicit problems – situation which, again, can be explained by the curricula. High school teachers concentrate on clarifying basic calculus concepts, like limit, convergence, monotony and for all these explicit problems are proper. As the difficulty of the problem has to increase, they move towards the types “sum” and “non-linear recurrence”. These problems, when analyzed, showed that teachers still focused on theorems and criteria present in textbooks (just as in case of easy problems with explicit general term), but asking for skillful application of them. By “skillful application” we mean that no advanced techniques are needed, but rather good knowledge of algebra (identities, inequalities) or typical examples and sequences (like in case of applying the majoring criteria).

This later is the main aspect that differentiate students' and teachers' problems. As it can be seen in the above table, students prefer implicit or recurrent definitions of the sequences. It is also interesting that many students pose "contextual problems", that is problems in which sequences appear as a collateral issue: the main focus is on another mathematical object so that the problem can't be seen as strictly relating to introductory analysis.

These results suggest that students see problem posing as a self-referenced activity focused on problems and with no specific audience. Problem difficulty is judged based on the ability to solve the problem and use of techniques, meanwhile teachers build their problems with their students in their mind. When speaking about the problem posing process they mention that the addressee is their classroom and difficulty is judged based on curricular indications and classroom experience. The case of the (posed) difficult problems is interesting: where students ask for specific transformations (usually beyond the textbook's reach) or use non-familiar contexts, teachers concentrate on situations about which they know that the application of the usual theorems can be problematic. Therefore, they prefer problem types (like non-linear recurrence or explicit) that can be solved with text-book theorems and the difficulty relies in identifying the instances that satisfy the conditions of application. In these terms, teachers problem posing can be seen as a constraint based process, where constraints arise from their classroom experience.

Questions' analysis

Some interesting conclusions about the posing process were reached by the analysis of the task specified by the problem, that is, by the analysis of the problems' questions. We defined four principal categories. In the first category we included questions related to the *verification of the concepts*, that is the question refers to the statement of some definitions or theoretical results, recognition of some property, construction of examples or counter-examples. In the second category are the *demonstration* tasks, those that ask for justification (through mathematical reasoning) of some facts of algebraic or analytic nature. In these cases, the problem statement is imperative and the facts to be demonstrated are explicitly stated. A third category contains *exploration* tasks. These can ask for the verification, study or observation of a property, identification of a sequence's pattern given by some terms and/or generation of following terms, discussions of the results on the value of parameters or different representations of a mathematical object. The questions from this category are characterized by doubt, meaning that *a priori* one can obtain several answers. The last category of questions – of *computations* – include tasks that ask for the application of some formula (in case when the expression of the general term is given), computation of the general term, of a limit, sum, or the determination of a parameter's value such to have some conditions satisfied.

In table 2 the statistical results are shown (in percentage for the questions types), for the four category of questions (tasks) and the three category of participants. The total

number of problems and questions appears at the end of the table and a ratio of question/problem is computed.

Table 2. Statistical data on questions

	Students			Secondary			High school		
	E	A	D	E	A	D	E	A	D
<i>Verification</i>	-	3	3	-	-	-	-	-	-
<i>Proof</i>	29	26	23	22	22	27	3	11	24
<i>Exploration</i>	10	20	23	35	26	20	48	22	30
<i>Computation</i>	61	51	52	43	53	53	48	67	47
#Questions	31	35	31	63	58	59	29	27	34
#Problems	24	25	21	42	41	37	20	20	22
Ratio	1.29	1.4	1.48	1.5	1.41	1.6	1.45	1.35	1.55

The data from table 2 leads to some interesting conclusions. A first one is that none of the participant categories seems to be interested in problems that aim the verification of concept understanding. There are only two problems asking for construction of examples, but these are in a special context in which very complex properties are required. A possible explanation of such situation can be the fact that these types of questions are not very common in textbooks, evaluation exams, although probably they are quite common in everyday class activities. Still, teachers and students do not seem to give them importance as stand-alone problems.

A second, surprising, conclusion is that high school teachers seem to be less interested in demonstrations and exploration in favour of computation, when compared with the other two participant category. More, high school teachers, tend to put problems of demonstration type more as difficult ones (24% in difficult against 3% of easy problems). In the meantime, the distribution of demonstration type questions is more equilibrated in case of students and secondary school teachers. Such results can be related to the tendency toward an algorithmic training, as preparation for end school exams, observed in the Romanian education lately (Pelczer *et al.*, 2008).

We also identified a certain disposition of teachers (independently of the school level that teach) for questions that refer to passing sequences from one representation into another, aspects lacking from students' problems. This suggest that teachers know and pay attention to the importance of multiple representations of a concept; passing a sequence between different representational forms has a high pedagogical value. It is interesting that teachers consider exploration as proper, mostly, for easy problems.

As far the ratio between questions and problems is concerned, we see a small tendency of teachers to pose more questions than students. The tendency is even more

visible when we count all the questions (even those that are of the same type). Such situation is explained by the fact that teachers generate problems with an audience in their mind (their own class), an audience that is made up of problem solvers; therefore, their tendency for multiple questions reflects their way of acting in the class. We even found problems with more than 5 questions for it. In conclusion, we see that teachers create, through the posed problem, a context for learning in which, on the same problem statement multiple skills can be practiced.

Problem formulation

The first aspect refers to the adequacy of the question with the context of the problem and the difficulty level. In any context there are several questions that can be asked; the context with the question gives a particular instance. By considering that we are interested in classroom problem posing, we study these instances from the point of view of their pedagogical value (Baker, 1991). This attribution is subjective, based on the experience of the authors of the present article. Adequacy with the difficulty level refers to the correspondence between the attributed difficulty and the elements of the problem. In particular, it means to analyze the selection of the question (from a possible set of questions that can be formulated in that context) and whether there were better alternatives. Then, problems are analyzed from the point of view of well-formulatedness: are all the elements necessary for solution mentioned in the problem? The last aspect refers to the solvability of the problem: can the problem be solved under the given specifications?

As pedagogical value of the problems is concerned it can be told that there are some common goals between the three categories of participants, for example, the verification/ application of concepts of monotony, boundedness or convergence. However, there are two interesting results. First, no student posed a problem that would require the identification of the sequence's pattern nor asked for exploration of different situations. Second, students tend to pose problems (especially, when it comes to difficult ones) that require the application of algorithms or techniques that are not in the textbook. This tendency is explained by their vision of difficult problem: one that is out of their own (or most students) reach. However, it is important to underline that such a perception goes beyond of difficulty appreciation; it reflects, partially, their view of a well-prepared student: one that has an extensive knowledge of algorithms and techniques.

It has to be remarked that neither teachers pose problems that aim to check whether there is a deep understanding of the concepts involved with sequences. Above, we already described a possible explanation for this situation. Still, teachers tend to ask for exploration and their problems can be solved just by methods shown in the textbook. This aspect turns us back to the difficulty issue: students make more difficult problems by involving techniques that are beyond the textbook or by transforming the context of the problems, meanwhile teachers involve algebraic knowledge in the expression of the problem such to remain strictly related to the

topic. With regard to difficulty, students also have problems in finding the proper question in a context – the question that would turn a problem in a difficult one. Teachers' problems are more typical, the questions that could be asked in a specific situation (and the mathematical object on which focuses the question) are the standard ones, so they choose from a more restricted set of questions and are more familiar with the setting. Students, meanwhile, often create richer settings, but do not necessarily know how to choose a good question.

In other situations, students do not formulate properly the question. We give two examples from students.

Example 1. Let $(a_n)_n$ be a sequence given by $a_1 = 1$, $a_2 = 1$, and $a_{n+1} = \sin(a_n) + \cos(a_{n-1})$. Study if this sequence has a finite limit.

Example 2. Let $(a_n)_n$ be the sequence defined by $a_1 = 12$, $a_2 = 288$, and $a_{n+1} = 24a_n - 144a_{n-1}$, $n \geq 2$. Calculate $b_n = \sum_{k=1}^n a_k$ and examine the monotony and the convergence of the sequence $(b_n)_n$.

In the first example (Example 1, given as difficult problem), the student's question (the "finite" word) suggest that he had not paid enough attention to the expression of the general term: the limit, if it exists, obviously it can't be infinite. In the second example (given also as difficult problem), the second question refers to the monotony and convergence of a sequence defined from the previous one. Once the general term a_n is determined, it is "obvious" the monotony and the divergence of the second sequence (its general terms is positive and major to 1).

Our main conclusion to this first part of the analysis is that teachers' problems are typical ones that require only textbook material for solving and have specific pedagogical goals; their approach is shaped by their classroom and teaching experience: they pose problems having a specific audience in their mind (their own classroom) and think of curriculum as the main guide for the type of knowledge that must be used.

By well-formulated problem we mean a problem in which all the elements necessary for solution are given and there is no contradiction between the given elements. Textbooks, problem books always contain well-formulated problems, a situation which at its turn can lead to the case that students don't know what it is and how they could check a problem from the point of view of formulation. Exactly this situation make well-formulatedness an important factor in the evaluation of the problem posing results.

Solvability, another characteristic, refers to the possibility of finding a solution for the problem with a certain set of knowledge. As in the case of well-formulatedness, students experience in classroom is limited to solvable problems, which gives them a bias when it comes to evaluate the posed problem: often this aspect will not be

considered. However, it is true that students frequently do not know to decide whether a problem is not solvable or is just that they can't solve it. Still, in the problem posing context it is natural to expect to pose problems that are solvable, even if not by the author of the problem. It also needs to be underlined that well-formulatedness affects the solvability of the problem, therefore there will be always less solvable problems than well-formulated ones.

In the analysis we carried out there were no cases of ill-formulated or non-solvable problems at teachers. However, at students this appears in few cases. Ill-formulated problems can be grouped as problems that have not enough elements in their statement (like “under formulated”) and ones that have contradictory information in their statement (in some cases, over-formulated). We consider two relevant examples.

Example 3. Consider the following recurrence formula: $a_{n+1} = 2a_n - a_{n-1}$. Calculate the general term a_n .

Example 4. If $(a_n)_n$ a sequence such that $\frac{a_n}{a_{n-1}} > 1$ and $\frac{a_n}{a_{n+1}} > 1$, decide if it is convergent.

In example 3 we illustrate the case of under-specification: without specifying the first terms, the general term can't be computed. Example 4 shows a case of contradictory information, that makes that the problem has no sense under the current specification.

Why do teachers create well-defined and solvable problems? We argue that these problems can serve to reach the pedagogical goals they envision, and that they have the mathematical knowledge and teaching experience that allow them to verify their posed problems (or, from the beginning, to restrict themselves to problems that are “worthy” to be done). Whether teacher's choice for well-defined problems is result of the use of textbooks and exams practices or, rather, it is a conscious decision remains a question on which we shall not delve in this paper. On the other hand, students often are not aware of this aspect or are not considering it when reviewing their own problems – a fact that can be (partly) explained by the fact that since they had no particular receiver in their mind during the generation they didn't “looked” at the problem from the solvers' point of view.

As overall conclusion, we can say that differences between teachers' and students' generated problems can be identified at every level (problem types; questions types; meta-characteristics of the problems – well-formulatedness, solvability and adequacy) and the differences can be explained by teacher's classroom and pedagogical experience, on one hand, and mathematical knowledge, on other hand.

CONCLUSIONS

The analysis of the posed problems leads to the conclusion that there is a specific trait for each participant group. This can be underlined by different ways.

In the first place, teachers (secondary and high school) seem to be strongly influenced in the choosing of the problem type and question formulation by the curriculum and the subject usually given at final exams (mostly national scale examinations). High school teachers seem to concentrate on the development of computing abilities, meanwhile secondary teachers pay equal attention to demonstrations, exploration and calculations. Students seem to be interested in extra-curricular contexts and solution techniques. We explain this situation by the fact that teachers have permanently an audience in their mind at the moment of generation and they employ their pedagogical and mathematical knowledge such to adapt the problems to an envisioned concrete classroom situation (known from their classroom experience).

The explanation is congruent with the next conclusion, too. Teachers seem to be guided by diverse pedagogical goals and take into consideration their class when adapting the difficulty level. On contrary, students see problem posing as a self-referenced activity focused on the problems with no specific audience. There are two further arguments in this line. On one hand, a teacher starts, in general, from a specific idea of problem generation and formulates (in average) more tasks (or questions). On other hand, teachers pay much more attention to the formulation of the problem, in comparison with students: many of students' generated problems have an unclear statement or the proposed solutions are erroneous which very rarely occurs at teachers.

The analysis we carried out has several benefits. First, sheds light on what students and teachers do perceive as important in teaching, evaluating and knowing about sequences. Second, the analyses proves interesting for pre-service teacher education. Some time after beginning their careers as teachers, these students will start to choose or pose the problems with a focus on their audience, but maybe it would be beneficial to explicitly train them, before getting into the classroom, to think on meta-characteristics of the problems and to identify and use techniques that help building them. We consider that our conclusions are in favour of using a problem posing approach or training in pre-service teacher education.

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ADVANCED MATHEMATICAL KNOWLEDGE: HOW IS IT USED IN TEACHING?

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For the purpose of the study reported here we define Advanced Mathematical Knowledge (AMK) as knowledge of the subject matter acquired during undergraduate studies at colleges or universities. We examine the responses of secondary school teachers about the ways in which they implement their AMK in teaching. We find an apparent confusion between what teachers perceive as difficult or challenging for their students and what is ‘advanced’ according to our working definition. We conclude with a call for a more articulated relationship between AMK and mathematical knowledge for teaching.

Research reported here is the beginning of our journey aimed at identifying explicit relationships between school mathematics and university mathematics, as perceived by secondary school teachers. We first describe the relationship (or lack thereof) between teachers’ knowledge of mathematics and the achievements of their students, which led researchers to posit a need for ‘specialized’ mathematical knowledge for teaching. Then we describe different kinds of teachers’ knowledge and provide a working definition of *advanced mathematical knowledge* (AMK) and its relation to advanced mathematical thinking (AMT). Acknowledging the existing gap between secondary and undergraduate mathematics we illustrate suggestions for reducing this gap. We then describe the views of several secondary school mathematics teachers about their usage of AMK in their teacher practice.

SUBJECT MATTER KNOWLEDGE AND TEACHING

While teaching is unimaginable without teachers knowing the subject matter, it is unclear how “knowledge *for* teaching” can be measured. The most used measure, the number of mathematics courses taken by a teacher, did not lead to conclusive results. Begle (1979) found that students’ mathematical performance was not related neither to the number of university courses their teachers had taken, nor to teachers’ achievement in these courses. However, Monk (1994) found a minor increase in secondary students’ achievement associated with the number of college courses in mathematics taken by mathematics teachers. Further, “researchers at the National Centre for research on teacher education found that teachers who majored in the subject they were teaching often were not more able than other teachers to explain fundamental concepts in their discipline” (NCRTE, 1991, quoted in CBMS, 2001, p. 121).

More recent studies recognized the inherent complexities with these kind of results, mainly that the degree held and number of courses taken by a teacher are not appropriate measures of mathematical knowledge. Following a comprehensive literature review, Hill, Rowan and Ball (2005) concluded that measuring teacher's mathematical knowledge more directly – by looking at scores on certification exams or exam items related to a specific topic – generally revealed a positive effect of teachers' knowledge on their students' achievement.

Struggling with the question of what kind(s) of teachers' knowledge benefit teaching and learning, researchers realized that mathematics knowledge *for* teaching (Ball, Hill & Bass, 2005) may be a special 'register' of knowledge, a special combination of content and pedagogy, that relies on deep understanding of the subject and awareness of obstacles to learning. This specialized knowledge has received some attention at the elementary level (e.g., Ma, 1999), and it has been shown that such specialized knowledge for teaching was significantly related to students' achievement at elementary grades (Hill, Rowan & Ball, 2005). However, the issue has yet to be explored in detail at the secondary level. We believe that achieving this specialized knowledge for teaching at the secondary level is impossible without sufficient exposure to advanced mathematical content.

TEACHERS' KNOWLEDGE

Epistemological analysis of teachers' knowledge reveals significant complexities in its structure (e.g., Scheffler, 1965; Shulman, 1986; Wilson, Shulman, & Richert, 1987). Addressing these complexities and combining different approaches to the classification of knowledge, Leikin (2006) identified three dimensions of teachers' knowledge, as follows:

Kinds of teachers' knowledge: based on Shulman's (1986) classification where *subject-matter knowledge* comprises teachers' knowledge of mathematics, *pedagogical content knowledge* includes knowledge of how students approach mathematical tasks, as well as knowledge of learning setting; and *curricular content knowledge* includes knowledge of types of curricula and knowledge of different approaches to teaching mathematics.

Sources of teachers' knowledge: based on Kennedy's (2002) distinction, *systematic knowledge* is acquired mainly through studies of mathematics and pedagogy in colleges and universities, *craft knowledge* is largely developed through classroom experiences, whereas *prescriptive knowledge* is acquired through institutional policies.

Forms of knowledge: based on Atkinson and Claxton (2000) and Fischbein (1984) distinction, *intuitive knowledge* determines teacher actions that cannot be premeditated, and *formal knowledge* is mostly connected to planned teachers' actions.

In these terms, we investigate connections between teachers' systematic formal subject matter knowledge, within and beyond the secondary curriculum, and its possible transformation into their pedagogical content knowledge or mathematical knowledge for teaching.

ADVANCED MATHEMATICAL KNOWLEDGE

We study teachers' advanced mathematical knowledge (AMK) rather than advanced mathematical thinking (AMT). We define AMK as *systematic formal mathematical knowledge beyond secondary mathematics curriculum, likely acquired during undergraduate studies*. We acknowledge that existence of different curricula makes this definition time and place dependent, however, sufficient similarities among the curricula make it useful for our pursuits.

Coordinators of the WG-12 at CERME-6 suggested two interrelated perspectives on AMT: According to *mathematically-centred perspective* AM-T is related to mathematical content and concepts approached at the upper secondary and tertiary levels. According to *thinking-centred relativistic perspective* A-MT is addressed through focusing on students with high intellectual potential in mathematics.

This study is performed within the context of mathematically-centred perspective on AMT. The notion of AMT is receiving continuous attention in mathematics education. The seminal volume *Advanced Mathematical Thinking* edited by David Tall (1991) was a landmark that positioned AMT as an area of research in mathematics education. It also intensified conversations on what constitutes AMT, and how it can be identified and supported. Tall (1991) characterised AMT as a transition "from describing to defining, from convincing to proving in a logical manner based on definitions" (p. 20). Tall also suggested that advanced mathematical thinking must begin in early elementary school and should not be postponed until postsecondary studies.

The difference in perspective on what constitutes AMT shifted the focus, or at least the description of the research area, from AMT to tertiary mathematics (Selden & Selden, 2005). As such, our definition of advanced mathematical knowledge (AMK) accords with this shift: AMK is knowledge related to topics in tertiary mathematics.

There are significant gaps between secondary school mathematics and tertiary mathematics. The discontinuity of experience appears not only at the level of presentation of mathematical content and lack of readiness for challenges but also in unresponsive styles of teaching and assessment (Goulding, Hatch & Rodd, 2003). These gaps have two significant outcomes relevant to mathematics education: (1) students, even those identified in school as high-achieving students, experience unexpected difficulties in entry-level undergraduate mathematics courses, and (2) many teachers perceive their undergraduate studies of mathematics as having little relevance to their teaching practice. The latter issue is of our interest in this paper.

Our goal is to examine teachers' ideas of how AMK is implemented, both actually and potentially, in teaching secondary mathematics.

PROCEDURE

The study included two stages.

At the first stage we interviewed several secondary school teachers. During the interviews the teachers were asked to reflect on their teaching and to share experiences in which they used their advanced mathematical knowledge. Following the difficulty our interviewees had responding on the spot, and because of the vagueness of some responses, we designed and implemented a formal written questionnaire that attempted to elicit specific and detailed responses.

At the second stage 18 in-service mathematics teachers were asked to complete the written questionnaire. It included the following questions:

1. To what extent are you using AMK in your school teaching?
2. Provide 3 examples of mathematical topics from the curriculum in which, in your opinion, AMK is essential for teachers. In each topic specify the usage of AMK.
3. Provide 3 examples from your personal experience of a teaching situation (such as classroom interaction, preparing a lesson, checking students' work, etc.) in which you used AMK. Provide detailed description of each case.
4. Provide 3 examples of mathematical problems or tasks from the school curriculum in which AMK is necessary or useful for a teacher. In each case describe the usage of AMK.

The time for completing the questionnaire was not limited and the teachers could consult any resources they found appropriate. The questions were preceded with a definition of AMK, consistent with our above working definition:

In this questionnaire we refer to knowledge acquired in Mathematics courses taken as part of a degree from a university or college as "Advanced Mathematical Knowledge"

In the context of CERME WG12 – Advanced mathematical thinking – we report on the results from secondary-school mathematics teachers only (n=6).

RESULTS

Most participants in our study, in responding to Question #1, acknowledged the importance of AMK in secondary teaching. They indicated that they are or have been using AMK in preparation for teaching, in supporting students' solutions and in generating pedagogical examples. However, exemplifying such usage with detailed descriptions proved to be more challenging.

In responding to Question #2, most topics that participants mentioned related to Calculus. Teachers mentioned definition and usage of derivative, limits, and asymptotes. These topics further featured in teachers' examples provided in response

to questions #3 and #4. This is hardly surprising, as the topics of Calculus are the last ones taught in high schools for a selected population of students and are usually the first ones encountered in undergraduate studies of mathematics. Of note is a response of one participant, Gal, who acknowledged his explicit attempt to avoid Calculus related topics, as those examples were in his opinion “obvious, taken for granted”. His three examples of topics included geometrical representation of equations and inequalities, normal distribution and linear programming. We appreciate his attempt to avoid the ‘obvious’, but we also note that his first example is not really ‘advanced’, and the other two examples mentioned topics that were introduced to the Israeli curriculum relatively recently. Though Gal was exposed to these topics at the university, they would not be considered ‘advanced’, according to our definition, to a recent high school graduate.

In teachers’ oral responses, and on written responses to Question #3 and #4 we identified the following themes (1) connection to the history of mathematics, (2) meta-mathematical issues, (by “meta-mathematical” we mean cross-subject themes, such as definition, proof, example, counterexample, language, elegance of a solution, etc.) and (3) mathematical content. Within issues related to mathematical content we further differentiated between responses that identified mathematical tasks or situations clearly related to AMK, responses that mentioned ‘extra-curricular’ tasks with solutions requiring AMK, and descriptions of complicated tasks or problems with solutions based on the mathematical content from the school curriculum, rather than AMK.

In what follows we exemplify each theme with illustrative examples.

Connection to history

Tanya noted that she learned in a university that logarithms were invented independently from the exponential function. As such, while the local curriculum introduces logarithms as the “inverse” of exponential notation, she introduces logarithms consistent with their historical development, building a relation between geometric and arithmetic sequences.

Greg noted that he learned in a university course about the Pythagoreans and their decision to keep secret their discovery of irrational numbers. He often uses this story to motivate students when he teaches the topic of irrational numbers.

We note that though both experiences exemplify pedagogical content knowledge and describe valuable teaching situations, they do not really rely on advanced mathematical content.

Meta-mathematical issues

Proof: Paul noted in his interview that he finally understood the meaning of mathematical proof after failing a first course in analysis. He claimed this made a

profound impact on how he teaches ‘proof,’ but he was not able to articulate this claim with any examples.

Language: Nadia stated that undergraduate mathematics made her very sensitive to mathematical language, and this influences her teaching in not allowing students to use sloppy expressions. As an example, she shared a recent exchange in which a student said, “these angles make 180” and she asked him to rephrase, aiming for an expression like “the sum of the measures of these angles is 180 degrees”.

Precision and Aesthetics: Donna wrote: “The importance of mathematical discourse connected in my mind to my studies in the university. I pay attention to the preciseness of mathematical language used in my classroom and explain to my students differences in the precise and imprecise mathematical formulations. I also am aware of the aesthetics that exists in mathematics and try to bring to my classroom examples of beautiful solutions and encourage students finding beautiful solutions”.

Many responses focused on meta-mathematical content and referred to appreciations of meaning or of elegance, understanding versus procedural fluency. This tendency identifies a clear connection between AMT and AMK.

Mathematical content

Examples related to school curriculum and AMK

In her interview Rachel described that when working with low achieving students on solving a system of two linear equations, she wanted the results to be integers. To achieve this, without building the equations by substituting the solutions, she relied on her knowledge of determinant and inverse matrix algebra, acquired in a linear algebra course. She showed that when the determinant is 1 or (-1) the values of unknowns are integers. She exemplified this using the parametric form of equations:

If $ax + by = c$ and $dx + ey = f$, then $x = (ec - fb)/(ae - bd)$ and $y = (fa - cd)/(ae - bd)$
As such, in building equations she chose $\det \begin{bmatrix} a & b \\ d & e \end{bmatrix} = ae - bd = \pm 1$.

Pat recalled that when the task was to find the coordinates for the vertex of a parabola, Grade 11 students, not exposed to Calculus, had to find the roots of the related polynomial, where the midpoint between the roots was the x-coordinate, and then use the equation for a parabola to find the y-coordinate. She could quickly check their solution using Calculus, finding the derivative and, with the help of derivative, finding the extremum point.

The task Michelle chose was to prove that $2^n \geq n$ for all n , by induction or in any other way. Usually in the framework of school mathematical curriculum students learn proofs by induction without formal learning of Peano Axioms. Michelle’s solution included use of this axiom. Michelle provided a precise solution of the task (that we do not display herein) and then wrote:

Peano axiom (In each subset of natural numbers there is a minimal element) serves as basic assumption for the set of Natural numbers. The other one is the axiom of induction. This topic belongs to the Number Theory. Use of Peano axiom makes solutions shorter many times and makes solutions possible at all.

In these three examples we identify three different ways in which AMK can be implemented: Rachel described a situation of *creating a task* for her students, in which she applied her knowledge of Linear Algebra. Pat mentioned a teaching situation in which she was able to *check students' solution* rather 'fast' using her knowledge of Calculus. Michelle's example included a specific task from Grade 12 curriculum, for which she was able to *produce a proof* using her AMK of Number Theory, in addition to the 'standard' proof expected in school.

Whereas our request, both in the interviews and in the written questionnaire, invited respondents to draw connections between their AMK and teaching or curriculum, in many cases it either received no attention or was misinterpreted in two different ways: examples of AMK without relation to teaching or school curriculum, and teaching/curriculum related examples without AMK.

Examples related to AMK beyond school curriculum

Searching for tasks that require AMK or are related to AMK, some teachers provided examples of tasks that are out of the scope of the secondary mathematical curriculum, in its most advanced stream. For example Kevin's task was "Find $\int xe^x dx$ ". His solution included integration by parts which exemplifies his AMK, but does not attend to the request to provide examples related to teaching situation from personal experience or tasks related to school curriculum.

Donna's example also relied on content beyond school curriculum:

Given a sequence of numbers $a_n = \frac{5n-3}{2n+1}$, prove that for this sequence $\frac{2}{3} \leq a_n \leq 2\frac{1}{2}$ for any n . In the proof provided in her written work she relied on the calculation of a limit, a notion that is not explored in the current curriculum. As in the example provided by Kevin, her choice demonstrated her AMK, but did not attend to teaching or curriculum.

Examples of curricular mathematical content without AMK

Ivan suggested the following tasks:

1. Given two points A(7,5) and B(3,1). Write the equation of a circle with diameter AB
2. Let us take for example the rational function $y = \frac{-x^2}{x^2 - 4x + 3}$ and go through the steps: (a) What is the range and the domain of the function? (b) What are the asymptotes? (c) What are the extremum points? (d) Sketch the graph.

Both examples provided by Ivan belong to the high school curriculum and are not explored further in undergraduate mathematics courses. In a classroom conversation with peers Ivan noted that these tasks were difficult for his students and thus were considered as related to AMK. We note that while exploring a rational function and sketching its graph is not an easy assignment, it is not beyond the reach of a student who learned this topic within the school curriculum.

Comments on teachers' examples

An appropriate response to our request, both in interviews and in a written questionnaire, is an example of knowledge that a teacher would possess and use in an instructional situation, but to which a good student would not have an access, within the considered curriculum. As mentioned above, responses provided by Rachel, Pat and Michelle – that we judge as ‘appropriate’ – exemplify implementation of teachers’ knowledge beyond the specific curriculum content presented to their students, but which is applicable in a teaching situation. Kevin and Donna attend to AMK, but ignore curriculum, while Ivan attends to curriculum, conflating AMK with “what students find difficult”. As such, we consider their examples as ‘inappropriate’. However, based on the available data it is impossible to determine whether the examples these teachers provided result from their inability to exemplify what was requested, or from their misinterpretation of our request.

We would like to note that Questions #3 and #4 of the questionnaire were designed in order to avoid vague general claims that we encountered in the interviews and anticipated in participants’ responses to Questions #1 and #2. That is why in creating the questionnaire we explicitly asked participants to exemplify specific problems, and to determine a connection between the presented situation or task and the AMK. However, in 18 situations and 19 task examples generated by 6 secondary-school teachers in their written responses, only 5 situations and 8 task examples were formulated concretely and accompanied by solutions. The other 13 situations and 11 tasks suggested by the teachers provided only an outline for the mathematical content.

Further, among the written responses, Michelle’s was the only one that explained explicitly the relationship between the tasks and problems that she generated and AMK. Her ability to connect the content learned in school with the content learned in the university is an important feature of her mathematical awareness. Further research, based on a combination of written responses with follow up clinical interviews, is necessary to determine whether this ability is a rare gift of only a few teachers or whether specific prompting is needed to bring this ability to surface.

CONCLUSION

While undergraduate content requirements for secondary teachers exist almost everywhere, it has not been investigated *how* mathematical knowledge acquired at the undergraduate level – referred to here as AMK, “advanced mathematical knowledge”

– is manifested in teaching practice. In this paper we report on our first steps in this investigation.

The results of our preliminary exploration indicate that teachers tend to conflate the usage of AMK in teaching practice with either demonstrating their AMK in general or with attending to curricular content that is perceived as difficult. Given the small size of both groups of participants we focused on identifying repeating themes in their responses, rather than providing a precise account of occurrences. Further research will determine to what extent the identified themes persist within a larger and more diverse population.

Our study calls for identifying explicit connections between AMK and mathematics taught in school. An explicit awareness of these connections and an extended repertoire of examples will inform the instructional design in teacher education.

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URGING CALCULUS STUDENTS TO BE ACTIVE LEARNERS: WHAT WORKS AND WHAT DOESN'T

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We report an on-going design experiment in the context of a compulsory calculus course for engineering students. The purpose of the experiment was to explore the feasibility of incorporating ideas of active learning in the course and evaluate its effects on the students' knowledge and attitudes. Two one-semester long iterations of the experiment involved comparison between the experimental group and two control groups. The data were collected from observations, research diary, course exams, attitude questionnaire and two additional questionnaires designed to explore patterns of students' learning behaviors. The (preliminary) results show that active learning can have a positive effect on the students' grades on condition that the students are urged to invest considerable time in independent study.

Key words: active learning, achievements, attitudes, calculus, design experiment

THEORETICAL BACKGROUND

Research on undergraduate mathematics education convincingly argues that active learning is more beneficial for students than learning in traditional mode (e.g., Artigue, Batanero & Kent, 2007). Following Sfard (1998), we refer here to *active learning* as learning through participation based on engaging in problem solving and collaborative activities, and to *traditional learning* – as learning through acquisition based on listening to a teacher exposing theoretical material or demonstrating problem-solving approaches. We learn from the research literature that active learning can help either in developing positive attitude to mathematics (e.g., Tall & Yusuf, 1999) or in improving students' grades in undergraduate calculus, algebra and statistics courses (e.g., Burmeister, Kenney & Nice, 1996).

Teaching in accordance with the principles of active learning is not an easy endeavour. There is a growing body of research that explores pitfalls of active learning, either from academic staff' or students' perspectives. For instance, Pundak & Rozner (2008) reviewed the reasons why academic staff frequently resists innovative teaching and suggest that adopting by the lecturers and TAs active learning paradigm heavily depends on:

...(1) teaching staff readiness to seriously learn the theoretical background of active learning, (2) the development of an appropriate local model, customized to the beliefs of academic staff; (3) teacher expertise in information technologies, and (4) the teachers' design of creative solutions to problems that arose during their teaching" (p. 152).

Solow (1995), cited in Roth-McDuffie, McGinnis & Graeber (2000), found that active learning oriented faculty were anxious about resistance and negative reaction from their students who did not want their teachers "to shake their comfortable relationship with math, no matter how distasteful that relationship may be" (p. 226). In summary, existing students' and teachers' beliefs and perceptions about mathematics teaching and learning are pointed out as the major barriers to spreading active learning methods (e.g., Roth-McDuffie, McGinnis & Graeber, 2000).

Are there more barriers? Apparently, yes, and it seems reasonable that some of them are embedded in the current educational system. For instance, the aforementioned study of Yusof & Tall (1999) reported success in implementation of active learning in a problem solving course with a flexible syllabus, in which some topics could apparently be omitted, and the released time could be used for learning in more depth the remaining topics. Such flexibility is rarely allowed. In another aforementioned study reporting success, by Burmeister, Kenney & Nice (1996), the students were provided practically unlimited assistance, and, even more importantly, they were ready to accept it. Again, such a situation is rather a lucky exception from what is observed in many colleges and universities.

We found rather a surprising lack of research that takes into account the apparent tension between what active learners are expected to do and what they can do, given the entire burden of college study. Our on-going study contributes to addressing this lacuna. In this paper, we describe an experiment aimed at incorporating active learning in a compulsory calculus course for engineering students and focus on the following questions:

1. How do engineering students cope, in terms of time and effort, with requirements of calculus course, in which tutorials and assignments are organized to promote active learning?
2. How does the promotion of active learning, under given constraints, affect the students' grades and attitudes towards the subject?

METHOD

The research setting

The experiment is conducted at ORT Braude Engineering College, in the contest of a multi-variable calculus course given for second-semester undergraduate students. The syllabus of the course consists of the following topics: vector-valued functions, differentiation of scalar functions, maxima and minima, double and triple integrals, integrals over paths and surfaces, the integral theorems of vector analysis and applications. The course is compulsory for the students; its syllabus is compulsory for the teachers. The students take the course in continuation of a one variable calculus course. We will refer to the first-semester course as CAL1, and to the second-

semester course as CAL2. CAL2 is taught 6 hours a week: four hours of lectures in groups of 40-60 students and two hours of tutorials in groups of 20-30 students.

The study design

The study was initially designed as a one-semester quasi-experiment with a control group (Cook & Campbell, 1979). It then evolved into a design experiment (Cobb, 2000; Cobb et al., 2003) of several one-semester long iterations. This paper is written after the second iteration and before the third one. The purpose of a quasi-experiment was to find out the effect of implementation of active learning ideas, in terms of the course grades. The need in continuation of the study in the form of design experiment emerged from the lack of satisfaction from the results of the first semester and from our thinking how to refine the teaching and to capture various effects of active learning. For these reasons we decided to keep comparing the experimental group (G1) and the control groups (G2 and G3) within every iteration.

Participants

Overall numbers of students (NS) in G1, G2 and G3 groups and the numbers of tutorial classes to which each group was divided (NTC) are given in Table 1. The groups G1 and G2 consisted of all second-semester students of the Department of Software Engineering. At the beginning of every semester, the students were given brief information about two different styles of tutorials, active and traditional. Based on this information, some students chose to join G1, and the rest – G2. Group G3 consisted of all the students of the Department of Electrical and Electronic Engineering. They were not given the choice and were taught in a traditional mode (see *Theoretical Background* section).

	G1		G2		G3	
	NTC	NS	NTC	NS	NTC	NS
Iteration 1	1	25	2	40	3	62
Iteration 2	1	20	2	46	4	94

Table 1: The sample

Groups G1 and G2 were taught by Ludmila Shvartsman, one of the authors of this paper, who conducted both lectures and tutorials. Group G3 was taught by a team of lecturers and TAs, including another author of this paper, Buma Abramovitz. All the lecturers and TAs involved in the experiment were of comparable teaching experience and of similar level of teaching achievements. Specifically, their past students, on average, achieved similar grades in the course and gave similar feedback.

The mathematical content of the lectures, as well as the problems and exercises given to the students in the tutorials, were the same in all the groups. All the students had access to the same theoretical materials and examples published at the course website. Also, the students were given the same midterm and final exams. The difference

between G1 and the rest of the groups was in the styles of conducting tutorials and in the use of homework assignments, as will be described below.

The research tools

The experiment is described in detail in the research diary written by Ludmila. It includes descriptions of and reflections on all tutorials in G1, a protocol of a lesson in G2 compared with a lesson in G1 based on the same problems, and protocols of more than 10 meetings of the research team. One lesson in G1 was videotaped. The information about teaching in G3 was collected from Buma who taught there and from many meetings and conversations with the other lecturers and TAs of G3. We also developed and run a student questionnaire in all the groups. We call it Tutorial Styles Questionnaire (TSQ). The questions concerned the students' opinions about tutorials and patterns of their participation in the tutorials. The questionnaire was validated in 8 interviews with G1 students at the end of the first iteration.

During the first and the second iterations, G1 students' final grades in CAL2 and CAL1 were compared with grades of G2 and G3 students. The variance in CAL2 final grades was explained using stepwise multiple regression analysis, in which CAL1 grades and the variables indicating to which group a student belonged served as independent variables.

After the first iteration we developed and implemented two additional multiple-choice questionnaires. The first one concerns the students' attitudes to multi-variable calculus and solving problems. It is adapted from Yusof and Tall (1999). We call it Attitudes Questionnaire (ATQ). The second one was developed to estimate effort that students invest, or can invest, in studying the course before and after the lessons. We call it Effort Distribution Questionnaire (EDQ).

RESULTS AND ANALYSIS

Iteration 1

During the first semester active learning in the experimental group was promoted, but not urged. The G1 students were required to read relevant theoretical material and to approach problems, published on the course website, before every tutorial lesson. The solutions were also published. In addition, all the students were invited to get help from Ludmila during her office hours. The tutorials' content and conduct were built on the assumption that the students would come to the lesson being familiar with the basic problems.

During the lessons, the students were given more advanced problems than those published on the web. The students were given some time to think and discuss these problems in small groups, and then their ideas were presented to the whole class. Finally, the solutions emerged from these discussions and presentations. The teacher acted more as a mediator of the discussions than as an authority providing the

solutions. The G1 classroom supported such interactive and collaborative activities (see Pundak & Rozner, 2008, for a detailed description of this special classroom).

All G1, G2 and G3 students were given an optional once-a-week Webassign homeworks of 4-5 exercises, the answers to which were to be submitted and checked electronically (see www.webassign.net for details). G1 students in pairs were also offered an opportunity to solve additional, more challenging, homeworks. These homeworks were commented and graded by the teacher every week. The purpose of these additional homeworks was to further promote interactive and cooperative learning. We call the former type of homework *Webassign homeworks*, and the latter one – *Commented homeworks*. Both types of homeworks could be resubmitted for one time to improve the grades.

The components of final course grades are presented in Table 2.

Group	Final exam	Midterm exam	Webassign homeworks	Commented homeworks
G1	70%	20%	5%	5%
G2, G3	70%	20%	10%	

Table 2: The structure of final grades in the first semester

Midterm exams, Webassign homeworks and Commented homeworks were optional, that is, it was up to the students to include or not the homework grades into a final course grade. The final grade of the students who did not take part in midterm exam and/or did not submit homeworks was fully determined by the final exam.

The reality appeared to be more complicated than our expectations. Most of G1 students appreciated the new for them style of the tutorials, but only about half of the group actually followed the requirements (it was evident from TSQ, the diary and the interviews). We observed that some G1 students indeed came prepared for the tutorials, and others did not. Some were engaged in cooperative problem solving, and some remained the consumers of the solutions demonstrated by others. Some students had benefited from the feedback on the homeworks, and others had ignored them.

Ludmila became more satisfied with the conduct of the tutorials and the students' collaboration at the second half of the semester. Generally speaking, the desired style of the tutorials has been finally achieved in G1, and it indeed was different from the traditional style in G2 and G3. This was evident from the comparative analysis of two lesson protocols and TSQ. However, the desired change in out-of-class study was not achieved. In particular, G1 students devoted less time to homework than it was expected: from 30 to 60 min instead of 2 hours a week. G3 students, on average, also invested in the homeworks from 30 to 60 min a week, and G2 students – less than 30 minutes.

Comparative analysis of the final course grades was also not in favour of G1. The mean and SDs were: 63.9 (19.5), 66.0 (22.7) and 76.0 (15.9) for G1, G2 and G3, respectively. A stepwise multiple regression analysis revealed that belonging to G3 was beneficial even after neutralizing the fact that, on average, CAL1 grades in G3 were higher than in G1 and G2 (72.15 (11.98) in G3, 70.32 (12) in G1, and 69.72 (12.34) in G2). Let us remind that G1 and G2 were taught by the same teacher, and G3 was taught by other teachers.

At the end of the semester, we summarized the findings and designed the second iteration. We decided:

- To urge students to work more out of the class by changing the structure of the course final grade.
- To control more aspects of the experiment. In particular, we decided to measure the students' attitudes towards the subjects (see the Research Tools section).
- To check feasibility of the requirements to learn actively by taking into consideration the students' overall burden of study.

Iteration 2

The second iteration was started six month after finishing the first one. The in-between time was used for validating TSQ, developing EDQ, piloting new elements of teaching and refining the evaluation tools.

First, challenging preparatory problems were published on the web without solutions. These problems were discussed at the beginning of each tutorial during 10-15 min. The rest of the lesson was conducted as in the first iteration.

Second, Webassign homeworks that included technical exercises were cancelled for all the students. The Commented homeworks became compulsory for G1 students, and remained optional for G2 and G3 students.

Third, a new compulsory test was offered in addition to an optional midterm exam and a compulsory final exam. This test was composed from two out of about 150 preparatory problems and the problems that appeared in the Commented homeworks; we call it *Homework test*. All the students were aware of its structure and the source from which the tasks were to be chosen. The components of a final grade of the course are presented in Table 3.

Group	Final exam	Midterm exam	Homework test	Commented homeworks
G1	65%	20%	10%	5%
G2, G3	65%	20%	15%	

Table 3: The structure of final grades in the second semester

For those students, who decided not to take the midterm exam, the weight of the final exam was 85%.

These changes worked as follows. At the beginning of the semester, about three quarters of G1 students were ready for the tutorials and actively participated in the discussions. Less than half of the students remained active learners in the middle of the semester. They explained that they merely did not have enough time to properly prepare themselves for the tutorials, so we decided to try something else. Ludmila started asking different pairs of students to take a lead during the lesson. Naturally, the leading students had to invest more time in preparations. This made the lessons more interesting and, in a way, showed the rest of the class that they can do the same.

As in the first iteration, TSQ results enlightened the difference between tutorial styles in G1 and the other two groups, however, the levels of satisfaction of G1, G2 and G3 students from the tutorials were about the same. The attitudes towards the subject, in terms of ATQ, were also not different in all the groups.

EDQ data showed that G1 students devoted more time to out-of-class study than G2 and G3 students (on average, 6.24 (2.43) hours in G1 vs. 4.98 (1.75) hours in G2 and G3 a week, $t=1.97$, $df=41$, $p<0.05$); about 60% of the time was devoted to doing the homework in G1, and 47% - in G2 and G3. Note that, according to our estimation, an average student needs about 8 hours a week to fully cope with the requirements. EDQ also showed that G1 students studied systematically during the semester, whereas G2 and G3 students increased the time of independent study towards the end of the semester.

In addition, the students were asked in EDQ: "Given the general load of your study and time constraints that you have, which minimal grade in CAL2 course would you accept as satisfying?" and then "How much additional time are you ready to invest per week in study in order to obtain a 10% higher grade than that you have indicated in the previous question?" Surprisingly, the responses of G1, G2 and G3 students to these questions were very close. We interpret this finding as follows. First, learning motivation of G1 students was not significantly higher than that of G2 and G3 students. Second, the expectation that an average student should invest about 8 hours a week in out-of-class study was not beyond of what the students said they could do (on average, the students of all the groups were ready to invest 4 additional hours).

This time G1 students did better than their peers in terms of the course final grades. The mean and SDs were: 71.5 (16.3), 52.4 (26.6) and 65.2 (26.7) for G1, G2 and G3, respectively. A significant regression equation showed that belonging to G1 was beneficial in comparison with belonging to either G2 or G3, even after neutralizing the differences in CAL1 grades (71.6 (12.7) in G1, 64.4 (9.2) in G2, and 73.8 (11.4) in G3).

Thus, we can report success, in terms of course grades, of an experimental style of conducting tutorials. However, the students' attitudes to the subject did not change and remained relatively low. It should also be noted that our expectations about the students learning behaviors were only partially fulfilled. Specifically, we succeeded

more in urging the students to do their after-the-lesson homeworks than in convincing them to solve recommended problems before the tutorials.

We are going to deal with these issues in the future iteration(s). In particular, we consider publishing more problems on the course website before the lesson, and asking students to choose which problems they are interested to discuss during the lesson. We hope that the students will take more responsibility for their learning outcomes (cf. Brousseau, 1997). This may encourage them to invest more time in preparation for the tutorials and have more influence on the content of the course. In turn, this may affect their attitudes to the subject.

DISCUSSION AND CONCLUSIONS

The main lesson that we have learned from the first two iterations of the experiment can be put in words of Latterell (2008): "Students do what is expedient, and not necessarily what professors think they should" (p. 12). So, for us, the crucial issue was how to make active learning of calculus expedient for the students. The first iteration of the experiment showed that conducting tutorials in interactive and cooperative mode is not sufficient in order to obtain traceable improvements in the students' achievements and attitudes. It has become evident that fulfillment of our expectations requires changes also in the students' learning behaviors out of class, and that these requirements should be supported by appropriate modification of the structure of a course grade. This idea was realized during the second iteration and appeared feasible, in terms of time and effort, for the students. The second iteration resulted in significant advantage of the experimental group in comparison with two control groups. Is the observed effect due to incorporated innovations? We believe that it is, for the following reasons:

- The experimental group did better not only in comparison with G2 control group, taught by the same teacher, but also in comparison with G3 control group taught by the others. The teachers were aware of competitive nature of the experiment. They all were of comparable experience and past achievements in teaching, so it is unlikely that the observed advantage of the experimental group can be just attributed to the differences in the teachers' professionalism or enthusiasm.
- The mathematical content of the course was exactly the same in all three groups.
- We admit that random assignment of students to the experimental and control groups would be preferable. Even though it could not be realized under the conditions embedded in practice of college education, the achieved effect cannot be attributed just to the differences in students' learning motivation or mathematical background. This claim is supported by EDQ data and by the regression analysis. Note that our way of dealing with the issue of non-random assignment is in line with what is done in some other studies (cf. Schwingendorf, McCabe & Kuhn, 2000).

We are aware, of course, that the reported effect may be due to some combination of the aforementioned factors or to some uncontrolled in our experiment ones. This adds us motivation to keep running the experiment. Currently, we see the process of educating undergraduate students to learn actively as a multi-stage enterprise, in which many factors are involved. Some of them, for instance, beliefs of students and teachers, are extensively explored (Pundak & Rozner, 2008; Roth-McDuffie, McGinnis & Graeber, 2000). Others only recently deserved attention of the mathematics education research community.

The distinction that Harel (2008) made between intellectual and psychological needs involved in learning mathematics is particularly relevant to discussion of our findings. The intellectual needs, such as the need to construct new knowledge in response to a perturbing problem that otherwise cannot be solved, are in the focus of contemporary mathematics education research. Psychological needs, such as the need to be competent and secure in relationships with others, frequently remain peripheral. However, the latter needs are crucially important in our and our students' real lives and must be taken in consideration when one requires his or her students to be active learners, and thus, to put more time and effort in study. As a matter of fact, one difference between the first and the second iteration of our experiment can be explained in these terms: the first iteration was focused on intellectual needs of the students, whereas the second one was organized so that the students could be more successful when conforming to the requirements of active learning. In a way, this distinction calls for balance between active and traditional learning modes, as suggested by some theorists (e.g., Sfard, 1998) and practitioners (e.g., Tucker, 1999) since the active learning mode relies mostly on the students' intellectual needs, and the traditional mode – on their psychological needs.

The last comment is about content dependency of the presented findings. Because of our intention to outline a long study in a brief paper, examples of calculus problems from the tutorials and examples from the questionnaires are not included. It may create an impression that the reported findings are not exclusive for the chosen mathematical context. Perhaps, they are not indeed. We hope to discuss this topic in the oral presentation and in the future publications.

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FROM NUMBERS TO LIMITS: SITUATIONS AS A WAY TO A PROCESS OF ABSTRACTION

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Abstract: When they enter the University, students have a weak conception of real numbers; they do not assign the right meaning to a writing as $\sqrt{2}$, or π , but neither x or parameters. This prevents them to have a control about formal proofs in the field of calculus. We present some situations to improve students' real numbers understanding; these situations must lead them to experiment approximations and to seize the link between real numbers and limits. They can revisit the theorems they were taught and experience their necessity to work about unknown mathematical objects.

SIGNS AND SITUATIONS IN THE PROCESS OF TEACHING CALCULUS

Noticing that mathematical work in the field of Calculus is usually very difficult for even good students when they are entering the French University, we have studied the transition between the secondary mathematical organisation in teaching (pre)calculus, and the University one. Our questions address the problem of the links that can be built between the intuitive approaches of Upper Secondary School and the formal one that is predominant in University. This research led us not only to analyse students' productions in the field of calculus, but to try to design situations to make them do the required step between the two levels of conceptualisation.

The theoretical frame we use is due to Brousseau, for the Theory of Didactical Situations (TDS), and C.S. Peirce for its semiotic part.

According to Saenz-Ludlow (2006), "For Peirce, thought, sign, communication, and meaning-making are inherently connected. (...) Private meanings will be continuously modified and refined eventually to converge towards those conventional meanings already established in the community. (...) "... A whole sign is triadic and constituted by an object, a 'material sign' (*representamen*), and an *interpretant*, the latter being an identity that can put the sign in relation with something – the object. A very important dimension in Peirce's semiotics is that interpretation is a *process*: it evolves through/by new signs, in a chain of interpretation and signs. The *interpretant* – the sign agent, utterer, mediator – modifies the sign according to his/her own interpretation. This dynamics of signs' production and interpretation plays a fundamental role in mathematics where a first signification has always to be re-arranged, re-thought, to fit with new and more complex objects.

Peirce – who was himself a mathematician – organised signs in different categories; briefly said, signs are triadic but they are also of three different kinds. We will strongly sum up the complex system of Peirce's classification (ten categories,

depending on the nature of each component of the sign, representamen, object, interpretant: see Everaert-Desmedt, 1990; Saenz-Ludlow, 2006) by saying that we will call an *icon* a sign referring to the object as itself – like a red object refers to a feeling of red. An *index* is a sign that refers to an object as a proposition: 'this apple is red'. A *symbol* is a sign that contains a rule. In mathematics all signs are symbols to be interpreted as *arguments*, though they are not exactly of the same complexity; and so are the language arguments we use in mathematics for communication, reasoning, teaching and learning. The semiotic theory will help us to identify the kind of sign produced in teaching-learning interactions, and the appropriateness (with regard to the situation) of how students interpret the given signs. Then we use the theory of didactical situations to build situations appropriate to knowledge.

Signs and situations

Mathematics aims at definition of 'useful' properties that can help to solve a problem or to better understand the nature of concepts. A strong characteristic of these properties is their invariance: they apply to wide fields of objects – numbers, functions, geometrical objects, and so on. This implies the necessity of flexibility of mathematical signs and significations. To grasp the generality and invariance of properties, students have to do many comparisons – and mathematical actions – between different objects in different notational systems. While the choice of pertinent symbols and different classes of mathematical objects is necessary to reach this aim, it is not sufficient. To produce knowledge, the situation in which students are immersed is essential. By 'situation', we mean the type of problems students are led to solve and the milieu with which they interact. Brousseau's Theory of Didactical Situations (Brousseau 1997) claims that to make mathematical signs 'full of sense' – which means that signs have a chance to be related to conceptual mathematics objects – it is necessary to organise situations that allow the students to engage with validation, that is, to work with mathematical formulation and statements. In Bloch (2003), we explained how we build situations where the aimed knowledge appears as a condition to be satisfied in a problem. In Bloch (2007b) we illustrated how such a situation – the Pythagoras's lotto – could be carried on to restore the meaning of multiplication in specialised classes.

In the present paper, we first explain how students' difficulties can be lightened by using Peirce's system and how this system helps us to identify the needs of the subsequent teaching; then we present three situations that were experimented with students of first year of University. We try to make it clear how these situations could lead students from a rather iconic or indexical point of view about numbers and limits to the aptitude to an argumentation.

FROM LIMIT ALGEBRA TO FORMAL PROOF

In our main studies we chose the concept of limit because it is the first analytic concept students meet, and it is possible to build a very rich and contrasted corpus of

tasks about limits, from the Premiere and Terminal – in upper secondary school for scientific students in France – to the first year of University.

At the entrance to the University, almost all exercises carry the structural conception of the notion of limit. These exercises are based on general conjectures; their resolution requires a perfect adaptation of students to the formal definition of the limit, whereas at the high school, the limit notion is conceived as a process. Its representations appear to be more susceptible of operational interpretations. In a previous study (Bloch & Ghedamsi, 2004) we proposed to identify didactical variables that are pertinent to characterise the extent of the rupture. These variables are the degree of formalisation in the domain of the analysis; the setting of validation, the limit algebra or the analysis one, the degree of generalisation; the number of new notions introduced in the limit environment; the type of tasks (heuristic or graphic or algorithmic); the choice of techniques, the degree of autonomy solicited; the mode of intervention of the notion, process status or object one; the type of conversion between the semiotic representation settings.

The identification of these variables allows us to detect global ruptures at the passage from the secondary teaching institution to the superior one. At each level, the values given to these didactic variables are seen as mutually exclusive. We can observe that almost all the variables change, and that the rate of change is considerable. Students are confronted with a global revolution in the required work and of their means of work. By this conceptual "jump" students are supposed to (Peirce's levels are in italic):

- Work with general notations ($x, f\dots$) and no more with specific numbers or well known functions: *overtake the indexical idea of numbers and functions to assume a symbolic one;*
- Be able to achieve reasoning on generic mathematical objects: *produce signs as right symbols and arguments;*
- Know calculus theorems and how they can be useful: *link taught arguments and personal ones;*
- Deduce specific properties from general reasoning about sequences, functions, limits: *go back from a general argument to an index.*

And then:

- Achieve reification about the concept of limit;
- Gain the unifying formalism (definition with ε, N), and by this way generalise the notion of limit and be able to use formal tools to prove.

NUMBERS AS TOOLS TO DO CALCULUS

The use of formal tools includes the manipulation of 'generic numbers', written x : teachers at University usually do not even notice that this could be a problem. For instance, these exercises are considered as rather plain:

Find the limit in 0 of: $x \rightarrow x \times \sin(1/x)$

Solve an equation as $f(x) = x$ (with the limit of a sequence)

Find the limit of a sequence with a parameter in the function, as (x_n) : $x_0 = I$ and

$$x_{n+1} = a \sin x_n + b$$

However, in our studies we can notice that even good students at University have an uneasy use of real numbers' notation, and not only with an x , but also with a number as $\sqrt{2}$ or π . This difficulty prevents them to be able to assign the right meaning to a letter in a mathematical writing, as $a \sin x_n + b$. The status of a , b , x , n is not clear for them. The number π , for instance, is seen as a 'notation' – that is, an icon or an index in Peirce's system – but not really a number because numbers are 'well known' – for students the common model of numbers is a rational number, or even better an integer. In a previous study (Bloch & al. 2008) we noticed that the field of numbers students met at secondary school was very narrow: the main reason is that when a new notion is introduced, teachers present it with familiar numbers to avoid an increase of difficulties. It follows that students meet occasionally some irrational numbers when they are told these numbers exist, but they never use them to calculate on vectors, functions, limits, derivatives...

Signs as \exists , \forall , or even parentheses are not well understood; students often say they are in a mathematical sentence to indicate something about the variables, but they do not know exactly what; they do not know either why they should be in an order more than in another (Chellougui, 2007). These signs are clearly iconic for them.

As we intended to build situations about the concept of limit, we thought it necessary to reintroduce a work about numbers; students need numbers to experiment and prove and it is not possible they master formalism about numbers if they do not know what numbers are.

As said in Bloch & Schneider, 2004:

Building situations for learning the concept of limit must then take into account the kind of semiotic representatives that is used; and we must not forget that a proper mathematical knowledge, especially including proof, is built only if the selected semiotic representatives and the milieu allow adequate reasoning, *and* if students can seize these tools of control.

We observe then that in the work about limits students cannot seize the numerical tools of control. For this reason we planned to build situations about the concept of limit, those situations including a students' work about approximations, nature of numbers – rational, irrational, and transcendent (even if the question is obviously not to prove the transcendence at this level). We have experienced these situations with classes of students – two classes for the von Koch snowflake, one for each of the two others. This is a clinical experiment; we do not talk here of the reproducibility, but the thorough a priori analysis that is performed for each situation guaranties the

experimental reproducibility. Of course the actual one depends on the conditions in each class and it could not be else. Séances were videotaped or registered.

THREE SITUATIONS ON LIMITS

1. The Von Koch snowflake

This situation takes place with scientific students, 17 years old. The aim is to study a shape – a fractal – which perimeter is infinite as the area is finite: this dialectic between two types of limits aims at making them build reasoning to decide on which condition a limit can be infinite or finite. A first experiment is to be done with a pocket calculator; students can then make a conjecture about the perimeter and the area (see annex for the schemas).

The formula for the perimeter is $P_n = P_0 \times (4/3)^n$ so $\lim_{n \rightarrow +\infty} P_n = +\infty$

It will be proved with the Euler's inequality $(1+a)^n > 1+na$. We observe that half of the students think that the perimeter is finite, and half of them think that it is not: so it is not evident.

The area is $A_n = A_0 + \frac{3}{5}A_0 [1 + (\frac{4}{9})^n]$ so $A_\infty = \lim_{n \rightarrow +\infty} A_n = \frac{8}{5}A_0$

Notice that if we start from a equilateral triangle of side a , $A_0 = a \sqrt{3}/4$, so it is irrational. It is an important value of a didactical variable, because it prevents students to try to 'catch' the limit with decimals: they have to carry out a reasoning to know if the area is infinite or not. To prove the result it is possible to introduce the logarithm function and show that $(\frac{4}{9})^n$, which is the functional term in this formula, tends to zero: it can be made smaller than every 10^{-p} , for any value of p :

$n \log(\frac{4}{9}) < \log 10^{-p}$ gives $n > -p / \log \frac{4}{9}$ because, of course, $\log \frac{4}{9} < 0$.

According to their first opinion, half of the students think that the area is infinite, one of them saying: "Anyway the area does the same as the perimeter". We also observe that the symbol of a function incorporated in the area formula is not seen by a lot of students. They have to work a long time before some of them become able to identify this symbol. The other ones seem to think the formula as a whole, a kind of icon of function. Sequences acquire a clearer meaning of "a way to attain a number", but the link between a sequence and its limit is however still indexical: they appear to be disconnected in a way. It's just that the sequence refers to the limit.

All this work eventually leads students to reasoning about sequences, functions, ways of experimenting and proving. It is a real entrance into the way of reasoning in Calculus, but it does not make students necessarily link their knowledge about \mathbb{R} and the limits. This is why we tried to build and experiment the two other situations.

2. The Euclidean algorithm of $\sqrt{2}$

In her thesis, I.Ghedamsi (Ghedamsi, 2008) makes students – in a course of first year at University – experiment the construction of a sequence of rational numbers tending to an irrational number \sqrt{d} , where d is an integer, $d \geq 2$; d is not a square number as $d-1$ is. For instance, the antiphérèse of $\sqrt{2}$ leads to a development of $\sqrt{2}$ in a sequence of unlimited continued fractions, the condition to get a finite development being that the number would be rational.

We assume that $(\sqrt{d} - \alpha) = \frac{1}{\sqrt{d} + \alpha}$ allows to give a development of \sqrt{d} in a sequence of

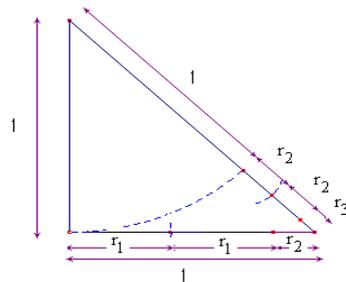
unlimited continued fractions, $\sqrt{d} = \alpha + \frac{1}{2\alpha + \frac{1}{2\alpha + \frac{1}{2\alpha + \text{etc.}}}}$;

and the sequence converging to $\sqrt{2}$ is given by : $u_0 = 1$ and $u_{n+1} = 1 + \frac{1}{2 + u_n}$

And finally:

$$\sqrt{2} = 1 + \frac{1}{r_1} = 1 + \frac{1}{2 + \frac{r_2}{r_1}} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{r_3}{r_2}}} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \text{etc.}}}}}}$$

...where r_1, r_2, \dots are the remainders that appear in a geometric way in the following rectangle triangle:



The work on the sequence leads students to realize that they can find a 'good' approximation of $\sqrt{2}$, as good as they decide. Students' work can lean on the geometric illustration, which gives a reality to the number. Students say that before, they thought $\sqrt{2}$ was a kind of 'notation' – an icon – and now they realize that it is a real number, in both meanings! Notice that at the same time they have enhanced their calculation ability on sequences and they become able to make a link between mathematics theorems (existence of a limit) and an already known number. They also

conceive now what it means that \mathbf{Q} is dense in \mathbf{R} . We observe that they become able to really link the existence of a number and the sequence that 'gives' the number.

Nevertheless, they now just consider numbers as $\sqrt{2}$, which is not sufficient to get into the idea of numbers that cannot be 'seen' or 'calculated'. This is why another situation is necessary: it must compel students to cope with numbers we reach only through the use of mathematical theorems as the nested intervals theorem, the limited development of a function, or the Newton's method to find a fixed point. Of course this progression is also a mathematical one, from algebraic numbers to other irrational ones. It is also a semiotic process from numbers as writings and theorems as abstract rules to numbers as mathematical objects and theorems as useful statements to work about these objects, theorems as tools of the mathematical work. Theorems become arguments to do the work.

3. The fixed point of cosine

The cosine function is continuous in $[-1,1]$ and maps it into $[-1, 1]$, and thus must have a fixed point. This is clear when examining a sketched graph of the cosine function; the fixed point occurs where the cosine curve $y = \cos(x)$ intersects the line $y = x$. Numerically, the fixed point is approximately $x = 0.73908513321516$ (thus $x = \cos(x)$); but students cannot have an spontaneous idea of this value.

The aim is to make students work about a number they do not know, and cannot 'represent' except in a graphical way – but the curve of cosine is not a calculator. We do not describe the situation here (for details see Ghedamsi 2008), we just say that the problem is to compare two approximation methods to reach the fixed point: dichotomy and the Newton method.

Students are really surprised not to 'find' the number, as can be seen below:

"S₁: $u_3 = \cos u_2$ and $u_2 = \cos u_1$ and... we have to choose an u_0 ...

S₂: u_0 is in the interval $(0,1)$...

S₁: but finally... it's the same! We cannot find the exact value???

S₃: even with good software?! As for e ... (the basis of exponential function).

Teacher: How does software proceed to calculate a number?

S₁: I think they use sequences and calculate how many terms they need...

S₂: It means that the fixed point of cosine has no exact value... it exists because we find a sequence...

Teacher: Is it the same with $\sqrt{2}$?

S₃: $\sqrt{2}$ has an exact value because its square is 2

Teacher: and how do we call a number like this? It is transcendental. And what do you propose to calculate this number?

S₁: We could use sub-sequences... " (Then students work about adjacent sequences)

We observe that the progression of the situations leads to cope first with an idea of limit, the fact that we need theoretical tools to attest that a sequence has got a finite or infinite limit; then they work about density of \mathbf{Q} in \mathbf{R} ; and finally they are led to use

theorems they were taught to become able to speak of a number "that cannot be seen". The meaning of these theorems appears: the function of Analysis theorems is to allow the work on unknown objects, but it supposes that we can make a verification that theorems fit to find the unknown number.

Then this last situation compels students to become aware that the conditions of a theorem are of some interest and that they cannot neglect them.

CONCLUSION

Situations based upon a numerical heuristic work confirm to be efficient to engage students into a proof process. We noticed that they had to become able to achieve reasoning on generic mathematical objects: situations aim at doing a connection between their previous numerical knowledge and the notion of real number, which must be linked with the use of theorems.

In order to link heuristic and formal work, situations were organized in three steps: 1) first meetings with the tools of calculus; 2) an investigation about algebraic well recognised numbers that allow to experiment and give examples or counter examples; 3) finally a situation that needs the use of theoretical means.

We can conclude that:

- The use of approximations allows identifying mathematical objects which existence is only formal; it is a work about mathematical symbols – arguments and no more kinds of indexes of a knowledge.
- Situations organize comings and goings between intuitions and formalism;
- Situations were built with the concern of a balance between the values of the macro-didactic variables: more or less formalisation, generalisation; limit algebra or the use of theorems.

We can attest that the work in these situations creates an epistemological change in students' conceptions. They are made able to consider real numbers with their true nature, that is, conceptual objects in relation with other coherent objects in a mathematical theory. They eventually accede to the argumental nature of mathematical objects and do not see them anymore as icons drawn by the teacher.

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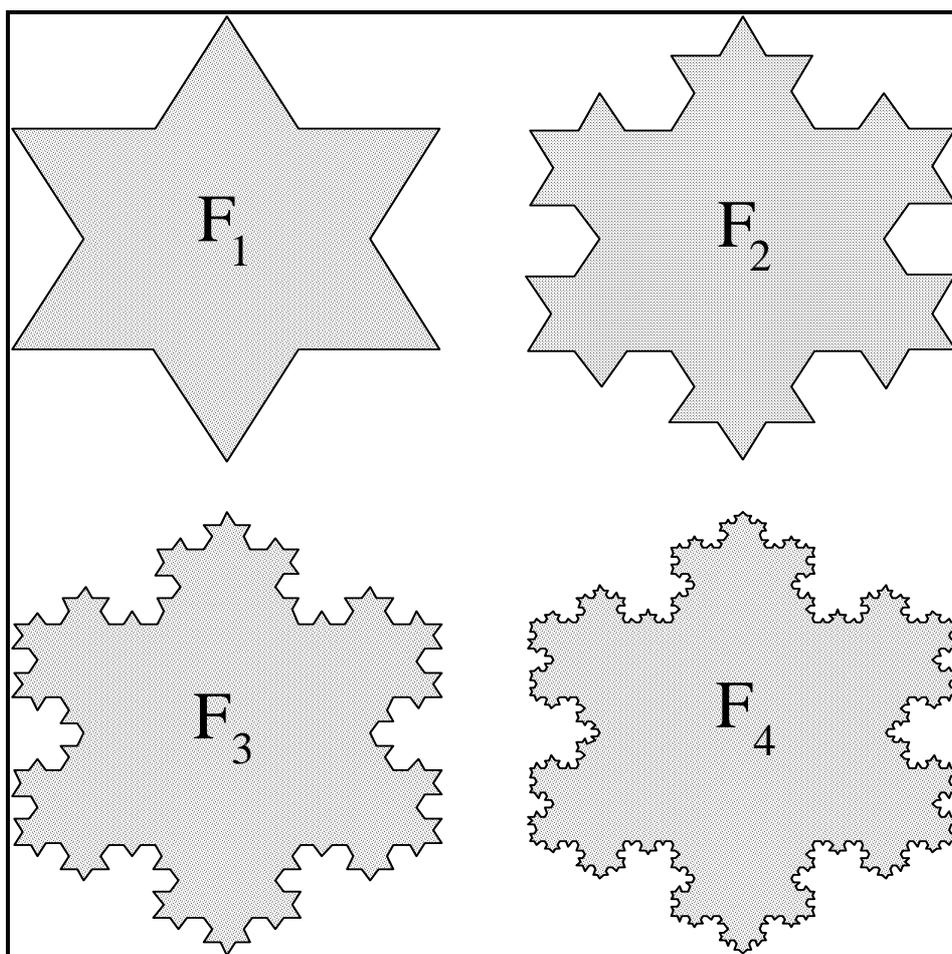
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ANNEX

The Von Koch snowflake, F_1 to F_4



What are the perimeter and area of F_∞ , the final fractal?

FROM HISTORICAL ANALYSIS TO CLASSROOM WORK: FUNCTION VARIATION AND LONG-TERM DEVELOPMENT OF FUNCTIONAL THINKING

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ABSTRACT : We present the outline and first elements of the second phase of our work on mathematical understanding in function theory. The now completed first phase consisted in a historical study of the differentiation of viewpoints on functions in 19th century elementary and non-elementary mathematics. This work led us to formulate a series of hypotheses as to the long-term development of functional thinking, throughout upper-secondary and tertiary education. We plan to empirically investigate three main aspects, centring on the notion of functional variation : (1) “ghost curriculum” hypothesis; (2) didactical engineering for the formal introduction of the definition (3) assessment of long-term development of cognitive versatility.

Key-words: functional thinking, concept-definition, cognitive versatility, AMT, historical development of mathematics.

NON-STANDARD QUESTIONS EMERGING FROM HISTORICAL STUDY

In 2006, the history of mathematics group of the Paris 7 Institute for Research on Mathematics Education (IREM¹) completed a study on the “multiplicity of viewpoints”, with funding from the French Institute for Research on Pedagogy (INRP). The challenge was to combine historical and didactical investigations, and the main results were published in (Chorlay 2007(a)) and (Chorlay & Michel-Pajus 2008). On the basis of this theoretical work, we engaged in 2007 in a second research phase which involves field-work and deals with issues of AMT² and teaching of mathematical analysis at both upper-secondary and tertiary levels.

The first phase started when we became aware of possible interactions between historical and didactical work : on the one hand, R. Chorlay was engaged in a dissertation of the historical emergence of the concepts of “local” and “global” (Chorlay 2007(b)); on the other hand, didactical work was being conducted on similar issues with regard to teaching at upper-secondary (Maschietto 2002) or tertiary levels (Praslon 1994, 2000), under the supervision of Pr. Artigue and Pr. Rogalski. We

¹ <http://iremp7.math.jussieu.fr/groupestravail/math.html>

² (Tall 1991) and (Artigue, Batanero & Kent 2007).

engaged in a historical study, centred of 19th century elementary and non-elementary mathematical analysis, so as to gain insight into the explicit emergence and differentiation of the four “viewpoints” which didactical work on mathematical analysis had distinguished : point-wise, infinitesimal, local and global.

Our work centred on the history of several *hot-spots* where the viewpoints interact strongly : definition of “maximum”, use of the two-place “ function f is [*property*] on [*domain*]” syntagm, proofs of the mean value theorem, proofs of the theorem linking the variation of f and the sign of its derivative, proof (if any) of the existence theorem for implicit functions. The interactions with typically AMT issues occurred at four different levels : (1) in terms of mathematical concepts : function concept³, real numbers, limits and continuity⁴, proofs in calculus, use of quantifiers; (2) in terms of curriculum, we focused on typically higher-education maths topics and transition from secondary to tertiary education stakes; (3) we centred on issues of cognitive flexibility⁵, in particular the ability to change viewpoints, levels of abstraction, theoretical frames, and semiotic registers⁶ in an autonomous manner; (4) the explicit use of *meta*-level terms to describe abstract viewpoints (such as “local” or “global”) raise many questions in terms of transmission (implicit/explicit classroom use, transmission by definitions or by paradigmatic examples) and efficient use (effective problem solving or proof design based on *meta*-level knowledge)⁷.

This work left us with a few unexpected and unanswered questions, though. The historical work on the notion of function, maximum or domain showed us that some of the aspects that we thought would be the least problematic evolved at a different pace from that of apparently more sophisticated ones. To be more specific : notions of domain, maximum, and function variation seem to be of a rather elementary nature. In the French curriculum they are the first notions to be taught (in the first year of upper-secondary education) when the notion of function is first introduced, one year before students begin calculus. From a didactical viewpoint, these notions depend only on the point-wise and global viewpoints; they are compatible with a mere proceptual view of functions. Thus we were puzzled by the discovery that the notion of variation, for instance, only came to be defined⁸ in Osgood’s 1906 course on mathematical analysis (Osgood 1906). The characteristics of this non-elementary textbook are analysed in (Chorlay 2007(b), chapter 7) : it helps document the strict *co-emergence* of (1) the notion of domain in elementary analysis, (2) the explicit use

³ (Vollrath 1989), (Artigue 1991), (Dubinsky & Harel 1992), Carlson’s paper in (Dubinsky and Kaput 1998), or for more recent developments (Stölting 2008).

⁴ (Tall & Vinner 1981), (Cornu 1991).

⁵ (Robert & Schwartzenberger 1991), see also Robert’s and Rogalski’s papers in (DIDIREM 2002).

⁶ See Duval’s paper in (DIDIREM 2002)

⁷ See Robert’s and Artigue’s papers in (Baron & Robert 1993).

⁸ To the best of our knowledge, that is.

of “local” and “global” as *meta*-level descriptive terms, and (3) the point-wise definition of formerly undefined functional properties, such as variation. The not-so-elementary epistemological nature of these notions is also documented in Poincaré’s work : he listed them among “qualitative” properties of function which, he claimed in 1881, form a new and difficult field of inquiry (Poincaré 1881); needless to say Poincaré’s notion of “qualitative” study encompasses more than intuitive or graphical aspects.

It turned out that these unexpected historical facts echoed teaching problems which we had experienced over the years, as teachers of mathematics (at upper-secondary and tertiary levels) and pre-service or in-service teacher trainers. I engaged in a new study, centring on the (elementary ?) notion of function variation, with a few epistemologically founded hypotheses on its role in the long-term maturing of functional thinking. Small-scale empirical study conducted in 2007-2008 helped me specify the lines of inquiry; larger scale empirical study is now to consider. I would like to present here three related aspects of this work.

THE “GHOST CURRICULUM” HYPOTHESIS

Let us present some elements of the French syllabus for upper-secondary students who major in science. For our purpose, it is interesting to separate notions in two families, depending on whether they use “elementary” or “sophisticated” concepts⁹ :

For the sake of brevity we only presented in this table the list of notions, but it is absolutely necessary to complement it by an analysis of their ecology, an analysis for which the tools from Chevallard’s praxeology theory (task / technique / technology / theory) seem to us to be the relevant ones (Chevallard 1999). At university level, students usually start with a big recap of all they (are supposed to) know, with formal definitions and proofs of everything; then they move on to typically higher-education topics : Taylor series, Fourier series, differential equations etc.

Our hypotheses are :

- An analysis of *tasks* can show that, at high-school level, there is actually very little interplay between the two columns.
- The poor cognitive integration of the “basic” point-wise aspects of the “elementary” column (in particular : domain and variation) may be rather harmless at high-school level but turns into an obstacle (of mixed epistemological and didactical nature) in the secondary-tertiary transition. Empirical evidence is already available in (Praslon 2000).

⁹ For the sake of clarity : though we want to question the “elementary” nature of some concept (or, more precisely, conceptual elements of a body of knowledge), we will not choose the easy way out by saying “in the end, every mathematical concept is sophisticated and thorny” ... end of the story. The question of function variation is interesting because *there are good reasons* to consider it to be elementary (point-wise, proceptual etc.).

- The case of function variation is a typical case in which an element of the *concept image*¹⁰ is integrated early on and proves remarkably stable over the years, but the formal *definition* hardly plays any part¹¹.

Year	“elementary”	“sophisticated”
1 age 15/16	Basic notions/vocabulary about functions : function as abstract mapping, domain, graph, maximum and minimum, variation. Properties of basic functions : $x \mapsto ax+b, x^2, 1/x.$	
2 age 16/17	Composition of functions; theorem on the variation of composite functions.	Definition of the derivative, of tangents. Theorem (without proof) linking the variation of f and the sign of f' . Limits : intuitive notion for functions, formal notion for sequences. Sines and Cosines as functions.
3 age 17/18		Limits : formal definition for functions; definition of continuity. Exp and Ln functions. Integral calculus (based on a semi-intuitive definition of the integral). Completeness of the set of real numbers; proof of intermediate value theorem.

To be more specific, French students are taught the following definition : function f , defined over interval I , is an increasing (resp. decreasing) function over subinterval J if, for any two elements a, b of J , $a \leq b$ implies $f(a) \leq f(b)$ (resp. $f(a) \geq f(b)$); “increasing” means order preserving, “decreasing” means order reversing. Our hypothesis as to the poor integration of the *concept definition* in the *concept image* is twofold :

- Poor integration of the definition, even in the long term. We have two ways to test this empirically. The obvious one is to ask students (from high-school 2nd

¹⁰ We consider the notion of variation to be an element of the function concept.

¹¹ See, for instance, Vinner’s paper in (Tall 1991); or, for recent work on definitions (Ouvrier-Bufferet 2007)

year to University 3rd year) to define “increasing function”. We will also test students’ ability to recognise and name the concept they’re working with; in particular, at the end of an exercise in which, in several steps, it is established that inequalities of the $a \leq b$ type imply inequalities of the $f(a) \leq f(b)$ type, students will be asked to sum up in words what they have just proved.

- Easy integration in the concept image, from the outset. For instance, we would like to assess to what extent 1st year high-school students succeed when faced with the following task : given the graph of a function, compare $f(1)$ and $f(1,0001)$. This is a slightly unusual question (compare $f(1)$ and $f(2)$ would be a standard question), which reflects the intuitive perception of order preservation or reversing. Our hypothesis is that a high proportion of students do well when asked this question *even* before the formal definition is given, and that the proportion doesn’t change dramatically *after* the definition is given. This would mean that the fact that “variation has to do with order” is a strong *cognitive root*, but that it is not accepted *as a definition*. We have historical evidence in 19th century analysis that it can be considered obvious that variation has consequences in terms of order, without it being *defined* in terms of order (or defined at all, for that matter).

From the theoretical viewpoint, this work should contribute to the general reflection on the role of visual imagery in the building of formal concepts¹².

It is this large set of hypotheses, regarding both sets of tasks (and their evolution in upper-secondary and tertiary education) and issues of cognitive integration (or lack thereof) that we label the “ghost curriculum” hypothesis.

DIDACTICAL ENGINEERING

Our historical work on the 19th century allowed us to document a great variety of ways of expressing and dealing with function variation. We selected three of them on which to base didactical engineering for the introduction of the definition in the 1st year of high-school. All three rest on the “cognitive root” hypothesis, that is : it can be made intuitively clear to most students that variation (a word which they manage to use properly in semi-concrete or graphical contexts) “has something to do with order”.

Definition A : the official definition in the French curriculum (see above).

Though this definition relies only on the point-wise viewpoint and is consonant with a purely proceptual view of functions, the (somewhat hypocritically !) hidden double universal quantification is certainly a major obstacle. The other two definitions that

¹² See, in particular (Pinto & Tall 2002), where the understanding of quantifiers is also discussed. It should be noted that, with its two existential quantifiers, the definition of *functional variation* has different mathematical and cognitive properties from that of *limit*.

we're coming up with have to satisfy two criteria : (a) try to avoid this quantification problem (b) be equivalent to definition A (which is, eventually, what students are to learn).

Definition B : “function f is an increasing function over interval J ” means : whenever a list of numbers from J can be ordered $x_1 \leq x_2 \leq x_3 \dots \leq x_n$, then the images are similarly ordered : $f(x_1) \leq f(x_2) \leq f(x_3) \dots \leq f(x_n)$.

This definition clearly satisfies criterion (b), but it seems to be even harder to swallow in terms of quantification ! This may be true from a technical point of view but we have reasons to think it is not from a cognitive point of view. For one thing, it echoes ordering tasks which are familiar to students (as from primary school), thus adding the new abstract notion to the list of methods for ordering numbers. We have deeper epistemological reasons to support our claim, though. Definition A fundamentally rests on the idea that a function is a map between sets, variation properties being properties of maps between ordered sets. There are ways to teach the notion of abstract map (e.g. potatoes and arrows) but these are not taught in the current curriculum. Studying 19th century mathematics showed us how professional mathematicians used efficiently other function concepts than the map-concept. In what we described as a World of Quantity model (Chorlay 2007(a), 2008), the basic notions are not “set” and “map” but “variable quantity” and “dependence between two quantities”. To make a long story short, a single quantity can “vary”, and two dependent quantities x and y have dependent variations. This different conceptual frame leads to different definitions and different proof-styles; it also rest heavily on a specific semiotic register (DIDIREM 2002) which we called the “narrative style”. Our definition B was suggested by both this theoretical frame and semiotic register, thus resting to some extent on the idea of a variable quantity which we feel the long $x_1 \leq x_2 \leq x_3 \dots \leq x_n$ chain expresses in a discrete fashion : it should smooth out the transition from the purely intuitive grasp of (continuous) variation of a *single* quantity to the purely discrete mapping-between-ordered-sets formulation of definition A (which expresses no idea of “variation” whatsoever). The extent to which definition B really reflects what is found in the 19th century is a deep question, but we have no time to go into that here. Let us move to

Definition C : “ f is increasing on interval $[a,b]$ ” means that for every number c between a and b , $f(c)$ is the maximum of f on interval $[a,c]$.

Again, this definition satisfies criterion (b) (a two-line proof based on transitivity of order does the trick); it satisfies criterion (a) since we are down to one universal quantifier instead of two : it can thus help us asses to what extent the *double* quantification of definition A is a specific obstacle. The cognitive root this time is not that of “continuously variable single quantity” but that of maximum, which is part of

the official curriculum¹³. Actually we worked out this definition on the basis of Cauchy's conception of function variation¹⁴.

We should start testing teaching *scenarios* based on definitions B and C as steps towards definition A with 1st year high-school students next academic year, though we still have engineering work to do.

LONG-TERM ASSESMENT OF COGNITIVE VERSATILITY

This work on *definitions*, their formulation and their integration in the concept image, is not the only relevant aspect; understanding, remembering and identifying (whether proactively or retroactively) a definition are not the only necessary skills for a versatile thinker : devising counter-examples for incorrect assertions, recognising and proving the equivalence of different formulations of the same concept, understanding complex proofs, devising simple proofs ... are also essential skills, especially in the transition from secondary to tertiary education. We have several leads regarding these aspects, some of which we started testing in 2007-2008. Let us mention three.

The first two rest on a list of pairs of statements, from which we give three examples here : f is a function which is defined over $[0,1]$

	True	False
If f increases on $[0,1]$ then $f(0) \leq f(1)$		
If $f(0) \leq f(1)$ then f increases on $[0,1]$		

	True	False
If f increases on $[0,1]$, then $f(x)$ decreases as x decreases		
If $f(x)$ decreases as x decreases, then f increases on $[0,1]$		

	True	False
If f increases on $[0,1]$ then, for any two distinct numbers a and b (between 0 and 1), $\frac{f(b) - f(a)}{b - a}$ is positive		
Reciprocal of the former		

¹³ However, this formulation might cause cognitive dissonance : students usually come across maxima which are also local maxima, what is not the case in this definition.

¹⁴ See (Cauchy 1823), p.37. Cauchy's viewpoint was local, but we opted for a global formulation.

We have a list of 12 such pairs in which levels of abstraction, cognitive roots, and semiotic registers vary. This pool of (pairs of) statements can be used in at least two different ways. We used it last year to ask 2nd year high-school students to devise graphical counter-examples when they deemed the statement to be false. This work on graphical counter-examples is interesting since it promotes a deeper understanding of the concept without trying students' ability to devise formal written arguments using quantifiers (and negations of implications, and the like). In contrast, we will use some of these pairs (or definitions A, B and C) with more advanced students in order to assess their ability to devise written formal arguments for the statements they deem to be true : these should be tested with senior high-school students, undergraduate university students, and pre-service maths teachers. Using the same pool of statements at different levels in upper-secondary and tertiary education should help us gain insight into stages of cognitive maturity.

The third lead concerns the proof of the following theorem : Let f be a differentiable function, defined on interval I ; if f' is positive on I then f increases on I . The proof which is usually taught at university level first appeared in the 1850s¹⁵ but we documented many other “proofs” in the 19th century, most of which are flawed. We were quite fascinated though by Cauchy's proof, which is not flawed yet differs significantly from our standard proof, both in proof-pattern and view of function variation. What field-work is to be based on this material is yet to be determined.

CONCLUSION

We presented the outline of a new research project which, to some extent, is the sequel of a former historical and epistemological work¹⁶. We identified a series of questions which directly bear on issues of teaching and learning at upper-secondary and tertiary levels; they naturally fit within the research field on AMT in terms of maths topics (mathematical analysis) and didactical issues (cognitive versatility, proof, concept image / concept definition dialectics). The specific topic of function variation is but a tool to assess the conditions for successful learning of function theory, conditions which we assume partially rest on the understanding of seemingly elementary (point-wise, procept-compatible) notions. Exciting field work is now ahead of us.

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¹⁵ But its links with properties of the real line such as completeness or local compactness became clear only after Weierstrass' work on the maximum theorem (Chorlay 2007(b)).

¹⁶ It can be emphasised that the link between history and pedagogy in our projects (either the former one or the new one) is not one of the standard and well-identified links (see, for instance, (Barbin 2000)).

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EXPERIMENTAL AND MATHEMATICAL CONTROL IN MATHEMATICS

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This paper talk about a problem which can put students in the role of a mathematical researcher and so, let them work on mathematical thinking and problem solving. Especially, in this problem students have to validate by themselves their results and monitor their actions. The purpose is centred on how students validate their mathematical results. I also present the first results of my experimentations. So, this paper is related to learning processes associated with the development of advanced mathematical thinking and problem-solving, conjecturing, defining, proving and exemplifying.

BACKGROUND

The maths à modeler team (www.mathsamodeler.net) is developing a type of problem for the classroom called RSC [1] (Grenier & Payan, 1998, 2002 ; Godot, 2005 ; Ouvrier-Bufferet, 2006). The aim of a RSC is to put students in the role of a mathematical researcher. Grenier and Payan (2002) define a RSC as a problem which is close to a research one and, often, only a partially solved problem. The statement is an easy understandable question which is situated on the outside of formal mathematics. Initial strategies exist, there are no specific pre-requisites. Necessary school knowledge is, as much as possible, the most elementary and reduced. But, many strategies to put forward the research and many developments are possible for the activity and for the mathematical notions. Furthermore, a solved question, very often, postponed to new questions.

A RSC seems very interesting for gifted students because it is a challenging problem where they can find new results and be confronted with uncertainly and doubt. However, a RSC was not developed to be used only by gifted students, a RSC is for all the students and the goal of a RSC is not only to challenge students but, firstly, to make them work on mathematical thinking and especially “transversal knowledges and skills” which means: Experimenting, Conjecturing, Modelling, Proving, Defining...

So, in a RSC, students are confronted with an open-field where they have to make their own investigations and validate by themselves their results and actions. They have also to manage their research, for example by trying to solve sub-problems or easier ones instead of the initial problem. Moreover, it can also be a way for students to develop their problem solving skills as it can be considered as a “non-routine” problem.

In French handbooks, it seems that problems do not give the responsibility of the validity of their results to the students. Whereas, it is important for students to be

confronted with uncertainty and doubt in mathematical problems because first, they have to control their results to be sure that they are true. Second, they have to convince themselves and their colleagues that their results are true. So, even if they do not give a mathematical proof, they enter in a phase of argumentation which can let them give mathematical arguments like counter-examples. Third, they have to monitor more carefully their actions as they do not know a solution or a plan to solve the problem.

So, a RSC is a type of problem which can give responsibility to the students. But a RSC can also let students work on definition (Ouvrier-Bufferet, 2006), modelling (Grenier & Payan, 1998), experimental approach (Giroud, 2007) and more generally on transversal knowledges and skills.

In this paper, I present a RSC, *the game of obstruction*, which is a discrete mathematics optimization problem. This problem is only partially solved. I propose this problem for 2 reasons: let students work on mathematical thinking and problem solving, and in his quality of very challenging problem.

I give a mathematical and didactic analyses of the problem. I also propose results of my experimentations that will be centred on how students control their mathematical results, especially with these types of control:

Different types of results control in mathematics

The experimental control: Dahan (2005) claims that there exists 2 types of experimentations in mathematics: generative experimentations, which are experimentations that we carry out to generate facts when we have no idea of the result ; and checking experimentations that we carry out to check an hypothesis [2] or a conjecture. So, the checking experimentation can be a way to control the results. But unfortunately, even if a result is experimentally checked as true a lot of time, it can be false. In mathematics, we need a proof. However, we can use the experimental validation before going to the proof stage to convince ourselves that the result is true.

For example, if we do not know whether the Goldblach conjecture: *all even number superior to 2 can be written as the sum of two prime numbers*, is true, we can control this proposition by carrying out checking experimentations on 2, 4, 6, 8, 1284... And as we seen that each times it works, it can convince us that the conjecture is true.

The mathematical control: the mathematical control is what we call proof. We can not have a “better” control.

It is essential to have a proof to name a fact theorem, for example the Goldblach conjecture is true for all even number higher than 2 and lower than $4 \cdot 10^{14}$ (Richstein, 2000) but we can not call it theorem because we do not have a proof for all even numbers.

We have also others types of control, for example if an analogue problem is known to be true.

But here, the 2 types of control that I will consider are the experimental and mathematical control.

These 2 kinds of control, mathematical and experimental, do not contradict each other. Considering Polya's distinction between plausible and demonstrative reasoning (1990), it appears that the experimental control is part of the plausible reasoning whereas the mathematical control is part of the demonstrative reasoning. And as Polya (1990) claimed:

Let me observe that they do not contradict each other; on the contrary, they complete each other.

Indeed, in mathematics both are useful, we can use the experimental control to estimate the plausibility of a result and we need the mathematical control to be completely sure.

Now, I present the theoretical framework that I use to make my analysis.

THEORETICAL FRAMEWORK

I recall briefly what is a didactic variable. For Brousseau (2004), a didactic variable of a problem P is a variable which can change the solving strategies of P and which can be used by the teacher. So, by using the didactic variable the teacher can change the knowledge in game in P for the students.

I also use the notion of research variable (Grenier & Payan, 2002 ; Godot, 2005). A research variable of a problem P is a variable of P which is fixed by the students. The didactic choice for the teacher is to choose which variables of P will be used as research variables. This choice is made by considering the questions, conjectures, proofs that these variables could generate. In a RSC, there are research variables as it can let students manage their research.

The notion of didactic contract (Brousseau, 2004) is also used. The didactic contract corresponds with the implicit relations between the students and the teacher. An example in French classrooms is when students learn the factorization of polynomials, when the teacher asks a student to factorize $4X^2+4X+1$, the answer that the teacher wishes is $(2X+1)^2$ not a factorization like $4*(X^2+X+1/4)$ which is, even, a right factorization but not a factorization in irreducible polynomials which is implicitly asked.

And to analysis the experimentations, I use the framework developed by Schoenfeld (2006) to analysis mathematical problem solving behaviour:

the key elements of the theory are:

- knowledge;
- goals;
- beliefs;

- decision-Making.

The basic idea is that an individual enters any problem solving situation with particular knowledge, goals, and beliefs. The individual may be given a problem to solve – but [...] what happens is that the individual establishes a goal or set of goals – these being the problems the individual sets out to solve. The individual's beliefs serve both to shape the choice of goals and to activate the individual's knowledge – with some knowledge seeming more relevant, appropriate, or likely lead to success. The individual makes a plan and begins to implement it. As he or she does, the context changes: with progress some goals are met and other take their place. With the lack of progress, a review may suggest a re-examination of the plan and/or re-prioritization of goals. [...] This cycle continues until there is (perceived) success, or the problem solving attempt is abandoned or called to a halt.

THE GAME OF OBSTRUCTION

The situation was suggested by Sylvain Gravier. In order to present the problem we will need some useful definitions. A (n, c) -card game (or for short *card game*) is a set of cards having n lines, each of which contains a color in $\{1, \dots, c\}$.

Given a (n, c) -card game, the color of the i^{th} line of a card C will be denoted by C_i . A *bad line* in a set of 3 cards C, C' and C'' is a line i for which either $(C_i = C'_i = C''_i)$ or $(C_i \neq C'_i \neq C''_i \text{ and } C_i \neq C''_i)$.

An *obstruction* is a set of 3 cards such that **all lines are bad**.

Now the problem can be stated as follows:

Given two integers n and c , find the largest (n, c) -card game which does not contain an obstruction. (P1)

Some examples:

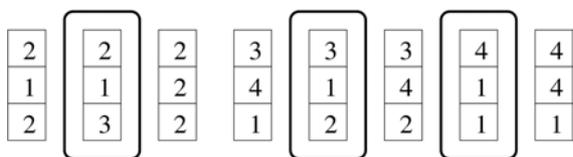


Figure 3: A $(3,4)$ card game containing an obstruction

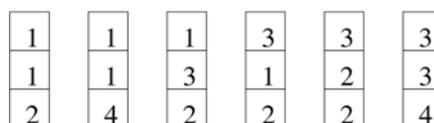


Figure 4: An obstruction-free $(3,4)$ card game

First, one can observe that: *one may consider a card game for which all the cards are distinct*. Indeed, given an obstruction-free card game of cardinality m for which all the cards are distinct, by duplicating each card, we obtain an obstruction free card game of cardinality $2m$. Conversely, there are no 3 copies of the same card in an obstruction-free card game.

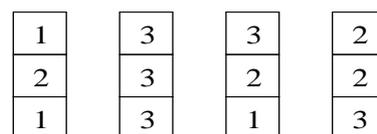


Figure 1: A $(3,3)$ card game

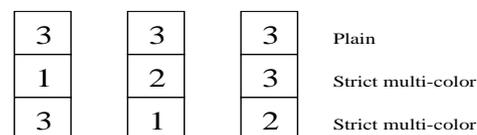


Figure 2: An obstruction

According to that, we will now only consider card games for which all the cards are distinct. The cardinality of a largest (n, c) -card game with no duplicated cards will be denoted by $Max(n, c)$.

Mathematical analysis

It is worth noticing that (P1) is still an unsolved problem so before trying to solve it one may study a weaker version: (P2) *How can we build a set without obstruction?* (P2) problem suggests determining an efficient method (algorithm) to check if a given set of cards contains an obstruction. I will denote this problem by (P3).

Another way of simplification will be to fix n and/or c . To work on optimization problems, we need to consider the following problem: (P4) *How can an upper bound be found?*

(P2) and (P4) split (P1) into the two aspects of an optimization problem: lower and upper bounds.

Unfortunately, since (P1) is still not solved, we do not have yet a general strategy to solve (P4) efficiently. Mainly, a strategy (SP4) to answer (P4) is based on *enumerating all possible obstruction-free card games*. For a low value of n , an easy enumerating argument shows that theorem:

Theorem 1: *For any integer $c \geq 2$, we have $Max(1, c) = 2$ and $Max(2, c) = 4$.*

Now, I present some strategies to solve our problems. First, concerning (P3), a “naïve” way would be to check all sets of 3 cards among a given card game. Nevertheless this strategy fails when the number of cards m is large since it requires $O(m^3)$ cases to be explored. Nevertheless, a strategy based on the structure of the given card game exists. For i in $\{1, \dots, c\}$, the i -block of a card game G is the subset C^1, \dots, C^t of G such that $C^1_1 = \dots = C^t_1 = i$.

(SP3) *First check that each block does not contain an obstruction (you can apply this strategy recursively). Secondly, search obstructions that have at most one card per block.*

In general, this strategy is no more efficient than the “naïve” way. Nevertheless, it appears that for large obstruction free card game G , the colours are recursively and equitably distributed on each block, therefore (SP3) checks in $O(\text{Log}_c(m)^3)$ steps that G has no obstruction.

Another interest for using (SP3) is that it allows first results on $Max(n, c)$ to be obtained. Indeed, consider an obstruction-free (n, c) -card game, then each block is at most $Max(n-1, c)$ in size. Therefore $Max(n, c) \leq c \cdot Max(n-1, c)$, which gives an answer to (P4).

Moreover, from an obstruction free $(n-1, c)$ -card game G of cardinality t , one can build an obstruction free (n, c) -card game of cardinality $2t$. Indeed, for $i=1, 2, \dots, c$, consider the obstruction free (n, c) -card games G_i obtained from G by adding a line to

each card and assigning color i to this new line. The set $G' = G_1 \cup G_2$ is an obstruction-free (n, c) -card game of cardinality $2t$, which gives an answer to (P2).

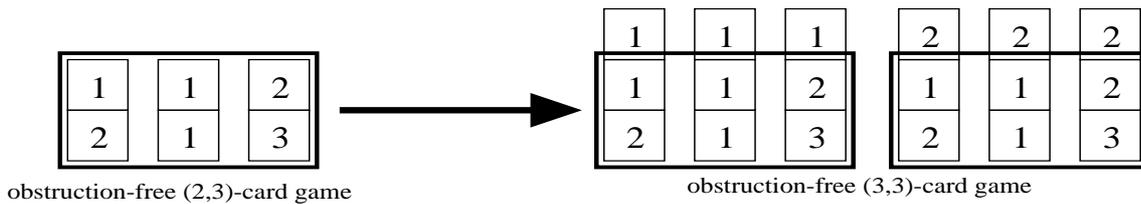


Figure 5: An example of the inductive construction based on SP3

These two remarks lead to:

Theorem 2: *Given integers n and $c \geq 2$, we have that:*

$$2 \cdot \text{Max}(n-1, c) \leq \text{Max}(n, c) \leq c \cdot \text{Max}(n-1, c).$$

Observe that for $c = 2$, we get: $\text{Max}(n, 2) = 2^n$. Notice that this result can be proof without theorem 2 by giving an inductive proof.

Nevertheless, when $c \geq 3$, one can find obstruction-free card game of larger cardinality than $2 \cdot \text{Max}(n-1, c)$. To find such obstruction-free card game one can apply “greedy” strategies:

(S1P2) *Start from an obstruction free card game G (it can be empty) and add a card C such that $G \cup C$ is still obstruction-free until there is no such card.*

(S2P2) *Start from a card game G and while there is an obstruction in G , remove a card from this obstruction.*

Observe that these two strategies give $\text{Max}(n, 2)$ since there is no obstruction in a $(n, 2)$ -card game. In general, an obstruction-free maximal card game G is built (i.e. for every card C not in G , $G \cup C$ contains an obstruction). It is worth noticing that (SP3) produces also obstruction-free maximal card game G , but this requires additional arguments. If one chooses an appropriate order for eliminating cards one can find an optimum of (P1) using (S1P2) or (S2P2). Of course, finding such an order remains an open problem. Nevertheless, when n is ‘large’, one may use a suitable order which ensures that one considers all possible cards ; for instance the lexicographic ordering. Unfortunately, even when $n=3$, the lexicographic ordering gives a maximal obstruction free $(3, 3)$ -card game of cardinality 8. However, by applying (S1P2) or (S2P2) with other orderings, one can find an obstruction free $(3, 3)$ -card game of cardinality 9 ($> 2 \cdot \text{Max}(2, 3)$). Similarly, one can exhibit an obstruction free $(4, 3)$ -card game of cardinality 20.

Moreover, by applying a (SP4) strategy one can prove:

Theorem 3: *$\text{Max}(3, 3) = 9$ and $\text{Max}(4, 3) = 20$.*

Didactic analysis

I decided to use n the number of lines and c the number of colours as research variables (Grenier & Payan, 2002 ; Godot, 2005). Since they can lead to new

questions like: what is the link between a n -line game and a $n+1$ -line game ? Trying to solve this question would provide an inductive construction of obstruction free card games which can be seen as an inductive proof. Moreover, it can let students generalize some results, especially with 2-colours. So, students can use these variables to manage their research.

There exists a more general problem than (P1), in which the size of an obstruction is a variable of the problem, but here, I decided to use it as a didactic variable by fixing its value to 3. I choose a size of 3 because for 1 or 2, the situation is very easy. It becomes sufficiently complex from 3.

Through mathematical analysis one can determine the following knowledge involved in solving (P1):

- The definition of an obstruction requires the understanding of **logic quantifiers**.
- (S1P2) and (S2P2) suggest using an **algorithmic** approach to solving (P2) using **eliminating ordering** (for example **lexicographic ordering**). Moreover, since these strategies build a maximal obstruction-free card game, one can discuss **local /global maximum**. Therefore, these strategies will produce solutions which can be conjectured as optimal.
- (SP3) allows a card game to be **modelled** which can be reinvested to (partially) solve (P2) and (P4) as shown in proof of Theorem 3. Moreover, (SP3) applied on (P2) gives an **inductive construction** of obstruction-free (n, c) -card game based on two copies of an obstruction-free $(n-1, c)$ -card game.
- (SP4) is an **enumerating** approach for solving (P4). To reduce the number of cases to be considered it will be convenient to use **variables** for the enumerating.
- The distinction between problems (P2) and (P4) is related to **lower and upper bounds** on an optimization problem (P1) which is closely related to **necessary and sufficient conditions**.
- Solving (P1) with $c = 2$ provides all possible 2^n cards in a card game on n lines to be **counted**.

OUR EXPERIMENTATIONS

Two experiments were carried out, one with a “seconde” (tenth grade) class, E1, and another with a “première scientifique” (eleventh grade) class, E2. Pupils worked in groups of 3-4. In each class, we let them search for 2 hours. The E1 experiment was carried out before the E2 one. We filmed one group in each experiment.

The problem was presented orally with examples on the blackboard. We gave to them some material with which they can experiment. In E1, we gave plain circles of 4 different colours and in E2, we added n -line cards with no colours and $n=1, 2, 3, 4$.

But in both experiments the problem is posed generally as (P1), we did not ask students to only use the number of colours or the number of lines that is given materially.

Results of experimentations

First, my analyses are focused on how one group of the tenth grade class tried to solve (P3), that is to say, how they control the presence of an obstruction in a card game.

They started by building an obstruction free card game with 3 lines and 4 colours with the additive strategy (*SIP2*). They built a card game G1 of cardinality 4 and then they added a card C. Then they searched obstructions in G1UC by trying to check “randomly” all triple of cards. They did not find any obstructions but they were not sure to have tested all triple. Here, the knowledge of how to find all triple is missing. Then, they formulated this question (P3a): *How can we know if all triples of cards were checked ?* They tried to answer (P3a) during one minute but they did not find a solution. After that, they concluded that they checked all triples of G1UC although they did not. Thus, they decided to give (P1) a higher priority than (P5). Seeing that they could not solve (P5) quickly and believing that their experimental control based on “checked all triples” is sufficiently efficient, they decided to rely on the experimental control.

During all the session they relied on the experimental validation for the obstruction's property although, I showed them obstructions in their card games. They did not decide to re-examine their plan by searching an other strategy to solve (P3) than “check all triples”. Despite that, they observed that this strategy is *too difficult to do* and that the experimental control based on this strategy was not efficient.

So, it seems they gave (P3) a lower priority than (P1). It joins Schoenfeld (1992) observations that students are more concerned about the initial problem than to sub-problems, although sub-problems can be key elements. And here, (P3) is key element to make progress on (P1). The group said 11 times that a card game was obstruction-free and it was true only once.

In the two experimentations, none of the group seemed to search an efficient method to answer (P3), they only used strategies based on “checked all triples”, although many of them were confronted to (P3). So it seems that students decided to rely on the experimental validation and not on the mathematical validation for the obstruction free property. An interpretation could be that students did not find a solution so they decided to rely on the experimental control to progress in (P1). However, for the group above, it seems, as they only search for one minute, that they decided to not spend too much time on (P5). So, they did not recognize the role of (P5) and (P3) for solving (P1).

Summarize of the experimentations:

It appears that the use of material during experiments E1 and E2 led pupils to carry out their own experiments in mathematics. Students started to manipulate and carry out experimentations to solve (P3) and (P2). Even if (P3) was identified, they stayed in the experimental control. Consequently, there were some group which did not obtain results on 3 lines. But, they made hypotheses or conjectures that they checked with experiments like “*this card game is maximum*”, “*by using this strategy, we build an obstruction free card game*” or “*with only 2 colours on each card, there are no obstructions*”, which allowed them to find counter-examples. Here, students are responsible of deciding the validity of their propositions. But for one group, it was not the case, they made an experimental control of the obstruction free property of their card game and after called us to validate their results. They did not take the responsibility of the result's validity. There was a problem in the didactic contract.

They proved $\text{Max}(n, 2)$ for $n=1, 2$ and 3 . But only one group generalized this result and this group made the 2 experimentations.

They used at most 4 colours and did not try to generalize with more. Moreover, they tried to use all the colours. Here, we can see a consequence of the didactic contract: use all that is given and not more. So, the didactic contract has to be changed to let students manage their research.

The concept of variable useful in an enumerating strategy like (SP4) was not discussed. Similarly no good eliminating ordering was proposed by the pupils ; they remained in a ‘naïve’ strategy.

BRIEF CONCLUSION

This situation was experimented with “ordinary” students and show that this problem can let students take the role of a mathematical researcher. Although they did not use the variables of the problem to try to solve easier sub-problems, they carried out experiments to try to answer their own questions, formulated conjectures and made proofs. Moreover, it seems, as in Schoenfeld (1992) studies, that contrary to an expert they have some difficulties to identify one of the key element to solve (P1) ; although they identified (P3), they relied on the experimental control.

Students did not work on all knowledges identified in the didactic analysis, especially the concept of variable which is a powerful abstract concept. We tested this situation on a longer time (18 sessions during one year). In this context, strategies (SP3) and (SP4) were developed and their corresponding results were obtained.

NOTES

1. RSC: Research Situation for the Classroom.
2. Here the definition of hypothesis used is: a proposition that we enunciate without opinion. It is not the same as the usual definition of a mathematical hypothesis,

3. In France, seconde corresponds at a tenth grade class, it is a general section. Première scientifique corresponds to a eleventh grade class and it is the scientific section.

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INTRODUCTION OF THE NOTIONS OF LIMIT AND DERIVATIVE OF A FUNCTION AT A POINT

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This paper contains the results of a pedagogical research devoted to the understanding of the notions of finite limit and derivative of a function at a point. In case of teaching limits, the effort spent by a teacher is not effective because for students the notion of a limit is very formal. This claim is supported by our pedagogical research using graphs of functions. We present also a concept of differentiable functions and derivatives. The notion of a differentiable function f at a point x is based on the existence of a function φ such that $f(x+u) - f(x) = \varphi(u)u$ for all u from some neighborhood of 0 and φ is continuous at 0. We show applications of this concept to teaching basic calculus.

INTRODUCTION

At present, the notions of limit and derivative of a function at a point is taught according to the Slovak curriculum in the last year of secondary school. In future, according to a new curriculum, this part of mathematics will be taught only at universities. In this article we will present some results of our pedagogical experiment with students at secondary school and university students - future teachers. We carried out the experiment at St Andrew secondary school in Ružomberok during the school year in the regular class according to official curriculum. Analogously, we carried out our experimental teaching of calculus to freshmen at the Pedagogical Faculty of Catholic University in Ružomberok during the regular calculus tutorial classes.

We base our didactical approach on the calculus teaching concept by Professor Igor Kluvánek. He was a well-known Slovak-Australian mathematician. He prepared a new course of mathematical analysis during his 23-rd year stay at the Flinders University in Adelaide, South Australia. Even though Kluvánek was a renowned researcher, an essential attribute of his lectures was his effort to present the calculus to students in a clear and simple way.

THEORETICAL BACKGROUND

In the field of Mathematics Education there is abundant literature discussing the problems of teaching and learning limit and derivative of a function at a point. The notions of limit and derivative are taught at Slovak secondary schools in the (senior) last year. In a Slovak textbook Hecht (2000) the notion of derivative is introduced in several parallel ways. One of them is via the tangent of a function at a point. This approach is according to Hecht static and it is based on finding of the tangent with the help of secant, which has two common points with the graph of the function. The first is the point of tangency and the second point is “in the limit movement” to the point of tangency. Hecht (2000) at this point introduced also the notion of the

functional limit. According to Tall & Vinner (1981) the limit of the function is often considered as a dynamic process, where x approaches a , causing $f(x)$ to get close to c . Conceptually, the differentiation may include a mental picture of a chord tending to tangent and also of the instantaneous velocity. The intuitive approach prior to the definition is often so strong that the feeling of the students is a *dynamic* one:

as x approaches a , so $f(x)$ approaches L

with definite feeling of motion.

Klůvnek (1991) in his concept of calculus teaching used the notion of continuity as a base notion. Klůvnek proposed to teach first the notion of continuity and with this notion he defines the notion of limit:

„It is not suitable to teach first the notion of limit of continuous variable and after this to define the continuity. Logically, it doesn't matter what of notions is first. However, there exists from pedagogical point of view a great difference. Each experienced teacher underlines that the limit of the function is not the value of the function at this point. The reason for this teacher's activity is: The teacher will not have problems by explaining the notion of continuity. The students cannot differentiate limit of the function at a point and study continuity of the function at a point.”

In case of teaching limits, the effort spent by a teacher is not effective because for students the notion of a limit is very formal. At this stage of teaching calculus, a teacher does not have big chances to use the notion of a limit as a prime notion of calculus. The next advantage of the continuity is the number of quantifiers. The definition of the limit of the function at a point can be written in the form:

A number k is said to be a limit of the function f at a point x if for every real number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every x satisfying the inequality $0 < |x - a| < \delta$ we have $|f(x) - k| < \varepsilon$.

This definition has four quantifiers and the definition of continuity has three quantifiers:

A function f is continuous at a point a if for every real number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every x satisfying the inequality $0 < |x - a| < \delta$ we have $|f(x) - f(a)| < \varepsilon$.

Klůvnek comes on and shows the following formal definition of continuity:

A function f is continuous at a point a , if for every neighbourhood V of the point $f(a)$ there exists a neighbourhood U of the point a such that for every $x \in U$ we have $f(x) \in V$.

This definition is possible to formulate with two quantifiers:

A function f is continuous at a point a if for every neighbourhood V of the point $f(a)$ there exists a neighbourhood U of the point a such that $f(U) = \{f(x) : x \in U\} \subseteq V$.

Now suppose we are given a function defined at every point of a neighbourhood of a point a with the possible exception of the point a itself. We may try to find a number k such that, if it is declared to be the value of the given function at a , then the function becomes continuous at a . Such a number k is then called the limit of the given function at the point a . Let us state the definition of limit more clearly and precisely.

Definition 1. Given a function f , a point a and a number k , let F be the function such that

1. $F(x) = f(x)$, for every $x \neq a$ in the domain of the function f ; and
2. $F(a) = k$.

The limit (left limit, right limit) of a function f at a point a is the number k such that the function F , defined by the requirements 1 and 2 is continuous (left-continuous, right-continuous, respectively) at a .

Similarly as in the case of limits, Kluvánek (1991) introduces the differentiation of a function at a point via continuity:

Definition 2. Let f be a function defined in some neighbourhood of a point x . A function f is said to be differentiable at a point x if there exists a function φ , continuous at 0, such that for every u in a neighbourhood of 0 we have $f(x+u) - f(x) = \varphi(u)u$. The value $\varphi(0)$ is called the derivative of f at the point x .

Klúvák shows also more practical interpretations of this definition. If the function $f(x)$ is interpreted as the law of motion of a particle on a straight-line, then x and $x+u$ represent instants of time and the values $f(x)$ and $f(x+u)$ the corresponding positions of the particle. The difference $f(x+u) - f(x)$ is the displacement of the particle during the time-interval between the instants x and $x+u$. The particle moves at a constant velocity given by the function $\varphi(u)$. The velocity is the rate of displacement.

Let $f(x)$ be the costs of producing x units of the given commodity, f is the costs function of this commodity and $\varphi(u)$ is the marginal costs.

Let $f(x)$ be the amount of heat needed to raise the temperature of a unit mass of the substance from 0 to x (measured in degrees). Then $\varphi(u)$ is the amount of heat needed to raise the temperature of a unit mass of the substance by one degree; $\varphi(u)$ is the specific heat of the substance.

Temperature extensibility can be approximated by linear function $l=l_0(1+\alpha\Delta t)$. The value of the function $\varphi(u)=l_0\alpha$ describes the change of longitude of a solid according to the unit change of temperature.

These definitions 1 and 2 of the limit and derivative of the function we use in our experimental teaching.

In Kluvánek's opinion, more proofs in calculus can be carried out easier and he criticised the proof in the course of pure mathematics in Hardy (1995), because Hardy used the limits instead of continuity.

Theorem. *If a function f is differentiable at a point x and a function g is differentiable at the point $y = f(x)$, then the composite function $h = g \circ f$ is differentiable at the point x and $h'(x) = g'(y) f'(x)$.*

Proof. Since f is differentiable at x , there exists a function φ continuous at 0 such that $\varphi(0) = f'(x)$ and $f(x+u) - f(x) = \varphi(u)u$ for all u in a neighbourhood of 0. Since g is differentiable at y , there exists a function ψ continuous at 0 such that $\psi(0) = g'(y)$ and $g(x+v) - g(x) = \psi(v)v$, for all v in a neighbourhood of 0.

Hence,

$$\begin{aligned} h(x+u) - h(x) &= g(f(x+u)) - g(f(x)) = \\ &= g(f(x) + (f(x+u) - f(x))) - g(f(x)) = g(f(x) + \varphi(u)u) - g(f(x)) = \\ &= \psi(\varphi(u)u) \varphi(u)u \end{aligned}$$

for every u in a neighbourhood of 0.

Let $\chi(u) = \psi(\varphi(u)u)\varphi(u)$ for every u such that $\varphi(u)u$ belongs to the domain of the function ψ . By properties of continuous functions, the function χ is continuous at 0 and our calculation shows that $h(x+u) - h(x) = \chi(u)u$ for every u in a neighbourhood of 0. Hence, the function h is differentiable at x and

$$h'(x) = \chi(0) = \psi(0) \varphi(0) = g'(y) f'(x) \quad \square$$

Kronfellner (1998) proposed to integrate history of mathematics in the teaching process. This is possible also in case of a derivative. Kronfellner (2007) used the next example of the derivative of x^3 according to Isaac Newton (1643 – 1627) from his “Quadrature of Curves”:

“In the same time that x , by growing becomes $x + o$, the power x^3 becomes $(x+o)^3$, or

$$x^3 + 3x^2o + 3xo^2 + o^3$$

and the growth or increments

$$(x + o) - x = o \text{ and } (x + o)^3 - x^3 = (x^3 + 3x^2o + 3xo^2 + o^3) - x^3 = 3x^2o + 3xo^2 + o^3$$

are to each other as

$$1 \text{ to } 3x^2 + 3xo + o^2$$

Now let the increments vanish, and their “last proportion” will be 1 to $3x^2$, whence the rate of change of x^3 with respect to x is $3x^2$.”

Popp (1999) presented Fermat's method of searching of extremes. This method is based on the fact that the difference between functional values $f(x)$ and $f(x + h)$ is small, because the number h is “near to zero”. We apply this to the quadratic function $f(x) = ax^2 + bx + c$:

$$\begin{aligned}
 f(x) &\approx f(x+h) \\
 ax^2 + bx + c &\approx a(x+h)^2 + b(x+h) + c \\
 ax^2 + bx &\approx ax^2 + 2ahx + ah^2 + bx + bh \\
 0 &\approx 2ahx + ah^2 + bh \\
 0 &\approx 2ax + ah + b
 \end{aligned}$$

Now if $h = 0$, then $0 = 2ax + b$ and $x = -\frac{b}{2a}$.

If we will find the derivative of a function f by this method, we can use the interpretation of derivative as a slope of the tangent of the function f . For this reason we use the function $g(x) = f(x) - sx$. Now we calculate the derivative of the function $f(x) = x^2$. In this case $g(x) = x^2 - sx$. We use now similar algorithm than by quadratic function:

$$\begin{aligned}
 g(x) &\approx g(x+h) \\
 x^2 - sx &\approx (x+h)^2 - s(x+h) \\
 x^2 - sx &\approx x^2 + 2hx + h^2 - sx - sh \\
 0 &\approx 2hx + h^2 - sh \\
 0 &\approx 2x + h - s
 \end{aligned}$$

Now if $h = 0$, then $0 = 2x - s$ and $x = \frac{s}{2}$ or $s = 2x$. This result is very similar to $y' = 2x$.

The problem of Fermat's method is that it is partially not correct. The number h is used in different senses. First, it is the finite number which we use for division. After the division we suppose $h = 0$. Popp expect that this problem solved in the history of mathematics Gottfried Wilhelm Leibniz, but the complex solution is provided by the nonstandard calculus.

EXPERIMENTAL TEACHING

Barbé J., et al. (2005) described two basic didactical aspects of teaching limits. The first is *algebra of limits*. It assumes the existence of the limit of a function and poses the problem of how to determine its value – how to calculate it – for a given family of functions. This aspect prevails in Slovakia. Unfortunately a lot of students calculate the limits mechanically without understanding.

The second aspect *topology of limits* emerges from questioning the nature of “limit of a function” as a mathematical object and aims to address the problem of the *existence of limit* with respect to different kind of functions. This aspect is seldom used in Slovakia. Similar situation is also when teaching of derivatives.

We carried out an experimental teaching devoted to understanding by students the notions of finite limit and derivative of a function at a point. We will stress to

students not to calculate the limits and derivatives mechanically. We stress to students the existence and non-existence of limits and derivatives. We use in our experimental teaching the calculus concept developed by Professor Igor Kluvnek. Our experimental group consisted of 27 students of the St Andrew secondary school in Ruomberok.

The goal of the research was also to analyze the students' mistakes and to find their roots. The problems we have solved with students are usually not contained in typical mathematical textbooks. In this article we describe qualitative research using excerpts from student answers in the framework of *field notes* method.

The notion of the limit we introduced by the definition 1 via continuity of the function at a point. We used this definition for the examples, which we solved with students using graphs. For this approach we have been inspired by Habre & Abboud (2005). They show that the students have a better capability of handling the difficulties with derivatives, if they assimilated the notion of derivative visually.

$$\text{Dominik: } \lim_{x \rightarrow 3} (2x + 3) = \quad D(f) = R \quad F(x) = \begin{cases} 2x + 3 & \text{for } x \neq 3, \\ L & \text{for } x = 3. \end{cases}$$

Teacher: Sketch the graph of the function F for $x \neq 3$.

(Dominik sketched the graph, see Figure 1)

Teacher: What we have to do in order that this function becomes to be continuous?

Miroslava: We fill the circle.

Teacher: Which functional value at the point 3 do we use? What does it mean for the limit of the function at the point 3?

Dominik: 9 and so $\lim_{x \rightarrow 3} (2x + 3) = 9$.

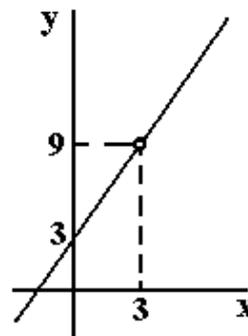


Figure 1

$$\text{Erika: } \lim_{x \rightarrow 3} \frac{1}{x-3} = \quad F(x) = \begin{cases} \frac{1}{x-3} & \text{for } x \neq 3, \\ L & \text{for } x = 3. \end{cases}$$

Teacher: Is it possible to find the value $F(3)$ so that this function becomes to be continuous?

More students from the class: It's impossible.

Teacher: What does it mean for the limit of the function at the point 3?

Erika: It doesn't exist.

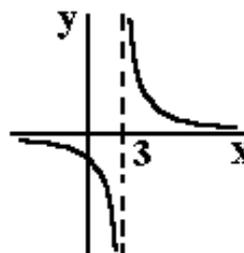


Figure 2

In the similar way the students calculate with the help of graph the limit $\lim_{x \rightarrow 3} \frac{2x^3 - 54}{x - 3}$. After this example the students calculate the limits without graphs and this teaching unit we ended by the following example:

Example 1. Which of the following functions has limit at the point 1? Describe your argumentation.

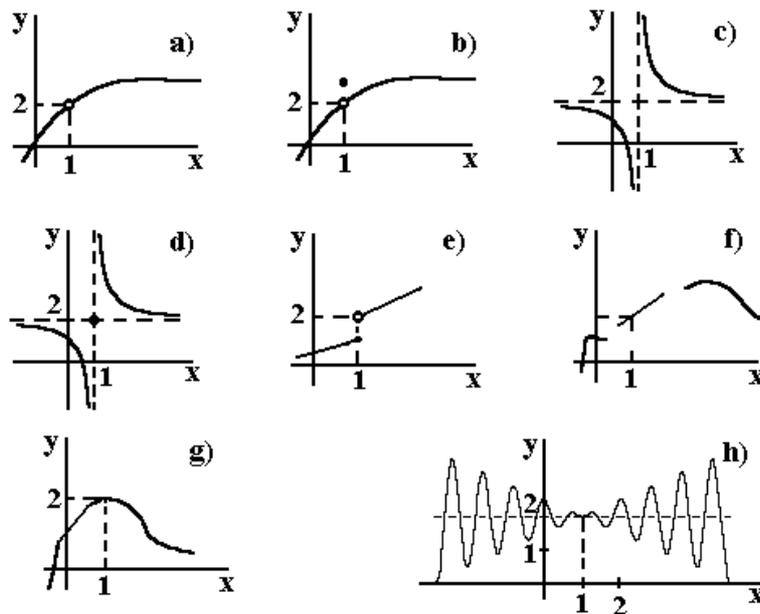


Figure 3

Every student made some mistakes. One half of them wrote, that the function in a) has limit. In b) only 3 students did so. It was difficult for students to understand that if the function is not continuous at one point and has some functional value at this point, then this function can have a different limit at this point. Three quarters of students wrote the correct answer that the function in c) does not have a limit. One student wrote that the function in d) has a limit because this function is defined at the point 1. Similar mistake committed 20 percent of students in e). In f) and g) 25 percent of students wrote that these functions are continuous at the point 1 and wrote nothing about the limit. The function in h) was difficult for three quarters of students. They wrote that this function hasn't a limit at the point 1, one student wrote that this function is not continuous at the point 1.

Similar conception to build a notion in calculus teaching via continuity was used when we introduced the derivative of the function at a point. The function $\varphi(u) = \frac{f(x+u) - f(x)}{u}$ from Definition 3 was replaced by the function of the slope of chord given by formula $s_{f,a}(x) = \frac{f(x) - f(a)}{x - a}$. We illustrate our procedure in next example.

Teacher: Calculate the derivation of the function $y = x^2$ at the point 1 from the definition!

$$\text{Robert: } y = x^2, a = 1. \quad s_{f,1}(x) = \begin{cases} \frac{x^2 - 1}{x - 1} & x \neq 1, \\ k & x = 1. \end{cases} \quad \frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1} = x + 1$$

$$s_{f,1}(x) = \begin{cases} x + 1 & x \neq 1, \\ k & x = 1. \end{cases}$$

Teacher: Do you know to describe the graph of the function $y = x + 1$?

Robert: The line.

Teacher: More precisely.

Robert: The straight line.

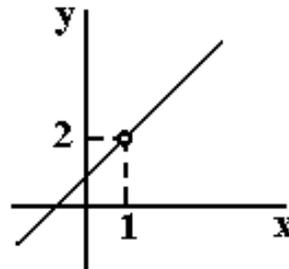


Figure 4

Teacher: What is it possible to add so that the previous function becomes continuous?

Miroslava: We have to fill the circle.

Teacher: How?

Ivan: By number 2.

Teacher: What does it mean for the value of derivation of the function $y = x^2$ at the point 1?

Robert: It is equal to 2.

Teacher: We considered functions with derivation at every point of the domain. Now, we are going to deal with functions having no derivation at least at one point.

$$\text{Pavol: } f'(2) = |x - 2| \quad f'(2) = ? \quad s_{f,2}(x) = \begin{cases} \frac{|x - 2|}{x - 2} & x \neq 2, \\ k & x = 2. \end{cases}$$

$$x \in (2; \infty): \frac{|x - 2|}{x - 2} = \frac{x - 2}{x - 2} = 1 \quad x \in (-\infty; 2): \frac{|x - 2|}{x - 2} = \frac{-(x - 2)}{x - 2} = -1$$

Teacher: Is it possible to extend the function (to define its value at 2) so that it becomes continuous?

Lukáš, Lucia: No, it isn't.

Teacher: What does it mean for the derivation at the point 2?

Pavol: It doesn't exist.

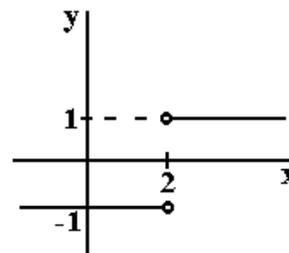


Figure 5

We worked now with derivative of polynomial functions and after we give the students following example:

Example 2. Which function of the next functions (see Figure 6) has the property $f'(3) = 2$?

Only 15 percent of student correctly solved this example. The correct answer in a) had 90 percent of students, but incorrect answer in b) had 60 percent and incorrect answer in d) had 40 percent of students. The correct answer f) had 25 percent of students. Nobody had incorrect answers c) and e).

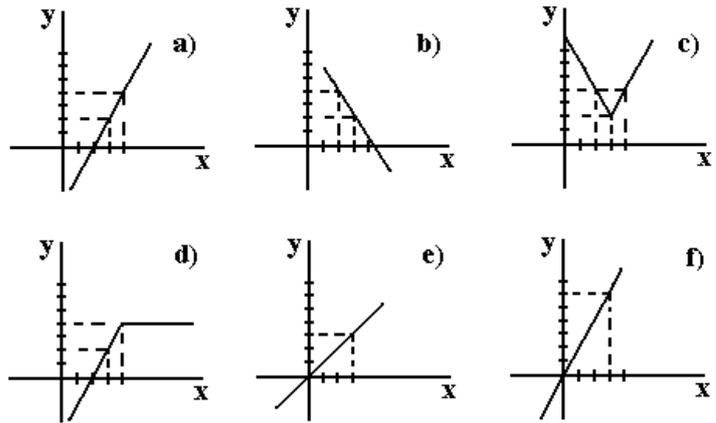


Figure 6

CONCLUSIONS

At the end we borrow few lines from Kluvánek (1991):

“If the reader does not value mathematics and mathematical analysis more than a comfortable feeling that the way calculus is taught at his and other famous universities is essentially all right, then for him the present paper does not have much to say.”

We feel that the quality and the amount of intellectual activities needed to transform the mathematics understood (limit and derivation of a function at a point) into the mathematics suitable for teaching should never be undervalued. The effort needed to understand mathematical knowledge matches the effort to invent it. If one wants to write a good mathematics textbook, he has to carry out a mathematical research in the usual sense of the word. In our paper we wanted to follow the idea cited above. From the historical point of view very similar approaches is possible to find by Karl Weierstrass (1815 – 1897), because in his lectures of 1859/60 gave *Introduction to analysis*.

We believe that practically there is not sufficient effort to understand problems related to the existence of a limit and a derivation of a function at a point. Our approach makes teaching basic notions and solving problems easier. Students are able to solve most of problems applying the before mentioned method.

The exploitation of graphs provides opportunity to solve and calculate limits and derivations of a function at a point without mechanical calculations. Graphs of functions not only provide easy specification of the value of limit and derivation of a function at a point, but they lead to visual understanding of its nonexistence, too.

We are agree with results in Tall D. et al. (2001) in the sense that teaching limits and derivatives should be done in the wider context of learning mathematics through arithmetic, algebra, calculus and beyond. We show that it is possible to build the notions not mechanically, but with understanding. In our experimental teaching we

also carried out an output test which shows that the visual representation of limits and derivative helps students to solve the examples devoted to understanding the notions in question (especially existence and non-existence of limits and derivative).

Visual representation of calculus notions is important in the international studies such PISA and TIMSS. Interesting research about using graphs in the teaching process can be found in Cooley, Baker, & Trigueros (2003).

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FACTORS INFLUENCING TEACHER'S DESIGN OF ASSESSMENT MATERIAL AT TERTIARY LEVEL

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We study the process of design of examination papers in the first year of French university and identify some institutional constraints and some teachers' beliefs that influence this process.

Keywords: university expectations, teacher's collective work, documentary genesis, assessment material

INTRODUCTION

Numerous research works considered the difficulties met by the universities' first-year students. These works identify various reasons for those difficulties, offer various interpretations and develop various means of didactic action. The attention of researchers was initially centred on the new knowledge met and was then devoted to the new reference consisting in the practices of the expert mathematicians. It eventually moved upon new institutional expectations (see for a synthesis Gueudet on 2008). It led in particular to observe that students' private work is focused on learning how to mimic techniques, whereas teachers expect that students develop a real mathematical autonomy (Lithner 2003, Castela 2004).

The researchers who made those reports highlighted a difference between teachers' expectations and institutional expectations, the latter being particularly visible through the exam subjects. Those would in fact be organized around the mimicking of methods studied during the tutorials. As teachers of the tutorial write the examination texts, the latter would choose to question students on simple contents, such as exercises similar to those studied and corrected in class, notably to avoid a too important failure. Yet, the impact of evaluations on the work of students is very important (Romainville 2002). Besides many innovative teaching designs propose new assessment modes, such as group projects with oral examinations (Grønbaek and Winsløw 2006).

Here we do not wish to suggest an innovation, but simply to investigate whether examinations are really related to the mimic of methods. In the case of a positive response, we try to understand why university teachers propose such evaluations. This preliminary study will allow us to propose other modes of assessments.

This paper is directly related to the themes of CERME 6 group 12, adopting a mathematics-centered perspective about the teaching at tertiary level, and considering the important part of effective teaching settings constituted by assessments.

The development of an examination text is a documentary work, implying various resources, generally carried out in a collaborative way by a team of teachers. The

documentary approach of didactics (Gueudet and Trouche in press) showed that such a work was influenced by beliefs, expectations, etc., of teachers, and that documents resulting from this work influenced in return these beliefs (Cooney 1999). This process of both development of documents (here of examination texts) and evolution of teachers' beliefs depends strongly on the institutional context. The institution indeed influences its actors through a system of conditions and constraints which can be very general or related to precise contents (Chevallard 2002) and which shape the knowledge within the institution.

Considering this point of view, we chose to study a first-year mathematics course in a French university, for which we followed the development processes of the examination texts. In section II, we present this tutorial and our methodology. We noted that the assessment relates only to the mimics of techniques. Thus, our central question here is the following one:

Which institutional conditions and constraints and which beliefs of the teachers control the choices carried out during the development of the examination papers?

We give some elements of answer by analyzing in section III the institutional constraints, conditions, and beliefs of teachers who lead to the choice of a specific exercise. In section IV, we illustrate the consequences of these constraints through the successive evolutions of the statement of a given exercise, and also show the phenomena of inertia related to the manner in which the examination papers are developed.

Finally, we conclude by evoking possible clues for an improvement of the assessment practices that could foster the students' mathematical activity.

CONTEXT AND METHODOLOGY OF THE STUDY

We study more particularly a mathematics course from the first semester in a French university. This course is devoted to students graduating in physics.

During the first semester, students follow six courses, only one being in mathematics. Our choice came from the author's involvement in the course. We initially thought that the context (teaching mathematics to Physics students) could lead to exercises coming from physics situations in the examination papers. We quickly noted that it occurred neither in the tests, nor in the sheets of exercises. We will not improve this question here.

To help with the secondary-tertiary transition, this course - like all those of the first semester - is organized in small groups of about thirty students (five groups), each group having a unique mathematics teacher. The course is 4 hours a week over 12 weeks. To ensure coherence between the various groups, a blow-by-blow program (the topics studied are specified, as well as the time that should be devoted to them) is given to each teacher and the sheets of exercises are the same for every group. Both

program and sheets of exercises come from the background of the teachers involved in this course during the two previous years.

The contents were chosen according to the mathematical tools necessary in the other courses: complex numbers, study of functions, Riemann integrals, first and second order linear differential equations. It thus contains secondary level knowledge in each of the first three topics, with each time a deepening and new knowledge: n^{th} roots of a complex number, inverse of trigonometrical functions, change of variables in an integral... All these topics are introduced to solve some kinds of differential equations.

The assessment consists of two one hour-long exams at the end of week 5 and of week 9 and of a two hours-long final one at week 12 (just after the end of teaching).

The mark of a student is the maximum mark between the final exam and a weighted average of the three tests (1/4 for each one hour exam, 1/2 for the last one). Indeed this topic should deserve a specific study and we will not study it in this article. Students who don't succeed have a resit, but we focused on the three tests that gave the first final mark.

The development's work of examination texts is shared out at the beginning of the course among the teachers: the first exam (CC1) was entrusted to Omar and Georges, the second one (CC2) to Omar and Thierry while the final examination paper (E) was prepared by Marc (responsible for this course), Thierry, Georges and Marie-Pierre (author of this paper). In the three cases, the appointed teachers initially worked together before proposing an almost finished text to the other ones.

The data were gathered through interviews (appendix A) of teachers involved in a same exam, initially before the development work to question them about their intentions, then to discuss their choices afterwards. We paid attention on the following points: coordination between the teachers and supports used for the development of the text, choices for the contents of this one and objectives that guided these choices.

We now will present the analysis of the gathered elements.

CONSTRAINTS AND BELIEFS: REASONS FOR IMPLEMENTATION OF METHODS

The examination texts given since September 2004 (i.e. during 4 academic years) are mainly composed of exercises aiming to the use of methods learned during this course. In this section we detail various aspects of this choice, and the reasons for it, by illustrating our point with an exercise, which seemed to us emblematic.

Texts of assessment: agglomerates of short exercises

Each examination paper is made up of a list of short exercises: it never relates to one or two long problems. Various reasons lead to this choice. First, the duration of

exams (1 h or 2 h) is limited (the mathematics exam of the French end of secondary school certificate for scientific students, "Baccalauréat S", lasts 4 hours). This duration is an institutional constraint of general level; in particular, the 3 hours examinations were gradually removed at the University of Rennes 1 in order to make possible two examinations in the same half-day: it optimizes the occupancy of the rooms of examination and the working time of the university porters. This optimization is crucial because of the increase in the number of exams. Indeed, it is observed "the bursting of the academic year in semesters and the courses in units of teaching involved an increase of the number of evaluations" (Gauthier & al 2007)

Beyond this *time constraint*, a big factor emerges from our interviews, factor which deals with the objectives that the teachers assign to assessment, and thus of what we name under the generic term of belief: an evaluation must include all the parts of the previous program, particularity that we will name the *belief of exhaustiveness*. Omar stresses that an assessment must make it possible for the student to have a diagnosis of his knowledge: any gap could then be filled before the following tutorial. This diagnosis must thus be complete. This argument is not valid any more for the final examination; however, Marc regards as very important the fact that the examination paper covers all the contents, on the one hand to force the students to revise everything, and on the other hand "to draw a distinction between those who have been working enough and those who have not". However, the content of this course is divided in five chapters: this is also an institutional constraint, which relates more directly to the mathematical contents and which we name *constraint of the knowledge organization*. Now, the final examination paper generally consists of five exercises (or four exercises, with one in two sections)

Moreover, assessment never consists in long problems because of the importance attached to the success rate: teachers fear a "snowball effect" (Omar) of a mistake because of linked questions. We will return now to this fundamental factor.

Exercises of detailed implementation of methods

Let us consider the following exercise, resulting from the final examination paper (December 2007):

1. Determine the square roots of $3+4i$.
2. Solve, in \mathbb{C} , the equation $z^2 + 3iz - 3 - i = 0$.

We want to underline some important points about this exercise. It applies the method of resolution of quadratic equations with complex coefficients, method learned during the tutorial. The intermediate calculation of square roots is the subject of the first question. Thus the student can check the result in question 2), since they have to find the value given into 1) (it is a typical effect of contract didactic, Brousseau 1997). In addition, all the numerical values are whole numbers, never exceeding two digits, which allows the student to check very easily, and even allows a relatively effective method by trial and error in question 1.

However, this exercise is emblematic of such assessment. The same kind of exercise is found in each subject of the first exam and of the last one for the 4 last years.

The use of whole numbers is an institutional constraint specific to mathematics in the first year at the University of Rennes 1: the *constraint of ban on calculators*. This constraint is associated with the teachers' beliefs of the need for the students to understand calculations that a software can carry out automatically: this topic requires a specific study, which we will not undertake here.

The primary reason that explains the choice of such an exercise is the objective of a sufficient success rate. This clearly appears in the exchanges of emails, when this exercise is proposed, following remarks on the fact that “it misses complex numbers” (Georges); “one could have put a short exercise, but easy, on the complexes” (Thierry). Marc then suggests the exercise saying: “It should easily improve their marks. What do you think about it?” The other teachers approve: “this exercise seems very fine to me” writes Georges. “I agree with Georges, as that will increase the chances of the students” Thierry adds. In his interview, Marc recognizes that question 2 could have been only asked, but, according to him, question 1 ensures that the intermediate stages will be visible in the writing of the students, thus making it possible “to give points”.

The *constraint of success rate* is crucial in the choices of examination papers on all school levels, but perhaps even more in universities in scientific studies, victim of disaffection. The average mark for a given course cannot be under 10. This exercise provides any student who attended the course with 2 valuable points. The degree of freedom left to teachers for the development of the assessment is restricted by these constraints and beliefs. This, however, is not enough to explain the astonishing similarity of the examination papers year after year.

RULES IN ACTION: GENESIS OF AN EXERCISE

We saw in previous section some very strong constraints and beliefs: time constraint; belief of exhaustiveness associated with the constraint with the knowledge organization; constraint/belief of ban on calculators; constraint/belief of success rate. We will now see their influence upon the development of one of the exercises of the second exam.

Work in each group of the appointed teachers always started by the choice of the contents to evaluate. These contents are divided into exercises, and each teacher then assumes the wording of some of these exercises.

During their first meeting, Omar and Thierry identify four contents of knowledge to be evaluated in the second examination: integration with, on the one hand its definition and on the other hand calculations, then two topics on functions. The exercise that we will study was relating to the definition of the Riemann integrals, i.e. by the integral of step functions. Omar was in charge of its drafting.

A non-standard exercise is proposed

The first text proposed by Omar is given in appendix B. The announced objective was the approximation of $\ln(2)$ by integrals of step functions "In the first questions, the objective is to make them calculate the integral of step functions, then, in the last one, to see that it is convergent, therefore to make them apply what they learned". Omar is a young teacher (PhD student): he proposes a relatively non-standard exercise.

He wanted to give sense to the calculations usually requested from the students by showing that these calculations yield the approximation of $\ln(2)$.

Omar submits this exercise to Thierry thinking it is too long (*time constraint*) and that the only first three questions will be kept. The exercise looking indeed too long to Thierry, he decides, after having spoken about it with Marc, to remove the last two questions "it is a little long, it is necessary to remove the question which embarrasses more the students, therefore n ". One thus finds the *constraint of success rate* to which one could add a belief of the teachers that calculation with parameters are too difficult for students. We will not speak about this didactic difficulty, which does not enter within the framework of our study.

Change of aim

Thierry will not be satisfied with the simple shortening. He will return it strongly modified to the great distress of Omar: the idea of approximation (chosen to give sense to calculations) completely disappeared. There remains only the calculation of integrals of step functions. The values remain the same ones with two exceptions: the value of f on the interval $]4/3, 5/3[$ became negative and f takes a different value in point $4/3$. This second change is, according to Thierry, "to see whether the students understood that integration is independent of the choice of the value in a point". The change of sign allows the calculation of the integral of f , then of its absolute value. The set aim is, always according to Thierry, to evaluate a usual error: "there are people who are also mistaken, [thinking that] the absolute value of the integral is the integral of the absolute value".

In both cases, the aim is not to check the understanding of the implementation of a method, but rather of mathematical concepts. In the first case, the question illustrates a concept, whereas, in the second one, it illustrates some properties of this concept.

This exercise is also non-standard in the choice of the numerical values. If the choice of these values had a mathematical reason at the beginning (approximation of the function $1/x$), they were kept in the final version, in spite of a relative opposition of the other teachers. Marc will ask for example: "do you really want all those $1/3$...?" He will add, at the end of the module, that: "the colleagues for the second control were a little creative, which resulted in the average not being good". One finds again the *constraint of success rate*, here joined however with the belief that to propose non-standard exercises (that is to say exercises not present in the sheets of exercises) will not answer the institutional constraint of success rate. However here, this exercise,

that Marc qualifies the “creative one”, did not induce a specific failure of the students contrary to the opinion that he expresses.

There thus still exists a certain degree of freedom in the design of the subjects, but it seems to be exploited only by young teachers (Thierry has been teaching only for 4 years). It would be interesting to follow their later evolution.

The effect of documentary geneses

Our observations show that the documentary geneses constitute an important factor of inertia. All the teachers consulted past papers: either for the contents of the exercises by changing only some values, or in the structure of the evaluation with the choice of the exercises' number and of the selected topics. "The reasons for which I thought of making 4 [exercises], it is that the last time, they were 4" tells us Omar who will recognize: "I nevertheless looked at past papers" and "I looked at the exercises' sheets to give exercises which are not completely new". Marc will be more positive on this point: "the exam is rather standard; examination papers always have 5 exercises out of the 5 topics. [...] I asked people to send exercises on the 5 topics". Past papers are distributed to students before each exam and are corrected during the course. Students interpreted thus these texts as matching to the didactic expectations of the teachers.

The teachers looked at these former subjects in their development of a new examination paper because they made it possible to obtain the average expected by the institution. "The average [with CC2] was not good and so I absolutely wanted to make again a [standard] subject" will acknowledge Marc

Which didactic actions can one consider following this study? We give hints in the conclusion below.

CONCLUSION AND PROSPECTS

Our study deals with the teachers' activity, and more precisely with a part of this activity which goes on apart from the class. It must not be forgotten that the students and their learning constitute the central objective of our work. We stressed the importance of the questions of didactic contract in the teachers' choices of assessment. However the didactic contract involves teachers as well as students, and fixes the responsibilities for each one concerning the knowledge. The past papers constitute for the student a central reference, determining the institution expectations. Exam texts are composed of short exercises, consisting most of the time of the implementation of techniques: thus the private student's work turns naturally to the mimics of techniques.

Beyond this consequence on students' work, one observes an influence of the assessment on the teaching contents, and on the evolutions of these year after year. This extract of Marc's interview seems extremely significant to us in this respect:

"The more I teach this course, the more I... for example last year [...] I defined the integral [...] This year I said: listen, it has something to see with the area [...] if I teach that still 2,3

years I do not know what will remain. I make really more and more recipes by requiring nevertheless more rigor than in the physics tutorials.”

“According to you, what leads you to teach more recipes? ”

“The level of the students and the expectations of the students.”

Marc gives us the worrying description of a teaching emptied little by little of its contents, because of the “level of the students” (perceptible by their marks) and their expectations; however these expectations are largely determined by the didactic contract, and thus by the examination texts.

Thus to leave the present situation, to escape in particular inertia related to the documentary geneses, seems to us a real need.

To master methods is important in mathematics. Part of the assessment could be officially turned towards this objective. It would even be possible to make pass such an exam on computer by using e-exercise bases (such as WIMS, Cazes et al. 2007). Indeed, the implementation of methods is hardly the requirement object of wording: assignments were not corrected.

An exam on computer, directly providing a mark, could make it possible to free up time for another mode of assessment, based on a real problem solving, and to give place to a written work. Must this work have a time limit; must it be completed by an oral examination? The precise organization has to be specified.

In addition, in particular for a course involved in the mathematical tools for physics, the use of a calculator seems absolutely necessary to us. Indeed, the use of whole numerical values is clearly out of touch with the physical situations. Our study shows that a change of assessment, and even a joint change of the pedagogic resources and practices, are essential if the mathematics teaching at University must contribute to the increasing of students' mathematical autonomy.

The context of our work was a course for Physics students: what about assessment in the case of Mathematics students? We conjecture a similar development - testing rather methods - but a precise study has to be done.

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APPENDIX A

Questionnaire before the development of the examination paper (written answers)

1. What coordination is planned between the teachers dealing with the conception of the examination paper (meetings, mail exchange...)?
2. What coordination is planned with the other teachers of the tutorials (contents of the assessment, proof reading...)?
3. Which resources do you expect to use (exercises books, past papers of this tutorial or of another one...)?
4. Which a priori shapes do you think to give to this exam (exercises, problems, multiple-choice questionnaire)? Why?
5. What do you want to assess in this exam?

Questionnaire after the test (interview guide)

1. Presentation of the teacher and his teaching experiences.
2. Looking back on the first questionnaire: Has the conception of the examination paper happened as expected? Otherwise, what have been the changes, and why?

3. Analysis of the examination paper, exercise by exercise. Details of choices and expectations. As far as the intermediate exams are concerned: which exploitation during the next tutorials?
4. In general about the reasons for the choices made in the conception of an examination paper in this course:
 - To give something close to exercises made in the tutorial
 - To give something which allows to adapt the teaching according to the results of the test
 - To test all the studied contents
 - To test the most important points (which one ?)
 - To test what will be useful for the following tutorial
 - To respect the time-frame
 - To give a subject quick to correct

APPENDIX B

First version

Let I be the value of the integral $\int_1^2 \frac{1}{x} dx$ and f the step function defined by:

$$f : x \mapsto \begin{cases} f(x) = 1 & \text{si } x \in [1, \frac{4}{3}[; \\ f(x) = \frac{3}{4} & \text{si } x \in [\frac{4}{3}, \frac{5}{3}[; \\ f(x) = \frac{3}{5} & \text{si } x \in [\frac{5}{3}, 2]. \end{cases}$$

- 1) Plot on the same graph f and the mapping $x \mapsto \frac{1}{x}$.
- 2) Calculate $\int_1^2 f(x) dx$, and deduce an estimation of I obtained by the left rectangle method with a regular subdivision into 3 intervals.
- 3) Prove that the estimation of I obtained by the left rectangle method with a regular subdivision into 4 intervals is equal to $\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$.
- 4) Prove that the estimation of I obtained by the left rectangle method with a regular subdivision into n intervals is equal to: $\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1}$.
- 5) Calculate I . Deduce the approximation

$$\ln(2) \sim \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1}$$

Final version

Let f be the mapping defined by:

$$f(x) = \begin{cases} 1 & \text{si } x \in [1, \frac{4}{3}[; \\ -2 & \text{si } x = \frac{4}{3}; \\ -\frac{3}{4} & \text{si } x \in [\frac{4}{3}, \frac{5}{3}[; \\ \frac{3}{5} & \text{si } x \in [\frac{5}{3}, 2]. \end{cases}$$

- 1) Calculate $\int_1^2 f(x) dx$.
- 2) Calculate $\int_1^2 |f(x)| dx$.

DESIGN OF A SYSTEM OF TEACHING ELEMENTS OF GROUP THEORY

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In order to teach on the basis of the genetic approach, one should undertake an analysis consisting of the following two stages: 1) a genetic elaboration of the subject matter and 2) an analysis of the arrangement of contents including a consideration of various ways of representing it and its effect on students. The genetic elaboration of subject matter consists in the analysis of the subject from four points of view: historical, logical, psychological and socio-cultural. Also important is the epistemological analysis of the subject. We describe here the design of the system of study of the concepts of group theory.

Keywords: tertiary mathematics education, teacher education, group theory, genetic approach, genetic teaching.

1. INTRODUCTION.

In this paper, we describe the design of the system of teaching of the concepts of group theory using the genetic approach. Recently, teaching of group theory was discussed in the number of papers, and modern textbooks on the subject appeared, see, e.g., Armstrong (1988), Burn (1985), Burn (1996), Dubinsky, Dautermann, Leron & Zazkis (1994), Dubinsky & Leron (1994), Leron & Dubinsky (1995), Zazkis & Dubinsky (1996).

However, in the textbooks created by M. Armstrong and R. Burn, only geometrical sources of group theory are emphasized and used for motivating the learning. Articles are mainly restricted to using constructivist teaching or APOS theory (Dubinsky & McDonald, 2001).

Our approach based on the genetic principle combines historical and epistemological elaboration of the subject matter with psychological and socio-cultural aspects and allows to construct effective system of teaching the subject.

In preparation of the system of teaching, we also use the principles of concentricism and of multiple effect (Safuanov, 1999).

The *principle of concentricism* requires the following means in teaching a subject: the *preparation* and, in particular, the *anticipation*; the *repetition* on the higher or deeper level and the *increase*; the *fundamentality* (the deep and strong study of the carefully selected foundations of a discipline).

The *principle of multiple effect* (on students) states that the essential educational result can be achieved not with the help of one means, but many, directed to one and the same purpose. For example, the following means of expressiveness may be used

in teaching undergraduate mathematics: the *variation*, *splitting* (subject matter into smaller pieces), the *contrast*.

2. SYSTEM OF TEACHING BASED ON THE GENETIC APPROACH

In (Safuanov, 2005) the genetic approach in the teaching of a mathematical discipline (a section of a mathematical course, an important concept, or a system of concepts) is described. Its implementation requires two parts: 1) a preliminary analysis of the arrangement of the content and of methods of teaching and 2) the design of the process of teaching.

The preliminary analysis consists of two stages: 1) the genetic elaboration of the subject matter and 2) the analysis of the arrangement of contents, the possible ways of representation, and the effect on students. The genetic elaboration of the subject matter, in turn, consists of the analysis of the subject from four points of view:

historical;

logical;

psychological;

socio-cultural.

The purpose of the historical analysis is twofold: 1) to reveal paths of the origin of scientific knowledge that underlie the educational material and 2) to find out what problems generated the need for that knowledge and what were the real obstacles in the process of the construction of the knowledge.

For the construction of the system of genetic teaching, it is very important to develop problem situations on the basis of historical and epistemological analysis of a subject.

The major aspect of the logical organization of educational material consists in organizing a material in such way that allows the necessity of the construction and of the development of theoretical concepts and ideas to be revealed.

The psychological analysis includes the determination of the experience and the level of thinking abilities of the students (whether they can learn concepts, ideas and constructions of the appropriate level of abstraction); and the possible difficulties caused by beliefs of the students about mathematical activities. The analysis also has the purpose of planning a structure of the students' activities related to mastering concepts, ideas, and algorithms, of planning their actions and operations, and also of finding out the necessary transformations of objects of study.

One more purpose of the psychological analysis of the subject matter is finding out ways to develop the motivation for learning.

The socio-cultural analysis allows us to establish connections of the subject with the natural sciences, engineering, with economical problems, with elements of culture, history and public life; to reveal, whenever possible, non-mathematical roots of mathematical knowledge and paths of its application outside mathematics.

During the second part of the analysis, considering the succession of study, it is necessary, in accordance with the principle of concentricity, to find out, on the one hand, which concepts and ideas studied before should be repeated, deepened and included in new connections during the given stage, and, on the other hand, which elements studied at the given stage, anticipate important concepts and ideas, which will be studied more deeply later.

The principle of multiple effect on students requires also the search for the possibilities of multiple representation of concepts under the study, possibilities of using three modes of transmission of information (active, iconic and verbal-symbolical) and other means of effect on students (the style of the discourse, emotional issues, elements of unexpectedness and humor).

After two stages of analysis, it is necessary to implement the design of the process of study of the educational material. We divide the process of study into four stages.

1) *Construction of a problem situation.* In genetic teaching, we search for the most natural paths of the genesis of processes of thinking and cognition.

2) *Statement of new naturally arising questions*

3) *Logical organization of educational material*

4) *Development of applications and algorithms.*

According to principles described above, we present here the design of a system for the teaching of the concepts of group theory.

3. THE PRELIMINARY ANALYSIS.

1) Genetic development of a material.

a) Historical analysis.

F.Klein, who had brought in the essential contribution to the development of the group theory due to “Erlangen program” of the study of geometry through the study of groups of geometrical transformations, argued that “the concept of a group was originally developed in the theory of algebraic equations” (Klein, 1989, p. 372). Thus, groups, in his opinion, have arisen as groups of permutations. However, such fundamental concept as a group had also other roots in mathematics. As indicated in “The Mathematical encyclopedic dictionary” (1988, p. 167), sources of the concept of a group are in the theory of solving algebraic equations as well as in geometry, where groups of geometrical transformations have been investigated since the middle of the 19-th century by A. Cayley, and in number theory, where in 1761 L.Euler “in essence used congruences and partitions into congruence classes, that in the group-theoretic language means decomposition of a group into cosets of a subgroup” (ibid.). However, abstract groups were introduced by S.Lie only at the end of the 19-th century.

The main conclusion from this historical analysis is that the theory of groups has grown out of the development of many diverse ideas and constructions in mathematics and serves to the generalization and more effective theoretical consideration of these ideas and constructions.

b) Logical and epistemological analysis.

For the introduction of the concept of a group, the preliminary knowledge of a lot of set-theoretical and logical concepts and constructions is necessary which can be seen from the detailed logical and epistemological analysis of the homomorphism theorem (Safuanov, 2005. p. 260). In turn, the group-theoretical concepts are used in the subsequent sections. Abelian groups are used in the definition of vector spaces, rings, ideals and fields. The cosets of a subgroup and quotient groups are used in the definition of cosets of ideals and quotient rings. The groups are used also in geometry, in the study of groups of linear, affine and projective transformations. At last, groups will further occur in useful for the future teachers special courses on Galois theory, on geometry of Lobachevsky etc.

From the point of view of epistemology, groups serve for the organization of ideas connected to permutations, bijections and symmetries, therefore, examples connected to these ideas will serve to the good formation of the concept of a group in students' minds.

c) Psychological analysis.

School graduates are not actually prepared for mastering such abstract concept as a group. They can not operate with general concepts of algebraic operations and even with mappings. Therefore, in particular, they can not freely investigate geometrical transformations and their compositions.

On the initial stage, in our view, it is inexpedient to motivate the introduction of the concept of a group by examples of sets of transformations (for example, translations or rotations), because, as the experience of teaching geometry to the first year students of pedagogical universities shows, the geometrical imagination of many students (and spatial imagination in general) is very poorly developed. One more serious complication is bad understanding of quantifiers. On the initial stage the weaker students perceive quantifiers formally, poorly understanding and confusing their sense; they try to learn formulas with quantifiers by rote, confuse the arrangement of quantifiers in the formulas. As a result, the sense of the definition of a group becomes deformed, when the students try to reproduce the definition: it turns out, for example, that for any element of a group there is a distinct neutral element or, on the contrary, for all elements of group there is a common inverse. For the elimination of these difficulties it is necessary to offer the students special exercises, performance of which would reveal the role of the arrangement of quantifiers.

As the majority of the school graduates perceive mathematics mainly as actions with numbers, it is necessary to use these representations at the initial stage of the

construction of group-theoretical concepts. Besides, the school graduates remember such rules as associativity and commutativity of addition and multiplication, and these properties anticipate associativity and commutativity of group-theoretical operations.

According to the activity approach (Leontyev, 1981, p. 527-529), in order to operate with group-theoretical concepts (for example, groups, subgroups, cosets), it is necessary that intellectual operations (say, finding out the structure of a group, construction of cosets of a subgroup etc.) were carried out at first as actions, i.e. as purposeful procedures. It accords also to Ed Dubinsky's APOS (action - process - object - scheme) theory of the learning of concepts. Therefore it is necessary to plan skills which should be acquired by students at intermediate stages of learning group-theoretical concepts. It is necessary to design actions, which should precede mastering these skills. For example, before the study of the general way of construction of cosets (as results of the "multiplication" of the entire subgroup to an element of a group), the students should get experience of construction of concrete cosets of finite and infinite subgroups.

One more remark of the psychological character. It is well-known that the concept of a group isomorphism is narrower than the concept of a homomorphism and, moreover, in some sense more difficult, as it includes rather complex requirement of the bijectivity of a mapping. However, the teaching experience shows that, nevertheless, at the initial stage it is expedient to acquaint the students only with the concept of an isomorphism, as it is easier to be interpreted as the "similarity" of groups in some sense (for example, the similarity of the multiplication tables of finite groups); it is easier and more natural also to consider various examples of isomorphisms than those of homomorphisms.

d) Analysis from the point of view of possible applications.

The concept of a group since several decades became rather popular part of the cultural property of mankind. For example, the psychologist J. Piaget tried to use this concept for theoretical study of the psychological theory; the experts in the quantum mechanics believed that the group theory can be used for solving any problem. The group theory turned out to be extremely useful in the search of elementary particles and in the study of the structure of chemical molecules. Of great interest are the consideration of symmetry groups of geometrical figures and the use of groups for the research of patterns. Good examples of the applications of the group theory are the investigation of the "Fifteen puzzle" and graceful group-theoretical proofs of number-numerical theorems of L. Euler and P. Fermat.

2) Analysis from the point of view of the arrangement of a subject matter, of the opportunities of use of various means of representation of objects, concepts and ideas and of the influence on students.

Using results of the genetic elaboration, it is possible to offer the following version of the arrangement of a subject matter and of the use of means of influence.

As the theory of groups has grown out of generalizations of diverse ideas and constructions, we offer also to use some lines leading to group-theoretical concepts from the different perspectives: numbers, cosets, bijective transformations and permutations.

In accordance with the official abstract algebra syllabus, we devote to the study of groups several (four) stages at different places of curriculum, and such arrangement allows to effectively use elements required by principles of concentricism and multiple effect. As a result, students cumulatively acquire the necessary knowledge and skills, not losing their interest and motivation to the learning from the beginning to the end of the study of group theory.

The first stage: already at the introductory lecture it is possible to suggest to the students to consider systems of integers under the addition and non-zero rational numbers under the multiplication, to recollect properties of these arithmetic actions. It is expedient to help the students to reveal the properties of associativity, of the existence of neutral and inverse elements in the system of integers, and the students will be able to reveal independently by analogy the same properties in the system of non--zero rational numbers. Further it is necessary to try to lead the students to the idea that it would be useful to study properties of arithmetic actions based on the revealed fundamental properties and abstracting from the concrete number systems considered above. Here is “the moment of truth” (Safuanov, 2005) where axioms of group should be formulated. Note that the moment of truth is similar to the act of reflective abstraction (as the interior co-ordination of operations of the subject in a scheme) in the theory of Piaget (Dubinsky, 1991), and also to a moment of reification (Sfard, 1991). Such organization of teaching may be difficult and not always completely possible. Therefore, sometimes the appropriate help of the teacher may be useful.

In the ideal case, students should do it independently. Nevertheless, most likely, on this stage the teacher will have to formulate axioms of group himself or to offer the students to find the definition in a textbook.

At this first acquaintance the concept of a group will not be quite strict, as it will be based only on students’ intuitive representations about binary algebraic operations (“actions on elements of sets”), and the possibility of non-commutativity of an operation is not emphasized at all. In effect, this preliminary concept serves only as the anticipation of more detailed acquaintance at the following stages.

The second stage: after the consideration of the addition of cosets and the addition tables for small modules (for example, 2, 3, 4), it is possible to raise the question about the performance of addition in a set of cosets modulo arbitrary $n > 1$. Properties will be similar to properties of the addition of numbers. The students can guess the fulfillment of laws of associativity and commutativity, the existence of neutral and inverse elements, and even in some extent to participate in proving these properties. After that it is possible to introduce a stricter definition of a group, beginning with the

definition of ordered pairs and binary algebraic operations (as the rules putting in correspondence to every ordered pair of elements of a given set a certain element of the same set - at this stage students are not yet familiar with the concept of a direct product of sets). Here it should be underlined that the considered groups of cosets under the addition, as well as groups of integers under the addition, are Abelian (commutative), though there are also examples of non-commutative groups.

The third stage: preliminary, but already quite strict statement of elements of the theory of groups after the consideration of elements of the theory of sets, direct products, mappings, including bijective ones, and permutations. At this stage all formal definitions of concepts necessary for the strict introduction of group-theoretical concepts are available as well as sufficient amount of motivating and illustrating properties and examples. At this stage, after the introduction of the formal definition of a group and proof of the elementary properties, it is expedient to consider symmetry groups of geometrical figures. It is useful also for the maintenance of interest to the theory of groups and for the accumulation of the necessary amount of interesting and useful examples for the illustration of further constructions. Just at this stage the examples of non-commutative groups (symmetry groups and groups of permutations) are considered.

At this stage the concepts of a subgroup and isomorphism of groups should be strictly introduced, but in detail they should not be studied yet: they only anticipate systematic study of group-theoretical concepts and constructions at later stages, after studying linear algebra.

The group-theoretical knowledge acquired at the third stage, is used at the construction of concepts of rings, fields (in particular, of the field of complex numbers) and vector spaces.

The fourth stage: systematic study of elements of the theory of groups (including generalized associativity, cosets, normal subgroups, Lagrange's and homomorphism theorems). This knowledge already is sufficient for further study of quotient rings, Galois theory etc.

As to means of influence on students, in the teaching of elements of the theory of groups it is possible to use various evident ways of representation of a subject matter, considering, for example, permutations, symmetry of geometrical figures, geometrical transformations. Among ways of representation of groups it is possible to employ, in case of finite groups, lists of elements, multiplication tables etc. Among other means of influence one can mention the contrast (examples of groups versus semigroups which are not groups, normal subgroups versus subgroups that are not normal), variation (Abelian and non-Abelian groups, additive and multiplicative ones etc.).

3. DESIGN OF THE PROCESS OF STUDY OF GROUP-THEORETICAL CONCEPTS.

In the designing process of teaching we take into account all the results of the preliminary analysis, and thus the task of designing becomes considerably facilitated. Note that after designing and checking the intended system of study of a theme in practice, using a feedback, results of the control and assessment, it is necessary to bring in corrective amendments, sometimes essential, to the designed system. So, for the study of the theory of groups we at the third stage (after studying permutations) at first intended to prove the generalized associativity. However, the experience has shown that this rather short inductive proof nevertheless requires from students the well-developed logic reasoning and inordinately large efforts for mastering. Therefore, we have transferred this proof to the last, fourth stage devoted to systematic study of algebraic systems.

1) Construction of a problem situation.

As is already shown, for the successful construction of a problem situation it is necessary to organize it (including new questions, naturally arising from it) so that in a certain time there would occur the “moment of truth” when the students independently or with the minimal help of the teacher would open for the new concept for themselves.

For the first time such moment of truth arises already during the introductory lecture, when the preliminary version of the concept of a group arises as a generalization of properties of arithmetic actions in sets of integers (addition) and non-zero rational numbers (multiplication). At further stages this preliminary version of the definition forms the basis for the motivation of the consideration of the concept of a group, basis for its stricter study. So, for example, studying properties of the addition of cosets or multiplication of bijections of a set, permutations of a finite set, symmetries of a geometrical figure, the students already can find out that each time they deal with groups – and thus new moments of truth arise.

2) Statement of new naturally arising questions.

For example, when constructing a problem situation at the third stage (when passing to types and elementary properties of groups), one can use questions of the following kind: whether are groups under consideration commutative? Whether there exists an infinite non-commutative group? Is the neutral element of a group unique? For a given element of a group, is an inverse element unique? Is it possible to solve equations in groups? At the fourth stage (systematic study of more complicated group-theoretical concepts) the questions are pertinent: do the right and left cosets coincide? Do cosets of a normal subgroup form a group under multiplication? etc.

3) Conceptual and structural analysis and logical organization of educational material.

Conceptual and structural analysis and logical organization of group-theoretical concepts is rather complicated, as is seen, e.g., from the genetic decomposition of the homomorphism theorem (Safuanov, 2005. p. 260). This process is not straightforward, but rather long and, moreover, often occurs in several stages divided in time. From group axioms the properties of groups are deduced, and at final stages of study of groups a number of rather difficult theorems is proved.

4) Development of applications and algorithms.

Despite the importance of the theory of groups, its applications are too non-trivial: so in an obligatory course it is problematic to consider such major applications, as the Galois theory or, say, geometrical applications, which are more appropriate for considering in detail in a geometry course. Nevertheless, it is important to consider such simple and interesting examples of applications as the fifteen puzzle, group-theoretical proofs of number-theoretical theorems of L.Euler and P.Fermat, symmetry groups of geometrical figures etc.

The students also should learn such procedures as construction of the multiplication table of a finite group, finding cosets of a normal subgroup (i.e. construction of a quotient group) etc.

Concerning the development of cognitive strategies note that, according to the genetic approach, it is important to teach the students to construct analytical proofs, i. e. such ones that start from the statement that must be proved, and include the search of the facts necessary for the proof of the final statement. Then one searches how to find these necessary facts etc. It resembles going from the end of the proof to the beginning (in computer science such approach is referred to as “backtracking”) (see Goodman&Hidetniemi, 1977). The theory of groups gives such opportunities.

4. IMPLEMENTATION.

This system of teaching was successfully implemented in practical teaching at the pedagogical universities of Ufa and Naberezhnye Chelny for two decades. The students studying abstract algebra course by this system constantly show much better achievements and, most important, more positive attitude and interest to the subject than students studying the discipline by traditional deductive and “definition – theorem – example – exercise” approach.

Of course, the genetic approach can be applied for teaching other mathematical topics and mathematical disciplines.

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CERME 6 – WORKING GROUP 13

COMPARATIVE STUDIES IN MATHEMATICS EDUCATION

TABLE OF CONTENTS

Introduction.....	2447
<i>Eva Jablonka, Paul Andrews, Birgit Pepin</i>	
Comparing Hungarian and English mathematics teachers’ professional motivations.....	2452
<i>Paul Andrews</i>	
Spoken mathematics as a distinguishing characteristic of mathematics classrooms in different countries	2463
<i>David Clarke, Xu Li Hua</i>	
Mathematical behaviors of successful students from a challenged ethnic minority.....	2473
<i>Tiruwork Mulat, Abraham Arcavi</i>	
A problem posed by J. Mason as a starting point for a Hungarian-Italian teaching experiment within a European project.....	2483
<i>Giancarlo Navarra, Nicolina A. Malara, András Ambrus</i>	
A comparison of teachers’ beliefs and practices in mathematics teaching at lower secondary and upper secondary school.....	2494
<i>Hans Kristian Nilsen</i>	
Mathematical tasks and learner dispositions: A comparative perspective.....	2504
<i>Birgit Pepin</i>	
Elite mathematics students in Finland and Washington: access, collaboration, and hierarchy	2513
<i>Jennifer von Reis Saari</i>	
International comparative research on mathematical problem solving: Suggestions for new research directions.....	2523
<i>Constantinos Xenofontos</i>	

INTRODUCTION

COMPARATIVE STUDIES IN MATHEMATICS EDUCATION

Organisers

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AIMS AND SCOPE OF THE WORKING GROUP

The call for papers for the 2009 meeting of the working group set out with a description of the scope and aims of comparative studies in mathematics education. These include studies that document, analyse, contrast or juxtapose similarities and differences in mathematics education at different levels, such as:

- cross-cultural or cross-national comparison;
- comparison between sectors of school-systems;
- comparison between groups that share specific characteristics (for example, gender, language, social and economic background, cultural affiliation or other demographic features);
- comparing mathematics education with other school subjects.

There were no restrictions in the aspects of mathematics education that can be usefully addressed in a comparative study. These might, for example, include: Intended curricula; tools, teaching materials and resources; specific mathematical activities or the enactment of distinct mathematical topics; learning environments; teachers', student teachers' and students' aspirations, goals and values; student achievement and participation; features of classroom practices or features of teacher preparation programs.

The aims of the working group included to:

- share findings and outcomes of empirical studies that adopt a comparative approach;
- further develop research methodologies that are specific to comparative studies;
- identify ways in which macro-level survey studies and micro-level case studies can productively interact;
- develop a better understanding of how various theoretical approaches and conceptual frameworks shape the goals and the design of comparative research;
- consider how comparative studies can inform teaching and learning practices.

The group invited contributions with an empirical, methodological or theoretical focus. Papers with a methodological or theoretical focus could, for example, address issues of comparability of culturally-grounded practices, challenges of interpreting

outcomes of large-scale international achievement studies, methods of data aggregation in quantitative studies, technicalities of classroom-video studies, issues of cultural bias in coding or any other problématique that is specific to comparative studies.

PAPERS AND POSTERS

As the working group brings together researchers who share an overall approach rather than a focus on a set of topics, we find an interesting range of aspects of practices in mathematics education that were subjected to comparison in the research reports and posters. The participants' studies, some of which are ongoing projects, addressed mathematics education in different places of the world. The countries and regions include Australia, China, the Czech Republic, Finland, France, Germany, the Hong Kong Special Administrative Region, Hungary, Israel, Italy, Norway, the Slovak Republic, Syria, the United Kingdom of Great Britain and the United States of America. The titles of the papers and posters indicate the variety of aspects of mathematics education that were subjected to a comparison (presenting authors are underlined):

Paul Andrews, United Kingdom: *Comparing Hungarian and English mathematics teachers' professional motivations*

David Clarke and Xu Li Hua, Australia: *Spoken mathematics as a distinguishing characteristic of mathematics classrooms in different countries*

Tiruwork Mulat and Abraham Arcavi, Israel: *Mathematical behaviours of successful students from a challenged ethnic minority*

Giancarlo Navarra, Nicolina A. Malara, Italy; András Ambrus, Hungary: *A problem posed by J. Mason as a starting point for a Hungarian-Italian Teaching Experiment within a European project*

Hans Kristian Nilsen, Norway: *A comparison of teachers' beliefs and practices in mathematics teaching at lower secondary and upper secondary school*

Birgit Pepin, United Kingdom/ Norway: *Mathematical tasks and learner dispositions: A comparative perspective*

Jennifer von Reis Saari, United Kingdom: *Elite mathematics students in Finland and the Washington: Access, collaboration, and hierarchy*

Constantinos Xenofontos, United Kingdom: *International comparative research on mathematical problem solving: A framework for new directions*

As the posters are not included in the proceedings, short summaries are given in the following:

Maha Majaj, France: *Comparative study of the place of elementary number theory in the programs and the textbooks in the middle school between France and Syria*

The teaching of elementary number theory has undergone changes in the French and Syrian education systems. In Syria, its place changed with the evolution of the textbooks about five years ago and in France it was reintroduced, after fifteen years of absence, in 1998 (grade 12), 1999 (grade 9) and 2001 (grade 10). The study compares elementary number theory in the programs and textbooks, topic by topic, by taking into account a distinction between tool and object and identifies the didactical transposition choices and their effects on the design of textbooks. An initial study indicated that the choices of the Syrian educational system can be seen as corresponding to the French program since the beginning of the 20th century. This observation led to including an analysis of the evolution of the French program and textbooks from the reform in 1902 onwards.

Jan Sunderlik, Slovak Republic: *Intrinsic motivation and student teaching practice at universities from Great Britain, the Czech Republic and the Slovak Republic*

The study in progress sets out to investigate pre-service teachers' teaching practice in Great Britain, the Czech Republic and the Slovak Republic with a focus on their strategies for motivating students. It is to understand how the accumulated body of research on students' motivation may be useful for classroom teachers struggling with the issue. The notion of motivation is complex and, for example, described as linked to social needs, beliefs, behaviour and affect. One challenge of the research is to describe motivation in observational terms.

SNAPSHOTS AND CLOSEUPS FROM THE DISCUSSION

The groups at the CERME adopt a mode of working that assumes that all papers have been read before the start of the conference. The presenters in our group were invited to draw our attention to specifics and to expand on one or two points in order to provide us with 'an experience' for entering the discussion. The productive work and stimulating discussion lived on the continuous engagement of all participants, which made it possible to allude to a wide range of topics. In the following, a summary of some issues, which were not specific to a particular research report, is given.

Agendas and modes of comparison

The group agreed that although comparative studies serve to achieve a variety of goals, comparison does not itself constitute the goal of a comparative study. Comparison was seen as being always of interest because looking at practices from another culture (see below "units of comparison") provides a new 'lens' for looking at our own; it helps to make the familiar look unfamiliar. For the activity of describing similarities and differences in the empirical findings, the metaphor of "collecting stamps" was introduced. Synthesis was seen as a more far reaching goal of a comparative study than a mere description of similar and different aspects, and comparison was described as "the fuel of synthesis". A comparative approach can also aim at assisting theory construction. It is useful for this purpose especially because the emergence of differences supports cultural explanations, while similarities suggest structural (sociological) interpretations. While the improvement

of “home” teaching practice was seen as an important goal for a cross-national or cross-cultural comparative study, the members of the group agreed that not all research in mathematics education has to be advocacy.

“Units of comparison”

Acknowledging that all empirical research has a comparative aspect, one recurring point in the discussion concerned the question, are there ‘units’ for comparison that are too small or too big for allowing a study to be described as comparative. Agreement was reached that comparison has to be between aspects of “social conglomerates”, between two cultures (with shared discourse and identities). Just the fact that members of a group share an attribute does not mean that their membership of the group is related to that attribute, neither as a condition for or a consequence of that membership.

Examples of “units for comparison” discussed in relation to the research reports were: curriculum, ideologies in education, schools, processes of change, students’ productions, lesson structure, lesson events, groups of students in different institutional cultures, groups of successful and unsuccessful students from the same culture.

Methodology and Methods

Many problems identified in the discussion are not specific to comparative research, but the challenge of working across cultures makes them more visible. The research designs in the comparative studies presented in the group comprise a variety of approaches for creating accounts of the practices to be compared. The discussion focused on three approaches: documentation, cross-national intervention study (a “perturbation of practices”) and on the comparison with a different teaching practice (with a different pedagogy) as a quasi-experimental design.

Interpreting “silence in the data”

This discussion emerged out of an example of interview transcripts with students from two different cultures. The participants did not say anything after a prompt from an interviewer. In the group we created several interpretations of this fact: Silence is a normal part in any conversation – it is a thinking pause; silence is a sign of cultural or social alienation; silence is a general cultural behaviour; silence is an individual’s preference.

In the course of the discussion, “silence” was used metaphorically for missing aspects of a practice. These silences go unrecognized from within the practice and thus comparison can fill the gap left by silence.

To what extent are the outcomes comparable and can be synthesised?

Group members observed that the cultural differences sometimes are so fundamental that comparison is impossible. The results can then only be juxtaposed. The question

was also asked to what extent psychological frameworks could be useful in comparing groups from different cultural contexts.

Cultural affiliation of research personnel (interviewers, transcribers)

Group members were aware that inter-researcher reliability is a problem in all studies, but it is likely to be exacerbated in a cross-cultural comparative study or a study of different institutional cultures or any other social conglomerates with a shared discourse. Some methods were suggested and discussed. “Member checking” includes exchanging the accounts between the different communities (both the “researched” or the researchers’) and letting them check from their lens. One interesting example was provided in a study in which teachers in one country had been asked to read the accounts from teachers in other countries of what they do and why they do it.

How to avoid a culturally biased interpretation?

Group members shared the observation that interpretations are loaded with values from our own teaching tradition as well as research tradition. Researchers may project their home-grown categories into the other culture’s data, which amounts to a culturally biased gaze. Researchers might as well be at risk to produce an ‘idealistic’ description of their own practice, or alternatively (depending on the culture!), provide an account that is too critical of the home practice and celebrates the other.

The group found that exploiting different conceptual frameworks might help to identify the blind spots of each. The French “praxeology” served as an example. Some found that ‘contextualised tasks’ were not given attention as a category because the French curriculum does not include those as a characteristic element. In an approach that is more focused on the empirical material and does not set out with theoretical categories, the interpretative accounts for one set of data from one site maybe considered as the framework for interpreting the other (and vice versa). This approach is reminiscent of constant comparison as a standard method in qualitative data analysis.

All agreed that language matters, also within a culture, e.g. as a sociolect, as difference between formal and informal language use. This point draws attention to how to deal with translated transcripts; the choice of language into which protocols are translated is already a source for a cultural bias. The group pointed to the need of defining the cultural frame of each report.

Eva Jablonka, Paul Andrews, Birgit Pepin

COMPARING HUNGARIAN AND ENGLISH MATHEMATICS TEACHERS' PROFESSIONAL MOTIVATIONS

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In this paper I present qualitative analyses of interviews undertaken with English and Hungarian teachers of mathematics. One aim of the interviews was to elicit teachers' professional motivations – what were their subject-specific reasons for teaching mathematics? I frame the analyses against the altruistic, intrinsic and extrinsic motivational framework found widely in the literature before discussing its limitations and proposing refinements to highlight substantial differences within superficially similar sets of culturally located espoused motivations.

INTRODUCTION

A frequently cited reason for undertaking comparative education research is that “studying teaching practices different from one's own can reveal taken-for-granted and hidden aspects of teaching” (Hiebert et al., 2003, 3). In part this is because:

teaching and learning are cultural activities (which)... often have a routineness about them that ensures a degree of consistency and predictability. Lessons are the daily routine of teaching and learning and are often organized in a certain way that is commonly accepted in each culture (Kawanaka 1999, p. 91).

Explanations for such routines draw on beliefs that cultures “shape the classroom processes and teaching practices within countries, as well as how students, parents and teachers perceive them” (Knipping 2003, 282), to the extent that many of the processes of teaching are so “deep in the background of the schooling process ... so taken-for-granted... as to be beneath mention” (Hufton and Elliott 2000, 117). Thus, it is probably not surprising that a substantial proportion of comparative mathematics teacher research has focused on explicating the mathematics teaching script (Andrews, 2007a; Hiebert et al., 2003; Stigler et al., 1999), with a number of other studies having investigated particular contributory factors. For example, text books have been scrutinised (Haggarty and Pepin, 2002; Pepin and Haggarty, 2001; Valverde et al., 2002), teachers' mathematical content knowledge has been analysed (An et al., 2004; Delaney et al., 2008; Ma 1999); as have teachers' mathematics-related beliefs (Andrews and Hatch, 2000; Andrews, 2007b; Barkatsas and Malone, 2005; Cai, 2004; Correa et al., 2008). However, a largely ignored field in comparative teacher research concerns teachers' motivations for their professional activity: what stories do they tell to warrant their roles as teachers of mathematics? This paper is a first explicitly comparative examination of mathematics teachers' professional motivation.

According to available evidence, teachers' professional motives fall into three categories: altruistic, intrinsic or extrinsic (Kyriacou and Newson, 1998; Kyriacou

and Coulthard, 2000; Moran et al, 2001; Andrews and Hatch, 2002), although recently Cooman et al (2007) have highlighted a fourth - interpersonal. An altruistic motive presents teaching as a socially worthwhile act related to a desire to facilitate the development of both the individual and society at large. An intrinsic motive includes, inter alia, a person's desire to work with children or their subject specialism, while an extrinsic motive pertains, for example, to salary, conditions of service, holidays or status. Lastly, interpersonal motives refer to “social interactions commonly present in a teaching job” (Cooman et al, 2007, p. 127). An individual's personal motivation to teach is likely to be an amalgam, in varying proportions, of these factors (Moran et al, 2000) although there is evidence that preservice teachers in developed countries are motivated by both intrinsic and altruistic factors, while in developing countries extrinsic motivations (or mercenary) appeared more prominent (Bastick, 2000). Indeed, in respect of the former, intrinsic motives appeared dominant for preservice teachers in the US (Serow and Forrest, 1994) Greece (Doliopoulou, 1995), England (Reid and Caudwell, 1997; Priyadharshini and Robinson-Pant, 2003; Whitehead et al, 1999), Northern Ireland (Moran et al., 2000) and Australia (Manuel and Hughes, 2006).

In respect of the professional motivations of mathematics teachers there is little research (Reid and Caudwell, 1997). In respect of the UK, students following a post-graduate mathematics teacher education programme were less intrinsically motivated than those of other subjects, showing, in their greater enthusiasm for teaching as a good career, a more extrinsic perspective (Reid and Caudwell, 1997). Also, in contrast with mathematics undergraduates, who privileged intrinsic factors such as being able to use their subject knowledge or working with children (Kyriacou & Newson, 1998), post graduate teacher education students were less enthusiastic about sharing their knowledge, continuing their subject interest, improving children's life chances than their non mathematical colleagues (Reid and Caudwell). In terms of serving teachers, Andrews and Hatch's (2002) study showed that few people espoused either altruistic or extrinsic reasons, with most citing motivations intrinsic to either mathematics itself or teaching as a profession.

In sum, the totality of the above highlights the extent to which the tripartite framework has been used in different research contexts. However, with so little comparative work, and with most studies drawing on different instruments, we know little about the extent to which it adequately represents the motivations and beliefs of teachers in different contexts. In this paper we examine this issue by means of an initial comparative examination of mathematics teachers' professional motivations.

METHOD

Many of the studies cited above used survey approaches to explore teachers' motivations. Of these, many exploited factor analytic techniques to identify or confirm, depending on the type of analysis, motivational constructs. However, such approaches rely, essentially, on predetermined categorisations of motivation and may

miss not only subtle variations within the three dimensions but, importantly, components hitherto unconsidered. In this paper we attempt to let teachers tell their own stories and, to do so, look to narrative research. Narrative research is of interest due to its “potential to access the research subjects’ voices and to offer deeper, sensitive and accurate portrayals of experience that have escaped positivist quantitative research and less sensitive, objectivist qualitative research” (Swidler, 2000, p. 553). It “is probably the only authentic means of understanding how motives and practices reflect the intimate intersection of institutional and individual experience in the postmodern world” (Dhunpath, 2000, p. 544). Narrative researchers believe that teachers construct stories to make sense of their professional world (Swidler, 2000; Drake, 2006). That is, stories, “as lived and told by teachers, serve as the lens through which they understand themselves personally and professionally and through which they view the content and context of their work” (Drake et al. 2001, p. 2). Moreover, “these stories are subject-matter-specific and may differ greatly from subject to subject” (ibid).

With this in mind, 45 teachers from two regions of England, and 10 from Budapest, Hungary, were interviewed in the months following a questionnaire study of their conceptions of mathematics and its teaching. In both countries colleagues were drawn from a variety of institutions, which, as shown by various indicators, were representative of state schools in the different regions. The interviews, which were intended to elicit details about informants' professional life histories, were semi-structured and invited colleagues to describe how their careers had developed and to discuss the key episodes, “critical events” (Woods, 1993) or “critical incidents” (Measor, 1985) that had informed or transformed their professional lives. In order to frame their stories, colleagues were invited, fairly early in their interviews, to explain why they had decided to become teachers before being asked to consider the place of mathematics in the curriculum and their personal justification for both its curricular inclusion and their teaching it. Interviews, which were conducted in colleagues' schools, were tape-recorded and transcribed. Transcripts were posted to them for agreement as to their content although not one was queried. The method of constant comparison (Glaser and Strauss 1967, Strauss and Corbin 1998) necessitated that transcripts were read and re-read to identify categories of response. As new categories were identified, previously read transcripts were re-read to see whether or not the new category applied. The two sets of data, English and Hungarian, were analysed separately to ensure that culturally located differences were not obscured.

RESULTS

The reader is reminded that this paper draws on, in many cases, informants' recollections of events of many years earlier. Thus, it is not improbable, particularly acknowledging the temporal shift between events, that for some teachers, recollections concerning decisions about career choice may have been vague and romanticised. In particular, it is not improbable that recollections drew on colleagues'

affective responses to the profession which had dominated their lives. In some cases, but clearly not all, these would have been positive and, possibly, a little heroic. Consequently, some caution should be exercised when interpreting informants' utterances. In the following, all names are pseudonyms. Due to constraints of space, only a partial analysis is reported, which draws on the same three substantial categories of response that emerged from the data of each country. These focused on personal pleasure, the extrinsic properties of mathematics and the intrinsic properties of mathematics. These were not exclusive categories with most teachers alluding to at least two of them.

English teachers: Personal pleasure

Twenty seven English teachers indicated that their professional motives were located in the pleasure they gained from working with students. Jane, typical of most, described an enjoyment located explicitly in their students' mathematical success. She said:

I just enjoy teaching it (mathematics)... I can't explain it. I enjoy teaching it. I enjoy watching children who can't do maths suddenly discover they can add up. You know, children for whom it's not made sense all of a sudden this...“Oh that's why it works”, “Oh now I understand”. And I think it's that, and it doesn't matter what level that is. Whether it's down at the bottom end or it's up at the top end, it's that discovery that it works. That's what I enjoy doing. I enjoy seeing children make that leap. Sometimes it happens more often than others; with some children it's very slow, you know, the understanding, but when it comes it's like light dawning and they're so pleased and I think that's what it is.

For the others, like Hazel, their pleasure seemed less altruistically focused. She said:

I think I would always have ended up as a teacher. I loved being around little kids when I was a child... I'm a maths teacher because that's what I was good at and if I'd been good... at... French then I think I would have been a French teacher.

English teachers: Extrinsic properties of mathematics

Forty-two teachers commented that they were teaching mathematics to prepare students to manage successfully a world beyond school. The explicit foci of these comments varied but the underlying message was essentially the same; a child who cannot understand mathematics would struggle to make sense of the *real world* or *everyday life*. James commented:

I feel that maths is a tool and that if students... are to be fully prepared for what the modern world is to throw at them... I think that it's very important that they are... able to handle all the things that can be thrown at them.

For others this was explicitly linked to employment. Jack, who had previous work experience in cotton mills and council offices, suggested that:

It's there all throughout isn't it? I mean, at basic levels, the practical jobs, measurement and things like that through to, yes, obviously people want to be well qualified to go on and do, you know, industrial engineers or civil engineers, work that involves high powered mathematics as well.

Susan indicated yet another utilitarian perspective. She said that:

I think my main reason for supporting maths is because I think it's a support subject for other subjects as in you can't take your science or computers nowadays or anything further, if you, if students want to, without a basic knowledge of maths. So you can't do a lot of things further and develop knowledge that way. So I see it being a support subject for other subjects.

English teachers: Intrinsic properties of mathematics

Fifteen teachers offered statements indicative of their justifying their teaching of mathematics as a consequence of its intrinsic properties. Jean commented that "I always got a... buzz out of solving particular problems...especially when you've worked on them for quite some time. And so it's that enjoyment of the subject that I like to try and put across to children". Judy, in addition, discussed wanting her students to become critical thinkers:

I want children to feel the need to solve a problem. I give them the skills and help them to think through how to achieve that, even if it's a very, very simple idea, I always give them a reason why... I always say, don't ever be satisfied with well that is how it is, always ask and if I can't give you a reason then I should go away and find you a reason because I won't expect you to believe it just because I say so.

In similar vein Frank, discussed his belief in the importance of mathematical reasoning. He said that "the one area of maths that I really enjoy working with students is, is trying to get them to explain things, I suppose, explain, justify, prove along some sort of continuum there".

Hungarian Teachers: Personal Pleasure

Eight of the ten Hungarian teachers talked about pleasure gained from their professional activity. For the most part, this drew on students' mathematical successes. Vera commented that, "It feels good to teach the children to think", although most indicated that their pleasure derived from their students understanding of mathematics. Emese, for example, said that when "I tell them something new...and although they would probably have learnt about it without me, not only do they know it but they also understand it".

Two teachers located their comments on student understanding within the domain of problem solving. Ilona commented that:

I would like my students to understand and think about smaller or bigger problems in mathematics with joy... And I think it's the greatest thing in the world that I can teach mathematics because it's a fantastic way for educating children... when I see twenty kids

sit down and think and wrinkle their foreheads, and they put their heads in their hands and they turn the small wheels around until they get to some solution independently.

Hungarian Teachers: Intrinsic properties of mathematics

Every Hungarian teacher commented in ways indicating that, for them, mathematics possessed important and, essentially, intrinsic qualities. Emese noted that “students have to see that in mathematics you have to think logically”. Robert, expressing a similar theme, commented that “it is important that a child learns a particular thinking scheme and can solve problems with this method... how you can make a child to become a thinking child”.

At an explicitly philosophical level, Eva commented that “I like to quote an aphorism which more or less determines my life. Leonardo said *mathematics is the most important tool for understanding the truth everywhere and in everything* and this is my philosophy” while Robert added that mathematics “was a spiritual adventure and this was what attracted me so much (to the teaching of the subject)”.

Unlike the English data, three intrinsic subthemes emerged from the analysis. These concerned mathematics as problem solving, mathematics as a connected body of knowledge and mathematics as experientially learned.

Mathematics as problem solving

Nine Hungarian teachers discussed the importance of problem solving in their conceptualisation of mathematics and its teaching. Vera, outlined a view that teachers should alert students to

... certain types of problem which come up again and again ..., they should know the typical problems that they have to go through. And then it's also good if there are problems, we give them problems, which don't have completely unique solution so they should find them in other ways.

Emese, in addition, acknowledged the affective domain as part of the problem solving experience. She commented that:

We should teach them how to recognise the problem, develop ideas for the solution, put them into a logical order, and this way you reach the solution... The most beautiful and simple thing in the world is when you solve a problem and you realise that you were able to solve it...It can help you with a little more self confidence too.

Mathematics as a connected body of knowledge

Five teachers commented explicitly on mathematics as a connected body of knowledge. Rita, talking about number theory and geometry, commented that:

Within number theory, for example... you can take the numbers apart. Think of numbers and how they are built up. This building up is very important. And with other topics too, in geometry it's important to be able to build up things... This taking apart, building up, and often the building up is at least as important as taking apart.

Robert offered a more abstract perspective, commenting that “mathematics is built in such a way that it states certain things and it calls them *axioms* or *statements* which are considered as true and then I start to build up something and I wonder how far you can get from it”.

Mathematics as an intellectual challenge

Five alluded to mathematics as an intellectual challenge, something for which learners should expect to struggle. Eva commented that students:

shouldn't get everything ready-made but should have to look for the truth, to search for it. I mean it's more the research than the experience. I, for example, like geometry very much when they have scissors in their hands and they're folding and cutting papers and getting experiences... Still you can research to look for different solutions. We get to the same truth in different ways.

Kati, commented in similar vein, that children should experience the “joy of research... I think that one of the most important things is that children should be brave and should be able to get close to an unknown problem. And it's also very important that this love of adventure shouldn't be spoilt by me”. Zsolt, commented that “they have to get experiences. No matter what topic of mathematics they're learning, they should get as much experience as possible”.

Hungarian Teachers: Extrinsic properties of mathematics

Five teachers commented explicitly that mathematics provided key skills for a world beyond school. Vera noted, briefly, that it “has an influence on their whole life; the rational way of thinking”, while Ilona said:

I think we teach mathematics to help children find their way in life more confidently. Whatever they become, a cleaning lady, a banker, a doctor or anything... mathematics is a logical skill. Facts and things thought over a logical way will help them make their way more confidently.

In similar vein Emese commented that children “should be able to calculate the change in the shops and I want them to understand all it's good for in everyday life... I think they have to see that mathematics is about life”, while Rita said that it's “good if they can count. If they can look through how much is how much. Estimating is very important... I always say, you cannot read a book if you have to think about each letter”.

DISCUSSION

The above, albeit limited, results show that when located alongside their subject specialism, teachers of mathematics in England and Hungary report intrinsic motivations, although the three categories of response comprise embedded altruistic, intrinsic and extrinsic characteristics respectively. Thus, on the one hand, it could be argued that English and Hungarian teachers of mathematics present similar subject-related professional motivations. On the other hand, the widely differing proportions

of teachers reporting these categories indicate something profoundly different. Of course, the differences can be explained against a variety of cultural frameworks. For example, the dominance of the mathematically intrinsic motivations of Hungarian teachers and mathematically extrinsic motivations of English teachers reflect the underlying rational encyclopaedist and classical humanist traditions of Hungary and England respectively (Andrews and Hatch, 2000). But such explanations offer little by way of highlighting differences other than in the frequencies of the three dimensions. Therefore, the following is a tentative revision of the framework drawing on notions of rhetorical and warranted motivations.

Firstly, in respect of mathematically altruistic motivations, English teachers talked, in an unspecified manner, of motivations linked to mathematical understanding, while their Hungarian colleagues spoke of understanding-informed mathematical thinking and problem solving. Thus, on the one hand, around half the English sample presented rhetorical altruistic mathematical motivations, while, on the other, almost all the Hungarian teachers articulated a warranted altruistic mathematical motivation. Secondly, in terms of mathematically intrinsic motivations, English teachers tended to articulate a perspective concerning problem solving and the logical skills necessary to solve them, while the Hungarian teachers presented a variety of perspectives concerning not only problem solving but also the structural properties of mathematics and the intellectually challenging nature of the subject. Thus, the English teachers presented a weakly warranted, almost rhetorical, intrinsic mathematical motivation when compared with the Hungarian teachers' robustly warranted intrinsic mathematical motivation. Thirdly, in respect of mathematically extrinsic motivations, almost every English teacher and half the Hungarian teachers discussed mathematical success as a necessary prerequisite for employment or the learning of other subjects. In this regard, both groups of teachers presented a moderately warranted extrinsic mathematical motivation.

In summary, the qualifiers of rhetoric and warrant allow us to distinguish between the two sets of motivations and understand more fully the ways in which mathematics teachers' professional motivations are products of the cultures in which they live and work. A speculative conclusion would be that while both sets of teachers present a moderately warranted wider-world (extrinsic) justification for the teaching of mathematics, the English tend towards rhetorically-based motivations while the Hungarian tend towards warranted motivations.

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SPOKEN MATHEMATICS AS A DISTINGUISHING CHARACTERISTIC OF MATHEMATICS CLASSROOMS IN DIFFERENT COUNTRIES

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This paper reports research into the occurrence of spoken mathematics in some well-taught classrooms in Australia, China (both Shanghai and Hong Kong), Japan, Korea and the USA. The analysis distinguished one classroom from another on the basis of public “oral interactivity” (the number of utterances in whole class and teacher-student interactions in each lesson) and “mathematical orality” (the frequency of occurrence of key mathematical terms in each lesson). Our concern in this analysis was to document the opportunity provided to students for the oral articulation of the relatively sophisticated mathematical terms that formed the conceptual content of the lesson. Classrooms characterized by high public oral interactivity were not necessarily sites of high mathematical orality. The contribution of student-student conversations also varied significantly. Of particular interest are the different learning theories implicit in the role accorded to spoken mathematics in each classroom.

Key words: Spoken mathematics, classroom research, international comparisons

INTRODUCTION

The Learner’s Perspective Study (LPS) sought to investigate the practices of well-taught mathematics classrooms internationally. Data generation focused on sequences of ten lessons, documented using three video cameras, and interpreted through the reconstructive accounts of classroom participants obtained in post-lesson video-stimulated interviews (Clarke, 2006). The post-lesson interviews address the challenge of inferring student conceptions from video data (Cobb & Bauersfeld, 1994). The LPS approach of conducting case studies of classroom practices over sequences of at least ten lessons in the classes of several competent eighth grade teachers in each of the participating countries offers an informative complement to the survey-style approach of the two video studies carried out by the Third International Mathematics and Science Study (TIMSS) (Hiebert et al., 2003; Stigler & Hiebert, 1999). The criteria for the identification of the competent teachers studied in the LPS were specific to each country, in order to reflect the priorities and values of the school system in that country. In this paper, we report analyses of lessons documented in classrooms in Australia, China (Hong Kong and Shanghai), Japan, Korea, and the USA.

The complete research design has been detailed elsewhere (Clarke, 2006). For the analysis reported here, the essential details relate to the standardization of transcription and translation procedures. Since three video records were generated for

each lesson (teacher camera, student camera, and whole class camera), it was possible to transcribe three different types of oral interactions: (i) whole class interactions, involving utterances for which the audience was all or most of the class, including the teacher; (ii) teacher-student interactions, involving utterances exchanged between the teacher and any student or student group, not intended to be audible to the whole class; and (iii) student-student interactions, involving utterances between students, not intended to be audible to the whole class. All three types of oral interactions were transcribed, although type (iii) interactions could only be documented for the selected focus students in each lesson. Where necessary, all transcripts were then translated into English. All participating research groups were provided with technical guidelines specifying the format to be used for all transcripts and setting out conventions for translation (particularly of colloquial expressions).

In this paper, our unit of analysis is the utterance and we distinguish *private* spoken student-student interactions from whole class or teacher-student interactions, both of which we consider to be *public* from the point of view of the student. Our major concern in this analysis was to document the opportunity provided to students for the oral articulation of the relatively sophisticated mathematical terms that formed the conceptual content of the lesson and to distinguish one classroom from another according to the manner in which such student mathematical orality was afforded, promoted, constrained or discouraged in both public and private arenas.

STUDYING SPOKEN MATHEMATICS IN THE CLASSROOM

This paper reports four stages of a layered attempt to progressively focus on the significance of the situated use of mathematical language in the classroom. In our first analytical pass, an utterance is taken to be a continuous spoken turn, which may be both long and complex. We restricted our second-pass analysis to those mathematical terms and phrases that referred to the substantive content of the lesson (usually designated as such in the teacher's lesson plan and post-lesson interview). The third and fourth passes repeated the focus on utterances and then mathematical terms, but in the context of student-student (private) conversation.

We take the orchestrated use of mathematical language by the participants in a mathematics classroom to be a strategic instructional activity by the teacher. In this paper, we invoke theory in two senses: (i) the (researchers') theories by which the actions of the classroom participants might be accommodated and explained, and (ii) the (participants') theories implicit in the classroom practices of the teacher and the students. A particular focus is the role of the spoken word in both. The instructional value of the spoken public rehearsal of mathematical terms and phrases central to a lesson's content could be justified by reference to several theoretical perspectives. Interpretation of this public rehearsal as incremental initiation into mathematics as a discursive practice could be justified by reference to Walkerdine (1988), Lave and Wenger (1991), or Bauersfeld (1994). The instructional techniques employed by the

teacher in facilitating this progression could be seen as “scaffolding” (Bruner, 1983) and/or as “acculturation via guided participation” (Cobb, 1994).

The oral articulation of mathematical terms and phrases by students could be accorded value in itself, even where this consisted of no more than the choral repetition of a term initially spoken by the teacher. Teachers and students in some of the classrooms we studied clearly attached value to this type of recitation. In other classrooms, the emphasis was on the students’ capacity to produce a mathematically correct term or phrase in response to a very specific request (question/task) by the teacher. In such classrooms, both of these activities accorded very limited agency to the learner and the responsibility for the public generation of mathematical knowledge seemed to reside with the teacher. By contrast, in other classrooms, the instructional approach provided opportunities for students to “brainstorm” or to generate their own verbal (written or spoken) mathematics, with very little (if any) explicit cuing from the teacher (e.g. the classrooms in Tokyo).

The role of student-student spoken interactions also varied widely among the classrooms studied. The teacher’s posing of particular mathematical tasks (Mesiti & Clarke, in press) could prompt (and even promote) certain forms of individual, dyadic or small group mathematical behaviour and even monitor and guide that behaviour during classroom activities such as Kikan-Shido (Between-desks-instruction) (O’Keefe, Xu, & Clarke, 2006). However, within these constraints, students have significant latitude and agency in their use of spoken mathematics. The frequency of occurrence of student-student utterances varied from zero in some lessons (eg. Seoul) to as many as 100 distinct student-student utterances per lesson by individual students in classrooms in Australia and the USA. In each classroom, the activity of speaking mathematics was performed differently.

The results that are reported in this paper certainly suggest that the teachers in this study differed widely in the opportunities they provided for student spoken articulation of mathematical terms, whether in public or in private, and in the extent to which they devolved agency for knowledge generation to the students. The demonstration of such differences (and we would like to argue that these differences are profound and reflect fundamental differences in basic beliefs about effective instruction and the nature of learning) in the practices of classrooms situated in school systems and countries that would all be described as “Asian” suggests that any treatment of educational practice that makes reference to the “Asian classroom” confuses several quite distinct pedagogies. This observation is not to deny cultural similarity in the way in which education is privileged and encountered in communities that might be described as “Confucian-heritage.” But, the identification of a one-to-one correspondence between membership of a Confucian-heritage culture and a single pedagogy leading to high student achievement is clearly mistaken, and cultural similarity is not a sufficient indicator of those instructional practices that might be associated with the educational outcomes that we value.

THE USE OF MATHEMATICAL TERMS

In this paper, “utterance” and “mathematical term or phrase” require clear specification (below). Our analysis of public and private classroom interactions has restricted its attention to key and related (primary and secondary) terms, however the analysis of the post-lesson student interviews also considered ‘other’ terms used by students in interview to explicate the lesson’s content or in reflecting on the nature of mathematical activity in general. This paper focuses on analysis of public and private classroom interactions. Consideration of student use of spoken mathematics in the post-lesson interviews will be reported in another paper.

Figure 1 shows the number of utterances occurring in whole class and teacher-student interactions in each of the first five lessons from each of the classrooms studied in Shanghai, Hong Kong, Seoul, Tokyo, Melbourne and San Diego. An utterance is a single, continuous oral communication of any length by an individual or group (choral). Used in this way, the frequency (and origins) of public utterances constitute a construct we have designated as public oral interactivity. This does not take into account either the length of time occupied by an utterance or the number of words used in an utterance (problematic in a multi-lingual study like this one). Figure 1 distinguishes utterances by the teacher (white), individual students (black) and choral responses by the class (e.g. in Seoul) or a group of students (e.g. in San Diego) (grey). Any teacher-elicited, public utterance spoken simultaneously by a group of students (most commonly by a majority of the class) was designated a “choral response.” Lesson length varied between 40 and 45 minutes and the number of utterances has been standardized to 45 minutes.

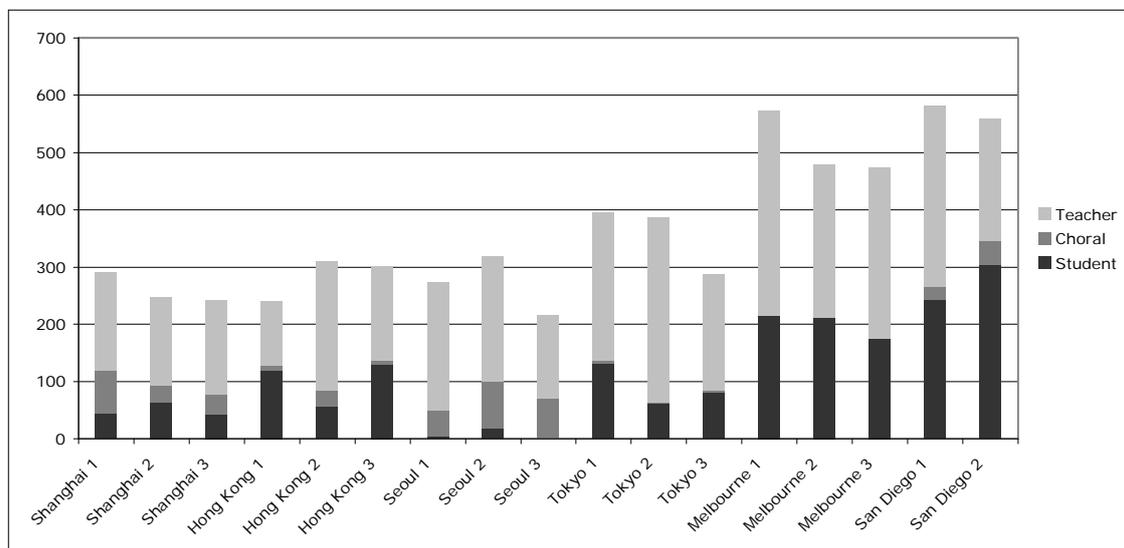


Figure 1: Number of Public Utterances in Whole Class and Teacher-Student Interactions (Public Oral Interactivity)

Figure 1 suggests that lessons in Melbourne and San Diego demonstrated a much higher level of public oral interactivity than lessons in Shanghai, Hong Kong, Seoul, or Tokyo. There were also substantial differences in the relative frequency of teacher,

student and choral utterances. It is worth noting that both teacher and student utterances in Shanghai tended to be of longer duration and greater linguistic complexity than elsewhere.

The classrooms studied can be also distinguished by the relative level of public mathematical orality of the classroom (that is, the frequency of spoken mathematical terms or phrases by either teacher or students in whole class discussion or teacher-student interactions) and by the use made of the choral recitation of mathematical terms or phrases by the class. This recitation included both choral response to a teacher question and the reading aloud of text presented on the board or in the textbook. For the purposes of this paper, those mathematical terms were coded that comprised the main focus of the lesson's content.

Figure 2 shows how the frequency of public statement of mathematical terms varied among the classrooms studied. In classifying the occurrence of spoken mathematical terms, we focused on those terms that could be related to the main lesson content (e.g. terms such as “equation” or “co-ordinate”). This meant that our analysis did not include utterances that constituted no more than agreement with a teacher's mathematical statement or utterances that only contained numbers or basic operations that were not the main focus of the lesson.

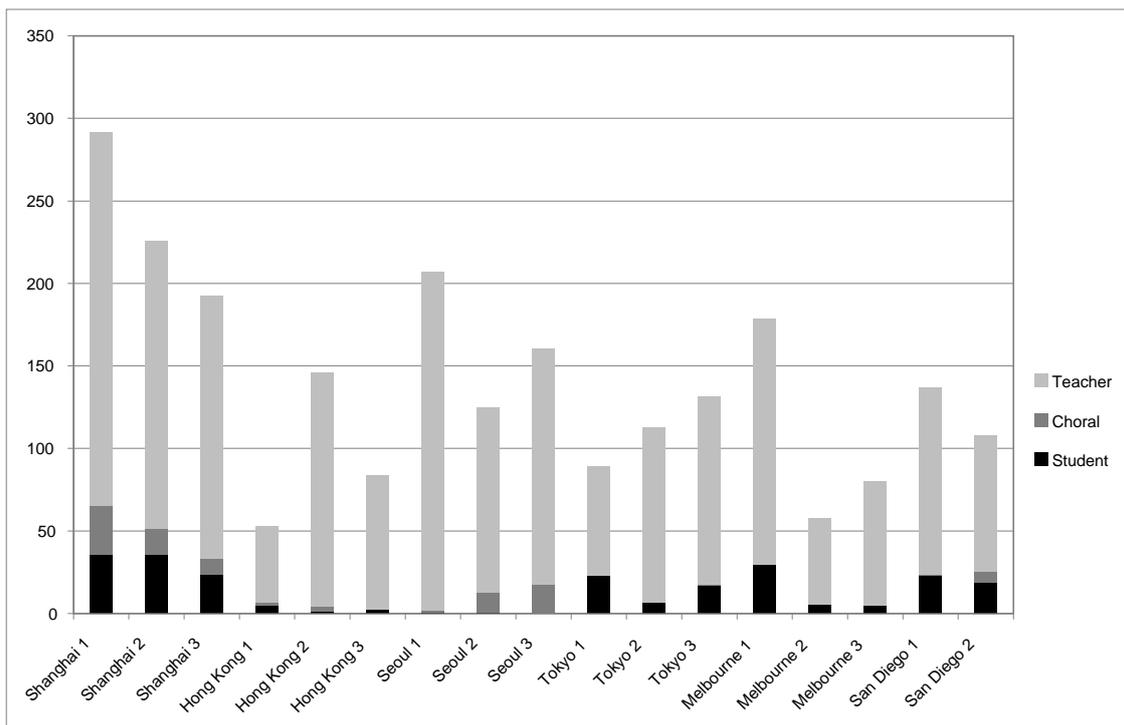


Figure 2: Frequency of Occurrence of Key Mathematical Terms in Public Utterances (Mathematical Orality)

In the case of the Korean lessons, the choral responses by students frequently took the form of agreement with a mathematical proposition stated by the teacher. For example, the teacher would use expressions such as, “When we draw the two equations, they meet at just one point, right? Yes or no?” And the class would give

the choral response, “Yes.” Such student statements did not contain a mathematical term or phrase and were not included in the coding displayed in Figure 2. Similarly, a student utterance that consisted of no more than a number was not coded as use of a key mathematical term. It can be argued that responding “Three” to a question such as “Can anyone tell me the coefficient of x ?” represented a significant mathematical utterance, but, as has already been stated, our concern in this analysis was to document the opportunity provided to students for the oral articulation of the relatively sophisticated mathematical terms that formed the conceptual content of the lesson. Frequencies were again adjusted for the slight variation in lesson length.

The most striking difference between Figures 1 and 2 is the reversal of the order of classrooms according to whether one considers public oral interactivity (Figure 1) or public mathematical orality (Figure 2). The highly oral classrooms in San Diego made relatively infrequent use of the mathematical terms that constituted the focus of the lesson’s content. By contrast, the less oral classrooms in Shanghai made much more frequent use of key mathematical terms and phrases. Since a single utterance might contain several such terms, and it was terms that were being counted in this analysis, Figure 2 provides a different and possibly more useful picture of the Chinese lessons, where both teacher and student utterances appeared to be longer and more complex than elsewhere.

Comparison between those classrooms that might be described as “Asian” is interesting. Key mathematical terms were spoken less frequently in the Seoul classrooms than was the case in the Shanghai classrooms. Even allowing for the relatively low public oral interactivity of the Korean lessons, the Korean students were given proportionally fewer opportunities to make oral use of key mathematical terms in whole class or teacher-student dialogue. In contrast to the teachers in Shanghai and Tokyo, the teachers in the Hong Kong and Seoul classrooms did not appear to attach the same value to the spoken rehearsal of mathematical terms and phrases, whether in individual or choral mode. It should be noted that Hong Kong 3 used English as the instructional language, while Hong Kong 1 and 2 used Cantonese, so any common features of the Hong Kong classrooms are likely to reflect dominant pedagogical practices, rather than be a specific result of the use of the Chinese or English language. The teacher in Hong Kong 2 appears similar to the three Shanghai teachers in the sense that he conducted his teaching most frequently in the form of whole class discussion. But his lessons show no signs of the pattern, evident in all three Shanghai classrooms, where the students were systematically ‘enculturated’ into the language of school mathematics. In particular, despite similarities between the public oral interactivity of Hong Kong 2 and Shanghai 1 (for example), the frequency of student use of mathematical terms in Hong Kong 2 was much lower.

While the overall level of public oral interactivity in the Tokyo classrooms was similar to those in Seoul, the Japanese classrooms resembled those in Shanghai in the consistently higher frequency of student contribution, but with little use being made of choral response. The value attached to affording student spoken mathematics in

some classrooms could suggest adherence by the teacher to a theory of learning that emphasizes the significance of the spoken word in facilitating the internalisation of knowledge. The use of choral response, while consistent with such a belief, could be no more than a classroom management strategy. The Hong Kong classrooms offered students least opportunity to use spoken mathematical terms of all the classrooms studied and student spoken mathematical contribution, whether individual or choral, was extremely low, even though the student component of general public oral interactivity of the Hong Kong classrooms was at least as high as in Shanghai.

THE RELATIVE SIGNIFICANCE OF STUDENT-STUDENT INTER-ACTIONS

While the private conversations recorded in any one lesson were only those of the Focus Students, it was possible to compare the public oral interactivity of these students with their private oral interactivity and, similarly, their public and private mathematical orality. From the outset, it must be noted that six classrooms stood out because of the virtually complete absence of student-student interaction: those in Shanghai and Seoul. In these six classrooms, student-student conversation can be discounted as an instructional strategy (or as a subversive practice by students). For example, in Seoul classroom 1, there were no instances of student private talk in the first four recorded lessons and only two private utterances from one of the focus students in lesson five. The first utterance was “That’s yours” and the second was “No.” Obviously, neither involved any technical mathematical terms.

In reporting the results that follow, we have put both Shanghai and Seoul to one side. The role played by private student-student interactions in the remaining classrooms is particularly interesting. In Table 1, the figures quoted for both public and private Oral Interactivity and Mathematical Orality are per focus student per lesson and have therefore been averaged over the spoken contributions of around 10 students per classroom. This should minimize the effect of individual student timidity or extroversion, although awareness of being recorded will have been a common characteristic of all focus students (and of their teachers). In reading the ratio columns of Table 1, it is simplest to think of the results as indicating, for example, that focus students in Hong Kong class 1 used a mathematical term on average once every eight public utterances but only once every 48 private utterances.

It seems a reasonable hypothesis that student use of mathematical terms would be less likely in private contexts than in public teacher-orchestrated contexts. For seven of the 11 classes reported in Table 1, this was clearly the case. It is all the more interesting, therefore, that in all three Japanese classrooms and one of the Hong Kong classrooms the focus students were at least as likely to use mathematical terms in private conversation as they were to use them when participating in teacher-orchestrated public discussion. Hong Kong 2 seems anomalous in its very low number of student utterances per lesson, both private and public. With such small

utterance numbers, slight variations in count may have the effect of inflating the ratio of private utterances to privately spoken mathematical terms.

Table 1: The use of spoken mathematics by students in public and private contexts

Schools	Oral Interactivity (utterances per focus student per lesson)		Mathematical Orality (mathl. terms per focus student per lesson)		Public Ratio (utts./term)	Private Ratio (utts./term)
	Public	Private	Public	Private		
Hong Kong 1	4.21	22.59	0.52	0.47	8.10	48.06
Hong Kong 2	2.84	7.15	0.41	1.30	6.93	5.50
Hong Kong 3	2.39	23.80	0	0.83	n.a.	27.67
Tokyo 1	6.13	14.79	0.28	2.24	21.89	6.60
Tokyo 2	2.08	33.85	0.23	9.46	9.04	3.58
Tokyo 3	6.92	11.67	0.61	0.99	11.34	11.79
Melbourne 1	16.16	99.14	2.85	5.59	5.67	17.74
Melbourne 2	14.36	83.75	0.18	0.30	79.78	279.17
Melbourne 3	15.78	73.51	0.17	5.63	92.82	13.06
San Diego 1	12.69	6.64	1.36	0	9.33	n.a.
San Diego 2	9.31	55.33	1.12	3.56	8.31	15.54

The Japanese result remains interesting; suggesting that Japanese students have a fluency in spoken mathematics that persists even across the public/private interface. It is also clear that student-student mathematical exchange was a feature of the Tokyo mathematics classrooms studied to a much greater extent than for the classrooms in Shanghai and Seoul.

CONCLUSIONS

It appears to us that the key constructs Public Oral Interactivity and Public Mathematical Orality distinguished one classroom from another very effectively. Particularly when the two constructs were juxtaposed (by comparing Figures 1 and 2). The contemporary reform agenda in the USA and Australia has placed a priority on student spoken participation in the classroom and this is reflected in the relatively high public oral interactivity of the San Diego and Melbourne classrooms (Figure 1). By contrast, the “Asian” classrooms, such as those in Shanghai, were markedly less oral. However, this difference conceals differences in the frequency of the spoken occurrence of key mathematical terms (Figure 2), from which perspective the Shanghai classrooms can be seen as the most mathematically oral. However, students in the Tokyo classrooms used spoken mathematics in both public and private situations. The relative occurrence of spoken mathematical terms is one level of analysis. We should also distinguish between repetitive oral mimicry and the public (and private) negotiation of meaning (Cobb & Bauersfeld, 1994; Clarke, 2001).

Despite the frequently assumed similarities of practice in classrooms characterised as Asian, differences in the nature of students' public spoken mathematics in classrooms in Seoul, Hong Kong, Shanghai and Tokyo are non-trivial and suggest different instructional theories underlying classroom practice. Any theory of mathematics learning must accommodate, distinguish and explain the learning outcomes of each of these classrooms. Consideration of the non-Asian classrooms is also interesting. With frequent teacher questioning and eliciting of student prior knowledge, the students in the Melbourne classrooms were given many opportunities to recall and orally rehearse the mathematical terms used in prior lessons. In terms of overall public mathematical orality and level of student contribution, Melbourne 1 resembles Shanghai 1 (without the use of choral response). In Melbourne 1, this public orality was clearly augmented by small group discussions, in which students drew upon their mathematical knowledge to complete tasks at hand. Such student-student conversations occurred much more frequently in the Melbourne classrooms. Student use of mathematical terms in situations not directly orchestrated by the teacher can be taken as a reasonable indicator of both the perceived need and the capacity for the purposeful employment of the technical language of mathematics. The relative infrequency of mathematical terms in student-student interactions in Melbourne 2 compared with the other two Melbourne classrooms suggests that these indicators are reflective of teacher influence.

To summarise: Students in the mathematics classrooms in Seoul have few opportunities to speak in class (either privately or publicly) and seldom employ spoken mathematics. Students in the Hong Kong classrooms are publicly and privately vocal, but make very little use of spoken mathematical terms in either context. Students in the mathematics classrooms in Shanghai are guided through the public orchestrated rehearsal of mathematical terms by their teachers, but seldom speak to each other in private during class time. Students in the mathematics classrooms in Tokyo participate orally in both public and private discussion and employ mathematical terms to a significant extent in both. By comparison, the students in Melbourne classroom 1 are highly vocal in both public and private contexts, and make more frequent public use of mathematical terms than any of the three Japanese classrooms, but less frequent use of mathematical terms in their private conversations. These different combinations of oral interactivity and mathematical orality represent at least five distinct pedagogies.

The next question is, of course, whether or not students are advantaged in terms of their mathematical achievement and understanding by classroom practices that afford the opportunity to develop facility with spoken mathematics. The implicit assumption in the classrooms studied in Hong Kong and Seoul seems to be that the employment of spoken mathematics is not to the students' benefit. Classrooms studied in Melbourne, Tokyo and Shanghai, despite differences in implementation, seem to make the opposite assumption. The post-lesson interviews may provide evidence of a connection between classroom mathematical orality and student learning outcomes.

This analysis is currently underway. We suggest that the empirical investigation of mathematical orality (in both public and private domains) and its likely connection to the distribution of the responsibility for knowledge generation are central to the development of any theory of mathematics instruction.

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MATHEMATICAL BEHAVIORS OF SUCCESSFUL STUDENTS FROM A CHALLENGED ETHNIC MINORITY

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This study explored the mathematical behavior of resilient students of Ethiopian origin (SEO), members of an underrepresented and challenged ethnic group in Israel. Using qualitative methodologies, we examined six SEO, three in an advanced secondary school mathematics track and three in a pre-academic course while working on non-routine mathematical tasks. The mathematical behaviours and views of these students were found to be highly consistent with their professed beliefs and behaviors, which we explored in a previous study. Success was attributed to beliefs enacted during problem solving and was accounted for by neither giftedness nor special ethnic characteristics, but rather by high motivation, self-regulation, and persistence driven by positive identities, personal agency and ethnic identification.

Key words: Mathematical behavior, beliefs, self-regulation, resilient, ethnic identity.

INTRODUCTION

In many countries all over the world, immigrants and ethnic minorities often face barriers at school resulting from various factors. Many researchers and educators believe that differential student learning, achievement, and persistence along ethnic and racial lines is one of the most troubling issues in mathematics education and in education in general (e.g. Martin 2000, 2003). In the case of Israel, educators and researchers have done much to describe and classify social, cultural, educational, and other societal difficulties encountered by different groups of immigrant Jews and in particular, those students of Ethiopian origin (SEO, more than half of whom are second generation). A range of studies have documented the overall academic underachievement, the relatively high dropout rates, and the high representation of SEO in special education programs (e.g. Lifshitz, Noam, & Habib, 1998; BenEzer, 2002; Levin, Shohami, & Spolsky, 2003; Wolde Tsadik, 2007). In mathematics, SEO are significantly underrepresented in the advanced tracks towards Matriculation. For example, during the years 1999-2003, among all SEO who were eligible for the 'Bagrut', the Matriculation exam taken at the end of grade twelve in different subjects, only 2% studied mathematics in the advanced track [1], compared with 17% of the entire student population.

In different countries, some groups of immigrants and ethnic minorities achieve well academically; sometimes they even outperform mainstream students. Several studies have focused on explaining differential achievements between various minority groups and within certain minority groups (e.g. Ogbu, 1991; Martin, 2000, 2003; OECD, 2006). Most findings challenge the belief that the disadvantages and difficulties created by being an immigrant or a member of a minority prevent students from excelling in education.

Researchers are increasingly linking motivational, cognitive, and social environmental aspects of learning. Many studies have provided new insights into why individuals choose to engage in different learning activities, and how their identities, beliefs, values, and goals relate to their engagement and mathematics achievements (Steele, 1997; Nasir, 2002; Martin, 2000, 2003; Sfard & Prusak, 2005). It is argued that students' problem-solving processes are influenced by beliefs about the self, about the nature of mathematics knowledge, the task at hand, and its context (e.g. Schoenfeld, 1983). Moreover, implementing self-regulation during problem solving is regarded as an important variable affecting the quality of the solving process: self-regulated learners analyze tasks and set appropriate goals to accomplish these tasks, monitor and control their behaviors during performance, make judgments of their progress and alter their behaviors according to these judgments (Zimmermann, 1989). Social cognitive theorists, assume that self-efficacy is a key variable affecting self-regulated learning and performance (Bandura, 1986); self-regulated learning is believed to occur to the degree that a student can use personal (i.e. self) processes to strategically control and direct both his/her behavior and the immediate learning environment (Bandura, 1986; Zimmermann, 1986).

Based on the personal and environmental factors identified by research in mathematics education and especially based on the findings related to the success of individuals from populations at risk of academic failure, we sought to understand the success factors of SEO, students enrolled in the advanced mathematics track towards Matriculation. We focused on these students' views about their personal experiences in learning mathematics and the perceived impact of the personal and environmental variables on their persistence and success [2]. The conceptual framework used to guide our inquiry is based on the assumption that there are certain malleable personal and environmental factors that play significant roles in these students' academic resilience, defiance of the odds and their ultimate academic achievement. We adhere to the claim that, as opposed to studies of failure (regardless of their academic depth), studies of success constitute a more promising way of understanding and eventually increasing the circle of successful students (Garmezy, 1991; Martin, 2003). In our studies we sought to understand what enables some SEO to succeed despite the potential obstacles they face. We attempt to answer the following questions:

1. To what perceived personal/environmental variables do SEO in Israel attribute their success in mathematics?
2. What are the salient mathematical behaviors of SEO when working on mathematical tasks? How do they view, and reflect upon, their own behaviors?
3. How do the perceived variables, the enacted mathematical behaviors, and the students' views of these behaviors relate to each other?

In a previous study we explored the first question, through students' self-reports obtained using semi-structured interviews (see below a summary of this study). In the present study we present findings concerning the second and third questions.

FINDINGS FROM THE PREVIOUS STUDY: STUDENTS' SELF-REPORTS

A diverse group of SEO enrolled in the advanced mathematics track towards Matriculation were interviewed and followed up. The group consisted of fourteen students aged 17-19 (seven males and seven females), of which nine were high school students from four different cities and the other five were students enrolled in a special pre-academic program in a prestigious technological university in Israel (each from a different city). All were 'solos', i.e., the only SEO in the advanced mathematics track in their cohort at their schools, which is the optimal situation in most high schools. Our goal was to better understand how these students interpret their experiences and academic achievements within the advanced track in mathematics, in high school and in the university preparatory program, where the presence of students of Ethiopian origin is scarce. Using the qualitative methodology of a collective case study (Yin, 1984; Shkedi, 2005), we analyzed interview transcripts using a grounded approach and employing open coding techniques (Strauss & Corbin, 1990). Data were also triangulated with other sources such as classroom observations and interviews with other students, teachers, and parents. The key elements of success we identified were organized under three major categories (Mulat & Arcavi, submitted):

- (1) Motivational variables related to mathematics (e.g., mathematics identity, personal agency, productive attribution beliefs, academic goals, ethnic identification, and social goals activated by a positive cultural model)
- (2) Actions and strategies – perceived behavior (e.g., fostered use of academic self-regulation and coping strategies)
- (3) Immediate environmental variables (mathematics classrooms, teachers, and parental support)

The central finding of the study was that the synergy among students' motivational variables, their academic self-regulation and coping strategies, shaped and supported by their interaction with the environment, appeared as the key to their success in mathematics.

THE PRESENT STUDY

The aim of the study reported here is to explore the mathematical behaviors and the task-related views of a subgroup of the participants in the previous study, and to examine how the findings of the two studies relate to each other.

METHODOLOGY

Subjects: Six SEO from the previous study participated in this study. Three of them (Eden, Melka, and Jacob) were high school students, and the other three (Selam,

Ronnie, and Danny; all pseudonyms) were students in the pre-academic program. The selection of these participants depended upon the availability of extensive data relevant to this study.

Tasks: The students worked on five mathematical tasks, selected especially for this study according to the following criteria: The tasks had alternative solutions; they varied in their level of difficulty; their content level was rather basic and accessible to high school students, yet they were non-routine, challenging, and required some planning strategies. The problems were previewed by mathematics educators who agreed on the mathematical appropriateness for high school students.

Data collection and analysis: The data consisted of students' written work, the interviewer's recorded observations, the protocols of the dialogues, questions and reflections that emerged during task completion, and the transcripts from follow-up interviews. In the interviews, all students were asked to describe their solution approaches and their thinking processes in completing the tasks and to describe their perspectives. These tasks were also given to students' peers in the lower mathematics tracks of the secondary schools. A qualitative descriptive methodology was used to analyze the combined data (Shkedi, 2005).

FINDINGS

A description of students' solution processes, along with the observed behaviors and views for three of the tasks are given, followed by a summary of the significant findings.

Problem 1.

Find the equation of the line parallel to the given pair of parallel lines and that lies exactly midway between them: (1) $3x-2y-1=0$

$$(2) 3x-2y-13=0$$

Task completion: All participants efficiently completed this problem. The task was characterized by all of them as non-routine since its formulation was seen as different from what they usually encountered at schools, yet it was perceived as easy and accessible by available tools or algorithms.

All subjects showed confidence in their ability to complete this task, and had completed it easily; appearing to be satisfied with their ability (two had minor computational errors). However, despite the existence of alternative ways to solve the problem, both the high school and the pre-academic students applied the 'slope-point formula' procedure they learned at school. Accordingly, the common stages in the students' solution procedures were in this order:

- Transformation of the equations to an explicit form
- Identification of the common slope
- Identification of the y-intercepts (some solved for the x-intercepts)

- Finding the midpoint between the intercepts (using formula or graphs)
- Writing the answer - equation of the line

The participants attributed their success to their rich experience and mastery of similar school tasks. Yet, this task was found to be difficult to many students in the lower tracks, who blindly tried to solve the pair of simultaneous equations in search of a point, after they found the common slope, the first two stages above.

Although both the high school students and the pre-academic students were equally successful in solving this task, we detected a difference in their use of a heuristic and the perception of its necessity. Two of the high-school students drew the graphs of the lines to find the midpoint of the intercepts, whereas all of the pre-academic students did not, claiming that they do not need the graphs to solve this problem and if they do, they can imagine them. The following quotations exemplify these differences among students:

Instead of visualizing in your head, it is already in your notebook and it is hard to get confused that way. (Jacob)

Here I do it in my head. You see that they have the same slope...when I can't see things with my imagination, I use sketches. But here you know the question leads you to the solution. (Selam)

Problem 2.

ABC is a right-angled triangle, $\angle ABC=90^\circ$.

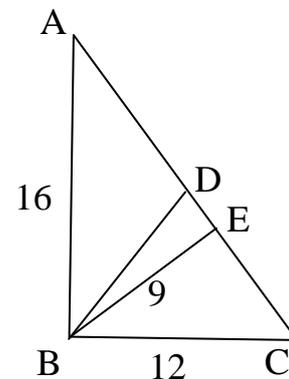
AB=16; BC=12 and BE=9; BD is the median to AC, and BE is the altitude to AC.

There is an error in one of the given numbers.

(a) Show that there is an error (report all your processes).

(b) Change only one of the numbers (9, 16, or 12)

to correct the error.



Task completion: Students showed different performance levels on this task. All students started by marking the given numbers on a triangle they drew and by calculating the length of the hypotenuse $AC=20$ (one made a computational error). Five of the students also marked $AD=BD=DC=10$, referring to the theorem about the median to the hypotenuse in a right triangle, but only three used this information to produce their solutions later. Only three of the students completed both parts of the task independently showing ease and confidence (but one had computational as well as other major errors and thus got a wrong answer). The other three students had difficulties in devising a plan and an effective strategy to proceed with the task; they were stuck for a long time; two of them said that they checked whether there is a side with a length greater than the sum of the other two sides. These students were confused and disturbed since they did not know how to plan their solution procedure

and were uncertain about their understanding of the question. After some unsuccessful trials, they quit and proceeded with the other questions and returned to complete the task after receiving supporting clues and prompts from peers and from the interviewer.

In the first part of the task, students used different strategies to show that a triangle having the given sizes is not possible. Two showed that they got two different areas for the same triangle, three showed two different sizes for a side of the triangle; another student showed that the corresponding sides of similar triangles are not proportional. Five of these students used the same strategies they used for the first part to answer the second part of the question. One chose to use a trial and error method. Half of the students mentioned the possibility that the error could be corrected by changing any one of the three numbers. Since there were different ways to show that there is an error, the error could be corrected by changing any one of the three numbers, implying different ways and possibilities to answer the second part of the question. Yet all participants decided to change 9 (which was a good choice); four students (two of them with support) completed the problem successfully. The other two students, one who used a trial and error method and another who made a major error in her computations to change 9 got wrong results.

All students characterized this task as non-routine, saying that it is not like school tasks that they usually solve with great ease and success, and that here they could not just apply known algorithms to obtain a solution. Danny, who completed all the tasks successfully, characterized this task as 'a deceptive question'. Jacob said:

This is a question in geometry, but never, at least I never encountered questions like this, saying that there is a mistake, correct a mistake. Usually they give you exercises that have solutions at the very beginning, and if you work by the book, you succeed, but here you have to think more.

Melka also referred to her school experiences:

We are not used to such kind of questions; they never tell us to correct mistakes; they always provide us with given objects and ask to do other things and not to correct mistakes.

In sum all the students (some with probing), completed the first part of this task successfully by using different strategies. While four of them also succeeded with the second part, the other two students used ineffective strategies and got wrong answers.

Problem 3.

Given is an array of natural numbers arranged under four columns, A, B, C, and D, as shown here.

(a) Under which letter does the number 101 appear?

(b) Under which letter does the number 1001 appear?

A	B	C	D
1	2	3	4
8	7	6	5
9	10	11	12
	...	14	13

(c) Answer questions (a) and (b) above, for a five-column array of numbers with the same pattern

Task completion: This task seemed to be more difficult than the other tasks especially for the high school students. It also took most students more time than each of the other tasks. Only two students (both pre-academic) found effective rules and gave correct solutions with clear explanations.

As a first step towards finding a solution to this problem, all students added more numbers to the list following the given pattern. All of them tried for quite a long time to find a possible pattern or rule to solve this task (unlike students in the lower tracks, who tried to answer it by listing numbers to reach 101 without looking for a rule). As stated above, only two students (both pre-academic) proposed a similar rule: even and odd multiples of 4 can be found in alternate lines of the outer left and outer right columns (A and D), respectively. The other pre-academic student, however, did not recognize the sequences' pattern on the extreme columns and added to the list in a wrong order. Consequently, she did not succeed in completing the task, but she refused to hear a solution method and asked to complete the task at home by herself. Three of the high school students did not write their rule clearly, and their answers were mostly wrong or not justified, although two were certain they had obtained a working rule. The other student, who seemed less confident, said that she solved it logically, using her common sense, and that she did not know how to communicate her method.

This task was characterized as difficult by all participants. One of the students even commented that it is not a mathematical question; the other said it is a 'thinking' question that challenges the mind, and that schools do not offer such questions.

SUMMARY

As stated above, this study explored the mathematical behaviors displayed by successful SEO, and analyzed the relationships between these behaviors and the professed beliefs and reflections found in a previous study in which the students also participated. Some of the findings from the previous study (e.g., ethnic identification, social goals, and parental support) were not salient in the present study due to their very nature; these categories are rarely captured while students work on mathematical tasks. However, in other findings we found consistency between the 'professed' beliefs and behaviours and the 'enacted' mathematical behaviors, as described in the following.

Motivational beliefs: Students showed a variety of behaviors and performances. Although some students lacked confidence when they had no handy effective strategies, their behaviors were consistent with their professed efficacy beliefs and their confidence in their ability to solve the problems. They said that they have the mathematical knowledge necessary for completing the tasks and shared their enjoyment and satisfaction of being engaged in questions that demand thinking. They attributed their difficulties in solving these problems to a lack of previous experience

with non-routine questions. They expressed their expectations that schools should provide opportunities to encounter and practice such tasks that require 'thinking'.

Self-regulation strategies: Self-regulation is one of the characteristics that we had identified in the previous study as playing a prominent role in these students' success in school mathematics. The students expressed their belief that what it takes to succeed in school is planning and evaluating their own actions and strategies by investing time and effort to study what is taught at school. When these students failed to solve some of the non-routine tasks of this study, they attributed it to not having the right tools, since their learning efforts were directed to what school had taught them. Thus, cognitive regulation and retrieval of the appropriate knowledge and the strategic tools needed for the tasks in this study were difficult for them. Many of the students quit after some unsuccessful trials, moved on to other questions but still returned to the unsolved tasks later. We took this willingness not to give up as yet another manifestation of these students' good self-regulation strategies applied to difficult situations for which they were unprepared. This strategy was found to pay off for some students, since with some probing they succeeded to complete the tasks.

Solo learning: Though students were told that they can work with their peers (three of them had opportunities to do so) and also that they can ask for support from the interviewer at any time while working on the tasks, they did not use these opportunities productively. Suggestions to support the students when they were stuck at certain stages were all initiated by the interviewer. The preference of students to work alone was also in line with these students' professed 'solo learning' characteristics.

Perceptions about the tasks: The tasks were characterized as non-routine, including the first question that all could easily solve, yet they expressed their enjoyment and satisfaction in performing such tasks. The students were very critical about mathematics lessons at schools that do not offer students opportunities to face challenging tasks.

Differences within groups: Though the tasks are appropriate for any high school student, overall, the pre-academic students showed (a) greater confidence in completing the tasks (even when they were not always successful), and (b) better communication skills to write and explain clearly their solution processes. These differences could be attributed to the pre-academic students' self-reports that in contrast to high school teachers, the teachers in the program have exposed them to meaningful mathematics learning, which also developed their confidence, intrinsic interest in mathematics and mathematics identity.

DISCUSSION AND CONCLUSION

Whereas these SEO's success in school was, to some extent, due to learning by playing well the school rules, which are mostly rehearsing and following algorithms, completing the tasks of this study engaged these students with a quite different experience. Thus, since these students were not especially gifted and their knowledge

resources come only from school, their success can be attributed to their mathematics identity, motivation, and self-regulation skills; all these were supported by their other professed beliefs and views in relation to the tasks. Moreover, the heterogeneity of solution approaches and strategies observed in this study is proposed as a further confirmation how resilient and minded to success these students are, each of whom mustered resources and alternatives from his/her own to solve the tasks.

In sum, neither exceptional cognitive ability nor common cognitive characteristics of a certain "ethnic" group are variables that play significant roles in analyzing success (or failure) of these SEO. It is their determination, personal identities and support that shape their self-regulation, persistence and beliefs that shape their behaviours and ultimately their success. From this and related findings, we argue that educational systems that want ethnic minorities to succeed academically have much to learn from these and related findings regarding the roles of identities, self regulation, enhancement of motivation and support of learning which can take place in collaboration with peers.

NOTES

1. In the Israeli education system not all students are eligible for Matriculation; eligibility is determined according to the students' prior achievements. In mathematics, those eligible have taken one of three levels: basic (3 units), intermediate (4 units), and advanced (5 units).
2. In this study success refers to enrolment in the advanced track towards Matriculation

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A PROBLEM POSED BY J. MASON AS A STARTING POINT FOR A HUNGARIAN-ITALIAN TEACHING EXPERIMENT WITHIN A EUROPEAN PROJECT

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The paper reports on a collaborative project involving Italy and Hungary, within the European Project PDTR [1], and presents an analysis of its implementation and outcomes. The work stemmed from a problem about the exploration of regularities, proposed by John Mason, scientific advisor of the project. We start from the preliminary analysis of the problem carried out by the two teams, present re-elaborated versions, planning of the activities and modalities for implementing them in the classroom in the respective countries, discuss the outcomes of the experiment, final reflections made by experimenting teachers and general ones made by the teams about the materials elaborated during the activities.

Key words: Arithmetical Regularities, Early Algebra, Teachers Professional Development, Teaching Experiment, Teaching-Research

INTRODUCTION

The central aim of PDTR project has been to engage teachers of mathematics in the process of systematic, research-based transformation of their classroom practice so to initiate, using teaching-research as the leading methodological agent, the transformation of mathematics education towards a system which, while respecting the standards and contents of the national curricula, would be more engaging and responsive to student's intellectual needs, promoting independence of thought, and realizing fully the intellectual capital and potential of every student and teacher.

The teachers' work, in a first phase, addressed issues and questions of the PISA test, with particular reference to the promoted competencies, some of them - such as argumentation, posing and solving problems, modelling and representations – are clear indicators of a new way of conceiving the mathematical teaching and classroom activity. In a second phase, the PDTR apprentices and IT designed teaching experiments, collected data, observed their pupils with a new investigatory eye, analyzed and discussed the data with their team members.

In this context, some teaching experiments were carried out with the aim of promoting a direct exchange between the teams on the ways of implementing common activities in the participating countries. The richest exchange occurred in the Hungarian-Italian Bilateral Teaching Experiment (HIBTE), which was developed in the field of the algebraic and pre-algebraic thinking (Malara & Navarra, 2003).

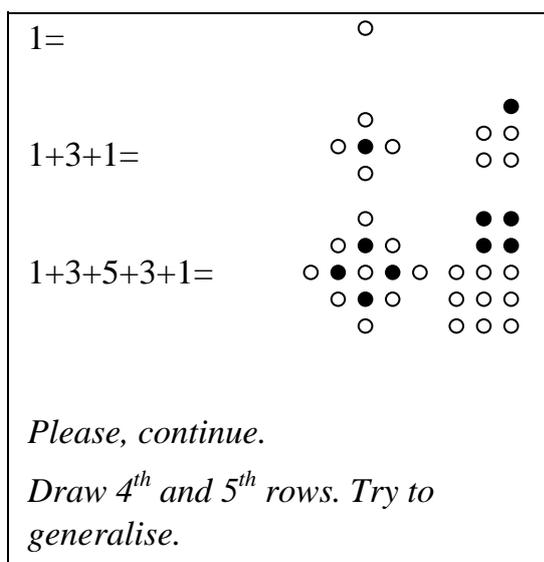
METHODOLOGY

Meaningful increasing research in mathematics education points to the renewal of its teaching through a linguistic and socio-constructive approach in the sense of early algebra with pupils of k-8th gr. In this perspective, teachers come to play a complex role in the classroom and they need to face a number of unpredicted and not easily manageable situations. Regarding this, several scholars highlight the importance of a critical reflection by teachers on their activity in the classroom (Mason, 2002; Ponte 2004) so that they can also become aware of the macro-effects on classroom activities caused by their (sometimes not appropriate) micro-decisions. To promote this attitude in teachers, within the Italian Team (IT), a complex written activity of critical analysis of classroom transcriptions, in which the teachers, their mentors, the mentor coordinator and the academic researcher cross their comments, has been enacted. It is called *Multi-Commented Transcripts Methodology* (MCTM) (Malara, 2008). The methodology of work between the two teams developed in 5 phases: 1) Adjustment by HT of the proposal made by John Mason, PDTR expert, to the Hungarian Team (HT) teachers; 2) didactical transposition of the adjusted proposal in HT classes (9th-12th gr.), evaluation of the results; 3) analysis, adjustment of HT proposals by IT and transposition in IT classes (6th-7th gr.); 4) implementation of MCTM; 5) analysis by HT of IT transcripts; 6) cross reflections.

DISCUSSION

The original proposal by John Mason to Hungarian PDTR teachers

During his lecture in Debrecen (Hungary), Mason asks the participants (about 30 pre-service mathematics teacher and about 30 secondary school mathematics teachers) to solve the following problem (Fig.1). After 10-15 minutes, it is clear that such type of problems are very uncommon to Hungarian teachers and students, most of them cannot do anything. Seeing the difficulties, Mason numbers the rows and sketches the fourth row in the shape of a 'cloud' which hides the sum (Fig.2).

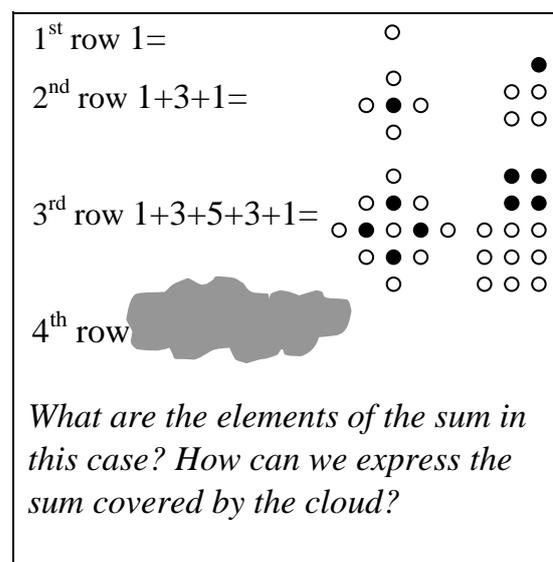


1 = ○

1+3+1 = ○ ● ○ ○ ○

1+3+5+3+1 = ○ ● ○ ● ○ ○ ○ ○ ○ ○ ○ ○ ○

*Please, continue.
Draw 4th and 5th rows. Try to generalise.*

1st row 1 = ○

2nd row 1+3+1 = ○ ● ○ ○ ○

3rd row 1+3+5+3+1 = ○ ● ○ ● ○ ○ ○ ○ ○ ○ ○ ○ ○

4th row [cloud]

What are the elements of the sum in this case? How can we express the sum covered by the cloud?

Fig.1: Mason's problem

Fig.2: Mason's problem adapted

Reactions by Hungarian students and teachers

In November 2007 Mason's problem, its *a priori* analysis, HP₁, HP₂, HMP₃ and the commented outline of the results obtained in Hungarian classes are sent out to the IT, together with comments like the following:

"... The first experiences with Mason's proposal are very negative. The Hungarian students are not used to open problems, to visual representations, to induction and generalization".

The Italian team in turn analyses the problems. The coordinator writes to the Hungarians:

"... The teachers reacted to these problems by saying that it is nonsense to bring this task into a class, independently on the plan of work, because this proposal requires a lot of time (time for the students' individual and/or small group exploration, for assessing students' results, for organizing and realizing in the class the discussions on the students' contributions)."

The teams are stuck. Both students and teachers react to the experiment with either a sense of frustration or hostility. An in-depth reflection on the HIBTE is then enacted, and the discussed themes start from the Mason-episode to widen up.

FIVE KNOTS

Five central issues emerge from the analysis:

1) What are HIBTE's objectives? The first answer, provided by both Hungarian and Italian teachers, was: *to look at if/how students explore/solve the three problems*. But **the main issue** is: were these Mason's objectives, or those which HT and IT attributed to Mason's proposal and consequently to HIBTE?

2) Who is HIBTE's referent? There are three possible answers: the students, the teacher-researchers, the researchers. The answer 'the student' was the first one and brought about problems to both Hungarian and Italian teachers: unusual problems, classes not prepared to tackle them, missing pre-requisites, activity not included in a planning which requires a lengthy time (particularly if the class has not experienced similar activities). But is it true **that students** were the main referents of the HIBTE?

3) What are the needed competencies? Are the mathematical ones the only or main ones? The question is: perhaps the needed competencies are wider and the mathematical ones are only a subset?

4) How can the problem proposed by Mason be set in the class' teaching and learning context? Mason's proposal may be viewed as a *virtual* proposal. He provided an input and it was up to the single countries to compare it to their own cultural reality, their school systems, their teacher training programs and their usual behaviours. In the prior analysis, HT and IT needed to give *a sense* to the proposal, with relation to their specific theoretical frameworks, for instance: in the prior analysis HT focused on

didactical-mathematical aspects and on *students*, whereas IT focused on *methodological* aspects and on *teachers*. So: actually setting the problems out in the classes, is this the *sense* of the proposal?

5) Why studying sequences and regularities? The answer is: Mason meant to be *provocative*. He perfectly knows that the theme is highly important (modelling, generalizing and so on) but he also knows that its underlying spirit is completely, or at least largely, stranger to the school systems of many countries. His proposal means: do not think of setting the problem in the class immediately, get really engaged with this question, and think about what might/should happen in your class, *and therefore* in your way of thinking, *and therefore* in your school system *and therefore* in your country's teacher training system, so that *these problem situations and activities may become components of the spine of a different way of conceiving mathematics teaching, as well as of implementing it.*

Let us get back to our initial questions: who is the referent of HIBTE? Which are the objectives? If we think that students are the referents and their competencies in mathematics the objectives, we would break an open door: given the premises, a negative outcome would be easily predictable. *The actual referents are trainee-teachers-researchers and researchers.* The objectives are not 'only' mathematical knowledge and the strategies to enact it, but rather *reflection* – initially individual and then shared – *on methodological issues that, appropriately set, can make this type of problems feasible and meaningful in the class.* It is in this line that IT opens up the theoretical umbrella under which the HIBTE will develop. It is decided that an initial experiment will be carried out by Navarra [2], with his class (6th grade) and later by some other *trainee teacher-researcher*, in 6th- 7th grade classes, on the basis of HP₁ and HP₂. Mason's problem is left aside, because teachers consider it as unsuitable for the expertise of pupils of this age.

1) The teaching experiment in Italy

The transposition of Hungarian problems in two 6th and 7th grade classes

Navarra's class could be defined as 'expert' since pupils have in their background (K-5th gr) more than five years activities on the study of regularities in an *early algebra* setting (40-50 hours with Navarra teaching together with the class teacher). The class is used to working in an ArAI environment and therefore to verbalizing, arguing and constructing knowledge socially. Navarra proposes a new version of HP₁ (**Fig.6**):

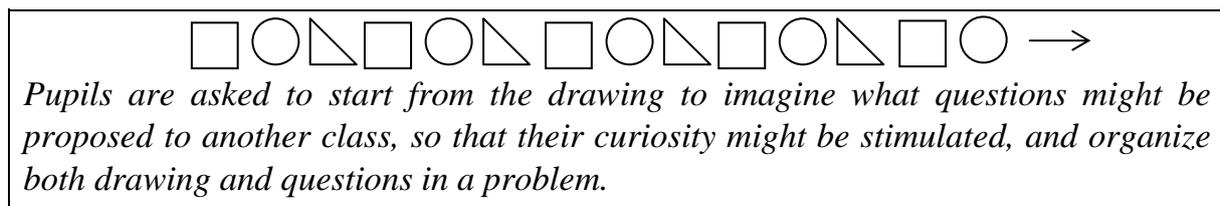


Fig.6: HNP₁ – initial problem situation, HP₁ version

Turning an input into a problem is not a new practice. Pupils, divided in groups, elaborate 36 questions and then reduce them to 13, through a large collective discussion. The first 6, out of the 13 questions, are defined ‘ice-breaking questions’ purposefully organized for a ‘non expert’ class; 4 are defined ‘opening questions’; the last 3 questions (‘difficult questions’) are, in fact, the same as in HP₁ (Fig.7).

A. Ice-breaking questions	B. Opening questions	C. Difficult questions
<p>□○△□○△□○△□○△□○▷</p> <ol style="list-style-type: none"> 1. What does the arrow mean? 2. Which is the module? 3. How many figures is a module made of? 4. How does the sequence carry on? 5. If I repeat the module 50 times, how many times is the circle repeated? And the square? And the triangle? 6. When triangles will be 345 how many modules will there be? 	<p>□○△□○△□○△□○△□○▷</p> <ol style="list-style-type: none"> 7. The squares are at places 1, 4, 7, 10, 13. What about circles and triangles? 8. Is every type of figure at even places? Only at odd places? Both at even and odd places? 9. In 23 modules how many figures are there? 10. Were the shapes 100, how many modules would we have? 	<p>□○△□○△□○△□○△□○▷</p> <ol style="list-style-type: none"> 11. Explain how you can find the figure at place 34. And place 95? And 243? 12. Explain how you can find out in what position are the 56th triangle, the 192nd square, the 368th circle? 13. Can you arrange general formulae to find out at which position is any odd square, circle or triangle?

Fig.7: Questions proposed by pupils

Pupils themselves solve the questions, during discussion, analyzing, comparing, modifying and eliminating them. Altogether, eight hours of work in class; four diaries drawn from four digital recordings. The class goes through the experience productively because they set it in a familiar context. Warning: one does not say ‘extraordinary context’, but rather ‘familiar’; one means a suitably constructed context, with an internal consistency pupils were aware of, undertaken when they were five years old.

The problem of analyzing pupils’ questions is proposed by Navarra in a 6th grade class of a colleague of his. Pupils’ reactions to the first six questions are of confusion, and make Navarra realize that, before tackling them, he needs to broach, although in a short time, with some very delicate *methodological questions coming well before the solution*, that is: pupils are scarcely used to talking about mathematics, have an initial block when they need to explore a problem situation, are not familiar enough with competencies like verbalizing, arguing, controlling and comparing different languages and translating from one language to another; focus more on ‘results’ than on strategies and thinking processes. Moreover: the approach to generalization and modelling are nearly unknown; there is a stereotype about the impossibility of a creative and functional attitude in the production of mathematical expressions; there is a weak control over mathematical contents such as: multiplicative structures, divisibility, division algorithm, properties of operations, use

of letters, etc.; there is a poor use of tables to explore and compare data as well as to analyze what is constant and what varies. One could say that it is a standard class, with standard pupils, a standard teacher, standard programs.

The ‘ice-breaking’ questions allow groups to produce mathematical expressions that are reported on the blackboard, compared and selected in a search for the most correct, consistent and the clearest. The first 10 questions turn out to be effective, and the outcomes of the activity in this second class (8 hours) are globally satisfactory.

The eight hours of work in the first class on the first task produce four diaries, drawn from four digital recordings. The transcripts, commented by Navarra, are sent out to other components of the IT who comment them in turn, following the multi-commented transcripts methodology. After this, HP_2 (Fig.4) is analysed and then structured in three worksheets A, B, C [3] so that the difficulties may be diluted. The worksheets are meant to favour a representation through letters: (A) of the relation between the last addendum (a) and the ranking number (n) of the n^{th} row ($a=2n-1$); (B) of the relation between the ranking number (n) and the sum (s) of the n^{th} row ($s=n^2$); (C) of the sum of the first n odd numbers. The protocols relating to Navarra’s experiment are analyzed and classified by IT. Based on the outcomes, the worksheets are refined with some changes and then proposed to a 7th grade class, with teacher Marco Pelillo, novice trainee researcher.

Classification of the results is based in particular on the following aspects: (i) identification of how different perceptions of written expressions and of drawings influenced the related algebraic or ‘pseudoalgebraic’ expressions produced by pupils (i.e. many interpreted the two graphical representations, seeing the first, as representing the operations of sum of odds indicated, and the second, as representing the result of the sum; this interpretation was encouraged by the fact that a dot was missing in the first line of the second representation); (ii) strategies and consistency used by students to develop their explorations up to the identification of general forms and ways to express them in either natural or algebraic language; (iii) analysis of pupils’ verbal representations’ like “The line number is always doubled by 2 and decreased by 1”; “The difference between the line number and the last term of the sum is always equal to the number of the previous line; adding up the line number to the number of the previous line you get the last term of the sum as result”; (iv) identification and analysis of algebraic expressions that could be reduced to $a=2n-1$ like: $a=n+n-1$, $a=(n+1):2$, $a=n\cdot 2-1$, $a=n+(n-1)$ (a = ‘last addendum’ and n = ‘row number’); (v) analysis of written expressions that could be reduced to $s=n^2$ or to $s=n\times n$ (s = ‘sum’ and n = ‘row number’); (vi) analysis of written expressions to be reduced to $1+2+3+\dots+2n-1=n^2$ or $n\times n$, to test pupils’ capacity to spot the equality between the sum of the first n odd numbers and the square of n . At the end of the experience Pelillo makes the following comment:

“...It was very hard to make pupils represent the equality, since they were not able to express the sum of the first n odds in general terms, despite the hard work made to

represent the last term... I produced a justification of that equality in a recursive way, on the basis of geometric remarks, and representing the odd number to be added to the subsequent line of data with the gnomon of the square corresponding to this one... Many pupils immediately grasped the regularity. The identification of the result of the sum of the first n odds was easy, whereas more problematic was the representation of the sum of the first n odds... The linguistic aspects turned out to be problematic. A basic difficulty was evident in pupils' linguistic expression... We might talk about *a proximal use of the Italian language*.”

In February 2008 the Italian versions of the problems, the commented transcripts by Navarra (32 pages), the classifications of protocols are sent out to HT.

2) The teaching experiment in Hungary

HT analyses materials sent by the IT and, on the basis of this, decides to carry out a teaching experiment in two classes (5th and 6th grade, Béla Kallós, novice teacher researcher trainee). In July 2008 HT sends to IT the synthesis of the work carried out at Kallós on HP₁ and HP₂ together with the teachers' remarks on the Italian materials.

Comments by Béla Kallós

“... The students were divided into two groups. The groups received the task sheet. I asked the students to read the text carefully, if they did not understand something, they could ask me. I have planned 25-30 minutes for the pair work. In the last 10-15 minutes we discussed the solutions with the whole class... The students did not understand the problems in all cases... We have seen that at this age some students can express their solution using formal language”

“Some reflections on myself as a teacher. In PDTR J.P. da Ponte formulated four main phases in the development of the teacher-researchers: teacher; good teacher; researcher; teacher researcher. I am a very young teacher yet, not with much experience. I am just on the way to be a good teacher. Most of my teaching actions are intuitive, based on my personality and some experiences as a student, teacher student and teacher. Until now my main aim was to teach mathematics and science as might as possible effectively. These two experiments are my first trials in research in mathematics education... I was socialized by the traditional Hungarian education. Mathematics has a high prestige in Hungary, the competitions, the fostering of talented students are in the centre. We in Hungary are focused on teaching mathematics and not on children.”

“About my teaching style: I audio-recorded my lessons first time and it was a surprise for me to hear myself. I need to develop my articulation, my construction of sentences. I should have given more time for the students to think about the solution of the problems. I need to have more tolerance to the students' misconceptions and mistakes.”

Use of open problems: “We have seen how much difficulties the most open formulated version caused for Hungarian students. In my experiment I modified the task sheet into such small concrete questions that the originally open problem became a closed one. It is

clear that in such a case the students do not have too much freedom to be creative, flexible. I think I should use more time for problem posing, problem variation.”

Some Hungarian teachers’ reflections on Navarra’s transcript

“As for the used teaching method: the students of 9th, 11th, 12th worked in groups, they got about 15 minutes to solve Mason’s problem... In Hungary the group work is very rare, the teacher’s leading role is very strong and is based on the ideology that everybody must achieve the same high level.”

“In the Italian commented transcript the activity contains very detailed analysis of students’ products. In Hungary, we usually close the discussion after some minutes, very fast with the right result!... From the point of view of handling the mistakes, for us it was interesting to observe how tolerant the teacher was with the students’ mistakes. We must accept the effectiveness of the Italian style: *the students need to explain the source of the mistakes*. For example, Navarra says to the pupils: ‘It is important for you to understand the mistake’ and, in one case: ‘What is more important for you in this moment, focusing on the tenth at the division, or on the remainder?’ In Hungary the written division algorithm is taught in 4th grade, in higher classes our teachers don’t consider this question necessary to handle anymore, because ‘everybody must know it’.”

“In developing the students’ way to form arguments and explanations, it is fascinating to observe how the teacher tended to improve students’ arguing: ‘Please, make your thinking method understandable!’... It is typical for this age that pupils cannot express themselves: ‘I can do it, but I cannot explain why!’ Very often students repeat the process they used as explanation. We can only agree that to develop the PISA competence ‘mathematical communication’ is a long process, and we must do it consciously”.

“Varying the figures of the unit is a good possibility to check the understanding of the students both of the process vs. product and of the general rule. The younger students tend to concentrate only on the product and not on the process... Simply, the Hungarian mathematics teachers do not care for this problem.”

“We wondered how many children participated in the communication at this problem, changing the number of figures in one unit, changing the type of figures, using reverse problems... Navarra always summarized the results and the pupils analyzed them on the whiteboard. In our opinion for this age group the clear visual explanation is important.”

CONCLUSIONS

Enacting International collaborative projects in the educational field requires great involvement by all participants. But enacting *meaningful* forms of collaboration, regarding issues with a shared value, requires the *construction of a common ground*, where conceptions (of mathematics and its teaching) and educational values might be questioned and the cultural and environmental operating conditions are made explicit. In the case of HIBTE, the will to engage in a single task and communicate methods and results, provided a basis for important in-depth analysis, far from the initially predicted one. The original proposal by Mason was lived as a stimulus to lead

teachers to reflect upon many issues, very important from general points of view: the role and the way of being in the class, the capacity of anticipating the class' behaviours as a reaction to teaching proposals; the need to acquire a range of competencies to enable improvisation in the classroom. Therefore, more than carrying out an in-depth analysis of mathematical aspects, which is in the 'natural spirit' of the exploration of problem situations like the ones we proposed, in our case, exchanges occurred under a *methodological, before being mathematical*, theoretical umbrella. The main referents were teachers, well before students; the main questions concerned linguistic and social competencies, well before cognitive aspects. The meaningful part was the fact that teachers acknowledged how much verbalization, argumentation and dialogue with peers may be productive to promote the mathematical construction, as well as to produce conscious and meaningful learning in pupils.

NOTES

1. The European PDTR project, *Professional Development of Teacher-Researchers*, involved seven teams of mathematics teachers, apprentices in the craft of teaching-research, from: Hungary (Debrecen); Italy (Modena, Naples); Poland (Rzeszów, Siedlce); Spain (Barcelona) and Portugal (Lisbon).
2. G. Navarra is a teacher-researcher sharply involved in teachers education in early algebra. He is responsible with N.A. Malara of the teaching experiments and production of the ArAl teaching materials. In PDTR Project he has been mentor of the Italian team (leader N.A. Malara).
3. Due to space constraints, worksheets A, B, C can be found in www.aralweb.unimore.it.

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A COMPARISON OF TEACHERS' BELIEFS AND PRACTICES IN MATHEMATICS TEACHING AT LOWER SECONDARY AND UPPER SECONDARY SCHOOL

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The focus of this paper is a comparison of lower and upper secondary teachers' beliefs regarding teaching mathematics in general. This is linked to a research project concerning the transition from lower secondary to upper secondary school and the learning and teaching of functions. In Norway the transition from the 10th to the 11th grade always involves these separate institutions. The results presented here are based upon interviews with teachers at both lower and upper secondary level of schooling and some interesting differences in their views of mathematics teaching are uncovered. Hopefully, these preliminary findings could give rise to meaningful discussions related to how a qualitative approach to the transition issue might be carried out.

Keywords: mathematics teaching, transition, lower secondary, upper secondary

INTRODUCTION

In Norway, the transition between different phases of schooling, particularly in relation to the learning and teaching of mathematics, is an area where little research has been done and the major part of the international research in this field concerns the transition from upper secondary school to university/university college (often denoted as the secondary-tertiary transition) (Gueudet, 2008; Guzmán et al., 1998). My own experiences as a student and a teacher, at both lower and upper secondary school levels have led me to believe that the traditions and beliefs in these institutions differ in ways which in turn might affect students' learning. As a PhD student (in my second year), I have chosen this transition as the focus of my research. It is important to note that in Norway, upper secondary schooling is divided in two main programmes: the vocational programmes, which are orientated towards practical professions and the general study program, which aims to prepare students for higher education. The curriculum is different in these programmes and is considered to be more 'theoretical' at the general study program. This is also the case for mathematics as a subject. Both of these programmes are included in this research. Further, I have chosen to focus on *functions* as this is an area highly relevant to both levels of schooling, and personally I find the development of students' conceptual understanding of functions to be an interesting research area. It is also possible to expand this area of research, for example by taking the universities/university colleges into the consideration, as the learning and teaching of functions is an important issue in several of these study programmes. However, in this paper I will focus on mathematics teaching in general (not only teaching related to functions).

RESEARCH QUESTIONS

I pose the following research questions, relevant for this paper:

What are the differences in the didactical approaches related to mathematics teaching, in lower secondary versus upper secondary school? How are such possible differences perceived by the teachers at both these levels of schooling?

To approach the first question, I compare the lower and upper secondary teachers' views and practices concerning the teaching of mathematics in general. Concerning the second question, I present the lower secondary teachers' statements related to how they think upper secondary teachers perceive the teaching of mathematics in lower secondary school. These statements are then being compared to the actual statements of the teachers at upper secondary school.

THEORETICAL BACKGROUND

An established and well-documented argument within educational research is that teachers' beliefs are one of the best indicators of the decisions teachers make throughout their career (Pajares, 1992). The link between beliefs and actions, therefore, motivates for many of my interview questions. As indicated by Mosvold (2006, p. 37) research shows that many of these "beliefs are shaped from the experiences of those who taught them". What often seems to be conflicting interests, or even paradoxes, experienced in teachers everyday practice, is described by Mellin-Olsen (1987; 1991) as characteristics of a 'double bind'. According to Mellin-Olsen, double bind can be recognized at many levels. One aspect of this can be that the individual is tightly connected with his environment, and consequently left with few individual choices. Often this relates to the 'didactical contract' which in its simplest form means that "the teacher is obliged to teach and the pupil is obliged to learn" (Mellin-Olsen, 1987, p. 185). Hidden (or in some countries even explicit) competition between teachers at the same time as they need to cooperate can be an example of a double bind. The confidence the teachers often express that they feel in traditional teaching, for example the early introduction of standard algorithms without giving their students 'permission' to use alternative methods, can be another example. Such 'permission' could, from the teachers' point of view, imply a break in the didactical contract. In turns this could lead some teachers into what they consider as 'safe' and effective curriculum-oriented teaching, preparing students for an oral or written exam. According to Mellin-Olsen (1987, p. 150), a double bind "is due to the handling of *metaknowledge* about the control caused by the taxonomies." Based on information found in some of my interviews, I have reasons to believe that at least some of the teachers on different levels experience what could be described as aspects of double binds. Some, especially recently educated teachers, state that their "ideals of teaching" often have to be set aside because of their obligations to the curriculum and the upcoming exam.

As my observations in the classroom concern the teaching of functions I find it relevant to include the Leinhardt et al. (1990) quote: “There is no proven optimal *entry* to functions and graphs” (p. 6). It is therefore, in my view, important to be aware of the multitude of different didactical approaches and to be conscious about the various conclusions.

METHODOLOGY

Five different classes in five different lower secondary schools participated in this research. Two of these schools are private schools while the other three are public. The private schools were included in an attempt to seek some diversity in the sample, while the public schools were somewhat randomly selected, with the only criteria being that they, due to practical reasons, were located within a ‘reasonable’ distance from my working place. As the Norwegian school system is quite homogenous I believe that these schools are representative to their area. The headmasters were contacted via telephone and their school was invited to participate. The number of students willing to participate from each class varied from three to ten. In total 33 students participated and I am currently conducting follow-up research on ten of these as they have now entered upper secondary school. I have chosen the follow-up students on the basis of three criterions: equal gender distribution, students at both vocational and general study programmes, and variations of ‘skills’ (on the basis of their marks). The purpose is to gain a rich material with some diversity. My data collection at lower secondary school mainly consisted of five “phases”: Observations of the teacher teaching, recorded conversations with the students engaging in mathematics in the classroom, interviews with the students, collection of students’ handwritten material and an interview with their teacher. This provides me with a diverse and rich data material which allows me to study mathematics education from various perspectives. The data collection at upper secondary school is done in a similar way, and I consider the fifth phase (teacher interviews) to be most valuable for this paper, as this relates to both teachers beliefs and practices. My use of research instruments did vary somewhat from school to school, primarily due to the fact that some teachers imposed restrictions for example on my use of a video camera. By the use of semi-structured interviews I aim to seek information mainly about teachers’ beliefs. However, I also try to get a broader picture of their teaching practice, by asking them to estimate the use of different teaching methods. They were interviewed for about 45 minutes, and in addition to their teaching practice they were asked about their views on ‘good teaching’ in general. They were also asked to provide some personal background information. I have aimed to design the interview questions in accordance with Kvale (1997, p. 77), suggesting that “The questions should be easy to understand, short and free for academic terminology” [1].

It was also important for me to formulate questions that would make it possible to compare teachers’ beliefs and ideas in lower and upper secondary education. These interviews were all recorded with a Dictaphone.

EXAMPLES AND ANALYSIS

Teachers at lower secondary school

I will start this section by presenting excerpts from teachers own statements regarding what they consider as good teaching in mathematics. These first three statements are excerpts from the interviews with the teachers at the 10th grade at lower secondary schools.

In your opinion, what characterises good teaching in mathematics? [2]

Jon: Good teaching...eh...variation, organised towards the individual student...eh..., adjusted according to different teaching styles, and that you go through the given exercises with this in mind.

Interviewer: Could you please go into some details about how you organise teaching towards the individual student in your practice?

Jon: Yes, this can be done by different tools, we might use the blackboard as a medium, and we might use the computer as a medium. We can do some practical exercises, where we work in a physical way, or we can make some problem solving exercises. We can do this interdisciplinary along with other subjects.

.....

Sue: Good teaching in mathematics...eh...ideally, good teaching in mathematics, the start of a lesson...eh...it should be some repetition from the last time, in terms of “what did you learn?” Eh...maybe about five minutes, “what did you learn the last time?” Then a period in which you go through new content on the blackboard. And maybe a longer period, where the students can do some exercises.

.....

Ann: In general, I think it is important that the individual student is making progress from his or her own starting point, within the subject that we are dealing with. Of course this has to be done in accordance with the curriculum, and so forth. But you have to achieve this. That is what I think.

Interviewer: Do you have any concrete ideas related to how this might be carried out?

Ann: Well, this has to do with differentiation. You know...eh...it is a very big gap, and you have to motivate students to make progress from where they stand, actually. But this is difficult to achieve. This can be done by giving different levels in the tasks given at the students’ working plans. We also try to differentiate in the tasks given in the folder. [3]

We notice that their answers are not quite univocal, and the three teachers’ views on “good teaching in mathematics” seem to differ in some ways. Jon seems to give an account of some general aspects of good teaching, and Sue seems to relate the question to a concrete situation, like a recipe of a good lesson. Common for both Jon

and Ann is the importance of differentiation. The tables below show the teachers' suggestions of how frequent different teaching methods are used. The time measured in minutes estimates the time used in each lesson. The three schools all have 4 lessons a week, each 45 minutes. These numbers are only based upon what they have done related to the class participating in this research.

Teacher	Lectures-blackboard	ICT	Homework Discussions	Individual Exercises	Pair/group-work	Problem solving
Jon	1-3 lessons a week 15-20 min	1-3 lessons a week 30 min	1 lesson a week 10 min	Almost each lesson 30 min	2 lessons a month Whole lessons	Sometimes (hard to establish)
Sue	Each lesson 30 min	6 lessons (this year) Whole lessons	Each lesson 5-10 min	2-3 lessons a week 15 min	Not organised[4]	Never
Ann	2 lessons a week 30 min	Sometimes (hard to establish)	2 lessons a month 5-10 min	2 lessons 15 min + 2 whole lessons	2 lessons a month Whole lessons	A few Times (hard to establish)

Table 1: The frequencies of different teaching methods (assumed used most frequently)

Teacher	Interdisciplinary Projects	Excursions	Outdoor Activities
Jon	2-3 weeks a year	Never	Never
Sue	2 weeks a year (together with Art and Design)	Never	Never
Ann	Never	Never	1 day a year

Table 2: The frequencies of different teaching methods (assumed used less frequently)

The tables show for example that Sue states that she never uses 'problem solving' as a method of teaching and seldom uses ICT. She also seems to use the blackboard and discussions related to homework more frequent than the others. It is also interesting to notice Jon's relatively frequent use of ICT. The pre-assumed more rarely used methods, as interdisciplinary projects, outdoor activities, and excursions appear with quite similar frequencies.

The idea behind the next question is to grasp one aspect of the teachers' beliefs concerning upper secondary school.

How do you think that the teachers at upper secondary school conceive of the teaching in mathematics at lower secondary school?

Jon: I do not really know – maybe they shake their heads and think “what in the world have we done at lower secondary school?” But I also think they have completely different pre-conditions for their activity.

Interviewer: In what way?

Jon: Well, you do not have “the herd” in an ordinary class at upper secondary school – they come there because they have applied for going there – but we have the average of the whole Norwegian population in one class!

.....

Sue: I am very convinced that the teachers at upper secondary school feel frustrated about the students at lower secondary school and their total lack of knowledge.

Interviewer: Ok...?

Sue: Well, maybe, and here they come at upper secondary school, and they can not add two fractions!

Interviewer: Mm...?

Sue: Here they come at upper secondary school and do not manage this! They have not learned anything...

.....

Ann: I do not really have any strong opinions here, but my impression was, when I worked there myself, that the teachers there were very different. I also think that there was a big difference among the students, related to which lower secondary school they attended before they started.

Although this question could be regarded being a bit speculative, since most of the answers are hypothetical, I was surprised by the level of consensus. As we can see, both Jon and Sue indicated some negative assumptions, while Ann was more neutral. Both of them seemed to share the worries that the teachers at upper secondary school, to some extent, are frustrated by the limitations of their students’ starting point. The negative assumptions were also shared by the two other teachers, not presented here.

Teachers at upper secondary school

I will now consider four of the teachers at upper secondary school answering the corresponding questions. The first two excerpts are from teachers at the general study programme.

In your opinion, what characterises good teaching in mathematics?

Tony: Well, maybe the most important aspect in such a subject dealing with systematics, is clarity. Clarity in the presentations and that one manages to simplify complicated issues. The teacher’s job, in a way, is to simplify the textbook for the students, because we observe that this is a subject that is very hard to study on your own and you are very dependent on going through the content.

.....

Mary: It must be teaching...eh...in such a way that the students understand what they are doing. Eh...and that they are motivated to continue to work with mathematics

Interviewer: Do you have any thoughts of how this can be done?

Mary: I think on this level, if they are mastering the mathematical content, this in itself is good enough for motivation. Helping them to master the exercises is very important, because most of the students like mathematics.

In this next excerpt, the same question is asked to a teacher at a vocational programme.

Lisa: I have some years with experience from the lower secondary school, and I think that working with concretes and go outdoors and do things is a good way of working with mathematics. Good teaching will be to organize such activities in a good way. Now at upper secondary school I almost only teach by giving lectures at the blackboard, in an old-fashioned way.

Interviewer: What is the reason for that, you think?

Lisa: It is another culture here. They are all working, determined to get the students through the textbook in an efficient way.

Interviewer: Why do you think it is difficult to teach the way you would like?

Lisa: Well, I am new here and I do not want to go against my colleagues.

It is interesting to notice Lisa's reflections on her own situation, probably much due to her background from lower secondary school. The two other teachers at the general study programmes do not express the same kind of worries. They both seem to share the value of good explanations and the importance of doing exercises from the textbook. Jon stresses the importance of clarity and Mary the importance of mastering the textbook content.

In the same manner as for the teachers at lower secondary, the teachers at upper secondary school were asked about their use of different teaching methods. The results are presented in the tables below. Tony and Mary's classes have five lessons a week and Lisa's has three.

Teacher	Lectures-Blackboard	ICT	Homework Discussions	Individual Exercises	Pair/group-work	Problem solving
Tony	Each lesson 15 min	2 lessons a month 30 min	1 lesson a week 10 min	Each lesson 30 min	Not organized	Never
Mary	Each lesson 15 min	5-10 lessons this year Whole lessons	1 lesson a week 10 min	Each lesson 30 min	Not organized	Never
Lisa	Each lesson 20-25 min	Never	A few times (hard to establish)	Each lesson 20-25 min	Not organized	Never

Table 3: The frequencies of different teaching methods (assumed used most frequently)

Teacher	Interdisciplinary Projects	Excursions	Outdoor activities
Tony	Never	Never	Never
Mary	Never	Never	Never
Lisa	Sometimes at the mechanical working rooms	Never	Never

Table 4: The frequencies of different teaching methods (assumed used less frequently)

As illustrated the use of methods assumed less frequently, are rarely/never used. The more common methods appear in quite similar frequencies, and the ‘typical’ lesson seems to be divided in two, with the first part consisting of a lecture at the blackboard and the second part consisting of individual exercises from the textbook. In general it seems like there are only small variations between these teachers and their use of methods.

The next question was posed with the intention to compare the upper secondary teachers’ statements with the lower secondary teachers assumptions.

Which thoughts do you have concerning mathematics teaching at lower secondary school?

Tony: It is always easy to blame the teacher responsible for the class, the previous year, but they have whole classes with enormous gaps between the students. Probably much time is used just to keep them quiet. So the students coming to us may not have got the follow-up which they should, from the lower secondary school. They take to easy on it [the students] and their efforts are not as they should have been.

.....

Mary: I think teaching at lower secondary school is very dependent on the personality of the teacher...eh...and this is of course also the case at upper secondary school. But in general I will assume that it is quite similar. Maybe it is more group work at lower secondary school.

.....

Lisa: I think the students get to work on their own to much, and they do not take that responsibility, they are not keeping up and they end up here. That being said I think the teachers vary their methods more, as I said before. I also think that much of the differences are due to the teachers’ background. At lower secondary school they are educated at general teacher education institutions, but here they are educated at universities.

By the exception of Mary being more neutral to the question, the other two seem to express some kind of worries. Common for these are the suspicions that the students do not get the required follow-up from their teachers. It is also interesting to notice

how Lisa is pointing to the teachers' background as a possible reason for different ways of teaching.

The comparison of these interview excerpts and the tables from the lower and upper secondary level of schooling, gives rise to some reflections. While at least two of the teachers at lower secondary emphasized differentiation and the importance of reaching the individual student, the teachers at upper secondary school tend to emphasize the importance of good explanations, techniques and individual task solving, mainly from the textbook. The exception here is Lisa, who expresses some frustration of being 'forced' into a teaching tradition which seems to go against her own principles. The tendencies expressed by these teachers are also to some extent reflected in the tables, and the overview of the teachers' use of methods in the classroom.

The lower secondary teachers' beliefs concerning the upper secondary teachers' perception of teaching in the lower secondary level showed some consensus. These were at most negative assumptions, and to some extent they were in accordance with what the teachers in upper secondary actually stated. Although their suspicion of the insufficient follow-up of the student was not actually stated among the lower secondary teachers, they shared the worries concerning their students' 'insufficient' mathematical knowledge. Despite these remarks, it is important to notice that the statements within the group of teachers at both lower and upper secondary school are far from univocal. This is also the situation if we study the interviews in a more holistic manner.

CONCLUSION AND FURTHER DISCUSSION

So what can we infer from the examples above? The teachers at lower secondary school related some of the challenges in teaching to their students' abilities, and the diversity within their group of students. This was also mentioned by some of the teachers at upper secondary school. I think that common to these, and similar statements, are the relation to what Mellin-Olsen (1987; 1991) denotes as a double bind. This is because the concerns of most of these teachers relate to what in their view are conceived of as conflicting issues. The obligations of getting through a given curriculum, and at the same time being able to teach in a fruitful way, for some, seemed to cause a dilemma. Apparently the teachers at upper secondary school feel that the most 'safe' way of coping with the demands of the curriculum is in terms of traditional teaching methods. One reason might be that there usually is a higher probability for the students in upper secondary school having to take an exam. Another reason, also indicated among both group of teachers, could be that there exists a view that students at upper secondary level have made a more specific choice related to their career, and the mathematics is in a way a part of that choice. Therefore it becomes important for the teachers that nothing is 'omitted', and hence few 'risks' are taken. Being aware that these are only speculations, I still think these could be important hypotheses to investigate further upon. In Lisa's case, being loyal to her

colleagues and at the same time manage to teach in a way that she considered as appropriate obviously constituted a dilemma.

As Lisa further mentioned, cultural issues such as the fact that teachers at upper secondary level tend to have a university background while teachers at lower secondary tends to come from general teachers education should also be considered, in an attempt to understand possible differences in their beliefs and practices.

NOTES

1. Translated from Norwegian by the author.
2. All the transcriptions are translated from Norwegian, with an attempt to preserve the teachers' original statements as authentic as possible.
3. This teacher regularly gave her students exercises which they were supposed to put into a folder. The folder was evaluated by the teacher. In total the folder counted as one third of their final marks in mathematics.
4. This means that the students were allowed to cooperate at their individual tasks, but no group work was organized by the teacher.

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MATHEMATICAL TASKS AND LEARNER DISPOSITIONS: A COMPARATIVE PERSPECTIVE

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Mathematical tasks in textbooks, their ‘mediation’ by teachers and the classroom environments in England, France and Germany are the focus of this study. The author claims that the different mathematical tasks in textbooks (in connection with their mediation by teachers) influence, to a large extent, the differences in activities and practices that are going on in mathematics classrooms, and that these in turn mediate different kinds of learner dispositions. The classroom culture, with its differing dimensions, is likely to set the scene for pupil development as ‘learners of mathematics’. The web of these connections is studied in this report.

Keywords: Mathematical tasks; learner identity; comparative education; socio-cultural; culturally figured worlds.

INTRODUCTION

Mathematical tasks in textbooks, learning opportunities and pupil dispositions

Students spend much of their time in classrooms working on mathematical tasks chosen from textbooks. In recognition of the central importance of textbooks, the framework of the Third International Mathematics and Science Study (TIMSS) included large-scale cross-national analyses of mathematics curricula and textbooks as part of its examination of mathematics education and attainment in almost 50 nations (Valverde et al, 2002). They claim that

Textbooks are the print resources most consistently used by teachers and their students in the course of their common work (ibid., p. viii).

Moreover, they comment on different learning opportunities being offered to students in different mathematics classrooms.

Clearly, one issue of pervading importance to the nations that participated in TIMSS was the quality of educational opportunities afforded to students to learn mathematics and science - and the instruments that optimise such quality (ibid, p. viii).

Textbooks are a major source of provision of these educational opportunities. Romberg and Carpenter (1986), for example, noted that the textbook was consistently seen (in the US) as “the authority on knowledge and the guide to learning”. (p. 25)

It appears that tasks in textbooks influence, to a large extent, how students experience mathematics. Textbooks provide children with opportunities to learn, and learn those things which are regarded as important by their government. Teachers mediate textbooks by choosing and affecting tasks, and in that sense student learning, by devising and structuring student work from textbooks.

It can also be argued that tasks, most likely chosen from textbooks, influence to a large extent how students think about mathematics and come to understand its meaning. Indeed, Henningsen and Stein (1997) assert that

the tasks in which students engage provide the contexts in which they learn to think about subject matter, and different tasks may place different cognitive demands on students Thus, the nature of tasks can potentially influence and structure the way students think and can serve to limit or to broaden their views of their subject matter with which they are engaged. Students develop their sense of what it means to “do mathematics” from their actual experiences with mathematics, and their primary opportunities to experience mathematics as a discipline are seated in the classroom activities in which they engage ... (p. 525)

Hiebert et al (1997) similarly argue that students

also form their perceptions of what a subject is all about from the kinds of tasks they do. ... Students’ perceptions of the subject are built from the kind of work they do, not from the exhortations of the teacher. ... The tasks are critical. (p. 17-18)

Moreover, they assert that

the nature of the tasks that students complete define for them the nature of the subject and contribute significantly to the nature of classroom life The kinds of tasks that students are asked to perform set the foundation for the system of instruction that is created. Different kinds of tasks lead to different systems of instruction. (p. 7)

It appears that mathematical tasks are central to student learning, their developing perceptions of what the mathematics is and what doing mathematics entails.

CLASSROOM ENVIRONMENT, MATHEMATICAL TASKS AND LEARNER IDENTITY

According to Lave and Wenger (1991), tools (and artefacts) constitute the resources, and students learn by participating in social practice using the tools. This also relates to ‘conceptual tools’, most likely reflected and used in tasks. If students use a conceptual tool, as perhaps advised by a worked example, or teacher’s exhortations, or an exercise, and if they use the tool actively, they are likely to build an increasingly rich understanding of the ‘usefulness’ of this tool in their mathematical world, and of the tool itself. Learning how to use a conceptual tool involves much more than the set of explicit rules it may describe. The occasions and conditions for the use arise out of the contexts of tasks and activities that students are expected to do, and they are framed by the ways the members of the community (e.g. textbook authors) see the world of mathematics.

Different practices in mathematics classrooms are likely to influence the development of different learner identities. For example, Boaler et al (2000) investigated the practices of secondary school teaching from a student’s perspective “in order to understand how they construct a sense of themselves in relation to mathematics” (p.

4). They argue that in the US and UK classrooms they studied there exists an “unambiguous vision of what it means to be successful at mathematics, and of what it means to be a mathematician” (p. 8).

According to Henningsen and Stein (1997) what it means to ‘do mathematics’, or to ‘behave mathematically’, for students, is largely dependent on the nature of the tasks and activities students are engaged in, and these in turn ‘colour’ their perceptions of the subject. Thus, doing mathematics, and developing certain perceptions of the subject, is likely to ‘produce’ particular ‘mathematical dispositions’ or a ‘mathematical point of view’ (Schoenfeld, 1988), as well as acquiring mathematical knowledge.

As Boaler (2000) emphasises, students do not just learn methods, or how to carry out a task or to apply algorithms, in mathematics classrooms, but they learn ‘to be mathematics learners’. Different classroom cultures, different constraints and affordances, provided by different settings and opportunities for engagement in mathematical practices, are likely to influence their perceptions of what it means to learn and do mathematics. Learning how to engage successfully with the mathematics means learning how to and identifying with the norms of the classroom community. Particular tasks in textbooks may reinforce practices initiated and propagate by the teacher, or vice versa.

Furthermore, Boaler and Greeno (2000) use the notion of identity formation in “figured worlds” (Holland et al., 1998) to explore pupil learning and the influence of pedagogies on their learning. Figured worlds are perceived here as places “where agents come together to construct joint meanings and activities” (p. 173). Mathematics classrooms can be regarded as such figured worlds, because students and teachers work together in these environments and construct meanings of the mathematics, and within that of themselves as learners of the mathematics. Holland et al (1998) is cited to draw attention to actors, and to interpretations by actors when asserting that figured words are socially and culturally constructed realms “of interpretation in which particular characters and actors are recognised, significance is assigned to certain acts, and particular outcomes are valued over others” (p. 52).

This is particularly interesting in terms of comparing “figured worlds” in different countries’ classrooms. Questions such as the following may arise: What is similar, or different, in mathematics classrooms in England, France and Germany? What are the rituals of practice? What kinds of tasks are pupils expected to perform, what kinds of activities do pupils, and teachers, engage in? What kinds of interpretations are made, what kinds of acts are respected, what kinds of outcomes are valued?

RESEARCH DESIGN

In a previous study (e.g. Pepin, 1999; Pepin, 2002) the author developed an understanding of practices in lower secondary mathematics classrooms in England, France and Germany, concluding that national educational traditions were a large

determinant and influence on what was going on in these classrooms. In a more recent study, Pepin and Haggarty have investigated mathematics textbooks in the three countries, and connected to that, the ways they were used, by teachers (e.g. Pepin & Haggarty, 2003). This not only supported some of the earlier findings, but also suggested that the use of curricular materials (such as textbooks), together with the selection of (mathematical) tasks, impacts to a large extent on the mathematical ‘diet’ offered to students.

The author thus re-analysed the amount of data collected over the years, in particular mathematical tasks in selected textbooks, in terms of potential pupil disposition and identity formation. Particularly relevant, and useful, was the work of Boaler and Greeno (2000) and the notion of pupil identity formation in ‘figured worlds’ (Holland et al, 1998). In terms of analysis a procedure involving the analysis of themes similar to that described by Burgess (1984) was adopted, which had already proved useful in other cross-national studies (e.g. Broadfoot & Osborn, 1993). However, due to the additional cross-cultural dimension, it was important to address the potential difficulties with cross-national research, in particular issues related to conceptual equivalence, equivalence of measurement, and linguistic equivalence (Warwick & Osherson, 1973; Pepin, 2002). In order to locate and understand teacher pedagogic practices and the classroom cultures in England, France and Germany, it was useful to draw on knowledge gained from earlier research (see above) which highlighted the complex nature of practices in mathematics classroom environments, and the value of comparing.

The main questions asked was: How may mathematical tasks in textbooks, teacher practices and classroom environment influence pupil identity construction as learners of mathematics in England, France and Germany?

DISCUSSION AND CONCLUSIONS

To connect tasks in textbooks to students’ developing identities as learners of mathematics is not a common link made. Textbooks are often frowned upon, and teachers do not wish to be seen to teach ‘according to the book’. However, for better or for worse, and as research indicates, textbooks are the main resources used in mathematics classroom all over the world (Valverde et al, 2002).

This is also true for England, France and Germany. Moreover, teachers choose tasks and exercises from those books, for pupils to complete, students learn from the kinds of work they do during class, and the tasks they are asked to carry out shape to a large extent the kind of work they do. Pupils learn the conceptual tools provided by the tasks in textbooks, by ‘legitimate peripheral participation’ (Lave & Wenger, 1991) in the practice of school mathematics. However, there are particular school mathematics practices in different countries, and within those countries differing practices in different school ‘streams’ and ‘sets’ that are supported by different textbooks for those groupings. Moreover, the types of tasks, the mathematical connectivity between tasks, the conceptual tools suggested for solutions, amongst others, reflect and

support a particular school mathematical culture. Pupils are socialised into these cultures, and as members of the cultures, develop dispositions and form identities as learners of mathematics. However, it would be difficult to claim that in each country there exists a homogeneous mathematical culture supported by textbooks. Instead, the developing ‘identities’ here are seen as those potentially emerging from the analysis of mathematical tasks in textbooks, and the mediation of those tasks by teachers, thus the tools used by teachers in their classroom practice.

What would pupils learn from the tasks provided by the textbooks analysed, and what kinds of work/activities would they do related to the tasks? In order to engage in the mathematics, pupils must find the task intriguing, something they would like to resolve. This assumes that students relate to the task in the sense that the contexts and situations make it real for them. On the basis of results from this study it is argued that in all three countries pupils are likely to be asked to do exercises and to complete tasks (from textbooks) that are presented in context- context embeddedness seems to be important- and these contexts are similar. Whether the contexts are relevant to pupils, whether they connect to their life experiences is beyond the scope of this study. What is different in the three countries is how the mathematics is linked to the contexts and what pupils are asked to do in those tasks. Whereas in German textbooks it appeared that context and mathematical concepts are connected in the tasks analysed, and links are forged between them, in the English textbook chapter pupils are asked to do contextualised tasks where context are chosen seemingly for their own sake, and with little logical progression or connection to the underpinning mathematical ideas. Most exercises could be done without knowing about concepts of the topic area. To what extent students may deduce concepts, by simply doing the exercises, is not clear. Interestingly, French textbook exercises studied appeared to use contexts as a pretence for introducing the mathematics, a Trojan horse to lead students to the ‘essential’ section, the ‘cours’, the mathematical concepts.

To ask what students would learn from these tasks also needs a more nuanced perspective. By addressing the mathematics at the conceptual level (e.g. ‘oppositeness’ in negative Numbers) one could argue that in France and Germany students would get more insights into the conceptual nature of mathematics, and perhaps its structure, than through English textbook tasks. A second type of ‘residue’ (Hiebert et al, 1997), it can be argued, may be given through the strategies or methods, for solving problems, provided. French textbooks are explicitly addressing this in a separate section (‘apprendre a resoudre’) and exercises are organised accordingly. Putting the three country’s textbooks on a continuum, it is argued that English textbooks leave it to pupils, or their teachers, to devise or identify strategies to solve problems, and this is likely to be with common sense, whereas in particular French textbooks are explicit about how to solve particular problems.

The message that students may therefore get is that (1) mathematics is simply there to be done (e.g. English *KM 7²*), and that contexts and concepts do not necessarily ‘talk to each other’; that (2) it is not the contexts that matter (e.g. French *Cinq sur Cinq*),

but the underlying mathematical concepts, and that there are strategies to ‘reduce’ the contextualised problems to ‘simple’ mathematical tasks; or that (3) concepts and contexts may be connected, and that the formally structured mathematics, including its strategies for solving problems, may be useful in real life problems (e.g. German Grammar school *LS7*).

In terms of teacher mediation of tasks it appears that one of the most important responsibilities for a teacher is to set appropriate tasks. Teachers in all three countries chose those tasks predominantly from textbooks. What was different were to what extent teachers initiated pupils into those tasks and the ways they chose to introduce the mathematical ideas necessary to do the exercises selected from textbooks. The picture that was painted was that whereas in one country (Germany) teachers introduced the mathematical notion in whole-class discussions and chose particular tasks to ‘consolidate’ the concept, in another (England) teachers gave relatively brief introductions or rules, and wanted a large number of straight forward exercises to practice. In another (France) teachers were provided with activity type tasks, from textbooks, to initiate pupils into the concept, and after explaining the ‘essentials’ (*cours*) teachers wanted differentiated exercises to attend to the perceived heterogeneous class.

To what extent teachers selected appropriate and related tasks, so that pupils could see the same mathematical idea from a different angle, or to chain tasks in such a way that opportunities are created to gradually increase pupil understanding was not clear. The literature (e.g. Hiebert et al, 1997) claims that tasks that are related in such a way increase the coherence of students’ mathematical experiences. Coherence here means that students would perceive the sequence of activities and exercises to fit together and make sense. This goes beyond the scope of this study, but it could be argued that, from the analysis of textbook tasks in selected English textbooks, and looking at the sequence of tasks in selected chapters, students are likely to be asked to do a series of individual, nearly random, tasks that are relatively disconnected and appear not to be leading anywhere. French textbooks provide exercises, graded with respect to the level of perceived difficulty and for particular areas within the topic.

In addition, results from a previous study (e.g. Pepin, 1999) show that French, and in particular German Gymnasium teachers chose exercises, that were perceived to exemplify the idea well and to be ‘difficult’, for solving in class, and sometimes in whole-class discussion, whereas ‘easy’ routine exercises were assigned for homework. English teachers said that most of their students needed ‘much of the same’ to practice.

In terms of classroom environment and culture teachers have a great influence, and this was true for England, France and Germany. Within the limits of the system, whether students were taught in mixed classes (*collège* France), whether they were setted (England) or streamed (Germany), teachers had some freedom to select tasks that could potentially guide their instruction and they could mediate those tasks in ways they thought best. To what extent teachers created cognitive conflict, in order to

challenge pupils' ideas, is beyond the scope of this investigation, but in terms of tasks in textbooks this may potentially be provided by selected cognitive activities (*activités*) in French textbooks (Pepin & Haggarty, 2003). Moreover, allowing mistakes, perhaps even inviting them for pupil learning, or asking open questions would be another way of influencing the mathematics classroom culture. Looking at tasks in textbooks, there were no open questions in the English textbook chapter analysed, and hardly any in the French and German textbooks. Teacher pedagogic practices, however, may be interpreted as going some way towards that goal: all teachers, but particularly German teachers, used mistakes in homework exercises as a site for deepening pupil understanding (Pepin, 1998). These were discussed in detail and at times over an extended period of time.

In summary, it can be argued then, albeit from this limited research, that the dispositions that pupils are likely to develop as learners of mathematics, are linked to the textbook tasks provided by teachers, the practices that pupils are engaged in when doing those tasks, and the environment they work in and experience in class during engagement- and these are different in the three countries. Whereas in all three countries one could argue that pupils are 'conditioned' to become 'conformists'- hardly any negotiation about the mathematics and its learning is provided-, in England the mathematical diet in textbooks may also offer learners to become 'common sensers'. Can one say that in France the 'instrumentalist' identity may be favoured, and in Germany the 'connector', in addition to the 'conformist'? If this link was seen to be strong, one would need to consider to what extent pupils are 'trapped' in these identities, for better or worse, according to what they are offered by their teachers. What kinds of opportunities would need to be provided for change to be possible?

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ELITE MATHEMATICS STUDENTS IN FINLAND AND WASHINGTON: ACCESS, COLLABORATION, AND HIERARCHY

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This paper draws from a small scale study of elite mathematics students' beliefs, motivations and access in Finland and Washington State. In particular, students' experiences with extracurricular mathematics, collaborative learning, and their elite peer groups are examined.

INTRODUCTION: FINLAND, WASHINGTON AND ELITE MATHEMATICS

Large scale international comparisons exert seemingly unavoidable influence on educational systems. Such numerical comparisons of performance are often read as competitions; the results become lists of winners and losers, focusing attention on the high-scoring educational systems. However, even if large scale international comparisons can tell us where to look, they cannot tell us what to look for.

Within Mathematics and Science education, Finland has recently drawn such attention for its success in the PISA studies. One of the most striking features of the Finnish educational system is the lack of tracking, or separating students according to perceived ability, until the end of lower secondary (*yläaste*), at roughly age fifteen or sixteen. This has drawn the attention of de-tracking reformers (see e.g. Oakes, 2008).

While the efficacy of tracking has been questioned (e.g. Rothenberg, McDermott & Martin, 1998 or Boaler, 2002), de-tracking may have negative consequences for high-achieving students (Terwell, 2005, p. 663). In this paper, I focus on those students who would be expected to benefit most from tracking: students enrolled in the highest possible track available, whom I call *elite mathematics students*. In Finland, these students have enrolled in an academic upper secondary, and then in Long Mathematics (*pitkä matematiikka*) instead of Short (*lyhyt matematiikka*). In Washington, where tracking may begin as early as third grade (age 8 or 9) these students are taking courses classified as Honours, Advanced Placement (AP), or International Baccalaureate (IB). All would reach at least Calculus by graduation.

Participation in elite tracks has been shown to have lasting negative effects on students' mathematical self-concepts (e.g. Marsh, Trautwein, & Lüdtke, 2007), best known as big-fish-little-pond effect. Structure, then, seems to effect the development of students' beliefs and identities as mathematics learners, influencing students' academic decisions. It seems worthwhile, then, to ask how elite mathematics students' identities and beliefs, as well as opportunities to learn within a partially de-tracked system, Finland, compare to those of students in a heavily tracked system, Washington State [1]. Osborn (2004, p. 265) cautions against the "...growing tendency to 'borrow' educational policies and practices from one national setting where they appear to be effective and to attempt to transplant these into another, with little regard for the potential significance of the cultural context..." The object of this

study, however, is not to set policy, but to illuminate, through the juxtaposition of two systems, features of each.

RESEARCH DESIGN

The research described in this paper was a small scale study designed to explore elite mathematics students' identities, beliefs, and access to learning in Finland and Washington State conducted with the help of Jasu Markkanen from the University of Turku. The study consisted of 13 student interviews conducted in Spring 2008 in Finland and Washington State. Markkanen conducted four interviews (at Pääjanne and Keitele). While many themes emerged from these interviews, in this paper, I will briefly focus on three specific questions:

What extracurricular mathematic experiences have these students had, or had access to?

What are the students' experiences with and views on cooperative learning?

What are students' characterisations of their peer groups, which were cited by participants from both countries as a key benefit of elite mathematics tracks?

These are a combination of prefigured themes and themes that emerged during the interviews. While questionnaires already exist regarding students' beliefs and motivations, (Malmivuori & Pehkonen, 1996), and are being refined to function internationally (Diego-Manecón, Andrews & Op't Eynde, 2007), they are not focused on the particular population of elite mathematic students I wished to examine, hence the need for an exploratory study.

Semi-structured interviews were chosen to allow opportunity for participants to impact the research, while considering the need for some comparability across interviews. Students were intended to be interviewed in pairs, but sometimes were interviewed in groups of three; extra students who turned up for the interviews were not turned away. Paired interviewing was inspired by its use in other studies (Boaler 2008, Evens & Housart, 2007). The interview schedule was piloted with two Finns and one Washingtonian, all who had studied mathematics at the tertiary level.

When analysing the data I have attempted to consider that ‘...there are clear dangers in saying that the interviews simply tell us more about the answers of the individual, as this ignores the presence of their interview partner.’ (Evens & Houssart, 2007, p. 22). I see the students’ words as public statements, at times inspired, supported, or edited by the presence of peers in the interview setting. I also acknowledge that the interviews may also have served as much in constructing or clarifying certain beliefs as in recording them.

The Selection of Cases and Participants

Eisenhardt (1989, p. 537) writes that while “...cases may be chosen randomly, random selection is neither necessary, nor even preferable.” Here, I have chosen cases with an eye towards both comparability and capturing a diverse population. These highlighted characteristics of the schools make them more identifiable, and so

to assure anonymity, Finnish and American English pseudonyms have are used for the cities as well as the schools and student-participants. The cities I shall call Jokimaa and Riverview are small metropolitan areas with a similar population (roughly 170 000 people), with higher than average immigration when compared with Finland or Washington at large, and containing at least one university.

From each community I chose one IB school with higher immigrant enrolment, and two schools considered strong in mathematics or mathematics related fields. A fourth school was added in Riverview as described later. In Finland, these schools were:

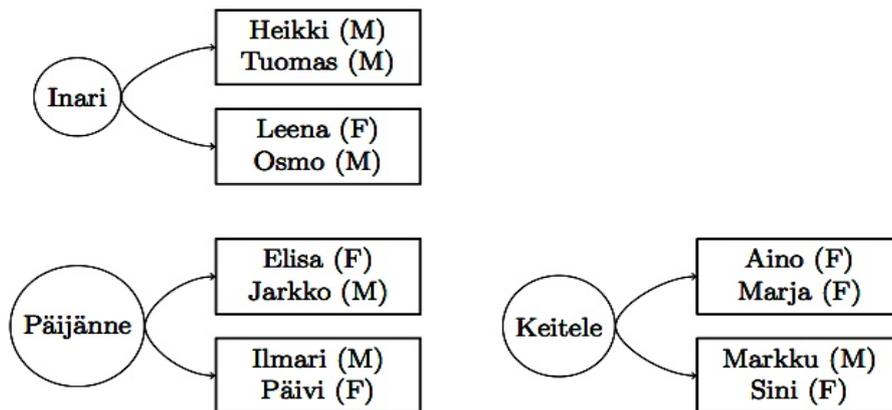


Figure 1: Interview Map for Jokimaa, Finland

Keitele Lukio, known for having a strong and extended mathematics programme

Inari Lukio, an IB programme in an area of high immigration for the Jokimaa area

Päijänne Lukio, offering a special IT line including university level courses

In Washington these schools were:

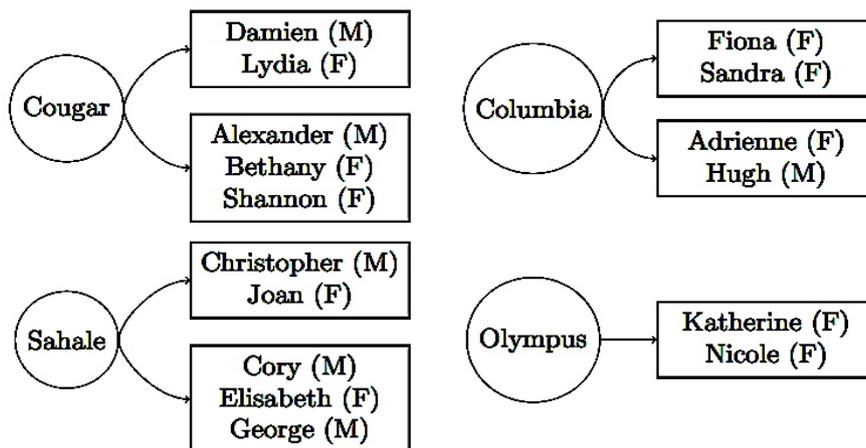


Figure 2: Interview Map for Riverview, Washington

Columbia High, known for strong performance in academic competitions and state exams and offering the most advanced AP mathematics course

Sahale High School, an IB programme with a higher minority enrolment rate

Cougar High is the most affluent high school in Riverview

Students from a fourth school approached me to be included in the study:

Olympus High has the lowest state tests scores, and is majority Latino/Hispanic.

RESULTS FROM THE INTERVIEWS

In this section I will discuss the development of three themes: extracurricular involvement, collaborative learning, and the conceptions of the elite mathematics peer group, first in Finland, then in Washington. The quotes below are selected to illustrate general themes (or exceptions) throughout the interviews.

Jokimaa, Finland: Extracurricular mathematics

Students interviewed from Jokimaa had no experience with extracurricular mathematics besides sitting for an optional national exam. Neither did they seem to be aware of any opportunities such as mathematics clubs. However, when explicitly asked, students did not seem to regret the lack of opportunity:

Saari(JS): Do you think you would have used the opportunity if there'd been some kind of extra-curricular mathematics?

Tuomas: Well, maybe not. [Laughing]

Heikki: [Laughing] To be honest no!

JS: And why, why not?

Heikki: Well, I, uh, value my other leisure activities more, perhaps.

Jokimaa, Finland: Collaborative Learning

Similarly, most Jokimaa students seemed to have little experience with collaborative learning, either formally or informally. For example Äinö said "...usually I've just done things by myself, and haven't cooperated with anyone."

While collaborative learning was described as mostly positive, when there was a mismatch in the level of achievement, it becomes. For example, while Leena enjoys the group work assigned in her IB mathematics course where collaborative work '...benefits, because if you know something and the other one knows something else then you can combine those and maybe understand it better.', she found it frustrating in other contexts, for instance in lower secondary prior to tracking:

Leena: Well, not in that case cause they were the easy problems that I had already solved and other ones asked me all the time that "how can you do this?" and stuff and... yeah I didn't like it. [laughs a tiny bit]

JS: Okay, so you didn't really feel like you were getting any academic benefit?

Leena: Yeah, I was just telling them how to do it.

Neither informal nor formal collaborative learning seemed to play a large role in the students' experiences, and perceptions of collaborative learning were ambivalent.

Jokimaa, Finland: Elite Mathematics Peer Groups

Among the students interviewed, the community of peers within elite mathematics courses in Finland was considered a key benefit of the course. Students believed their peers to be more interested and focused on mathematics, and that this enriched the course. For example, from Marja:

On the Short, there are many people there who study it because they have to, because maths is obligatory, and there is an atmosphere that maths isn't fun, even though there may also be people there who have just wanted to choose short maths [...] it's my experience that on the Long Maths, there are many who really want to invest in the subject and are able to listen during the lesson and all.

Students considered that the nature of the peer group allowed for deeper and more worthwhile content:

Jarkko: Yeah, I think I sort of feel, like, in principle, when the study group in long mathematics consists of the people who are interested in mathematics, at least, then the environment is easily more pleasing than the short mathematics study group where you can have many people who simply aren't interested in anything mathematical. So it is more encouraging as a study environment, and also in that you get deeper into all the things, you don't- it's like- you can see things as wholes and not only get small bits.

Elisa: Yeah, I actually agree... that at least is an advantage- that those who only take the courses and aren't at all interested, those people aren't there. And that when you have interested people you get to go deeper.

Jokimaa students seemed to emphasise that peers' interest and willingness to learn mathematics was a key asset for their own learning, and a mechanism of selection into elite courses. Students did not portray peer groups as a reason for retention in mathematics. This coincides with Jokimaa students' choice of elite tracks in accordance with future plans, as well as a greater independence from peer and family influence in school and track choice when compared with Riverview.

Riverview, Washington: Extracurricular Mathematics

All of the Riverview students had ample access to mathematics related extracurricular activities and most participated. However, they did not seem to consider involvement as an influential factor in their mathematical careers. One exception was Cory, who had an intention of pursuing mathematics at the tertiary level:

I feel like I'm almost entirely developed on the outside. Cause like, I have my classes which I kind of just do...like not just do it like C's but I mean, I do and I do good and I um- But like usually I find- cause I don't- I don't know, sometimes I don't feel challenged in a lot of my classes anyways.

Elsewhere, students revealed a lack of real enrichment in these activities, such as when I questioned two of the most accomplished students about a mathematics competition they had been involved in for several years, Math is Cool:

- JS: Okay, so, hmm... did you do anything related to number theory?
- Sandra: Um.
- JS: Have you- have you guys seen-
- Sandra: What is number theory?
- JS: Well have you seen like modular arithmetic? I'm just curious.
- Fiona: Oh! Modu- okay like
- Sandra: Yeah
- Fiona: Modular arithmetic
- JS: I'm not asking you what it is I'm just- just wondering if-
- Sandra: Like mod, like that thing, with the dividing?
- Fiona, JS: Yeah
- Sandra: That's in Math Is Cool.
- Fiona: It's in Math Is Cool, like, it's a really challenging- but we don't actually know what it is, just if you give us one simple type of problem with that we'd be able to do.
- Sandra: We'd be able to do it. We don't understand it, but we could do it. [Laughs]

While students were exposed to mathematics to which they would otherwise not have had access, it did not often seem to facilitate deeper understanding.

Riverview, Washington: Collaborative Learning

Many of the students interviewed in Riverview had strong collaborative networks outside the classroom. Such students considered these networks crucial in their success and persistence in elite mathematics. Students, such as George and Elizabeth, created lasting partnerships with daily mathematics collaboration.

As in Finland, however, there were students who found the idea of collaboration compelling, but frustrating in practice. For example Adrienne said:

Well, to teach someone something you have to really understand it, so... you learn it better and you have to remember it more, because you have to figure out exactly what you are talking about before you can help them understand it.

However, her experience was dissonant with this ideal. Again from Adrienne:

Well, sometimes it's frustrating because I'm not exactly patient, so if a person has trouble understanding something that I think is really obvious then I have to keep trying to find different ways to explain it to them and that's kind of tiresome...

While in general, collaboration was discussed positively, as in the Jokimaa case where there was a mismatch in achievement, actual encounters could be negative.

Collaborations were also limited by hierarchy, which Sandra describes legal terms: 'There's like this kid John, who's like the smartest kid, and then we're like the second, legally, or third'. Hierarchy determines collaboration as Fiona says, "It's more like among the smart people we ask each other questions".

While intensive collaborations were more evident in Riverview than in Jokimaa, they did not seem to regularly extend past a tight sub-group of peers.

Riverview, Washington: Elite Mathematics Peer Groups

As in Jokimaa, elite mathematics students enjoyed their peer groups, and emphasised that such a community was a strong motivation for staying in elite mathematics tracks. Riverview students also defined themselves against other students in order to explain the benefits of their elite tracks. Here Bethany and Alexander use their experience with a 'regular' or mixed-ability class:

Bethany: And there was- half the people would not care at all, they were just- they- Some of them were just going to drop out of high school right there, but there were some people who actually cared, they wanted to learn what was, the teacher was trying to teach, and as the AP honours classes are introduced, it's the people who care about what they... get in a high school or want to go to college and need good grades and good classes, those are the people that go on to the AP classes. So instead of being held back by a group of trouble makers-

Alexander: [overlapping] Oh it's so hard to learn- [laughing]

Bethany : or potential drop outs, [Alexander: sound of disgust]- instead surrounded by people who keep on wanting to learn more who are kind of the driving force of the class, and you're all about the same level throughout it.

Throughout the interviews, the peer groups' positive characteristics were a motivation to continue in elite mathematics, and separation from struggling, ill-behaved, or unmotivated students a key benefit. Furthermore, access was believed to be mediated by character. Hard work and desire were the necessary prerequisites, even when students discussed significant parental involvement in track placement.

Nicole and Katherine were the only students who questioned the sorting mechanisms:

Katherine: [It] kind of makes you wonder. [...] It makes you wonder if-

Nicole: The racism is really gone.

Katherine: Yeah. And then you see in your class when you're a class of almost-

Nicole: Thirty

Katherine: All Caucasian people [In a majority Latino/Hispanic school] talking about Affirmative Action it's kind of like, how...

However, while questioning the visible sorting at Olympus in several instances, Nicole and Katherine also see access to elite courses as a question of character. Nicole said: “It has a lot to do with work ethic. And if they want to be pushed or if they just wanna breeze right through.”

DISCUSSION AND CONCLUSIONS

There were stark contrasts in access to extracurricular mathematics in Jokimaa and Riverview; Jokimaa students had no opportunities for sustained involvement, whereas Riverview students had diverse choices, and almost all of them had been involved in mathematics related activities. Most Riverview students downplayed the effects of such involvement. However, for at least one student, Cory, involvement was key to his interest and persistence in mathematics.

In Finland, participation in mathematics competitions such as Math Olympiad is used as a signifier of talent (see e.g. Nokelainen, Tiiri, and Merenti-Välimäki, 2002). Yet, the students I interviewed had no access to this, or other, enrichment programmes. So, while PISA finds evidence of equality in Finland's performance, it may be masking inequality of access at the top.

In neither Jokimaa nor Riverview was there evidence of the sort of collaborations described by, for example, Boaler (2008). While collaborative learning is often associated with de-tracking, the Finnish students seemed to have less experience with peer-supported learning. Students from both communities had ambivalent feelings about collaboration where there was a mismatch in achievement. There seems to be room in both communities for further exploration of modern collaborative learning.

For both Jokimaa and Riverview students, an elite group of peers was a positive aspect in mathematics tracking. However, the descriptions used by Riverview students were more hierarchical, and attributed blame to low performing students. Their characterisations seemed close to Sayer's (2005, p. 233) description of belief in the ‘moral well-orderdness’ of the world, where:

...[T]he extent to which individuals' lives go well or badly is believed to be a simple reflection of their virtues and vices. It refuses to acknowledge the contingency and moral luck which disrupt such relations arbitrarily.

George said “...it kind of disgusts me to see the people that sit there and just ‘Oh- I have a D in this class and I’m taking Algebra for the fifth time because I don’t do my homework’” That such descriptions seem common among elite mathematics students in Washington, but seemingly not in Finland, is notable. They would arguably be more appropriate in Finland, where there is greater intergenerational class mobility (see Pekkarinen, Uusitalo & Pekkala 2006 or Breen & Jonsson 2005). Furthermore, these themes have resonance with Zevenbergen’s (2005) study of Australian students within a tracking system, where the discussion of classroom ethos and mathematical *habitus* using Bourdieu presents a possible way to deepen future work on this project.

The strong positive characterisations of elite peer groups in both Finland and Washington (also seen in Zevenbergen's (2005) study), and their place in improving learning and retention in elite mathematics, raises questions about how elite students might reply to the big-fish-little-pond concept or the possibility of de-tracking.

Limitations and Conclusions

There are several limitations to this study: more students were interviewed, and interviewed for slightly longer in Riverview, generating richer data from Washington, the linguistic aspects of the research are rough, and there were differences in interview styles between Jasu Markkanen and myself. The students' responses are thoroughly embedded not only in their schools, but their wider communities. However, important reforms, such as universal education and desegregation have involved changes in culture; culture is not fixed.

Regarding elite mathematics students, this study suggests a potential benefit from conducting international comparisons beyond the focus of studies such as PISA. Equality of provision may look different depending on the questions asked, and a comparative lens may clarify where to focus our attention.

NOTES

1. Education is governed mostly on the state level in the US. Washington is a better unit of comparison (than the US) with Finland in terms of population and resources and in addition, recently revised its mathematics curriculum through comparison with Finland, see Plattner (2007).

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INTERNATIONAL COMPARATIVE RESEARCH ON MATHEMATICAL PROBLEM SOLVING: SUGGESTIONS FOR NEW RESEARCH DIRECTIONS

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This paper is divided in two sections. In the first part, three problem solving views are discussed (problem solving as a process, as an instructional goal and as a teaching approach). In the second part, four research dimensions for international comparative studies on problem solving are proposed: (a) the research trends on problem solving in different countries-the researchers' perspective; (b) the curricular importance and justification of problem solving-the policy-makers' perspective; (c) teachers' beliefs, competence and practices in problem solving-the teachers' perspective; (d) students' beliefs and competence in problem solving-the students' perspective.

PROBLEM SOLVING-A MULTIDIMENSIONAL CONCEPT

Within the domain of mathematics education, the words *problem* and *problem solving* are extensively used. However, there is no consensus upon definitions, since many people use these terms to mean different things. The apparent agreement on the importance of problem solving does not say much about what problems and problem solving mean. In fact, it may mask very different views of what constitutes a problem and what kinds of problem solving abilities are desirable, teachable and evaluable (Arcavi & Friedlander, 2007). In respect of 'problems', there is evidence of polarisation, with some labelling problems as routine exercises that provide practice in newly learned mathematical techniques and others reserving the term for tasks whose difficulty or complexity makes them genuinely problematic (Schoenfeld, 1992; Goos et al., 2000). Furthermore, problem solving has been mostly viewed as a goal, process, basic skill, mode of inquiry, mathematical thinking, and teaching approach (Chapman, 1997). It appears, however, that the main perspectives on problem solving are those seeing it as a process, as an instructional goal and as a teaching approach.

Problem solving as a process

Various writers have developed frameworks for analysing problem solving as a process. Polya (1945), as the inaugurator of the research in the field, suggested four phases for the problem solving process: understanding the problem, devising a plan, carrying out the plan, and looking back. Polya's model comprised the basis on which other models were developed, for instance the six-phase one proposed by Kapa (2001): identifying and defining the problem, mental representation of the problem, planning how to proceed, executing the solution according to the plan, evaluation of what the problem solver knows about his/her performance, reaction to feedback.

However, ‘Polya-style’ models are often misinterpreted as a linear application of a series of steps, either because of the way they are presented in numerous textbooks (Wilson et al., 1993) or because they are perceived as such by most teachers (Kelly, 2006). In recognising the above deficiency, Mason et al. (1985) analyse three phases for the process of tackling a question; Entry, Attack and Review. It could be argued that Mason’s phases are parallel to those of Polya. This is partly true, since there are obvious similarities between the *Entry* and *understanding the problem*, the *Attack* and the *devising and carrying out the plan*, the *Review* and the *looking back*. Nevertheless, Mason et al.’s (1985) *attack* phase appears not to necessitate a predetermined plan in the manner of Polya’s *devising and carrying out a plan*.

Problem solving as an instructional goal

Mathematics proficiency, according to Kilpatrick et al. (2001), refers to successful mathematics learning and has five strands (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition). *Strategic competence* is defined as the ability to formulate, represent and solve mathematical problems. For many educational systems, the strategic competence in problem solving has a central role in mathematics teaching/learning and has been set as a fundamental instructional goal. For instance, problem solving has been identified as one of the five fundamental mathematical process standards along with reasoning and proof, communication, connections, and representations, by the National Council of Teachers of Mathematics (NCTM, 2000). For NCTM, mathematics teaching and learning and problem solving are synonymous terms; therefore the building of new mathematical knowledge through problem-solving should be in the centre of mathematics education. Similarly, in the context of China, Cai and Nie (2007) argue that the activity of mathematical problem solving in the classroom is viewed as an important focus of instruction that provides opportunities for students to enhance their flexible and independent mathematical thinking and reasoning abilities.

Problem solving as an instructional approach

Kilpatrick (1985), in a retrospective account of research on problem solving between 1960 and 1985, has identified five instructional approaches in *teaching mathematical problem solving* (osmosis, memorisation, imitation, cooperation, reflection). Despite the differences on how mathematical problem solving is approached in each of these categories, there is a common element: Problem solving is viewed as a cluster of skills students should acquire. From a different perspective, Nunokawa (2005) proposes four types of problem solving approaches in teaching mathematics. These approaches equate problem solving and mathematics teaching/learning. The first type refers to *emphasizing the application of mathematical knowledge students have*, through which students are expected to enrich their schemata of the targeted mathematical knowledge. This corresponds to ‘teaching for problem solving’. The second type is about *emphasizing understanding of the problem situation*. As Nunokawa points out, “what is important in this type is deepening students’ understanding of the situations that they are exploring using their mathematical

knowledge” (p. 330). The third type regards *emphasizing new mathematical methods or ideas for making sense of the situation*. In other words, the teaching of mathematics occurs *via* problem solving. The teacher should select problematic situations that are appropriate to bring to light informal or naïve approaches from students, some of which can be formulated into the targeted mathematical knowledge. Finally, the fourth type is about *emphasizing management of solving processes themselves*. This corresponds to ‘teaching *about* problem solving’; what students should obtain is “the wisdom concerning how to treat problematic situations, manage their solving processes, and put forward their thinking” (Nunokawa, 2005, p. 334).

THE NEED FOR INTERNATIONAL COMPARATIVE RESEARCH ON PROBLEM SOLVING

The diversity and interactivity of the international mathematics education community provides both the opportunity and motivation for comparative studies. Comparative research can claim to be a useful tool towards a better understanding of the educational process in general and in one’s own system in particular (Grant, 2000); it is not necessarily meant to supply answers to questions but rather to enable planning and decision-taking to be better informed (Howson, 1999). Comparative research could be about the mutual benefits of sharing good practice and about the adaptive potential of the policies and practices of other educational systems to our own (Clarke, 2003).

Challenges confronting the international research community require the development of test instruments that can legitimately measure the achievement of students who have participated in different mathematics curricula, research techniques by which the practices, motivations, and beliefs of all classroom participants might be studied and compared with sensitivity to cultural context, and theoretical frameworks by which the structure and content of diverse mathematics curricula, their enactment, and their consequences can be analysed and compared (Clarke, 2003, p. 144).

Comparative studies in mathematics education can be distinguished as two types: large-scale (mostly quantitative) and small-scale (mostly qualitative) studies. Large-scale studies such as TIMSS and PISA, have had much criticism. In my opinion, their biggest weakness is that they implicitly promote the idea of a *global mathematics curriculum* (a curriculum to which all school systems would subscribe), an idea based on the awareness of the world as one (Andrews, 2007b). Additionally, they are increasingly interpreted as competitions with inevitable winners and losers. Small-scale studies usually compare only two or three educational systems in relation to mathematics (Kaiser, 1999). They “share a common characteristic of seeking insight into the ways in which mathematics is systemically conceptualized and presented to learners in different countries” and generally celebrate cultural differences and identify the adaptive potential of one system’s practices for another, by acknowledging culturally located traditions (Andrews, 2007b, p. 489).

During the 1980s and 1990s, problem solving has been the subject of extensive research in the U.S.A. The results of these studies have influenced the research and curricula development in many countries, such as in China (Cai & Nie, 2007), Australia (Clarke et al., 2007), Japan (Hino, 2007), Brazil (D'Ambrosio, 2007), Singapore (Fan & Zhu, 2007) and so many others. However, despite the US's influential research and curricular lines, problem solving research in many countries has evolved differently. Not only does the term *problem solving* mean different things in different countries, it has often changed dramatically in the same country (Torner et al., 2007). This has to be taken into consideration by comparative researchers in the field of problem solving, because many attempts to make international comparisons across countries fall into the trap of assuming that things with the same name must have the same function in every culture (Grant, 2000).

There is a lack of small scale studies on problem solving in the whole gamut of international comparative research. Taking all the above into account, I propose four distinct but also overlapping dimensions that comparative research on problem solving could focus on. Studies regarding these four dimensions should aim at in-depth investigation and analysis of how mathematical problem solving is being conceptualised in different educational settings. Nonetheless, studies of this kind should be approached and interpreted as efforts of the international mathematics education community towards international cooperation and national improvement. In the following pages I describe each of the four dimensions briefly.

a) The research trends on problem solving in different countries - The researchers' perspective

Comparative studies, from this point of view, should aim at comparisons between the research interests of mathematics educators and the research produced in each system. Comparing evidences from single-national studies around the world reveals that the problem solving research produced in different countries varies enormously. From the Australian perspective, for instance, Clarke et al. (2007) describe problem solving research in terms of three themes (obliteration, maturation, generalisation). Similarly, with respect to Portuguese research, Ponte (2007) states that the interest has now moved from mathematical problems to mathematical investigations and describes three research themes: the development of students' ability to do investigations, the promotion of students' mathematics learning, the influence of these activities on students' attitudes and conceptions. Other countries have not developed problem solving as a separate area of mathematics education research for various reasons. In the context of France, didactic research is influenced both by the Theory of Didactic Situations and the Anthropological Theory of Didactics (Artigue & Houdement, 2007). In both theories, problem solving has a central role; therefore the didactic research on mathematics is not separated from research on problem solving. In Brazil, however, this phenomenon appears for a different reason: problem solving is not examined as a separate area of mathematics education, but as part of the current reflection on Education and Cognition (D'Ambrosio, 2007).

b) The curricular importance and justification of problem solving - The policy-makers' perspective

Comparative research in this area should examine the explicit and/or implicit emphasis on problem solving in intended curricula and how problem solving within them is cultivated. By *intended curricula* I refer to “documents or statements of various types (often called guides, guidelines, or frameworks) prepared by the education ministry or by national or regional education departments, together with supporting material, such as instructional guides, or mandated textbooks” (Mullis et al., 2004, p. 164). In his paper, Xie (2004) compared the cultivation of problem solving between national mathematics standards issued by the National Council of Teachers of Mathematics (NCTM) in the U.S.A. and the Ministry of Education (MoE) of China. Both NCTM and MoE consider problem solving abilities to be the main goal of mathematics education. The definitions they offer of problem solving seem to be related to similar goals. However, there are certain differences between their goals. In NCTM, the term “problem-solving” is used to refer both to an end and an approach; while in MoE, problem-solving is seen mainly as a goal. Unlike the NCTM, the MoE does not mention students learning on their own but rather that they should apply the learned mathematics language to think or communicate mathematically. Differences do not only exist cross-nationally. In their single-national study in Israel, Arcavi and Friendlander (2007) interviewed the managers of different curriculum development projects. Despite the similarities on the participants views and approaches to problem solving (i.e. its importance, recognising the existence of different sorts of problems, etc) there are noticeable differences among the different theoretical and practical approaches to problem solving, even within the same community (of curriculum developers), focusing on the same target population (elementary schools) within a centralised system (in Israel) with a uniform syllabus.

c) Teachers' beliefs, competence and practices in problem solving - The teachers' perspective

International comparative studies about teachers' mathematics related beliefs (i.e. Whitman & Lai, 1990; Correa et al., 2008; Santagata, 2004; Andrews & Hatch, 2000; Andrews 2007c) and practices (i.e. Leung, 1995; Andrews, 2007a; Givvin et al., 2005) suggest that these two factors are more similar to each other within single countries than they are across countries. While there are some single-national studies about teachers' problem solving beliefs and practices, as for example in Australia (i.e. Anderson et al., 2008) and Cyprus (i.e. Xenofontos & Andrews, 2008), I am not aware of any cross-national studies in this area. From a different starting point (examining English and Hungarian teachers' beliefs about mathematics teaching), Andrews (2007c) concludes that English teachers tended to view mathematics as applicable number and the means by which learners are prepared for a world beyond school, while Hungarian teachers perceived mathematics as *problem solving* and logical thinking and independent of a world beyond school. Taking all the above into

account, the similarities and differences of teachers' problem solving beliefs, competence and practices could be another dimension of the international comparative research in the field.

d) Students' beliefs and competence in problem solving - The students' perspective

Students' beliefs, competence and performance have traditionally attracted mathematics education researchers all around the world. Problem solving literature is, in my opinion, dominated by papers from students' perspective (i.e. Mason, 2003 in Italy; Nicolaidou & Philippou, 2003 in Cyprus; Op'Eynde & De Corte, 2003 in Flanders; Goos et al., 2000 in Australia, Cooper & Harries, 2002 in England and so on). International comparative studies, such as TIMSS (Mullis et al, 2004) and PISA (OECD, 2003) have examined students' problem solving performance in different countries. Particularly, PISA included *mathematical literacy* in its mandate (Clarke, 2003) and looked at mathematics in relation to its wider uses in people's lives (OECD, 2003). Mathematics literacy in PISA is measured in terms of students' capacity to recognise and interpret mathematical problems encountered in every-day life, translate these problems into a mathematical context, use mathematical knowledge and procedures to solve problems, interpret the results in terms of the original problem, reflect on the methods applied, and formulate and communicate the outcomes (Clarke, 2003). Both TIMSS and PISA were large-scale projects. What is needed in researching students' beliefs and competence in problem solving are small-scale qualitative studies that compare two or three educational systems.

CONCLUSIONS

The importance of mathematical problem solving in mathematics teaching and learning is internationally well defended. By acknowledging and investigating the cultural diversity of problem solving in different educational systems with respect to the four dimensions proposed above could be beneficial. The creation, promotion and establishment of a problem solving culture around the world is, in my opinion, important for better mathematics teaching and learning. International collaborations and comparative research could be the vehicle towards this direction.

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CERME 6 – WORKING GROUP 14

EARLY YEARS MATHEMATICS

TABLE OF CONTENTS

Introduction.....	2535
<i>Patti Barber</i>	
Girls and boys in “the land of mathematics” 6 to 8 years old children’s relationship to mathematics interpreted from their drawings.....	2537
<i>Päivi Perkkilä, Eila Aarnos</i>	
“Numbers are actually not bad” Attitudes of people working in German kindergarten about mathematics in kindergarten	2547
<i>Christiane Benz</i>	
Learning mathematics within family discourses.....	2557
<i>Birgit Brandt, Kerstin Tiedemann</i>	
Orchestration of mathematical activities in the kindergarten: the role of questions.....	2567
<i>Martin Carlsen, Ingvald Erfjord, Per Sigurd Hundeland</i>	
Didactical analysis of a dice game	2577
<i>Jean-Luc Dorier, Céline Maréchal</i>	
“Tell them that we like to decide for ourselves” Children’s agency in mathematics education.....	2587
<i>Troels Lange</i>	
Exploring the relationship between justification and monitoring among kindergarten children	2597
<i>Pessia Tsamir, Dina Tirosh, Esther Levenson</i>	
Early years mathematics – The case of fractions.....	2607
<i>Ema Mamede</i>	
Only two more sleeps until the school holidays: referring to quantities of things at home.....	2617
<i>Tamsin Meaney</i>	
Supporting children potentially at risk in learning mathematics Findings of an early intervention study.....	2627
<i>Andrea Peter-Koop</i>	

The structure of prospective kindergarten teachers' proportional reasoning.....	2637
<i>Demetra Pitta-Pantazi, Constantinos Christou</i>	
How can games contribute to early mathematics education? – A video-based study	2647
<i>Stephanie Schuler, Gerald Wittmann</i>	
Natural differentiation in a pattern environment (4 year old children make patterns)	2657
<i>Ewa Swoboda</i>	
Can you do it in a different way?.....	2667
<i>Dina Tirosh, Pessia Tsamir, Michal Tabach</i>	

INTRODUCTION TO WORKING GROUP 14: EARLY YEARS MATHEMATICS

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The working group met for the first time at CERME 6 and we found many similarities but also considerable differences in our countries and individual contexts. Most countries represented were reappraising Early Years' education and due to recent research (Clements and Sarama 2007a) were also re-considering the curriculum offered to the youngest children in mathematics.

One of the most significant changes observed in Germany, UK and Israel has been to look at the ways in which children are being taught and what they are being taught. A few years ago, mathematics did not play an official role in German kindergarten. Learning mathematics was reserved for school. Kindergarten teachers were not confronted during their job training with mathematics education. Now different documents in matters of educational policy are raised, where mathematics learning now is included. But the curricula of the single federal states of Germany differ in the explicitness of the statements made concerning mathematics. It ranges from very in-depth descriptions of mathematical contents in kindergarten, to others, where mathematics does not play an important role. Schooling for 3-6 year olds is not compulsory and is paid for.

In England there is full time free education for all children from the age of four and part time for all children from three. There is now a prescribed curriculum for this age group containing problem solving, reasoning and numeracy ? as the mathematics strand of the new curriculum document named as 'The Early Years Foundation stage' for ages from 0-5. The curriculum is compulsory but there are no specific ways of doing it. Training for the teachers is seen as very important largely due to research (The Effective Provision of Pre-School Education (EPPE) Project: Final Report A Longitudinal Study Funded by the DfES 1997-2004) highlighting that the best practice in Early Years settings was with qualified teachers.

In Israel school is compulsory from the age of 6 and the new curriculum here is compulsory. It covers the basic ideas in maths with some free play but is also teacher orchestrated.

In Denmark the thinking about mathematics is similar to the German thinking. The philosophy is on the development of the whole child. There are no specific goals for children and the emphasis is on play but there is a movement towards a specific curriculum. There is a raising awareness of mathematics pedagogy and how to it but there are problems with the cost.

In New Zealand children begin school at 5. The curriculum document for 0-8 is Te Whariki and it advocates a holistic approach to teaching and learning

In Finland all teachers have Masters in early education. There is pre-school until 6. Skills are taught to develop mathematical thinking.

In Portugal education for 3-6 year olds is not compulsory but the majority attend.

In Poland there are not enough pre school places for those who want them and it is not obligatory. Fees are paid for pre school therefore there are financial reasons why some children do not attend. Children attend school at 7. In the 0-6 kindergarten there is preparation for school. In mathematics this consists of numbers, counting, and shapes. There is no special training for pre school teachers but all teachers are educated with masters.

Cyprus has a system where children attend Nursery from 3 years old. The formal curriculum begins between 5 and 6. The EY maths curriculum consists of free play, building structures, numeracy, and geometric shapes. All teachers have to have a degree and maths education is part of this. There are a huge number of people who want to do the job.

In Norway 80-90% from 1yr. at 3 yrs more than 90% of children attend the kindergarten. It is felt that all children should be able to go to kindergarten. School begins at 6 years old. In 2006 there were official documents mentioning mathematics – numbers, space and form. The training is 3 yrs at university.

There were many papers submitted and we organised them into the following themes

- Discussion of theoretical concepts and models and how they are used in analysis
- Research methods/methodologies: discussions on how very young children are able to articulate their understanding of mathematics/mathematical thinking e.g. drawings, gestures and recordings (written notations).
- Discussion on how parents can contribute to our perspective of what children are doing.
- Our challenges: we are working in different paradigms, a discussion on what we mean by learning to make that explicit in our papers and discussions
- Many perspectives are observed: very young children, teachers, other adults

After discussion of the papers the following challenges emerged for the group in the future:

- Impact on policy makers
- Cooperation and collaboration between members of the group
- Gender! Teachers (salary, role models, social standing) Children (differences in teaching and learning outcomes)
- What is mathematics in the early years and what does it look like?
- How can we support children's mathematical thinking in the early years?

Clements, D.H. and Sarama, J. (2007a) Effects of a preschool mathematics curriculum : summative research on the building blocks project in *Journal of research in Mathematics Education* 2. 136-163.

**GIRLS AND BOYS IN “THE LAND OF MATHEMATICS”
6 TO 8 YEARS OLD CHILDREN’S RELATIONSHIP TO MATHEMATICS
INTERPRETED FROM THEIR DRAWINGS**

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In this paper we highlight 6 to 8 years old children’s relationship to mathematics. For this task we use children’s drawings. Children were asked to imagine themselves in math land. We describe, reduce, and interpret to organize our analyses of gender differences. Theoretical basis lies on theoretical knowledge of math learning, and interpretation of children’s drawings. We found that there are meaningful connections between gender, children’s developmental level, emotions, and math productions.

METHODOLOGICAL INTRODUCTION

This paper is based on our multidisciplinary research project “Children and Mathematics”. We have gathered data from 6 to 8-year-old children ($n = 300$) by our pictorial test (Perkkilä & Aarnos, 2007a). Pictorial test has two parts: a picture collection presented to children and children’s drawings of themselves in the math land. In this paper we concentrate on children’s drawings. Drawings give children another language with which to share feelings and ideas. Our goal is to reach the usefulness of multidimensional approaches for understanding children’s drawings. The main aims are:

1. To describe math contents and impressions girls and boys produced in their drawings.
2. To reduce results towards the core meaning of math and contextual basis for math learning.
3. To interpret girls’ and boys’ mathematical and psychological needs for math learning environment.

The interpretative framework we use to organize our analyses of gender differences in children’s drawings “Me in the Math Land” is shown in Figure 1.

	Mathematical Perspective	Psychological Perspective
Description	Children’s productions	Impressions
Reduction	Meaning of Math	Contextual basis for math learning
Interpretation	Math needs	Psychological needs

Figure 1: Framework for analysing girls’ and boys’ drawings

As the column headings “Mathematical Perspective” and “Psychological Perspective” indicate, the analytical approach involves coordination two distinct theoretical viewpoints on mathematical activity. In our analysis we’ll take three steps: description, reduction and interpretation. The entries in the column under mathematical perspective indicate three aspects of children’s relationship to

mathematics, and the entries in the column under psychological perspective indicate three related aspects of individual basis for children's math learning.

The drawings were analysed by an open method; all the contents, colours, and impressions were classified. We found from the data following categories:

1. "Me" (person in the picture) with two subcategories: a) activities, and b) social situations,
2. Real life contents with four subcategories: a) wild nature, b) animals, c) buildings, and d) vehicles,
3. Mathematical contents with five subcategories: a) amounts of numbers, b) quantity of numbers, c) arithmetical problems, d) geometrical forms, and e) mathematical talk, and
4. Impressions with five subcategories: a) human expressions, b) colours, c) emotional expressions, d) creativity, and e) maturity.

The background variables were gender and grade.

PERSPECTIVES ON MATHEMATICS LEARNING

Hersh (1986) has answered to the question "What is mathematics?" as follows: "It would be that mathematics deals with ideas. Not pencil marks or chalk marks, not physical triangles or physical sets, but ideas (which may be presented or suggested by physical objects). The main properties of mathematical knowledge, as known to all of us from daily experience, are:

- 1) *Mathematical objects are invented or created by humans.*
- 2) *They are created, not arbitrarily, but arise from activity with existing mathematical objects, and from the needs of science and daily life.*
- 3) *Once created, mathematical objects have properties which are well-determined, which we may have great difficulty in discovering, but which are possessed independently of our knowledge of them.* (Hersh, 1986, 22.)

The nature of mathematics comes up especially then when you try to develop mathematical model from every day situation, and to apply mathematical system for example in the problem situation to another new every day situation (Ahtee & Pehkonen, 2000, 33-34). The daily life problems are increasingly emphasized in recent mathematics curricula in various countries. For example an illustration of the daily life problems in arithmetic could begin by having children use their own words, hands-on-materials, pictures, or diagrams to describe mathematical situations, to organize their own knowledge and work, and to explain their strategies. Children gradually begin to use symbols to describe situations, to organize their mathematical work, or express their strategies. (Singer & Moscovici, 2007, 1616.)

Mathematical knowledge cannot be revealed by a mere reading of mathematical signs, symbols, and principles. The signs have to be interpreted, and this interpretation requires experiences and implicit knowledge – one cannot understand these signs without any presuppositions. Such implicit knowledge, as well as attitudes

and ways of using mathematical knowledge, are essential within a culture. Therefore, the learning and understanding of mathematics requires a cultural environment. (Steinbring 2006, 136.) According to Berry and Sahlberg (1995, 54) many children have preconceptions about modelling which are based on interpretations of real models. They argue that it is worth to utilize these preconceptions in school mathematics. According to Presmeg (1998) there is strong evidence that traditional mathematics teaching does not facilitate a view of mathematics that encourages students to see the potential of mathematics outside the classroom. Although some reports indicate that children are involved in many life activities with mathematical aspects, they continue to see mathematics as an isolated subject without much relevance to their lives.

EARLY MATHEMATICS LEARNING AND GENDER ASPECTS

According to Aunio's (2006, 10) research review there are contradictory research results in children's mathematical performance and gender. For example Dehaene's (1997), Nunes & Bryant's (1996) research results show that girls and boys possess identical primary numerical abilities. Carr and Jessup (1997) have reported that during the first school year, boys and girls may use different strategies for solving mathematical problems, but there is no difference in the level of performance. Whereas Jordan, Kaplan et al. (2006) found in their research small but reliable gender effects favouring boys on overall number sense performance as well as on nonverbal calculation.

According to Ojala and Talts (2007), we can better understand why girls in school and afterward usually achieve their learning goals better. Their study shows that gender differences in learning are probably emerging early before school starts. The gender differences were present in most areas of learning expect language, mathematics, and science. (Ojala & Talts, 2007, 218.)

According to Geist and King (2008) to support excellence in both boys and girls we must design experiences and curriculum that meet the needs of both boys and girls by understanding their uniqueness. Most teachers would never consciously treat boys and girls differently; however assumptions about gender roles and myths about learning mathematics can sometimes lead to us treating boys and girls differently without even realizing it. This is what is know as the "self-fulfilling prophesy." (Geist & King, 2008, 44-50.)

According to Muzzatti and Agnoli (2007), gender differences exist also in gender stereotyping of mathematics. Despite the lack of gender differences in actual mathematics performance, girls evaluate themselves as being less competent, and as they grow older, both boys and girls lose confidence in their ability and perceive this subject matter as more difficult and as less likeable. (Muzzatti & Agnoli, 2007, 757.)

Interpreting and understanding children's drawings

The children are telling us in pictorial language how they feel about themselves and the determining influences in their lives. They are also telling us how they need other persons. An attempt to interpret child art within a single theoretical framework can only result in frustrating oversimplification. More productive than a single-minded approach is an eclectic one that draws upon disciplines that have contributed significantly to our understanding of the infinite variety of human behaviour. (DiLeo, 1983, 214-216.) In this paper such an eclectic approach will draw upon mathematics learning and teaching, educational and developmental psychology.

The first representation of the human form has been observed wherever children's drawings have been studied. During the preschool years, spontaneous drawings tend to be more elaborate with the inclusion of other items of significance, notably houses, trees, sun, and other aspects of nature. Human figures in particular are regarded as valuable indicators of cognitive growth. A qualitative as well as a quantitative change occurs at about seven or eight years when "intellectual realism" gives way to "visual Realism", a change that finds its correspondence in the Piagetian concept of a shift from the preconceptual (preoperational) to the concrete operational stage. These terms express, in substance, a metamorphosis in thinking from egocentricity to an increasingly objective view of the world. (DiLeo, 1983, 37.)

Two developmental stages of drawing are especially relevant to our research: intellectual and visual realism (see fig. 2). According to Malchiodi (1998, 1) drawing has been undeniably recognised as one of the most important ways that children express themselves and has been repeatedly linked to the expression of personality and emotions. Children's drawings are thought to reflect their inner world. Although children may use drawing to explore, to problem solve, or simply to give visual form to ideas and observations, the overall consensus is that art expressions are uniquely personal statements that have elements of both conscious and unconscious meaning in them and can be representative of many different aspects of the children who create them. (cf. fig. 2)

Age	Drawing	Cognition
4-7	<i>Intellectual realism</i> Draws an internal model, not what is actually seen. Draws what is known to be there. Expressionistic. Subjective.	<i>Preoperational stage (intuitive phase)</i> Egocentric. Views the world subjectively. Vivid imagination. Fantasy. Curiosity. Creativity. Functions intuitively, not logically.
7-12	<i>Visual realism</i> Subjectivity diminishes. Draws what is actually visible. Human figures are more realistic. Colours are more conventional.	<i>Concrete operations stage</i> Thinks logically about things. No longer dominated by immediate perceptions. Concept of reversibility.

Figure 2: Intellectual and visual stages related to Piaget's stages of cognitive development according to DiLeo (1983, 37-38.)

According to Malchiodi (1998) phenomenological approach is a way to understand children and their drawings. Understanding children's creative work is attractive because it entails looking at drawings from a variety of perspectives, including among others developmental and emotional influences. (Malchiodi, 1998, 35-40.)

Themes of children's drawings may also be gender-related. General differences in the themes of boys' and girls' drawings, observing that "the spontaneous production of boys reveal an intense concern with war fare, acts of violence and destruction, machinery, and sports contents, where as girls depict more tranquil scenes of romance, family life, landscapes, and children at play". Girls use fairy tails images such as kings and queens and animals such as horses as the subjects of their drawings. Whether this, tendency to portray specific subjects by boys and girls is developmental or the result of parental or societal influences or both remains as an unsolved question. (Malchiodi, 1998, 186-187.)

Vygotsky (1978) viewed drawing as a way of knowing, as a particular kind of speech, and emphasized the critical role of drawing in young children's concept development; particularly because the drawing event engages children in language use and provide an opportunity for children to create stories.

RESULTS

Descriptions

Children drew themselves in rich forms, produced math contents and informal contents (e.g. nature and buildings). Most children were standing alone in the math land. Most girls were smiling and some of the boys seemed to be involved in action. Girls and boys equally expressed numbers and arithmetical problems. Besides children themselves wild nature was the main content of the pictures.

Mathematical productions

	Girls (%)	Boys (%)
None	23,2	28,3
Numbers	76,8	71,7

Table 1: Number expressions

	Girls (%)	Boys (%)
None	65,8	65,5
Arithmetical problems	34,2	34,5

Table 3: Arithmetical problems

	Girls (%)	Boys (%)
Numbers (≤ 10)	44,5	40,0
Numbers (> 10)	32,3	31,7

Table 2: Number quantities

	Girls (%)	Boys (%)
None	12,9	15,2
Numbers with forms	29,0	29,7
Other forms	58,1	55,2

Table 4: Forms

There were no differences in girls' and boys' math expressions (Tables 1- 4). These results have similarities with some other researches e. g., Nunes and Bryant (1996), Carr and Jessup (1997), Perkkilä and Aarnos (2007a).

In figure 3 drawers are practicing their number sense which is essential part of early math curriculum. Still there is a worry that this kind of number practicing is not enough in children's early math learning.



Figure 3: First-grader boy's and first-grader girl's drawings demonstrating huge number productions



Figure 4: Second-grader boy's and second-grader girl's drawings demonstrating creative use of numbers

These children also are practicing their number sense but in a more creative way than children in figure 3. However, we have to accept that it is difficult to conclude any differences only by the pictures. Concerning to this challenge, we sustained trustworthiness by comparing these differences to children's other responses in our pictorial test, and by finding parallel results.

Emotional expressions

	Girls (%)	Boys (%)
Sad	4,5	19,3
Neutral	42,6	60,0
Joy	52,9	20,7

Table 5: Emotional impressions ($\chi^2=41.8***$)

Statistically significant gender effect can be seen in girls' and boys' emotions (Table 5). Most girls express in their drawings joyful attachment for mathematics whereas it was hard to see clear emotional expressions in most boys' drawings, and so they were interpreted to have neutral attachment for mathematics. We wonder if results have basis in either the differences in girls' and boys' development (e.g. Bornstein et al. 2006) or early gender stereotypes (e. g. Steele 2003; Golombok et al. 2008).

Reduction

	Girls (%)	Boys (%)
Alone	73,5	63,4
With others	7,7	9,7
With fairy	15,4	14,5
None	3,2	12,4

Table 6: “Me” in Math Land

	Girls (%)	Boys (%)
Standing	67,1	62,1
Moving	22,0	18,0
Housing	3,2	1,4
None	7,7	18,5

Table 7: “My Action” in Math Land

The meaning of math for these children seems to be “being alone, silent, producing numbers and arithmetical problems”. Most children seem to be at level of intellectual realism (see Fig. 2). Contextual basis for math learning is for most children in this research outside school buildings, mostly in wild nature (Table 8).

	Girls (%)	Boys (%)
Wild nature	80,6	62,1
Animals	36,1	23,4
Buildings	36,1	44,8
Vehicles	3,2	13,1

Table 8: Contents of Math Land

Typically, in boys’ drawings there were few more buildings and vehicles whereas girls produced few more animals and wild nature (e.g. Malchiodi 1998, 186-187). The buildings in the drawings were towers, cottages, castles, home houses etc.



Figure 5: First-grader boy’s and first-grader girl’s drawings demonstrating no numeric content

In these drawings (Fig. 5) children seem to practise early mathematical skills e.g. classifying, grouping, and making series. In general, these skills develop in early years.

Interpretation

Different kinds of needs can be interpreted from children’s drawings “Me in the math land”. Children have both mathematical and psychological needs. Concerning the math learning we could find three different groups of children: “traditional school mathematicians” (Fig. 3), “wild and creative mathematicians” (Fig. 4), and beginning mathematicians” (Fig. 5). These groups need differentiations in math teaching (cf. Geist & King, 2008). In order to collect the main gender effects, three main scales were counted of the categories presented earlier: emotions, developmental level, and

math productions. The connections were analysed by t-test (gender differences), and by correlations (dependences between scales). Concerning the psychological needs there are great discrepancies in children's developmental level and emotional basis. Still there can be seen gender views (Fig. 6).

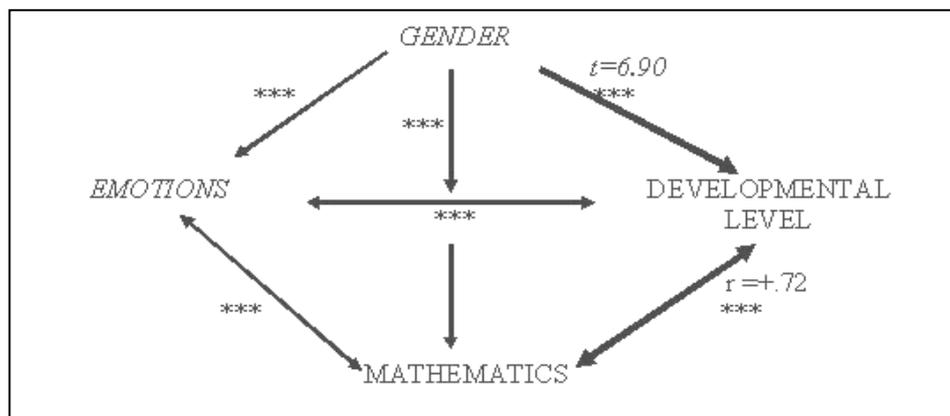


Figure 6: Statistically meaningful connections between gender and basic scales interpreted and counted in children's drawings

All connections between gender and three scales (emotions, developmental level, and math productions) are statistically significant, favouring girls. The most powerful connection is between gender, children's developmental level, and math productions. Furthermore, children's mathematical skills have strong effect in their mental development. Therefore children need mathematical inspirations in their growing environments.

We found a strong cumulative circle between children's developmental level, mathematics productions, and emotions (fig. 6). Aunola et al. (2004) have shown that children's mathematical skills develop in a cumulative manner from the preschool to the first years of school, even to the extent that the initial mathematical skills in beginning of preschool were positively associated with their later growth rate: the growth of mathematical skills was faster among those who entered preschool with already higher mathematical skills. Aunola et al. (2004) also showed that by the end of grade 2 children have problems both in attachment for mathematics and in math learning.

According to Geist and King (2008), when boys enter school they are often less able than girls to write numbers correctly or align numbers for tasks such as adding and subtracting on paper. Girls, on the other hand, find writing and completing worksheets much easier. (Geist & King 2008, 45-46.) Boys' weaker fine motor skills were also seen in children's drawings. As shown in tables 1 to 4 there were no gender differences in math expressions themselves. While interpreting profoundly the data we have looked at the issues behind math expressions e.g. emotions and developmental level.

Many teachers believe that girls achieve in mathematics due to their hard work, while boy's achievement is attributed to talent. These differing expectations by teachers and

parents may lead to boys often receiving preferential treatment when it comes to mathematics. Children may internalize these attitudes and begin to believe what their teachers and parents believe. As a result girls' assessment of their enjoyment of mathematics falls much more drastically than boys' assessment as they move through the grades. These attitudes may shape the experiences that children have as they are learning mathematics. (Geist & King 2008, 44-45.)

Concerning the need for learning environments, children's math land is mostly in the nature. They spontaneously combine the informal and formal mathematics. Boys seem to need more lively actions and constructions in their learning environments. Girls' expectations towards mathematics learning environments are more positive than boys'. Teachers and other educators should recognize how powerful out-of-school learning experiences could be in math learning. Mathematical experiences are essential parts in children's world from very early of life. The child's focusing on numerosity produces practice in recognizing and utilizing numerosity in the meaningful everyday context of the child.

CONCLUSIONS

The description and interpretation of children's drawings gave us insights into children's math experiences and needs. Children's drawings can be an effective of evaluating important basis of math learning, e.g. their relationship towards mathematics. This method also allowed children, who found written reporting and recording difficult, a better opportunity to reveal their understanding the nature of mathematics and their inside needs for the learning situations. (cf. DiLeo, 1983; Malchiodi, 1998; Vygotsky, 1978)

The Finnish curriculum (2004, 17) is giving more attention to the following aspects: Special needs of girls and boys; Equal opportunities for children to learn and to start school; Strengthening children's positive self-concept and their ability to learn skills; Having children learn to understand the significance of a peer group in learning; and Having children learn to join learning and to face new learning challenges with courage and creativity.

According to Perkkilä and Aarnos (2007b, 3), in school children have to learn formulas, exact proofs, or formalized definitions. Without real life connections this kind of math learning may restrict the talk about math in to formal mathematics. In present research children drew themselves mostly in real life situations. Daily life problems and narratives in learning situations could promote early math learning (cf. Singer & Moscovici, 2007; Presmeg, 1998).

The gender variations found in children's drawings are important to think about. We suggest that early math learning environments should be child centred and gender sensitive.

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“NUMBERS ARE ACTUALLY NOT BAD”

Attitudes of people working in German kindergarten
about mathematics in kindergarten¹

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The following article deals with the results of a questionnaire survey, in which attitudes and beliefs of German kindergarten teachers² about “mathematics”, “teaching and learning of mathematics” and “mathematics in the early years” were evaluated. After a quantitative analysis it can be stated that a schematic view of mathematics of kindergarten teachers prevailed and active and constructive learning of mathematics was highly agreed upon. The answers of the open question about learning goals revealed a broad range. With the help of the results, consequences for pre-service and in-service kindergarten teacher education are shown.

Key words: early years, kindergarten teachers, attitudes, competences, kindergarten teacher education

INTRODUCTION AND BACKGROUND

The interest in mathematics learning and education for the early years has increased immensely in the last years. A few years ago, mathematics did not play an official role in German kindergartens. Learning mathematics was reserved for school. Kindergarten teachers were not confronted during their pre-service education with mathematics education. Recently, different educational policy documents have begun to include references to mathematics learning. But the curricula of the single federal states of Germany differ in the explicitness of the statements made concerning mathematics. It ranges from very in-depth descriptions of mathematical content to be used in kindergartens, to others, where mathematics does not play an important role. In most of the curricula, there are very vague statements about learning goals. Therefore it depends heavily on the knowledge, attitudes, values and emotions of the people who are working in the kindergarten if and how they do mathematics together with the children. The kindergarten teachers play an important role because they create and influence the contexts for learning mathematics in kindergarten. “They are the architects of the environment, the guides and mentors for the explorations, the model reasoners and communicators and the on-the-spot evaluators of children’s performances” (Greenes 2004, p. 46).

¹ In Germany the pre-school institution is called kindergarten (for children from year 3 to 6).

² In German language the expression teacher is not used for people working in kindergarten, they are called educator. For this article I use the expression kindergarten teacher according to the English expression nursery teacher.

Results of the research of belief domain confirm that beliefs are behind teachers' behaviour in their classroom and act as a filter to indications of curriculum (Leder, Pehkonen & Toerner 2002). We can see this in the description of beliefs of Furinghetti and Pehkonen (2000, p.8): "Beliefs form a background system regulating our perception, thinking and actions; and therefore, beliefs act as indicators for teaching and learning". Skott (2001) also describes the consistency between beliefs and practice. Ngan Ng, Lopez-Real & Rao (2003) revealed in their study the strong influence of beliefs especially for kindergarten teachers. They noticed that there were more consistencies between beliefs and practices in kindergarten teachers compared with primary grade teachers. The big influence of prior knowledge, attitudes, emotions and individuals' understanding is also emphasized by the representatives of the cognitive-constructivist psychology of learning (Seel 2003) and the neurobiology (Roth 1997).

The construct "belief" consists of different components. One component is the view of mathematics. Mathematics as a science has different dimensions. According to Grigutsch, Raatz & Toerner (1998), there are four different aspects. Grigutsch et al. conducted an empirical study with over 300 math teachers and validated four aspects through different statistical tests: formalism, scheme, application and process. The aspect of formalism characterizes mathematics strictly by logical and precise thinking in exactly defined subject terminology with exact reasoning. Mathematics as a collection of calculation acts and -rules, which precisely indicates how to solve problems, describes the aspect of scheme. The aspect of application describes that mathematics has a practical use or a direct application. Mathematics also can be seen as problem-related process of discovery and understanding. Freudenthal (1982) describes the aspect of process very clearly, by defining mathematics as human activity in contrast to ready-made mathematics.

Next to the different aspects of mathematics, the belief about how mathematics should be learned and taught influences our exposure to children and to mathematics. Here, two contrasting positions can be described: "The assumption that the goal of mathematics instruction is to transmit knowledge to students and the view that students construct mathematical knowledge by active reorganizing their cognitive structures" (Cobb 1988, p. 87). The constructivist view of learning is generally accepted in mathematics education. Many research reports and even official documents represent a view of children who actively construct mathematics.

In conclusion it is obvious that the emotions and conceptions of kindergarten teachers about mathematics and mathematics education are important factors which influence their actual practice of doing mathematics in kindergarten. It is important to know some aspects of their conceptions and emotions related to mathematics education when discussing basic and advanced training of kindergarten teachers.

DESIGN

A questionnaire survey was conducted in the Karlsruhe area³ with 589 kindergarten teachers (Benz 2008) in order to evaluate the conceptions of kindergarten teachers. With the questionnaire it was examined, which attitudes, experiences and prior knowledge kindergarten teachers have concerning “mathematics” and “mathematics education”.

At the beginning of the year 2007, 550 questionnaires were distributed in kindergartens, of which 281 were returned. Moreover, 308 prospective kindergarten teachers of 2 vocational schools were surveyed. Of the 589 respondents, 554 were female and 35 were male. None of the kindergarten teachers that were working in a kindergarten at the time of the survey had had “mathematics in kindergarten” as part of their vocational education. Only the prospective kindergarten teachers who started to work after 2008, dealt with the topic of “mathematics in kindergarten” during their education to be a kindergarten teacher. The gradual changes in the education policy led to changes of the curricula.

The single items of the questionnaire were differently constructed. In the first part, the kindergarten teachers could express their feelings towards mathematics in multiple answers. In later questions, they could give their agreement to single statements on “mathematics”, “learning of mathematics” and “mathematics in kindergarten” with the help of a rating scale from 1 (does not apply at all) to 4 (applies completely). Which competences children should gain in kindergarten was asked in “open questions”. “Open” questions were used in order not to restrict or influence the answers too much.

RESULTS

Feelings about mathematics are better than their reputation

In the questionnaire, four adjectives were given, that could be seen as emotionally neutral (*useful, important, abstract, useless*). Four emotional positive items (*challenging, interesting, clearly understandable, fascinating*) and four negative adjectives concerning emotions (*confusing, frightening, boring, incomprehensible*) were listed too. Table 1 set out the results from the questionnaires.

useful	63%	confusing	35%	frightening	15%
important	59%	incomprehensible	24%	clearly understandable	9%
challenging	52%	abstract	21%	boring	7%
interesting	40%	fascinating	19%	useless	3%

Table 1: Feelings towards mathematics in percentages

³ There are kindergarten teachers working in the city of Karlsruhe (280 000 inhabitants) and also kindergarten teachers who are working in suburbs and villages around Karlsruhe.

Adjectives that could be described as neutral feelings with a positive value judgement, like useful and important, were chosen more frequently than any other terms. This is in contrast to the often cited public bad images of mathematics. The next most frequently chosen words were *challenging* and *interesting*. This concerns adjectives, which could be linked to positive feelings. Then follow two negative feelings like *incomprehensible* and *confusing*. *Incomprehensible* expresses that mathematics cannot be understood at all, while *confusing* can relate to a part of mathematics. This could be the reason why *confusing* was chosen more often than *incomprehensible*.

Thus, it must be noted that, concerning mathematics, positive emotions are more often predominant than negative emotions. Still, it is not to underestimate that one third of all kindergarten teachers regard mathematics as confusing.

Schematic view of mathematics prevails

The kindergarten teachers got a variety of statements where they could show their agreement in a multilevel rating scale from 1 (does not apply at all) to 4 (applies completely) in order to see which aspect prevails. In each case, 5 answers could be related to the aspect of scheme and formalism (e.g. mathematics demands formal accuracy), the aspect of process (e.g. solving problems is a main part of mathematics) and the aspect of application (e.g. mathematics trains abilities that are useful in everyday life). In order not to confront the kindergarten teachers with too many items the aspect of scheme and the aspect of formalism were jointed together. Grigutsch et al. (1998, 45) pointed out a very strong correlation between these two factors: “The formalism and scheme aspects positively correlate with one another and represent both aspects of a static view of mathematics as a system. They stand in opposition to the dynamical view of mathematics as a process”⁴.

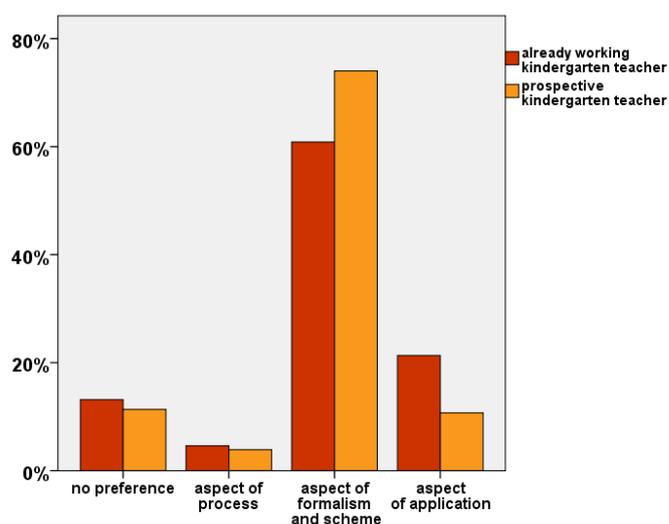
The mean values of every aspect for every person were calculated⁵. Then it was looked on which aspect the kindergarten teachers preferred. The results can be seen in Figure 1. 68% of all kindergarten teachers, agreed mostly to statements of the aspect of scheme and formalism. 16% agreed mostly to the aspect of application and only 4% agreed mainly to the aspect of process. For the remaining 12%, one prevailing aspect could not be determined.

Currently employed kindergarten teachers responded differently to these questions than did pre-service teachers. The pre-service kindergarten teachers were more likely to choose the aspect of scheme and formalism. Kindergarten teachers who are currently employed are more are more likely to choose the aspect of application.

⁴ The new categories were verified through a factor analysis. 44% of the common variance can be explained with these three factors. Cronbach's alpha for the aspect of formalism and scheme is 0.58, for the aspect of process 0.60 and for the aspect of application is 0.74. For every factor there is a very significant intercorrelation between each of the items of the factor.

⁵ The mean value for all kindergarten teachers for the aspect of process is 2.5; for the aspect of application it is 2.7; and the aspect of formalism and scheme it is 3.2.

The low part of kindergarten teachers choosing statements of the aspect of process is probably due to their own experiences in school. Mathematics was not experienced as a lively science, in which problem solving, creating of own solution strategies and personal ideas was common. Grigutsch et al. (1998) show the opposite tendency. They noticed in their study that the aspect which math teachers agreed mostly was the aspect of process. The aspect of application was also highly agreed upon whereas the aspect of scheme and the aspect of formalism was least agreed upon.



Prevailing aspects of mathematics in %		
	Already working N= 281	Prospective teacher N= 308
no preference	13.2	11.4
process	4.6	3.9
formalism scheme	60.9	74.0
application	21.4	10.7

Figure 1: Prevailing aspects of mathematics

Active and constructive learning of mathematics gets high agreement

After the statements of different views about mathematics, the respondents were confronted with statements concerning the acquisition of mathematical knowledge. Thereby, five statements had related to transmission, for example: “mathematics is best learnt when model solutions are demonstrated“ and five statements related to constructivist learning theory, such as “children should discover new knowledge on their own, I just give the hints”⁶. The answers concerning more a view of transmission had a mean value of 2.8. Statements that are based more on constructive learning theories achieved a mean value of 3.3.

As before, after calculating the mean value, the answers of the kindergarten teachers were sorted according to the prevailing aspect. The results can be seen in Figure 2.

⁶ The categories were verified through a factor analysis. The scree test showed an extraction of two factors. 41% of the common variance can be explained with these two factors. Cronbach's alpha for the aspect of transmission is 0.57 and for the constructivist aspect it is 0.76. For every factor there is a highly significant intercorrelation between each of the items of the factor.

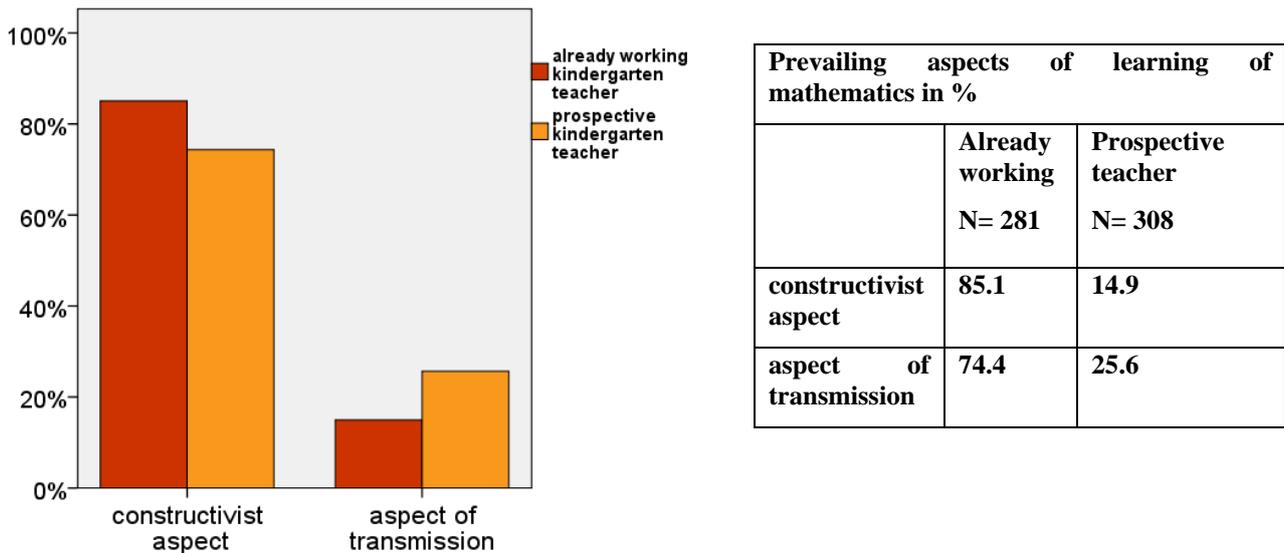


Figure 2: Prevailing aspects concerning the acquisition of mathematical knowledge

In doing so, it becomes clear that the kindergarten teachers, which are already working, set a higher value on constructivist aspects and less value to the aspect of transmission. Looking on the mean value of single items the tendency can be demonstrated too. Kindergarten teachers ($M=3.32$; $SD=.75$) already agreed more to the constructive statement “mathematical tasks can be solved in different ways” than prospective kindergarten teachers ($M=2.98$; $SD=.86$).

A constructivist conception of learning includes a certain awareness of mistakes: Mistakes are thereby an essential part of the way of learning and a normal aspect of the exploring learning process. They are not a blemish that should be deleted. Only a person, who learns, makes mistakes. The person, who does not make mistakes any longer, has stopped learning. In order to know what kindergarten teachers think about mistakes, there were two items concerning mistakes. The quite low mean values of 2.5 (“The most important thing is to achieve correct results” see figure 3 left) and 2.3 (“avoiding mistakes is important“ see figure 3 right) of the negatively formulated items show a positive attitude towards mistakes.

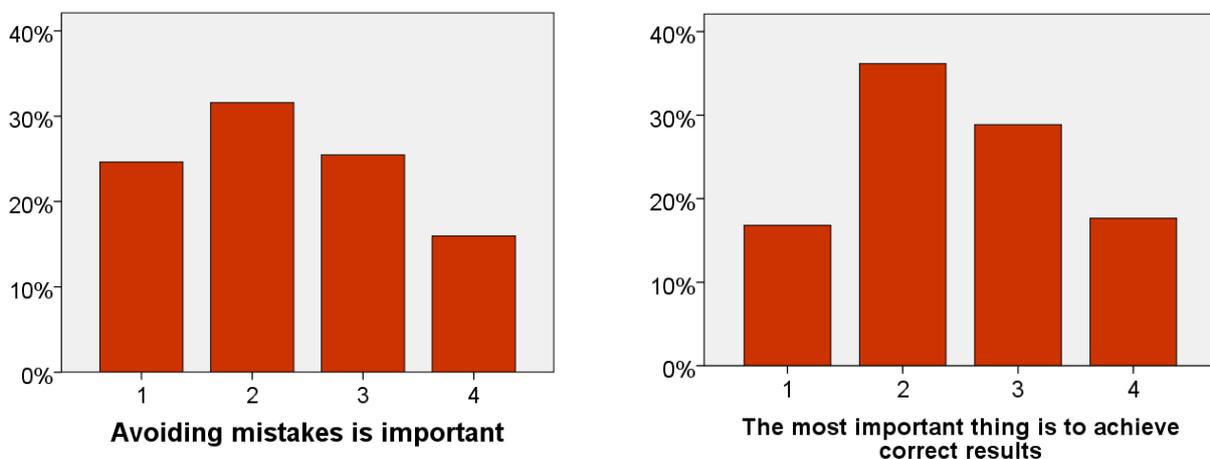


Figure 3: Attitude towards mistakes

But more than 25% of the kindergarten teachers chose “3” of the rating scale and 15% chose the top agreement “4” for both statements. So many kindergarten teachers think that errors should be avoided. This shows that a positive attitude concerning mistakes is not yet completely prevailing for all kindergarten teachers.

Broad spectrum of desired competences

As already mentioned in the introduction, there are not many concrete learning goals with respect to content in many curricula, which children should have acquired at the end of their kindergarten time.

There was an open question about what kindergarten teachers believed that children should learn. The answers were summarized in the following categories. The frequency of statements to each category is illustrated in Table 2 (Percentage of the kindergarten teachers making a statement to the respective content).⁷

counting	48%	reading or writing of numbers	29%
sets	38%	geometry (building, shapes, patterns)	26%
calculating	36%	measures (length, weights, time, volume)	17%

Table 2: Expected competences

The range of content was very broad. Very few kindergarten teachers noticed “nothing” or “mathematics should be learned at school and not in kindergarten”. But most of the kindergarten teachers wrote some competencies. Many content topics from primary school mathematics were mentioned. Counting as well as handling of sets was brought up most often. According to the kindergarten teachers, the children should also already learn simple arithmetic problems, often with the additional comment “embedded in situations” or “with objects”. Mathematical competencies concerning measures were rarely mentioned. This is astonishing, because the reference to everyday activities is very obvious concerning measures.

It makes one thoughtful when reading some statements about very high expected competences of the children such as “conceptual knowledge up to 100”, “numbers up to 100”, “counting up to 100”, “all basic operations like addition, subtraction, division and multiplication”, “multiplication tables”.

CONCLUSION

Due to the illustrated tendencies, the following components seem to be meaningful and essential for a pre-service and in-service teacher education in the area of preschool mathematical education:

⁷ One kindergarten teacher wrote: “Numbers are actually not bad, so children should learn numbers in kindergarten”.

Focus on the aspect of process with regard to mathematics

Because most of the kindergarten teachers preferred the schematic view of mathematics, it is important that mathematical components should be included in Kindergarten teachers' education. Kindergarten teachers should have the possibility to make their own mathematical experiences and thus experiencing the aspect of process and problem-solving of mathematics. Similar to an important goal of elementary teacher education, the important goal of mathematical components in and for preschool teacher education is to:

contribute to breaking a vicious circle. Many (prospective) teachers do not feel confident with mathematics due to their own prior negative learning experiences. Thus, they are likely to perpetuate their limited understanding to their own students. In this context, (prospective) teachers' encounters with mathematics play a crucial role, as they offer opportunities to encourage them to develop a lively relation to the activity of *doing mathematics*. (Selter, 2001, p.198)

Focus on active construction of knowledge with the consequence for doing mathematics with children

Although there was a high agreement to statements which can be referred to a constructivist view of learning, there were quite a lot of mostly prospective kindergarten teachers who showed a higher agreement to statements according to the aspect of transmission. So another important aspect for the basic and advanced kindergarten teacher education are the fundamentals of the cognitive-constructivist learning theory like e.g. the active meaningful construction of the knowledge. It is also important to concretise this with the help of learning environments to provoke children's curiosity and to enable individual exploration. Thereby, an important aspect is the role of the kindergarten teacher as a learning companion, who is able to inspire and support the children's own constructions. In addition to providing learning environments, it is also important that kindergarten teachers can use children's daily experience. Everyday situations can provide rich mathematical experiences quite often. Therefore, kindergarten teachers should develop a view for opportunities of learning mathematics in order to see this in everyday kindergarten activities.

Valuing children's own construction

When children construct their own knowledge, not standardised generalisations and analogies are included. They occur as spontaneous systematic errors. A child which construct the counting sequence, twenty-seven, twenty-eight, twenty-nine, twenty-ten do overextend the pattern it has noticed (e.g. the twenties are formed by combining the term twenty with each number in the single-digit counting series one, two, three ...nine, Baroody & Wilkins 2004). As already stated 25% of the kindergarten teachers chose "3" of the rating scale and 15% chose the top agreement "4" for the statements "it is most important to achieve a correct result or "it is important to avoid

mistakes”. Therefore it is important that learning mathematics take place in an environment where errors do not have to be avoided. So the valuing of child’s own constructions and patterns they have explored is one basic component of pre-service and in-service kindergarten teacher education.

Focus on content regarding learning goals

As could be seen in the open question, the range of learning goals was very broad. Many content topics from primary school mathematics were mentioned, even As Steinweg (2008) mentions, it is essential, to talk about helpful basic competences that help the children in the transition from kindergarten to school. Concerning these basic competences, it is important to keep in mind that the learning goals from school should not transferred into the kindergarten and thus pressurising kindergarten teachers and children. Therefore learning goals should be one aspect of the discussion of mathematics education in the early years.

In summary, the important aim of the early learning of mathematics is that children have the possibility to playfully explore mathematics as a lively science. It is the challenge of people involved in mathematics education to provide opportunities for all kindergarten teachers so that they can explore and develop to be learning companions who are creative, curious and imaginative.

In addition to consequences for pre-service and in-service kindergarten teacher education, the research results point out that further research is needed. One aspect to focus on is the first sight minor differences between prospective kindergarten teachers and kindergarten teachers who have practical experiences already. Another question is to investigate the actual practice of doing mathematics in kindergarten. Furthermore it is interesting if at all and how a kindergarten teacher education that focuses on the mentioned components influences the practice.

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LEARNING MATHEMATICS WITHIN FAMILY DISCOURSES

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In our research, we are concerned with early mathematical learning processes embedded in family discourses. Thereby, the focus is on interactional patterns which shape the mathematical experiences of preschoolers. What kind of mathematical discourse do preschoolers become familiar with? And what conceptions of mathematics arise from such everyday discourses?

In this paper, the centre of attention is the research design of a study in progress. Thus, we present our theoretical framework and underlying methodological considerations. Additionally, we complete this article with some data from preliminary studies in order to illustrate our approach.

Keywords: home mathematics, support structures, enculturation, acculturation

INTRODUCTION

In mathematics education research, the understanding of mathematics as a human product, which cannot be separated from its cultural context, is more and more prevalent. Regarding this culturality of mathematics, two complementary views of learning mathematics can be recognised. On the one hand, learning mathematics means that one becomes a part of the mathematical culture which permeates one's social environment (Bishop, 1988). On the other hand, mathematical learning processes are also an intended acquirement of an apparently unchangeable faculty culture with its specific set of terms, structures and principles (Prediger, 2003). In our opinion, these two descriptions supplement each other and correspond with the fundamental distinction between enculturation and acculturation (Bishop, 1988 & 2002; Frade & Faira, 2008). In both conceptions, mathematical learning is embedded in discursive processes between one generation and the next.

Against this background, we are interested in early mathematical learning processes. Toddlers and preschoolers already make varied experience with mathematics in different social activities. Thereby, discourses with their parents are of prime importance. Thus, our main research question is: What kind of mathematical discourse from the familial context is familiar to the child entering school? We want to pursue this question in an empirical and qualitatively laid out study, which is in line with the interactionistic research paradigm (Cobb & Bauersfeld, 1995).

In the following pages, we shed light on the picture of mathematics as a cultural property and clarify the implications for our conception of learning mathematics. Subsequently, the methodological approach derived from this framework will be presented and, finally, be illustrated by data from our preliminary studies.

THEORETICAL FRAMEWORK

In looking back at children's experiences with mathematics, we necessarily do so with a certain preconception of mathematics. „Mathematics is an intellectual instrument created by the human species to describe the real world and to help in solving the problems posed in everyday life.” (D'Ambrosio, 2001, p. 67) For our theoretical framework, we adopt this idea from the research in ethnomathematics: mathematics is no entity existing outside human experience, but a human product (Prediger, 2003; Street, Baker & Tomlin, 2005).

This assumption about the nature of mathematics affects our conception of learning mathematics. Thus, children do not encounter mathematics itself, but a cultural practice that is recognized as mathematical by capable members of the belonging culture (Sfard, 2002). For this reason, not only is mathematics a social construction, but learning mathematics is as well. Therefore, Bishop demanded as early as 1988: “[...] a mathematical education must have at its core the assumption of being a social process.” (Bishop, 1988, p. 13) Consequently, learning mathematics means that a child participates in a practice to an increasing degree. This idea of learning is explicitly exhibited in Sfard's theoretical work. She defines learning mathematics as “becoming fluent in a discourse that would be recognized as mathematical by expert interlocutors.” (Sfard, 2002, p. 5) Pursuant to this latter definition, adults are of prime importance for the child's development due to the fact that they can spur mathematical discourses.

In line with this approach to mathematical learning, we focus on the emergence of mutual understanding and coordination in discourses between a child and an adult as expert interlocutor in a certain degree.

Home Mathematics

With regard to early mathematics and its conjunction with school mathematics, van Oers states: “In fact, students are from the beginning of their life a member of a community that extensively employs embodiments of mathematical knowledge. The school focuses attention on these embodiments and their underlying insights, and by so doing draws young children into a new world of understanding.” (van Oers, 2001, p. 59) Subsequent to this claim, we focus in our research project on the type of constitution of these “embodiments of mathematical knowledge” emerging in the familial environment of preschoolers. According to our theoretical fundamentals presented above, we assume that the individual conditions under which the children enter the “new world of understanding” are fundamentally different according to their cultural experiences at home.

For children, family is a place of experience beside others such as the nursery school or peer groups. In spite of being just one component of the child's life-world, family has an extraordinary relevance, with its own values, rules and practices.

With regard to our research focus “learning mathematics within family discourses”, we refer to Bishop's differentiation between enculturation and acculturation (1988 &

2002). These two conceptions contain two different perceptions of learning mathematics. In the first one, learning mathematics is seen as the induction, by the cultural group, of young people into their culture (Bishop, 1988). Pursuant to this point of view, mathematics is a natural part of the everyday life that is shared with the young. By contrast, Bishop (2002) delineates learning mathematics as a process of acculturation. Following Walcott, he defines acculturation as a “modification of one culture through continuous contact with another” (Bishop, 2002, p. 193f.). So, in this case, mathematics is regarded as a separate culture which is, for a start, disconnected from children’s everyday life. With regard to our field of observation, we don’t commit ourselves to one of Bishop’s opposed conceptions. In fact, we like to identify the degree to which home mathematics learning can be thought as an enculturative or acculturative experience (Fade & Faria, 2008).

Furthermore, mathematical discourses practiced at home are of particular importance not only because they carry certain pictures of mathematics, but because they familiarize children with particular interactional patterns (Street et al., 2005). An empirical study conducted by Street et al. (2005) shows that children’s experiences of these discourses are dramatically different. In terms of mathematical discourses at home and at school, the researchers explain that, for some children, there is a gulf between these contexts: “The school replicates the Primary Discourse of middle class homes whilst it presents children from other backgrounds with a Secondary Discourse.” (Street et al., 2005, p. 7) At this point, we can clearly see the connection between early mathematics, discourse practices at home and their relation to mathematics education. According to the study just cited, many children are restricted in their prospects to succeed in mathematics education because they are confronted with a problem of language: the switch between home and school discourses can be a source of difficulty because of different values, rules and patterns. In line with those conclusions, but without relating her research to classes, Sfard exposes interactional patterns that are especially similar to school discourses. “This structural similarity can be seen mainly in the type of questions presented to the children, in the parent’s fine-tuned scaffolding actions, and in their tendency for repeating one kind of tasks several times, until the children show evidence of some mastery.” (Sfard, 2005, p. 249; see also Street et al., 2005).

Support Structures

This view on early learning processes is related to our idea of support structures in child-parent-discourses and to the general discussion about the decisive role of adults for children’s development (Vygotsky 1978, Bruner 1983, Rogoff 1989). Vygotsky delineates learning as a process in which children internalize skilled approaches from their participation in joint activities with more skilled partners. These joint activities that would be impossible for the child on its own define the so-called “zone of proximal development” (Vygotsky, 1978). With this theory of development Vygotsky realizes the integration of individual learning in social and cultural context. In another manner, Bruner (1983) does the same. He conceptualises learning with regard to a

support system provided by capable interlocutors. The child is induced in a certain “format”, which contains the idea of increasing autonomy and responsibility for the child. An advancement of these two theories was introduced by Rogoff (1989). With regard to Bruner, she pushes the interactional equality of adults and children closer to the spotlight: “The mutual roles played by children and their caregivers rely both upon the interest of caregivers in fostering mature roles and skills and on children’s own eagerness to participate in adult activities and to push their development.” (Rogoff, 1989, p. 209) According to this basic assumption, she describes the learning process as a “guided participation”. Thereby, she replaces Vygotsky’s idea of internalization by that of “appropriation”. In the process of appropriation, the children “can carry over to future occasions their earlier participation in social activity.” (Rogoff, 1989, p. 213) In other words, in her opinion, learning is a process of transformation of individual participation in cultural activities. Because of this analogy to interactionistic fundamentals, we regard the concept of guided participation as especially valuable for our theoretical framework. What kind of guided participation shapes the child’s early mathematical experiences? And, in more detail, what picture of mathematics do young children become familiar with?

Pursuing these key questions, we plan to explore the different forms of guided participation in German families between the two poles of enculturation and acculturation.

METHODOLOGY

Our main focus is on everyday mathematical discourses between preschoolers and their parents. In order to achieve a well-rounded picture of early mathematical learning processes in families, we plan to collect different types of data, which will be related to each other via the help of data triangulation. Hence, we will collect basic data of the family (age, siblings, educational background, etc.), data of interaction and data from parent interviews. This need not mean that we use the diversity in data to mutually check their validation, but rather to shed light on the subject matter – namely processes of enculturation or acculturation within the family – and, as such, gain a more multi-faceted than inherently consistent image. We lay out our study as a comparative set of case studies, which means that we will collect data in several families and, after analysing them case by case, we will compare different families on the one hand and insights from different kinds of data on the other.

In the following, we will describe the main data types - “interaction processes” and “guideline interview” - and illustrate them with examples from our preliminary studies.

Interaction processes

To get access to interaction processes which are of interest within the scope of our research project, we have chosen two impulses which we consider as more or less typical for the familial context: picture-books and games. Therefore, we would like to ask a child of preschool age and its parent in each case to take a look at a picture

book, or to play a game together. These situations will then be videotaped for later analysis.

The reason we regard picture books and games as adequate for initiating mathematical discourses is because of their value in the child's everyday life: „The underlying thought of using picture books for mathematics education is that they can offer a meaningful context for learning mathematics and can offer a ‘cognitive framework’ with ‘cognitive hooks’ to explore mathematical concepts and skills. Picture books are also ascribed an important role for the development of mathematical language.” (Heuvel-Panhuizen, Boogaard, Scherer, 2007, p. 831) In our opinion, games can be of similar relevance for learning mathematics.

In order to initiate mathematical discourses, we chose picture books and games that offer varied mathematical contents. In addition, we will invite the participating families to present a book or game they are familiar with. In each case, the participants may choose the place as well as the book or game and, finally, stop reading or playing whenever they wish to. Thereby, we assume that everyday practices and discourse structures emerge even in contact with potentially strange material. Analysing such discourse structures referring to mathematical learning processes, we focus on emerging support structures.

In order to identify support structures in these initiated discourses, we will conduct an analysis of interaction which refers to the interactional theory of learning (Cobb & Bauersfeld, 1995). This method was devised by a working group around Bauersfeld, in reference to ethnomethodological conversation analysis. Focusing on the evolvment of the topic(s) and patterns of interaction, this analysis serves as a foundation. Thus, an analysis of participation follows which focuses on the issues of “responsibility and originality that one can ascribe to a person's utterance” (Krummheuer, 2007, p. 67; Brandt, 2007).

Interview

These interactional situations are to be complemented by semi-structured interviews taken with each parent at the beginning of the study, thus, nearly a year before the start of school, and also at the end, a few weeks after the child's first day at school.

The interviews are based on problem-centered guidelines (Patton, 2002; Witzel, 2000). The first interview is to shed light on the parents' ideas of mathematics, of mathematical and general learning processes, the families' practices concerning books and games and the preparation for the forthcoming school start. In the final interview, however, different priorities are set. So, the focus is rather on the experiences made with our materials during the preceding months, on the potential impact that the study has on the family's everyday life, and on the experience with school start.

In line with the conception of the problem-centered interview, the respondents are always considered as “experts of their orientations and actions” (Witzel, 2000). For this reason, the interview guidelines just serve as a basic checklist during the

interview to make sure that all relevant topics are covered. In fact, the most important point is that the interview situation provides “a framework in which respondents can express *their own* understandings in their own terms” (Patton, 2002).

In order to find the basic ideas outlined by the parent, we will conduct the qualitative content analysis devised by Mayring (Mayring, 2000). We will use this generally accepted method in a certain form which includes two central steps: “inductive category development and deductive category application” (Mayring, 2000, p. 3). The scope for the category development will be the distinction between mathematics as a social practice in everyday life and as a fixed faculty culture and in this sense learning mathematics as enculturation or acculturation.

EXAMPLES FROM PRELIMINARY STUDIES

In order to illustrate our research design, we will present examples of the main data types and first conclusions in the following.

Example 1: Florian – mathematical discourse

This first episode is extracted from a reading session with Mrs. Gerlach, her 5-year-old son Florian and her 2-year-old daughter Loni [1]. They look at the picture book “365 Pinguine” [2].

- Mrs. G. Every morning, a new penguin arrives. How many are there?
 Florian Hum.
 Loni Two!
 Mrs. G. 31 plus 28 equal?
 Florian Hum, I don't know.
 Loni (*citing the book*) Ring! Ring!
 Florian Oh.
 Mrs. G. That's rather difficult.
 Florian Yes, but it is... Well, 20 plus 30 equal, oh, 50. Then, plus 8 is 58. Yeah, it is 58.
 Mrs. G. You did it really well. However, you missed one.
 Florian 59.
 Mrs. G. Fif, and here is the solution (*points at the solution presented in the book*).

In this short sequence, a mathematical matter arises from reading. Entering into that question, Mrs. Gerlach doesn't push her son for an answer. By emphasising the intricacy of the problem at hand, she opens the situation for him. From now on, he can fail to answer the question without losing face. Against this background, Felix uses the opportunity to exhibit his mathematical capacity. He ventures to enter a

mathematical field with which he isn't familiar yet. Thereby, he decomposes the problem into two steps. The second step of calculation is not affirmed by Mrs. Gerlach. She refers to the solution presented in the book instead. Altogether, Felix is responsible for the solution process; in terms of the analysis of participation, he is the "author" which means that he expresses his own ideas in his own words (Krummheuer, 2007).

Example 2: Linus – mathematical discourse

This second episode is from a reading session with Mrs. Bultmann and her 5-year-old son Linus. They look at the picture book "Es fährt ein Boot nach Schangrila" [3].

Mrs. B. At pier 6, the woodpecker starts feeling sick. For this reason, five koalas immediately complain to the captain. Five bears, small and grey. Do you know where they are?

Linus (*tips a koala in the picture*)

Mrs. B. One. Point a finger at the koalas! Look here, one (*points the finger at another koala in the picture*). With the finger, Linus!

<Linus (*points at all the five koalas one after another*)

<Mrs. B. One, two, three, four, five – great!

In this episode, Mrs. Bultmann reads the text out at first. Subsequently, she sets a specific structure, asking Linus to find the koalas. Instead of answering verbally, he points at a koala in the picture. This nonverbal answer is marked as inadequate by Mrs. Bultmann. Thus, she gives the number word and asks Linus to point at the koalas, although he already did the latter. By this means, she specifies how to perform the fixed algorithm she demands: pointing and pronouncing the number words at the same time and step by step. In the following, she initiates the counting process once again, starting with another koala. Linus continues pointing at the koalas, whereas his mother pronounces the number words. Altogether, the mother insists on a specific structure, in which Linus' action is integrated; in terms of the analysis of participation, Linus is a „relayer“, which means that he "claims no responsibility neither for the syntactical nor for the semantic aspect of his statement" (Krummheuer, 2007, p. 67).

Example 3: Different ideas of mathematics - interview

In addition to the reading sessions, we interviewed all parents. Here are three answers to the question: What comes first to your mind when you hear the word mathematics?

Mrs. Gerlach: Hum, mathematics? Well, logic, structures. Hum... Hum, and everyday life as well, so, the relevance for the everyday life, thus, there are a lot of things which have to be calculated. So, it is of great importance on all levels and, it is, yes, I think, it is really important.

Mrs. Bultmann: When I think about math? Oh, my God... Everything with plus, I would say. So, spontaneously, I would think about everything with plus.

Mrs. Yoritomo: Mathematics, so, systematic thinking. And very useful. And, for me, with the piano, it is especially important, no, the foundation of course. It's really counting and playing at the same time. This is really of prime importance.

These three answers shed light on the diversity of views on mathematics. While both Mrs. Gerlach and Mrs. Yoritomo spontaneously emphasise rather abstract ideas of mathematics, Mrs. Bultmann names the concrete operation of addition – but as a strange idea without connection to her everyday life. Against the background of the complete interviews, this difference between the answers will be even more obvious. While Mrs. Gerlach and Mrs. Yoritomo regard the mathematical basic operations (like addition and subtraction) as part of their everyday lives, Mrs. Bultmann constricts useful mathematics to counting. Her larger distance from mathematical matters comes to the fore as well, when she describes situations in which her son encounters mathematics within the family's everyday life. In this regard, she speaks about proportionality, whereas her son just copes with counting up to ten in the reading situation. By contrast, Mrs. Gerlach's and Mrs. Yoritomo's examples concerning the same topic are more concrete. They report on kitchen activities, playing shops or games of dice, planning holidays or taking interest in mathematical basic operations. It is an astonishing notice that Marc, Mrs. Yoritomo's 4-year-old son, spontaneously names preparing jam as something with relation to counting. Quite afterwards, his mother explains this concrete kitchen experience and the embedded mathematical activities.

Summary and Conclusions

As a summary, we will relate the presented diversity in the parents' views on (home) mathematics and in forms of support structures to our basic idea of learning mathematic as enculturation or acculturation.

Firstly, the ideas of (home) mathematics, reported in the interview, shed light on different levels of familiarity with mathematics. For instance, Mrs. Bultmann regards even mathematical basic operations aside from counting processes as strange and disconnected from her everyday life. Consequently, her son may adopt this distance to mathematics, experiencing elementary calculations in an acculturation process. The other two families treat mathematical topics as more common and integrated in their everyday discourses. This is discernable in Marc's spontaneous insertion during the interview mentioned above and in the short interaction sequence with Mrs. Gerlach and her two children: Not only Florian's participation, but also Loni's reaction shows understanding of the problem at hand: Although "two" is a wrong answer regarding the number of penguins, the utterance is thematically adequate. In contrast to Linus, the children in these families become familiar with mathematical practices within an enculturation process.

These expositions can be supplemented by a deeper examination of the reading sessions. Within these sequences, different kinds of support structures emerge. More precisely, we can see the space given by the conception of “guided participation” (Rogoff, 1989). While one support structure focuses on the child’s involvement in a fixed practice, the other one emphasises the child’s role as a competent interlocutor who produces ideas on his own. We assume that, by these different kinds of participation, the children get different ideas of how to learn mathematics: adopting a fixed structure or probing a flexible tool according to individual ideas. On a more theoretical level, the first form conforms to an intended acquirement of an apparently unchangeable faculty culture, thus, to an acculturative experience. By contrast, the second form corresponds the conception of enculturation, which includes mathematics as a natural part of everyday life.

NOTES

1. Transcription rules: This font marks text read from the picture book. < marks persons speaking simultaneously.
2. “365 Penguins”. Fromental, J.-I. & Jolivet, J. (2008). *365 Pinguine*. Hamburg: Carlsen Verlag.
3. “A boat goes to Shangrila”. März, L. & Scholz, B. (2006). *Es fährt ein Boot nach Schangrila*. Stuttgart/Wien: Thienemann Verlag.

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ORCHESTRATION OF MATHEMATICAL ACTIVITIES IN THE KINDERGARTEN: THE ROLE OF QUESTIONS

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The aim of this study is to address the subtleties in the process of how kindergarten teachers orchestrate mathematical activities with a group of children. Drawing on a sociocultural perspective on learning and development, talk-in-interaction, emerging from naturally occurring data, has been analysed to get insight into how a kindergarten teacher orchestrate mathematical activities. The analyses show that the kindergarten teacher's use of questions, which we categorise into six groups, played a significant role in the orchestration of children's learning process. Through the use of questions and a pair of scales, verbal and non-verbal responses were engendered, relevant mathematical terminology was offered, and an inquiry approach towards measuring as a mathematical topic was initiated.

Keywords: kindergarten teacher, orchestration, teacher questions collaboration, inquiry

INTRODUCTION

During the recent years, mathematics in the kindergarten has been on the agenda with respect to the content of Norwegian kindergartens and their role in the society. In particular, this is emphasised in the curriculum for kindergarten (KD, 2006), where mathematics for the first time is explicitly mentioned as a topic with which children are supposed to be engaged. These societal demands of the kindergarten have put to the fore questions such as “What are we supposed to do with regard to mathematics in the kindergarten?” and “How do we do it?”.

A research project called Teaching Better Mathematics (TBM¹) has been initiated at the University of Agder. In this project, we are collaborating with several schools and kindergartens to promote learning and development in mathematics teaching. This paper reports from a case study situated within this project, analysing an activity in one kindergarten.

In this study, we use the notion of orchestration to describe a kindergarten teacher's actions when the children worked with measuring tasks. This includes an emphasis on the role of the kindergarten teacher's questions and comments to children's responses in the conversation. We also include the preparations made ahead of the sessions as being part of the orchestration, that is planned tasks, use of a pair of scales as well as the framing of the learning environment and number of children involved

¹ The TBM project is supported by the Research Council in Norway (NFR no. 176442/S20) and is managed by didacticians at UiA. The TBM project is based on collaboration between didacticians and teachers, kindergarten teachers and their leaders in two local councils and the local county where UiA is situated. The TBM project aims to promote development of mathematics teaching in schools and kindergartens, including participation in workshops arranged by didacticians at UiA, and research into these processes.

in the activity. Teachers' actions and arrangements during sessions are included in what Kennewell denotes as "supporting features" in teachers' orchestration:

The teacher's role is to orchestrate the supporting features – the visual cues, the prompts, the questions, the instructions, the demonstrations, the collaborations, the tools, the information sources available, and so forth... (Kennewell, 2001, p. 106).

From our collaboration with the kindergarten teacher, the following research question has been formulated: What roles do a kindergarten teacher's questions play in interaction with children when orchestrating mathematical activities?

THEORETICAL FRAMEWORK

In this study we adopt a sociocultural perspective on learning and development, that is we view learning as a social and situated process of appropriation where individuals make concepts, tools, and actions their own through collaborating and communicating with others (Rogoff, 1990, Säljö, 2005; Wertsch, 1998). In the process of appropriation, the role of tools is significant, in particular language in interaction with other psychological as well as physical tools (Vygotsky, 1978, 1986). The reason for adopting this theoretical position is our aim of describing and making sense of institutionalised interaction and learning activities among adults and children in the kindergarten. This perspective is useful for our emphasis on the orchestration of participation in social, mathematical activities. In adopting such a perspective when analysing our data, we aim at making sense of how adults and children are engaging in interaction by using verbal and non-verbal actions.

The experience the children do with measuring at various points and in different settings, altogether constitutes the basis from which the children are making shared meanings (Rogoff, 1990). By orchestrating a mathematical activity, the kindergarten teacher creates a learning environment for the children to engage and participate with ideas and arguments.

The theoretical stance of our study is in accordance with the TBM project's theoretical perspective in general (cf. Jaworski, 2007), where inquiry is a main theoretical notion. An intention from the didacticists' point of view in the project has been to study and promote development of mathematics teaching through inquiry (Jaworski, 2005; Wells, 1999). According to Wells (1999), inquiry is a process described as "a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them" (p. 121). The nature of the collaboration with respect to the inquiry process is in accordance with how Wagner (1997) describes a co-learning agreement:

In a co-learning agreement, researchers and practitioners are both participants in processes of education and systems of schooling. Both are engaged in action and reflection. By working together, each might learn something about the world of the other. Of equal importance, however, each may learn something more about his or her own world and its connections to institutions and schooling (p. 16).

We acknowledge that didacticians (researchers) and teachers (practitioners) bring different expertise and engage in inquiry together to inform and develop their different practices.

In the study we aim to consider how the kindergarten teacher's orchestration promotes inquiry in learning and teaching. This is done through an emphasis on how the kindergarten teacher and the children explore mathematics together. The questions posed by the kindergarten teacher and the actions resulting from those questions are the unit of analysis in this study.

Studies have documented that whole-class interaction often is dominated by teachers' questioning to control and support their teaching (Barnes, Britton, & Torbe, 1986; Kirby, 1996; Myhill & Dunkin, 2005). Although several of these studies report that teachers also want to support students' investigations and reflections, their use of factual questions, or what Kirby (1996) calls simple questions, inactivated the students. Kirby argues that the way children interpret a story is heavily dependent on the kind of questions used by teachers. Kirby focused on the amount of information contained in the questions, and he found that use of simple questions was dominating. The lack of more complex questions used by the teachers prevented the children to make sense of the story text.

We want to argue with Roth (1996), that questions per se are not "universally good but need to be evaluated in terms of their situational adequacy" (p. 710). In accordance with what Roth argues, we are not treating the kindergarten teacher questions alike and categorise them indistinguishably. We are interested in the role these questions play, with respect to context, content, and children responses, "in student-centered, open-inquiry learning environments" (op. cit., p. 710).

ANALYSIS AND RESULTS

In this study we have collected empirical material through the use of video camera as well as field notes from one kindergarten. Our data consisted of a video tape of 27 minutes which was transcribed in full. Naturally occurring talk-in-interaction has been captured on an occasion when a kindergarten teacher has been engaging in measuring activities together with several children. In this case, the kindergarten teacher called Unni orchestrated a mixed-aged group of children who were participating in a measuring activity through interaction and communication. They were engaging with a pair of scales to measure which were heavier of various things with different size and weight.

In the activity, Unni interacted with six children 3-4 years of age, two girls and four boys. In Figure 1, a picture from the activity is presented.



Figure 1: The children and the kindergarten teacher engaging in the activity

Unni was a well experienced kindergarten teacher, with a background of more than ten years from working in a kindergarten. The measuring activity orchestrated in this case had previously been introduced to the kindergarten teacher in a workshop at the university. The introduction to the activity was made by didacticians at the university, but only as an example of an activity that might be possible to orchestrate in a kindergarten. No explicit guidelines were given with respect to how to orchestrate the activity and it was the total enterprise of Unni the measuring activity observed.

Thematically, we divided the data material into two parts. In the first part, the orchestration and interaction are about the weight of a toy crocodile and a box including plastic bears of various sizes and weight. The comparison of weights between these was made by all children both when holding them in their hands and with the use of a pair of scales. The second part concerned comparing the weight of small plastic bears of different sizes and weight. The children were challenged by the Unni to reason about the weight of the largest bear in comparison with the smaller ones. Both these activities were tightly orchestrated by Unni.

In analysing the transcribed material, we observed over 150 questions asked by Unni (cf. Table 1 below). We do not find the exact number of questions significant. Rather, we found it interesting to register that the communication and interaction between the kindergarten teacher and the children were fundamentally oriented around those questions and the children's verbal and non-verbal responses to them. With this as a background, we were able to categorise the questions into six different kinds of questions, and we analysed what kind of responses the various types of questions initiated. Some categories of questions were dominating more than others and some categories initiated more responses from the children than others. We are aware that others have categorised teacher questions as well (cf. Barnes et al., 1986; Myhill &

Dunkin, 2005; Roth, 1996; Wood, 1988). Roth, for instance, developed a typology of questions asked by one teacher with respect to their content. However, this typology of questions does not immediately fit with the categories we have forwarded. We focus on the role the questions played in the communicative practice and not exclusively on their content. Thus, our categories are elaborated with respect to the children's responses (Roth, 1996).

Table 1: Frequency table of the six categories of questions

Suggesting action	30
Open	71
Asking for argument	12
Problem solving invitation	12
Re-phrasing	19
Concluding	10
Total	154

In the following we will give a description of the six categories of questions. We will continue our analysis by going deeper into the role the different categories of questions played in the kindergarten teacher's orchestration. We consider what kinds of responses we observed from students, both verbal and non-verbal, to questions in the different categories.

Suggesting action: Questions within this category are characterised by their feature of initiating physical actions among the children, and not solely as initiating an oral answer. Typical questions in this category were: "Stein, can you feel?", "But do you think that it will go up if we put more into that?", and "Can you count them, and see if it is as many as this?".

Open: Almost half of the questions were categorised as open. Questions within this category inquired into the children's knowing with respect to the problem they studied. For instance, "Do you think this one weighs the most?", "How can we decide which one of them are the heaviest?", and "What has happened now?".

Asking for argument: This category includes the questions asked which follow up on an utterance from a child. The content of these questions includes that the child is asked to give reason(s) for his or her answer or opinion. Examples of this kind of questions are: "Why do you think that?", "How can we know that they have the same weight?", and "Why wasn't it equal this time?".

Problem solving invitation: Some of the questions included a problem or a challenge. These questions initiated opportunities for reasoning as well as being motivating with regard to experimenting and solving the problem. For instance, Unni

challenged the children by asking questions such as: “Is it possible to estimate how many such bears we need for them to be as heavy as a large one?”, and somewhat later “If I put two large bears into this one (puts two large plastic bears in one of the scales), what do you have to do to make it even?”. These questions are different from Suggesting action questions in that the former do not suggest any concrete actions to do to solve the challenge or problem.

Re-phrasing: At several occasions Unni re-formulated the children’s utterances into coherent sentences and questions. Very often the children responded with single words or short utterances, which were re-phrased as questions by Unni. Firstly, the questions set forth a mode of wondering among the children. When one boy called Tore said “this is heaviest”, Unni responded with “Do you think that one is the heaviest?”. Secondly, in these questions Unni took the opportunity to introduce new concepts, for instance the concept of weighing. When a boy called Arild said “That is the largest, therefore it is the heaviest”, Unni responded with a confirmation and a new question: “That is largest, but which one weighs the most?”. This is coinciding with Roth (1996), that teachers elicit specific content knowledge through questions.

Concluding: This category is used to describe those questions where the kindergarten teacher promotes a mathematical relationship or observation. The aim of those questions seem to be the children’s approval or for them to acknowledge a specific issue. For instance, in the following question Unni argues for adding more plastic bears in one of the scales: “That has to be heavier so that it can come further down, doesn’t it?”. Moreover, later she makes the point that “And then they have the same weight?”. The conclusions are given in the questions, but she wants the children to reason and conclude for themselves.

In the initial phase of working with the measuring tasks, Unni often asked *suggesting action* questions. In these questions, the children were asked to do actions with the pair of scales. In approximately all cases, such questions were followed by physical actions by the children instead of verbal responses. It is worth mentioning here, that it is possible to doubt if the questions are genuine questions (cf. Roth, 1996) or if they are invitations to what the kindergarten teacher Unni wants the children to do. However, those questions signal to the children that it is up to them to decide whether to do something or not.

In her orchestration, Unni’s use of these questions typically was followed by posing *open* questions. We observed that the *open* questions created attention to the practical activities that the children were involved in. For instance, when Unni asked “What happened now?”, the purpose with the question was probably to focus the children’s attention on the measurement activity. At several occasions, the *open* questions also served as a follow-up on questions from other categories. It seems as if the *open* questions were necessary to (a) keep their conversation going, (b) to engage and motivate the various children in their problem-solving efforts, and (c) to make them having a shared focus of attention.

The *open* questions challenged the children to respond verbally. Typically the open questions resulted in short replies such as “yes” or “no. Unni often continued with *re-phrasing* questions or *asking for argument* questions. By doing that, Unni seemed to have further initiated verbal responses from the children.

The *re-phrasing* questions were tools for adjusting the children’s use of mathematical language. Unni never explicitly corrected them, but through her re-phrasing, she emphasised the preferable terms to use. This issue is exemplified when Unni rephrased Arild’s utterance “And now they are equal of size” into “Are they equally heavy?”.

Re-phrasing questions were responded to by the children with affirmative replies such as “yes” or with comments such as “that” and pointing with fingers if they were asked to decide which of two things were heavier. In order to challenge students more verbally, Unni continued with *asking for argument* questions or by way of new *open* questions. When students responded successfully to *asking for argument* questions, it often led to *concluding* questions. If students did not succeed replying verbally to the *asking for argument* questions, Unni usually continued with some *open* questions, but also sometimes with *suggesting action* questions in her orchestration. To use those kinds of questions seemed not to have been a preferable choice by Unni, but questions she utilised when students did not manage to succeed with their argumentation.

We have already emphasised that the session we observed consisted of two parts. In the second part the children worked with the plastic bears and Unni started to use *problem solving invitation* questions. These questions usually invited the children to propose actions or to accomplish actions. Unni then followed up with *open* questions or *asking for arguments* questions which challenged the children verbally. Occasionally, she also used *suggesting action* questions to follow up the problem solving invitation questions. When a new sequence was initiated by a problem solving invitation question, the conversation usually fell into a similar sequence of questions as discussed above.

The *concluding* questions often occurred as a result of a previous discussion of a phenomenon. These questions occurred in three different settings. In one setting the questions concerned what they observed, such as “And when the scale is down, it is heaviest?”. In a second setting the questions concerned what the children were supposed to do. The questions included suggestions to actions, but the suggestions were assumed by Unni to be the correct thing to do. The question “Should we remove one from this scale too?” is an example of this setting. The third setting concerned mathematical conclusions. Questions used within this setting we interpret as being an important step in the kindergarten teacher’s efforts to facilitate the children’s process of appropriation. The question “So, if we take out two of the same size, we will restore balance again, if we take one from each?” exemplifies her effort to achieve a shared focus of attention among the children with respect to a certain mathematical

relationship. After different questions have been posed and responded to, the concluding questions may help the children to achieve a shared meaning for various terms and actions.

DISCUSSION

As argued above, the children's actions and utterances are divided into verbal and non-verbal responses. Concerning the children's responses to the questions, only a few questions resulted in inadequate response or no response from the children. Most often, they were able to give relevant verbal responses or they responded with pointing gestures or actions with respect to the given artefacts in order to answer the kindergarten teacher's questions.

The verbal responses were often supported by different types of gesturing. The children did rarely answer questions with complete sentences. This is, however, not surprising, thinking of their age (3-4 years). This observation might also be explained by studying the way the kindergarten teacher posed the questions. Many of the questions were formulated in ways that initiated short responses. On the other hand, when the kindergarten teacher used questions that from our perspective initiated more elaborated responses, the children still gave short responses.

Since the questions were so closely linked to the practical activity, the children were able to respond to several questions in a non-verbal way. They answered lot of questions by pointing, shaking their heads or by moving the artefacts. For instance, in working with balancing the scales, the kindergarten teacher asked about how they could lift one of the scales so that they restore balance. In stead of verbally answer the question, Kari put a brick in the highest scale. Occasionally the children also combined verbal and non-verbal responses. This observation, we argue, signifies the importance of including physical artefacts as tools in orchestrating mathematical activities.

The complexity in the interaction is illustrated in the kindergarten teacher's use of different categories of questions, and we observed a sequence in her use of these categories. Such a sequence typically was initiated by using a *suggesting action* question (occasionally *problem solving invitation* question). Then she continued with an *open* question, followed by either an *asking for argument* question or a *re-phrasing* question. The sequence ended with one or several *concluding* questions. This finding that the kindergarten teacher has an aim for the activity which was supposedly reached by her sequencing of questions coincides with Roth (1996). He also found that the teacher controlled the communicative practice among her students, not through a classical IRE² sequence, but by means of a sequence of queries.

² IRE is an abbreviation of a communicative pattern found in traditional classrooms: The teacher takes Initiative, the students give Response, and the teacher Evaluates the response

We argue that the kindergarten teacher played a significant role in the children's learning process. Kirby (1996) claims that lack of complex questions prevented the children to make sense of mathematical ideas. However, we believe that the kindergarten teacher, in her orchestration tied the mathematical ideas together through her frequent use of questions, in a way that made it possible for the children to participate. Thus, the children were involved in a joint activity where they achieved shared foci of attention, and opportunities for achieving shared meanings were given (Rogoff, 1990; Wertsch, 1998). It seemed as if that the kindergarten teacher expected short answers and never went empty for new questions to ask in order to bring the learning process forward.

An aim of the TBM project is for the kindergarten teachers' to develop inquiry as a way of being in teaching. Indication of this development is in Jaworski (2007) described in the following way: "So, developing inquiry as a way of being involves *becoming, or taking the role of, an inquirer; becoming a person who questions, explores, investigates and researches within everyday, normal practice*" (p. 127). We argue that the kindergarten teacher's orchestration of the activity, with her use of questions to promote investigation and reasoning, is exemplifying inquiry as a way of being. Our observations suggest that questions represent an effective tool in order to engage a group of children in learning activities. In accordance with Kirby's (1996) findings, the children did not pose questions. Therefore it might be objected whether the children made sense of the mathematical issues in this case. However, we believe that the joint participation and collaboration created a mathematically goal-directed activity, from which the children made shared meanings for concepts, terminology, and actions. From an analytical point of view, not every question may be characterised as genuine questions. For instance, some of the *suggesting action* questions and *concluding* questions are hidden suggestions or instructions. This is in accordance with what Myhill and Dunkin (2005) found, that teachers often "had a set answer in mind" (p. 424) even when they asked open questions. Nevertheless, it is likely to assume that the children perceived these questions as real since they both verbally and non-verbally actively participated in the activity. Our study thus shows that through the use of questions, the kindergarten teacher created a milieu of inquiry (Wells, 1999), and they were a substantial part of her orchestration.

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DIDACTICAL ANALYSIS OF A DICE GAME

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Abstract: *in this paper, we analyse an activity for 1st grade students, taken from the official pedagogical material for mathematics in French-speaking Switzerland. This activity is part of the curriculum about addition and comes in the form of a dice game. After some succinct considerations about games in mathematics education, we give an a priori analysis (according Brousseau's theory of didactic situations) of the activity. We then give account of an experimentation we made in Geneva, first with the teacher in her class and then with two duos of students outside the class. Finally, we suggest some modification in the didactical design in order to make this activity more pertinent.*

INTRODUCTION

In the whole of French-speaking Switzerland, for mathematics teaching, there is a single common official set of pedagogical material, including text-books and files for students and a teacher's book with curriculum and didactical commentaries. Like in many other countries, especially for lower grades, many of the mathematical activities are presented in the form of games.

The interest for games in mathematics teaching is nearly as old as mathematics. Huizinga (1989) refers to Piaget (1945), who put forward the importance of games with rules in opposition to fiction games for education. Caillois (1951) claims that a game is rather a challenge than just an exercise: "A Child does not train for a specific task. He acquires through games a wider capacity for overcoming difficulties." (p. 319). The virtues of games are widely recognised in mathematics education especially for lower grades (Milliat & Neyret 1990). Nevertheless, some critical voices can be heard about certain excesses (Valentin, 2001). Indeed, games may be a very good means for learners to acquire mathematical knowledge, yet, it is not always easy to match the game's stake with a precise mathematical goal. In this sense, we recall here some basic principles of Brousseau's theory of didactic situations:

Doing mathematics is only possible by solving problems, yet, it should be reminded that solving a problem is only part of the work at stake; finding good questions is as important as finding their solution. [...] In order to make possible such an activity, the teacher should therefore imagine and offer to students, situations that they can apprehend, in which knowledge appears as the optimal reachable solution to the given problem. (Brousseau 1986, 35) or (Brousseau 1998, 49).

Therefore, when setting up a mathematical activity in the form of a game, one needs to analyse the adequacy of the game's finality with the potential for acquisition of the specific intended mathematical knowledge as an optimal solution to win the game.

In a survey about the use of the official pedagogical material by teacher in French-speaking Switzerland, Tièche-Christinat (2001) noticed that games are usually chosen in reference to the pleasure they are supposed to give to students, while the mathematical content is secondary. It is also well-known that some students do not like games at school. In this research work, we analyze and experiment an activity in the form of a game proposed in the official pedagogical material for the first year of primary school in Geneva. Some work in this sense, but about other activities, had already been done during a one-day seminar organised by the Institute for Pedagogical Research (IRDP) in Neuchâtel (Jaquet & Tièche-Christinat, 2002).

A PRIORI ANALYSIS OF THE ACTIVITY “TURN THE DICE”

This activity is part of the official material for 1P (first year of primary school, age 6) in French-speaking Switzerland. It is located in module 3. *Problems to get to know sums*, in a sub-section entitled: *Add and subtract in situation* and refers to the objective: *Getting to 20 by adding numbers*. Here is a translation of the text of the activity as it is found in the teacher’s book:

Turn the dice

Description 2 students / One dice

- *Rules* : One student rolls the dice and says loudly how many points he got. The other turns the dice on one of the lateral sides and adds the points to the preceding total. The game follows on this way: each player, in turn, turns the dice on one of the lateral sides and adds the numbers. The first who gets to 20 wins.

Possible extension: starting with 20 to reach 0. The first who overcome 20 wins...

The first goal of an a priori analysis is to look at an activity from a more distant viewpoint in order to localise some blind spots and elucidate some hidden goals. In this sense, Brousseau’s theory of didactic situations (see (Bessot, 2003) for a basic yet enlightening introduction) provides some tools in order to interpret an activity as a special case of a more general set of didactic situations. Describing such a set means revealing didactical variables and their different possible values, such that the activity correspond to a particular choice of value for each variable. A didactical variable correspond to a potential (yet often implicit) choice for the teacher that modifies the accessibility of different strategies for solving the problem. Thus, a different choice of value for any didactical variable changes the nature of the learning and correlatively the meaning of the knowledge at stake. Such a methodology consists in revealing implicit choices made against other possible ones. Therefore, it reveals what is usually hidden because implicit. Listing possible students’ answers, which is what an a priori analyses is too often reduced to, is only one part of the analysis and is only fully valuable when one knows how to interpret different strategies in the whole set of possibilities. In this sense, the activity “Turn the dice” can be seen as a specific element of the set of situations in form of a game with two players:

In turn, each player chooses or picks up at random (this may vary at each turn) a number in a set E_i (i^{th} turn): The number is then added to the preceding total. The winner is the player who reaches first a certain predetermined value N .

We define six didactical variables:

- *two about the general rule of the game:*

Vov = “yes” or “no”, depending whether the final value N can be overcome or not.

VN = N , the value to be reached or overcome in order to win.

- *two variables that can change at each turn:*

Vrand = “yes” or “no” according to the fact that the number is respectively picked up at random or chosen by the player.

VEi = E_i , the set of possible numbers to be chosen or picked up at random at the i^{th} turn.

- *two variables that deal with the material used for the games:*

Vrep: determines, in relation to the material used, the type of representation for the numbers (side of a dice either with dots or numerals, cards with numbers written with letters, numerals or constellations, etc., tokens, spoken numbers...)

Vwrit = “yes” or “no”, depending whether the players can write their sums or not.

Of course, this list of variable is only partial and partly subjective. This is why we have to justify our choices by showing how the subsequent a priori analysis is relevant for our observation. We distinguish two levels: the knowledge at stake locally at each turn of the game, and the global strategy of the game.

Making sums (local knowledge)

Regarding competencies for addition in 1st grade, the value of VN cannot really exceed 20, and the numbers in the sets E_i are also limited to 5 or 6. Moreover, in 1st grade, many students still counts on their fingers and make additions by over-counting one-by-one from the first number of the sum (to do $4+3$, the student count loudly or in his head raising fingers three times: “five, six, seven”). The memorised repertory is still very limited, which means that very few sums are known by heart. $Vwrit$ is quite important in this game, not only because students can actually make the addition using written devices, but also because writing the sums at each turn reduces the effort of memorisation. In the activity “Turn the dice”, the value of this variable is left to the teacher’s choice. In our experimentation, the teacher chose not to let students the possibility to write. Furthermore, the various possible values of $Vmat$ modify the possible techniques for making sums. Dice (with spots), cards with constellations, tokens... make possible, even promote, techniques using one-by-one over-counting. On the opposite, numbers in numerals, letters or just spoken promote other techniques like recalling a repertoire or “calcul réfléchi” or necessitates to use fingers or written techniques if $Vwrit$ =yes. In the activity “turn the dice”, the type of representation of the numbers on the side of the dice is not specified. In our experimentation, the teacher chose a dice with spots. However, one of the objective

in 1st grade is to progressively bring students to abandon techniques using one-by-one over-counting. They should start memorising the repertoire and use “calcul réfléchi”.

This first analysis shows that the choices for the activity “Turn the dice” are coherent with the level of 1st grade students. The game is possible. Yet, regarding the learning of addition, there are some contradictions with the goals at this level of education. Moreover, the game does not provide a milieu with possible feedback for the learning of sums. Indeed, nothing in the game offers a possible feedback to a mistake in a sum, except the control of the other player, or the teacher if s/he is watching at the right time. In other terms, if one student gives a wrong result for a sum and if the other player does not react and the teacher is not watching, the game can go on without the mistake being corrected. Therefore, “making sums” is a knowledge necessary for the game to be played, but is not subject to a control and certainly not the main tool for an optimal winning strategy. Therefore, if we refer to Brousseau’s quotation given above, we can see that there is an inadequacy here between the game’s stake and the didactical objective: *Problems to get to know sums*. In order to play correctly, students have to know how to make sums correctly. If they do not, they may play anyway, but nothing in the milieu organised through the game gives any feedback. Nothing is organised didactically for them to learn sums, they have to know, but they can make errors without being corrected, except if the other player knows better or the teacher is here to correct. Furthermore, we have seen that the use of a dice with spots is likely to promote the basic technique “over-counting one-by-one”, which is supposed to be progressively banished in 1st grade. Such an activity is therefore not especially good in order to train 1st grade students to do sums. At most, if they have a reliable technique, this game may help them memorizing sums, but the excitement of the game is likely to overcome this goal!

Game’s stake (global strategy)

At this level, the values given to VE_i , V_{rand} and V_{ov} are crucial.

For the choice $V_{rand} = \text{“no”}$ and $VE_i = \{1,2\}$ at each turn and $V_{ov} = \text{“no”}$, the game is called the “race to 20” and has been analysed by Brousseau (1998, 25-44). Such a game has a winning strategy, corresponding to the series of winning numbers 2, 5, 8, 11, 14, 17, 20, that can be discovered by subtracting 3 to 20 repetitively down to 2, or by dividing 20 by 3, the rest being 2. Brousseau showed how such a situation can be used to make 4th grade students discover the Euclidean division and debate about a general strategy for being sure to win. In the case of the activity “turn the dice”, there is no such strategy. Even if a strategy for winning is possible, it is far from being reachable by 1st grade or even much older students.

In the opposite, if $V_{rand} = \text{“yes”}$ at each turn, this is just a game of chance, which, therefore, doesn’t call for any strategy, at least in relation to any mathematical content. Moreover, dices are often related to games of chance, it is therefore likely that students act just as if “turn the dice” is only a question of chance, especially considering the fact that on the first go, the player rolls the dice.

Is there a possible strategy to win the game “turn the dice”? If yes what can 1st grade student catch from it? The main difficulty of this game is that E_i changes at each turn. Moreover E_i depends on the choice made at the $(i-1)^{\text{th}}$ turn, therefore by the other player. Two opposite sides of a dice always add to 7. This gives the rule for possible choices with regard to the last chosen number. Each turn can be represented by the number “ i ” (order of the turns), the name of the player who just played (P1 or P2) and $S(n)$, S being the last sum calculated and n the last side chosen.

For instance [3 , P2, 12(5)] means that it is the 3rd turn, P2 has turned 5 which adds to a total of 12. At the 4th turn, P1 must therefore choose in $E_4 = \{1,3,4,6\}$.

- If P1 chooses 1, the status is 13(1) and $E_5 = \{2,3,4,5\}$. if P2 chooses to turn 4, the status is 17(4). Since 3 is not possible, and numbers over 3 are too big, P1 must choose 1 or 2 and P2 wins at the next turn. Thus, 1 is not a good choice for P1.
- If P1 chooses 3, the status is 15(3), and P2 can turn 5 and wins.
- If P1 chooses 4, the status is 16(4), P2 cannot win but if he turns 2, the status is 18(2), so P1 has no other choice than turning 1 and the game is blocked.
- If P1 chooses 6, the status is 18(6), P2 can turn 2 and wins.

This example shows that the strategy is quite complex. A player must anticipate all the possibilities and short time anticipation may be fatal. Moreover if $Vov = \text{“no”}$ like in the original game, some games may lead to a dead-end. This is far too complicated for 1st grade students. Indeed, at this level, students are likely to be unable to just anticipate the result of the next turn. Indeed, this requires more than just addition, but also knowledge about complements to 20, which is a first step toward subtraction: “how much is it from 14 to 20?”, etc.

In conclusion, the game’s stake does not have to do just with adding numbers (no more than 6) to reach 20, but also being able to anticipate the next (one possibly two or more) turn(s). One mathematical knowledge needed is then to be able to anticipate the effect of adding a number and knowing the complements to 20, from at least 14. It is therefore impossible to hope that 1st grade students develop a strategy that leads to victory in each case. At most, they can anticipate one or two turns when the sum gets over 12, or a bit more. Therefore some important didactical questions are: “what knowledge can be aimed at through such an activity?”. “Are 1st grade students sufficiently knowledgeable to do their sums without mistakes?”. “Can they do more than play at random and develop some strategy at least towards the end of the game, involving some abilities for anticipation on sums, and complements to 20?”.

In order to answer these questions, we organised an experimentation of this activity in a 1st grade class near Geneva.

EXPERIMENTATION

The class counts 22 students of average level, in a village near Geneva. The teacher has only 3 years of practice and teaches 1st grade for the first time, she also uses this

activity for the first time and we did not exchange with her about it before. The experiment took place in March. The teacher decided to explain the game to the whole class for about 10 minutes, before splitting the class in two. One half plays (5/6 duos), while the rest of the class has to do some work individually in autonomy. Each half-class played for about 15 minutes. In the end, a conclusive session with the whole class is organised. Our observation is based on a video recording of the whole class (beginning and end) and for each half-class, on a video recording of one duo, plus an audio recording of another duo.

Devolution

The teacher reads the rules of the game and asks questions. Some students comment with their own words. Then, the teacher chooses two students to play a game, which is summarised in the following table:

Player	Marie	Renan	Marie	Renan	Marie	Renan	Marie	Renan
Number chosen	3	6	2	3	2	1	2	1
Total	3	9	11	14	16	17	19	20

Neither Marie, nor Renan take time to think about what they choose (except Renan at the last turn!). This validates our hypothesis that, for them, it is like a game of chance. At each choice of a new number, the teacher asks for the total and several students raise their hands, and there is a quick general agreement on the result. Renan starts with a big number, in order to get near 20 quickly. Marie is more careful and on the contrary chooses the smallest number she can, in order to prevent Renan from getting too near to 20! At the 5th turn, Renan chooses 3, getting to 14(3). That can lead Marie to win if she chooses 6! Yet, she does not and nobody notices. She chooses 2, getting to 16(2), Renan can win by choosing 4, but he chooses 1 (nobody notices) getting to 17(1). Again Marie can win, but she chooses 2 (nobody notices either), getting to 19(2). Renan cannot do anything else than win!

This shows clearly that the game's stake is not accessible to the students straightaway. They concentrate on their sums and do not see the goal. Getting to 20 is the criterion to stop but not a goal to reach first. The teacher does not try either any devolution of the stake. One student spontaneously says. "Marie always makes 2 and Renan 1". The teacher interprets: "Oh why do they always choose small numbers?" and Marie instantly replies: "This way it is easier to count!". Clearly the students are concentrated on their sums and reduce the difficulty without care for the game's stake. Therefore, in this collective phase of devolution (5 min.), all is about sums and nothing about the game's stake is debated.

The games

In this paper, we cannot analyse in detail all the games we observed, we only give some general comments (see Dorier & Maréchal (in press) for more details). Some students did not understand that they had to choose one side after the first go, instead they rolled the dice at each turn. This validates again our hypothesis that they play

like a game of chance. Something we did not anticipate lead to some unnecessary noise and excitation. Indeed, to turn the dice, many students pressed the edge of the top-side. As a result, the dice often rolled several times or even off the table. The students play fast, which is a sign that they do not choose really their numbers. Globally, they tend to choose big number at the beginning and small ones near the end. This is a sign that the game's stake is taken into account at a basic level. However, several times, students made a choice that allowed the next player to win, while it could have been avoided. Some duos did not respect the fact that 20 should not be overcome. Systematically, students looked around the dice to check the possible choices. Most of the time, they over-counted one-by-one, pointing each spot on the side. Some counted on their fingers and very few recalled memorised results. This validates our analysis and shows that the choice of a dice with spots favours an elementary technique for making sums. Some mistakes on the results of sums (even with the elementary technique) occurred and were usually not corrected by the other player. Some duos have great difficulties in memorising the totals or even making the additions. Nevertheless, some duos show that they tried to anticipate the results near the end of the game. However, the complements to 20 did not seem to be known by heart and students usually counted on their fingers or directly on the sides of the dice. No duo anticipated two turns. No example of a game coming to a dead end had been observed. However, the students were happy, they had play!

Conclusive phase – Whole class

Spontaneously, the students tell stories about their games “I won twice and he won three times!” , “we did not manage to finish...”... This has nothing to do with strategies or even sums, it is all centred on social aspects of the game. In order to re-direct the debate, the teacher asks: “Do you think that, in this game, there is a technique to win? Something that would help to win... more easily?”. One student suggests that it is good to choose big numbers. A short debate starts on the effects on the game of choosing big or small numbers. After some discussion, Pierre suggests that choosing alternatively big and small numbers allows to win. In response, the teacher asks Pierre to play against her. At the fourth turn, the status is 14(6) and it is the teacher's turn. She realises suddenly the difficulty and ask the students what she should play. 2 and 5 are given as answers, she chooses 5, getting to 19(5), which allows Pierre to win. The teacher's conclusion is that Pierre's technique only works if the other player follows his rule! One of the observer then ask what would have happened if the teacher had chosen something else than 5, like 3. The teacher agrees and turns 3, the status is then 17(3). She asks Pierre what he would do. He stays silent for quite a long time and finally says he would choose 3. Obviously at this stage, the teacher is not quite sure of herself, so she closes the discussion by saying: “Is there only one technique?”. There is quite a long silence before a student starts talking again. But even then, nothing really interesting happens. Finally the teacher says: “Is there a time in the game, ... maybe one number... from which you know you can win... maybe...for instance if you get to 10, can you be sure to win?”. We can hear a

few “no”... the teacher goes on: “Is there a number, that you can say: ‘if my friend put this number, I am able to win if I turn the right side?’.” No answer. Then Marie claims that she has a technique: “In fact I... at first, I choose nothing special... and then, toward the end when it is a bit more difficult, eh.. I look around the dice and I count the sums, and...”. The teacher goes on: “You look around the sides and you look which comes to 20. Did you all think about looking at the possible sides before you turned the dice?”. Around 8 students raise their finger. “Did that help you to win?”. One student answers: “There was not the side I wanted, because it was underneath.” At this moment the bell rings and the class is finished.

The conclusive phase shows that the teacher struggles with her goals and the students’ reactions. She had probably under-estimated the difficulty of the game. Of course, this is a lack of questioning from her part, but this is also due to the difficulty of the situation itself and the lack of didactical analysis in the official pedagogical material, in order to help teachers lead this activity. Our a priori analysis shows that the milieu of the situation is not suitable to give sufficient feedback to the students on the validity of their sums. It also shows that, without any other didactical device, students are likely to play by chance and develop very few strategies. At most, they try big numbers at the beginning of the game and small ones at the end. Our observation confirms these conclusions. It also confirms that students use only one-by-one over-counting strategies and do not use more elaborate techniques for their sums. Nevertheless, some students do try to anticipate the results of their choices toward the end of the game and try to guess the complement to 20, mostly by counting on the visible sides of the dice. Yet, without stronger motivation, they fail to really develop a strategy, and do not anticipate more than one turn. Our observation also shows that students do not spontaneously reflect on the reason that made them loose, by analysing the last turns of the game they just played. They do not try other choices, to see what could have changed. In our experimentation, the teacher did not try to make students do so. Moreover, when one of the observers tries to initiate such an analysis in the collective conclusive part, the teacher finally gives up.

New experimentation with duos out of the class

Even if this experimentation allowed us to validate our a priori analysis, we wanted to see what kind of behaviour students may have, if they were asked to reflect on the end of a game they just played, and anticipate the effects of other choices. Therefore, a few weeks later, we asked the teacher if we could work individually with a couple of duos. She accepted and we organised a new experimentation during an hour with two duos of students, in a separate room, while the teacher stayed with the rest of the class. We do not have space here to analyse what happened then, so we will only give a short account (see (Dorier & Maréchal, in press) for more details).

Globally, this experimentation shows that when asked to reflect on the last turns of a game they have just played, the students we observed are able to anticipate the two or even three next turns. They understand that they have to find the complement to 20 and anticipate the possible choice for the next player. Once this type of reflection is

initiated, they play more carefully the following games, and develop some anticipating strategies, that make them reflect on the complements to 20 and possible issues. Moreover, this experimentation showed that students knew their sums by heart, and were able to give up the “one-by-one over-counting strategy”, if they were asked to, or when they had to anticipate and therefore were not able to use the dice to count. This confirms the fact that in its basic version the activity “turn the dice” promote a technique that students can overcome using a more expert one. It also shows that making them anticipate the next turns, induces them to switch technique.

CONCLUSION

Our observations have been limited, thus, we have to be careful about the conclusions we can draw. Globally, the experimentation in class with the teacher confirms the conclusion of our a priori analysis, that such an activity is likely to be reduced to a game of chance, which means that students do not learn much. The second experimentation shows, on the contrary, that on certain specific conditions, students can be led to reflect on the way they play and develop some more expert strategies, and in particular, acquire some knowledge about complements to 20. In this sense, “turn the dice” may be seen as a consistent mathematical activity accessible to 1st grade students. However, the conditions of our second experimentations are too particular to be reproduced as such in normal conditions. Therefore, we need to find a didactical device in order to make the realisation of this activity possible in “normal conditions” and proper to induce a consistent learning. Using a dice with numbers written in numerals rather than spots, could be a solution in order to block the one-by-one over-counting strategy, but then it is impossible to use it to check sums in case students fail. Therefore, this solution is only possible, if students do know their sums by heart. Therefore, this activity should not be given in the beginning of 1st grade, but rather at a time when most students have memorised sums with little numbers.

Letting the students play a few games at the beginning is quite important in terms of devolution, even if they just play by chance. During this phase of appropriation, it is important to check that all the rules are understood (the dice is rolled only at the beginning, it is forbidden to exceed 20, it is important to control the turning of the dice...). It may also be possible to tell students that they can (should?) use other techniques than one-by-one over-counting on the side of the dice (or this can be debated in the next phase only).

After this first phase (as short as possible) a first time in common can be organised by the teacher. After asking the students what they did, two can be chosen to play a game in front of the class. Then, the teacher can organise a collective reflection on the last turns of the game and analyse the effects of alternative choices. This should produce a change in attitude for most students (like what we observed in our last experimentation). This can be repeated once or twice, before students are asked to play again in duos, 8 games each. It is important to limit the number of games and to

give sufficient time, to prevent students from going too fast trying to play as many games as possible, like we observed in the beginning of our experimentation. Each time a player wins he gets one point. The totals are to be compared at the end. This gives a bit of competition in the games, in order to favour the search for a strategy and not just chance. A final collective debate should lead to the institutionnalisation on the strategies as well as complements to 20.

Of course, a new experimentation is necessary to see if this new proposition inspired by our first analysis would lead to a more satisfactory lesson.

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“TELL THEM THAT WE LIKE TO DECIDE FOR OURSELVES” – CHILDREN’S AGENCY IN MATHEMATICS EDUCATION

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Interviews with primary school children about their lived world of school mathematics, unanimously and strikingly revealed that the practical/creative school subjects were their favourites. These subjects granted them agency and modes of bodily expressions that were not available in mathematics and the other academic school subjects. The interviews are analysed from a perspective of school mathematics education as a social practice that draws attention to and valorises the children's perspective. The question is raised whether the children's preferences reflect a genuine perception of postmodern life conditions that should be taken seriously.

Keywords: children’s agency, embodied agency, children’s perspectives

INTRODUCTION

If learning is assumed to involve intentional action (Skovsmose, 2005), then students’ agency in mathematics teaching and learning is an important issue. Yet, studies on agency in mathematics classrooms (e.g. Boaler & Greeno, 2000; Klein, 2001b) have rarely considered the perceptions of primary school children. In high school classes and teacher education situations, agency has been discussed in terms of students’ opportunities to make choices and to have authorship within the discourse around mathematics. Interviews with 10-year-old children in a Year 4 class in Denmark also revealed restrictions on agency in mathematical activity in these respects. As well, the children perceived their bodily actions as being restricted. When asked about their preferred school subjects, almost unanimously, the children pointed to design (needlework), visual art, physical education, and swimming as the subjects, they liked the best. These subjects provided opportunities for creative, physical, and/or playful forms of agency. This was in stark contrast to the subjects they considered to be the most important subjects, i.e. Danish, mathematics and English where they experienced very little, if any, agency and much tighter bodily control. They felt that they had to do what the teachers requested and could hardly imagine the situation being any different, i.e. what agency could be in these subjects.

The children's preferences could be a reflection of the long-term effort of learning mathematics and the challenges involved, as opposed to the immediacy of the practical/creative subjects, or they could be a voicing of popular notions of so-called academic schools subjects as tedious. Regardless of their validity, these explanations to children’s views seem unlikely to be exhaustive, and troubling questions remain. Could it be that the children's preference for practical/creative school subjects – with their space for creative playful whole-body agency – reflect a valid perception of

what is important for them to develop in order to grow up as competent citizens in a postmodern world [1]? What does the perceived absence of agency do to their perception and learning of mathematics? Are children in difficulty in learning mathematics especially affected by this apparent lack of agency?

THE NOTION OF AGENCY

The Oxford English Dictionary defines *agency* as “the faculty of an agent or of acting; active working or operation; action, acting”. *Agent* comes from Latin *agere*, to act, or to do. An agent acts or exerts power, as distinguished from the *patient* and the *instrument*; the agent acts upon the patient/instrument. Hence, in sociology and social sciences, *human agency* denotes the faculty to act deliberately according to one’s own will and thus to make free choices. A central issue in these sciences is the relation between *structure* and *agency*; i.e. how social and cultural factors such as social class, religion, gender, ethnicity, customs, etc. shape the opportunities that individuals have, and how does human agency change these factors.

Schooling, and mathematics education as part hereof, constitute a major social and societal arena in the organisation and rhythm of children's daily life as well as their future lives as independent adult. In this arena of mathematics teaching and learning, children's agency could be seen to involve three aspects. The first is based on an assumption of *children as social actors* (Højlund, 2002; James, Jenks, & Prout, 1998; Kampmann, 2000). Consequently, they make sense of their experiences in school mathematics irrespective of the agency granted to them at school. They ascribe meaning (Skovsmose, 2005) from a ‘global’, holistic life world perspective (Kvale & Brinkmann, 2009) that integrates their experiences in mathematics learning with their future life perspectives (Lange, 2008a). The second aspect concerns the organisation of their *mathematical activity*, which may leave them more or less agency in the sense of opportunities or expectations to (co-)create mathematical concepts, discuss mathematical ideas, make choices, think for themselves, etc. as part of their learning process (Boaler & Greeno, 2000). The third aspect relates to *embodied agency* (Benner, 2000; Shilling, 1999) in that school norms impose physical restraints on students’ bodily freedom such as requiring them to sit on their chair at their desk, keep quiet, have their mobile phones turned off, etc. As is discussed later, children are very aware of these restraints.

Interviewing high school students in advanced calculus classes in USA, Boaler and Greeno (2000) found that ‘traditional’ mathematics education, dominated by instruction in and training of procedures to find the one correct answer to diverse mathematical problems, afforded virtually no agency to students, but required them to “surrender agency and thought in order to follow predetermined routines” (p 171). Boaler and Greeno discussed students’ agency with reference to the notion of *figured worlds*, a key term in Holland, Lachicotte, Skinner and Cain’s (1998) discussion of social systems. Within this framework, agency is conceived in terms of authorship and as a prime aspect of identity. Seeing mathematics classrooms as figured worlds

and agency as authorship, draws attention to the children's/students' and teachers' interpretations of the rituals of their shared practice and their positions and roles, and to the shaping of their sense of self, their identities, in the social practices of mathematics education. Boaler and Greeno (2000) found that:

[i]n the schools in which the students worked through calculus books alone, the students appear to view the domain of mathematics as a collection of conceptually opaque procedures. The majority of students interviewed from the traditional classes reported that the goal of their learning activity was for them to memorize the different procedures they met. Such a figured world of didactic teaching and learning rests on an epistemology of received knowing. In this kind of figured world, mathematical knowledge is transmitted to students, who learn by attending carefully to teachers' and textbook demonstrations (Boaler & Greeno, 2000, p. 181).

In order to be successful, students in 'didactic' classes needed to "assume the role of a received knower and develop identities that were compatible with a procedure-driven figured world" and be willing "to build identities that give human agency a minimal role" (p. 183). The students saw success as requiring "a form of received knowing, in which obedience, compliance, perseverance, and frustration played a central role" (p. 184). Some students, girls in particular, rejected mathematics because

they were not prepared to give up the agency that they enjoyed in other aspects of their lives, or the opportunities to be creative, use language, exercise thought, or make decisions. ... [T]hey wanted to pursue subjects that offered opportunities for expression, interpretation, and agency (p. 187).

Referring to Pickering's (1995) discussion of agency in mathematics and science Boaler and Greeno concluded that the students only had opportunities to learn what Pickering termed "the agency of the discipline" which is the agency aspects of mathematics, in which human agency play the least role, thereby seriously distorting their perception of mathematics as a scientific discipline.

While Boaler and Greeno criticised procedural teaching for its reduction in students' agency, Klein (2001a; 2001b) criticised pedagogical practices that base mathematics education on conjecture, reasoning, investigation and inquiry. Writing from a poststructuralist position, she claimed that current practices are framed by humanist notions of rational, autonomous learners. These notions take students' agency for granted, overlook always present power relations, disregard that identity and agency are discursively constituted and not an individual disposition, and hence do not recognise that students' agency needs to be considered in every learning encounter (Klein, 2001a). Like Boaler and Greeno (2000), Klein discussed agency in terms of *authorship*, but with reference to Bronwyn Davies:

[S]tudents can experience a sense of agency in a discourse where they have a knowledge of themselves as respected and competent in (a) speaking and writing the commonly accepted truths of the discourse, in (b) enacting established ways-of-being, and in (c)

going beyond these to forge something new (Davies, 1991). Agency has to do with authority, not in the sense of control over but in the sense of *authorship*; authorship of voice and action in a community conversation. All pedagogic discourses, regardless of whether we see them as transmissive, child-centred, constructivist or social constructivist, support agentic behaviour to the extent that they impart a robust knowledge and skills base and authorise student initiated constructions and ways of making sense of experience (Klein, 2001b, p. 340).

Boaler and Greeno (2000) looked at high achieving high school students perceptions of agency in USA, and Klein analysed agency in an Australian teacher education context. I am exploring young children's perspectives (Lange, 2008b) on agency in a Danish *folkeskole* (public primary and lower secondary school). These children also seem to experience restrictions on expressing their agency in their mathematics lessons. However, apart from illustrating their perceptions of lack of choice and ability to author discourse, I discuss how bodily aspects of agency may be particularly relevant for smaller children. My contention is that the children seem to be suspended between two conflicting experiences. On the one hand, they experience joy and engagement arising from spaces of agency in the practical/creative school subjects that they do not believe is important. On the other hand, they think of mathematics as a school subject that are important for their future, but the agency they value so much is virtually absent in their perception of their learning experiences in this subject.

METHODOLOGY

The empirical material for this paper comes from interviews with children about 10 years old in a Danish Year 4 class. I observed their mathematics classes for almost a year and interviewed students in groups, pairs and individually. The aim of the research was to explore children's knowledge about their mathematics education, especially the meaning they ascribed to and the sense they made of their experiences with being in difficulty in learning mathematics (Lange, 2007). As I took the children's meaning ascriptions to be in a narrative form, my conversations with them invited them to tell about their experiences. Hence, the interviews I conducted were semi-structured life world interviews, i.e. interviews that "seek to obtain descriptions of the interviewees' lived world with respect to interpretation of the meaning of the described phenomena" (Kvale & Brinkmann, 2009, p. 27).

There were twenty children in the class. All but one participated in one of three group interviews early in the school year. Half of the children were interviewed in pairs or individually a little later, and again near the end of the school year, with some overlapping of the two groups. The interviews took place at the school, lasted 30-45 minutes, and were audio recorded; the group interviews were also video recorded.

Taking *children's agency* to be a theoretical construct, only "visible" in the interviews from theoretical perspectives, I wanted my interpretative activity to be as transparent

as possible. This was especially necessary because my empirical material was interviews with young children whose life world and linguistic universe are rather different from mine. I contend that children's meaning ascriptions, the “web of logic”, the discourse in which they embed their experiences with school mathematics, are to be found in stories about their lived school mathematics world. The children's narratives that I was looking for were rarely found as rounded well-formed stories ready to be copy-pasted into research papers. More often they unfolded as dialogues involving my active listening and questions (Kvale & Brinkmann, 2009). Consequently, a longer transcript is given rendering an example of the children's voices. The following interpretation shows the analytical process. For reason of space, extracts from other interviews are summarised within the interviewees' horizon of understanding and such condensates are used as a points of departure for the interpretation (Kvale, 1984; Lange, 2008a).

WE LIKE TO DECIDE OURSELVES

In an interview in October 2006, Maria and Isabella (all names apart from mine are pseudonyms) expressed that they liked the school subjects of design, swimming, physical education and visual art. Recently Maria had also started to like maths. When asked to comment on my observation that all the children seemed to like these subject the dialogue went as follows [2].

- 1 Maria ... because in design we do something creative and such. I like that and in physical education it is not only think, think, think, think, think, think, think, think all the time ...
- 2 Isabella It is also more that you, for instance in design we are allowed to decide ourselves how it [a teddy bear] should look like, how it should be, and also in physical education and such we sort of run around and play. (*She explains the different ball games they play assisted by Maria*)...
- 3 Troels Ok. And some of the good things [about visual art and design] is that you are allowed to decide more yourself?
- 4 Isabella Yes I think so because
- 5 Troels Yes, is it so that in mathematics and Danish and English you are not allowed to decide very much?
- 6 Maria I don't think so
- 7 Isabella No, yes but (Maria: you are not allowed so much) we are not allowed like decide (Maria: ourselves how) we must just like do the problems we get and
- 8 Maria And then we must do them and we may decide ourselves the way we do it, just that it is right. And that, then I like better some (Isabella: yes some) subjects where you just “Ah, what sh[ould]? How? Oh, I think I will do like this.”
- 9 Isabella Yes for instance you decide (Maria: how you yourself also) if you are going to draw a drawing if it should be a face or it should be, yes then you decide yourself and then. Yes it is like more, you can just sew

- 10 Maria Also where you can come up with ideas yourself. You cannot really do that, 'cos you cannot really come up with ideas. _ I don't think _ I just think it would be a good idea if like this sum came in because it was more difficult or a little easier because you cannot just
- 11 Isabella No decide just like that
- 12 Maria Here you can come up yourself, because when we should sew those teddy bears then you figured out yourself. I figured out myself that mine should have dots and that it should have such long legs
- 13 Troels So it is important that about deciding for yourself?
- 14 Maria Yes
- 15 Isabella Yes I like that

By the end of the interview Maria and Isabella asked me for what I was going to use the interview and if it was because I wanted to become a teacher. I told them that I was a “teacher teacher”.

- 16 Maria So you can see what you should do to make your class better?
- 17 Troels You may say so. It is because I would like to know how children think about mathematics
- 18 Maria Are you only teaching mathematics?
- 19 Troels Yes that is I teach how student teachers, people who want to become teachers, I teach them how they should teach mathematics
- 20 Maria And then you can tell it to them
- 21 Troels Yes
- 22 Maria And then they can do it and then they can see that you like to decide for yourself
- 23 Troels Yes
- 24 Isabella Yes
- 25 Maria I think that is good

Maria likes design because they do something creative (1; numbers refer to the transcript lines). She also likes physical education because it not only about thinking (1). Isabella likes that in design they may decide how a teddy bear should look like and that in physical education they run and play ball games (2, 4). In mathematics, they must do the problems they get (7); they may decide how they do them as long as they get them right (8), but they cannot really come up with their own ideas (10, 11). They like to use their imagination (8-12) and find it important to be able to decide for themselves as they can in visual art and design (13-15). This is the message they want me to bring to my teacher education students (16-25).

Interpreting the interview excerpt from my adult, research perspective, Maria and Isabella express that they appreciate when school subjects make space for their creative imagination (1, 8, 9, 10, 12) and decision making (2, 4, 9, 12-14, 22-25) and/or the presence of their whole playful body (1, 2). They experience these spaces in design, visual art, and physical education but not in mathematics (7, 10, 11). Here they are given problems that they have to get right (7, 8), and they cannot imagine

how ideas of their own could come into play (10, 11). They do not talk about getting a right *answer*, which would presuppose that there was a question. In Danish, Isabella talks about “lave opgaver” (“do problems”; 7), which is common “school mathematics” Danish. Nonetheless, it is a linguistic mix between the older phrase from the days of arithmetic “lave regnestykker” (“do sums”) and the language of the more recent reform curriculum “løse opgaver” (“solve problems”). There is a linguistic consistency between how they describe their activity as *doing* problems (7) and getting them *right* (8) – as opposed to *solving* problems, or *answering or exploring* questions as stipulated in the curriculum – and their experience of not being able to *come up with ideas* (10).

The other children interviewed in the same round of interviews as Maria and Isabella also liked practical/creative subjects and by and large for the same reasons: that they could use their imagination, do something with their hands, decide something, or engage in playful, physical activity often with competitive elements. They also thought that they did not make decisions in mathematics. The following paragraphs add more details to the picture drawn from the interview with Maria and Isabella.

Asked about differences between the subjects, in regards to what the children could decide, some children, all of immigrant background, said that there were no differences. After all, children cannot say no to what the teacher says (Hussein and Kamal); the teacher tells them what to do and then the children do it (Sahra and Bahia). Responding to the question, Kamal said that in history they are told off the least. Sometimes, they may decide a little in swimming. In maths, they are not allowed to decide anything and they are not told off so much either. Jette [the maths teacher] gives many five-minutes [short breaks]. An interpretation of this statement could be, that in the absence of agency in learning situations, what becomes of interest is how the teacher control is exercised (amount of telling off) and the allowance for time and space that is free of teacher control.

In school discourse, the academic subjects, in particular Danish, mathematics, and English (as a second language), are positioned and resourced as more important than the practical/creative. The children have incorporated this in their meaning ascription to their school experiences. Mathematics is important because being good in mathematics gives access to education which is a prerequisite for at future of their own choice (Lange, 2008a). Some children are explicit about the different valorisation of school subjects. Bahia and Sahra said that apart from mathematics, Danish was also an important subject; visual art not so much, design a little bit, and physical education was there in order to have fun. Kalila reflected the valorisation indirectly. When I asked which subjects she liked, she said that she liked mathematics and Danish, and asked, “Is it not that kind of subjects you are thinking of?” In reality, of all the subjects, she liked design and swimming the best. “That is more like something for me, I think”.

Many of the children described physical and bodily restraints imposed on them at school. Kalila in particular gave a vivid and heart-felt description of this and of her joy of using her imagination: In design, the teacher explains something if you keep your mouth shut. After that, you may run around, get up, talk and jump. In Danish, you must remain seated and not talk to your neighbour. In swimming, you may talk and be together and you cannot do that in maths. In design you make your own imagination of a doll, for instance, one crooked and one long eye, no nose, eyebrows – you may decide yourself. It is good to use your imagination. Kalila imagines her doll while the teacher tells about it. In Danish and maths, you cannot use your imagination. You must calculate in maths and not make your own numbers. After school, the smaller children in the recreation centre cannot go out and then come back whereas in the club for the bigger children like her you may go home and come back, go to the kiosk, bring lollies and have your mobile phone open. Children are generally very aware that they are growing. Agency is an important marker in this process; as Kalila explained older children have more physical freedom to move and to decide for themselves than younger children.

Thus, the subjects that the children like because of the agency, imagination and bodily freedom they are allowed, are positioned as not important, and the subjects positioned as important grant them little agency, space for choice or creativity, and exert a tight control of their bodies.

I DON'T LIKE MATHS WHEN I DON'T KNOW WHAT TO DO

These children grow up in a society where it is highly unclear which experiences of the older generations are valid, where the faculty to choose in almost every issue of life is paramount, and where creativity is highly valued in public discourse about present and future needs of individuals and society. Choice making and creativity are prime examples of agency, and the children in this research really appreciated when such features were part of their learning. The practical/creative subjects, thought of in the school discourse as recreational, seem to have more to offer in this respect, than mathematics and the other subjects positioned as the most important.

When making sense of their experiences, the children perceived no agency for them in school mathematics learning, and they could not imagine what it could be either. You are not supposed to make up your own numbers, as Kalila put it. Like the much older US high school students that Boaler and Greeno (2000) wrote about, these much younger student in a Danish comprehensive school were ascribed identities with minimal human agency. In the terminology of Klein (2001b), they did not perceive invitations and support to develop their authorship of mathematical constructions and ways of making sense. They did make sense – the sense seen in the interviews, but their sense-making was not part of their “official” mathematical activities. These sense-making processes are active undertakings on part of the children in which they contribute to the construction of the discursive field embedding mathematics education and thus need to be seen as an aspect of children's

agency. As such, they are co-creators of the social practices of mathematics education, even when these social practices lead to a restriction on agentic behaviour.

The “no agency” experience of mathematics learning is problematic for several reasons. It gives a distorted picture of academic mathematics, and it reinforces instrumental learning rationales (Mellin-Olsen, 1981). Such rationales are not conducive to the learning of students in difficulty with mathematics (Lange, 2008a) – if they were, they would not be in difficulty. When such children do not succeed in “getting it right” in what to them seem unrelated tasks, void of inherent meaning and agency, they are left with having to cope with unproductive and awful feelings of helplessness. Maha expressed these feelings when she said that she hates Sudokus and metre and centimetre, and that she does not like mathematics when she does not know what to do, and nobody comes to help her, and she just sits and waits and waits.

NOTES

1 I understand *postmodernity* as “a social condition, comprising particular patterns of social, economic, political and cultural relations” (Hargreaves, 1994, p. 38)

2 The Danish transcript is rather detailed and forms the basis of the interpretation together with the audio recording. The translation into English is a compromise between a direct translation, an attempt to retain some of the linguistic features of children's spoken language, and a light approximation to written language by removing some of the repetitions and incomplete sentences.

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EXPLORING THE RELATIONSHIP BETWEEN JUSTIFICATION AND MONITORING AMONG KINDERGARTEN CHILDREN

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This paper investigates the types of justifications given by kindergarten children as well as the monitoring behavior exhibited by these children as they work on number and geometry tasks. Results showed that kindergarten children are capable of using valid mathematical procedures as well as the critical attributes of geometric figures in their justifications. Children also exhibited monitoring behaviors on both tasks. The study suggests a possible reciprocal relationship between giving justifications and monitoring behaviors in young children.

INTRODUCTION

According to the Principles and Standards for School Mathematics (NCTM, 2000), "Instructional programs from prekindergarten through grade 12 should enable all students to recognize reasoning and proof as fundamental aspects of mathematics" (p. 122). It is important to note that reasoning and proof are not relegated solely to the upper elementary and high school grades. These aspects of mathematics may be nurtured and should be nurtured from a young age. Two fundamental components of children's reasoning processes are justifications and metacognition (Tang & Ginsburg, 1999). In this article, justification refers to the act of defending or explaining a statement. Metacognition includes monitoring one's work. In this article, monitoring refers to those managerial skills which guide the problem solving process. This study is an initial investigation into kindergarten children's reasoning process and the possible relationship between giving justifications and monitoring.

THEORETICAL FRAMEWORK

In analyzing the long-term cognitive development of different types of reasoning, Tall and Mejia-Ramos (2006) described three mental worlds of mathematics: the conceptual-embodied, the proceptual-symbolic, and the axiomatic-formal. The thought-processes of early childhood are said to be embedded in the first two worlds and may be used to describe the types of reasoning displayed by young children as they develop geometrical orientation and number concepts. The first world focuses on objects and begins with perceptions based on the physical world. Through the use of language, children refine their mental perceptions by focusing on the object's properties, leading to the use of definitions which in turn are used to make inferences. This world is particularly apt for describing the development of geometric reasoning as described by the van Hiele levels (van Hiele & van Hiele, 1958). The proceptual-symbolic world builds on actions or procedures. These are encapsulated into symbols that function both as "processes to do and concepts to

think about" (Tall, 2004, p. 285). For example, the act of counting leads to the concept of number.

Different types of justifications are an outgrowth of the different cognitive worlds. "Initially, something is true in the embodied world because it is 'seen' to be true" (Tall, 2004, p. 287). Later on, justifications are based on definitions such as used in Euclidean geometry. In the proceptual world, something is true because some procedure shows it to be true. As reasoning in this world develops, justifications are given using symbolic manipulations. Yet, knowing how to use some procedure or knowing the definitions of some concepts are not always enough. Mason and Spence (1999) differentiated between knowing-about the subject and knowing-to act in the moment. They claimed that students do not always appear to know-to use what they have learned and that it is essential to raise students' awareness of their behaviors.

Awareness and expression of one's thinking and behaviors, as well as recognition of mistakes and adaptability contribute to students' success in problem solving (Pappas, Ginsburg, & Jiang, 2003). Schoenfeld (1992), building on Poya's (1945) work of problem solving, pointed to several important aspects of monitoring: the ability to plan, assess progress "on line," act in response to this assessment, and look back. Research has shown that secondary school students, as well as undergraduate students, exhibit few monitoring behaviors during the problem solving process (Jurdak & Shahin, 2001; Lerch, 2004). At the elementary level, Nelissen (1987) reported significant differences in monitoring behaviors between high-achieving and low-achieving students. Preschool children were shown to have little awareness of mistakes and little ability to select appropriate strategies without adult assistance (Pappas, Ginsburg, & Jiang, 2003). All in all, students of different age levels were found to encounter difficulties with monitoring. Yet, since these processes are important, they should be an integral part of mathematics instruction (NCTM, 2000). Being that mathematics is part of the kindergarten curriculum, we should also look for ways to foster monitoring among young children. It has been suggested that, for school-age students, the act of explaining and justifying one's responses may facilitate monitoring (Pape & Smith, 2002). Is this true also for young children? And is this relationship reciprocal? May the act of monitoring provide an impetus for children to justify their responses?

This paper focuses on justification and monitoring among kindergarten children. Specifically we investigate (1) the types of justifications given by young children, (2) the existence of monitoring among young children, and (3) the possible relationship between justification and monitoring among young children.

METHOD

Fourteen preschool classes in low-socioeconomic neighborhoods participated in this study. Each class consisted of approximately 30 pre-kindergarten and kindergarten children between the ages of four and six years old. In this paper

we focus on different types of monitoring and justifying responses given by the kindergarten children (between the ages of five and six years old), to two tasks. These children were expected to enter first grade in the upcoming school year.

Two main focal points of the kindergarten curriculum are number concepts (counting objects, identifying number symbols, and comparing the number of items in different sets) and geometry (identifying different two-dimensional and three-dimensional geometrical shapes). In this paper we describe the children's responses to two tasks. Each child sat with the researcher in a quiet corner of the class. Verbal responses as well as gestures were recorded by the researcher.

Task one: Which has more? Two bunches of nine and 12 bottle caps, respectively, were placed on a table before the child. All the bottle caps were of the same shape and size. Each bunch was placed by the fistful on the table, keeping the caps bunched together, without any set order of placement. The child was asked two questions: (1) Which bunch has more bottle caps? (2) Can you check? The questions which accompanied this task were designed to assess children's ability to estimate amounts as well as their ability to check their estimation. The request for monitoring (Can you check?) came from the researcher. Our aim was to investigate if this request would lead the child to justify his answer and if so, what type of justification would the child give.

Task two: Is this a pentagon? For this task, children were shown six cards, two cards, each with a drawing of a pentagon, and four cards, each with a drawing of a non-pentagon shape. Children were asked two questions: (1) Is this a pentagon? (2) Why? The questions which accompanied this task were designed to assess children's ability to identify a pentagon as well as their ability to use the critical attributes of a pentagon in their justifications. Reasoning based on critical attributes indicates a more mature level of reasoning than merely visualizing the whole shape (van Hiele & van Hiele, 1958). In this activity, the researcher asked for a justification. Our aim was to investigate if the request for a justification would then lead the child to monitor his answer.

Analyzing the results. Students' responses were assessed on two levels. First, the type of justifications given were analyzed according to Tall's (2006) theory of the three mental worlds of mathematics described previously. Second, the types of monitoring behaviors exhibited by the children were analyzed with a focus on the following behaviors: (1) expression of one's thinking, (2) planning, (3) assessing progress "on line", (4) awareness of mistakes, and (5) looking back.

RESULTS

In this section we offer a sample of the justifications and monitoring exhibited by kindergarten children in the tasks described above. Samples were chosen in order to illustrate typical responses as well as to demonstrate the range of justifications and monitoring exhibited by these children.

Task one: Which has more?

We begin by presenting children who offered correct estimations, with valid and invalid justifications. We then present a child who offered an incorrect estimation.

Correct estimations and valid justifications. One of the strategies used to check which bunch had more bottle caps was counting. Counting the number of bottle caps in each separate bunch was considered a valid justification.

C1: (The child counts the bottle caps in each bunch separately.) I told you that I know there are more bottle caps here (pointing to the bunch of 12 caps).

C2: We can count. (The child proceeds to count the bottle caps in each bunch separately and smiles in recognition of her correct estimation.) I was right!

C3: We can count. (The child proceeds to count the bottle caps in each bunch separately.) Here (pointing to the bunch of 12 bottle caps) there are more. Twelve is bigger than nine.

The reasoning exhibited by all three children was embedded in the proceptual-symbolic world. All of the above children took action upon being requested to monitor their estimation and each had a valid procedure used to justify their estimations. C1 and C2 both followed their actions with an assessment of their initial estimations. In other words, an external request for monitoring was followed by a justification, which in turn was followed by monitoring (looking back). Yet the quality of their monitoring had a subtle difference. C1's response, "I told you", hints at the child's response being directed outward, toward the interviewer. C2's smile, along with his response "I was right" was directed inward and hints at the possibility that the outside request for monitoring led to a more introspective form of monitoring. C3 had a method for monitoring his estimation (counting) which was followed by a justification (12 is bigger than nine). This justification indicates that the child has possibly abstracted the bottle caps to numbers and can now compare the number concepts without reference to the physical objects at hand. Both C2 and C3 expressed their thoughts ("We can count") before plunging into actions. Yet, C3 does not look back.

One child was unsure of how to apply the counting procedure:

C4: (The child counts the smaller bunch first, stops, and looks at the second bunch.) Should I continue from here? (C4 considers if he should continue the counting sequence by counting the second bunch starting from 10.) Or should I start from the beginning? (C4 does not wait for an answer but proceeds to count the second bunch of 12 caps correctly, starting from 1 and concluding with 12.) Here (pointing to the bunch of 12) there are more.

C4 is developing his reasoning ability within the proceptual-symbolic world. He knows he ought to use a counting procedure. He monitors his procedure "on line" by stopping mid-way and thinking of how to proceed. C4 is struggling to

connect the procedure with the concept. By monitoring his actions he switches from *doing* mathematics to *thinking* about mathematics.

Not all children responded immediately to the question of which bunch had more bottle caps. Instead, when asked which bunch had more, one child responded, "I need to count." Only after she was told to answer first without counting did she choose the bunch with 12 bottle caps as having more than the other. In other words, this child had a plan which she wished to implement before answering the question.

Other than the counting procedure, children relied on the principle of one-to-one correspondence to compare the amount of bottle caps in each bunch:

C5: (The child lines up each bunch in two separate rows, making sure that each cap touches the next. He then compares the length of each row.) This one is longer.

C5 compared the lengths of the two rows of bottle caps. As the caps were all of the same size and each cap touched the following one, this was a valid method. For C5, the procedure of lining up the bottle caps led to a reflection on the concept of length.

Correct estimations but invalid justifications. Some children estimated correctly which bunch had more but replied with invalid justifications stemming from improper use of the counting procedure.

C6: (The child counts the smaller bunch first, 1...9, and proceeds to count the second bunch, 10...21.) There are 21 bottle caps in this bunch (pointing to bunch of 12 caps).

Unlike C4, who had thought about counting both bunches together but did not, the counting activity of C6 may be considered a rote procedure divorced from conceptual meaning.

An invalid justification sometimes left the child unable to assess the correctness of his estimation:

C7: (The child counts the bunch of 12 bottle caps but does not count the bunch of nine bottle caps.)

Researcher: And how do you know that there are more in this bunch than in the other bunch?

C7: I don't know.

Other children, although correctly estimating which bunch had more, did not respond with justifications based on mathematical procedures or concepts:

Researcher: How do you know which bunch has more?

C8: Because we see.

Researcher: Can you check?

C8: Yes.

Researcher: How?

C8: With the eyes.

This child seems to be reasoning within the conceptual-embodied world instead of choosing an action or procedure. His correct estimation was based solely on his visual perception. The outside call for monitoring did not trigger a switch to an appropriate mathematical procedure.

Incorrect estimation but correct conclusion. The opportunity to monitor one's thinking was noticeable when a wrong estimation was given. For example, one child incorrectly estimated that the bunch of nine bottle caps had more caps than the bunch of 12 bottle caps. When asked to check, he responded:

C9: (The child counts each bunch separately and smiles.) Oh! This bunch (pointing to the 12 bottle caps) has more.

For C9, the external request for monitoring was followed by a valid action and justification, which in turn was followed by the awareness ("oh!") that a mistake was made.

Task two: Is this a pentagon?

Children were shown six different shapes and asked to identify the shapes as pentagons or non-pentagons and to justify their identification. At times, their initial identifications remained unchanged and at times children's final identifications differed from that of their initial identifications. In this section we review typical responses to one pentagon shape and to one non-pentagon shape (see Figure 1).

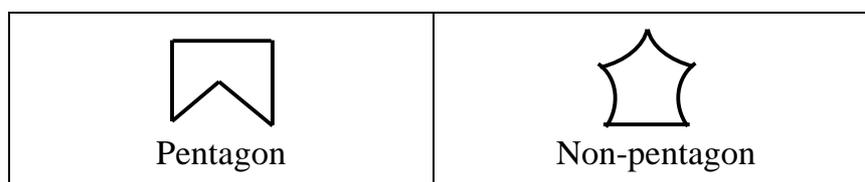


Figure 1: Two shapes presented to children for the pentagon task

Correct initial and final identifications with critical attribute reasoning. Regarding the pentagon, children who identified this shape correctly often justified their identification by referring to critical attributes of the pentagon.

C10: It has five vertices, it's a closed shape, and it has five straight lines.

Regarding the non-pentagon, some children who correctly identified this shape as a non-pentagon referred in their justifications to "crooked" or "rounded" lines. One child justified his correct identification by saying, "It's not (a pentagon) because it has two rounded sides... actually it has four rounded sides... it doesn't matter." This child assessed his justification "on line". At first he noticed two rounded sides. Then he took a closer look and noticed four rounded lines. However, he realized immediately, that in fact it does not matter how many rounded sides the shape has, because even one is sufficient to nullify the shape

as a pentagon. This child exhibited monitoring, not of his solution (which was correct) but of his justification. As he was justifying his conjecture, he monitored the correctness and perhaps quality of his justification.

Regarding both shapes, some children first counted the vertices or sides and only then responded to the question of identification. Such children thought about how to go about identifying the shape, acted on their plan, identified the shape and then justified their identification.

Incorrect initial identification but correct final identification with critical attribute reasoning. Children who corrected their initial incorrect identifications, typically referred to the critical attributes of a pentagon in their justifications.

Regarding the pentagon:

C11: It's not a pentagon. Let's check. (The child counts the vertices.) It is a pentagon because it has five sides and five vertices and it's closed.

C12: It's not a pentagon. The line here points to here (referring to the concaveness of the pentagon). (The child counts the vertices.) It is a pentagon.

C11 immediately went to check his conjecture, even before the researcher had a chance to ask him why he claimed the shape was not a pentagon. In other words, he initiated the monitoring (when he declared "let's check" and counted the vertices) which in turn led to a correct identification based on a correct justification. C12 initially used a justification based on a non-critical attribute (the direction of the line). This justification was followed by monitoring (counting the vertices) which in turn led to a correct identification. Both C11 and C12 exhibit reasoning which integrates both the conceptual-embodied world with the proceptual-symbolic world. They begin by using perceptual reasoning. This reasoning is monitored by using the counting procedure and number concepts of the proceptual world, which ultimately leads back to reasoning based on properties and critical attributes.

Regarding the non-pentagon, one child claimed at first that this shape was a pentagon. When asked why he thought it was a pentagon, he proceeded to count the points and said, "Yes... uh... no. It has five vertices but it's not straight." In this case, justifying the conjecture led to self-initiated monitoring.

Incorrect initial and final identification with critical attribute reasoning. At times, children gave incorrect identifications along with critical attribute reasoning. For example, regarding the pentagon:

C13: It's not a pentagon. It doesn't have five sides. (There was no indication that the child had counted the sides.)

It seems that C13 gave a verbal justification without carrying out any action. Although he gave a justification befitting his (incorrect) identification, the request for justification did not lead this child to monitor his response. He did not look back and was not aware of his mistake.

Unchanging identifications (correct and incorrect) with visual reasoning. Not all children justified their identifications using the critical attributes of a pentagon. Regarding the pentagon:

C14: It's a pentagon because it looks like a pentagon.

C15: It's not a pentagon because it looks like a tooth.

C16: It's not a pentagon because it doesn't have the shape of a pentagon.

Regarding the non-pentagon:

C17: It's not a pentagon because it looks like a circus (tent).

C18: It's not a pentagon because it's not in the shape of a pentagon.

The above children used visual reasoning in their justifications. Within the conceptual-embodied world, their reasoning has not advanced past their perceptions. Both C15 and C17 embodied the rather abstract concept of a pentagon into a more familiar physical entity. C14, C16, and C18 have a mental image of a pentagon which does not fit the shape on the card. These justifications accompanied both correct and incorrect identifications and were not accompanied by monitoring.

Some children gave justifications that were a mix of perceptual reasoning along with reasoning based on attributes. Regarding the non-pentagon:

C19: It's not a pentagon because it has five vertices but it doesn't look like a pentagon.

C19 is a child in transition. Previously, he had correctly identified the pentagon noting only its five vertices. His justification regarding the non-pentagon takes note of the five points (they are not vertices as they do not connect straight lines), but disregards them because the shape "doesn't look like a pentagon." In other words, he realizes that the attribute of "vertices" is worthy of notice but he may not have the knowledge or words to describe that the sides need to be straight lines. Instead, his final justification relies on his visual perception. In a sense, C19 exhibits monitoring. He clearly has a strategy by which he checks if a shape is a pentagon (counting vertices) but "on line" rejects that reason in favor of relying on his mental image of what a pentagon should look like.

DISCUSSION

This paper has shown that young children are able to justify their conjectures by using appropriate mathematical procedures, such as counting, or by reverting back to critical, geometrical attributes. Some children, capable of giving complete mathematical justifications, also exhibited monitoring behaviors. A child who knows a pentagon must have five straight sides as well as five vertices is ultimately better equipped to monitor both his answer, as well as the quality of his justification.

Some of the justifications given by children were based on visual reasoning. For these children, operating at the first van Hiele level of reasoning, a visual

justification is a convincing justification. They "see" with their eyes that one bunch has more than another and either feel no further need to verify their perception or do not have the knowledge to do so. Although we may value and encourage visual estimation, justification and proof are about necessary and sufficient conditions that validate or refute a mathematical assumption. Furthermore, children who base their justifications solely on visual reasoning, claiming that something looks like or does not look like something else, have limited recourse when it comes to monitoring their answers.

Referring back to Schoenfeld (1992), this paper suggests that young children are able to plan a strategy in advance (counting the vertices before identifying the shape), monitor their progress "on line" (change from visual reasoning to reasoning based on critical attributes), as well as act in accordance with this assessment. When encouraged to do so, children are able to express their thinking. This paper has also shown that justification and monitoring may have a reciprocal relationship. A request for monitoring may encourage justification which in turn may encourage further monitoring. At the same time, a request for a justification may encourage the child to monitor his actions, which in turn may improve the justification.

In this paper we presented two tasks which acted as springboards for children to monitor and justify their responses. More research is needed to examine how different tasks, activities, and games, and the questions which accompany them, may be used to promote both monitoring and justification among young children. At this young age, we are interested in children developing a proving attitude (Simpson, 1995), where they value the opportunity to convince themselves and others. This paper focused on the relationship between an individual's monitoring behaviors and justification. We call for more research in the area of monitoring and justifications among young children.

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EARLY YEARS MATHEMATICS – THE CASE OF FRACTIONS

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This paper describes children's understanding of order and equivalence of quantities represented by fractions, and their learning of fraction labels in part-whole and quotient situations. The study involves children aged 6 and 7 years who were not taught about fractions before. Two questions were addressed: (1) How do children understand the order and equivalence of quantities represented by fractions in quotient and part-whole situations? (2) Do children learn fraction labels more easily in one type of situation than another? Quantitative analysis showed that the situations in which the concept of fractions is used affected children's understanding of the quantities represented by fractions; their performance in quotient situations was better than in part-whole situations regarding order, equivalence and labelling.

This paper focuses on the effects of part-whole and quotient situations on children's understanding of the concept of fraction. It explores the impact of each of this type of situation on children's informal knowledge of fractions.

Framework

The Vergnaud's (1997) theory claims that to study and understand how mathematical concepts develop in children's minds through their experience in school and outside school, one must consider a concept as depending on three sets: a set of situations that make the concept useful and meaningful; a set of operational invariants used to deal with these situations; and a set of representations (symbolic, linguistic, graphical, etc.) used to represent invariants, situations and procedures. Following this theory, this paper describes a study on children's informal knowledge of quantities represented by fractions, focused on the effects of situations on children's understanding of the concept of fraction.

Literature distinguishes different classifications of situations that might offer a fruitful analysis of the concept of fractions. Kieren (1988, 1993) distinguished four types of situations – measure (which includes part-whole), quotient, ratio and operator - referred by the author as 'subconstructs' of rational number, considering a construct a collection of various elements of knowing; Behr, Lesh, Post and Silver (1983) distinguished part-whole, decimal, ratio, quotient, operator, and measure as subconstructs of rational number concept; Marshall (1993) distinguished five situations – part-whole, quotient, measures, operator, and ratio – based on the notion of 'schema' characterized as a network of knowledge about an event. More recently, Nunes, Bryant, Pretzlik, Evans, Wade and Bell (2004), based on the meaning of numbers in each situation, distinguished four situations – part-whole, quotient, operator and intensive quantities. In spite of the diversity, part-whole and quotient

situations are distinguished in all these classifications. These situations were selected to be included in the study reported here.

In part-whole situations, the denominator designates the number of parts into which a whole has been cut and the numerator designates the number of parts taken. So, $\frac{2}{4}$ in a part-whole situation means that a whole – for example – a chocolate was divided into four equal parts, and two were taken. In quotient situations, the denominator designates the number of recipients and the numerator designates the number of items being shared. In a quotient situation, $\frac{2}{4}$ means that 2 items – for example, two chocolates – were shared among four people. Furthermore, it should be noted that in quotient situations a fraction can have two meanings: it represents the division and also the amount that each recipient receives, regardless of how the chocolates were cut. For example, the fraction $\frac{2}{4}$ can represent two chocolates shared among four children and also can represent the part that each child receives, even if each of the chocolates was only cut in half each (Mack, 2001; Nunes, Bryant, Pretzlik, Evans, Wade & Bell, 2004). Thus number meanings differ across situations. Therefore, it becomes relevant to know more about the effects of situations on children's understanding of fractions when building on their informal knowledge.

Applying Vergnaud's (1997) theory to the understanding of fractions, one also needs to consider a set of operational invariants that can be used in these situations. It is relevant to know under what condition children understand the relations between numerator, denominator and the quantity. The invariants analysed here are equivalence and ordering of the magnitude of fractions, more specifically, the inverse relation between the quotient and the magnitude.

Thus this study considers a set of situations (quotient, part-whole), a set of operational invariants (equivalence, ordering of fractional quantities), and a set of representations (symbolic, linguistic, pictorial) used to represent invariants, situations and procedures. This study investigates whether the situation in which the concept of fractions is used influences children's performance in problem solving tasks. The study was carried out with first-grade children who had not been taught about fractions in school. Two specific questions were investigated: (1) How do children understand the order and equivalence of fractions in part-whole and quotient situations? (2) Do children learn fraction labels differently in these situations?

Previous research (Correa, Nunes & Bryant, 1998; Kornilaki & Nunes, 2005) on children's understanding of division on sharing situations has shown that children aged 6 and 7 understand that, the larger the number of recipients, the smaller the part that each one receives, being able to order the values of the quotient. However, these studies were carried out with divisions in which the dividend was larger than the divisor. It is necessary to see whether the children will still understand the inverse relation between the divisor and the quotient when the result of the division would be a fraction. The study reported here tries to address these issues focusing on the qualitative understanding of this inverse relation. The equivalent insight using part-

whole situations – the larger the number of parts into which a whole was cut, the smaller the size of the parts (Behr, Wachsmuth, Post & Lesh, 1984) – has not been documented in children of these age. Regarding equivalence in quotient situations, Empson (1999) found some evidence for children’s use of ratios with concrete materials when children aged 6 and 7 years solved equivalence problems. In part-whole situations, Piaget, Inhelder and Szeminska (1960) found that children of this age level understand equivalence between the sum of all the parts and the whole and some of the slightly older children could understand the equivalence between parts, $1/2$ and $2/4$, if $2/4$ was obtained by subdividing $1/2$.

In a previous study, Mamede and Nunes (2008) compared the performance of 6 and 7 year-olds children when solving equivalence and ordering problems of quantities represented by fractions after being taught fraction labels in quotient, part-whole and operator situations. They found out that children who worked in quotient situations could succeed in some equivalence and ordering problems, but those who worked in part-whole and operator situations did not, despite all of them succeeded in labelling fractions. This shows that children are able to learn fraction labels without understanding the logic of fractions. The results of this study suggested that quotient situations were more suitable than the others when building on children’s informal knowledge. Nevertheless, more research is needed regarding these issues.

Research about the impact of each of the situations in which fractions are used on the learning of fractions is difficult to find. Although some research has dealt with these situations with young children, these were not conceived to establish systematic and controlled comparisons between the situations. We still do not know much about the effects of each of these situations on children’s understanding of fractions. Nevertheless, if we find out that there is a type of situation in which fractions make more sense for children, it would be a relevant finding to introduce fractions to them in the school. There have been no detailed comparisons between part-whole and quotient situations documented in research on children’s understanding of fractions. This paper provides of such evidence.

METHOD

Participants

Portuguese first-grade children (N=80), aged 6 and 7 years, from the city of Braga, in Portugal, were assigned randomly to work in part-whole or quotient situations with the restriction that the same number of children in each level was assigned to each condition in each of the two schools involved in this study.

The children had not been taught about fractions in school, although the words ‘metade’ (half) and ‘um-quarto’ (a quarter) may have been familiar in other social settings.

The tasks

An example of a problem of equivalence and ordering presented to the children is given below on Tables 1 and 2.

Problems of equivalence of quantities represented by fractions	
Quotient situations	Part-whole situations
Two girls have to share 1 bar of chocolate fairly; 4 boys have to share 2 chocolates fairly. Does each girl eat the same, more, or less than each boy? Why do you think so?	Peter and Emma each have a bar of chocolate of the same size; Peter breaks his bar in 2 equal parts and eats 1 of them; Emma breaks hers into 4 equal parts and eats 2 of them. Does Peter eat more, the same, or less than Emma? Why do you think so?

Table 1: A problem of equivalence presented to the children in each type of situation.

Problems of ordering of quantities represented by fractions	
Quotient situations	Part-whole situations
Two boys have to share 1 bar of chocolate fairly; 3 girls have to share 1 chocolate bar fairly. Does each girl eat the same, more, or less than each boy? Why do you think so?	Bill and Ann each have a bar of chocolate of the same size; Bill breaks his bar into 2 equal parts and eats 1 of them; Ann breaks hers into 3 equal parts and eats 1 of them. Who eats more, Bill or Ann? Why do you think so?

Table 2: A problem of order presented to the children in each type of situation.

Regarding the labelling problems, there were two types: the ‘what fraction?’ problems, in which the child was asked to write the fractions that would represent the quantity; and the ‘inverse’ problem in which the fraction was given and the child was asked to identify the meaning of the numerator and denominator. An example of each type of labelling problems presented to the children is given below on Table 3.

Problem	Situation	Example
What fraction?	Part-whole	Paul is going to cut his chocolate bar into 4 equal parts and eats 3 of them. What fraction of the chocolate bar is Paul going to eat? Write the fraction in the box.
	Quotient	Three chocolate bars are going to be shared fairly among 4 friends. What fraction of chocolate does each

		friend eat? Write the fraction in the box.
Inverse	Part-whole	Anna divided her chocolate bar and ate $\frac{3}{5}$ of it. Can you draw the chocolate bar and show how she did it?
	Quotient	Some children will share some chocolate bars. Each child gets $\frac{3}{5}$ of the chocolate. How many children do you think there are? How many chocolates? Can you draw the children and the chocolates?

Table 3: An example of each type of labelling problems presented to the children in each type of situation.

Problems presented in part-whole situations were significantly longer than those presented in quotient situations. To reduce this effect, the interviewer made sure that each child understood the posed problem. All the problems were presented orally by the means of a story, with the support of computer slides. The children worked on booklets which contained drawings that illustrated the situations described. No concrete material was involved.

Design

At the beginning of the session, the six equivalence items and the six ordering items were presented in a block in random order. The children were seen individually by the experimenter. In the second part of the session, the children were taught how to label fractions with the unitary fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{5}$ and the non-unitary fraction $\frac{2}{3}$, in this order. After that, they were asked to solve three ‘what fraction?’ problems and one ‘inverse’ problem. All the numerical values were controlled for across situations.

RESULTS

Descriptive statistics for the performances on the tasks on quotient and part-whole situation are presented in Table 4.

Tasks	Problem Situation			
	Quotient (N = 40; mean age 6.9 years)		Part-whole (N = 40; mean age 6.9 years)	
	6 years	7 years	6 years	7 years
Equivalence	2.1(1.5)	2.95 (1.54)	0.6 (0.7)	0.6 (0.5)
Ordering	3.3 (2.1)	4.25 (1.3)	1.45 (1.4)	1.2 (0.83)

Table 4: Mean (out of 6) and standard deviation (in brackets) of children’s correct responses by task and situation.

A three-way mixed-model ANOVA was conducted to analyse the effects of age (6- and 7-year-olds) and problem solving situation (quotient *vs* part-whole) as between-participants factor, and tasks (Equivalence, Ordering) as within-participants factor.

There was a significant tasks effect, ($F(1,76)=18.54$, $p<.001$), indicating that children's performance on ordering tasks was better than in equivalence tasks. There was a significant main effect of the problem situation, ($F(1,76)=146.26$, $p<.001$), and a significant main effect of age, ($F(1,76)=4.84$, $p<.05$); there was a significant interaction of age by problem solving situation, ($F(1,76)=7.56$, $p<.05$). The older children performed better than the younger ones in quotient situations; in part-whole situations there was no age effect. There were no other significant effects.

An analysis of children's arguments was carried out and took into account all the productions, including drawings and verbalizations.

Based on the classifications of children's arguments when solving sharing problems (see Kornilaki & Nunes, 2005) and when solving equivalence problems in quotient situations (see Nunes et al., 2004), five types of arguments were distinguished attending to children's justifications solving equivalence and ordering problems in quotient situations, which were: a) invalid, comprising arguments that are not related to the problem; b) perceptual comparisons, the judgements are sustained on perceptual comparisons based on partitioning; c) valid argument, based on the inverse relation between the number of recipients and the size of the shares; d) only to the dividend (or numerator), based on the number of items to share and the shares, ignoring the inverse relation between the recipients and the shares; e) only to the divisor (or denominator), based on number of recipients and the shares, ignoring the number of items being shared.

Based on a classification of children's arguments on equivalence and ordering problems of fractions (see Behr et al., 1984), four arguments were distinguished also from children's justifications when solving equivalence and ordering problems, in part-whole situations. These four arguments were: a) invalid, comprising arguments that are not related to the problem; b) valid argument, based on the inverse relation between the number of parts into which the whole was cut and the number of parts eaten/taken, attending to the size of the shares; c) only to the dividend (or numerator), based on the number of parts eaten/taken, ignoring their sizes and the number of parts into which the whole was cut; d) only to the divisor (or denominator), based on the number of equal parts into which the whole was divide, ignoring their sizes and the number of parts eaten/taken.

Table 5 shows the children's arguments when solving equivalence and ordering problems and the rate of correct responses for problems in quotient and part-whole situations.

Children presented more valid arguments based on the inverse relation between the number of recipients and the size of the shares, when solving problems in quotient

situations. In part-whole situations, the valid arguments were based on the inverse relation between the number of parts into which the whole was cut and the number of parts eaten/taken. In part-whole situations the most frequent arguments used were based on the number of parts eaten/taken, ignoring their sizes and the number of parts into which the whole was cut.

Type of argument	Type of situation			
	Quotient (N=240)		Part-whole (N=240)	
	Equiv.	Order	Equiv.	Order
Invalid	0	.01	.01	.02
Perceptual comparisons	.03	.09	-	-
Valid	.27	.38	.03	.06
Only to the dividend (numerator)	.09	.14	.18	.13
Only to the divisor (denominator)	.03	.01	.05	.01

Table 5: Type of argument and proportion of correct responses when solving the tasks in quotient and part-whole situations.

These results show that, when solving ordering problems in quotient situations, almost 40% of the responses were correct and justified with an explanation attending to the numerator, denominator and the quantity. This was not achieved when solving the correspondent problems in part-whole situations.

Also the fraction labels were analysed for each condition of study. Descriptive statistics for the performances on the labelling problems on quotient and part-whole situation are presented in Table 6.

Tasks	Problem Situation			
	Quotient (N = 40; mean age 6.9 years)		Part-whole (N = 40; mean age 6.9 years)	
	6 years	7 years	6 years	7 years
Labelling	3.5(1.1)	3.5 (0.95)	2.3 (0.92)	2.4 (1.1)

Table 6: Mean (out of 4) and standard deviation (in brackets) of children's correct responses by task and situation.

In order to analyse the effect of situation on children's learning to label fractions, a two-factor ANOVA was conducted to analyse the effects of age (6- and 7-year-olds) and situation (quotient vs part-whole) as the main factors.

There was a significant main effect of situation, ($F(1,76)=25.45, p<.001$): children learned fractions labels more easily in quotient situations than in part-whole

situations. There was no significant age effect and no interactions. Thus it can be concluded that the children learned to label fractions more easily in quotient situations than in part-whole situations and that is not dependent on age.

Figures 1 and 2 show examples of children's drawings when solving the inverse problems in quotient and part-whole situations, respectively. Some incorrect solutions will be shown and discussed in presentation.



Figure 1: Children's solution of the inverse problem in quotient situation.



Figure 2: Children's solution of the inverse problem in part-whole situation.

These children were not taught about any strategies to solve the problems. In spite of succeeding in labelling problems in quotient and part-whole situations, only 30% of those who solved the inverse problem in part-whole situations drew the correct number of cuts and the correct number of parts taken. When dividing the chocolate bar, 37.5% of the children counted the number of cuts instead of the number of parts, ending up with the incorrect number of parts into which the whole was divided; 20% of the children drew incorrect number of cuts and incorrect number of parts taken, and 12.5% of the children could not to solve the problem. This contrasts with the 92.5% of children who successful solved the inverse problem in quotient situation, drawing the correct number of chocolates and the correct numbers of children; 2.5% drew the incorrect number of children but the correct number of chocolates, and 5% did not solve the problem.

DISCUSSION AND CONCLUSION

Children's ability to solve problems of equivalence and ordering of quantities represented by fractions is better in quotient than in part-whole situations. Children's arguments when solving these problems reveal that quotient situations are easier for the child to understand the relations between the numerator, denominator and the quantity. The levels of success on children's performance in quotient situations, supports the idea that children have some informal knowledge about equivalence and ordering of quantities represented by fractions. These results extend those obtained

by Kornilaki and Nunes (2005), who showed that children aged 6 and 7 years succeeded on ordering problems, in sharing situations, where the dividend was larger than the divisor. The results presented here showed that the children still be able to use the same inverse reasoning when dealing with quantities represented by fractions. The findings of this study also extended those of Empson (1999) who showed that 6-7-year-olds children could solve equivalence and ordering problems in quotient situations, after being taught about equal sharing strategies. The children of this study were not taught about any strategies.

Regarding the labelling of fractions, the children's performance in both situations reveals that quotient situations are easier for children to master fraction labels, understanding the meaning of the numbers involved, than part-whole situations. In part-whole situations, the majority of the children also succeeded in labelling problems and understood the meaning of the numbers involved clearly enough to identify them in a new situation. These results converge with those found by Mamede and Nunes (2008) who showed that children of 6-7-year-olds could successful learn fractions labels in quotient and part-whole situations, understanding the meaning of the numbers involved, without being able to solve equivalence and ordering problems in these situations, having difficulties in understanding the relations between the numerator, denominator and the quantity.

In spite of succeeding in labelling fractions in both situations, the learning to label fractions in quotient and in part-whole situations seems to involve different types of difficulties for the children. Whereas in quotient situations the values involved in the fractions could easily be represented by drawing, as they refer to different variables – number of recipients and number of items being shared-, in part-whole situations, as both variables refer to parts, partitioning (division of a whole into equal parts) may play an important role for some children in this task.

This study shows that part-whole and quotient situations affect differently children's understanding of fractions. These results suggest that quotient situations should be explored in the classroom in the first years of school. Nevertheless, more research is needed providing a deeper insight on the effects of situations in which fractions are used on children's understanding of fractions.

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ONLY TWO MORE SLEEPS UNTIL THE SCHOOL HOLIDAYS: REFERRING TO QUANTITIES OF THINGS AT HOME

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Children bring a wealth of mathematical knowledge from home to school but sometimes this knowledge may not be utilised in the most appropriate way. In this paper, one six/seven year old girl's home interactions over 20 weeks about measurable quantities are presented. It would seem that most of the interactions used terms to compare discrete amounts with an undiscussed norm, with only a few interactions involving units of measurement. There were no references to reading a scale, except in regard to time. Time was discussed in far greater detail than any other attribute. Although time is considered to be difficult to learn because of its abstract nature, it may in fact be an easier concept to start with when introducing the sense of how units of a quality are related to each other.

THE INTERCHANGE OF HOME AND SCHOOL MATHEMATICAL KNOWLEDGE

Many children arrive at school with significant mathematical understandings (Clemson & Clemson, 1994). However, the challenge is how to build on “this rich base of mathematical experiences in ways that acknowledge and support the family’s role” (Clarke & Robbins, 2004). In order to do this, we need to understand how mathematics is used in the home and how these experiences change as children become older. In this paper, I examine a six/seven year old child’s interactions at home around measurement ideas over the course of twenty weeks. Although she had been at school for two years, there was still frequent communication between home and school. For this child, amounts of different qualities were discussed in different ways. Discussions of time were some of the few occasions where units were used and the only occasions where units were compared and contrasted. Yet the unit concept is often considered something that should be taught in regard to other measurement attributes such as length, before introducing time units (NZ Ministry of Education, 2007). Consequently, there is a need to query assumptions about how to introduce measurement units that build on children’s home experiences.

Most research into mathematical practices at home has concentrated on young children, generally preschoolers, and number concepts (Vandermaas-Peeler, 2008; Gifford, 2004; Clarke & Robbins, 2004). Once children start school, although the influence of home activities is still acknowledged as being important, less is known about the types of activities done and how they could connect into formal school mathematics development.

Socio-cultural approaches about acquiring mathematical understanding at home are now seen as adding useful background to how children become mathematically

competent (Benigno & Ellis, 2008). Using socio-cultural ideas, Street, Baker and Tomlin (2005) developed the *ideological model of numeracy* so that they could better describe why there might be differences between home and school numeracy practices. Table 1 describes the four inter-related dimensions of the model.

Dimensions	Description
Content	The mathematical concepts, such as measurement.
Context	The situation in which a numeracy practice takes place.
Values and Beliefs	The participants beliefs about how numeracy practices should progress and how new skills and knowledge are taught within them.
Social and Institutional Relations	The overarching factors that channel what are seen as appropriate choices in the other three dimensions.

Table 1: Dimensions from the ideological model of numeracy (Street et al., 2005)

This model is useful as an analytical tool as it provides insights into whether a simple transfer of mathematical practices can occur between home and school, or whether explicit discussions about differences between home and school need to occur. For example, in an earlier paper, I discussed how the child seemed to have more control in her interactions at home than she did at school (Meaney, 2008). This may have been because different power relations exist in the home situation compared to those between a student and their teacher and even between mother and child in a school setting. The interactions discussed in that paper also showed how the power relations interacted with the values and beliefs of the participants about how mathematical practices should be conducted. Therefore, the dimensions of the model can provide useful insights into why differences occur and the sorts of discussions that are needed if home mathematical practices are to be acknowledged in school.

Although the influence of context, values and beliefs and social and institutional relations is reasonably well known (Benigno & Ellis, 2008), the influence of content is not so clear. Measurement concepts have not received any specific attention when considering mathematical practices in the home. This is despite the fact that there have been recent calls in Hawai'i to redesign the early years school mathematics curriculum so that it focuses on measurement ideas before introducing number (Dougherty, 2003). Although some measurement concepts do appear in the data of some projects (Clarke & Robbins, 2004 for example), these are not discussed explicitly in regard to the implications for formal school mathematics teaching. It may well be that as a consequence, teachers teach about measurement presuming that students have had certain experience at home, whilst at the same time ignoring the experiences that students may actually have. Therefore, exploring the measurement concepts used at home is a rich area for investigation.

METHODOLOGY

Research about home mathematical practices has tended to rely on parents' nominated examples (Blevins-Knabe & Musun-Miller, 1996) and to some degree on them documenting them through diaries or photos (Clarke & Robbins, 2004). These methods have raised concerns about parents' ability to recognise mathematical interactions (Bottle, 1999). In some cases, parents and children have been recorded in laboratory situations where they have been provided with toys and other props (Vandermaas-Peeler, 2008). This non-home setting may well have affected the data that was collected. Bottle (1999) used a video camera to film interactions as they happened in the home and felt that it allowed for more comprehensive data to be collected. She visited each family for approximately two hours every four months. However, she also acknowledged that the intrusive nature of the researcher's videoing activities may have influenced the activities that were recorded.

For this research, it was decided to audio tape the interactions of a six/seven year old child in order to investigate how she acquired the mathematics register at home and at school. Given the amount of recording that was done, video recording would not have been logistically possible. Although only one child was recorded, this was done consistently over half a year and produced an enormous amount of data.

The child was recorded for one day a week, for twenty weeks, in the second half of 2005. From when she woke in the morning until she went to school, the research child wore a lapel microphone connected to a digital voice recorder. During her mathematics lesson, she was again recorded and the class discussion captured on another voice recorder connected to a conference microphone. After she was collected from school, the child wore the voice recorder until she went to bed. The child's parents are Samoan speakers but English was the primary language spoken at home. The mother was the research assistant for this study and organised recording the child's interactions. Her mother listened to all of the recordings and sent to a transcriber those she believed were worth transcribing.

The mother's awareness of the purpose of the project could have influenced the types of activities done at home. However, most of the time the child seemed unaware of the microphone and that she was being recorded. Therefore, although the set of transcripts may not be a true representation of the mathematics interactions that occurred, they are a rich alternative source of data to that collected by other methods.

TALKING ABOUT AMOUNTS

In the transcripts, more interactions made reference to size or amounts of things than to number. The attributes discussed included height, depth, volume, space, mass, heat, speed, tightness, strength, loudness, and amount. However, these quantity references are not easily connected to what Buys and de Moor (2008) described as the "basic pattern of the learning-teaching trajectory" (p. 23) for measurement. This trajectory includes three stages:

- measuring through comparing and ordering
- measuring through pacing off using a measurement unit
- measuring through reading off with the help of a measuring instrument (p. 25)

Many of the interactions used measurement terms as specific amounts “big girl/little girl” (Week 3) where an implicit comparison was made to an undiscussed norm. This does suggest an order, but no examples of explicit ordering occurred in the transcripts. There were also no instances of comparisons between items using expressions, such as “bigger than” or “more than”. What was evident was that measurement terms often appeared in relationship to actions such as “turn the volume down” (Week 2). In the transcript from Week 3, a connection is made about the research child’s brother being too tall to walk under a table.

Mother: Oh come here, ah you bumped your head. Oh dear, oh dear. Did you see he bumped his head? Watch where you’re going. You’re tall, see you’re too tall to walk under that.

Research Child: Then he went on the ground, he went like this, mum.

Mother: Oh, he fell down. He used to be able to just walk under it because he was short but now

This extract shows that a comparison is made between the height of the table and the toddler, but the emphasis seems to be more on *walking under* than on the differences in height between the child and the table.

Sometimes, some of the terms suggested that there was a continuum of amounts; often this came through the addition of “bit” to an expression such as in “a bit chilly”. The following extract comes from Week 8 where the discussion is about how something’s mass could result in a cushion popping. Different animals are discussed, showing a sense of ordering the animals according to their varying masses. However, there is no explicit discussion of what is being compared and therefore no actual ordering of the animals. The lines indicate where speech was not clear enough to be transcribed.

Mother: I thought the one [activity] that you jump on the blue cushion would’ve been fun.

Research Child: Too bad you’re not a child.

Mother: ___ blue cushion.

Research Child: ‘Cause then you’ll pop it. [Mum laughs]

Mother: I’m not that heavy, it’s a big cushion. ___ after would pop it, not me, I’m not fat.

Research Child: ___ .

Mother: Who do you think? Maybe someone as big as a whale.

Research Child: A whale would really pop it.

Mother: If a whale jumped on it, it would definitely pop.

Research Child: And we'd all get hurt.

Mother: If an elephant jumped on it, it might pop.

Research Child: Then we might all get hurt.

Mother: What other animal do you think might pop it?

Research Child: Giraffes wouldn't. What about antelope?

Occasionally, units were used to describe the amount of something. Generally, these were whole units, "two, three big teaspoons" (Week 18) that could not be broken down into smaller units, even when discussing the unit of a half. The following extract comes from Week 6

Mother: If you're hungry you can have one of the mandarins.

Research Child: Then can I have a scone, half?

Mother: ___ half.

Research Child: Half is the same, half is a half.

Time

The exception in the interactions was in discussions about time. Of all the attributes, time was talked about more often and for longer periods. The discussions were around all three stages outlined by Buys and de Moor (2008). In regard to comparing and ordering, there were also examples involving an implicit comparison. For example in the Week 5 transcripts, the mother wants to go out.

Mother: What time does that program finish? Does it take long?

Research Child: No, not very long.

Mother: Good.

Although there were still no discussions about activities taking longer or shorter than other activities, there were occasions when the time taken for certain activities was discussed. The following comes from Week 7.

Mother: Alright, you do need to think Research Child, to stop us from being late all the time, what time do you think you should get up in the morning?

Research Child: 6 o'clock.

Mother: (Amazed and unbelieving sound) Six, but you don't have to be at school until 9? Wouldn't that be too early?

Research Child: Don't worry, just stay there until it opens.

Mother: That's three hours before 9 o'clock, it's too early.

Research Child: How about 7?

Mother: That's not too bad. How long does it take you to get ready, like, get your clothes on and brush your teeth?

Research Child: Well I'm not sure about 7 o'clock, 'cause that's the time when you get ready, and 8 o'clock was when it's only two things we do.

Mother: What?

Research Child: Just all we have to do is, you know, you do my hair and do my face.

Mother: What about breakfast?

Research Child: Yeah, we'd, it'd, um, 7 o'clock we do breakfast.

Mother: You don't eat breakfast until you're dressed.

Research Child: Yeah, then, dressed, break., I mean, brush your teeth, breakfast, ____ and then do my hair, face, yeah. Is that, is there anything else?

Mother: Shoes?

Research Child: Do my shoes up.

Mother: Pack your bag.

Research Child: Pack my bag and then go.

Mother: Alright, so then what time do you get up in the morning?

Research Child: Still 7 o'clock.

Mother: 7 o'clock. Are you sure you can do that?

Research Child: I'm not sure.

Mother: (laughs) You can try. Well if you can't, 7.30 is alright.

Research Child: Yeah, 7.30.

Mother: 'Cause it's not too early.

Research Child: Let's go at 7.30.

Mother: No that's when you wake up. Wake up at 7 or 7.30? I think 7.30 is realistic, 'cause we used to do that, and by the time it's 8.30 you'll just be eating and ready to go, and you would have finished eating.

There were several discussions around specific units of time – minutes, hours, days, weeks, months, seasons and years. Whilst watching television, during week 9, the Research Child says to herself “Only two more sleeps until school holidays”. She used units of time, ‘sleeps’, to think about an upcoming event.

Over the course of the twenty weeks, the mother began teaching her daughter how to read both an analogue clock and a digital clock. By the end of the year, the child had just about mastered being able to read an analogue clock. The following extract comes from Week 13.

Mother: Research Child, come and see what time it is by looking at the clock.

Research Child: Something to 9.

Mother: Good girl. How many minutes? Can you count?

Research Child: Mmm. Oh wait. Can I have it down because I can't see it properly.

Mother: You only ___ __ under 12. How many dots are in between that little space?

Research Child: 5?

Mother: Yeah – good! Now what does that tell you? 5 what. What does that mean?

Research Child: 5 to 9.

Mother: Good girl. 5 what to 9? 5 hours? 5...

Research Child: Minutes?

Mother: Good girl. 5 minutes to 9. Because what happens when the big hand gets to the 12?

Research Child: It means that it's 9 o'clock.

Mother: Good girl. See – you're learning fast. If the long hand was on the 1, it would be... and the little hand ___ ___.

Research Child: It would be 1 past 9.

Mother: Are you sure it would be 1 past 9? How many minutes is the gap?

Research Child: Oh no. That gap is... 5?

Mother: Yeah.

Research Child: 5 past 9.

Mother: What if the long hand was on the 2?

Research Child: It would be 10 past 9.

Mother: Good – and what if it was on the 3?

Research Child: Yeah, but 15 isn't on it.

Mother: No – you can't see 15, but each gap remember is 5. So it's like 5, 10, 15...

Research Child: Oh, so it does count 15?

Mother: Yeah!

Research Child: Oh. Is it 15 past 9?

From interrogating the data, it was clear that discussions about measurement were frequent with a range of different attributes. Although there were references to units, these were few and there were no references to reading measurements from a scale. Time was the major exception to this. It was discussed more often than any other

attributes and the way it was discussed included all three of the stages suggested by Buys and de Moor (2008) in their learning trajectory.

DISCUSSION

Buys and de Moor (2008) suggested that length is the most primary of physical quantities to measure. This is because “[n]ot only is it available to children’s perception, it is the most indicative quantity people want to find out about all sorts of objects” (p. 18). Time on the other hand is considered more abstract where the children need to develop a sense of time before they could learn to tell the time. It was therefore extremely interesting to find that in the twenty days of home discussions that time was much more prominent than length.

Street et al.’s (2005) ideological model of numeracy can provide insights into why time has such a prominent role in these home interactions. The *social and institutional relations* seem not to be different regardless of the content of the conversation. However, what is discussed at home is influenced by perceptions of what is “normal” to discuss in the home. The mother clearly believes that it is at home where the child should learn about time. Given the child’s facility with number and counting (as seen in Meaney 2008), this may no longer be considered something that needs as much attention at home. The other social and institutional relation that impacts on why time has become important is that the research child is constantly late for school which has implications for the child and her family and how they are perceived by the teacher and the school more generally. In order to continue being seen as a good family who supports their child’s education, attempts were made to improve the situation, such as the discussion from Week 7. For the child to take some role in ensuring she meets the expectation that she will arrive before the first bell, she needs to be able to read a clock and speed up her activities appropriately.

Having accepted the need for the child to learn about time and specifically how to read a clock, the mother makes some unconscious decisions about how to introduce it so that the child acquires the necessary knowledge. Although other units of time are used more generally, such as “sleeps” for example, the mother used a “school-like” discourse to teach her daughter how to read a clock. Given that the mother has experience as a teacher, albeit a secondary English teacher, then using formal instructions may well be something she can draw upon. However, it is interesting that reading a clock face is the only occasion where she chooses to use such skills.

In discussions that involve references to attributes, it is clear that context of being at home results in an emphasis on the actions related to the attributes. Even in the discussions about time, the context relates to actions – decorations come down because the child’s birthday is over but will go up again for Christmas. This is likely to be different to school where comparing items, such as the size of feet, is done for the sake of the comparison, not because it is related to another action. Context therefore does have an impact on how the activity is framed (Benigno & Ellis, 2008).

There are some interactions where an implicit comparison about time is made in the same way as there was in the discussions about other attributes. However, the nature of time means that it is actually difficult to discuss it without referring to specific units – years, months, weeks, days, hours, and minutes. Getting a sense of time (Buys & de Moor, 2008), actually means becoming familiar with units of time and how they are related. Content does interact with context, values and beliefs and social and institutional relations. This was the case for all the measurable attributes, but in the transcripts was particularly so for time.

USING HOME MATHEMATICAL PRACTICES IN SCHOOL

These transcripts come from interactions with one child over the course of 20 weeks and are not representative of what may occur in other households. However, these do raise questions about how to make use of home mathematical practices in school.

The transcripts suggest that a belief that length is the primary physical quality may not in fact match what children experience in their home situations where discussing time, in one form or other, is something that is discussed regularly. For this child, time was given prominence, probably because her continual late arrival at school meant that she and her family were not meeting societal expectations. For other children, it may be different circumstances that affect what measurement attribute is given prominence. Also for this child, interactions around measurable attributes were connected to actions. Schools need to talk with their students' families to find out whether measurement is connected to action in their homes so that teachers can take this into consideration when designing their teaching programmes.

For this child, there were many interactions that discussed the relationship between different units of time. If this is also the case for other children, this may provide a better context for introducing formal units than the more common one of length.

The data from this research suggest that home measurement practices cannot be taken for granted but instead must be investigated further. This will allow for greater discussions between families and teachers in which the school may better learn how to make use of the mathematical experiences that children have had at home.

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SUPPORTING CHILDREN POTENTIALLY AT RISK IN LEARNING MATHEMATICS – FINDINGS OF AN EARLY INTERVENTION STUDY

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Recent psychological studies as well as research findings in mathematics education highlight the significance of number skills for the child's performance in mathematics at the end of primary school. In this context, the three year longitudinal study (2005-2008) involving years K – 2 that provided the background of this paper seeks to investigate the influence of intervention based on number skills prior to school on children's later achievement in primary school mathematics. Following an overview of the theoretical background and the design of the study, quantitative findings from the first year of the study regarding the mathematical achievements of children potentially at risk learning school mathematics one year and immediately prior to them starting school will be presented and discussed.

BACKGROUND AND FOCUS OF THE PAPER

Children start to develop mathematical knowledge and abilities a long time before they start formal education (e.g. see Anderson, Anderson, & Thaubergger 2008; Ginsburg, Inoue, & Seo, 1999). In their play and their everyday life experiences at home and in child care centres they develop a base of skills, concepts and understandings about numbers and mathematics (Baroody & Wilkins, 1999). Anderson et al. (2008) recently reviewing international studies on preschool children's development and knowledge conclude that research

(...) points to young children's strong capacity to deal with number knowledge prior to school, thus diminishing the value of the conventional practice that pre-number activities are more appropriate for this age group upon school entry. (p. 102)

However, the range of mathematical competencies which children develop prior to school obviously varies quite substantially. While most preschoolers manage to develop a wide range of informal knowledge and skills in early numeracy, there is a small number of children who for various reasons struggle with the acquisition of knowledge about numbers (e.g. see Clarke, Clarke, Grüßing, Peter-Koop 2008). Furthermore, recent clinical psychological studies suggest that children most likely to develop learning difficulties in mathematics can already be identified one year prior to school entry by assessing their number concept development (Krajewski 2005; Aunola, Leskinen, Lerkkanen, & Nurmi, 2004). Findings from these studies also indicate that these children benefit from an early intervention prior to school helping them to develop a base of knowledge and skills for successful school-based mathematics learning. This seems to be of crucial importance as findings from the

SCHOLASTIK project (Weinert & Helmke, 1997) suggest that students who are low achieving in mathematics at the beginning of primary school in general tend to stay in this position. In most cases, a recovery does not occur. In addition, Stern (1997) emphasises that subject-specific previous knowledge is more important with respect to success at school than general cognitive factors such as intelligence. Thus, the study reported in this paper aims to investigate how children potentially at risk in learning school mathematics can be identified one year prior to them starting school and compares the effects of early intervention on one-on-one basis carried out by student teachers with that of small group interventions

DEVELOPMENT OF NUMBER CONCEPT

While pre-number activities based on Piaget's *logical foundations model* are frequently still current practice in the first year of school mathematics (Anderson et al. 2008), research findings as well as curriculum documents increasingly stress the importance of students' early engagement with sets, numbers and counting activities for their number concept development. Clements (1984) classified alternative models for number concept development that deliberately include early counting skills (Resnick, 1983) as *skills integrations models*.

Piaget (1952) assumed that the development of number concept is based on logical operations based on pre-number activities such as classification, seriation and number conservation and emphasised that the understanding of number is dependent on operational competencies. In his view, counting exercises do not have operational value and hence no conducive effect on conceptual competence regarding number.

However, since the late 1970s this theory has been questioned due to research evidence suggesting that the development of number skills and concepts results from the integration of number skills such as counting, subitizing and comparing. Studies by Fuson, Secada, & Hall (1983) and Sophian (1995) for example demonstrate that children performing on conservation tasks who compare sets by counting or using a visual correspondence are highly successful. Clements (1984) investigated the effects of two training sequences on the development of logical operations and number. Two groups of four-year-olds were trained for eight weeks on either logical foundations focussing on classification and seriation or number skills based on counting. A third group with no training input served as a control group. Instruments measuring logical operations and number abilities were designed as pre- and post-test measures. It is not surprising that both experimental groups significantly outperformed the control group in both tests, however, the children that were trained on number skills significantly outperformed the logical foundations group on the number test while there was no significant differences between these two groups on the logical operations test. Clements' results comply with and extend previous research that had indicated that number skills such as counting and subitizing affect the development of number conservation (Fuson, Secada, & Hall, 1983; Acredolo, 1982). Hence, he concludes:

(...) the counting act may provide the structure and/or representational tool with which to construct logical operations including classification and seriation, as well as number conservation. ... Not only may explicit readiness training in logical operations be unnecessary, but well structured training in counting may facilitate the growth of these abilities as well as underlie the learning of other mature number concepts. (Clements, 1984, 774-775)

An early training based on number abilities such as counting, comparing and subitizing may be especially important for children who are likely to develop mathematical learning difficulties. The longitudinal intervention study reported in this paper investigates the identification and subsequent enhancement of preschool children potentially at risk learning school mathematics prior to their first year at school.

METHODOLOGY

Based on current research findings reported in the previous section, the longitudinal study (2005 – 2008) that provides the background for this paper seeks

- to investigate young children's mathematical understanding in the transition from Kindergarten to primary school,
- to evaluate appropriate assessment instruments, and
- to explore how children potentially at risk learning school mathematics can be supported effectively in terms of their number concept development in early childhood education.

This paper focuses on the third aspect – exploring the effectiveness of early intervention based on the following two underlying research questions:

1. What are the effects of an eight months intervention program aimed at the development of number abilities for kindergarten children (five-year-olds) identified to be potentially at risk learning school mathematics upon school entry?
2. In how far has the early intervention a lasting effect with respect to their achievement in mathematics at the end of grade 1 and grade 2?

In this paper however, due to space restrictions only the first of the two research questions will be addressed by comparing the performance of the children potentially at risk learning mathematics from two groups before and after an eight months intervention prior to school entry.

Overall, 1020 five-year-old preschoolers from 35 kindergartens (17 in urban, 18 in rather rural regions) in the northwest of Germany took part in the first year of the study (September 2005 – August 2006). With the permission of their parents these children performed on three different tests/interviews conducted at three different days within a fortnight by preservice mathematics teachers from Oldenburg University who had been especially trained for their participation in the study:

- the German version of the *Utrecht Early Numeracy Test* (OTZ; van Luit, van de Rijt, & Hasemann, 2001) – a standardized test aiming to measure children's development of number concept conducted in small groups involving logical operations based tasks as well as counting related items,
- the *First Year at School Mathematics Interview* (FYSMI) [1] developed in the context of the Australian *Early Numeracy Research Project* (Clarke, Clarke, & Cheeseman, 2006) – a task-based one-on-one interview aiming at five-year-olds which allows children to articulate their developing mathematical understanding through the use of specific materials provided for each task,
- the *Culture Fair Test* (CFT1) – an intelligence test for preschoolers to be conducted in groups between four and eight children (Cattell, Weiß & Osterland, 1997) in order to be able to control this variable with respect to the children identified at potentially at risk learning mathematics.

A total of 947 children performed on all three tests. Their data provided the basis of the quantitative analysis based on the use of SPSS. While the majority of the children interviewed demonstrated elaborate abilities and knowledge as described by Anderson et al. (2008), 73 children (about 8 %) in the sample severely struggled with certain areas relevant to the development of number concept such as seriation, part-part-whole-relationships, ordering numbers and counting small collections. They were identified as 'children at risk' with respect to their later school mathematics learning on the basis of their performance at the OTZ and the FYSMI. 26 of these 73 children (35.6 %) came from non-German speaking background families. However, only 13.6 % of the children in the complete sample (n=947) had a migrant background. Hence, these children from migrant families were over-represented in the groups of children potentially at risk.

The intervention program for the children identified to be potentially at risk learning school mathematics was conducted in two groups: Children in group 1 had weekly visits from a pre-service teacher who had been prepared for this intervention as part of a university methods course. The pre-service primary teachers were introduced to the children as 'number fairies' who wanted to show them games and activities that they could later share with their peers. This was done to ensure that the children did not feel pressure and experience themselves as slow learners at a very early point in their education. The intervention program for the group 2 children in contrast was conducted by the kindergarten teachers within their groups. While the intervention in group 1 was done one-on-one at a set time each week, the kindergarten teachers working with the children in group 2 primarily tried to use every day related mathematical situations, focussing on aspects such as ordering, one-to-one correspondence or counting as they arose in the children's play or everyday routine, in particular challenging the children identified to be at risk in these areas. The kindergarten teachers completed a diary in which they described these situations, noted how often they arose and what they did with the children in the whole group (or a small subgroup as in a game situation) and with the children at risk in particular. Like in group

1 the children of group 2 were not aware of the fact that they took part in an intervention. However, the parents of all children that took part in the intervention had been informed and given their written permission. It is important to note that for ethical reasons it was not possible to establish a control group, i.e. children identified to be potentially at risk who did not receive special support in the form of an intervention as parents would not have agreed for their children to be part of this group.

In both groups the intervention was conducted over eight months, involving about 45 min a week and based on individual learning plans developed by the pre-service and kindergarten teachers. During the intervention the pre-service as well as the kindergarten teachers were supported by the researchers to the same degree to ensure comparability of the two groups. The activities were based on number work and counting activities following the skills integration model described above.

PRESENTATION AND DISCUSSION OF RESULTS

While it was to be expected that the performance of most children would increase from pre- to post-test due to age related advancement with respect to their cognitive abilities, the results of the study demonstrate that the total group of the children identified to be at risk in learning mathematics showed the highest increase. Figure 1 shows the means of the pre- and post-tests conducted in September/October 2005 and June/July 2006 comparing the complete sample with the children at risk. The analysis was based on the number of children that had completed all three tests in 2005 as well as the OTZ and FYSMI in 2006. Hence, the number in the complete sample decreased to $n = 715$ with 60 children (8.4 %) potentially at risk.

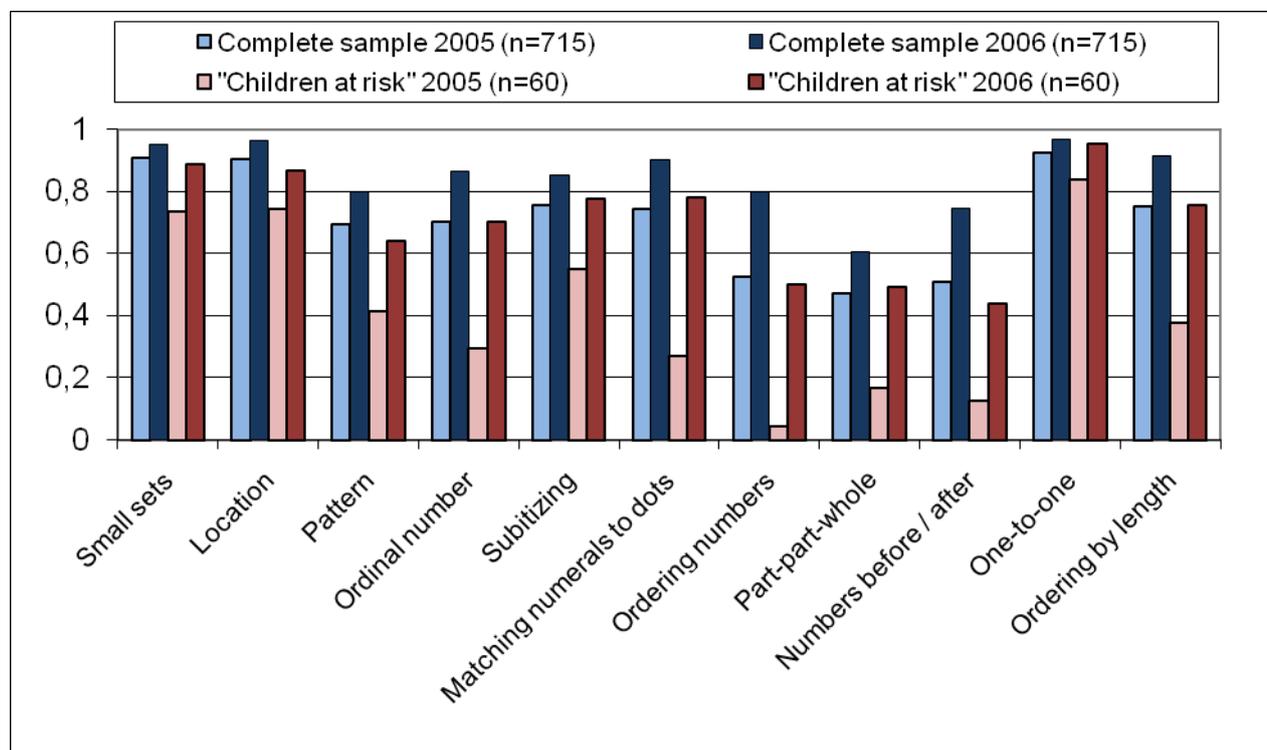


Figure1: Means of the pre- and post-test of the FYSMI

The data clearly shows that the children potentially at risk have in particular increased their competencies in those areas that were aimed at during the intervention, i.e. knowledge about numbers and sets as well as counting abilities, and performed significantly better in the post-test in the tasks related to *ordinal numbers*, *matching numerals to dots*, *ordering numbers*, *numbers before/after* and *part-part-whole relationships* [2]. However, it is important to note that due to the fact that for ethical reasons a control group was unavailable, a distinct effect of the intervention omitting other potential factors cannot be substantiated by this particular research design. Furthermore, ceiling effects hamper the comparison of the increase in mathematical competencies between the whole sample and the group of children identified to be potentially at risk in learning school mathematics. Despite this, the children potentially at risk undoubtedly demonstrated increased number knowledge and skills – domains which are seen as key predictors for later achievement in school mathematics (Krajewski 2005, Aunola et al. 2004).

Data from this study also suggests that children from non-German speaking background families show lower competencies in number concept development one year prior to school entry than their German peers. A comparison of the FYSMI pre-test data of the children with German as their first language and the children with a migration background based on a total of 947 children who completed the interview (see Fig. 2), shows a significant difference in achievement ($p < 0.001$) in the areas language of location, *subitizing*, *matching numerals to dots*, *ordering numbers* and *numbers before and after*.

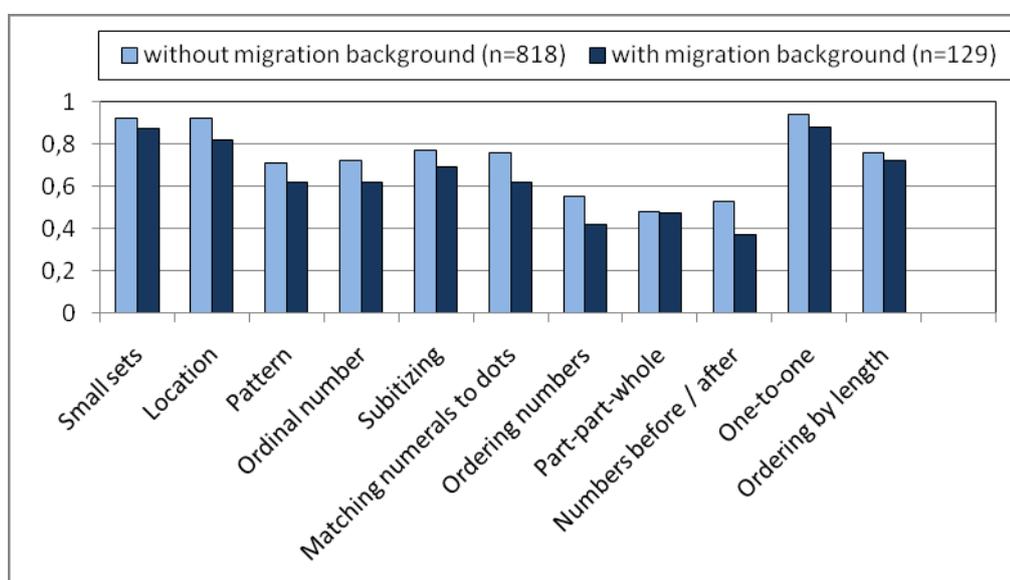


Figure 2: Mean scores of children with a migration background and German speaking background children in the FYSMI pre-test

Complying with these results, children with a migration background demonstrated significantly lower counting abilities with respect to the number related items in the OTZ. A detailed investigation of these results indicates that language related factors

play an important role. In the sub-group of the children from Turkish families [3] it was found that most of these children identified as potentially at risk in learning school mathematics, showed better performances in counting and number activities when they were encouraged to answer in Turkish (Schmitman gen. Pothmann, 2008). Thus, the intervention obviously proved beneficial with respect to their mathematical performance in the German language. The 23 children with a migration background in the group of 60 children identified potentially at risk demonstrated a clear increase in achievement in the post-test. While the achievement of both groups significantly increased ($p < 0.001$) within the test interval, these children on average demonstrated an increase of 3.6 points between pre- and post-test compared to an increase of 2.9 points in the remaining group of the 37 children from German families. However, the difference in achievement between these two groups is not significant ($p = 0,164$). In comparison, the growth in achievement in the group of children with migration background but without a potential risk factor in terms of their school mathematics learning is 1.3 points, while the mean score in this group of German children is 1.1. Again, the difference between those two groups ($p = 0,629$) is not significant (ibid, 161). Immediately before school entry the mathematical competencies of children with and without migration background obviously have converged – in some areas, i.e. *matching numerals to dots*, *ordering numbers* and *part-part-whole*, they even show slightly (however, not significantly) better results (ibid, 121).

And also another finding with respect to early intervention for preschoolers identified to be potentially at risk in learning school mathematics is encouraging. With respect to the substantial increase in achievement demonstrated by the 60 children with a risk factor in the FYSMI post-test, no significant difference between the group of 13 children who worked once a week with pre-service teachers introduced as number fairies (group 1) and the remaining 37 children who received remedial action within their groups by their kindergarten teachers (group 2) was found (Fig. 3).

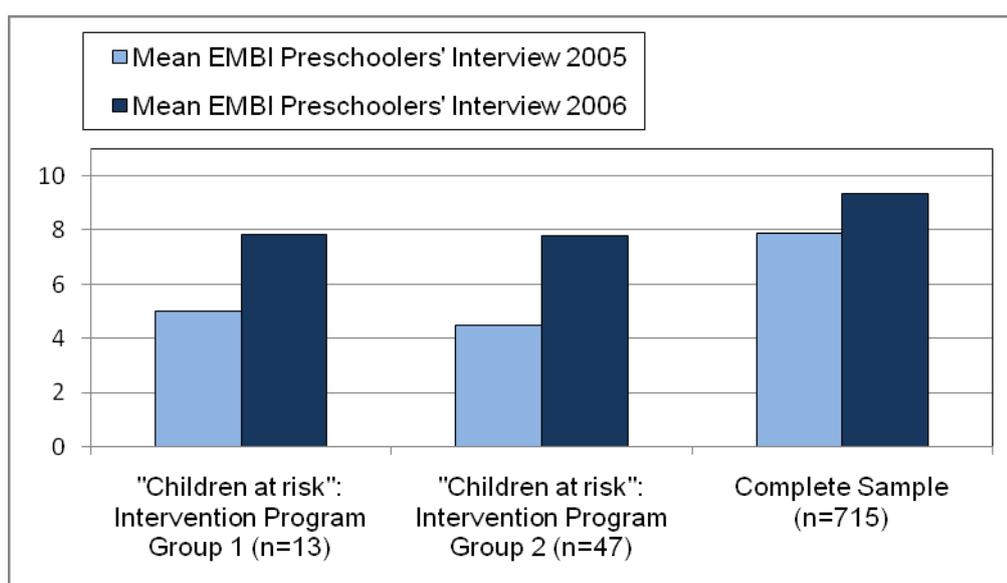


Figure 3: Mean score of the FYSMI comparing the two intervention groups

This suggests that an intervention in the everyday practice by the kindergarten teacher who had received professional development in this area is as effective as a weekly one-on-one intervention by a visiting and hence more cost-intensive outside specialist. In addition, Figure 3 shows a clear increase in achievement in both groups of an average 2.5 points in group 1 and even 3.2 points in group 2 which is clearly higher than the increase in the complete sample (see above).

IMPLICATIONS

The findings of the study suggests that preschoolers who had been identified as potentially at risk in learning school mathematics one year prior to school entry could benefit significantly from an eight months intervention program based on the enhancement of number knowledge and counting abilities. Data from the pre- and post-tests clearly indicate increased knowledge, skills and understanding of numbers and sets, i.e. particularly those areas of number concept development regarded as predictors for later achievement in school mathematics (Krajewski, 2005, Aunola et al., 2004). Further analyses suggest that for more than 50 % of these children this increase in their mathematical achievement prior to school entry proves to be of lasting effect at the end of grade 1 (Grüßing & Peter-Koop, 2008). In how far this will hold true at the end of grade 2 is currently under investigation.

Furthermore, there were no significant differences in achievement found in the post-test between the groups of children that had experienced a one-on-one intervention by the preservice mathematics teachers who had been particularly trained for this task, and the children that had worked with their kindergarten teachers within their home groups. While clinical studies had already shown positive effects of early intervention (e.g. Krajewski 2005), this study suggests that there is not necessarily a need to bring external specialists into the kindergarten to work with individual children [4]. A comprehensive screening and respective enhancement of preschoolers potentially at risk by their kindergarten teachers is possible – given that the kindergarten teachers are prepared for this task during their initial and/or inservice training.

In addition, the findings show that children with a migration background are not only over-represented in the group of preschoolers with a risk factor with respect to school mathematics, they also demonstrated the highest increase in mathematical achievement in the test interval. Hence, it appears to be important not only to focus on screenings that determine (German) language development prior to school as it is currently done in all German states, but also to investigate early mathematical abilities in order to identify children who need extra support in their number concept development. Since the PISA study has emphasized that the group of migrant children is overrepresented among the low achieving students at the age of 15 (Deutsches PISA-Konsortium, 2001) and findings from the SCHOLASTIK project (Weinert & Helmke, 1997) indicate that low achievers in mathematics at the beginning of primary school in general stay in this position, this seems of crucial importance. While the German version of the *Utrecht Early Numeracy Test* (van Luit et al., 2001) – the OTZ – showed clear ceiling effects and also proved to be very

difficult for non German speaking background children due to its demands on German language comprehension, his study suggests that the FYSMI (Clarke et al., 2006) is a suitable instrument for the collection of information on preschoolers' number concept development and the respective identification of children potentially at risk in learning school mathematics. This instrument allows children to articulate their developing mathematical understanding through the use of simple materials provided for each task in a short one-on-one interview that takes about 10 to 15 minutes for each child. Bruner (1969) has already highlighted the importance of material based activities for young children who for various reasons cannot yet verbally articulate their developing and sometimes already quite elaborate (mathematical) understanding.

NOTES

1. The FYSMI is designed to be conducted in the first year of school, which in Australia is the preparatory grade preceding grade 1. This preparatory year is compulsory for all five-year-old children. In Germany in contrast, formal schooling starts with grade 1 when children are six years old. While a majority of German five-year-olds attend kindergarten, this is not compulsory and involves fees to be paid by the parents.
2. The analysis of the data from the standardised OTZ showed clear ceiling effects. Over 40 % of the children reached level A which supposedly represents the top 25 % of the children in this age group. However, in level E representing the bottom 10 % of the scale, the test differentiated sufficiently with respect to the sample.
3. The majority of the children with a migrant background in the sample was from Turkish parents, followed by families from Russia, Kazakhstan, Lebanon and Iraq.
4. However, it is acknowledged that there might be cases in which a specialist based one-on-one training in addition to the help provided by the kindergarten teacher is expedient.

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THE STRUCTURE OF PROSPECTIVE KINDERGARTEN TEACHERS' PROPORTIONAL REASONING

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Lamon (1997) claimed the development of proportional reasoning relies on different kinds of understanding and thinking processes. The critical components she suggested are: understanding of rational numbers, partitioning, unitizing, relative thinking, understanding quantities and change, ratio sense. In this study we empirically tested a theoretical model based on Lamon's model, with data collected from 244 prospective kindergarten teachers. The analysis of the data provided support to this theoretical model and revealed that rational number, reasoning proportionally up and down and relative thinking are statistically significant predictors of proportional reasoning. These findings allow us to make some first speculations of which type of processes should be emphasized for the development of proportional reasoning in early years.

Key words: proportional reasoning, rational number

INTRODUCTION

Ratio, proportional thinking and reasoning abilities are seen as a corner stone of school mathematics; this observation is reflected in current syllabus documents, (e.g., National Council of Teachers of Mathematics, 2004) and by educators (e.g., Nabors, 2002). Researchers have often noted that the topic of proportional thinking can be challenging for schoolchildren (Fuson, 1988; English & Halford, 1995; Gelman, 1991; Steffe & Olive, 1991; Kilpatrick, Mack, 1995; Swafford, & Findell, 2001). Proportional reasoning is in essence a process of comparing one relative amount with another. From a psychological perspective, proportional reasoning is a late accomplishment developmentally because it entails second-order reasoning; inasmuch as proportions are relations between two quantities, comparisons between proportions entail considering relations between relations (Piaget & Inhelder, 1975). However, although there is indeed considerable evidence that a full understanding of proportional relations develops slowly (e.g., Moore, Dixon, & Haines, 1991; Noelting, 1980), the notion that reasoning about relations among relations is intrinsically beyond the capabilities of young children has been strongly questioned (Spinillo & Bryant, 1991). To develop young students' understanding, teachers should be aware of the critical components of understanding proportions. Thus, the main focus of the present study is to shed some light on the structure of kindergarten prospective teachers' understanding of proportional problems.

Until recently, we have had little understanding of how proportional reasoning develops. Based on previous research, we will develop and validate a framework of kindergarten pre-service teachers' thinking while they work on representations of

proportional problems. Lamon (1999, 2007) asserted that understanding rational numbers marks the beginning of the process of proportional reasoning. Thus, in the proposed framework we will articulate the understanding of kindergarten prospective teachers' on rational numbers, and related concept such as unitizing, partitioning, relative thinking, understanding quantities and change, ratio sense.

Specifically, in this study, we will propose a conceptual framework, which is mostly based on previous research on rational numbers (Kieren, 1988) and on the features of Lamon's (1999) model of proportional thinking. This framework constitutes an attempt to encompass the whole spectrum of kindergarten prospective teachers' understanding of proportional situations and problems. Furthermore, the study provides an empirical verification of the proposed model and traces the different types of thinking projected by kindergarten prospective teachers in the context of rational number and proportional tasks.

THEORETICAL BACKGROUND

Components of proportional reasoning

Lamon (1999, 2007) suggested that proportional reasoning is complex and to achieve it one has to master different kinds of understanding, thinking processes and contexts. Specifically, she proposed six areas that contribute to proportional reasoning: partitioning, unitizing, quantities and change, rational numbers, relative thinking and rate. Kieren (1988) claimed that the concept of rational number consists of four interrelated subconstructs, ratio, operator, quotient and measure, and part-whole permeates these four subconstructs. A short description of each proportional reasoning components and a brief definition of each subconstruct are provided below:

Relative thinking is a cognitive function which describes the ability to analyze change in relative terms. It is also called multiplicative thinking (Lamon, 1999).

Unitizing is the cognitive process of mentally chunking or restructuring a given quantity into familiar or manageable or conveniently sized pieces in order to operate with that quantity (Lamon, 2007).

Quantitative reasoning in visual and verbal situations is the ability to interpret and operate on changing quantities. Quantitative reasoning may or may not involve numbers. It may involve the comparison of numbers in standard form or qualitative judgments (such as more, less, etc) without actually having a quantity (Lamon, 1999).

The partitioning and part-whole subconstruct of fractions is defined as a situation in which a continuous quantity or a set of discrete objects are partitioned into parts of equal size (Lamon, 1999).

The ratio subconstruct of rational numbers is regarded as a comparison between two quantities. Thus, it is considered as a comparative index, rather than as a number (Carragher, 1996).

In the operator interpretation, rational numbers are viewed as functions applied to some number, object, or set (Behr, Harel, Post, Lesh, 1993; Marshall, 1993). One could conceive operator either as a single composite function that results from the combination of two multiplicative operations or as two discrete, but related functions that are applied consecutively.

The quotient subconstruct can be seen as the result of a division situation. In particular, the fraction x/y indicates the numerical value obtained when x is divided by y , where x and y represent whole numbers (Kieren, 1993).

In the measure subconstruct, a fraction is associated with two closely interrelated and interdependent notions. First, it is considered as a number, which conveys the quantitative personality of fractions, its size. Second, it is associated with the measure assigned to some interval. For example, $2/3$ corresponds to the distance of $2 (1/3$ -units) from a given point. This is the reason that this subconstruct is associated with the use of number lines.

Prospective teachers' subject matter and pedagogical knowledge

Although previous studies have examined teachers' abilities to solve proportionality problems (Post, Harel, Behr, & Lesh, 1991) and their ability to distinguish between proportional and non proportional situations (Simon & Blume, 1994) until now, no studies have described teachers' understanding of all the above mentioned components of proportional reasoning and whether they actually contribute to proportional reasoning. Since we encourage teachers to aim to a more conceptual understanding of mathematical concepts, we need to determine whether they have the necessary understanding of the concept and certainly its related components (Cramer, Post, & Currier, 1993).

There is no doubt that teachers' understanding of proportional reasoning also affects the way that they will present this topic to their students. In other words, the way in which a teacher will present proportional activities in her classroom is an indicator of what she believes to be more important and appropriate for students to learn, and hence, affects the way that their students understand mathematics (Thompson, 1992). The fact that mathematics in kindergarten may appear to some individuals as simple or trivial can be very misleading. Kindergarten teachers must know the mathematical concepts that students need to master and facilitate them to build necessary knowledge that these children are capable of, in those early years.

Proportional reasoning is a topic often introduced in the last years of primary school. Still, it is believed that it is not an all-or-nothing affair but various dimensions contribute to its construction which grows over a period of time (Lamon, 1999). During students' kindergarten years some of these dimensions may be addressed. It is important to clearly identify the contribution of these various dimensions to proportional reasoning and find ways that these may be introduced and addressed in

the kindergarten classroom. It is very likely that the exposure to one or some of these dimensions may provide a better in-road to proportional reasoning.

The Proposed Model

The model proposed in this article is based on Lamon's (1999) conceptualisation of different kinds of understanding and thinking process necessary for the development of proportional reasoning and Kieren's (1988) theory on the multifaceted personality of rational number (see Figure 1). Two modifications were made to Lamon's model. Firstly, we added the dimension "reasoning proportionally up and down". Reasoning proportionally up and down, involves students' ability to analyse the quantities in a given situation to determine that they are related proportionally and that it is appropriate to scale them up or down (Lamon, 1999). We felt that this dimension was necessary and was missing from the Lamon's model. Secondly, the rate dimension was taken as one of the four subconstructs of rational number and not an isolated dimension (Kieren, 1988).

The proposed model consists of nine first-order factors as shown in Figure 1. Figure 1, makes easy the conceptualisation of the way in which the nine first order factors are: unitizing, understanding quantities and change, relative thinking, ability to reason proportionally up and down, partitioning/part-whole, ratio, operator, quotient and measure. There are also two second order factors, rational number and proportional reasoning. The model suggests that proportional reasoning is related to students' abilities in unitizing, quantities and change, relative thinking, reasoning proportionally up and down and rational number. Rational number is presented as a multi-dimensional factor which is composed of four subconstructs: ratio, operator, quotient and measure, with partitioning/part-whole being the basis for the development of these four subconstructs.

METHODOLOGY

Purpose of the study

Drawing on Lamon's (1999) and Kieren's (1988) theoretical models and employing tasks used in previous studies, the present study aimed to examine prospective kindergarten teachers' proportional reasoning. In particular, the study aims to investigate the relationship amongst: partitioning, unitizing, understanding quantities and change, relative thinking, reasoning proportionally up and down, measure, rate, operator and quotient with proportional reasoning as they will be projected through prospective kindergarten teachers' responses.

Participants and tasks

To answer our research questions, a test on proportional reasoning was constructed guided by the criteria regarding the development and the measurement of the concepts embedded in the theoretical models described earlier. The test included 31 items measuring the participants' abilities in part-whole, unitizing, quantities and

change, rational numbers, relative thinking and reasoning proportionally up and down. For the measurement of rational number, the test included tasks on its four interrelated subconstructs: ratio, operator, quotient and measure. Most of the tasks that were used were taken from previous studies such as Lamon's (1999) and Charalampous and Pitta-Pantazi (2007).

The test was administered to 244 kindergarten pre-service teachers studying at three universities in Cyprus.

Scoring and Analysis

Students' fully correct responses were marked with 1 and the incorrect responses with 0. If a student gave a partly correct response, for example if s/he gave a correct answer but wrong justification, this again was marked with 0. The confirmatory factor analysis (CFA), which is part of a more general class of approaches called structural equation modeling, was applied in order to assess the results of the study. CFA is appropriate in situations where the factors of a set of variables for a given population are already known because of previous research. In the case of the present study, CFA was used to test hypotheses corresponding to Lamon's theoretical conceptualization of what constitutes proportional reasoning and Kieren's model of rational number subconstructs. Specifically, our task was not to determine the factors of a set of variables or to find the pattern of the factor loadings. Instead, our purpose of using CFA was to investigate whether proportional reasoning is a composite function of various types of understanding presented by previous research (Kieren, 1988; Lamon, 1999, 2007).

One of the most widely used structural equation modeling computer programs, MPLUS (Muthen & Muthen, 1998), which is appropriate for discrete variables, was used to test for model fitting in this study. In order to evaluate model fit, three fit indices were computed: The chi-square to its degree of freedom ratio (χ^2/df), the comparative fit index (CFI), and the root mean-square error of approximation (RMSEA) (Marcoulides & Schumacker, 1996). The observed values of χ^2/df should be less than 2, the values for CFI should be higher than .9, and the RMSEA values should be close to zero.

RESULTS

The results are presented in relation to the aim of the study. Figure 1, represents the model which best describes the theoretical model we proposed for proportional reasoning. More specifically, it illustrates that proportional reasoning is a result of abilities in partitioning, unitizing, understanding quantities and change, relative thinking, reasoning proportionally up and down and rational number. From a structural point of view, nine first order factors were included: unitizing, understanding quantities and change, relative thinking, reasoning proportionally up and down, part-whole, measure, rate, quotient and operator. Each of these factors

involved three to six tasks. There were also two second order factors: rational number and proportional reasoning.

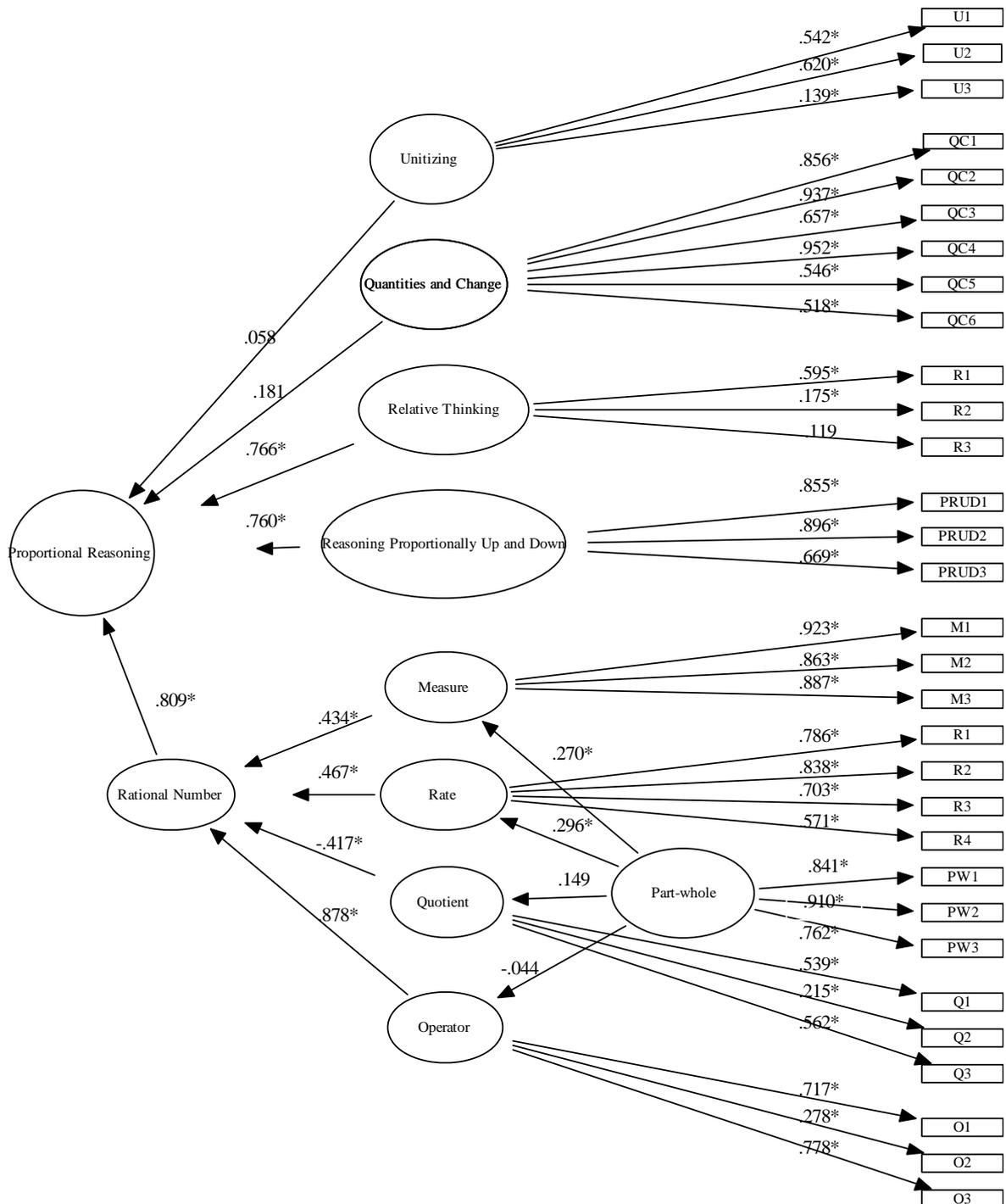


Figure 1: Model for proportional reasoning.

The numbers in the diagrams indicate the factor loadings and the * the values that are statistically significant

Confirmatory factor analysis (CFA) was used to evaluate the construct validity of the model. CFA showed that 30 out of the 31 tasks employed in the present study

significantly correlated on each factor, as shown in Figure 1. It also showed that the observed and theoretical factor structures matched the data set of the present study and determined the “goodness of fit” of the factor model (CFI=0.933, $\chi^2= 641.330$, $df= 418$, $\chi^2/df=1.53$, RMSEA=0.047), indicating that, unitizing, understanding of quantities and change, relative thinking, reasoning proportionally up and down and rational number can represent distinct function of prospective kindergarten teachers’ proportional reasoning.

The structure of the proposed model also addressed the predictions of unitizing, understanding of quantities and change, relative thinking, reasoning proportionally up and down and rational number, in proportional reasoning. First, the results obtained confirmed Kieren’s (1988) conceptualisation, that the concept of rational number is comprised by four subconstructs: ratio ($r=.467$ $p<0.05$), operator ($r=.878$ $p<0.05$), quotient ($r=-.417$ $p<0.05$) and measure ($r=.434$ $p<0.05$). The three subconstructs, ratio, operator and measure correlated significantly with rational number whereas the quotient subconstruct had a negative significant correlation with rational number ($r= -.417$ $p<0.05$). This may be due to the fact that the quotient task required division, a reverse type of thinking. It was also confirmed that the part whole/partitioning interpretation of rational number is related to the four subconstructs, ratio ($r=.296$ $p<0.05$), measure ($r=.270$ $p<0.05$), operator ($r= -.044$ $p>0.05$), and quotient ($r=.149$ $p>0.05$). However, only the relationships to ratio and measure subconstructs were statistically significant.

Second, the results obtained showed that to develop proportional reasoning different kinds of understanding, thinking processes and contexts are essential. The analysis revealed that the critical components of proportional reasoning are: unitizing, understanding of quantities and change, relative thinking, reasoning proportionally up and down and rational number. The loadings of each of these factors on proportional reasoning indicated that rational number ($r=.809$ $p<0.05$), reasoning proportionally up and down ($r=.760$ $p<0.05$) and relative thinking ($r=.766$ $p<0.05$) significantly predicted students’ performance in proportional reasoning. Performance in rational number was the strongest predictor for success in proportional reasoning. Unitizing ($r=.058$ $p>0.05$), and understanding of quantities and change ($r=.181$ $p>0.05$) although appeared to predict abilities in proportional reasoning, did not significantly contribute to proportional reasoning.

DISCUSSION

The present study aimed to empirically test a theoretical model based on Lamon’s (1999) conceptualisation of proportional reasoning, with prospective kindergarten school teachers. The results of this study confirmed the theoretical model and also indicated the extent of the impact that different components have in proportional reasoning. It was confirmed that part-whole, unitizing, understanding of quantities and change, relative thinking, reasoning proportionally up and down and rational number predicted prospective teachers’ abilities in proportional reasoning, with

rational numbers, relative thinking and reasoning proportionally up and down being the most significant predictors. The results of the study also lend support to Kieren's (1988) conceptualisation of the multifaceted construct of rational number, since this construct was significantly related to all four subordinate constructs measure, rate, operator and quotient. As a whole, these findings suggest that a profound understanding of rational number, unitizing, relative thinking, thinking about quantities and change, reasoning proportionally up and down are related to students' performance in proportional reasoning.

The findings of the study suggest that different thinking processes and contexts are necessary for the teaching of proportional reasoning. For instance, teachers may present children with situations which require relative thinking or scenarios where quantities and change need to be discussed. Students may be asked to compare extensive (the length of two ribbons) or intensive quantities (the sweetness of a drink when adding sugar) (Nunes, Desli, & Bell, 2004). Other teachers may decide to start with partitioning tasks, by asking students to share one item or a set of items to two or more individuals. Another possibility is to introduce activities where reasoning proportionally up and down is required. Previous research (Sophian & Madrid, 2003) has shown that young students are capable of this type of thinking. Such reasoning can be introduced through activities where students are required to carry out many-to-one correspondence. These processes allow young students to build an understanding of composite units, provide additive solutions which may later be linked to multiplicative solutions (Sophian & Madrid, 2003).

Obviously, designing instruction that will develop young students' proportional reasoning requires an understanding of young students' intuitive knowledge. It is very likely that from their everyday life, young students may develop a tendency towards certain ways of thinking which may make one of the abovementioned approaches to proportional reasoning more effective. It still needs to be investigated which teaching approach and emphasis on which one of these proportional reasoning dimensions can be more effective for students development of proportional reasoning in their early years of schooling.

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HOW CAN GAMES CONTRIBUTE TO EARLY MATHEMATICS EDUCATION? – A VIDEO-BASED STUDY

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In recent years early mathematics education has become an area of increased interest and research activity. Consequently, a growing number of educational programs and especially developed materials are published and used in kindergarten. Games, however, are an often underestimated yet promising approach for the early years. We asked if, how, and under what conditions early mathematics education (3- to 6-year-olds) can be organized with everyday materials, for example games. In a two-phase design, we first developed criteria based on didactical considerations to assess materials. In the following empirical study we videotaped children using selected materials. The research resulted in first descriptions of the conditions under which potentially suitable materials can develop mathematical potential in young children.

Keywords: number concept, arithmetic skills, early childhood education, kindergarten, learning materials, video study, grounded theory, games

1 THE CONSTRUCTION OF NUMBER CONCEPT

Since the late 1990s a growing research activity can be observed in the field of *early mathematics education*. Within this research there is a consensus about the *contents* that should be part of a *preschool curriculum*. The answers differ in detail but many authors focus on fundamental ideas or important aspects of mathematical thinking like number and quantitative thinking, geometry and spatial thinking, algebraic reasoning (patterns, relationships) or data and probability sense (cf. Ramani & Siegler, 2008; Peter-Koop & Grübing, 2007; Clements & Sarama, 2007a/b; Baroody et al, 2006; Lorenz, 2005; Balfanz et al, 2003; Krajewski, 2003; Arnold et al, 2002;). Some authors also mention process ideas like mathematization and communication or argumentation (cf. Perry et al, 2007; Clements & Sarama, 2007b, 463).

Our research relates to the *construction of number concept and quantitative thinking*, because “for early childhood, number and operations is arguably the most important area of mathematics learning. In addition, learning of this area may be one of the best developed domains in mathematics research” (Clements & Sarama, 2007b, 466). Consequently, there are not only a lot of games and materials for kindergarten which address this area, but there also exists a well-developed theory on the construction of number concept our research can be based on. Although our research concentrates on this area we know that early childhood education needs a broader approach and a widespread fostering of abilities.

In the past fifty years, the research on children's development of quantitative thinking and construction of number concept has seen a change from Piaget's *logical-foundation-model* to the current *skills-integration-model* (cf. Baroody et al, 2006; Clements, 1984; Peter-Koop & Grübing, 2007).

Piaget's developmental theory emphasizes that the construction of number concept depends on the development and synthesis of logical thinking abilities, especially of classifying and ordering (cf. Piaget, 1964, 50ff). According to this view counting does hardly benefit the construction of number concept but might rather be an obstacle. The logical thinking abilities are not available until concrete operational stage, that is at the age of seven (cf. Piaget, 1952, 74). Therefore the construction of number concept is not possible until primary school and activities to foster this goal do not make any sense in kindergarten. In the pedagogical practice Piaget's theory led to set theory that postponed teaching number and arithmetic concepts until preschool and primary school (cf. for example Neunzig, 1972).

Particularly since the late 1970s Piaget's theory has given rise to a lot of criticism. In contrast to Piaget, Gelman and Gallistel (1978) underline the meaning of counting for the construction of number concept. In their opinion counting principles are innate and therefore available in kindergarten. Starkey and Cooper (1980) demonstrated that even infants are capable of distinguishing sets of small numbers and Wynn (1998) even speaks of infants' sensitivity to numbers. Thus nowadays there is a wide consensus that *preschoolers* show *considerable informal arithmetic knowledge* in spite of the existence of large inter-individual differences (cf. Baroody et al, 2006; Schipper, 1998). A well-developed number concept is not naturally given but requires nurturing: Learning number words for example may help to construct an understanding of number. There is also agreement on the *skills-integration-model*. The following skills seem to be central for the years before school attendance (cf. Resnick, 1989; Gerster & Schultz, 2000; Krajewski, 2003; Lorenz, 2005):

- *Perceptual and conceptual subitizing*: Perceptual subitizing is the spontaneous recognition of recurrent configurations up to sets of four that are associated with number words; whereas conceptual subitizing allows the instant recognition of sets bigger than four. Conceptual subitizing requires visual structuring processes (numbers as units of units) (cf. Clements 1999).
- *Verbal and object counting*: Verbal counting extends from simply reciting the number line (string level) to skills like counting forwards, backwards, counting on, counting in steps (bidirectional chain level) (cf. Fuson, 1988, 34–60); object counting contains counting sets and naming the number word (cardinality rule); and counting out objects to a given number word.
- *Comparing and ordering sets*: Comparison and ordering of sets is possible on a perceptual level (more, less, even) and on a numerical level (5 is more than 3). For small sets it is possible by perceptual subitizing.

- *Part-whole-connections, composing and decomposing sets*: These skills are closely connected to conceptual subitizing and the numerical comparison of sets. Understanding that a number is composed of other numbers is seen as the central skill for the construction of number concept (cf. Resnick 1989).
- *Beginning addition and subtraction with material and in concrete contexts*: Children can use either counting procedures and/or visual structuring processes to solve first arithmetical problems.

In a longitudinal study Krajewski (2003) proved that some of these skills are of great importance for later school achievement and success. They even allow the statistical prediction of marks in primary school mathematics.

2 RESEARCH QUESTIONS

In recent years different approaches to early mathematics education have been developed. One can distinguish at least two types:

- *Course-like educational programs* in kindergarten, focussing on the purposeful construction of specific mathematical skills, sometimes even following a relatively strict curriculum (e.g. in Germany Preiß, 2004/05; Krajewski et al, 2007; in the USA Clements & Sarama, 2007a; Ramani & Siegler, 2008).
- *Implementation of games, educational materials and informal learning opportunities* in the daily kindergarten practice, subsequent to joint activities, realized in a playful way, aiming at a wide spread fostering of children's abilities (e.g. in Germany Hoenisch & Niggemeyer, 2004; Müller & Wittmann, 2002/04; e.g. in the USA Balfanz et al, 2003).

Our study refers to the latter approach which seems promising but often underestimated. Examples for materials can be

- well-known *commercially available games* like common board games, card games and dice games,
- special *educational games and materials* to foster arithmetic skills which can be either purchased or developed by the educational staff (and the children) themselves.

The goal of our study is to analyze the role of these materials in early mathematics education. In detail we ask the following research questions:

1. What (theoretical) potential for children's construction of number concept do these materials have in principle?
2. Under what conditions can potentially suitable games and materials can develop their mathematical potential?
3. In which way can games contribute to early mathematics education? Is it possible to organize early mathematics education, at least partially, with games?

3 RESEARCH METHODS AND METHODOLOGY

Our research follows a qualitative design. According to the research questions it is a two-phase design (cf. figure 1) that will lead to a (grounded) theory about the conditions for a substantial and rich mathematical learning environment (cf. Strauss & Corbin, 1996):

- The first phase is a *theoretical analysis of games and educational materials*. We established theory-driven criteria on the basis of didactical considerations (cf. section 1) to assess the suitability of materials for the construction of number concept (cf. Schuler, 2008).
- The second phase is an *empirical evaluation of selected, theoretically proved games and educational materials*. A theoretical study can never capture all aspects of a learning environment. Thus we started a video-based study in cooperation with the staff of a selected kindergarten to test the criteria's workability, to develop further and more detailed criteria and to develop learning environments with materials that meet the criteria's requests. In a first step of data inquiry we videotaped educators while playing with children during an open offer at several occasions with selected materials. In a second step the researcher took the role of an educator and offered games during free play at several occasions.

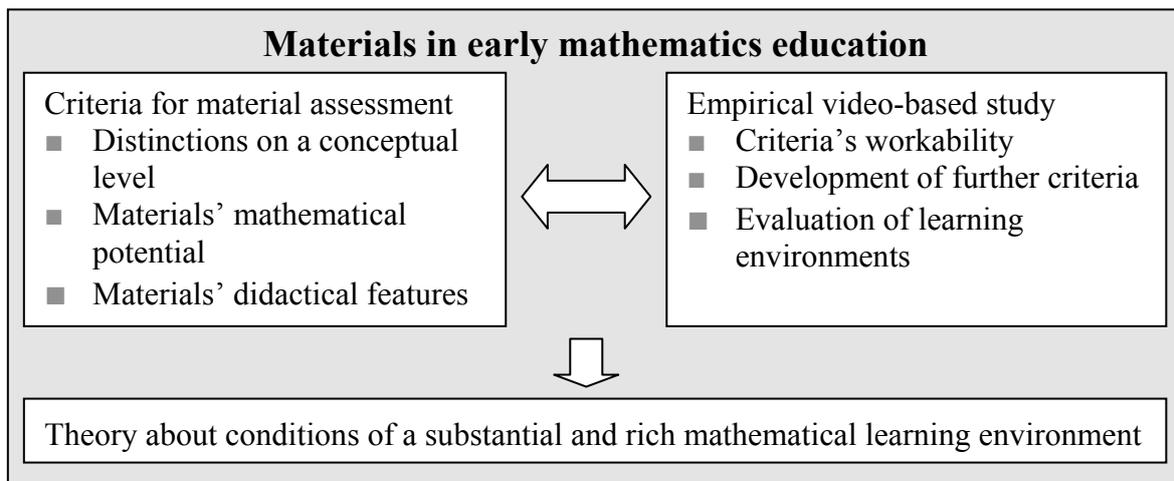


Figure 1: Two-phase research design

According to the *methodology of Grounded Theory* (Strauss & Corbin, 1996), which requires the ongoing change and interplay between action (data inquiry) and reflection (data analysis and theory construction) (cf. Mey & Mruck 2007, 13), the video-based study is still in progress. Basis of the data analysis are transcripts of video sequences. These transcripts do not include only verbal data but also the paraphrase of actions, gesture, facial expressions, as well as screenshots and a storyboard. The data analysis provided first answers to some of the earlier questions and led to further research activities following *theoretical sampling* (cf. Strauss &

Corbin, 1996, 148ff). Using the three most important tools in Grounded Theory methodology – theoretical coding, theoretical sampling, and permanent comparison – there was reason to believe that, aside from the material chosen, the educator’s role is crucial to the development of mathematical potential. It has become obvious that the initial criteria need supplementing because *the development of the mathematical potential is linked to conditions*.

4 RESEARCH RESULTS

4.1 Criteria for material assessment

During the past decade many suggestions for early mathematics education were published. Thus it seems necessary to develop criteria to assess these materials and to choose carefully (cf. Schuler, 2008).

1. In accordance with previous remarks, we first distinguished the materials from one another *on a conceptual level*.
 - Does mathematics appear as a part of kindergarten everyday life or is there the idea of a special class?
 - Does the material aim at support of at-risk children or of all children?
 - Does the material support one content idea (e.g. number) or different content ideas?
2. Following the skills-integration-model about the construction of number concept we asked what *mathematical content and potential* is inherent in the material. For the content idea “number and quantitative thinking” the skills mentioned in section 1 guide the analysis:
 - Does the material make it possible to compare sets on a perceptual and a numerical level?
 - Does the material support the construction of mental images of numbers (for example following the patterns of dice images)?
 - Does the material prompt counting activities (forward, backward, counting in steps, precursor/successor)?
 - Are composing, decomposing and first arithmetic activities possible?
3. Following the idea of an early mathematics education implementing mathematics in every day practices and fostering all children of different ages, we asked in addition the following questions:
 - Does the material meet different levels of previous knowledge?
 - Does the material allow access and challenge at different levels?

Mathematical content and potential	
Comparing and ordering sets	+
Constructing ideas of dice images (up to 6)	++
Constructing ideas of other images (up to 6)	++
Counting objects	++
Assigning sets to numerical symbols	+
Assigning numerical symbols to sets	+
Counting verbally	
Finding precursor/successor	
Composing and decomposing set images/numbers	+
Beginning addition and subtraction	+

+: possible ++: appropriate, highly supported

Table 1: Implementation of the criteria for the chips game

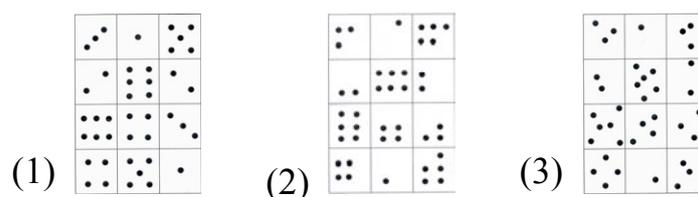


Figure 2: Boards for the chips-game

Games are one possible material to meet the conceptual needs. We want to illustrate the implementation of the criteria by an *example* (see table 1). The *chips-game* is played by two persons. Each person gets a board (three or more alternative versions, see figure 2) and chips of one colour. Throwing alternately one puts chips on the matching square. The person who covers all squares first wins. Variations take into account different levels of previous knowledge, access and challenge:

- playing and covering alone with or without a dice,
- boards with different images,
- two persons playing on one board with chips of different colours,
- covering the squares with number cards.

General mathematical skills like describing, giving reasons, arguing, forming hypotheses or making predictions are not material inherent. But data analysis showed that they can be stimulated by the educator's questions (see section 4.2). Thus *process ideas* can be described as *mathematical potential that develops in interaction*. One goal of the video data analysis is to generate more knowledge about how mathematical potential develops.

4.2 The video-based study

As mentioned above data inquiry, data analysis, and theory construction are still in progress. Therefore the following section reflects the contemporary status of the research process and the results we have got so far. In a first step coding and comparing sequences of the kindergartens educators on the one hand and the researcher taking the role of an educator on the other hand, led to *three preconditions on the part of the educator to develop a game's mathematical potential*:

- *Mathematical and didactical competence* contains the analysis, assessment, choice and presentation of materials and results in sensitivity for possibilities and variations in the games course.
- *Individual presence* emphasizes that the educator's actions and support depend on the individual child's needs and competences. The educator's presence can support affordance and lasting involvement with the material by creating game situations, explaining rules and goals, helping to follow the rules, to solve conflicts and to facilitate feelings of competence.
- *Conversational competence* means to develop the mathematical potential through comments on the game's course, questions that stimulate objective explanations, reflections on actions and thoughts, interchange between children, assumptions and hypotheses.

Concerning these three preconditions we observed difficulties on the part of the educators. Except for counting activities they were mostly not aware of the game's mathematical potential. They consequently could not stimulate other mathematical opportunities. Supporting presence during free play was often an organisational problem and aggravated the perception and realisation of individual needs. The educators questioning repertoire was mainly reduced to narrow questions like: How many are there? How many chips do you need? Where are five? Examples for questions to understand and stimulate the child's thinking are open and reasoning questions: How have you seen these are precise five? How do you know here are more/less than /just as many as there?

In a second step we started to investigate the mathematical opportunities during the game sequences. According to the differences in mathematical potential we distinguished different game sequences:

- introduction of a new game or material (1),
- game situation with fostering elements (2),
- game among children of similar age (3),
- game among children of different age (4).

Detailed analysis and open coding of transcripts of type (2), mostly a one-to-one-situation of educator and child, revealed so far the following characteristics:

- *Individual affordance* (cf. Lewin following Heckhausen 2006, 31, 105ff) *by optical or haptic features*: An example for optical affordance is a child's confusion and curiosity about differing set images in the chips game (see board 3 in figure 2). Haptic affordance can manifest in covering the set images with chips without using a dice.
- *Demonstration of skills and abilities*: In a game situation with fostering elements, children want to show what they already can. One can distinguish explicit ways of demonstration like "I can those." or "This is easy for me." from implicit ways that manifest in the child's increased gestural and verbal engagement.
- *Gestural and verbal explanation*: The chips game can be played on different levels of articulation – actions (having a throw, covering), gestural and verbal comments on actions (naming and showing dice and board images), gestural and verbal explanations (showing and explaining the differences and similarities between images of board 2/3 and dice images). The latter level requires the educator's purposeful questions and stimuli.

6 DISCUSSION

As we expounded in section 1 there is a wide consensus about contents in early mathematics education and about the importance of the construction of number concept and quantitative thinking. The theoretical analysis of selected games could show that games have a mathematical potential concerning the number concept. To identify this potential, central skills were reformulated for the analysis of kindergarten materials (see table 1).

Aside from contents, the question of *methods in early mathematics education* is an interesting and still little investigated issue: „little is known about preschool teachers' role in promoting math skills“ (Arnold et al 2002, 762). One can distinguish different statements about this subject:

- General statements about how children can learn mathematics emphasize the area of conflict between construction and instruction: "Early childhood educators face a balancing act – that is, an approach that is neither too direct nor too hands off" (Baroody et al, 2006, 203).
- A further discussion focuses on the role of playing and learning: "Play is not enough. [...] children need adult guidance to reach their full potential" (Balfanz et al, 2003).
- In addition, some authors stress the differences in content and method between kindergarten and primary school. "Early childhood mathematics should not involve a push-down curriculum" (Balfanz et al, 2003, 266) and kindergarten aims

at “preparing children for school but not by school methods” (Woodill et al, 1992, 77).

Our data analysis indicates so far that potentially suitable games need a competent educator with regard to didactical and conversational aspects. For one type of sequences – game situation with fostering elements – we phrased characteristics. These characteristics imply and allow more specific statements about an educator’s didactical and conversational competence. The educator has to discern the child’s individual approach to the material and has to consider the mathematically productive aspects. He has to make possible the demonstration of abilities and has to facilitate and challenge gestural and verbal explanations through suitable game materials, stimuli and questions.

For other types of sequences this work still is to come. We expect new findings from sequences where children play with other children of the same or of a different age and from sequences which have both elements – children playing together with selective educator’s interventions. Whereas we could find some answers to the still little investigated educator’s role in early mathematics education we do not know much about what children at this age can actually learn with and from each other. We also have to do further research on suitable ways of interventions to make a game mathematically productive without reducing the game’s idea and affordance.

Games can be described as one possibility to organize early mathematics education in correspondence with the daily kindergarten practice. But as we have seen this is not without requirements. These requirements simultaneously show the limitations of this approach.

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NATURAL DIFFERENTIATION IN A PATTERN ENVIRONMENT (4 YEAR OLD CHILDREN MAKE PATTERNS)¹

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Manipulation in learning geometry is a disputable topic because of different theoretical bases for creation of geometrical concepts. Some theories underline a great importance of visual information in forming the first level of understanding geometry. For children, such visual geometrical information could be provided by patterns. Assuming that visual information gives the first stimulus for creation of geometrical concept, I undertook the experiment to observe the possibility of going beyond visual states in early geometry, towards its dynamic images.

INTRODUCTION

Many children have a well-developed, spontaneous and intuitive mathematical competence before their school education (Clarke, Clarke, Cheeseman, 2006). Researches in this field put a great emphasis on early numeracy and competence in counting, although in some articles the topic of “spatial and geometrical competence and concepts” is described as well. In these attempts, “spatial development” is described by relations like: behind, beside, in front of...; concepts are usually limited to the basic geometrical shapes: triangles, squares, circles.

I strongly believe that guasi – geometrical activities can develop widely understood children’s mathematical competence. On one hand, since geometrical approach to mathematics is closer to children than arithmetical one, geometry can open doors to a world of mathematics. Geometrical cognition starts from a reflection upon the perceived phenomena and in this way correlates with the basic ways of learning among children. On the other hand, it gives a chance to develop such ways of thinking, that are typical for mathematical thinking. Skills like generalization, abstraction, perceiving relations, understanding rules are the base for this aim. Early geometry is in-between physical and abstracts worlds. By this, it enables to mathematize this world.

By stating an issue of enriching children’s mathematics by adding geometrical activities, we simultaneously pose a question: what such activities should include? Should they be focused on geometrical figures, or should they go beyond traditionally understood areas of children’s geometry? It seems, that geometrical regularities (patterns) are unexploited areas for such goals.

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Many educators are in opinion, that during the work with patterns, elements of mathematical thinking occur. A pattern is a form, a template, a model (or, more abstractly, a set of rules). It is a well-known fact that geometrical regularities rooted in patterns can be described by the language of geometrical transformations. My previous research confirm, that 4-7 year old children are capable of organizing the space and arranging it accordingly to geometrical relations in a spontaneous way (Swoboda, 2006). But these are static relations, represented visually, and connections between such grasping of relations and their dynamic representations are not scientifically proven.

PERCEPTION VERSUS ACTION IN EARLY GEOMETRY

Some theories stress the fact that geometrical knowing and understanding is created in a specific way. In those theories, the priority is given to perception.

The most popular theory of forming the geometrical concept comes from P.van Hiele. He describes the first level of understanding as “visual”, connected with non-verbal thinking. The emphasis is placed on the ability of recognizing shapes, which are judged by their appearance as the ‘whole’. Not much concerning the role of action is spoken, although a didactics conceptions suggest activities based on the action with objects. In J. de Lange's opinion (who comments van Hiele's theory), *a pupil who is on the visual level can obtain the first level of thinking when s/he is able to manipulate in domain of regularities.* (1987, p.78).

Some very interesting depictions related to geometrical understanding are present in conceptions worked out in Czech Republic by M. Hejný and P. Vopěnka. In their opinion, geometrical world is hidden in the real world, and it is emerging from the surroundings through the special intellectual activity which can be called “the geometrical insight” (Hejný, M. 1993, Vopěnka, P. 1989). At the beginning, there is no geometrical world nor geometrical object in a child's mind. Only objects from the real world exist. But we focus our attention on those objects in various ways. Sometimes we perceive „something”. Vopěnka (1989, p. 19) describes such a situation in the following way: *To see „this”, means to focus attention on “this”, to distinguish “this” from the whole rest. This, what can absorb the whole attention on itself, we call „phenomenon”.* Perceiving „something” creates the first understanding. For example, a child can focus his or her attention on a shape of an object or on a specific position of one object in relation to another. *Phenomena* open the geometrical world to a child. In spite of the fact that our attention is attracted by these phenomena, this first understanding is passive: stimulus goes from the phenomenon. In this depiction, the role of perception is large – the perception of „something” is the first step to creation of the child's own geometrical world.

In these depictions, the role of an action is lost. Results of psychological researches confirm that in understanding of shapes, the great importance lays upon the pictorial designate. But the next stage is needed. Acts of perception are important but are not

a sufficient source of geometrical cognition. Szemińska (1991, p.131) states that: *perception give us only static images; through these, we can catch only some states, whereas by actions we can understand what causes them. It also guides us to possibilities of creating dynamic images.*

Szemińska has worked very closely with Piaget and, widely known his results show that children (on the pre-operational level) have great difficulties in movements reproduction – they are not able to foresee a movement of an object in a space. The process of acquisition of such skills is lengthy and gradual. During manipulations, child's attention should be focused on *action*, not on the very *result of action*. It requires a different type of reflection than the one that accompanied his or her perception.

This short juxtaposition above shows that the relation between visual recognition of geometrical objects and actions which can lead to creation of dynamic images of those objects, need further investigations. They are still not recognized as an educational problem. For this reason I undertook the experiment to observe the role of manipulation in early geometry.

EXPERIMENT

In my experiment, as the basis I took Vopěnka' and Hejný's theories about the opening of the geometrical world. First of all, I based on the assumption, that the first understanding takes place when a child turns its attention on any geometrical phenomenon. I was interested in situations where children can manipulate. Results of my previous experiments showed that making patterns (arranging them out of blocks, folding out of puzzles, drawing), can fulfill our expectations.

In order to test the possibilities of creating a “path” from perception to manipulation, I prepared an experiment, which took place in March - April 2008. Children from a nursery school, aged 4, 5, 6 , were the subject of the series of observations. Clinical observation an interview with a small group of children was chosen as a methods.

Children were tested individually. As a research tool we used „tiles” (two types), shown on the right (Fig.1). The whole investigation of one child consists of two parts.



Fig. 1 – research tool

Part I, Stage I: A teacher makes a segment of the pattern (Fig.2).



Fig.2 – a segment of the pattern prepared by a teacher

On the table, there are also tiles arranged into two separate piles. Teacher says: Look carefully at this pattern and try to continue it. If a child doesn't undertake the task, the teacher will say: look how I do it. After that you will continue. If a child undertakes a task, then after having finished making the pattern, he/she will take part in the next stage of an investigation.

Part I, Stage II: Teacher says: Now, please close your eyes, and I will change something in your pattern. After that, you will say what has been changed. (Teacher exchanges one tile in the pattern, so that the regularity is distorted). Then, the teacher shows the pattern and asks a child: Is there something wrong here? Why? Regardless of the answer received from the child, the teacher says: and now try to correct the mistake I have just done.

Part II, Stage I: Teacher says: some days before we made a pattern by using these tiles. Do you remember? Now, try to build it again. If a child does not remember, the teacher starts to create the pattern and invites the child to cooperate.

Part II, Stage II: Teacher says: and now, I will invite your colleague and you will be the teacher for her. Firstly, you will show her how to work to make the pattern, and after that you will play with her in correcting it. You will do it just like we did it some days ago.

General aims of the experiment were to observe the possibility of awareness of results of different types of movements: translations and rotations (possible by using only one type of tiles) and mirror symmetry (which requires reverse copy of the shape). Additionally, for group of 4 year old children, I tried to find answers on these questions:

- How do children understand the task presented visually,
- How do they understand a verbal instruction related to the given task,
- How do they act by making and retrieving patterns.

RESULTS OF THE EXPERIMENT

In this paper I will present some results gathered in a group of 4 year old children and only from Part I. This educational and developmental level, in each of investigated domain, turned out very diverse. Children demonstrated both: various understanding of the task and various ways of its realization.

1. Reflection upon the visual information

Many children started to work spontaneously, just after hearing the command: *take a careful look at this pattern and try to continue arranging it*. From the command they depicted only the words: *try arranging it*. It is also possible that they acted in a spontaneous way: while seeing the fragment of the pattern and material for manipulations they started to play with them. The other group observed all that used to be on a table for a long time. Sometimes, they were taking and analyzing separate tiles. Therefore, different strategies were possible. It is showed by the following examples:

Strategy „helpless”. Here, a child did not actually know how to create motifs. It could act only when guided by the teacher. Left alone, the child could not follow these guidelines. According to Vygotski's theory, the creation of the whole motif is beyond the zone of proximal development.

Example: Kaja (girl)

- Teacher: Look carefully at this pattern and try to continue it5 seconds break... you can take it into hands.
- Pupil: She takes one tile, keeps it for 8 seconds without any movement. Finally she says: I don't know.
- Teacher: Look, put this tile here (the one in your hands), take another tile from the second pile, connect them – and what do you obtain? (a girl acts according to teacher's instructions). Could you continue your work in the same way? ...(10 sec. girl does not do anything). Take one from this pile, and from the second one ... (girl connects the motif in an upside-down position).

Strategy „trials and errors”. The beginning of work can be based on „blind” experiments: child has some materials for manipulation, but she/he doesn't know how to use it in order to obtain the aim. A child decides „to do something”. Manipulations can lead to interesting findings and frequently a child can draw conclusions from previous experiences.

Example: Oliwka (girl)

- Pupil: Quickly reaches for two tiles from one pile and tries to create a motif above the pattern. Although she manipulates and does not succeed, she accepts the arrangement consisting of two tiles of the same type, placed in an opposite way. She continues her work by taking tiles from the same pile again. This time she is not satisfied with the outcome so she takes two different tiles and creates a motif, which is upside-down. The last one she created was correct so she finished her work (Fig.3).



Strategy of a conscious creation of one motif by using two different types of tiles. Before starting the work, a child visually analyzed the whole pattern prepared by the teacher, as well the manipulative material. He/she could perceive the relation which enables them to continue the work without any trials preceding the right action. Sometimes only few manipulations support his/her decisions.

Example: Kuba (boy)

- Pupil: Observes...18 second motionless.
- Teacher: Go on. If you have any questions, you can ask. You can do whatever you want.
- Pupil: Takes one tile in his left hand, arranges it in a certain distance from the pattern as if he was planning to place the second one to match them. 10 sec break.
- Teacher: You started well.
- Pupil: 8 seconds. He takes a tile from the second pile and connects it to the motif. Then, he takes two tiles from the left pile, places them close to each other.

He manipulates them for a while but quickly puts them back and reaches for the other tile from the right pile. Next couples of tiles are arranged well. He continues the pattern from both sides.

Commentary: On this level actions from two distinct areas of activity exist: primal instinctive actions stimulated by a visual impulse and actions preceded by a reflection and a visual analysis of shape. Observations confirm that visual information is very important and many children can use it in a way, which is significant for 'geometrical seeing'. This means that children have the ability to analyze shapes, create a visual relation between the whole and the part, and perceive the relation of mirror reflection.

2. Various understandings of the instruction: try to continue.

Strategy „any nice motif”. In this situation, 4 year old children understand that tiles are a means to create a motif. They reach for them eagerly, and observe configurations of two tiles. Every interesting arrangement is a good solution for them.

Example: Stasiu (boy)

Pupil: He takes two tiles from one pile and he manipulates them in the corner of the table. He arranges them in a way which is shown at fig. 4 and, with satisfaction, looks at them.

Teacher: Is this like in our pattern?

Pupil: He puts tiles crookedly, trying to connect the line from tiles (fig.5).

Teacher: It is nice, but does it fit into our pattern?

Pupil: He manipulates again, exchanges a tile for another one but still of the same type. Then, he creates a configuration like shown at the fig.6. Very satisfied, he looks at the teacher.

Teacher: And again you have something different than we have here (the teacher shows the pattern). I will give you a small hint: try to take a tile from this pile.

Pupil: Quickly he reaches to the second pile and connects the motif (fig.7).

Teacher: So. And what do you think?

Pupil: He moves his motif to the pattern and says: this is a happy face.



Strategy „one, identical motif”. Among 4 year old children continuity does not necessarily mean infinity. This may mean that a child will create just one, identical motif. A child notices a rule but it is realized only by a simple duplication. This is rather a manifestation of the noticed rule than its continuity.

Example: Roksana (girl)

- Pupil: reaches for one of the motifs that were previously created by the teacher. She puts her hands on her knees, sits still and looks at the teacher.
- Teacher: So you moved one motif towards you. Now let us do the same with the second and the third one. And now try to continue. Try to make the pattern longer.
- Pupil: Simultaneously, she reaches for tiles from both piles, takes one out of each, checks the motif in the air and connects it to the pattern. She looks at the teacher.

Strategy „a lot of identical motifs”.

In this case, a child sees that the pattern consists of certain motifs and there is a large number of them. They do not necessarily have to match one another.

Example: Zuzia (girl)

- Pupil: First, she decides to arrange a motif using the same type of tiles but quickly she changes her strategy. She takes tiles from two piles, arranges a couple of separately placed motifs.

Strategy „one-dimensional continuation”. A child demonstrates the awareness that a pattern can be continued in both directions – to the right and to the left.

Example: Tomek (boy)

- Pupil: Immediately reaches for separate tiles from piles and correctly, in turns, he continues his work. Seeing that the space on the right side of table is finished, he continues his work on the left side.

Strategy „two-dimensional continuation”. A child wants to arrange tiles for as long as it is possible. If there is not enough space in a horizontal direction then it starts to build the next level, a vertical one. Nevertheless, the relation between the tiles is maintained.

Example: Ola (girl)

- Pupil: Immediately takes two different tiles in both hands and she places the connected motif close to the pattern. Without any hints she continues work in both directions – left and right. When there is no empty place in the line she asks: also here? (she shows the place over the pattern). She continues work as long as she has tiles.

Commentary: The possibility of manipulation may create occasions for something which P. Vopěnka calls ‘the first geometrical recognition’ - focusing attention on geometrical phenomena and specific relations of one object to another. A child may find satisfaction in searching for different configurations of two identical objects. But children at this age usually analyze patterns, search for repeated motifs. Finding and constructing motifs indicates a certain developmental level. In the framework of this period we may find examples of children that can spontaneously receive information from the pattern as an encouragement and challenge for making a whole

series of repeated motifs, for continuing them both in one and two-dimensional space. It is an action aimed at a rhythmical organization of infinite space.

Ad.3. Various methods of retrieving the „destroyed” pattern.

The correction of regularities progressed in two different ways:

A. A child rejected a „wrong tile” immediately and replaced it with the correct tile, taken from the proper pile – “replaced strategy”.

B. A child started to manipulate the „wrong tile”, trying at all costs to obtain the mirror position – “manipulative strategy”. Despite of his previous experience gathered while making the pattern, children undertook attempts of matching up two tiles of the same type. The strategy can be divided into three subcategories:

B1. A blind manipulation, simultaneous rotation of one or two tiles. Here, a child is convinced that two tiles don’t match each other but through a certain movement they could fit.

B2. A feeling that one tile is right but the second one is somehow wrongly placed. Therefore, manipulations, mainly rotations, are made with only one tile. Frequently a change order of tiles and their places occur.

B3. Going to the reverse side of the tile. Initial manipulations (rotations and translations) occur only in the area of a one-side oriented plane. After this stage, a child reverses the tile to its other side and checks the possibility of placing it in a different orientation.

Commentary: The occurrence of manipulation strategy suggest that there is a big conceptual gap between a static understanding of axis relation and its dynamic depiction. In the observed age group there was no crucial connection between the stage of making the pattern and the stage of correcting it. It seems that children treated the tasks as two totally different activities. As the children could not see any relation, they did not use the experience from the first stage. The first stage required only visual information. If they used it, they succeeded. The second stage introduced a false suggestion. Children recognized that the motif on the exchanged tile consists of a circle and arch configuration but they could not recognize the mirror symmetry in it. Because of obvious reasons, this manipulation strategy could not lead to success, but it seems that by these actions children gained many important experiences. For example, they became convinced that certain movements on a plane lead only to a limited range of final configurations. This type of movements will probably have a great significance for creating concepts of geometrical transformations or dynamic visual imaginations of geometrical objects.

The action, where a child uses a ‘replaced strategy’ could be interpreted dually. It is very probable that a child is well capable of benefitting from visual information. It is possible that a child sees the connections between two separate piles with tiles and the whole motif and can analyze shapes. In this case, when a child decides to replace

a tile, he/she chooses the „strategy of certainty”. The other interpretation is that a child knows only that two different piles exist, and by using tiles from both it is possible to be successful in some way. Those two interpretations do not give any answer about children’s intuitive knowledge regarding mirror symmetry as a transformation. The fact that some children immediately exchanged tiles for the proper ones does not necessarily mean that they were aware of the relation type or the type of the movement which is required for mirror translation. Such intuitions could only emerge during manipulations.

The table below contains the quantitative specification which shows the presence of these strategies in children’s work.

Replaced strategy (A)	Manipulative strategy (B)	Helpless	Other
4	13	1	1 ²

Table 1. Pattern correction strategies

SUMMARY

In the research, which I partially describe in this article, educational level of four year old children came out to be diverse. The results of investigations show different phases, activity levels in the framework of geometrical regularities.

Psychologists underline the great importance of visual information in early childhood. It is important for thinking development as perceived objects provoke a closer active recognition. Such direction should be obligatory when we speak about geometrical objects. The perceived geometrical phenomenon should be investigated by means of a spontaneous manipulation. Therefore, the direction should be as follows: phenomenon -> manipulation.

At this stage, manipulations are evoked by perception and are subordinated to perception. The manipulation itself is only a tool which enables to reach the aim. A child has a vague feeling that some kind of manipulations can establish an expected relation between objects, but has no idea what kind of movement is needed. While solving the problem, child does not consider what kind of manipulation he/she makes. In spite of this, these manipulations are important for further discoveries. The research showed that in this age group beginnings of behaviors that may be treated as a good basis for creating geometrical concepts in the future (dynamic images of geometrical transformations) take place.

Educational level of four year old children in this field may prove to be important. Observations in older age groups indicate a loss of dominance of a manipulative strategy to the advantage of a replaced strategy. Does it mean that the awareness of axis-symmetrical transformation increase? In my opinion, no. To my mind, it is the outcome of a higher ability to analyze shapes, to decompose a whole object into its attendants. A symmetrical object consists of two ‘identical’ halves, and older children find it easier to recognize them. But static relation of axis symmetry does

not mean that children understand transformations that change one half into the other.

A question arises: are these the following developmental steps of understanding these regularities or maybe they are the outcome of different relations between visual representations and actions? An overall glance on the course of individual children's work confirm that actions in the first phase do not give any reasons to forecast the way in which children will work in the second one. These problems require further investigations.

On the other hand – in this case, immaturity in visual analysis of shapes can be beneficial. Children do not make decisions on the basis of visual recognition of differences among tiles. They make most of their manipulations in a spontaneous way, and by this they gain experience which activates a dynamic understanding of geometrical relations.

The level of work with 4 year old children, for various reasons, is a very promising one. Every time, when a child is able to start the work, the outcome of undertaken actions can be treated as a springboard for a further discussion. None of chosen approaches towards the task can be understood as wrong and by this children do not suffer from the feeling of defeat. It gives a chance to compare results, discussion. It give a chance to function in the world of regularities, which is crucial for general mathematical understanding.

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CAN YOU DO IT IN A DIFFERENT WAY?

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In order to distinguish between two things one employs explicitly or implicitly a certain criterion. This criterion, being relevant to make the distinction in a given setting might be irrelevant in another setting. What counts as different in mathematics needs to be agreed upon. In this paper we analyze kindergarten children's different solutions to one task in order to learn about their ways of coping with multiple solutions and with multiple solution strategies. Our findings suggest that kindergarten children are able to suggest multiple solutions to this task and to apply several strategies to solve it, and that these abilities could be promoted by their engagement in related activities.

Let us start with a story about two kindergarten children, Nir and Jonathan, who were engaged in the Create an Equal Number (CEN) task. In this task, a child sat in a quiet corner of the kindergarten with an adult. He was presented with two distinct sets of bottle caps – three bottle caps were placed on one side of the table and five bottle caps were placed on the other (see Figure 1). All bottle caps had the same shape, size, and color. The child was asked: "Can you make it so that there will be an equal number of bottle caps on each side of the table?" After the child rearrange the bottle caps, the interviewer returned the bottle caps to their original arrangement (three in one set, five in the other) and asked the child, "Is there a different way to make the number of bottle caps on each side equal"? This rearrangement of the bottle cops (3 and 5) and the related question were repeated until the child said that there is no other way.



Figure 1: The initial stage of the CEN Task

The story of Nir: Nir looked closely at the two sets of bottle caps, and then he took out two caps from the set of five, and arranged each set of three in a similar position. In each set the caps were placed to formulate the vertices of an isosceles triangle. The interviewer then returned the caps to their original arrangement, asking Nir: "Is there a different way to make the number of bottle caps on each side equal"? Nir took out again two caps from the set of five, and this time he placed the caps in each set in a straight line, equally spread (see Figure 2).



Figure 2: Nir's second solution

Once more, the interviewer returned the setting to its original position, repeating his question. Again, Nir took out two caps from the set of five, rearranging the three caps in each set in a way similar to his first solution (isosceles triangles), but this time creating a larger distance between each pair of caps.

The interviewer rearranged the setting to its original position. Nir suggested a fourth solution, similar to his second solution (straight line), but this time with larger distances among the caps in each set (see Figure 3). In the following, and last iteration of the process, Nir provided the same solution as his first one.



Figure 3: Nir's fourth solution

The story of Jonathan: Jonathan looked closely at the two sets of bottle caps, and then he took one cap from the set of five, and added it to the set of three. This act resulted in two sets with four bottles caps in each. Jonathan disregarded the actual arrangement of the caps in each set. The interviewer, then returned the caps to the original arrangement, asking Jonathan: "Is there a different way to make the number of bottle caps on each side equal"? Jonathan asked: "may I take caps out?" the interviewer approved, and Jonathan took out one cap from the set of three, and three caps from the set of five, creating two sets of two caps each.

Once more, the interviewer returned the setting to its original position, repeating his question. This time, Jonathan removed all the caps from both sets, saying "two sets of nothing".

The interviewer returned again the setting to its original position, and posed the question. Jonathan took out two caps from the set of five, creating two sets of three caps each. In the next iteration, Jonathan took out two caps from the set of three and four caps from the set of five, creating two sets of one cap each. In the last iteration Jonathan said: "there are no other options". It seemed that for Jonathan the spatial arrangement of the caps on the table was insignificant.

What can we learn from these two stories? The two children were engaged in the task and each of them provided several solutions, attempting to fulfill the interviewer's request for different solutions. Nir based his solutions on spatial attributes and differentiated between them in two ways: the relative placement of the caps in each set (a line shape versus a triangle shape), and the relative distance among the caps in each set. Note that in each of Nir's solutions there were three caps in each set, i.e., equal numbers of caps. Jonathan's solutions differed in one way: the (equal) number of bottle caps for each solution.

The solutions of the children were based on two main criteria: the spatial placement (figural arrangement, distance); the number of elements. Within mathematics discourse, each of these criteria can be considered as relevant for differentiating among solutions in a given context. A triangle may be considered different from a line when sorting geometrical figures. The distance among elements may be considered as a relevant criterion when comparing lengths. The number of elements is a criterion for differentiating quantities. Thus, the relevance of a given criterion as a means to differentiate among solutions is related to the task at hand and to the norms related to

problem solving. These two issues are addressed in the theoretical background.

THEORETICAL BACKGROUND

During the last two decades there is a growing interest in early childhood mathematics education, and a growing recognition of its importance (e.g., NCTM, 2000; Sylva, Melhuish, Sammons, Siraj-Blatchford, & Taggett, 2004). NCTM recommends to provide children with activities aiming at promoting their mathematical thinking and understanding: "students understanding of mathematical ideas can be built throughout their school years if they actively engage in tasks and experiences designed to deepen and connect their knowledge" (NCTM, 2000, p. 21).

One way of promoting children's mathematical literacy is by engaging them in tasks with multiple solutions, and with a variety of related strategies: "opportunities to use strategies must be embedded naturally in the curriculum across the content areas" (NCTM, 2000, p. 54). The ability to identify differences and similarities among various strategies is context dependent and is by no means straight forward.

Yackel and Cobb (1996) highlighted the process of developing a common understanding of what counts as 'a different solution' in a classroom community. They claimed that "the sociomathematical norm of what constitutes mathematical difference supports higher-level cognitive activity" (p. 464). Establishing a socio-mathematical norm of what counts as different solution strategies is a key component in the creation of an autonomic learner.

Sfard and Levia (2005) analyzed a process in which Roni and Eynat, 4,0 and 4,7 year old, learned to interpret the term "the same" in a mathematical discourse with Roni's parents. Roni's mother presented the girls with two identical, closed boxes that contained marbles (the number of marbles could not be seen). She asked the girls "in which box are there more marbles? (p. 3)". To the mother's surprise, the girls chose one of the boxes, without attempting to count the number of marbles in the boxes. It was evident, from their reaction to the mother's later request to count, that both of them were capable of counting. When presented with two open boxes with the same number of marbles, upon the mother's request, the girls were able to count the marbles in each box, however did not use the term "the same" as an answer to the question "which box has more marbles?" Seven months later, the girls use counting as a strategy for comparing the number of marbles in the boxes on their own initiative, and they were also able to use

the term "the same". Sfard and Levia concluded that the use of words in a mathematical setting needs to be learned by children.

In the present study, we examined 5-6 year old children's perceptions of "what counts as different and what counts as the same" in the context of the CEN task (creating two equivalent sets when presented with two unequivalent ones).

SETTING

Two groups of 5-6 year old children participated in this study. The first group consisted of 81 children, who were taught by teachers participating in a two-year, *Starting Right: Mathematics in Kindergarten* program (this program was initiated in Israel, in collaboration with the Rashi Foundation. Details about *Starting Right: Mathematics in Kindergarten* can be found in <http://www.tafnit.org.il//pageframe.htm?page=http://www.tafnit.org.il/>).

The CEN task and other such tasks were discussed with the Project-K-teachers. The project children worked on tasks from various mathematical domains, such as geometry, measurement, number and operations. Some of the tasks involved pictorial mediators, and others involved physical mediators, like the CEN task. We bring here as an illustration one other task.

The task dealt with the concept of equality, oriented to promote the children's understanding of equivalent sets. Four children sit in a quiet corner with their teacher. Each child had a set of cards and a game board. Some cards had printed items on, and the others had the equal sign on. The number of items on each card varied from one to ten. The drawings on each card consisted of identical items. Each quantity was represented on four different cards and there were different pictures on each card (Card 1: two cars, Card 2: two pencils, Card 3: two balls and card 4: two flowers). Each child in turn was expected to place the equal sign on the board, and then to choose from among his cards two cards which displayed an equal number of objects. The child was then expected to place the cards on the game board on both sides of the equal sign, creating a "mathematical sentence". The other children were expected to confirm or to reject the correctness of the "mathematical sentence", and explain their decisions. It was also possible to place more than one card on each side of the equal sign, as long as the total number of items on each side was equal.

The second group consisted of 82 children, who were taught by teachers who did not participate in the program.

All the children learned were from low socio-economic backgrounds in the same town. Jonathan was one of the project-group children, while Nir belonged to the other group.

The CEN Task analysis

In the CEN task, a child was individually presented in the initial stage with two sets of identical items. The sets differed in the number of elements. In other words, in the initial stage, children were presented with two unequal sets. Then, they were asked to create two sets with the same number of bottle caps. After a child offered a solution, the caps were rearranged in the original setting, and s/he was asked once more to create two sets with the same number of bottle caps. This process continued until the child responded that there are no more solutions. The way the situation was presented, and the wording of the request, implied that the critical criterion for "different and same" is the number of elements in each set.

Two characteristics of the task at hand may be somewhat unusual. First, the task has more than one solution. In fact, the task has five different solutions. Also, several strategies can be used to solve the task. Some are one step strategies: (a) *Taking from both* sets a number of elements, obtaining the same number of caps in each set. This strategy led to one of the following solutions: ((1;1) - i.e., one element in each set), (2;2). (b) *Removing all* the elements from both sets. This strategy led to the solution (0;0). (c) *Taking only from the larger* set, which, in our case, meant taking two elements from the set of five, obtaining the solution (3;3). (d) *Shifting from one set to the other*, which, in our case, led to the solution (4;4). A two-step strategy is (e) *Collecting all* the elements, and then creating two new sets "from scratch". The *collecting all* strategy could result in each of the five solutions of the task.

RESULTS AND DISCUSSION

First we report on the children's solutions, then on their solution strategies.

Solutions. As mentioned above, this task has five solutions. Table 1 shows that while 45% of the non project children came up with no more than one solution, 56% of the project children offered at least four solutions.

Table 1: The numbers of solutions per child (in %)

	No solution	One solution	Two solutions	Three solutions	Four solutions	Five solutions
Project (N=81)	2	6	15	21	37	19
Non-project (N=82)	7	38	12	16	20	7

Table 2 indicates that, the percentages of project children who suggested each solution was larger than those of the non-project children. The percentages in Table 2 may also point to the level of difficulty of each solution: the solution (4;4) was the easiest, (3;3) was somewhat harder, (2;2) and (1;1) were evidently harder. The cognitively problematic solution, consisting of empty sets (Linchevsky & Vinner, 1998), was employed only by 27% of the project children and 9% of the non-project children.

Table 2: The solutions provided by the children (in %)

	(0;0)	(1;1)	(2;2)	(3;3)	(4;4)
Project (N=81)	27	52	65	80	88
Non-project (N=82)	9	38	39	67	72

Solution strategies. While analyzing the task, we relate to five strategies that were used by the children, namely *take from both*, *remove all*, *taking only from the larger*, *shifting from one set to the other*, and *collecting all*. Table 3 presents the percentages of children from both groups who employed each strategy.

The strategy of shifting one cap from the set of five caps to the set of three caps was the dominant strategy for the children in both groups. Collecting all the elements from the two sets into one large set, and then creating two new, equal-number sets with some of the elements, was the least popular strategy.

Table 3: The strategies used by the children (in %)

	Shifting from one set to the other	Take only from the larger	Take from both	Remove all	Collect all
Project (N=81)	80	73	74	27	17
Non-project (N=82)	70	51	40	9	6

Table 3 also shows that each strategy was used by larger percentages of project children than non-project children. The *remove all* strategy was employed by 27% of the project children. This strategy requires special thinking, since the sets remained empty.

The percentages presented in Table 4 may suggest that most children used more than one strategy while working on the task. Table 4 presents the percentages of the number of different solution strategies used by the children.

Table 4: The number of solution strategies per child (in %)

	no strategy	One	Two	Three	Four	Five
Project (N=81)	2	9	25	44	19	1
Non-project (N=82)	7	44	23	17	9	--

About 90% of the project children employed more than one solution strategy while working on this task, and only about 50% of the non-project children did so. Children's ability to approach the task from several angles and to use more than one strategy is impressive.

SUMMING UP AND LOOKING AHEAD

The main focus of our study involved examining 5-6 year old children's perceptions of "what counts as different and what counts as the same" in the context the CEN task. This task has multiple solutions and multiple solution strategies. A task may include an unspoken constrain –all the caps should be used while creating the two sets. Maybe Jonathans' first solution was base on

this constrain. When Jonathan was asked to find another solution, he explicitly asked "may I take caps out?" In this question, Jonathan might have expressed an understanding of the need to define the constraints of the task. Thus, he tried to find out the unspoken rules in this case. However, from Nir's behaviour we can learn that he did not have a similar constraints, and from his first solution he took out caps. Our data suggests that the project children outperformed their peers in the aspects we analyzed.

What could be concluded from the data presented here?

It seems that kindergarten children are capable of handling complex mathematical tasks, involving both multiple solutions and multiple solution strategies. The children provided creative solutions and employed creative solution-strategies. Silver (1997) argues that "mathematics educators can view creativity not as a domain of only a few exceptional individuals but rather as an orientation or disposition toward mathematical activity that can be fostered broadly in the general school population" (p. 79). He relates to three core features of creativity in the context of problem solving: fluency, flexibility and novelty. Problems that are characterized by many solution methods, or answers, have the potential, according to Silver, to enhance two core components of students' creativity: fluency and flexibility.

Our data suggests that young students at the age of 5-6 year-old may already be engaged in such activities. Yet, many students who did not take part in the project, gave many solutions, and used a variety of solution strategies. At the same time, some project children did not display such behavior. This raises the questions: What determines a child's ability to provide several solutions? and What kind of experience may foster creative behavior?

In our study the two sets were presented with concrete materials (identical bottle caps). Gullen (1978) studied K-2nd students' strategies while comparing the number of elements in two sets, but he presented them pictorially. He found strong dependencies between the strategy used to compare the sets and students' grade levels, and also dependencies between the numbers of elements in the sets and the employed strategies. His findings suggest that students' performance may be dependent on the task design.

More research is needed to identify parameters of tasks that may promote learning, i.e. presenting the task with concrete materials vs. presenting it pictorially? Starting from unequal, asking to create equal sets or starting with equal sets and asking to create unequal sets? Using homogenous elements or heterogeneous elements? Some other questions are: How many elements should be in each set? What other tasks can be presented to

kindergartens to elicit several solution and several solutions strategies? What types of tasks could encourage children to identify the critical mathematical criteria that apply for a given setting?

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CERME 6 – WORKING GROUP 15

THEORY AND RESEARCH ON THE ROLE OF HISTORY IN MATHEMATICS EDUCATION

TABLE OF CONTENTS

Introduction.....	2679
<i>Fulvia Furinghetti, Jean-Luc Dorier, Uffe Jankvist, Jan van Maanen, Constantinos Tzanakis</i>	
The teaching of vectors in mathematics and physics in France during the 20th century	2682
<i>Cissé Ba, Jean-Luc Dorier</i>	
Geometry teaching in Iceland in the late 1800s and the van Hiele theory.....	2692
<i>Kristín Bjarnadóttir</i>	
Introducing the normal distribution by following a teaching approach inspired by history: an example for classroom implementation in engineering education.....	2702
<i>Mónica Blanco, Marta Ginovart</i>	
Arithmetic in primary school in Brazil: end of the nineteenth century	2712
<i>David Antonio Da Costa</i>	
Historical pictures for acting on the view of mathematics.....	2722
<i>Adriano Demattè, Fulvia Furinghetti</i>	
Students' beliefs about the evolution and development of mathematics	2732
<i>Uffe Thomas Jankvist</i>	
Using history as a means for the learning of mathematics without losing sight of history: the case of differential equations	2742
<i>Tinne Hoff Kjeldsen</i>	
What works in the classroom Project on the history of mathematics and the collaborative teaching practice	2752
<i>Snezana Lawrence</i>	
Intuitive geometry in early 1900s Italian middle school.....	2762
<i>Marta Menghini</i>	
The appropriation of the New Math on the Technical Federal School of Parana in 1960 and 1970 decades	2771
<i>Bárbara Winiarski Diesel Novaes, Neuza Bertoni Pinto</i>	

History, heritage, and the UK mathematics classroom.....	2781
<i>Leo Rogers</i>	
Introduction of an historical and anthropological perspective in mathematics: an example in secondary school in France.....	2791
<i>Claire Tardy, Vivianne Durand-Guerrier</i>	
The implementation of the history of mathematics in the new curriculum and textbooks in Greek secondary education	2801
<i>Yannis Thomaidis, Constantinos Tzanakis</i>	

INTRODUCTION

THEORY AND RESEARCH ON THE ROLE OF HISTORY IN MATHEMATICS EDUCATION

Fulvia Furinghetti, Jean-Luc Dorier, Uffe Jankvist,
Jan van Maanen and Constantinos Tzanakis (the “joint chairs”)

In this working group, which was active for the first time in CERME 6, 23 papers were submitted. Four of them were rejected; four were accepted as a poster. In the end three of the accepter posters and one of the accepted papers were withdrawn. So in Lyon, 13 papers and one poster were presented.

If one takes into account that the working group has no tradition in CERME, and that those who submit and those who review have to find out what are the criteria for sound research about "Theory and research on the role of History in Mathematics Education", then the percentage of rejections is reasonable.

Especially the demarcation of the subject area was not always clear for the researchers who submitted. In one case, which extended to a whole series of papers, the joint chairs of the working group decided that the subject area should be defined in such a manner that these papers could be included, provided that they would have sufficient quality. Yet, originally, the subject area was described by the joint chairs in a narrower manner. The papers meant in this remark concern the history of mathematics education.

One could argue that these papers are about history and that their content may influence mathematics education, in the sense that the awareness about the nature of mathematics and its role in education that may be brought in by a study of issues of the history of mathematics education is important for pre- and in-service teacher education. Yet, this was not the manner in which the joint chairs had originally described "the role of history in mathematics education". The original idea was to assemble in the working group those colleagues who research the effects that integration of historical elements (problems, texts) in current mathematics education may have. The subdivision of the main theme in seven topics was clear in this respect, as may be seen from the list:

1. Theoretical and/or conceptual frameworks for including history in mathematics education
2. The role of history of mathematics at primary and secondary level, both from the cognitive and affective points of view
3. The role of history of mathematics in pre- and in-service teacher education, both from the cognitive, pedagogical, and affective points of view

4. Possible parallelism between the historical development and the cognitive development of mathematical ideas
5. Ways of integrating original sources in classrooms, and their educational effects, preferably with conclusions based on classroom experiments
6. Surveys on the existing uses of history in curriculum, textbooks, and/or classrooms in primary, secondary, and university levels
7. Design and/or assessment of teaching/learning materials on the history of mathematics

This is not aiming at having papers about the history of mathematics education. This means to work with students in current mathematics lessons and to find out how they respond to the historical elements in these lessons. Nevertheless, after some deliberation and also because some interesting papers were submitted, the joint chairs decided to add a new topic, n° 8, to the above list

8. Relevance of the history of mathematical practices in the research of mathematics education

and to review submissions in this area. In the preparations for CERME 7 it should be decided and clearly stated whether this topic 8 (briefly described as "the history of mathematics education") should be included or excluded from the programme.

Looking back on the proceedings of the working group during CERME6, we may conclude that there were two main streams of papers, one about the original theme of integration of history in current teaching (subtopics 1 to 7), and the other about how mathematics was taught in the past (subtopic 8). The two went together in a fairly harmonious manner.

The papers and the subtopics on which they focused are summarized in Table 1; the numbers refer to the above list of subtopics.

Ba & <u>Dorier</u>	1, 2, 8	Lawrence	2, 3, 5
Bjarnadóttir	8	Menghini	8
<u>Blanco & Ginovart</u>	5, 7	<u>Milevicich & Lois</u> (poster)	1, 4
Da Costa	8	<u>Novaes & Pinto</u>	8
Demattè & <u>Furinghetti</u>	2, 7	Rogers	3, 6
Jankvist	1, 2, 6	<u>Tardy & Durand-Guerrier</u>	1, 3, 7
Hoff-Kjeldsen	1, 2, 5	<u>Thomaidis & Tzanakis</u>	5, 6

Table 1. Main focus of the papers according to the 8 topics listed above

As to the working procedures, the time available for each paper was 45 minutes, which was equally divided between time for presentation and time for discussion. The discussions proceeded in a pointed and engaged manner, with input in the

respective aspects of the working group: research methodology, historical references, educational and mathematical points.

In the evaluation one important observation was made about the relation of this working group with another group which is active in the intersection of mathematics education and the history of mathematics, which is the affiliated study group of ICMI about the relations between the **H**istory and **P**edagogy of **M**athematics (HPM). We observed that HPM has contributions of more varied character. In this WG 15 we tried to work with a specific methodology (or maybe two methodologies: an educational research method - often influenced by historical research and methodology - for subtopics 1 to 7 and an historical methodology for subtopic 8), which as one of its elements includes a theoretical framework, in which the relevant literature is discussed.

Finally we propose for CERME7 to include this working group again, and to then name it:

"Historical dimensions and mathematics education: theory and practice
so as to include all 8 subtopics of the current working group 15.

THE TEACHING OF VECTORS IN MATHEMATICS AND PHYSICS IN FRANCE DURING THE 20TH CENTURY

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The work presented in this text is part of a doctoral dissertation in mathematics education (Ba 2006) about the teaching and learning of vectors, translations, forces, velocity and movement of translation in mathematics and physics. Here, we present the evolution of the teaching of vectors and vector quantities in mathematics and physics from the end of the 19th century up to now. We analyse this evolution in the light of the ecology of knowledge, as developed by Yves Chevallard (1994). This helps us understand the difficulties in recent periods, in order to create a successful interdisciplinary approach in the teaching of these notions in mathematics and physics.

INTRODUCTION

Vectors emerged during the 19th century at the border of mathematics and physics. We will not recall here their historical evolution (see e. g. CROWE 1967, DORIER 1997 and 2000, FLAMENT 1997 and 2003). Our interest is clearly into the history of their teaching in the curricula of both mathematics and physics in France since the end of the 19th century. Today, in France, vectors in mathematics occupy a small part of the curriculum of geometry in secondary education (8th to 12th grades), while vector quantities are taught in Physics in 11th and 12th grades. Introducing an interdisciplinary approach has been suggested in recent programs, but is yet not very successful, as shown by our study of textbooks and teachers' practices (BA 2006, BA & DORIER 2007). The bad effects of partitioning in curricula between mathematics and physics teaching has been pointed out, especially about vectors, by several authors (see LOUNIS 1989 for a review). In this context, our aim is to understand how such a partitioning has been made possible, in order to find a way to make the interrelation between mathematics and physics teaching better.

The ecological approach developed by CHEVALLARD (1994), is a theoretical tool proper to help us tackle this issue. Indeed, it allows to study the different positions and functions of vectors and vector quantities in the moving landscape of mathematics and physics teaching, with conditions and constraints for survival and development. The idea is to analyse the evolution of objects of knowledge in various (didactic) institutions like organisms in various ecosystems.

The ecologists distinguish, when referring to an organism, its habitat and its niche. To put it in an anthropomorphic way, the habitat is, in a way, the address, the place where it lives. The niche regroups the functions that the organism fulfils. It is, in a way, its profession in this habitat¹. (Op. cit., p. 142).

Following CHEVALLARD, ARTAUD (1997) analyses under which conditions new objects can emerge and live in an ecosystem.

For a new object of knowledge O to emerge in a didactical ecosystem, it is necessary that a certain milieu exists for this object, i.e. a set of known objects (in the sense that a non problematic institutional relation exists) with which O comes in interrelation. [...] A mathematical object cannot exist on its own; it must be able to occupy a specific position in a mathematical organisation, that has to be brought to life. The necessity for a milieu implies that a new mathematical organisation cannot emerge ex nihilo. It must lean on already existing mathematical or non-mathematical organisations¹. (Op. cit., p. 124).

The ecological approach consists therefore in bringing to light a network of conditions and constraints that determines the evolution of the positions that objects (vectors in our work) can have in the different periods corresponding to changes in the programs. In this perspective, we have to take into account various institutions (and their specific constraints): school in general, but also mathematicians and physicists.

We do it chronologically from 1852 up to today, according to various phases, corresponding to the main teaching reforms.

THE BEGINNINGS (1852-1925)

In 1852, techniques for obtaining the resultant of two forces is taught in physics in 11th grade (age 17). There is a reference to the parallelogram of forces, but no vectors as such, just a technique based on a geometrical pattern. The same year the term radius vector (*rayon vecteur*) is used in geometry. This comes from astronomy, where the radius vector designates the segment joining one of the foci of the ellipse describing a planet's trajectory to its position on the orbit. It has therefore not much to do with what we call a vector now.

Until 1902, vector and vector quantities are absent from French secondary teaching both in mathematics and physics. In 1902, the radius vector disappears, but the vector, as a directed line segment appears in the program of 11th grade in mechanics and kinematics, part of mathematics then. Meanwhile, in 11th grade too, in statics and dynamics, the scalar product is used to calculate the work done by a force. Therefore vectors enter the curriculum in 11th grade in the habitat of what we can call "paraphysics"¹, with a niche as representations of orientated quantities. This is

¹ This designates the topics at the border between physics and mathematics, a border that moved along the time and according to different countries.

coherent with their origin and use in science of that time. It is also coherent with the general aims of the 1902 reform, which promotes mathematics as the root of natural sciences. Moreover, the 1902 reform insists on collaborations between mathematics and physics teachers:

It would be good that [...] mathematics and physics teachers in the same support each other mutually. Physics teachers must always know at what stage of mathematics knowledge are their students and conversely mathematics teachers would gain in not ignoring some examples that they could choose, in the experimental knowledge already acquired, in order to illustrate the theories they have explained in an abstract way. (Introduction to Programmes du lycée, 1902, p.3)

The 1902 reform is quite ambitious and gives to the sciences and mathematics in particular a privileged position. A result of this ambition is that the curriculum is too important, therefore teachers complain that it is impossible to cover everything. In 1905, the ministry of education has to reduce the program. In this technical adaptation, vectors are moved from 11th to 12th grade and enter a new habitat, since they are now part of the geometry curriculum, where they have to be presented as tools for physics (their niche):

In mechanics, [...] teachers must avoid any development on purely geometrical aspect; it is in order to suppress any such occasion, that theorems on vectors have been reduced to a minimum and moved in the geometry curriculum, where they appear under their real aspect¹. (Instruction du 27 juillet 1905 relative à l'enseignement des mathématiques, p. 676)

Vectors are therefore transported from mathematical physics into geometry, in order to technically solve a purely didactical problem.

In 1925, without being explicitly in the program, vectors appear in the 9th grade, as a possible concrete representation of “algebraic numbers”, “concrete notions on positive and negative numbers”. This is a new potential habitat in arithmetics, as representations of one-dimensional orientated quantities (their niche). Here again, the reasons are mostly of didactical order.

In 12th grade, the content about vectors remains more or less the same than during the preceding period. Yet, vectors have migrated into trigonometry, for which they facilitate the didactical presentation. In kinematics, the use of vectors to represent velocity and acceleration is more systematic, like in mechanics, with forces. The habitat and niche in physics are therefore reinforced. Meanwhile, a comment in the program in 1925 is quite interesting:

In statics, the confusion that happened very often between the properties of systems of forces and those of associated systems of vectors, will disappear because of the general study of the latter.

Therefore the geometrical status of vector is reinforced, so is their niche in this habitat, due to the new connection with trigonometry.

In a bit more than 20 years, for purely didactical reasons, vectors initially hybrid objects at the border between physics and mathematics, acquired a geometrical status and a potential arithmetical one. Their use in physics is not anymore essential, since they have to be introduced separately.

A SLOW EVOLUTION (1937-1967)

In 1937, the use of vectors to represent algebraic numbers in 9th grade is made official, and the projection of parallel vectors on the same axis is suggested as a means to illustrate the multiplication of numbers with a sign. In the same vein, vectors are used in the presentation of homotheties. The arithmetical habitat is therefore reinforced.

The habitat in trigonometry remains but is moved down to 11th grade.

Habitats and niches are therefore identical. Clearly one-dimensional vectors live in arithmetic for the 9th grade, where multiplication by a scalar is important, while higher dimensional vectors are introduced in the 11th grade in trigonometry. The habitat in physics appears later, but more systematically, as an application. No mention of possible bridges between the different habitats is made, while difficulties in the use of vectors in physics are noticed officially.

In 1947, there are no major changes. For the first time, vectors are used to present a vector version of Thales' theorem in the 9th grade, following the use of vectors for homotheties. In the 11th grade, vectors are now a separate chapter in geometry, no longer part of trigonometry. The term of equipollent vector is introduced, and the link with translation is made.

Therefore, vectors have now gained an autonomous mathematical status. The dichotomy between arithmetics (one dimension) and geometry (higher dimension) still exists. Yet, Thales' theorem makes a bridge between the two habitats, and put forward the multiplication by a scalar, which originally was not very important in the use for physics.

In 1957, the potential bridge between the arithmetical and geometrical habitat is made. Vectors appear in the 9th grade, in geometry, in relation with homotheties and Thales' theorem: the arithmetic habitat has been absorbed into geometry. In the 10th grade, 3 dimensional directed line segments are introduced as part of the geometry curriculum, in relation with translations and analytic geometry. In the 11th grade, the distinction between directed line segments and free vectors is made. Applications to geometry and kinematics are important. Barycentres also appear for the first time and are linked to vectors. The geometric habitat is therefore stronger and has absorbed the arithmetic habitat, which only survive in a transitory phase in the 9th grade. In this enlarged geometric habitat, the niche is not anymore the representation of vector quantities from physics, but more an efficient tool for solving geometrical problems. For educational purposes, vectors have therefore become geometrical objects. They

are used to introduce analytic geometry and barycentres, two fields of geometry that historically existed before vectors!

In physics, in 12th grade, vectors are also used in magnetism, yet mostly through representation by coordinates. This, again, is quite ironical, compared to the historical development, when one recalls that Maxwell's formulae played an important role in the history of vectors, to impose the coordinate-free notations!

MODERN MATHEMATICS (1968-1985)

In the enormous changes brought by modern mathematics, geometry teaching was to be profoundly renewed. Vectors were introduced in 7th grade, very formally. In 9th grade, the axiomatic structure of vector space was defined, yet limited to finite dimensions. In his history of linear algebra, Dorier (1997 or 2000a) has shown that the model of geometrical space, as the Euclidean three-dimensional vector space has been promoted by Dieudonné (1964) because, in his mind, it was the best preparation for the Hilbert and more general function spaces, which were important in the curriculum for post graduates in mathematics. Indeed, promoters of modern mathematics (among whom Dieudonné was one of the most radical) had a descending view of mathematics education: students had to be trained as young as possible to ideas that were essential to professional mathematicians. In this perspective, introducing geometry through vectors made possible to introduce the structure of Euclidean vector space very early. "Geometrical vectors" became then the (quasi unique) prototype of Euclidean vector spaces. Yet, this is a reduction and a deviation from the historical genesis.

[...] the nature of the geometrical vector [...] is the outcome of a dialectical perspective between algebraic structure and geometric intuition. It has to be underlined here that the expression "algebraic structure" does not mean that the geometrical vector is essentially the emergence of the theory of vector space in geometry. Indeed, one should not be misled by the proximity of vocabulary. The theory of vector space is by nature axiomatic, algebraic vectors (elements of a vector space) are not constructed, they are given objects defined only by their properties as element of a structure. Geometrical vectors on the contrary are the result of a dynamic process of abstraction: the object is created through an algebraic elaboration in interaction with geometric intuition. Moreover, the roles of vector and scalar products have been essential in the genesis of geometrical vector, whereas the linear structure put forward the multiplication by a scalar, which is not essential with regard to geometrical vectors¹. (DORIER 2000b, pp. 76-77)

A totally new mathematical organisation took place in geometry, in which vectors were central. But the nature of vectors was also changed, they became mostly examples of linear algebra theory. Therefore, a new niche appeared in the habitat of geometry: preparation of students to linear algebra, which was taught from 10th grade, up to post-graduate level (functional analysis). Vectors were also used in Physics, but the gap between formal objects and applications got very important and many students had difficulties:

The coordination mathematics-physics is getting complicated: in addition to the time lag between mathematics teaching and the needs of physics teaching there is a gap between modern mathematics taught and applicable mathematics used in the teaching of physics. Thus, a group will be constituted at the junction between the Laguarrigue and the Lichnerowicz commissionsⁱⁱ.¹ (BELHOSTE, GISPERT & HULIN 1996, p. 112)

Research works in physics education in the seventies pointed out several difficulties in the use of mathematics in physics, especially regarding vectors. MALGRANGE, SALTIEL & VIENNOT (1973) for instance interviewed students entering university and pointed out that a correct use of addition of vectors about forces or velocities was a major problem.

However, it is well known that the reform was quickly criticised and rejected.

A reform conducted by tertiary education for its own sake and interest without any clear vision of missions specific to secondary education, was certainly bound to fail right from the beginning, whatever was its scientific legitimacy and its promoters' good will. (BELHOSTE, GISPERT & HULIN 1996, p. 37)

In the late seventies, some modifications were adopted, but it is only in the early eighties, that a total reconstruction of the curricula took place.

THE COUNTER REFORM (1985-2002)

Following the failure of introduction of modern mathematics, in 1985 the teaching of vector space theory disappears from secondary education, replaced by a more concrete approach to geometry. The new program specifies: "vectors should not be only algebraic entities; mastering their relations with configurations play an essential role in the solving of geometric problems".

This eludes the fact that vectors are intrinsically algebraic, and that this algebraic nature does not refer just to the theory of vector space. Operations on geometrical vectors are part of their constitution as objects :

- Magnitude is the basis of arithmetic since Ancient Greeks.
- Orientation on the same line is what allows considering negative entities, a decisive step towards addition.
- Direction finally comes from the necessity of multiplication.

This last idea is the most complicated to understand. But, let us look at what is vector multiplication. In Greek algebraic geometry, the product of two numbers (lines) is the rectangle's area. If one considers a parallelogram instead of a rectangle, the sine of the angle formed by the two lines has to be taken into account in the formula for the area, i.e. the relative position of the two lines (the idea of negative implies to take into account the orientation of the lines). Thus, like Grassmann (1844) underlines it, in his introduction to the *Ausdehnungslehre*, the parallelogram, not the rectangle, symbolises the true concept of multiplication, if one considers orientated entities in geometry. This brings to light the importance of direction of lines in the construction of the product¹. (DORIER 2000b, pp. 79-80).

As a consequence of the rejection of any formal viewpoint in the teaching of vectors, these appear as tools for solving geometric problems, and eventually for physics, but have no clear status as objects. Even the use of vectors to illustrate operation on one-dimensional orientated quantities has disappeared. After the rejection of modern mathematics, the teaching of vector is lacking of theoretical reference. The model of linear algebra has been banished but nothing came in the place. Yet, some residues remain in few places. For instance it is still common today in textbooks for 10th grade, to show that vectors have some properties, which are actually the axioms of vector space (but it is not explicit).

Since the counter-reform in France, vectors are introduced in a naïve way in relation with translation. This viewpoint is not new, it has been developed for instance by Jacques HADAMARD (1898) in his *Leçons de géométrie*:

If by all the points of a figure, one draws equal parallel lines with the same orientation, the end points of these lines constitutes a figure equal to the original. [...] The operation through which one passes from the first to the second figure was given the name of translation. One sees that a translation is determined when a line is given in magnitude, direction and orientation such as AA' , which goes from one point to its homologue. Thus a translation is designated by the letter of such a line: e.g. the translation AA' . (op. cit., p. 51).

The vector first introduced in the 8th grade, finally got introduced only in the 9th grade. Moreover, in recent years, the content about vectors has been reduced to a minimum. The link with physics is promoted in the programs. But, as our survey of textbooks and teachers' practices (BA 2007) showed, it is very limited and very often not effective. On the other hand, vectors are used in physics to represent forces and velocity, but physics teachers keep complaining that their students are not competent enough with vectors.

In this last period, the habitat of vectors has been reduced to a small part in geometry. They are presented as efficient tools to solve geometric problems and models for forces and velocity. These niches however have difficulty in surviving. Indeed, several research works in mathematics education (e.g. BITTAR 1998, LE THI HOAI 1997, PRESSIAT 1999) have shown the difficulty in convincing students of the power of vectors for solving geometric problems. On the other hand, the distance and partitioning between mathematics and physics teaching makes the interrelation difficult. In our work, we have studied this problem not only about vectors but also about translations and movement of translation (BA & DORIER 2007).

CONCLUSION

Despite the rejection of modern mathematics in the eighties, the model of linear algebra, even if it has disappeared from secondary education, remains implicitly the only algebraic model for vectors, influencing the mathematical organisation of the teaching of vectors. In this sense, the multiplication by a scalar is overestimated

while, on the contrary, the vector product is underestimated. The axioms of vector spaces appear implicitly, while algebraic aspects more specific to geometric vectors are eluded, like the link with Thales' theorem and one-dimensional orientated quantities. The vanishing of any algebraic habitat or niche is like something missing after the (well founded) rejection of linear algebra. A reflection on the true algebraic nature of geometric vector and its link with geometric intuition is totally absent of the teaching of vector, since the beginning, while it had been an essential aspect in the genesis of vectors.

The niche "efficient tool for solving geometric problems" is quite problematic. It is indeed difficult to find geometric problems, accessible to students in 10th grade, in which vectors appear really as more efficient than more basic geometric methods. Moreover, our study of the evolution of the teaching of vectors shows that the geometric habitat was not "natural" at the beginning. From its origin as hybrid objects between mathematics and physics, vectors have been transformed, in a didactical process of transposition, into geometric entities. We have shown that several changes between 1925 and the beginning of modern mathematics have been motivated by purely didactical (not epistemological) constraints. Ideology on teaching and practical reasons often (if not always) have surpassed scientific motives. The changes occurred during the reform of modern mathematics are even more obviously driven by ideology and subject to suspicion on epistemological grounds.

The niche "tool for physics' entities" remains throughout the century up to now. Yet, our analysis of the evolution of the teaching of vectors shows that the gap between habitats in mathematics and in physics has constantly grown bigger. Until the sixties, parts of mechanics and kinematics constituted a common ground between mathematics and physics where vectors were used. Even then, an artificial separation was made and vectors got "rejected" in geometry. In today's mathematics textbooks, the examples taken from physics to illustrate the use of vectors are mostly inaccurate and often wrong from a physicist's viewpoint, while physics teachers refuse to do mathematics and expect mathematical tools to be at disposal in time (BA & DORIER in press).

For the interrelations between mathematics and physics teaching to get better, changes in the curricula will be necessary, but it will not be sufficient. For each subject capable of strengthening the relations between mathematics and physics, an epistemological analysis has to be conducted in order to make the adequate changes. Our claim is that this study must take into account the historical evolution of the concepts at stake AND the evolution of the teaching of these concepts, with a description of the constraints of the educational context. Such analyses must be the bases for teaching experimented completed by didactical analysis. Finally specific teachers' training is necessary, in order to make the changes possible.

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ⁱ Our translation.

ⁱⁱ The official commissions in charge of designing the new teaching respectively in physics and mathematics.

GEOMETRY TEACHING IN ICELAND IN THE LATE 1800S AND THE VAN HIELE THEORY

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The first Icelandic textbook in geometry was published in 1889. Its declared aim was to avoid formal proofs. Concurrently geometry instruction was being debated in Europe; whether it should be taught as purely deductive science, or built on experiments and intuitive thinking. The policy of Icelandic intellectuals was to enhance strategies to lead their country towards independence and technical progress, which partly coincided with foreign didactic currents. The discussion on geometry teaching is connected to the van Hiele theory of the 1950s on geometric thinking.

INTRODUCTION

Iceland has a well recorded history of its educational and cultural issues since its settlement around 900 AD. A large collection of literature of various kinds exists from the 12th-14th century. This includes literature of encyclopaedic nature, which contains some mathematics, mainly arithmetic and chronology. There is, however, little evidence that geometry of the *Elements* was ever studied in the two cathedral schools in Iceland in the period from the 12th to the early 19th century, while astronomical observations and geodetic measurements were made in the 1500s, 1600s and 1700s by local people who had studied at Northern European universities.

Iceland became a part of the Danish realm by the end of the 14th century. The two cathedral schools were united into one state Latin School in 1802. Their goal was to prepare their pupils for the church, and for studies at the University of Copenhagen, which introduced stricter entrance requirements in mathematics in 1818.

From the middle of the 19th century there were growing demands for independence from Denmark. Detailed proposals were written on schools for farmers and a lower secondary school for the middle class, as ways of raising educational standards of a future independent nation. Classical geometry was to be provided for those aiming for university entrance, while practical measuring skills and geodesy were proposed for future farmers.

As a milestone towards independence, the Icelandic parliament became a legislative body in 1874; an event followed up with legislation in 1880 on teaching children arithmetic and writing, and the establishment of a public lower secondary school, run by the state, established in 1880 in Northern Iceland. The school was intended for future farmers and craftsmen. Its syllabus, however, became more theoretical over time, and from 1908 its final examination was recognised as a qualification for entrance into the Latin School, which remained the only school of its kind until 1928.

Several privately-run lower secondary schools, as well as technical schools, were established from the 1880s with some support from the state.

Along with the establishment of schools, textbooks in the vernacular were written and published. Among them was the subject of this paper, the first Icelandic textbook in geometry, published in 1889, *Flatamálsfræði/Plane Geometry* by the Reverend Halldór Briem, teacher at the new lower secondary school in Northern Iceland.

EUCLIDIAN GEOMETRY AS A MODEL FOR DEDUCTIVE SYSTEMS

The study of geometry was collected into a coherent logical system by Euclid in his *Elements* in 300 BC. The main goal of studying classical Euclidian geometry, with its logical deductive axiom system, has been regarded as to provide training in logical reasoning. The Euclidian system provided a model for creating various axiom systems in the 19th century, such as for the set of positive integers in the 1880s; and Dedekind contributed to a precise definition of the idea of a real number in the same period.

There were, however, several flaws in Euclid's system, e.g. an assumption concerning continuity, not explicitly mentioned. D. Hilbert published his *Grundlagen der Geometrie* in 1899, where he defined five sets of axioms, a complete set, from which Euclidian geometry could be derived. Hilbert's set of axioms contains two which concern the basic idea of continuity, where Euclid's tacit assumption is made explicit (Katz, 1993: 718–721).

THEORIES OF GEOMETRY LEARNING

According to the theory of Pierre and Dina van Hiele, developed in the late 1950s, pupils progress through levels of thought in geometry. Their model provides a framework for understanding geometric thinking (Clements, 2003: 152–154). The theory is based on several assumptions: that learning is a discontinuous process characterised by qualitatively different levels of thinking; that the levels are sequential, invariant, and hierarchical, not dependent on age; that concepts, implicitly understood at one level, become explicitly understood at the next level; and that each level has its own language and way of thinking.

In the van Hiele model, *level 1* is the visual level, where pupils can recognise shapes as wholes but cannot form mental images of them. At *level 2*, the descriptive, analytic level, pupils recognise and characterise shapes by their properties. At *level 3*, the abstract/relational level, students can form abstract definitions, distinguish between necessary and sufficient sets of conditions for a concept, and understand, and sometimes even provide logical arguments in the geometric domain, whereas at *level 4*, students can establish theorems within an axiomatic system.

According to Clements (2003), research generally supports that the van Hiele levels are useful in describing pupil's geometric concept development, even if the levels are too broad for some tastes. The van Hiele levels may e.g. not be discrete. Pupils

appear to show signs of thinking at more than one level in the same or different tasks in different contexts. They possess and develop competences and knowledge at several levels simultaneously, although one level of thinking may predominate.

GEOMETRY IN EUROPEAN SCHOOLS

The Euclidian axiomatic deductive presentation of geometry was the norm for the subject in secondary schools of the early modern age. When people began to talk about geometry teaching based on observation and experiments, by the end of the 18th century in Denmark, the idea was hard to fight for (Hansen, 2002: 106).

Planting the seed of a new era, Rousseau wrote in his *Émile* in 1762:

I have said that geometry is not within the reach of children. But it is our fault. We are not aware that their method is not ours, and that what becomes for us the art of reasoning, for them ought to be only the art of seeing (Rousseau, 1979:145).

This quotation is in agreement with the van Hiele theory; the children are still at *level I*, the visual level.

During the 19th and early 20th centuries, the prevailing view of geometry instruction and general education in England was challenged (Prytz, 2007: p. 41–42). Mathematicians resumed the criticism regarding tacit assumptions and lack of rigour in Euclid's *Elements*. Educators argued that geometry could be made more palatable to pupils, and others demanded that mathematics instruction should be adapted to practical matters.

German philosopher and pedagogue Herbart (1776-1841) argued in 1802 that intuitive skills are important in connection to geometry instruction. Textbook writers Treutlein (1845-1912) in Germany and Godfrey (1876-1924) in England were influenced by him. Both of them underscored the importance of developing intuitive thinking in connection to mathematics instruction (Prytz, 2007: p. 43–44).

Thus experimental and intuitive approaches to geometry instruction in secondary schools were discussed in Germany and England by the turn of the 20th century. In both these countries, official reports stressed the importance of such teaching methods and they were included in the first geometry courses at the secondary schools (Prytz, 2007: p. 43).

University study by Icelanders was confined to University of Copenhagen, and they may have been influenced by Germans through Denmark. Their contact with Anglo-Saxon culture was through mass emigration from Iceland to North America from 1880 onwards. Evidence exists that there were currents of changes there too: "In the 1890s (and probably the 1880s) a major movement existed to steer geometry in the direction of practical geometry [in Canada]. There were a couple of guys from New York ... who were spearheading this movement" (Sigurdson, 2008).

THE POLITICS OF MATHEMATICS EDUCATION IN ICELAND

In the first half of the 19th century, in 1822-62, the Latin School was served by mathematician B. Gunnlaugsson. He had won a gold medal at the University of Copenhagen and, working alone, achieved the feat of making a geodetic survey of Iceland, to create the outlines of the country's modern map. During his period classical geometry teaching was developed at the school according to the 1818 requirements of the University of Copenhagen. Gunnlaugsson had to use Danish textbooks, but in order to enhance the pupils' motivation he gave them geodesy problems (Bjarnadóttir, 2006: 90–93; National Archives, Bps. C. VII, 3a).

Secondary schools in Denmark were split in 1871 into a language-history stream and a mathematics-science stream. The Icelandic Latin School was subject to the same law, but had its own regulations. It was too small to be divided into two streams, so after some lobbying and compromises the school was classified as a language-stream school in 1877; mathematics was only taught for four years of its six-year programme (Bjarnadóttir, 2006: 112-118). This decision caused some dispute and conflict for several years. University student F. Jónsson, later professor of philology at the University of Copenhagen, wrote in 1883, criticising the school and its regulations:

... to teach mathematics without practical exercises ... is ... as useless as it can possibly be, ... *the worst has been the lack of written exercises*; ... all deeper understanding has been missing, all practical use has been excluded ... the new regulations have 1) thrown out trigonometry, 2) prescribed that mathematics is only to be taught during the 4 first years (previously all) and thereby dropped for the graduation examination, and 3) geometry is to commence straight away in the lowest class; these three items are as I conceive them equally many blunders; ...

...to leave out the trigonometry is to leave out what is the most useful and interesting in the whole bulk of mathematics ... that the [geometry] study is to commence in the first grade; in order to grasp it, more understanding, more independent thought is needed than those in the first grade generally have; [I] tutored two lads in geometry and both of them were not stupid, and not young children, and for both of them it was very difficult to understand even the simplest items; but the reason was that they neither had the education nor the maturity of thought needed to study such things, which is entirely natural (Jónsson, 1883: 115–116).

The pupils of the Latin School were sons of farmers, clergymen and officials. The clergy also made their living from farming, as did county magistrates, so the majority of the pupils came from farming communities where there were no primary schools. New pupils came to school prepared by clerics in Latin, Danish and basic arithmetic, having seldom met geometric concepts. Land was e.g. not measured in square units, but valued according to how much livestock it could carry.

In terms of the van Hiele theory, one may take the view that the pupils did not possess 'the maturity of thought' needed to study deductive geometry as presented in the Danish author Jul. Petersen's system of textbooks, written in the period 1863-78

and used at the Latin School at the time to which Jónsson refers. The pupils were expected to jump to *level 3* of geometric thinking without any preparatory training at lower levels. Petersen's obituary said:

It was first around the turn of the century people began to realise that the advantages of these textbooks were more obvious for the teachers than for the pupils ... the great conciseness and the left-out steps in thinking did not quite suit children (Hansen, 2002: p. 51).

A reviewer wrote about the introduction to Petersen's 1905 edition:

... one reads between the lines the author's disgust against modern efforts, which in this country as in other places deals with making children's first acquaintance with mathematics as little abstract as possible by letting figures and measurements of figures pave their way to understanding of geometry's content ...

Working with figures ... aids the beginner in understanding the content of the theorems, which too often has been completely lost during the effort on 'training the mind'. If the author knew from daily teaching practice, how often pupils' proofs have not been a chain of reasoning but a sequence of words, he would not have formed his introduction this way ... for the middle school, it [the textbook] is not suitable (Trier, 1905).

Petersen's textbook on introduction to geometry remained as an introductory course at the school for nearly a hundred years, to be discarded in the late 1960s (Bjarnadóttir, 2006: 320); and it may have disrupted the life of many a young pupil.

GEOMETRY BY HALLDÓR BRIEM

The Reverend Halldór Briem (1852-1919) published his *Flatamálsfræði/Plane Geometry* in 1889. Briem studied 1865-71 at Reykjavík School, where he benefited from the controversial mathematics teaching described above by Jónsson. Briem stayed during 1876-81 in the Icelandic communities in Manitoba and Winnipeg in Canada, where he was editor of an Icelandic journal and was ordained as pastor to the immigrants. He may have become acquainted with school mathematics there, but there is no record of this. H. Briem wrote textbooks on geometry, English, Nordic mythology, Icelandic grammar and Icelandic history, in addition to plays, and made various translations into Icelandic, e.g. of the story of Robin Hood.

In the foreword to the *Plane Geometry*, H. Briem declared his policy:

... no textbook in geometry in Icelandic has been available. I have therefore had to make use of foreign textbooks ... Other schools for the public in this country have not been in a better situation in this respect, and this shortage is the more severe, as knowledge of mensuration is completely indispensable in various daily tasks of farmers, carpenters and others, besides that it is an important aspect of general education ...

In composing it, my goal has mainly concerned what is the most important in general working life and therefore I have emphasised the main items concerning that as much as possible, and omitted other items that are less important to working life. The arrangement

of the content is therefore different from what is customary in this kind of textbook, where every sentence is supported by scientific proofs, but according to my policy that did not apply here (Briem, 1889: iii-iv).

H. Briem's brother, the Reverend E. Briem, was also a textbook writer. His *Reikningsbók/Arithmetic* (1869) was a dominant textbook for adolescents, also at the Latin School, from 1869 to the 1910s. The brothers were hardly much involved in didactic discussions such as those which took place in Europe, about mathematics as a discipline exclusively to train the mind. They declared that it was their first aim to meet the immediate needs of young people for practical knowledge. One might even conjecture that they saw the bother of proving self-evident facts as an intellectual luxury (or adversity) that was not to be foisted on educationally-deprived youth.

The introduction to H. Briem's *Plane Geometry* is devoted to basic assumptions, such as the attributes of a space, a body, a plane or surface, a line and a point, in this order. The body is not composed of planes, the author states, and the plane not of lines, as the planes have no thickness. The line has no width and it is not composed of points. However, he does claim that two lines meet in a point. If one thinks of a point moving from one spot to another, its track is a line. If a line moves in a direction perpendicular to itself, its track will be a plane and if a plane moves in a direction perpendicular to itself, its track will be a solid (Briem, 1889: 1–3).

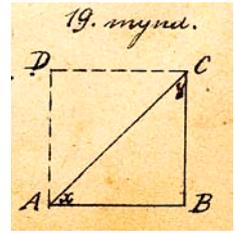
The great master, Gunnlaugsson, who had taught H. Briem's teacher and his brother at Latin School, also presented lines as tracks of points, planes as track of lines and bodies as the track of planes, but he did not mention that lines were not composed of points. However, a geometric plane could not be parted from the body of which it is a border, except in the mind by abstraction; nor could a geometric line be parted from the plane of which it is a border, or a geometric point be parted from the line of which it is a border, except in the mind by abstraction (Gunnlaugsson, 1868).

H. Briem seems to have thought of points as discrete objects and a line as a continuous track, not thinkable as made up of points. Briem had little opportunity to become acquainted with modern ideas of real analysis or the work of Dedekind in the 1880s. The work of Hilbert on Euclidian geometry, where Euclid's ambiguity about continuity was amended, had not yet appeared. But a clergyman teaching geometry to adolescents on the periphery of Europe felt a need to philosophise on his own, about the nature of lines and planes and their relations to points.

Briem continued with definitions: of parallel lines, an angle, of plane figures, such as triangles, various quadrilaterals, polygons, the circle and the ellipse and of similarity and congruence. The names of the shapes are in Icelandic with Latin in parentheses. Remembering the names must have been difficult, as this was the first Icelandic book on geometry. A score of exercises follow the definitions. Attached to the exercises are answers to them and explanations. This was necessary, as lower secondary schools were scarce and the textbook was to serve for home studies as well.

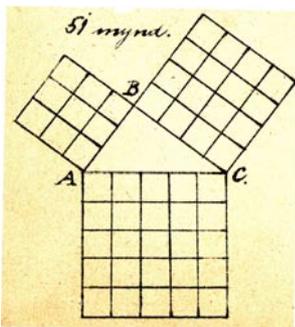
In connection to the definition of a triangle, its attributes are also investigated:

All the angles in a triangle are 180° in total. In the triangle ABC (diagram 19) CB is perpendicular to AB, therefore the angle B = 1R [R a right angle], furthermore CB is equal to AB; by drawing the triangle ADC equally large and similar to the triangle ABC [congruence had not yet been defined], one may see that x and y each are half of a right angle, therefore the sum of the angles in the triangle is 2R. The same applies to all triangles, as the larger or smaller one of the angles is, the others (one of them or both) become smaller or larger (Briem, 1889: 14).



In this text reference is made to a diagram; but because of the high printing cost, all diagrams are printed together as an appendix at the back of the book. Clearly the author appeals to the intuition of the reader to see that the angles x and CAD are complementary, as well as y and ACD. Furthermore, the triangle ABC is a special case of an isosceles right triangle, but the reader is invited to take its attributes as universal. The author had introduced parallel lines and their angles to a transversal line, and so he could have presented the regular proof of the sum of angles in a triangle, but preferred to do it this way.

The common reader, the future farmer or carpenter, may not have been expected to need more 'scientific' proofs. The fact that the sum of the angles in the triangle ABC is two right angles is more or less obvious from the diagram, but more credulousness is needed for believing that it applies to all triangles. Schools, through the centuries, have expected their pupils to believe what is stated in textbooks. This is not much different, except for the point of view that mathematics studies are expected to foster critical thinking among their students.



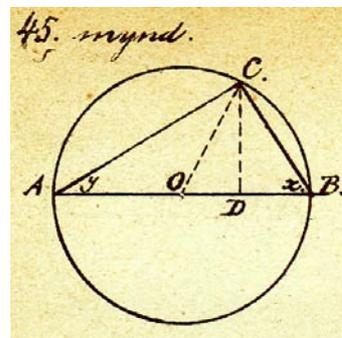
In continuation, the square root is introduced, as are common measuring units, which were fairly complicated before the introduction of the metric system in 1907. The next chapter concerns areas of parallelograms, squares, rhombi and triangles, with plausible explanations aided by the diagrams at the back of the book. The areas of a trapezoid and polygons are deduced from the area of a triangle. Heron's rule is introduced without proof or explanation, as is the Pythagorean Theorem, whose proof is stated to be too difficult for the readers. A

diagram of the 3 – 4 – 5 triangle (diagram 51) is presented as an illustration of the rule.

In a circle the perimeter is stated to be $3\frac{16}{113}$ times the radius, while later this and other values for π are said to be approximations to the true value, which may be reached as accurately as desired. The circle is conceived as composed of many small triangles, whose top-angles meet at the centre of the circle, from which the area of a circle was deduced. This continues with areas of sectors and annuli, and finally of an ellipse.

A chapter is devoted to proportions, which was probably difficult, as the pupils may not have had much experience in solving equations. When discussing proportions in the right triangle, the author reveals the algebraic proof of the Pythagorean Theorem.

In the final chapter, the author introduced constructions; to bisect a segment, to divide a segment into any number of segments, to construct a right angle, to double the area of a square and a circle, and to transform a rectangle to a square with the same area. This is illustrated in diagram 45, where the dimensions of the rectangle are AD and DB and the side of the square is the altitude CD. This is a consequence of proportions in the right triangle already introduced, and the author refers to it through diagrams. Earlier, the necessary prerequisite, that a periphery angle is half the centre angle of the same arc, had been illustrated for a right periphery angle, sufficient for this construction.



All things considered, the text, after the initial introduction of concepts, is readable, although concise, with sensible explanations of most of the formulas with the aid of diagrams, which regrettably could not be attached to the text in concern. The exercises were mainly computations of sizes of angles, lengths of sides in right triangles and various area computations, but no constructions. One may suggest that the level of the book was closer to van Hiele *level 2* than e.g. Petersen's textbook, but was certainly not *level 1*.

However, though it may be arguable that Briem's *Geometry* was based on observations of his diagrams, it can hardly be maintained that they concerned the pupils' real world. The problems seldom had content, and if so they were synthetic, in the sense that they asked to find areas that few would want to know. It was not customary to compute the area of land except to estimate the time needed to mow it, and few had reasons to determine the area of an ellipse-shaped dining table. The author was indeed faithful to the Euclidian content, but was unafraid to simplify proofs and appeal to intuition.

The author of *Plane Geometry* taught mathematics, Danish, singing and physical education in the state-run lower secondary school in Northern Iceland. *Plane Geometry* was used in that school and possibly in some other schools, but not at the Latin School, which adhered to law on Danish Latin schools. However, Briem's second geometry textbook on volumes (Briem, 1892), which was not as sensitive to rigour, was used there for some years.

In 1904 a learned mathematician, Dr. Ó. Daniélsson, graduated from Copenhagen University and returned to Iceland to teach. He completed his doctoral degree in 1909, with geometry as his special field. Until his time there was no mathematician with whom to debate geometry instruction. Dr. Daniélsson tried to use Briem's *Plane Geometry* in teacher training for one year, but gave up. He turned to foreign textbooks until he published his own, where he used e.g. the definitions of parallel

lines and their angles to a transversal line to prove that the sum of the angles in a triangle is 180° . He also proved the theorem of Pythagoras with the aid of geometric figures (Dánielsson, 1914).

DISCUSSION

Many pedagogues emphasise that learning is dependent on the cultural environment (see e.g. D'Ambrosio, 2001). It is notable that through the history of education in Iceland, trigonometry and geodesy stand out as being considered interesting and useful subjects, while no trace is found of rigid Euclidian geometry for any other purpose than fulfilling the entrance requirements of the University of Copenhagen.

H. Briem belonged to a generation of intellectuals who were much aware of the low status of education in Iceland, and who participated in the campaign for independence in order to be able to form own Icelandic educational policy. Briem was one of two teachers who were appointed to a new lower secondary school, of which people had great expectations that it would raise the level of education of the general public. The school was not restricted by any regulations on mathematics content, so Briem had freedom to form the mathematics instruction as he saw fit.

Briem's *Plane Geometry* may be seen as a reaction to the criticism of teaching in the Reykjavík School and of Petersen's textbook. Briem maintains that no foreign textbooks suited him as a model. However, his textbook seems to have been created according to international currents, promoting geometry teaching based on intuition and observation. This approach has resonance in the van Hiele theory, that pupils go through sequential levels of thought and have difficulties in reaching without preparation the abstract/relational level to understand or provide logical arguments unless they have been through lower levels of visualisation and description. One can hardly claim, however, that Briem was entirely successful in meeting the pupils' level of geometric thinking, but he did avoid bothering them with proving what they might have thought 'obvious facts'. His collection of exercises did not contain any pure deduction, but consisted of fairly approachable numerical exercises.

These were times of rapid change, from a stagnant agricultural society. Craftsmen were a rising class in the 1890s and the textbook was intended to introduce them to basic facts of geometry. It must have been of use in their trade, in view of the fact that no other text on the subject was available in the vernacular. Briem made a great effort to transform concepts from foreign languages into Icelandic, which had no tradition of geometry. It is, however questionable how far he succeeded in connecting the content to the Icelandic environment.

Briem's textbook was indeed an ambitious textbook for its time; and no comparable textbook intended for the non-college-bound general public, and reaching that level of complexity, has been published since in Iceland.

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INTRODUCING THE NORMAL DISTRIBUTION BY FOLLOWING A TEACHING APPROACH INSPIRED BY HISTORY: AN EXAMPLE FOR CLASSROOM IMPLEMENTATION IN ENGINEERING EDUCATION

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Abstract: *Probability and random variables turn out to be an obstacle in the teaching-learning process, partly due to the conceptual difficulties inherent in the topic. To help students get over this drawback, a unit on “Probability and Random Variables” was designed following the guidelines of the European Higher Education Area and subsequently put into practice at an engineering school. This paper focuses on the design, implementation and assessment of a specific activity of this unit concerning the introduction of the normal probability curve from a teaching-learning approach inspired by history. To this purpose a historical module on the normal curve elaborated by Katz and Michalowicz (2005) was adapted to develop different aspects of the topic.*

Keywords: probability, normal distribution, European Higher Education Area, teaching-learning materials on history of mathematics.

INTRODUCTION

Teaching probability and random variables turn out to be essential for the introducing of statistical inference in any undergraduate course in basic statistics. Statistics is one of the compulsory undergraduate subjects included in the syllabus of any engineering school. This subject, as developed at the School of Agricultural Engineering of Barcelona (ESAB) of the Technical University of Catalonia (Spain), primarily encompasses Data Analysis and Basic Statistical Inference. We believe that the very nature of the subject calls for special consideration in the teaching of the subject, especially with regard to the new European Higher Education Area (EHEA). Besides, the essentially biological profile of the ESAB seems to weaken interest in mathematical domains.

From our experience in teaching statistics at different engineering schools, we are well aware that probability and random variables represent a rather overwhelming obstacle for students, due to the conceptual difficulties inherent in the topic. To help students get over this drawback, a unit on “Probability and Random Variables” was designed following the guidelines of the EHEA. Subsequently, this unit was put into

practice at the ESAB. Throughout the module, the teaching-learning process was assessed using several evaluation techniques so as to analyse the learning outcome (Blanco & Ginovart, 2008). This paper focuses on the design, implementation and assessment of a specific activity of this unit concerning the introduction of the normal probability curve and some related aspects from a historical point of view.

Mathematical and statistical topics have been traditionally taught in a deductively oriented manner, presented as a cumulative set of “polished” products. Through a collection of axioms, theorems and proofs, the student is asked to become acquainted with and competent in handling the symbols and the logical syntax of theories, logical clarity being sufficient for the understanding of the subject. As a result, the traditional teaching of mathematics tends to overlook the mistakes made, the doubts and misconceptions raised when doing mathematics, detaching problems from their context of origin. However, since the construction of meaning is only fulfilled by linking old and new knowledge, the learning of mathematics, in general, and statistics, in particular, lies in the understanding of the motivations for problems and questions. In this respect, integrating the history of mathematics in education represents a means to reflect on the immediate needs of society from which the mathematical problems emerged, providing insights into the process of constructing mathematics (Tzanakis & Arcavi, 2000; Swetz et al., 1995).

How to introduce a historical dimension in our unit on probability and random variables turned out to be a challenge to our “standard” teaching activity, all the more so because first we had to determine which role history would play in the unit. Of the three different ways suggested by Tzanakis & Arcavi (2000) to integrate history in the learning of mathematics, the one that seemed to serve our purpose best was to follow a teaching-learning approach inspired by history. In the context of this paper history was integrated implicitly, since the main aim was to understand mathematics (statistics, in particular) in its modern form, bearing in mind, throughout the teaching process, those “concepts, methods and notations that appear later than the topic under consideration” (Tzanakis & Arcavi, 2000, p. 210). Accordingly, after having selected a historical module on the normal curve elaborated by Katz and Michalowicz (2005, pp. 40-57), we adapted it to develop different aspects of the topic. The aims of the activity were to:

Aim 1.- Show motivation for the topic.

Aim 2.- Show interrelation between mathematical domains, on the one hand, and mathematical and non-mathematical domains, on the other.

Aim 3.- Compare modern “polished” results with earlier results.

Aim 4.- Produce a source of problems not artificially designed for the purpose.

Aim 5.- Develop “personal” skills in a broader educational sense.

These aims are explicitly connected with the ones described by Tzanakis & Arcavi (2000, §§7.2. (a) and 7.2. (c1), pp. 204-206).

THE NORMAL DISTRIBUTION: AN INTRODUCTION INSPIRED BY HISTORY

Right at the beginning of the course our students are informed about the specified learning outcomes, classified according to Bloom's taxonomy (Bloom, 1956) into: Knowledge, Comprehension and Application. The learning outcomes regarding the normal distribution have been articulated as follows:

Table 1. Learning outcomes regarding the normal distribution.

After attending the course the student will be able to:

a) Define and recognize the normal (or Gaussian) distribution, as well as the standard normal distribution.	[Knowledge]
b) Convert an arbitrary normal distribution to a standard normal distribution.	[Comprehension]
c) Calculate probabilities of events when a normal distribution is involved, using the table of the standard normal distribution.	[Comprehension]
d) Describe the empirical rule 68-95-99.7.	[Comprehension]
e) Apply the rule 68-95-99.7 to assess whether a data set is normally (or approximately normally) distributed.	[Application]
f) Estimate the approximation of the normal distribution to the binomial distribution.	[Application]

To adapt the historical module it was first necessary to frame the activity within well-defined boundaries (Katz & Michalowicz, 2005). Therefore, we started selecting and later reflecting on some questions suggested by Pengelley (2002) for assessing historical material: (a) What is the purpose of studying the material? (b) How does it fit in with the curriculum? (c) Are there appropriate exercises, with an appropriate difficulty level and well chosen to demonstrate concepts? (d) Will it motivate students? (e) Will it help with something students have trouble with? Since the activity described in this paper was directed towards the learning outcomes mentioned above (see Table 1), question (b) was explicitly involved.

To show the original motivation for the topic of the normal distribution, the activity emphasized interrelation between statistics and health and social sciences, hence covering Aims 1, 2 and 4. Although the topic had already been introduced in the classroom, the teaching-learning process was able to benefit from the study of non-artificially designed problems. From Katz and Michalowicz's module we elaborated the material for the activity combining information about the historical development of the normal curve with some "appropriate" questions. There were no accompanying answer sheets as the activity was designed to be worked out in a two-hour computer lab session, individually or in pairs. Most of the students worked individually, whereas only few computers were shared by two students working together. The teacher acted as a consultant during the session. Students managed the time given over to every section of the activity themselves, according to their individual needs and skills. If they could not accomplish their work in the computer lab, they had the possibility to do it as homework. It is worth pointing out that the questions were

chosen not only to assess understanding of the information provided, but also to bring out the connection with other mathematical domains. Hence, students were asked to prove expressions and formulae, to use a spreadsheet to carry out elementary probability calculations and to represent data, and to investigate supplementary aspects regarding the contents of the activity. All these aspects were planned in order to cover Aims 3 and 5.

In connection with question (a) stated above, this activity attempts to introduce the normal probability distribution in its original context, and to help students to get acquainted with basic calculations involving the normal curve. The first section of the activity shows how De Moivre (1667-1754) obtained his discovery of the empirical rule 68-95-99.7. The second section gathers the discussion on the error curve in which Laplace (1749-1827) and Gauss (1777-1855) were involved. How Quetelet (1796-1874) calculated the table of the normal distribution from the approximation of the normal distribution by the binomial distribution is the target of the third section. To close the activity, the fourth section is centered on the first uses of the normal distribution in the real world, namely: i) analysis of the chest circumference of 5732 Scottish soldiers; ii) analysis of the heights of French conscripts to assess the normality of the distribution, revealing a significant figure of men who illegally avoided recruitment.

We interspersed the text with seven leading questions related to the topics discussed, conveniently placed after a specific topic, and not on a separate sheet at the end. *Questions 1, 4, 6 and 7* were directly inspired by the ones suggested by Katz and Michalowicz (2005) on pages 46, 55, 56 and 57, respectively. The rest were stated by us, to ensure that a particular point was fully understood. The questions were conveniently placed after a specific topic or a related result. The following paragraphs briefly describe each question, drawing attention to the educational aims served by each one.

Question 1: In an experiment in which 100 fair coins are flipped, about how many heads would you expect to see? What is the corresponding standard deviation? Find the limits (lower and upper) for the number of heads we would get 68%, 95% and 99.7% of the times.

This first question deals with direct manipulation of a binomial distribution, followed by a first encounter with the connection between the normal and the binomial distributions. This was intended to help students “warm up” by stating a link between the activity and a topic they had already learned in the classroom, thus relating to Aim 1.

Questions 2 through 4 are connected with Quetelet’s calculation of a symmetric binomial distribution. He considered the experiment of drawing 999 balls from an urn containing a large number of balls, half of which were white, and half black.

Question 2: Prove Quetelet's shortened procedure for the calculation of relative probabilities: $P(X = n + 1) = \frac{999 - n}{n + 1} \cdot P(X = n)$, where $P(X = n)$ represents the probability of drawing n black balls from the urn. Setting the value of $P(X = 500)$ to be 1, calculate the relative probabilities $P(X = 501)$ and $P(X = 502)$.

Students had to deduce this recursive formula from the probability function of the binomial distribution. This question was inserted to show the interrelation between mathematical domains, namely, probability and recursive proofs (Aim 2). In this case the interest lies in how to evaluate mathematical arguments and proofs, and to select and use diverse types of reasoning and methods of proof as appropriate (Ellington, 1998). Given that students often meet difficulties in proving recursive formulae, this exercise seems to be consistent with questions (c) and (e) suggested above.

Question 3: Using an Excel worksheet recalculate column A of Quetelet's table for the values 500 to 579 and graph the corresponding curve.

To get a deeper knowledge of the binomial-normal link, students were here asked to use a spreadsheet, in particular, the spreadsheet program Microsoft Excel. Since the activity was developed in the context of computer practical sessions, students had computers at their disposal. The computer practicals offer students the possibility to be actively engaged in the learning process, as well as to apply the concepts learnt to the prospective working practice. Since this topic turns out to be a usual source of difficulty, this exercise connects again with question (e). Besides, it helps not only to compare modern results with earlier ones, but also to develop "personal" skills such as how to manipulate a spreadsheet. Therefore, this exercise focuses on Aims 3 and 5.

Question 4: A discrete variable can be approximated by a continuous variable considering the following estimation:

$$P(x = k)_{discrete} \approx P(k - 0.5 \leq x \leq k + 0.5)_{continuous}.$$

For instance, $P(x = 500)_{binomial} \approx P(499.5 \leq x \leq 500.5)_{normal}$.

Using this information, recalculate the first four values in column A using a modern table of the normal distribution.

It can be assumed that the results of drawing balls out of the urn are normally distributed with mean of the number of black balls equal to 500 and standard deviation equal to $\frac{1}{2}\sqrt{999} \approx 15.8$. Compare these results with Quetelet's binomial table.

Understanding why we do things the way we do, and how mathematical concepts, terms and symbols arose, plays a relevant role in grasping the topic (Ellington, 1998). This question allowed the students to compare a modern table of the normal curve with the earliest table. Thus Aim 3 is again involved in the proposed activity.

Finally, *Questions 5, 6 and 7* concern some real world applications of the normal distribution.

Question 5: Read carefully Quetelet's procedure for determining whether the chest circumferences of the Scottish soldiers were normally distributed. Write down those points you do not understand completely.

Question 6: From the results in the example of the heights of French conscripts, discuss how Quetelet concluded there had been a fraud.

From the reading and through understanding of the example on the chest circumferences (*Question 5*) students were to draw conclusions in the case of the heights of French conscripts (*Question 6*). However, as we will see in the following section, since Quetelet's procedure proved to be difficult to understand, only a few students managed to answer *Question 6* correctly.

Questions 4, 5 and 6 contribute to Aim 3 in that they help to compare historical results with modern "polished" ones. Likewise, Aim 4 could be achieved, since these questions convey the idea that probabilistic tools represent a means to solve real-world problems, rather than just artificial designed exercises, framed in a theoretical context. By and large, this set of questions also fosters the practice of reading comprehension skills (Aim 5).

Question 7: On the Internet, browse for information on Galton's machine. What was the relationship between the inventor Francis Galton (1822-1911) and Charles Darwin (1809-1882)?

The intend of this last question was to help develop some "personal" skills, in a broader educational sense, such as reading, summarising, writing and documenting (Aim 5). Additionally, it was interesting to point out the interrelation between mathematical and non-mathematical domains, namely, between statistics and the theory of evolution put forward by Darwin (Aim 2). A fundamental part of this question involves the writing component and documenting. The incorporation of a writing component in statistics courses has been encouraged in recent years by Radke-Sharpe (1991) and Garfield (1994). Writing helps students to think about the assumptions behind statistical, graphical or instrumental procedures, to formulate these assumptions verbally, and to critically examine the suitability of a particular procedure based on its assumptions. The inclusion of documenting (i.e. browsing the Internet) facilitates student reading, understanding and summarizing from different sources. In short, reading, writing and documenting are tools that will serve students well in their future scientific or academic writing. Encouraging students to put concepts such as these into words will strengthen their understanding of those concepts.

ASSESSMENT OF THE TEACHING-LEARNING PROCESS

Among the questions mentioned above for assessing historical material, Pengelley (2002) suggests considering whether it will motivate students (question (d)). Though not the only source of feedback, student ratings provide an excellent guide for designing the teaching-learning process and, in particular, for assessing their motivation. Therefore, at the end of the activity students were asked to rate the activity thus:

- (1) Very good, (2) Good, (3) Satisfactory, (4) Poor, and (5) Very poor.

Figure 1 shows the results of this survey. Of the 60 students who took part in the activity, half of them regarded it positively (22 satisfactory, 6 good, 1 very good), whereas the other half rated it as poor.

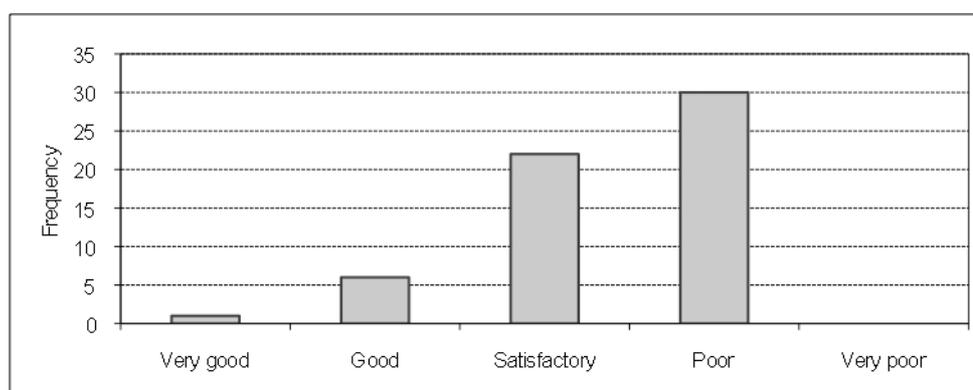


Figure 1. Student ratings on the activity.

Another aspect suggested by Pengelley (2002) for assessing historical material concerned the suitability of the degree of difficulty (question (c)). In order to determine whether the activity was appropriately difficult, we analysed in detail a random sample of size 20 drawn from the students who had handed in their answers. Every question (except *Question 5*) was marked with either Non-Answered, Poor, Fair or Good. From the graphics of Figure 2 regarding the assessment of the questions, it is clear that *Questions 1* through *4* are most frequently marked as “Good”. Surprisingly, all the students answered *Questions 1* and *2*, whereas the ratio of “Non-Answered” in *Question 6* exceeded the rest of marked ratios. As for *Question 7*, most of the students got “Fair”. This was partly due to the fact that students merely copied the information from the Internet and pasted it on their worksheets, thus showing no interest in summarising the information in their own words.

Relating to *Question 5*, from the comments given by our students we gathered that the construction of the table proved to be, in general terms, rather cumbersome.

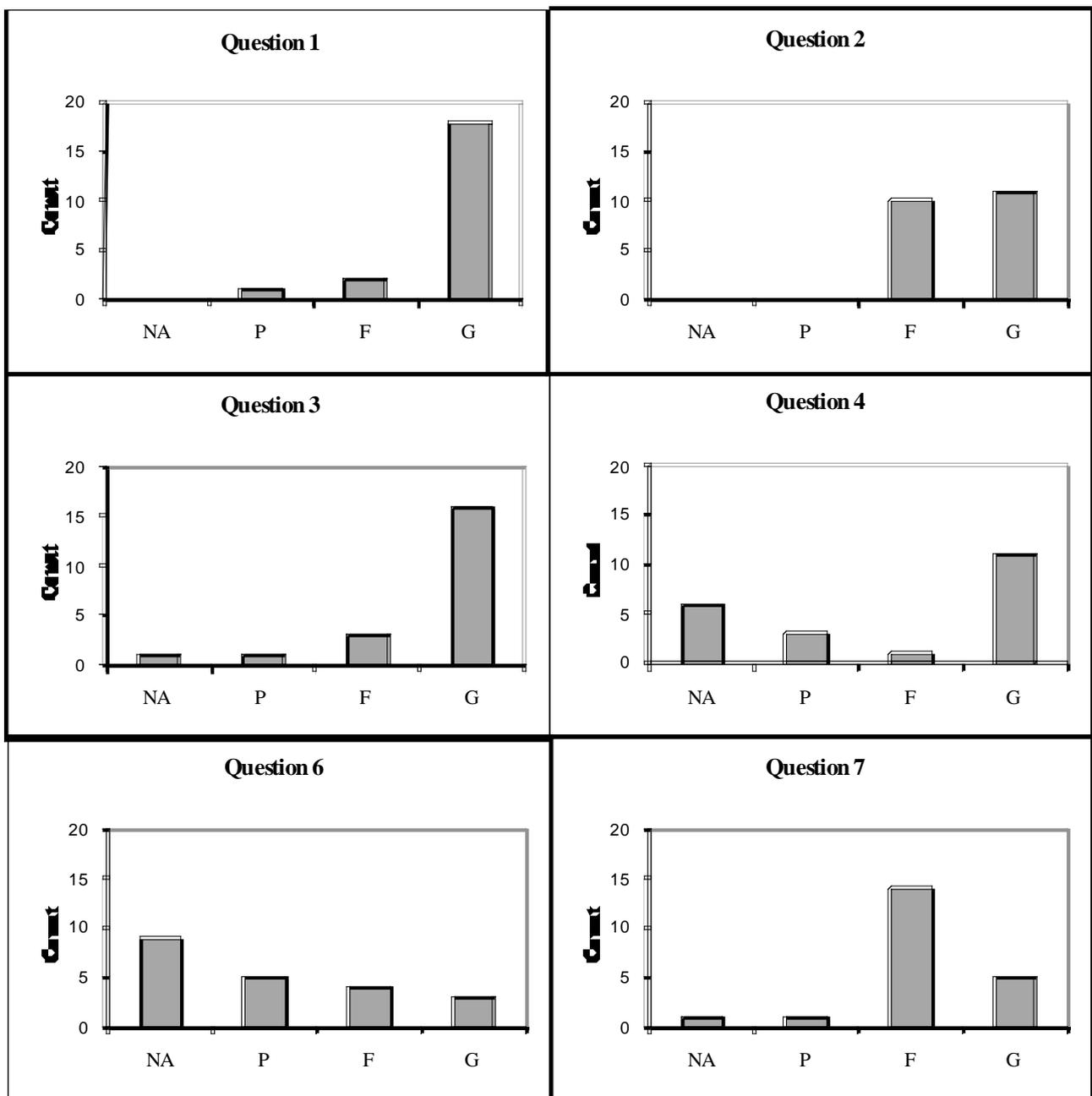


Figure 2. Assessment of the *Questions* of the activity with Non-Answered (NA), Poor (P), Fair (F) or Good (G).

FINAL REMARKS

As Fauvel and van Maanen (2000) point out, one should not underestimate the difficult task of the teacher to achieve a proper transmission of historical knowledge into a productive classroom activity for the learner. Given our lack of expertise in the field, in this first experience we were not able to foresee all the possible obstacles in the understanding process. Now we are aware of some difficulties inherent in the material (for instance, in *Questions 5* and *6*). First of all, the mathematical language

and form (notation, computational methods, etc) turned out to be rather confusing right from the beginning. In addition, the syllabus and a sense of lack of time made us cram the activity into a two-hour class. Likewise, we had a slight doubt about how useful the topic was for our students. Why not give the opportunity to appreciate the topic in itself, stressing the aesthetics, the intellectual curiosity, or the recreational purposes involved? Finally, we borrowed and adapted part of Katz and Michalowicz's historical modules on Statistics, but in keeping with our syllabus, more didactic resource material on this topic should be elaborated for future use.

On the whole, however challenging, the experience proved to be rewarding in the end. Not only did the activity supply a collection of non-artificially designed problems, but it also helped to develop further skills, such as reading, writing and documenting. Above all, it was a means to show the original motivation of the normal curve and hence, to render it more understandable. This experience has shown that probability cannot be regarded as a collection of "polished" products within a deductive structured system, but rather as a system with a peculiar life (expectations, false expectations and false starts), as Guzmán (1993) put it, determined and influenced by external factors and connected with mathematical and non-mathematical domains.

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ARITHMETIC IN PRIMARY SCHOOL IN BRAZIL: END OF THE NINETEENTH CENTURY

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The arithmetic is part of mathematical knowledge based on the idea of the number. The teaching of intuitive calculation in Brazil in primary education level at the end of the nineteenth century and early twentieth century seems to be influenced directly by the “Cartas de Parker”. These arithmetic charts based on the ideas of Pestalozzi, Froebel and Herbart were diffused in arithmetic textbooks and educational journals, testimonies of their strong influence in Brazil. This article is based on methodological presuppositions of the Cultural History, of the History of School Disciplines and the studies on the School Culture.

Keys-words: Arithmetic, Intuitive Calculation, Cartas de Parker, Grube’s Method, Elementary level.

INTRODUCTION

This article presents a partial result of the literature research related to a doctorate thesis, still under development. It aims to investigate the historical route Mathematical Education in Brazilian primary education teaching. It seeks to analyze the part “to count” of “the school of reading, writing and counting”; and includes understanding the process of its teaching by seeking answers to questions like, for example: which textbooks were adopted for the teaching of arithmetic at school? What was the role of Psychology in the evolution of the textbooks of arithmetic for primary education teaching? How were the contents of arithmetic school in the textbooks modified? What kind of modifications have the arithmetic’s textbooks been under to?

Considering the contributions of the Cultural History, the History of the School Disciplines and the studies on the School Culture, this research focuses the documentary sources such as textbook, school files, legislative texts relating to teaching as well as old daily materials (teachers’ personal files, pupils’ books, tests, periodic school magazines and exams questions)[1] .

According to Enfert (2003), unlike what occurred to the research of the French’s history of the primary education teaching, the history of the teaching of mathematics at this level did not receive the attention which it deserves. Except some cases of specialized studies, research, in a general way, mostly treated mathematics teaching at the secondary or higher level. The history of this discipline has not been treated as a whole (Arithmetic, Geometry, Geometrical Drawing, Algebra, Accountancy, etc), nor over its long duration.

In the History of the School Disciplines, Chervel (1998) defines a particular phenomenon called “vulgata”. At each time, the teaching given by teachers is, *grosso modo*, identical, similar for the same discipline and at the same level. All textbooks, or nearly all, say the same thing then, or almost. The concepts, the adopted terminology, the succession of the headings and the chapters, the organization of the corpus of the knowledge, even the examples or types of exercises performed are identical, except for some small variations. These variations justify the publication of new textbooks although they present only tiny variations.

The description and the analysis of the “vulgatas” are fundamental tasks for the School Discipline’s historian. If it is not possible to examine into the entire editorial production carefully, they must determine a sufficiently representative corpus of their various aspects. This is the only way that the historian can arrive at concrete and conclusive results.

The research in the teaching of mathematics in Brazil in primary education level at the end of the nineteenth century, particularly among textbooks of representative authors’ of their community, revealed a reference particular called “Cartas de Parker”. Their contents appears as a model and reference adopted by several textbooks published at the beginning of the twentieth century, and it seems to be like a “vulgata” and influences the teaching of the rudiments of calculus on this level of education.

INTUITIVE CALCULATION

According to Buisson (1880), intuitive calculation is a term which means a way of teaching the first elements of calculation. This methodology was borrowed from Germany and diffused in Russia, in the Netherlands, in Sweden and found a strong adhesion in the United States. This way of teaching was called Grube’s method.

In 1842, Grube published in Berlin the first edition of his *Leitfaden für das Rechnen in der Elementarschule nach den Grundsätzen einer heuristischen Methode* (Guide for calculation in the elementary classes, following the principles of a heuristic method). This “*Essai d’instruction éducative*”, as he called it, after causing warm discussions, was approved by membership of the class of teacher. His book was successfully in agreement with the new system of weight and measurements and got to its 5th edition in 1873. Many textbooks, in all the languages, were reproduced, imitated or applied the Grube’s method.

The Grube’s method consists in making the pupils to do themselves, by intuition, the fundamental operations of elementary calculation. Such method aims to make them known the numbers: to understand an object, which it is not only to know its name, but to apprehend it in all its forms, in all its states and in its various relations with other objects; to be able to compare it with others, to follow the transformations, to write it and measure it, compose it and break up of them, at their will.

By treating the numbers as unspecified objects that are familiar to the pupils, Grube opposes to the old long-established method in arithmetic which is calculated to teach the first four processes of addition, subtraction, multiplication, division, in the order in which they are named, finishing addition with small and large numbers, before subtraction is begun, and so on. An improvement on this method consisted in excluding the larger numbers altogether at the beginning and dividing the numbers on which the first four processes were taught, into classes, or so-called *circles*. The pupil learns each of the four processes with the small numbers of the first circle (i.e., from 1 to 10) before larger numbers are considered; then the same processes are taught with the numbers of the second circle, from 10 to 100, then of the third, from 100 to 1000, and so on.

Grube, however went beyond this principle of classification. He discarded the use of large numbers, hundreds and thousands, at the beginning of the course, as others had done before him; but instead of dividing the primary work in arithmetic into three or four circles or parts only, i.e., from 1 to 10, 10 to 100, etc., he considered each number as a circle or part by itself. He recommended that the pupil should learn each of the smaller numbers in succession, and all the operations within the range of each number, before proceeding to the next higher one, addition, subtraction, multiplication, and division, before proceeding to the consideration of the next higher number.

Treating, for instance, the number 2, Grube leads the child to perform all the operations that are possible within the limits of this number, i.e., all those that do not presuppose the knowledge of any higher number, no matter whether in the usual classification these operations are called addition, subtraction, multiplication, or division. The child has to see and to keep in mind that

$$1 + 1 = 2, \quad 2 \times 1 = 2, \quad 2 - 1 = 1, \quad 2 \div 1 = 2, \text{ etc.}$$

The whole circle of operations up to 2 is exhausted before the pupil proceeds to the consideration of the number 3, which is to be treated in the same way.

The four processes are the direct result of comparing, or “measuring”, as Grube calls it, two numbers with each other. Only when the child can perform all these operations, for instance, within the limits of 2, can it be supposed really to have a perfect knowledge of this number. So Grube takes up one number after the other, and compares it with the preceding ones, in all imaginable ways, by means of addition, subtraction, multiplication and division. This comparing or “measuring” takes place always on external, visible objects, so that the pupil can see the objects, the numbers of which he has to compare with each other.

This methodology does not only prepare the pupil to study the arithmetic, but it offers an advantage over the other methods about the necessary conditions to the promotion of mental calculation. The pupils subjected to this method do not become slaves of the numbers and pencils and their “armed operations”.

Soldan (1878) exposes the six most important points about the Grube's method of teaching:

a) *Language* - the language is the only way that the teacher will have access to what the pupil is thinking, because it is not requested any records of the calculations made by them. A complete answer must be required from the pupil, because it is only by doing it that the teacher will be able to evaluate what the pupil learned or not.

b) *Questions* - teachers should avoid asking too many questions. Such questions, moreover, as, by containing half the answer, prompt the pupils, should be omitted. The pupils must speak themselves as much as possible.

c) *Individual recitation and jointly with the class* - In order to animate the lesson, answers should be given alternately by the pupils individually, and by the class in concert. The typical *numerical diagram* [2] are especially fit to be recited in concert.

d) *Illustration* – Every process and each example should be illustrated by means of objects. Fingers, lines, or any other objects can be used to answer the purpose, but some kinds of objects must always be presented to the class.

e) *Comparison and measurement* – the operation of each new stage consist in comparing or measuring each new number with the preceding ones. Since this measuring can take place either in relation to difference (arithmetical ratio), or in relations to quotient (geometrical ratio), it will be found to comprise the first four rules. A comparison of two numbers can only take place by means of one of the four processes. This comparison of the two numbers, illustrated by objects, should be followed by exercises of fast-solving problems and a view of the numerical relations of the numbers just treated, in more difficult combinations. The latter offer a good test as to whether the results of the examination of the arithmetical relations of the number treated have been converted into ideas by a process of mental assimilation. In connection with this, a sufficient number of examples in *applied numbers* are given to show that applied numbers hold the same relation to each other that *pure numbers* [3] do.

f) *Writing of figures* – on neatness in writing the figures, the requisite time must be spent. Since an invariable diagram for each number will re-appear in all stages of this course of instruction, the pupil will soon become able to prepare the work for each coming number by writing its *numerical diagrams* on their slates.

The study of the Grube's methodology turns possible to hypothesize the influences of Grube's methodology into the publications of Mr. Parker.

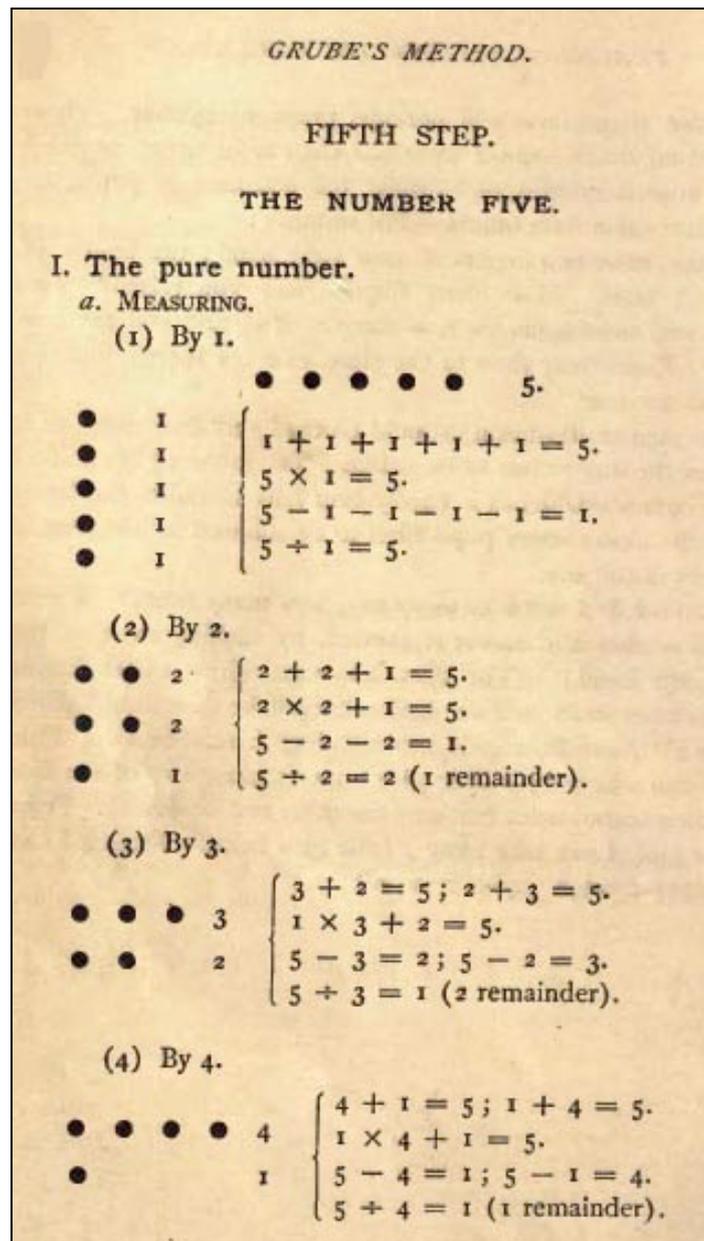


Fig. 1 – The Grube’s Method.

INTUITIVE’S METHOD AND THE “CARTAS DE PARKER” (NUMERICAL DIAGRAMS)

Research on the teaching of mathematics in Brazil in primary education level at the end of the nineteenth century through the sources, revealed a particular reference to Mr. Parker, this eminent American teacher, author of “Cartas de Parker”.

According to Montagutelli (2000), Francis Wayland Parker (1837-1902) developed an educational system which was recognized by John Dewey as the “father of progressive education”, also inspiring a few years later Granville Stanley Hall. Coming from a family of educators, Parker became a teacher when he was sixteen years, and later also served in the army at the time of the Succession War in the United States. At the end of the hostilities, he took the direction of a school in Ohio.

In 1872, he did a study trip in Europe: in Germany, he became familiarized with Herbart's pedagogy. It is possible that he took note of the Grube's method by this time. In 1875, he got back to the United States, where he became the supervisor of the schools of the town of Quincy, in Massachusetts. By this time, Parker develops the so-called "Quincy System". In an atmosphere without the rigid discipline imposed in the majority of the schools of this time, the pupils read newspapers or texts composed by their teachers; on the basis of knowledge, they approached the new concepts concretely followed by working groups besides also the practice of drawing and music.

Parker published five books on education: *Talks on Teaching* [4] (New York, 1883); *The Practical Teacher* (1884); *Course in Arithmetic* (1884); *Talks on Pedagogies* (1894) and *How to Teach Geography* (1885).

An important educational journal of the beginning of the twentieth century, "Revista de Ensino", created in 1902 by the Association of Public's Teacher of São Paulo (Brazil), devoted in several editions, in its section called Teaching Practice, several articles about the way of using the "Cartas de Parker".

According Pierre Ognier (1984), the educational journal, is one of vast documentary corpus, because it is a living witness evidence of teaching methods from an era and the conceptions of moral ideology, social and politics of a professional group. This makes it an excellent observatory, a picture of the ideology that governs.

Accordingly, it is a practical guide to everyday educational and school, allowing the researcher to study the pedagogical thought of one determined sector or a social group from the analysis of reported speech and resonance of the issues discussed within and outside the universe school.

This educational publication, "Revista de Ensino", over a number of editions, published about fifty charts, diffusing them in Brazil. These charts concretize the appropriation by Parker of the *numerical diagrams* stated in the Grube's method. They represent the way of treating the teaching of Arithmetic in an intuitive way. Moreover, they are presented like references for the development of textbooks of mathematics for the first levels.

By a heuristic process, i.e., a procedure which consists in discovering by the pupil what exactly wants to teach to him, the teacher questioned the pupil in front of the chart. Example extracted the fourth chart (see Figure 2): in the items *h*, *i* and *l* are representative drawings of the number *ten*. And by the observation, the pupil should give his answers or make remarks about this number formation. Thus, in the letter *h*, it is needed two *five* to have a *ten*; in letter *l* we find *three + three + four* to have a *ten*; in letter *i*, it is needed five times of *two* to have a *ten*. This way the pupil learned how to compose and break up the number into equal or unequal parts. The idea of the addition, subtraction, multiplication as of division is concomitantly subjacent with this process.

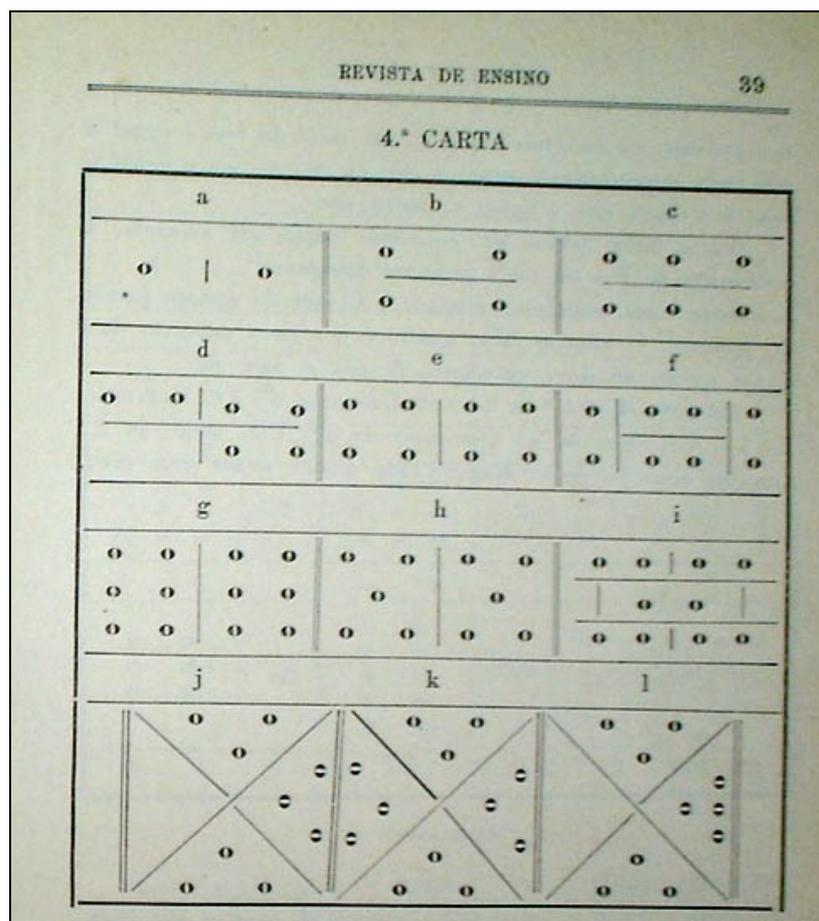


Fig. 2 – 4th Carta de Parker

In Brazil, in addition to the quotations and the articles of “Revista de Ensino” on “Cartas de Parker”, an important textbook of the beginning of the twentieth century, written by Arnaldo de Oliveira Barreto, *Série Graduada de Matemática Elementar*, published by the Salesians, in São Paulo, in 1912, quotes the name of Parker and the “Cartas de Parker” in the foreword signed by Oscar Thompson, director of the Normal School (Teacher School). There are also quotations in the presentation of the book and the final comments relating to the conferences pronounced by Parker.

The effective methodology of teaching during this time treated intuitive method which had been adopted in second half of the nineteenth century in the European, American and Brazilian schools; it was based on the ideas of Pestalozzi and Fröbel.

For Valdemarin (1998), the intuitive method was influenced directly by the current empiric of philosophy, carried by Francis Bacon and John Locke (seventeenth century) by determining the procedures of teaching based on the *observation*.

This method was presented in the form of a response to the abstract character and little utility of the instruction up to that point of use, by developing new didactic materials and a diversification of the teaching activities. It also brought others

innovations that were spread on successive Universal Expositions which were organized for the diffusion of teaching practices, like the ones held in London (1862), in Paris (1867), Vienna (1873) and Philadelphia (1876).

The presence of the intuitive method in teaching of arithmetic reveals a new teaching method which is opposed to the preceding way of teaching where the memorizing of the knowledge was privileged. The “Cartas de Parker” are the elements that made possible to associate the influence of this intuitive movement of the teaching of arithmetic in Brazil at this time. Evidences of dissemination of this methodology are present in articles in major educational journals such as the “*Revista do Ensino*” and of the textbooks like “*Aritmética Escolar*” of Ramon Roca Dordal [5] or “*Contador Infantil*” of Heitor Lacerda [6], among others.

CONCLUSION

According to Chervel (1998), the first task of the School Disciplines’s historian is to study the explicit contents of disciplinary teaching. The study of a “vulgata”, configured as “Cartas de Parker” enables us to connect the form and the contents of the teaching of mathematics in the primary education level at the end of the nineteenth century - beginning of the twentieth century in Brazil, becoming an important element of the writing of the History of Mathematical Education in Brazil.

Moreover, this study allows hypothesizing that the relation is given at educational backgrounds of the ideas that circulated in the late nineteenth century in Europe and materialize in Brazil on publications of textbooks and articles in educational journals. This seems to point towards the influence of intuitive teaching, conceived by their European authors as a pedagogical tool capable of reversing not only the inefficiency of school, but also reduce the existing economic development gap, since the emergent industrial labour demanded literate and think quickly and creatively individuals.

According to Valdemarin (1998) this inefficiency of school teaching was characterized by the formation of pupils with insufficient reading and the writing notions and also without satisfactory concepts of calculation, mainly because of the learning based exclusively on memory, giving priority to the abstraction, enhancing the value of repetition to the detriment understanding and impose contents without examination and discussion.

The explicit proposal of the “Cartas de Parker” appears to be consistent with the aspirations of a time that rejects the methods primarily based on the memory and develops the observation as a way of effective training of calculation.

It is through historical studies that we have access the way that great teaching thinkers thought about the teaching of mathematics and the way it echoes in Brazil.

NOTES

1. This research is subordinated to one of the thematic projects which are developed by the GHEMAT – Grupo de Pesquisa de História da Educação Matemática do Brasil (Group of Search for History of the Mathematical Education of Brazil): “A EDUCAÇÃO MATEMÁTICA NA ESCOLA DE PRIMEIRAS LETRAS, 1850-1950” coordinated by Prof. Dr. Wagner Rodrigues Valente and financed by the FAPESP. Through a financial support obtained from CNPq – Conselho Nacional de Desenvolvimento Científico e Tecnológico (National Council of Technological and Scientific Development), I have been developed my research of doctorate at INRP/SHE (Institut National Recherche Pédagogique, Service d’Histoire de l’Éducation – Paris – France) under supervision of Prof. Dr. Alain Chopin (05/2008 to 04/2009).
2. The *numerical diagram* of the Grube’s method will be presented later on in this article as “Cartas de Parker”.
3. A *pure number* also called an *abstract number*, which is that makes mention only quantity. Four, thirty, twelve are examples of *pure numbers*. Applied to an object, it will be called a *applied number* or *concrete number*. Thirty apples, four trees, three meters, are examples of *applied numbers* or *concrete numbers*.
4. This book was translated into Portuguese by Arnaldo de Oliveira Barreto in 1909 and edited by Livraria Francisco Alves: “*As Conferências de Parker*”.
5. See article Costa, D.A., Valente, W.R. (2007). Análise da Arithmética Escolar de Ramon Roca Dordal. In: Simpósio Internacional do Livro Didático, 2007, São Paulo. Livro Didático - Educação e memória. São Paulo: Centro de Memória da Educação – FEUSP, v.1.
6. See *Revista do Ensino*, 1902, p.146.

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HISTORICAL PICTURES FOR ACTING ON THE VIEW OF MATHEMATICS

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The article illustrates the underlying philosophy of an in progress book in which pictures taken from historical books are used to hint some fundamental ideas of the history of mathematics. Both epistemological and disciplinary issues are taken into account. The aim of the book is to let its potential readers know different aspects of mathematics as a science operating inside the socio-cultural context.

Keywords: historical pictures, original sources, mathematics view.

INTRODUCTION

This paper deals with the problem of the view of mathematics held by students and the means suitable to act on it. In previous works we have studied students' view of mathematics as a socio-cultural process with particular reference to the historical development, see (Demattè & Furinghetti, 1999). Our main conclusion was that this view was very narrow focused and based on common myths on mathematics. To answer the question "How to act on the image of mathematics held by students?" a book has been designed by one of the authors (A. D.) addressed to students of the final years of secondary school (16 years old onward) or readers who are interested in the popularisation of mathematics. The book is based on pictures taken from historical sources. Pictures have been largely used in history for communicating mathematical ideas, see (Mazzolini, 1993), and thus it is not difficult to collect materials for composing such a book. Words accompany pictures in order to create a unitary discourse and to focus on some aspects. Pictures strengthen what the verbal part say, like in a natural history museum where things and words, verbal and non-verbal communication coexist. Knowledge required for using the book in classroom (or elsewhere) is confined to elementary mathematics. As we will see in sections 3 and 4 some chapters are more suitable to develop mathematical topics *stricto sensu*, other are more oriented to raise reflections on historical-epistemological questions.

THE ROLE OF PICTURES

The idea of this book does not come out of the blue. We have already described in (Demattè, 2005; 2006a; 2006b) our work with pictures in the classroom. In particular, in the latter two papers we have discussed how students in front of a historical figure are able to mobilize some kind of narratives and to produce conjectures. This is due to the particular nature of the information provided by figures. Often images show supplemental details, which are not pertinent to the specificity of discourse. Readers can interpret these images in different ways. A discourse follows a logical track (sometimes very rigorous), a picture often permits freedom to the interpreter.

Therefore it is ‘friendly’ i.e. rich in possibility of reflections and personal reasoning. Our claim may be illustrated by some examples taken from the book.

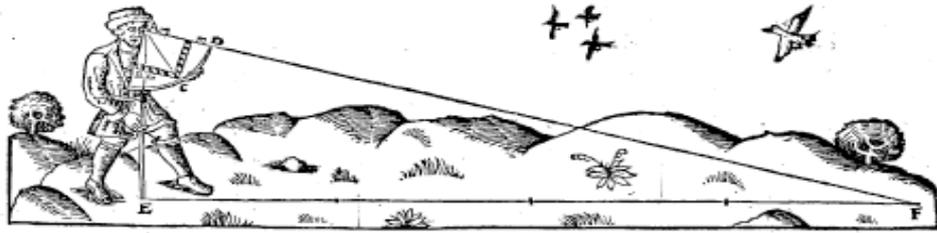


Fig. 1. Oronce Finé, *Protomathesis*, 1532

Pictures like Fig. 1 are aimed at showing how an instrument can be used, but the painter has added many details (hills, grass, trees, birds, elegant dress of the man) which make the scene realistic. The draw of the right-angled triangle and of the instrument (a “quadrant in a fourth part of a circle”) focuses on mathematical aspects.

To reflect on the use of the picture in Fig. 1 in classroom raises the following questions for the researcher: Can students appreciate these kinds of images? Do pictures like Fig. 1 make them want to use the facilities offered by mathematics? Do students see the relationship between the concepts and procedures shown in historical pictures and what they learn in school today? Maybe the answer is no, for each question. In any case the mathematics view suggested by this kind of pictures appears potentially positive in the fact that they address the attention to geometrical details and, in the same time, stimulate guessing the finalities of the action illustrated in the picture. A scene like the one in Fig. 1 suggests a simple story, a narration with a precise structure (some events happen before, some after, a goal of the action – including the implicit use of mathematics - is noticeable). (Demattè, 2006a; 2006b) report on an experiment where students were asked to write how they interpret Fig. 2.

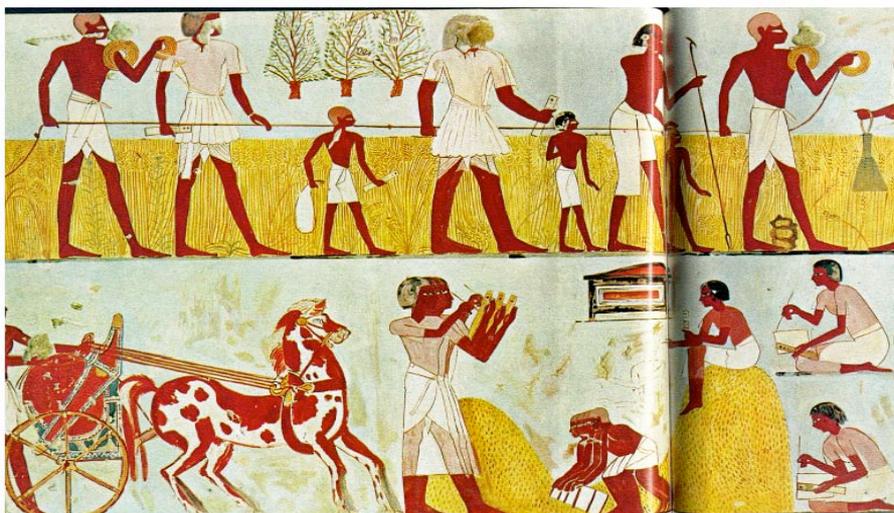


Fig. 2. A mural painted at Abd-el-Qurna, Egypt, around 1400 B.C

Some protocols show that they followed the pattern of a narrative. Because of the need to complete the story, students formulated also conjectures (e.g. the king's servants on the cart have the task of rewriting the data and, as the student write, “the aim of giving an account of them to the king”).

Students are rather naturally brought to formulate conjectures, which are coherent with context and with elements present in the scene, if they have adequate knowledge. To interpret mathematical aspects in the previous image from Finé's *Protomathesis* or in the following Fig. 3 the concept of similarity among triangles is required. But many other aspects require more knowledge: e.g. Why the square? Which is the purpose of the action of the man in the picture? etc.

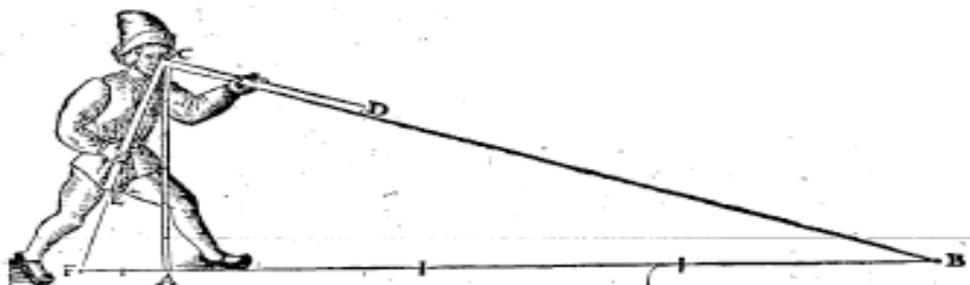


Fig. 3. Oronce Finé, *Protomathesis*, 1532

PICTURES AND MATHEMATICAL TOPICS

In the book the focus is on some grounding mathematical ideas that may be elaborated through the history of mathematics. These ideas regard the main chapters of mathematics (numeration, algebra, probability, etc., see Appendix). Some ideas are inherent to procedures and concepts: images suggest first of all the *incipit* of mathematical reasoning and its global structure. For example, the reader may reflect on the different ways of approaching the same theorem by considering the Chinese theorem of Pythagoras (Fig. 4) and what is done using Cartesian graphs.

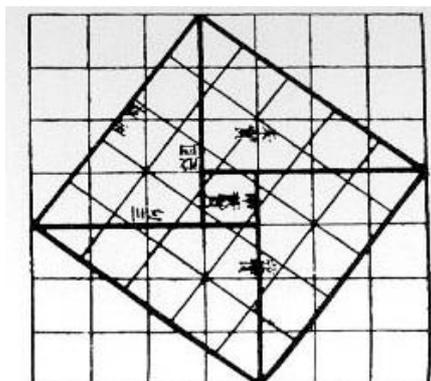


Fig. 4. ‘Pythagorean’ theorem from *Chou Pei Suan Ching*, about 500-200 b.C.

Moreover pictures, suggest at a glance some metacognitive information e.g. the level of complexity and the need of a detailed mathematical reasoning, as exemplified by

the Leibnizian graphs shown in Fig. 5 from *Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas, nec irrationales quantitates moratur, et singulare pro illis calculi genus* (A new method for maxima and minima as well as tangents, which is impeded neither by fractional nor by irrational quantities, and a remarkable type of calculus for this), see (Dupont & Roero, 1991).

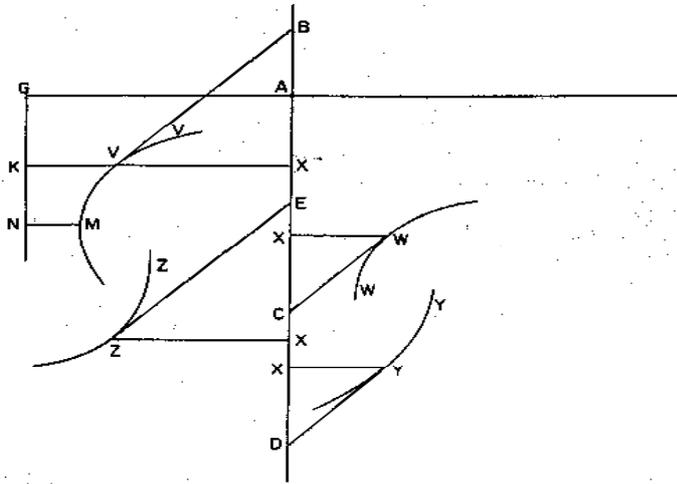


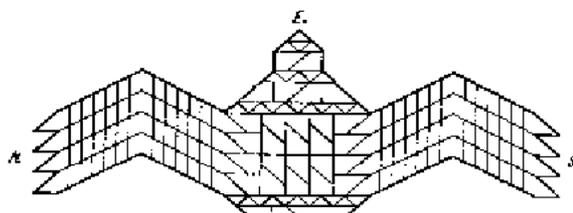
Fig. 5. Gottfried Wilhelm Leibniz, *Nova Methodus ...*, 1684

1. PICTURES AND HISTORICAL-EPISTEMOLOGICAL IDEAS

Some chapters address historical and socio-cultural aspects such as: reckoning and measuring as answers to problems of human activities. The students may perceive the hypothetical-deductive structure of mathematics as a model for other branches of the human knowledge such as philosophy and economy, or for every day life. Through these chapters some myths about mathematics may be discussed: the development of mathematics seen as a linear progress from ancient to contemporary times, euro centrism, independence from external factors.

In our previous papers, see (Demattè & Furinghetti, 1999; Furinghetti, 2007) we discussed how students and teachers may conceive the development of mathematics just as an evolutionary process. In doing that they loose the richness of the path of mathematical ideas that are lateral to the main stream of the development of mathematical concepts. Moreover we know that the intertwining and the reciprocal influence of internalist and externalist factors is a powerful perspective for studying mathematical concepts and its development, as shown in the paper (Radford, 2006). Mathematics has changed during the time but has become also different in different countries and cultural contexts.

Ethnomathematics (see a product in Fig. 6) is a fruitful branch of research in education. It is about learning mathematics connected to other areas, to social and



environmental problems (Joseph, 2003; Katsap, 2006). It lead to reflect on the fact that not only the European mathematics is the ‘true mathematics’

Fig. 6. The most elaborate altar from the Indian *Sulbasutras* (the first part probably was written in the 6th century B.C.). Many of the triangular and trapezoidal altars described in the *Sulbasutras* use then theorem of Pythagoras

Some external factors influence the daily work of researchers: relations among colleagues (well known ‘spy stories’ regarded 16th century Italian algebraists, see Fig. 7), salary (not ethically impeccable ‘involvements’ come from the fact that ancient and modern war requires a wide apparatus of mathematical knowledge), national policy pushed by the dominating class, see (Barnett, 2006; Swetz, 1987), etc. This is enough to confirm that context influences advancement of science.



Fig. 7. Italian mathematicians Niccolò Fontana (“Tartaglia”; 1499-1557) and Gerolamo Cardano (1501-1576)

MATHEMATICS VIEW

The ultimate aim of the book is to suggest a different mathematics view. Every chapter ends with a discussion about beliefs on the nature of mathematics, which are connected with the aspect treated in it. This part of the book regards factors that are not always made explicit in the classroom, but influence the personal relation with mathematics. We deem it is important to stimulate students’ awareness on these factors. In the book the pictures and the related comments show unusual, but in our opinion more realistic, aspects of mathematics. As discussed above, mathematics:

- is an historical construction which is socially and culturally bounded, therefore different cultural context have produced different forms of mathematics;
- is used in many professions and jobs; is present in the everyday life; has epistemological and also psychological aspects which are intertwined (such as the role of error and its acceptance by individuals);
- has relationships with other disciplines; requires debate, communication and involvement and may also originate wish to investigate.

We briefly recall some beliefs widespread among students and ordinary people that were detected in our study (Demattè & Furinghetti, 1999). These are some of the beliefs considered in the book with respect to the content of the chapters:

- it is better if I remember rules by heart and I don’t attempt to reason with my

brain;

- when I solve a mathematical problem I know that there is only one exact solution;
- mathematics learnt in school has not a practical use; not everybody has a ‘mathematical mind’;
- creativity is not necessary in mathematical reasoning; different topics, such as arithmetic, geometry, algebra, must be taught and learnt separately because they don’t have any connections; in mathematics approximated results are incorrect and do not give useful information;
- in mathematics errors are absolutely negative experiences;
- mathematics doesn’t depend on culture; I think that men have began to use the signs +, -, \times , : before Christ;
- if I study alone (not with mates) I’ll have better results in mathematics.

FINAL REMARKS

In a previous paper, see (Furinghetti, 1997) it is pointed out that there are two main streams in the use of history in the classroom: - to promote the image of mathematics, - to introduce mathematical contents. From our presentation it follows that our work is set in the first stream. Only a few parts of the chapters have been administered in the classroom. After completing the work it is planned to use it and to study students’ reactions. We expect to carry out empirical research that allows to answer questions such as the following:

- How will readers consider the kind of mathematics presented in the book? Will they establish connections with mathematics they learned at school or will they consider it an ‘extraneous entity’?
- What beliefs could change through learning the history of mathematics? What activities could be more useful?
- Learning history (in a broad sense) is also to remember facts and dates. What historical information could mathematics teacher require the students to remember? Could pictures create an opportunity to remember significant aspects of the history of mathematics?
- In our opinion, the citizen mathematics education requires new didactical choices. Could historical-epistemological analysis of mathematics replace some parts of traditional curriculum?

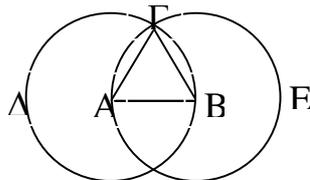
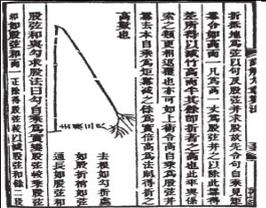
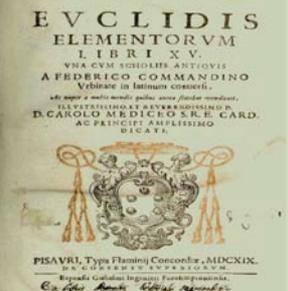
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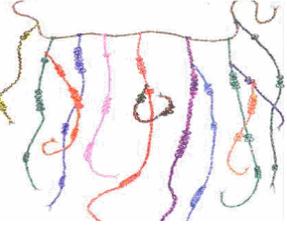
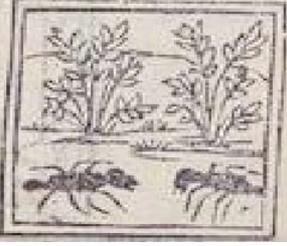
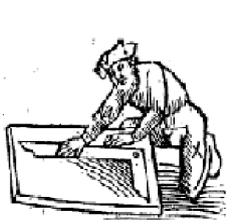
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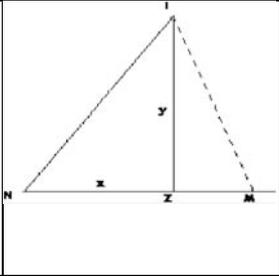
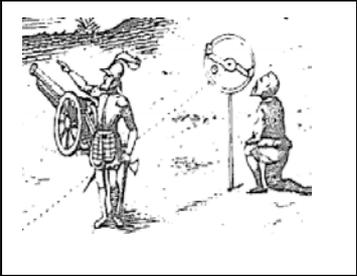
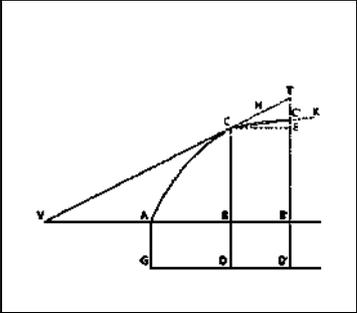
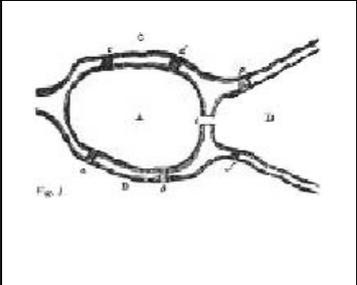
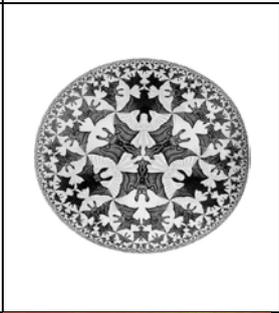
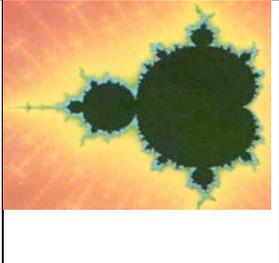
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APPENDIX. The structure of the book

In the book there is a preface explaining the aim and the rationale of the work and 30 chapters whose titles and some representative figures are shown below.

<p>Legenda E: Chapters mainly concerning historical or <i>Epistemological ideas</i>. M: Chapters over mainly concerning relevant <i>Mathematical topics</i>.</p>	<p>1. The first files of data (<i>M*</i>)</p>		
<p>2. Mathematics for administering a Nation (<i>E</i>)</p>	 <p><i>Egyptian peasants seized for non-payment of taxes... (Pyramid Age)</i></p>		
<p>3. Is mathematics we learn at school ancient? (<i>E</i>)</p>		<p>4. How to write a number (<i>M</i>)</p>	
<p>5. Algebra begins (<i>M</i>)</p>		<p>6. Mathematics is full of errors (<i>E</i>)</p>	
<p>7. Pythagoras in China (<i>M</i>)</p>	 <p>Il "problema del bambù spezzato" come viene illustrato nello <i>Hsüang Chieh Chü Chang Suan Fa</i>.</p>		

<p>9. What is geniality? (E)</p>		<p>10. Does it depend on material we have? (E)</p>	
<p>11. Mathematical knowledge doesn't "accumulate in layers" (E)</p>		<p>12. Recreational problems (M)</p>	
<p>13. Does an authority hold knowledge? (E)</p>		<p>14. Mathematics is culture (E)</p>	
<p>15. Masters of abacus (E)</p>		<p>16. Mathematics and trade (E)</p>	
<p>17. Geometry for builders (M)</p>		<p>18. Mathematics and politics (E)</p>	
<p>19. More recent than we think (E)</p>		<p>20. Is mathematics the same everywhere? (E)</p>	

<p>21. Problems of paternity (<i>E</i>)</p>		<p>22. Mathematics and war (<i>E</i>)</p>	
<p>23. Let's bet everything (<i>M</i>)</p>		<p>24. Calculus (<i>M</i>)</p>	
<p>25. Mathematics and other sciences (<i>E</i>)</p>		<p>26. Geometry of position (<i>M</i>)</p>	
<p>27. Beyond infinity (<i>M</i>)</p>		<p>28. Ethnomathematics (<i>E</i>)</p>	
<p>29. Past, present and future (<i>E</i>)</p>		<p>30. Imagine a mathematician (<i>E</i>)</p>	

STUDENTS' BELIEFS ABOUT THE EVOLUTION AND DEVELOPMENT OF MATHEMATICS

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The paper is an empirical study of students' beliefs about the history of mathematics. 26 students in an upper secondary mathematics class were exposed to a line of questions concerning the evolution and development of mathematics in the form of a questionnaire and follow-up interviews. In the paper it is argued that the existing literature on students' beliefs, in general, lacks a discussion of goals dealing with, for instance, desirable beliefs among students in order to provide them with a more coherent image of mathematics as a discipline. A couple of descriptions from the Danish literature and upper secondary regulations are provided as an example of such a dimension. The concrete student beliefs from the research study are evaluated against these descriptions.

KEYWORDS: History and epistemology of mathematics; students' beliefs and images; a goal-oriented dimension for students' beliefs.

INTRODUCTION

Beliefs about the history of mathematics is a topic which is touched upon from time to time in the literature on history in mathematics education, e.g. in Furinghetti (2007) and Philippou and Christou (1998). However, when scanning these samples, one soon finds that these concern the beliefs of in-service or pre-service teachers. Studies on students' beliefs about the history of mathematics seem to be rather poorly represented in the literature, if not altogether absent.¹ One reason for this that I can think of is that, in general, studies of beliefs in mathematics education are conducted with the purpose of improving mathematical thinking, learning, and instruction.² Beliefs, both cognitive and affective ones,³ are investigated in order to identify the 'ingredients' which do or do not make students capable of solving mathematical tasks or teachers capable of teaching differently and/or more effectively. Certain beliefs are identified as advantageous in the learning of certain mathematical contents, the solving of related tasks, etc., and educational studies are then conducted on how to change already existing beliefs into these more favorable ones. In this sense beliefs are regarded as means – or *tools* – to achieve understanding in the individuals' constructive learning process. Only rarely is providing students or teachers with certain beliefs, e.g. by changing existing ones, about mathematics or mathematics as a discipline considered as a *goal* in itself. And when this is done, the term 'beliefs' is usually not used. Instead mathematical appreciation, mathematical awareness, or providing students with a more profound *image* of what mathematics is, are the words or phrases more commonly used (e.g. Furinghetti, 1993; Niss, 1994; Ernest, 1998).

It seems to me that the beliefs discussion in mathematics education lacks a goal-oriented dimension. A dimension which addresses students' mathematical world view and proposes and evaluates some desirable beliefs in order to turn students into more critical citizens by providing them with intelligent and concerned citizenship and with some *Allgemeinbildung* in general (Niss, 1994). That is to say, to provide students with a more coherent image of mathematics as a discipline, the influence of mathematics in society and culture, the impact of society and culture on mathematics, and the historical evolution and development of mathematics as a product of time and space, to mention a few of the more 'pressing' ones. Occasionally researchers will touch upon these issues in the form of personal opinions, e.g. in curriculum development. However, a dimension about 'beliefs about desirable beliefs' – meta-beliefs we may call them – can only be addressed properly if the meta-beliefs are articulated as such, i.e. as goals in themselves.

In this paper I shall first present some extracts from the 2007-regulations for the Danish upper secondary mathematics program and the Danish report on competencies and learning of mathematics, the so-called KOM-report, which may serve as such a goal-oriented dimension for students' beliefs. Especially I shall focus on students' beliefs concerning the history of mathematics. Secondly, I shall report on a piece of empirical research in which a number of students were asked about their beliefs concerning the evolution and development of mathematics.⁴ Thirdly, these students' beliefs are analyzed and evaluated against the goal-oriented descriptions. The paper is ended with some final remarks and reflections on the presented empirical data and the larger research study which they are part of.

THE DANISH CONTEXT

Since 1987 history of mathematics has been part of the formal regulations for the Danish upper secondary mathematics program (see e.g. Fauvel and van Maanen, 2000, pp. 5-7), and with the newest reform and the present regulations of 2007 this part has become more dominant. Students are now expected to be able to “demonstrate knowledge about the evolution of mathematics and its interaction with the historical, the scientific, and the cultural evolution”, knowledge acquired through teaching modules on history of mathematics (Undervisningsministeriet, 2007, my translation from Danish).⁵ The official regulations for the Danish upper secondary mathematics program of 2007 are to some extent based on the Danish report *Competencies and Learning of Mathematics*, the so-called *KOM-report*, (Niss and Jensen, 2002, title translated from Danish) where it says the following about history:

In the teaching of mathematics at the upper secondary level the students must acquire knowledge about the historical evolution within selected areas of the mathematics which is part of the level in question. The central forces in the historical evolution must be discussed including the influence from different areas of application. Through this the students must develop a knowledge and an understanding of mathematics as being created by human beings and, in fact, having undergone an historical evolution – and not

just being something which has always been or suddenly arisen out of thin air. (Niss and Jensen, 2002, p. 268, my translation from Danish).

In the report, the focus of integrating history of mathematics is discussed in terms of a certain kind of overview and judgment which the students should acquire as part of their mathematics education.

The form of overview and judgment should not be confused with knowledge of ‘the history of mathematics’ viewed as an independent subject. The focus is on the actual fact that mathematics has developed in culturally and socially determined environments, and subject to the motivations and mechanisms which are responsible for this development. On the other hand it is obvious that if overview and judgment regarding this development is to have solidness, it must rest on concrete examples from the history of mathematics. (Niss and Jensen, 2002, p. 68, my translation from Danish)

The 2007-regulations describe the “identity” of mathematics in the following way:

Mathematics builds upon abstraction and logical thinking and embraces a long line of methods for modeling and problem treatment. Mathematics is indispensable in many professions, in natural science and technology, in medicine and ecology, in economics and social sciences, and as a platform for political decision making. At the same time mathematics is vital in the everyday. The expanded use of mathematics is the result of the abstract nature of the subject and reflects the knowledge that various very different phenomena behave uniformly. When hypotheses and theories are formulated in the language of mathematics new insight is often gained hereby. Mathematics has accompanied the evolution of cultures since the earliest civilizations and human beings’ first considerations about number and form. Mathematics as a scientific discipline has evolved in a continual interrelationship between application and construction of theory. (Undervisningsministeriet, 2007, my translation from Danish)

Thus, when the students are to “demonstrate knowledge about the evolution of mathematics” etc., as stated in the academic goals of the regulations, one must assume that it is within the frame of this “identity” that they are expected to do so. Another way of phrasing this is to say that one purpose of the teaching of mathematics at the Danish upper secondary level is to shape the students’ beliefs about mathematics according to the above description of identity. The purpose of including elements of the history of mathematics has to do with showing the students that mathematics is dependent on time and space, culture and society, that mathematics is not ‘God given’, that humans play an essential role in the development of it, etc., etc.

STUDENTS’ BELIEFS ABOUT THE ‘IDENTITY’ OF MATHEMATICS

In the beginning of 2007, I conducted a questionnaire and interview research study of second year upper secondary students’ (age 17-18) beliefs about the ‘identity’ of mathematics. A number of these questions had to do with evolutionary and developmental perspectives of mathematics, others had to do with sociological perspectives, and others again with perspectives of a more philosophical nature. In the following I shall present the students’ answers to three of these questionnaire

questions, one from each aspect. All in all 26 students answered the questionnaire. The students' questionnaire answers have been indexed in the following manner: one<few<some<many<the majority<the *vast* majority, a partition which roughly corresponds to the percentage intervals: 0-5%; 6-15%; 16-35%; 36-50%; 51-85%; 86-100%. Based on the questionnaire answers 12 students were chosen as representatives for the class in general, and these 12 students were interviewed about their answers. All quotes from the questionnaires and the interviews have been translated from Danish.

1. When do you think the mathematics in your textbooks came into being?

The majority believe that the mathematics in their textbooks came into being “some time long ago”. The suggestions concerning exactly when are, however, many and varied: “from even before da Vinci’s time!”; “when the numbers were invented”; “when we began using Arabic numerals”; “way before it says in the books”. Some points to antiquity and provide as argument that “the construction of, for instance, the pyramids must have required at least some mathematics”. One of the more interesting answers goes: “Long, long ago it all began and since then it has continued. But I am confident that the development goes more and more slowly, because you eventually know quite a bit.”

Out of this majority of students, some share the perception that mathematics has always existed, or at least has existed as long as human beings have been around. One says: “Mathematics in general has existed since the dawn of time, but highly developed [mathematics] has only emerged within the last 200-100 years.” Only one student believes the mathematics in the textbooks to be of a more recent date, and he is not afraid to fix this to “40 years ago”.

In the follow-up interviews, events in the history of mathematics were occasionally fixed within some not too unreasonable orders of magnitude, for instance, the beginning of mathematics to 4000-5000 years ago; Pythagoras to the first couple of centuries; and Fermat’s last theorem to “the Middle Ages or something”. But only few students were able to do this. Whether this is due to lack of knowledge about history of mathematics or lack of knowledge about history in general, or maybe both, is not to say. Finally, one of the students seemed very strong in her belief that it was impossible to practice mathematics without the Arabic numerals. When asked why, she answered: “the mathematics you do today, you wouldn’t have been able to do that... [without the Arabic numerals]”.

2. Do you believe that mathematics in general is something you discover or invent?

The majority of the students believe that mathematics in general is something you discover. Only a few believe that it is something you invent. Some students, though, believe that it might be a combination of the two. Examples of the discovery answers are: “Discover. I don’t think you can invent mathematics – it is something ‘abstract’

you find with already existing things.”; “Discover. Because mathematics is already invented. What happens today is only that you discover new elements in it.”; “Mathematics is all over – in our society, our surroundings and in the things we do. Therefore I do not believe mathematics to be something you invent, but on the contrary something you discover along the way. Of course, it might be difficult to say precisely, because where do we draw the line between discovery and invention?” Examples of students believing it to be a combination of discovery and invention are: “Many things might begin as an invention, but afterwards they are explored and people discover new elements in the ‘invention’ in question”; “Both, [I] think that you discover a problem and then solve it by inventing a solution or applying already known rules of calculation”; “You invent formulas after having discovered relationships”. One student’s answer touch upon the question of what mathematics ‘really’ is: “Good question... very philosophical. I think there are many different standpoints to this. I personally believe that it is something you discover. Numbers and all the discoveries already made are all connected. So for me it is more a world you enter into than one you make.”

In the follow-up interviews the student responsible for the last remark explained further: “Well, I see it as if mathematics is just there, like all natural science is, for instance, outer space. Outer space is there and now we are just discovering it and learning what it is. That’s what I think: It’s the same thing with mathematics.” When the remaining interviewees in favor of discovery were asked if the ‘exploration’ of mathematics corresponds to the exploration of the universe they all confirmed this belief. That is to say that they believed mathematics to always have existed, or as one student phrased it: “Mathematics has always been there, in the form of chemistry or something like that at the creation of Earth. And then we haven’t found out about it until later.” Or another one: “I think it has always been there, but I just think that the human beings are exploring mathematics more and more and are discovering new things.”

3. Do you think mathematics has a greater or lesser influence in society today than 100 years ago?

The vast majority of the students believe the influence is greater. This answer is in general based on the increased amount of technology in our everyday life in society. Answers as “definitely, more computer=more mathematics” and “everything develops and everything has to be high-technology” are often given. A few of those who believe that mathematics has a greater influence today also points to economic affairs as the reason, or that “the use of mathematics has become more advanced in our time”. Some think that mathematics has the same influence today as it had 100 years ago, and only very few believe that the influence today is lesser. One of the more ‘sensational’ answers of the latter kind is: “No, I don’t believe that, because even though we use mathematics a lot more in space etc. we have modern machines to do it.”

The follow-up interviews to a large degree confirm the beliefs described above. To the deepening question of why a student found the influence today to be greater, she answered:

Student: Because today you can, for instance, get an education at... or study mathematics at the university and things like that, and that you couldn't do a hundred years ago. [...]

Interviewer: *So it is something relatively new that you can study mathematics at the university?*

Student: No not new, but I do believe at a higher level. That is, you didn't know as many things back then as you do today.

Interviewer: *And you couldn't get an education as a mathematician in the same way, you think?*

Student: No.

The student who argued for lesser influence due to the use of modern machines is also given the opportunity to expand on her view in the interviews. She finds, amongst other things, that mathematics appears less present because we rely on technical aids to a great extent, and because the use of mathematics is mostly about “pushing some buttons”.

EVALUATING STUDENTS' BELIEFS AGAINST THE 'GOALS'

How do the above presentation of students' beliefs about the evolution and development of mathematics correspond with the goal-oriented description of overview and judgment in the KOM-report and the 'identity' of mathematics in the 2007-regulations? For example, are students able to “demonstrate [display] knowledge about the evolution of mathematics and its interaction with the historical, the scientific, and the cultural evolution”? Overall the students' answers to some of the questions appear rather diffuse, but let us look at the questions in turn.

In the answers to question 1 there seem to be an agreement that mathematics is 'old'. One student implies that da Vinci is old and that mathematics is older than him. However, only very few are capable of providing years on the origin of mathematics as well as on concrete mathematical results. That some students believe that mathematics only could come into existence by aid of the Arabic numerals does not strengthen the interpretation that the students possess knowledge about the evolution of mathematics in interplay with historical and cultural events either.

In question 2 the majority give expression to the fact that they believe mathematics in general to be discovered. In a Danish educational context this may appear surprising since, as Hansen (2001, p. 71, my translation from Danish) puts it: “it is clear that the strong position of constructivism in school circles fertilizes the ground for a more radical constructivist perception of the entire nature of mathematics. Because of the pedagogical constructivism in schools, children and young people are likely to have difficulties believing in special existence of mathematical quantities, figures, and

concepts.” Of course there are students who are inclined toward a view of mathematics in general as something invented, but they are few in number. The majority give expression to a Platonic stance. With the words of one of the students, it is “a world you enter into” – a world of ideas – where you explore the already existing mathematical objects in a similar way as we are exploring the Milky Way and the rest of the universe our planet is part of.

On the other hand, the students seem to have a quite good understanding of the fact that mathematics today has a much greater influence in society than it did 100 years ago (question 3). Again it is computers and other technology that are given credit for this. The fact that students only pay scant attention to economic affairs and political decision-making, e.g. based on mathematical models, may be seen as a consequence of the invisibility of mathematics in society (Niss, 1994). One student touched upon this when she said that mathematics appears less present due to use of technology. Another example is the student who in question 1 believed that the development of mathematics was happening at a slower and slower pace and who in the interviews explained herself:

Yes, but they just discovered more a long time ago, didn't they? It isn't very often you hear about someone who has discovered something new within mathematics, is it? Maybe it's just me who isn't enough of a mathematics geek to be told about it. But it just seems to me that nothing is really happening. Things are happening more often within natural science: now they have found a method to see the fetus at a very early stage by means of a new type of scanning or something.

This student seldom hears about new discoveries in mathematics, even though she is exposed to the subject several times a week, therefore she believes nothing is happening. Beside this, her remark also touches upon one of the differences between mathematics and the natural sciences: just because mathematics now is able to prove Fermat's last theorem or the Poincaré conjecture, then this is not something that will change our everyday or society neither tomorrow nor in 50 years (most likely), something which would be far more likely for discoveries in physics, chemistry, or biology – and to a larger extent for technology basing itself on these disciplines.

In general the fact that mathematics is driven by both outer as well as inner driving forces is not an aspect which the students seem to be very aware of. And concrete examples from the history of mathematics, in the form of the KOM-report's talk of “solidness” (cf. page 3), is not something which the students seem able to provide either.

FINAL REMARKS AND REFLECTIONS

According to Lester, Jr. (2002, p. 352), Kath Hart at a PME conference once asked: “Do I know what I believe? Do I believe what I know?” Lester's version of this question is: “Do students know what they believe?” Furinghetti and Pehkonen (2002) argue that one should take into consideration both the beliefs that students hold

consciously as well as unconsciously. But how to do this? Lester, Jr. (2002, pp. 352-353) sows doubt about some of the more usual methods for doing this: “I am simply not sure that core beliefs can be accessed via interviews [...] or written self-reports [...] because interview and self-report data are notoriously unreliable. Furthermore, I do not think most students really think much about what they believe about mathematics and as a result are not very aware of their beliefs.” Thus, the results above must perhaps be viewed in this light. However, other researchers (e.g. Presmeg 2002) argue that questionnaires, interviews, etc. are perfectly well suited to access students’ beliefs about mathematics as long as the usual precautions, for example the interviewee trying to please the interviewer, are taken into account.

In the research reported in this paper, the students knew nothing about my personal viewpoints on the evolution and development of mathematics; they were not familiar with the descriptions in the KOM-report, nor the ‘identity’-description in the regulations for that matter. So it seems reasonable to say that none of these views could have affected the students’ answers. Of course, they knew that the interviewer was a mathematician which might have led them to alter some of their views. Also, it is true that many students do not have a clear and conscious idea about their beliefs about mathematics, as Lester says. When asking the interviewees to deepen or expand their questionnaire answers some of them would have trouble remembering what they answered, some would be puzzled about their own answers, and some would take on different viewpoints in the interviews than what they had expressed in the questionnaire. Especially the question of invention and discovery was one that seemed to puzzle the students; often they would have difficulties in making up their minds. From an educational perspective, this is, however, the power of precisely this question: that there is no correct answer to it. It is a matter of conviction, whether you are a Platonist, a formalist, a constructivist, a realist, an empiricist, or something else. Thus, students will have to *reflect* about the question on their own in order to take a standpoint.

Especially reflection and the ability to perform reflection are considered to be major factors in changing beliefs (Cooney et al., 1998; Cooney, 1999). Thus, if the students who took part in the research presented above were to have their beliefs ‘molded’ or ‘shaped’ in such a fashion that they would fit the previously presented goal-oriented descriptions, then one way of doing this would be to set a scene which enabled them to perform reflections. In fact, the students’ questionnaire and interviews reported above are an initial part of a larger research study, one purpose of which was to provide the students with classroom situations in which they were expected to work actively with and reflect upon issues related to, amongst other questions 1, 2, and 3. More precisely, these situations consisted of two larger teaching modules which the upper secondary class was to engage in over a longer period of time.⁶ During and after the period of implementation, the changes in students’ beliefs were attempted evaluated through more questionnaires and interviews but also by means of videos of

classroom situations taking as the point of departure the ‘initial’ student beliefs as presented in this paper.⁷ A comparison of the questionnaire and interview results presented in this paper, i.e. those from before implementing the modules, with the later research findings, those from during and after the implementations, will be presented in Jankvist (2009).

As a very final remark, I shall point to my own belief that reflections ought not only be considered as a means for changing existing beliefs, or creating new ones. A students’ image of mathematics should include an awareness of mathematics as a discipline that consists of and gives rise to questions to which there are no correct answers (e.g. that of invention versus discovery), and for this reason the ability to reflect is equally important. That is to say that not only is the act of providing students with an image of, or a set of beliefs and views about, mathematics as a discipline a goal in itself, the act of making the students capable of reflecting about their images is a goal as well.

NOTES

1. An exception is a Danish study of Christensen and Rasmussen (1980).
2. A few examples are Schoenfeld, (1985) and Leder and Fortaxa, (2002).
3. I shall not here enter the discussion of defining ‘beliefs’. I do, however, implicitly base my understanding of beliefs on the definition given by Philipp (2007).
4. The full questionnaire consisted of 20 questions covering the three different aspects mentioned as well as more personal, affective matters of mathematics to be used in a larger study (Jankvist, 2009).
5. The word ‘demonstrate’ in Danish has a dual meaning; it may be used both as the word ‘prove’ and as the word ‘display’. Thus, students may only need to display knowledge.
6. Descriptions of and preliminary results from this research study may be found in Jankvist, (2008a) and Jankvist, (2008b).
7. E.g. beliefs on question 2 were evaluated by posing more specific questions relating to the cases of the two modules.

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USING HISTORY AS A MEANS FOR THE LEARNING OF MATHEMATICS WITHOUT LOSING SIGHT OF HISTORY: THE CASE OF DIFFERENTIAL EQUATIONS

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The paper discusses how and in what sense history and original sources can be used as a means for the learning of mathematics without distorting or trivializing history. It will be argued that this can be pursued by adopting a multiple-perspective approach to the history of the practice of mathematics within a competency based mathematics education. To provide some empirical evidence, a student project work on physics' influence on the development of differential equations will be analysed for its potential learning outcomes with respect to developing students' historical insights and mathematical competence.

INTRODUCTION

Fried (2001) argues that when history is used to teach mathematics the teacher must

either (1) remain true to one's commitment to modern mathematics and modern techniques and risk being Whiggish, [...] or, at best, trivializing history, or (2) take a genuinely historical approach to the history of mathematics and risk spending time on things irrelevant to the mathematics one *has to* teach. (Fried, 2001, p. 398).

Whig history refers to a reading of the past in which one tries to find the present.

The purpose of the present paper is to argue that this dilemma can be resolved by adopting (1) a competency based view of mathematics education, and (2) a multiple-perspective approach to the history of the practice of mathematics. Hereby, a genuinely historical approach to the history of mathematics can be taken, in which the study of original sources is also relevant to the mathematics one *has to* teach. To present some empirical evidence for this claim a student directed project work on the influence of physics on the development of differential equations will be analysed. The project belongs to a cohort of mathematics projects made over the past 30 years by students at Roskilde University, Denmark. Only one project is analysed in the present paper, but the reflections and discussions brought forward are based on knowledge about and experiences from supervising many of those projects.

First, mathematical competence and the role of history in a competency based mathematics education are presented. Second, a multiple-perspective approach to a history of the practice of mathematics will be introduced. Third, the chosen project work will be analysed and discussed with respect to specific potentials for the learning of differential equations within the proposed methodology. Finally, the paper ends with some conclusions and critical remarks.

MATHEMATICAL COMPETENCE AND THE ROLE OF HISTORY

In the Danish KOM-project (2000-2002) mathematics education is described in terms of mathematical competence. In this context mathematical competence means the ability to act appropriately in response to mathematical challenges of given situations. It can be spanned by eight main competencies (Niss, 2004). Half of them involves asking and answering questions in and with mathematics: (1) to master modes of *mathematical thinking*; to be able to formulate and solve problems in and with mathematics, i.e. (2) *problem solving* and (3) *modelling competency*, resp.; (4) to be able to *reason* mathematically. The other half concerns language and tools in mathematics: (5) to be able to handle different *representations* of mathematical entities; (6) to be able to handle *symbols and formalism* in mathematics; (7) to be able to *communicate* in, with, and about mathematics; (8) to be able to handle *tools and aids* of mathematics. In the discussion below, the possible learning outcomes of reading sources will be analysed with respect to these competencies.

History of mathematics is not one of the main competencies, but is included in the KOM-project as one of three kinds of *overview and judgement* regarding mathematics as a discipline. The first concerns actual applications of mathematics in other areas, the second, historical development of mathematics in culture and societies, and the third, the nature of mathematics as a discipline (Niss, 2004).

The KOM-understanding of the role of history in mathematics education has the honesty to history as an intrinsic part. In Danish secondary school this understanding of history is included in the curriculum (Jankvist, forthcoming). The objective of the present paper is to discuss in what sense such an understanding of history can be implemented in situations where the curriculum does not include history and does not assign time to teach history. Under such circumstances, history of mathematics is most likely going to play no role at all in the learning and teaching of mathematics unless it can also be used as a means to learn and teach subjects in the syllabus.

A MULTIPLE PERSPECTIVE APPROACH TO HISTORY OF MATH

How can we understand and investigate mathematics as a historical product? One way is to think of mathematics as a human activity and of mathematical knowledge as created by mathematicians. This has been the foundation for many recent studies in the history of the practice of mathematics (Epple, 2000), (Kjeldsen et al., 2004).

To study the history of the practice of mathematics involves asking why mathematicians situated in a certain society, and/or intellectual context at a particular time, decided to introduce specific definitions and concepts, to study the problems they did, in the way they did it. In this line of thinking, mathematics is viewed as a cultural and social phenomenon, despite its universal character. Studying the history of mathematics then also involves searching for explanations for historical processes of change, such as changes in our perception of mathematics, our understanding of mathematical notions, and our idea of what counts as a valid argument.

A way of answering such questions is to adopt a multiple perspective approach (Jensen, 2003) to history where episodes of mathematical activities are analysed from multiple points of observations (Kjeldsen, forthcoming). The perspectives can be of different kinds and the mathematics can be looked upon from different angles, such as sub-disciplines, techniques of proofs, applications, philosophical positions, other scientific disciplines, institutions, personal networks, beliefs, and so forth.

How can this approach be brought into play to ensure the honesty to history, in a teaching situation where the teacher wants to use history as a means for students to learn a specific mathematical topic or concept? It can be implemented on a small scale, by having students read pieces of original mathematical texts focusing on perspectives that address research approaches or the nature and function of specific mathematical entities (problems, concepts, methods, arguments), in order to uncover, discuss, and reflect upon the differences between how these approaches and entities are presented in their text book and the former way of conceiving and using them. In such teaching settings, the students have to read the mathematical content of the original text as historians, using the “tools” of historians, and answering historians’ questions about the mathematics. For such tools, see e.g. (Kjeldsen, 2009).

Through activities where students work with historical texts guided by historical questions, connections between the students’ historical experiences of the involved mathematics and their experiences from having been taught the text book’s version, can be created in the learning process. When students read historical texts from the perspectives of the nature and function of specific mathematical entities, they can be challenged to use other aspects of their mathematical conceptions in new situations. So, it is of didactical interest to analyse historical episodes of mathematical research with respect to their potential to challenge students’ mathematical conceptions.

A HISTORY PROJECT: PHYSICS AND DIFFERENTIAL EQUATIONS

In the following, the student directed project work will be analysed with respect to how and in what sense the students’ work with original sources provided potentials for the learning of differential equations – without losing sight of history.

The educational context: problem oriented student directed project work

The project report on physics influence on the development of differential equations was written by five students enrolled in the mathematics programme at Roskilde University (RUC). All programmes at RUC are organised such that in each semester the students spent half of their time working in groups on a problem oriented, student directed project supervised by a professor. The projects are not described by a traditional curriculum, but are constrained by a theme (Blomhøj & Kjeldsen, 2009).

The requirement for this project was that the students should work with a problem that deals with the nature of mathematics and its “architecture” as a scientific subject such as its concepts, methods, theories, foundation etc., in such a way that the status of mathematics, its historical development, or its place in society gets illuminated.

Among the cohort of project reports, constrained by these objectives, this particular project was chosen, because the students happened to investigate differential equations, which are included in the core curriculum of advanced high school mathematics and mathematics and science studies in universities. Hence, the project work could be analyzed with respect to the issues addressed in the present paper.

Analysis of the project work: learning outcomes and the competencies

The students formulated the following problems for their project:

How did physics influence the development of differential equations? Was it as problem generator? Did physics play a role in the formulation of the equations? Did physics play a role in the way the equations were solved? (Paraphrased from (Nielsen et. al., 2005, p.8)).

On the one hand, these are fully legitimate research questions within history of mathematics. They address issues about an episode in the history of mathematics seen from the perspective of how another scientific discipline influenced mathematicians' formulation of problems as well as the methods they used to solve the problems. On the other hand, these questions can only be answered by analysing the details of original sources that deal with this particular episode in the history of mathematics, studying how the differential equations were derived from the problems under investigation, how the equations were formulated, why they were formulated in that particular way, how they were solved and with which methods – issues which are also relevant for the learning and understanding of the subject of differential equations. Based on readings of three original sources from the 1690s, the students discussed these issues within the broader social and cultural context of the involved mathematicians, critically evaluating their own conclusions within the standards for research in history of mathematics. Hence, in this way of working with history in mathematics education history is neither Whiggish nor trivialized.

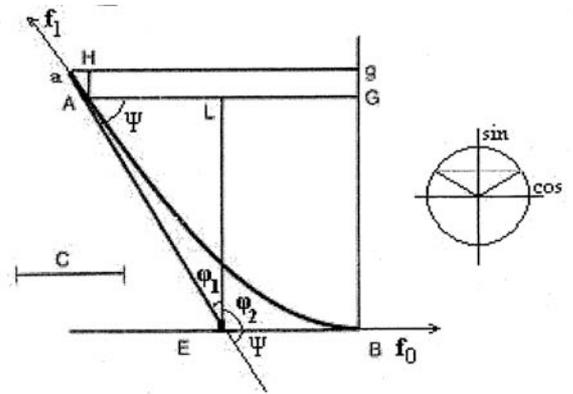
I will discuss three instances where the students – qua the historical work – were forced into discussions in which they came to reflect on issues that enhanced their understanding of certain aspects of differential equations in particular and of mathematics in general. The discussion will end with a short presentation of some of the learning outcomes with regard to the eight main mathematical competencies.

1: Johann's differential equation of the catenary problem. The catenary problem is to describe the curve formed by a flexible chain hanging freely between two points. The students read the solution that Johann Bernoulli presented in his lectures on integral calculus to the Marquis de l'Hôpital, supported by English translations of extracts (Bos, 1975). Bernoulli formulated five hypotheses about the physical system that, as he claimed, follow easily from static. For the students, of which none studied physics, to derive these assumptions was the first mathematical challenge in reading Bernoulli's text: "we had to derive most of them ourselves. We use 18 pages to explain what Johann Bernoulli stated on a single page" (Nielsen et. al., 2005, 19).

Below is one of the extract of Bernoulli's text (Bos, 1975, 36) that the students read. As can be seen from the text, Bernoulli used the five hypotheses to describe the catenary and the infinitesimals dx and dy of the curve geometrically and derived an equation between the differentials. The figure was produced by the students and is similar to a figure in Bernoulli's lecture, except from the sine-cosine circle.

Assuming these results, we go on to find the common Catenary as follows: Let BAA be the required curve, with its lowest point at B ; its axis, the vertical line through B , is BG . The tangent to the curve at its lowest point is the line BE , which will be horizontal. Let the tangent at any other point A be AE . We draw the ordinate AG and a line EL parallel to the axis. Let $BG = x$, $GA = y$, $Gg = dx$ and $Ha = dy$.

Because the weight of the chain is distributed evenly along its length we may put that weight equal to the length of the curve BA which we call s . Thus since an equal and constant force will always be required at the point B (by hypothesis (iv)) whether the chain BA is made longer or shorter, let this force be of magnitude a , expressed by the straight line C . Let us now imagine the weight of the chain AB to be concentrated and suspended at the point E where the tangents AE and EB meet. Then (by hypothesis (ii)) the same force is required at B to support the weight E as was previously required to support the chain BA . Indeed (by hypothesis (v)) the ratio of the weight E to the force at B is the same as the ratio of the sine of the angle AEB , or of its complement angle EAL , to the sine of the angle AEL , that is the ratio of EL to AL . Therefore, whatever position on the curve is taken for the fixed point A (the curve, by hypothesis (iii), being always the same) the ratio of the weight of the chain AB to the force at B is the same as the ratio of EL to AL , that is $s:a = EL:AL = AH:Ha = dx:dy$, and inversely $dy:dx = a:s$.¹



In their report, the students went through Bernoulli's text and filled in all the arguments. They were not familiar with this way of setting up differential equations from scratch so to speak, so the mathematization of the physical system was a major challenge for which they needed to consult some textbooks on static and to combine the physics with mathematical results about triangles and the sine-cosine relations.

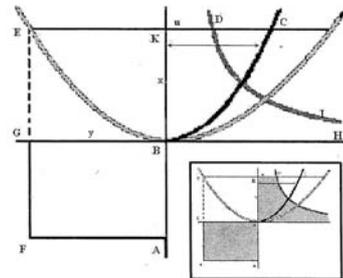
Bernoulli's arguments do not meet modern standards of rigour and that created cognitive hurdles for the students. Didactical, it is important to identify such hurdles because they create situations where the students, during their struggle with understanding the mathematical content of the original text, can be challenged to reflect upon the differences between our modern understanding and the one presented in the source, thereby enhancing their own understanding of the concept of, in this case, differential equations and the mathematical techniques and concepts underneath. A concrete example of this is Bernoulli's use of the infinitesimal triangle. In the text above he used similar triangles, to argue that $s:a = dx:dy$ but, as the students pointed out in their report, a does not lie on the tangent but on the catenary. Bernoulli also used the infinitesimal triangle later in the lecture, when he reformulated the differential equation derived above, using that $ds = \sqrt{dx^2 + dy^2}$. Again – as pointed out by the students – ds is a part of the catenary, not the hypotenuse of a right angled triangle.

This mixed use of geometrical arguments and infinitesimals in deriving and reformulating the differential equation was very different from the students' text book experiences of differential equations. The fact that Bernoulli's method worked in this particular case, despite its lack of rigour, provoked a discussion among the students and their supervisor (the author) about Bernoulli's use of the infinitesimal triangle

and his use of the infinitesimals, dx and dy , as actual infinitely small quantities. This made the students focus more systematically on the differences between now and then, questioning, at first, why we need to define a differential quotient as the limit (in case it exists) of difference quotients, then analysing the situation again to understand why Bernoulli's method worked fine for the catenary, and trying to picture situations where it would go wrong. This is an incidence where connections were created between the students' historical experiences and their experiences from modern mathematics which challenged them to examine their own understanding of the involved concepts. Through these discussions, the students built up intuition about infinitesimals and awareness about the reasons behind the construction of our modern concepts. Major differences were the lack, in the seventeenth century, of the concept of a function, of a limit, and the formalised concept of continuity. In this project work the historical texts provided a framework for discussions among the students and with their supervising professor, about what constitute the concept of a differential equation, and how we can read meaning into it. Through these discussions, which were triggered by the historical texts, the students came to reflect upon the concept of a differential quotient and the meaning of a differential equation on a structural level that went beyond mere calculations and operational understanding of the concepts. This is an example of what Jahnke et. al (2000) calls a reorientation effect of studying original sources.

2: Johann's solution of the catenary differential equation. Through some further manipulations Bernoulli reached the following formulation of the equation for the catenary $dy = adx/\sqrt{x^2 + 2ax}$ which he used to construct the curve geometrically. This puzzled the students and initiated discussions about, what it means to be a solution to a differential equation.

Let the normals AK and GH be drawn, meeting in B . Take $BA = a$ and describe an equilateral hyperbola BC with vertex B and centre A . Now construct a curve DI with the property that everywhere BA is the middle proportional between KC and KD , that is such that $KD = aa:\sqrt{(2ax + xx)}$. Now draw a parallel AF and take the rectangle AG equal to the area $HBKDI$. Prolong DK and FG , then their intersection point E will be on the required curve.¹



As can be seen from the above extract (Bos, 1975, 41), Bernoulli interpreted the integral geometrically, as the area below a curve. The students added an illustration of this in their figure, as can be seen above, with the two shadowed areas which are not present in Bernoulli's figure. This way of solving the equation by constructing the curve forced the students into discussions about conceptual aspects of solutions to differential equations. It made them articulate what constitute a solution in our modern understanding, an articulation that does not automatically manifest itself from solving differential equation exercises from modern textbooks. In order to follow Bernoulli's construction, the students were challenged to think about and use integration differently than they would normally do when solving differential equations analytically. They were also forced to use the properties of the curve

represented geometrically which they felt as a challenge. They were used to using the direct relationship between the analytical expression of a function and the coordinate system, to produce a graph. Here they went “the other way” and had to think of the curve as being represented by its graph instead of its analytical expression. Historically, they realised that what is understood by a solution to a differential equation has changed in the course of time.

3: Different solution methods of the brachistochrone problem. The brachistochrone problem is to describe the curve of fastest descent between two points for a point only influenced by gravity. Jacob and Johann Bernoulli published different solution methods to the problem in 1697. Johann Bernoulli interpreted the point as a light particle moving from one point to another. By using Fermat’s principle of refraction, he derived an equation for the brachistochrone, i.e. the cycloid, involving the infinitesimals dx and dy . Jacob Bernoulli considered the problem as an extremum problem using that, since the brachistochrone gives the minimum in time, an infinitesimal change in the curve will not increase the time.

The differences between Johann’s and Jacob’s solution of the brachistochrone illustrated for the students the power of mathematics. Johann’s solution was tied to the physical conditions of the problem and could not be generalised beyond the actual situation, whereas Jacob’s solution was independent of the physical situation and could be used on different kinds of extremum problems. Through the historical texts on the solution of the brachistochrone, the students experienced the characteristics of the nature of mathematics that makes it possible to generalise solution methods of particular problems. Thereby, they were able to understand why Jacob’s method could generate new kinds of questions that eventually led to a new research area in mathematics, the calculus of variations, and why Johann’s could not. For a didactical perspective on the brachistochrone problem see Chabert (1997).

Development of mathematical competencies. In the discussions above of episodes where the students through their work with the original sources used other aspects of their mathematical conceptions in new situations and discussions, some learning potentials regarding differential equations and the mathematical concepts underneath have already been emphasised, especially in the discussion of the students’ work with Johann Bernoulli’s text on the catenary. A more systematic analysis of the students’ report with respect to the KOM-report showed that the students, in their work with the historical texts, were challenged within seven of the eight main competencies. The students’ awareness of the special nature of *mathematical thinking* (1) was especially enhanced in their comparison of Johann’s and Jakob’s solutions of the brachistochrone as discussed above. The students’ *problem solving* (2) skills were trained extensively and in different areas of mathematics. As mentioned in the discussion of their work with Johann’s solution of the catenary problem, the students’ had to fill in a lot of gaps in order to understand Johann’s results. Each of these gaps required that the students derived intermediate results on their own about similar triangles using trigonometry, and solved mathematization problems. Through their

work with understanding the Bernoulli brothers' mathematization of the physical problems, parts of the students' *modelling* competency (3) were developed. The competency to *reason* (4) in mathematics was developed in all those parts of the project work where the students tried to make sense of the original sources by means of their own mathematical training and knowledge. (5) *Representations*: As exemplified in the discussion of the students' work with Bernoulli's construction of the solution to the differential equation of the catenary, the students were challenged so work with a representation of the solution to the differential equation that is different from the analytical representation given in modern textbooks. In the report, the students also solved the differential equation analytically and compared the analytical representation with Bernoulli's geometrical one. During their mathematization of the five hypotheses from static that Bernoulli took for granted, the students were trained both in working with different representations and in using the mathematical language of *symbols and formalism* (6). This competency was especially developed in the students' work with the two original sources on the brachistochrone problem in their struggle to understand Johann's mathematization of the path of the light particle and Jakob's use of the minimising property of the brachistochrone. The writing of the report (90 pages) in which the students, through a thorough presentation and analysis of the original sources, answered their problems for their project work within the historical context, developed their competency to *communicate* (7) in, with, and about mathematics in ways that go far beyond what normal exercises in solving differential equations requires. The competency to handle *tools and aids* (8) was not represented.

SOME CONCLUSIONS AND CRITICAL REMARKS

Based on their studies of the original sources and relevant secondary literature, the students concluded that physics did function as problem generator in the early history of the development of differential equations and played a decisive role in the derivations of the equations describing the catenary and the brachistochrone. They further concluded that physics played a significant role for Johann's solutions of both the catenary and the brachistochrone problem, but not for Jacob's solution of the brachistochrone problem. Jacob's arguments were not linked to the physical system; hence his method could be transferred to other problems of that type. This became the beginning of the calculus of variations. The students did not move beyond this in their project, but it is interesting to notice that the calculus of variation later became central in physics, providing an important feedback in the opposite direction.

The analysis of the chosen project has shown that, *if* we adopt a competency based view of mathematics education and evaluate learning outcomes not with reference to standard procedures and lists of concepts and results, but with respect to how and which mathematical competencies, the students have been challenged to invoke, and thereby develop, and *if* we let the students work with the history of the practice of mathematics studied from specific perspective(s) that address(es) significant issues

regarding the mathematics in question, then history can be used as a means to teach and learn core curriculum subjects without losing sight of history.

The above claims are further supported through analyses of other historically oriented mathematics projects that have been performed by students at RUC. A project on the history of mathematical biology, where the students read an original source of Nicholas Rashevsky on a mathematical model for cell division is treated in (Kjeldsen & Blomhøj, 2009) and analysed with respect to learning outcomes regarding deriving and understanding the general differential equation of diffusion, the students' understanding of the integral concept, and development of the students' modelling competency. Other examples of projects with substantial learning outcomes of core mathematics, in university mathematics education, are "Paradoxes in set theory and Zermelo's III axiom", "What mathematics and physics did for vector calculus", "Generalisations in the theory of integration", "Infinity and "integration" in Antiquity", "Bolzano and Cauchy: a history of mathematics project", "The real numbers: constructions in the 1870s", and "D'Alembert and the fundamental theorem of algebra". In the present paper focus has been on how history can be used for the learning of core curriculum mathematics without trivializing it or using a whiggish approach to history. The learning outcome of the above history projects can also be analysed with respect to *Mathematical awareness*, as explained by Tzanakis and Arcavi (2000), which includes aspects related to the intrinsic and the extrinsic nature of mathematical activity. These projects can then also be seen as empirical evidence for some of the possibilities history offers as referred to by Tzanakis and Arcavi (2000, 211). With respect to the KOM-report these aspects relate to the three kinds of *overview and judgement*.

It can be raised as a critic that only certain perspectives of the history are considered, and that e.g. to gain insights into historical processes of change, episodes from different time periods need to be studied. In the above project work, the students did not experience the historical process of change, but they did experience that the understanding of the involved mathematics in the 17th century was different from our understanding. The students did not solve a huge amount of differential equations, and they did not learn to distinguish between different types of differential equations.

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WHAT WORKS IN THE CLASSROOM - PROJECT ON THE HISTORY OF MATHEMATICS AND THE COLLABORATIVE TEACHING PRACTICE

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This paper describes the project that was undertaken in the South East of England, and which aimed to introduce the history of mathematics at the primary and secondary level. The project was conducted through collaborative teaching practice (peer based network of teachers collaborating on research, planning, teaching in teams, and assessing the outcomes of lessons) and was based on the premise that the history of mathematics can improve both the motivation and attainment when used as a contextual background in the teaching of mathematics at this level.

THE PROJECT BACKGROUND

The project described here was one of the first few projects awarded the support by the National Centre for Excellence in the Teaching of Mathematics (founded in June 2006). Aims of the project were to:

- Introduce the history of mathematics into everyday teaching in order to
 - Encourage students to begin making the connections between mathematical topics
 - Increase interest and motivation by setting the problems in historical context
 - Enrich mathematical understanding through historical explorations
 - Assess the role of the history of mathematics in setting the new curriculum
- Introduce collaborative teaching practice as a model of continuing professional development, at the same time adopting an inquiry-led learning approach to the lesson development thus raising issues about
 - Teachers learning with pupils (simultaneously in some cases) and the effects this may have on his or her professional role
 - Training preparation for teachers in an inquiry-led learning environment.

The answers to these questions will be provided in this paper in two-fold ways: through the personal reflections of teachers who participated in the project, and through a synthesis and explanation of methods used throughout the project. The latter is provided as a way of suggesting the model of continuing professional development for teacher groups and networks wishing to introduce the historical element into the teaching of mathematics through collaborative practice.

The project began in September 2006 and was completed in September 2008 with a national conference held at the London Mathematical Society at which experiences of the teachers involved were disseminated among the mathematics education community. Over the course of the project three secondary schools, with a total of fifteen teachers (two of whom were science specialists but taught mathematics to lower ability groups), and three primary schools with a total of three teachers have been involved. More than 450 pupils have been involved in the project at various times, spanning the age range between ten and fourteen (English Key Stages 2 and 3) and covering all ability ranges.

The project has been conceived and led by the author of this paper, and, as already mentioned, was supported by the National Centre for Excellence in the Teaching of Mathematics (UK). In the second year of the project the British Society for the History of Mathematics provided financial and organisational support; the University of Plymouth Centre for Innovation in Mathematics Teaching provided the training for all involved teachers in the principles of collaborative teaching practice, and the British Society for the History of Science provided extra funds for the final conference celebrating the project. An additional private consultant has been involved in the project in the second year, offering support in the matters of teacher training and the uses of the history of mathematics in development of mathematical pedagogy.

The new curriculum for England and Wales

The recent changes in the National Curriculum, and the new approach taken by the Qualifications and Curriculum Authority (QCA) introduced a certain amount of freedom for teachers, teacher teams, and consortia of schools to develop their own syllabus in all subjects. The modernising of the curriculum is driven by the need to take into account local needs and needs for different types of vocational training. One of the more positive aspects of this development may be seen in the fact that the local provision of education will have a degree of freedom (not yet defined), and that personalised learning, project based work and mentoring will all have a big role to play in this new vision of education. This opens a valuable opportunity for teachers to demonstrate that mathematics, like any other creative pursuit, is an area where exciting and useful contributions can still be made – both by teachers and by pupils. As such, the introduction of the historical element in the mathematics syllabus, although not sufficiently developed in the quote that follows, offers the *possibility* of developing teaching strategies which do not necessarily provide only historical context, but use the history of mathematics as a tool for discovering facts and exploring mathematical techniques. The new curriculum states that the students should recognise the ‘rich historical and cultural roots of mathematics’:

Mathematics has a rich and fascinating history and has been developed across the world to solve problems and for its own sake. Students should learn about problems from the past that led to the development of particular areas of mathematics, appreciate that pure mathematical findings sometimes precede practical applications, and understand that mathematics continues to develop and evolve.¹

Since the completion of the project, and based on the recommendations following from the project report, measures are being taken by the Joint Mathematical Council (UK) to define the ways in which history of mathematics can and should be deployed to help shape the future development of the curriculum, and the teacher pre-, and in-service training development and provision.

The current challenge now facing English teacher-training institutions will be to address the imbalance between the desire to introduce the historical element to the teaching of mathematics and a lack of the formal teaching in the subject area for the serving teachers. The project described can therefore, give a valuable insight into the types of issues facing teachers in this situation, with a view of defining some benchmarks on which it would be possible to base a programme of in-service training in the history of mathematics.²

METHODOLOGY, ACTIVITIES, DATA

Collaborative Teaching Practice and the History of Mathematics

The project has been pursued by practicing teachers with various degrees of experience in the teaching of mathematics (not all of whom are subject specialists), and therefore the question arose of how to create a professional learning environment which would be able to contain all levels of experience and mathematical ability in order to support their participation. Of major interest was the possibility of introducing a model of continuing professional development based on a set of principles which could be replicated elsewhere and which would help teachers develop a range of techniques, and introduce a new element which could help them structure their own learning at the same time as structuring their teaching programme.

We chose the model of collaborative teaching practice as one which would offer opportunities for teachers to develop their subject knowledge through research into the history of mathematics. Collaborative teaching practice was developed in different countries as far back as the 19th century (most prominently Japan, but recently also in the United States and England) and is sometimes also closely linked and/or referred to as ‘lesson study’.³ The collaborative teaching practice that was part of the described project as a way of peer-discussion and collective teaching tool was based on the simple cycle of planning - researching - sharing resources - teaching collaboratively - and finally assessing the outcomes of a lesson.

At the core of this envisaged professional learning model stood a belief that the interest and personal development can only be achieved in those situations and environments where the professionals themselves find an area of research they would like to pursue further.

Various mathematics educators have seen the different roles the history of mathematics can take through its introduction into the education of mathematics teachers - Freudenthal (1981) for example conceived it as giving a background to the teachers’ mathematical knowledge, while others concentrated on offering a

possible pathway to the deepening of teachers' reflection capabilities through an in-depth study of the development of mathematical concepts through history (see Arcavi, Bruckheimer, & Ben-Zvi, 1982, 1987; Swetz, 1995). One of the approaches, developed by Hsieh and Hsieh (2000), and Philippou and Christou (1998a, b) dealt with using the history of mathematics as a particular tool and context to develop beliefs and attitudes in mathematics.

The benefit of the use of history of mathematics however, in the context of the described project, can be best seen on the influence in which it created an opportunity for a *focus* of cooperation and collaboration as well as an impetus for the creation of a *conceptual landscape* which offered opportunities to teachers to develop their individual interests.

This highly individualist approach to the continual professional development of teachers can increase their subject knowledge and enable them, through the modern technologies, to share their experiences and knowledge with mathematics teachers and students from around the world. Our agreed aim was to adopt a creative and individualistic ethos in teaching, providing ample opportunity for bringing the history of mathematics alive to the present generation of school children. Eventually, in practical terms, the defined foci were enlarged to include, apart from the collaborative teaching practice and the individual research, the creation of a networking platform in the form of web-quests⁴.

Teachers' learning in an inquiry-led learning environment, and the collaborative teaching practice

The inquiry-led learning as developed through⁵ this project grew organically from the collaboration with similar-minded colleagues. The successful outcomes were produced in those instances in which a few necessary prerequisites were fulfilled - existence of full professional trust and exchange of information and knowledge had to be devoid of all performance management in participating groups of teachers. Collaborative teaching practice was described in the teacher reflections thus:

The students appreciated the teachers cooperating between themselves and being more relaxed and focused on learning rather than discipline.

It (this project) has certainly been a huge milestone in my professional development. Firstly, it has shown me the true value of collaborative teaching and the focus on the 'learning' rather than the 'teaching'. Secondly, it has made me question why I am teaching what I am teaching, and how to help the children answer the 'why' do we do this questions by giving them relevance and meaning to the maths. My next milestone experience will be to embed this into my teaching and more crucially into the teaching of my colleagues.

History of mathematics and the development of the curriculum

In the description of the other aspects of this project it is described how the history of mathematics helped shape the building of the professional learning environment

which then spilt over into the classroom. Historical dimension, apart from earlier mentioned benefits (see pages 1-4) was also important for teachers in terms of their involvement with the whole-school issues:

The maths becomes ‘embedded’ in the culture and life and is not seen as something totally dry and devoid of meaning. This also changed the perception of mathematics in my department... (by a science teacher)

There is a large scope in my school to bring about change in the mathematics curriculum and I am hoping to introduce an element of the History of Maths into the curriculum. ‘Using and Applying Mathematics’ is the common strand that is across the whole maths curriculum, and my experience on the project is that practical maths (in and out of the classroom) is a powerful medium by putting the children in the shoes of mathematicians from history so they can appreciate the ‘why’ and not just the ‘how’.

OUTCOMES - STRUCTURING THE SELF-REGULATORY CONTINUING PROFESSIONAL DEVELOPMENT THROUGH COLLABORATION AND RESEARCH

The project showed how the history of mathematics can set the ‘scene’ and act as a catalyst in creating a professional learning environment as well as giving a structure to endorse inquiry both in the student and in the teacher. In mathematics, this dimension is or can be, added to any such particular conceptual landscape.

The history of mathematics and the process of reorientation

As Furinghetti has shown (2007) some teachers tend to believe that the style of mathematics teaching they were affected by or exposed to must be reproduced in their own practice. In the case of the described project, this was most evident in the attitudes of teachers who were non-specialists in the subject. Furinghetti showed that the history of mathematics context allows for an exploration of topics in a new light and hence helps teachers construction of teaching sequences. While this was one of the added benefits of introducing the history of mathematics into the collaborative practice, we were also aware of the uses of history of mathematics in teaching, therefore allowing us to explore the various roles the history of mathematics can take in the classroom practice.

Whilst the history of mathematics in teacher education programmes has been described at some length by Furinghetti (2007), Schubring (Schubring et al., 2000), and Heiede (1996), little has been so far written about the in-service training of practicing teachers in this regard. This project aimed to begin the task by making a sketch of the possible influence the history of mathematics can have on in-service specialist and non-specialist mathematics teachers.

Therefore one of the project’s aims became to try to introduce what Furinghetti (2007) calls ‘reorientation’:

...the learners involved in the process ... are forced to find their own path towards the appropriation of meaning of mathematical objects.⁶

In this context, the acquisition of meaning was attempted through exposing beliefs about, and the partial understanding of, the concept in question with the new, ‘foreign’ meaning:

A meaning only reveals its depth once it has encountered and come into contact with another, foreign meaning: they engage in a kind of dialogue, which surmounts the closedness and one-sidedness of these particular meanings.⁷

In short, one of the teacher testimonies illustrates these described process thus:

... I was... astounded (by)... the depth there is in so many topics we have covered through this project. It has rekindled interest in mathematics in me; students find it interesting as well.

Scaffolding knowledge for non-specialist mathematics teachers

An increasing body of research shows that inquiry-based-learning helps create an environment in which the teacher may be required to act in manifold ways.⁸ These manifold roles of a teacher relate to the theory of ‘Knowledge Manifolds’, in which teachers are ‘promoted’ from teacher/preacher to teacher/consultant and teacher/resource type of roles. Naeve (2005) defined the ‘Knowledge Manifolds’ as ‘linked information landscapes (contexts) where one can navigate, search for, annotate and present all kinds of electronically stored information’.⁹ Such open information landscapes have developed with an exponential speed since the founding of Wikipedia (domain launched only in January 2001), and rest on fundamental principles of communal and self-governance in the same way in which Naeve suggests future ‘teaching landscapes’ will develop. This theory is in concordance with the network theories of knowledge as much as it is with the theory of ‘mobile learning’. The described project opted to further explore in practice such approach to teaching and learning in which teachers are as much learners as their pupils by making parallels between the sets of teachers with the sets of pupils. Some teacher reflections addressing this particular aspect are:

This project has developed my skills to be able to find resources and to try to relate things to the history.

Research was good for subject knowledge; because of the historical content, it widened our own perspective about mathematical topics, and gave us time to find about something in more depth.

Historical element shows you the different aspects of something in more depth; it allows for ‘scaffolding’ of the knowledge and easier transference to children. The historical element can also offer easier focus.

Furthermore, Naeve’s (2005) approach to knowledge which he identifies as that consisting of ‘efficient fantasies’ and learning as that consisting of ‘inspiring fantasies’ has a lot to offer in the context of creating a learning environment in which both teachers and students discover new facts and exchange ideas in a more elaborate, creative, and yet mathematically sound ways. Naeve’s description of fantasy has a lot to offer in terms of initiating a process of learning not only in the

here and now, but one that draws upon the initial interest in the ‘fantasy’ and how it (the fantasy) occupies a mind of a learner for a longer period of time, offering a prolonged urge to find ever increasingly new content about a subject matter. Teachers from the project spoke often about these ‘fantasies’ as most important in the initial stages of introducing a new mathematical topic or concept. The length of this paper does not, unfortunately, allow for further analysis on the subject matter in more depth.

What the conclusions teachers made however, agrees with Naeve’s suggestion that the education process consists in

...exposing the learner to inspiring fantasies and assisting her/him in transforming them into efficient fantasies.¹⁰

While Naeve somewhat exaggerated the view of the traditional ‘learning architectures’ being exclusively teacher-centric and consequently his concept of knowledge ‘pushing’ rather than knowledge ‘pulling’ may be lacking in subtlety, his intention to shift the focus onto the system of initiation into an interest field, whilst at the same time offering the system of skills to equip a learner with a set of tools to undertake the task of discovery and learning is at the centre of all: ‘collaborative’, ‘flexible’, and ‘personalised’ learning concepts.¹¹

So far, as in the case of Mariotti (2000), the focus on developing strategies to initiate ‘learning fantasies’ has been on the pupils. In the new type of learning environment, one in which ‘knowledge pulling’ rather than ‘knowledge pushing’ is taking place, teachers and pupils are learners and communicators of insights into mathematical facts at the same time, interchanging roles at different levels. From the experience of our project it became clear however, that some of the roles of the learner and some of the roles of the teacher are interchangeable, whilst others remain strongly rooted in the

- a) evolutionary roles and
- b) social roles these two groups represent.

CONCLUSION

Although no external evaluation had taken place to date, the internal, self-evaluation, concluded that this was an invaluable opportunity for all teachers involved in the project in terms of re-awakening their interest in the subject and increasing their self-awareness on their abilities in terms of subject knowledge, pedagogy and ability to conduct academic research. Additionally, teachers identified acquisition of skills in terms of ability to envisage their own CPD landscapes through building ‘knowledge patches’ and increased ICT competencies as further valuable benefits of their involvement in the project.

The nature of learning is a constantly changing environment, in which learners are often ahead in terms of their technological competencies than their teachers. The knowledge content does not move at such a great speed, but it’s presentation and availability is something that often lacks sophistication in the eyes of the learner. In

mathematics this is sometimes more often apparent than in subjects such as literature or history.

Mathematics learning has to gain an enormous amount from developing landscapes of knowledge patches that students can tap into through and because of their interests and abilities. This project began the process of enabling the teachers to be able to start developing these landscapes in collaborative environment, and having for a focus the wealth of resources that the history of mathematics has to offer.

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¹ Page 4 of the QCA Mathematics Curriculum, accessed 20th March 2008, <<<http://curriculum.qca.org.uk/subjects/mathematics/keystage3/index.aspx>>>.

² As this paper was being completed, the new module in the history of mathematics was being developed at the Open University UK, aimed at anyone interested in the history of mathematics.

³ See Lewis (1995), Lewis and Tsuchida (1998), Stigler and Hiebert (1999), and more recently Fullan (2004), (2005).

⁴ Self-contained websites offering materials for the study of particular mathematical topics. First webquest from this project is available from <http://www.webquests.mathsisgoodforyou.com/>.

5

⁶ Furinghetti (2007), 113.

⁷ Bakhtin (1986), 7, as reported by Radford, Furinghetti, and Katz (2007), 108.

⁸ Naeve describes these roles as that of “*knowledge cartographer* [who] constructs context maps, the *knowledge librarian* [who] fills the maps with content, the *knowledge composer* [who] combines the content into customised learning modules, the *knowledge coach* [who] cultivates questions, the *knowledge preacher* [who] provides answers, the *knowledge plumber* [who] routes questions and the *knowledge mentor* [who] provides a role model and supports learner self-reflection.” Described in Naeve (1997).

⁹ Naeve (2005), 6.

¹⁰ Naeve, (2005), 4.

¹¹ All part of the national strategies on ‘Every Child Matters’, ‘Personalised Learning’ and ‘Extended Schools’. See related sections at the <<<http://www.standards.dfes.gov.uk>>>.

INTUITIVE GEOMETRY IN EARLY 1900S ITALIAN MIDDLE SCHOOL

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A distinction between intuitive and rational geometry formally appeared in the Italian school programmes after the Italian unification of 1861. This distinction, that is not just an Italian issue, loosely corresponds to the points of view also adopted in the current geometry school programs both at a primary (6-10 and 11-14) and at a secondary (14-19) level. It is not difficult to define rational geometry: Although it has been approached with various methods, it is undeniable it arises from Euclid's elements. On the contrary, it is more complex to give a definition of intuitive geometry and to understand in which way it leads to rational geometry. This paper will illustrate the interpretation given to intuitive geometry by the school programs and by the many authors of textbooks at the end of 1800s and beginning of 1900s in Italy. This analysis can help to discuss today's curricular issues.

Key – words: Intuitive geometry – curriculum – history – school books.

INTRODUCTION

The term *rational geometry* first appears in the Italian school programs in 1867, a few years before the complete Italian reunion, which occurred in 1871. A school reorganization brought in Euclid's *Elements* as *the* geometry textbook aimed to teach the subject in the Gymnasium-Lycée.¹

In 1881, intuitive geometry comes to life to be taught in the first three years of the Gymnasium (the “lower Gymnasium” corresponding to the present middle school). Previously, geometry was not part of the school programs for students in this age.

As we will see forward, intuitive geometry was explicitly introduced as an introductory (*propaedeutic*) subject to let students better understand the rational geometry studies.

It was not just an Italian issue to make a distinction between intuitive and rational geometry. Although with a different interpretation, references to intuitive geometry

¹ Secondary education was divided into a first and a second level. To cover classical secondary education, a law of 1859 had introduced the Gymnasium and the Lycée - The Technical School and the Technical Institute were set up for technical secondary education.

The Gymnasium and the Technical School were preceded by four years of primary school. The Technical School thus covered the same age range as the present-day middle school (11–14) while the Gymnasium lasted for five years and hence included the first two years of high school followed by three years of Lycée.

can be found also in the German and English literature of the same period (Fujita et al., 2004). In the textbooks of Treutlein (1911) and Godfrey & Siddons (1903), intuitive geometry - still an introduction to rational geometry – is identified with the ability to perceive a shape in a space, partially aiming to provide the basic elements which explain the real world, and partially aiming to develop logical skills. Accordingly, Fujita et al. describe intuitive geometry as “the skill to ‘see’ geometrical shapes and solids, creating and manipulating them in the mind to solve problems in geometry”. This definition surely does not correspond to the characterization given by the Italian legislators at the end of the 19th century.

It is not difficult to give a definition for rational geometry. The term *rational*, as opposed to intuitive, is meant to refer to any aspect of the logical and theoretical organization of the geometry (Marchi et al. 1996); although rational geometry can be approached in different ways, Euclid’s Elements always remain at the foundations of this subject. On the other hand, it is more complex to define intuitive geometry and to analyze the way it is linked to rational geometry. Many researchers in mathematics education tackled this issue; a particular example is given by the theory of the Van Hiele levels (cfr. Cannizzaro & Menghini, 2006).

The lack of a formal definition and of a detailed description of the tasks of intuitive geometry caused continuous role changes in the Italian school programs. We believe it is important to discuss and analyze the reasons and the episodes which led to the introduction of intuitive geometry in the Italian school programs in the period between the 19th and the 20th centuries.

SCHOOL PROGRAMMES

In 1881, elementary geometry and geometrical drawing were introduced in the first three years of the Gymnasium. An earlier intuitive experimental approach was considered a good help for students to overcome the difficulties caused by rational geometry and by the logical deduction of Euclid’s textbook. Geometrical drawing should also contribute to overcome these difficulties. Intuitive geometry had to

give to youngsters, with easy methods and, as far as possible, with practical proofs, the first and most important notions of geometry, ...useful not only to access geometry, but also to let the students desire to learn, in a rational way, the subject throughout the Lycée.

Moreover, rational geometry was postponed to the Lycée, skipping the two years of the higher Gymnasium, in order to avoid all the difficulties caused by its study.

Three years later, the new minister, following a suggestion of the mathematician Beltrami, abolished the study of intuitive geometry from the lower Gymnasium and moved down rational geometry to the 4th year of the Gymnasium. This decision was a consequence of a lack of clear boundaries, and of the fear that teachers could not emphasize in the right way the experimental-intuitive nature of geometry being tied to the traditional logic-deductive aspect of rational geometry (Vita, 1986 p.15).

In the following years, only a few changes were introduced concerning the beginning

of the study of rational geometry - which could be moved down to the third year of the *Gymnasium* - and the learning approach to Euclid's books. According to Vita (1986, p.16), "the oscillation reflects a clear didactic anxiety and the desire of finding the most psychologically adequate time to teach *The Elements* by Euclid, with all its logical-deductive layout, to the 13-15 year old pupils".

In the 1900s a new program was broadcast: intuitive geometry was restored in lower *Gymnasium*, but, to prevent past problems, the programme included only elementary notions such as the names of the easiest geometrical shapes, the rules to calculate lengths, areas and volumes and also basic geometrical drawing. Some instructions specify that the new studies "were an introduction to rational geometry". Moreover, they underline that these new studies were "a review and an expansion of the notions acquired by the students at the elementary school", and required a practical approach, amplified by the teaching of geometrical drawing. With regard to rational geometry, the new programmes gave more freedom in the choice of the textbook, as long as it followed the "Euclidean method" (cfr. Maraschini & Menghini, 1992).

INTUITIVE GEOMETRY TEXTBOOKS IN EARLY 1900S

Since the program dated 1881 was effective for a very short period, we cannot find textbooks of intuitive geometry in those years. Instead, they appeared right after 1900. One of the first was the textbook by Giuseppe Veronese (1901). In Veronese's book we can easily notice the effort made to follow the ministerial programmes², considering the main properties of the geometrical shapes using simple observation, rather than intuition. Veronese wants to deal only with "those shapes that have an effective representation in the limited field of observation". Initially, not even the straight line, the plane and unlimited space are the subject of his dissertation, given that they need an abstraction process. Furthermore, Veronese believed it is dangerous to introduce concepts that will need to be amended at some stage in higher studies.

In the *Preliminary Notions*, Veronese gives examples of *objects* (table, house..) and of their *properties* (colour, weight..). Material points (grains of sand) lead to the abstract concept of point, and material lines (a cotton thread) lead to the abstract concept of line, which is defined, both with practical examples (a pencil line) and as a *linear set of points* (an anticipation of what students would find in his textbook for the Lyceé).

All the authors of intuitive geometry books of this period introduced the straight line using the idea of a stretched string, and explain later on the way it can be drawn using a ruler. Veronese 'surrendered' to the temptation of stating the reflexive, symmetric and transitive properties of the equality relation for the segments in a more abstract way. Afterwards, he explained that the congruence of the segments could be verified

² Index: preliminary notions; line; plane; equal shapes; plane polygons; circle; perpendicular lines and planes; polyhedra; cone – cylinder – sphere; sum, difference and measure of segments and angles; measure of segments and angles; surface areas, volumes; exercises. Drawing tools; basic constructions; Line, plane and unlimited space.

using a ruler or a compass. Here is an example on how the classical distance axiom was interpreted from the *observer's* point of view:

Assuming that the extension of the field of observation is appropriate, it is possible to verify that: On a straight line r , given a point A and a segment XY , two segments exist CA and AB having the same direction and length of XY . The axiom can be proved using a piece of paper marked with a segment of the same length of XY , and sliding it along the line r in the direction showed by the arrow $\underline{C \dashrightarrow A \dashrightarrow B \quad X \quad Y}$ (p. 9).

The textbook included only one simple proof. After the definition of symmetric points about a given point O (central symmetry), Veronese stated the following:

The shape symmetric to a line about a given point is another line.

Let ABC be a line and $A'B'C'$ the shape opposite to ABC about a point O . Using a compass, or copying the shape AOB on a piece of drawing paper and turning the paper up side down so that OA corresponds to OA' and OB to OB' , we can verify that the point C' is on the line identified by B' and A' ... (p.13).

We positively consider the fact that geometric transformations were considered suitable for an intuitive introduction to geometry: as a *tool*. Motions can in fact be carried out experimentally. We will find this use of geometrical transformations also in other books.

To avoid infinity, Veronese stated that two lines are parallel when they are symmetric about a point, and explained how to verify that two lines are parallel manually (p.14). He listed elementary definitions for triangles, quadrilaterals, other polygons and for the circle without stating any property of these shapes.

Throughout his book, Veronese included simple drawing exercises, meant to be done by hand (to draw a dotted line, to duplicate a segment marking some corresponding points, to draw symmetric shapes using a specific point as centre of symmetry). Only at the end of the book did he introduce some geometrical constructions, "aiming to improve, with practice, the intuitive perception of geometrical shapes, whose structure will be later analyzed using logical proofs". The chapter, describing geometrical constructions (of a triangle given three sides, of the bisector of an angle and other more complex constructions) which are not linked to the previous chapters, tacitly used theorems never illustrated earlier in the book (especially those concerning the congruence of triangles). Some instructions precede this chapter, explaining how to execute a clear drawing and how to test the quality of rulers, squares, rubbers and pencils. Although Veronese made a good work of keeping the manuscript simple, we have to note that no intuitive or rational effort was required from the student.

Frattini's textbook (1901) has a structure which is similar to book by Veronese. He only gave less importance to the preliminary notions, more weight to the properties of polygons, and he also added some minor practical proof. In the introduction, Frattini underlines that a "geometrical truth" exists, and it comes from "an immediate observation of the things, which is the essence of the intuitive method". In Frattini's

book, lines and planes are unlimited from the beginning and parallel lines characterization changes to the one that everyone knows (parallel lines never meet). Let us see the characteristics of some of his proofs.

There is exactly one perpendicular line through a given point to line on a plane (p.21). Let us bend a plane, imagine an immense piece of paper, and shape right angles so that one folding follows the line we want to draw the perpendicular to, and the other folding must include the point where the perpendicular passes through. Let us reopen the paper, it will be possible to see the trace of the perpendicular through the point and the line.

On their hand, perpendicular lines are defined basing on what can be seen in a folded paper, with a “correct” *informal* definition.

To state that “the sum of the three angles of any triangle is equal to two right angles (p.29)”, Frattini uses the classic proof, based on the congruence of alternate angles. This congruence, anyway, is introduced without a proof (“the student can find a reason”). Veronese does not write about this property, not even about its consequences.

The diagonals of a parallelogram mutually bisect (p.33). Suppose we cut out the parallelogram from a piece of paper, we would have, then, an empty space which could be filled either placing the parallelogram back in the same position or placing the angle A, marked with an arc, on top of the equivalent angle C, the side AD on the equivalent side CB and the side AB on CD. In this way the diagonals of the shape, though upside down, would be in the previous position, the same for their crossing point. The two segments OC and OA would switch their positions: this means they are the same length.

We note again the use of geometric transformations, in this case really *introductory* to the proof that will be given within rational geometry.

With regard to geometrical constructions, they were placed at the end of the book, just as in Veronese’s book. However, when it is possible, Frattini tries to explain them using the properties of polygons.

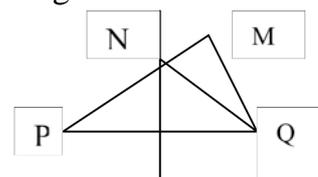
In 1907, a book by Pisati was published. In the preface he slightly dissented from the structure of the programmes as follows:

it seems proved that, in lower middle school, it would be a big mistake to leave the formal aspect of the subject completely apart. Pupils’ intellect, in the previous years of their life, has a formal nature..... Certainly, intuitive teaching of geometry is not easier than formal teaching;

In fact, his book started by stating the concepts such as axiom, postulate, theorem, corollary and problem. In his textbook, we can find explicit theorems and proofs. In example, Pisati introduced the idea of reflection about a line and proved that:

Theorem - All points on the perpendicular bisector of a segment, and no other points, are equidistant from the endpoints of the segment.

Proof. The first part of the statement follows from the properties of the axis of symmetry. To prove the second part, we see that, when the point M does not belong to the axis of the segment PQ, one of the line segments MP, MQ must intersect the axis (see fig.). Let us suppose that MP is the segment intersecting the axis and N the point of intersection. Consequently, we have $NP=NQ$. Thus $MP = NP + NM = NQ + NM$. Since $NQ + NM > MQ$; we have $MP > MQ$.



The theorem which states that the sum of any two sides of a triangle is always greater than the third is justified by considering the line as the shortest distance between any two given points. This contested metric definition of the line, which was also used by Frattini, will never be used again in any geometry textbook for the secondary Italian school. The theorems proved by Pisati, allow him to explain all geometrical constructions stated at the end.

The title “intuitive geometry”, which is not in Pisati’s book anymore, completely disappeared from middle school textbooks, and will only reappear with Emma Castelnovo’s book in 1948.

FURTHER DEVELOPMENTS

In 1905, the Minister Bianchi felt the need to remind us to “escape from abstract statements and demonstrations” adding, on the other hand, to use “simple inductive reasoning” to teach the “truths required by the school programmes”. In 1923, the reform made by Gentile turned the clock back. In the first three years of the *Gymnasium*, geometry studies “must only aim to keep alive all geometrical notions that the pupils have learnt at the primary school and to fix the terminology properly in their memory”. Therefore, there are fewer requirements than in the provisions dated 1900. Amongst the books published right after the reform of Gentile, we have to mention Severi’s textbook (1928) which includes a preface by the Minister of Public Education. In spite of the good comments given in the preface, it is difficult to say that the book follows the school programmes guidelines. Over the years, middle school geometry had lost its experimental-intuitive nature, or even its terminological function, becoming more and more rational. Textbooks were almost independent from the school programmes –which were in fact very brief and without any particular didactic connotation. The book by Severi is surely not an exception (although his book for higher school has always been appreciated for the experimental approach to theorems). It includes many theorems (also those regarding the angles at the centre and the angles at the circumference of a circle), with the most traditional proofs, except for using transformations (rotation and symmetry) as a support to the proofs and for avoiding the word “theorem”.

In 1936 and 1937, a couple of reforms introduced only minor variations, which allowed some simple deductive analysis in the lower *Gymnasium*.

In 1940, the first three-years of the *Gymnasium*, of the Technical school and of Istituto Magistrale³ were unified to form the middle school. With reference to geometry, although its intuitive nature was confirmed, it was suggested to emphasize the evident properties “by means of several suitable examples and exercises, which, sometime, can also assume a demonstrative connotation...”. So, we can find a bigger change compared to the small ones introduced in 1936: the purpose is to start from an intuitive way of thinking to go towards a more abstract logical nature.

An interesting book by Ugo Amaldi (1941) followed this reform. Amaldi completely stopped the process of “rationalization” of geometry. His textbook is similar to Frattini’s book, but it contains some new important changes: measurements and geometrical constructions are not illustrated in separate chapters but they are integrated with the other parts of the book, providing a useful didactic tool. We find many figures and references to real life (i.e. an opening door gives the idea of infinite planes all passing through the same straight line, paper bands illustrate congruent segments...), which had completely disappeared in the meantime. So, given the instructions to draw the axis of symmetry of a segment using a ruler and a compass, Amaldi suggests to check the construction by folding the paper and verifying that the circumferences, used for the construction, overlap. To know the sum of the angles of a triangle, he suggests cutting the corners of a triangle drawn on paper, to place them next to each other and to check that they form an angle on a line (but let us note that in this way the action is not introductory to a formal proof). Similarly, he suggests cutting and folding techniques to verify the properties of quadrilaterals.

At the end of the world war in 1945, a Committee, named by the Allied Countries, deliberated some programmes which were later adopted by the Italian Minister. The middle school programme reverted to practical and experimental methods, but the methodological guidelines for the higher *Gymnasium* are particularly interesting: it is suggested to leave more space to intuitive skills, to common sense, to the *psychological and historical origin* of theories, to physical reality, ... to use spontaneous *dynamic* definitions which fit the intuitive method better.

Vita observes that “unfortunately these suggestions appear to be disjointed from the school programmes that do not show any peculiar innovation”. An innovation is, indeed, represented by the book of intuitive geometry by Emma Castelnuovo (1948). In her book, Castelnuovo follows in Amaldi’s footsteps, using drawings, pictures, cross-references to reality and integration of constructions and measurements. In addition to this, her book, for the very first time, interacts with the student, not only to let him follow a logical deduction or a proof but also she also raises questions in his mind.

What is the meaning – you would question – of the statement that there is only one line passing through two distinct points A, B? How can the contrary be possible? It is true: it

³ Training school for primary school teachers.

is not possible to imagine two or more distinct lines passing through A and B. It is possible, however, to draw with a compass several circles passing through two points...

The book starts with paper folding, and goes on with ruler and square constructions. As Amaldi does, she re-uses the idea of the stretched string to introduce the properties of segments and straight lines; a method already used by Clairaut, who was Castelfnuovo's inspiration. Simple tools are made-up, as a folding meter to show how to transform a quadrilateral into a different one, and to analyze the limit situations.

CONCLUSIONS

Our analysis clearly shows the difficulty of finding an equilibrium between the notions that a pupil is supposed to learn, and the notions which he can accept by means of a non rigorous argumentation. It could seem that geometrical constructions were a real nuisance for early 1900 authors, due to their hidden theoretical content. Around the twenties, the problem seemed to be overcome by amplifying the rational aspect of geometry. It was only in the forties that the books of Amaldi and Emma Castelfnuovo succeeded in the attempt to integrate constructions in the intuitive geometry textbooks, reducing their number and their technical aspect. We have to admit that most authors, starting from Veronese and Frattini, as Amaldi and Castelfnuovo, perceived the need to reduce the dissertation: books are concise, authors are not eager to complete all topics, on the contrary, everybody tends to prefer a specific aspect of the subject.

Anyhow, the very aspect that seems to be relevant for approaching geometry in a really intuitive way is the active learning role of the student. Programmes tried, several times, to deny this role, and it was interpreted in different ways by authors. Emma Castelfnuovo foresaw and opened the door to the use of concrete materials.

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THE APPROPRIATION OF THE NEW MATH ON THE TECHNICAL FEDERAL SCHOOL OF PARANA IN 1960 AND 1970 DECADES

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The subject of this text is the appropriation of the New Math on the Technical Federal School of Parana in 1960's and 1970's. From a historical perspective, founded by Certeau (1982), Chartier (1990) and Julia (2001), this study sources from scholar documents, located on ETFPR files. The study concludes that the ETFPR did not prioritize in its Course Plans, the teaching of the New Math. In this period, the scholar culture of ETFPR was marked by teacher initiatives directed to elaboration of didactic material suited to the technical courses which were, in that moment, engaged in approaching the scholar mathematics to the technical culture, transforming it in a useful tool for the urgent need of forming the necessary work force to the industrial and technological development of the country.

Since 1960, the international New Math Movement (NMM) has penetrated several countries schools, seeking to introduce a new language into the scholar Mathematics as well as trying to adjust it to the new challenges brought by scientific and technological development that emerged in this period.

In Brazil, the movement has increased its force through actions of countless math teachers, like the ones triggered by the Group of Study of Mathematics Teaching (GEEM). The GEEM was created in São Paulo – Brazil and coordinated by teacher Osvaldo Sangiorgi, one of the most enthusiasts members of the NMM in Brazil.

In Brazilian educational context, the technical industrial teaching had a fundamental role in society economic projects, essentially in 1960 and 1970 decades. At that time, the increasing of education levels, especially for poor people, had the main objective of preparing the taskforce for industries, as well as absorbing imported technologies from rich countries. The Federal Technical School of Paraná (ETFPR) [1] carries out a main role, at that moment, of forming taskforce to technological and industrial development in Paraná State.

Considering the importance of local studies for understanding the national history of the NMM, recognized as a major change applied to Scholar Mathematics in a World level basis, the present study aims to understand how the New Math was appropriated by the ETFPR, in 1960 and 1970 decades. According to Valente (2008, p.665):

The NMM constitutes a fundamental reference to the Mathematics Education as a Research Field. The associated historical moment had triggered the organization and the systematization of scientific activities related to the teaching and learning of Mathematics. In other words: The NMM made the emerging of the Mathematics Education Research Field.

Oriented by a cultural and historical perspective, the study uses as sources the theoretical-methodological approaches of Certeau (1982), which conceives history as an “operation” that requires for its writing, as a practice activity, of a scientific approach. Besides, Certeau uses the concept of “Appropriation”, from Chartier (1990), with the objective to understand the use that scholar agents have made of the New Math, disseminated by the Movement in a scholar culture (Julia, 2001). The study arise questions about changes occurred in the Mathematics discipline offered by the ETFPR, in the NMM discussion period.

The study sources were based in files archived in the Nucleous of Historical Documents (NUDHI) and the General Files of Federal Technological University of Paraná State (UTFPR), in Brazil. In those files, some documents were consulted, such as: Professors Council Proceedings, Class Diaries, Courses Plans, Curricular Grades, Math Books and normative documents.

To confront the date related to the NMM reception, in the scholar practices of the investigated institution, some interviews were conducted with three teachers and an ex-student, which were witness of the teaching, and learning process that took place at ETFPR in 1960 and 1970 decades.

THE PROFESSIONAL TEACHING IN BRAZIL

Professional teaching, in Brazil, has begun in the Imperial time when the first nucleous of professional formation were founded, in Jesuitical colleges and residences. They were called “factory-schools of artisans and other professions” (Manfredi, 2002, p.68). In that period, the most part of manual and manufacturing jobs were done by slaves. In first Republic, when Brazil was entering a new stage in terms of economical and social development, the professional schools gained a new role, becoming truly technical schools networks. The teaching system of those schools then takes the objective of teaching people in great Cities. This type of schools, at that time, were directed essentially to poor people, and due to this considered as a second category school. There was also a great problem of scholar evasion. The most part of the professions that were offered were manual or artisan type, like joiner, shoemaking and tailor’s workshop.

After the 1930 revolution, with the large scale industrial development model adopted by the president Getúlio Vargas, that superseded the agro-exportation model, the factory-schools of artisans and other professions, which were initially the responsibility of agriculture ministry, became part of the new created Education and Health ministry.

In the New State Period, the professional education has the same role of the previous period, which was directed to poor classes. On the other hand, the secondary course was directed to elite classes. This duality was strongly discussed in the “Pioneers Manifest”, in 1932, which makes the proposal of the organization of academic and professional courses in the same institution as well as the adaptation of schools to

regional interests. In spite of that, only in 1942 the pioneers concerns were accepted by Gustavo Capanema Minister, whose Organic Laws, among other things, rebuild the Industrial Teaching. According to Cunha (1977, p.55), one of the main factors of the new organization was the Second World War economical context. According to the author, the countries that were involved in the war drastically decreased the exportation of manufactured products to Brazil. One great change proposed by the Organic Laws was the definition of the Industrial Teaching as a secondary course, destined to professional preparation of workers to the industry. With that, the industrial courses students could enter superior courses related to the corresponding professional course.

In the same period, complementary legislation in professional teaching, the edict-law 4.048 of 22nd of January, 1942, created a professional teaching system which was “parallel” to the official system, sustained by enterprises. This new system, nominated National Service of Industrial Learning (SENAI), was supported by the Industrial Confederation and had the finality of organizing and administrating the Industrial Learning Schools of SENAI all over the country. The motivation to the creation of SENAI was that, due to the extinction of the “factory-schools of artisans and other professions”, the old tasks of those schools then became an obligation of the Industries. So, professional enterprises assumed the task of preparing their own taskforce through SENAI and became, gradually, the inspiring model to the technical education for Brazil in later years.

Organized in two cycles (gymnasium and collegial), the first, brought by the Industrial Schools and second, by the Technical Schools, and systematized through the Organic Laws, technical education remained as a branch of education leading to the formation of professional demanded by the production system, therefore, a terminal branch of education. In the 1950’s, through the 1821 Act, the forming students from technical, industrial, commercial and agricultural secondary courses were able to access university courses, provided if they submit to the demands of college entrance examination.

At the end of 1950, with the new National order “education for development”, in the administration of Juscelino Kubitschek, occurred the reform of Industrial Education. With the Law 3552/59, federal technical schools have been given own legal personality, introducing administrative, educational, technical and financial autonomy and leaving them to constitute a uniform system, with organization and similar courses.

According to Cunha (1977, p.81), despite the autonomy given to technical schools, the control was taken by the Ministry of Education. This control was even increased by the Direction of Industrial Education (DEI) fixing the minimum required curriculum for technician’s certificates in specific areas. Among other functions, DEI was responsible for development of curriculum guidelines, the evaluation system, examinations and promotions, besides the development of teaching materials, courses plans and school performance indicators.

At that time of growth and improvement of the Brazilian industrial chain, the spirit of the technique has been widely sown in industrial schools throughout the country. The work of the technical, according to Cunha (1977, p. 30), "begins to depend more on their knowledge than their manual skill or ability of direction"

With the Law of Guidelines and Bases of Education (LDB), which restructured the education in three Degrees: primary, middle and high, technical education began to be offered in three ways: industrial, agricultural and commercial. It was only with this Law that in fact the entry to high education was consolidated for students of professional education.

From 1960, more and more young people were seeking high education as a mean of social ascension, as the economic model of concentration income left no other alternatives. According to Cunha (1977), in that decade, the social-economic profile of students in technical courses was changing. The number of technicians enrolled in high education during the period between 1962-1966 (about 33%), showed that students of the technical industrial courses hoped that the function of the courses were propaedeutic, an instrument of social ascent.

THE MATHEMATICS DISCIPLINE IN ETFPR, AT NMM PERIOD

According to the Information Bulletin of the Brazilian-American Commission of Industrial Education (CBAI, 1960e, p. 4) [2], the qualified professional is:

[...] the professional who knows the technology, the practice and still has sufficient basis for progressing into the professional field [...] needs of the concepts of general education as math, drawing, as well as extensive knowledge of technology related to their profession for the development of new techniques and improving of his work.

Considering Mathematics as a basic discipline for the technique culture of students, the biggest challenge that was presented to the teachers of technical courses was to contextualize the content, from problems of practical applications in technological world.

According Clemente (1948, p. 86):

[...] it is usual to say that mathematics teaches reason and, in industrial education, this proposition assumes a broader character. It's the Math that plays the most important role in the mental training of specialists. Therefore, follows that the teacher of mathematics has, perhaps, the most important part in the sum of knowledge that will form the expert Professional.

In this article, Arlindo Clemente proposes that the teacher of mathematics workshop must bring the factory into the classroom and seek to solve real problems of the job, replacing abstract mathematical problems by concrete ones.

The mathematical reasoning is the element that will transform the older worker, empirically formed, in the modern workman much more capable, with a

greater intellectual capacity. And, no doubt, this parcel of culture is one that will give the worker the possibility of connecting his brain to his hands. This is the function of mathematics in the education industry. (Clemente, 1948, p. 87)

The main concern of Clemente was the practical application of mathematical concepts to technical disciplines of industrial education and the choice of essential and minimum contents, necessary for the training of technicians.

The article by Martignoni (1951, p.695), "The Mathematics in Practice and Education," published in the Bulletin of CBAI, in July 1951, also highlights the importance of mathematics to bring the workshops and cut the superfluous. His speech, full of pragmatism, questioned the need to study contents that were not directly related to the practical application. He stated that math science is the reason for scientific progress and also that more elaborate math should be left for advanced studies because it will not meet the purposes of technical courses under the guidance of CBAI. In this context, Math should have a strong character practical and utility. Meanwhile, the Federal Technical School of Parana, already in late 50's, faced major problems with teachers of Industrial Technical Education, focusing on courses' quality. Then, Director of Technical School of Curitiba, Dr. Lauro Wilhelm, indicated in 1959 two major factors for technical courses low quality: the poor training of all kind of teachers and the lack of control over teacher' activities.

In the end of 1950's, the discussion on the mathematics in industrial technical courses had national repercussions. In III Brazilian Congress of Mathematics Education (Ministério da Educação e Cultura, 1959), held in Rio de Janeiro in 1959, coordinated by the Campaign for Improvement of Secondary Education and Broadcasting (CADES), the Industrial Education, whose committee was directed by Arlindo Clemente who presented for discussion, a Program dedicated to the teaching of mathematics in technical courses, highlighting the math in the workshops and the correlation of mathematics disciplines culture technique (Ministério da Educação e Cultura, 1959, p. 28).

NEW MATH TRACES OF ETFPR

The modernization of Mathematics was associated with betting on technical progress. For Valente (2006, p. 39), "the Math was valued as part of a scientific training that would have continuity in Higher Education and to do so was needed an approximation between approaches of mathematics in Higher Education and in secondary, considering conceptual terms, methodology and language". This approach to the mathematics of Higher Education was expressed on the main features in NMM: Accuracy, precision of language, deductive method, a higher level of abstraction, use of contemporary vocabulary, thought axiomatic among others.

However, even taking the Technical School teachers to participate in the preparation of textbooks of New Math of the group's Center for Research and Dissemination in

Mathematics Teaching (NEDEM) in Parana's State College (CEP), these actions do not seem to result in an upgrade of Mathematics programs. In the "Daily Class" (document 6) [3] of 1967 and 1972, teachers of the ETFPR Industrial Gymnasium do not show any trace of New Math.

In oral testimony, the teacher E1 [4] reported that mathematics' books, used in industrial Gymnasium, at end of the 1960's, were Marcondes (1969). The collection was divided into three volumes: algebra, arithmetic and geometry. Referring to the edition of 1969, there was not any New Math content.

It is important to remember that some Mathematics teachers, employed by ETFPR in the second half of 1960's, were still students in the Course of Mathematics at the Federal University of Parana (UFPR), and had no authority over his colleagues to propose changes in programs and in the textbooks adopted. The new teachers were in contact with contents of Modern Mathematics. In despite of that, they kept using programs developed by old teachers. Their independence was conditioned by a specific technical school culture which was the rule for many years.

Also, at the beginning of 1970's, new Mathematics teachers were minority. This is confirmed by the testimony of a former student from Industrial Gymnasium: They had some new teachers, but 70% were most experienced teachers (E3).

The teacher E1, in testimony to the researcher, reported that the first time he heard Theory of Sets was in 1967, when his teacher asked him an option to work on this topic. In 1970, when he graduated in Mathematics, by UFPR, he began working in the State Network for Teaching and ETFPR, teaching Mathematics belonging to gymnasium's course. According to E1, the network state of education first adopted the Mathematics book of NEDEM and later Oswaldo Sangiorgi's book. He said he came to work a full year in the State Network with Theory of Sets. In ETFPR he taught some notions of collections, but that was not intensive (E1).

In 1966, teacher Ricardo assumed the direction of ETFPR. The entry of this new director gave new direction to the teaching-learning organization of the school. He brought in baggage more than the experience of CBAI, the coexistence with the Americans and the commitment with institution and students. The strong American influence received by the new director was largely responsible for the ideas of method, rationality, profficiency that came with greater intensity. In his testimony, Professor Ricardo Luis Knesebeck reported that first, as coordinator of instruction, and after as Director, implanted in a draconian way a program of education for all teachers. To him, was an something absurd to teach and don't commit with anything.

The document "Content to be determined" (document 11) [6], prepared by Mathematics teachers and approved by Didactic Coordination, in 1969, showed that the program was based on the contents sequence of Quintella's books' collection in (1966), which until 1970 did not have any trace of New Math, Theory of Sets, relations, matrices, etc. as specified in the "Pilot Program" (document 12) [7] published by GEEM, in the year 1968.

In oral testimony E1 said that teachers closely followed the book (the first to the last page) and the Head of the Department selected the book's exercises that the teacher should do. In his opinion, this hand method worked very well. In Mathematics Program in first years (document 11), we found a topic: "General Review of the 1st cycle of matter." This may be an indication of teachers concern about maintaining a certain quality of education. In their opinion, the low quality in Mathematics taught in the gymnasium could be problem.

In the analysis of the goals of textbooks delivered to students, called "Auroras", observed in 1973, compared with the program of 1969, the complex numbers and trigonometric equations were removed, as well as the study of vectors and orthogonal views was simplified. We also note a greater emphasis given to trigonometric functions.

In the "Auroras" program in 1975 some contents were evaluated:

- I - SET - Goal 1: Operating with sets. 1.1 - Determine the union of sets. 1.2 - Determine the intersection of sets. 1.3 - Determining the difference between two sets . 1.4 - Determine the complement of a set. 1.5 - Correctly use the symbols of the theory of sets. II - NUMBERS (NUMERICAL SETS BASIC) - Goal 2 - Understand the fundamental numerical sets (...). III - RELATIONS AND FUNCTIONS. Goal 3 - Represent graphically relationship and function. (...)
- 3.3 - Determine the Cartesian product between two sets. (...)

This portion of student's evaluation manual confirms the evidence E1 of the introduction of theory of sets for students in secondary technical course and the new approach to function concept according to modern mathematics. Notion of variation and functional dependence of functions was virtually forgotten over the NMM that adopted the design of structural function of Bourbaki.

In the year 1975, the term "field of existence" has been replaced by "dominion" and "image" of trigonometric functions, which was the term used in the book Iezzi et al. (1973) [8]. Making a comparison between the "Pilot Program" (document 12), prepared by GEEM in 1968, for the first two years of secondary education, noted that ETFPR's program, although more extensive, included topics such as the trigonometric functions and triangles resolution, suggested by São Paulo's group.

In 1975 ETFPR made a complete revision of Algebra programs (Math I). With adoption book Iezzi's et al., (1980), the topics turn to a deal with sets, sets numerical key, full study of the functions of the 1st and 2nd grade, depending Exponential, logarithmic function, the study inequalities of 1st and 2nd grades, exponential and logarithmic. The subjects were addressed in accordance with the "Pilot Program" (document 12) suggested by GEEM in 1968.

Iezzi's book presented the contents of duty by a graphical approach. Separating each chapter, there was an example of mathematics application in today's world. There

was a concern with the formal mathematics, but not so exaggerated. At the end of the book, there were several references about Modern Mathematics.

We noticed that probably the book's Iezzi *et al*, (1980) deals with the theory of sets to meet a market need, as Kline (1976, p.135) warned "Other texts begins with a chapter on the theory of sets, It was then back to the traditional math and would henceforth no longer refer to the theory of sets or any other topic in modern mathematics".

The book's Gelson Iezzi *et al* have consolidated a discussion of teachers in curriculum modernization, which was common among ETFPR Mathematicians. In his testimony, teacher E1 said that he and his colleagues in the early of 1970 began to define functions as a particular case of the relationship between two sets (a structural design adopted by the NMM) rather than as a functional dependence as was discussed of Ary Quintella's book. According E2, a teacher of ETFPR the 1960's, the technical course did not give much emphasis to the theory of sets, considering it was an education more focused on practice. One possible explanation for slow integration of Modern Mathematics in ETFPR could be one of the goals for Educational System in ETFPR "(document 4) [9] as defined in 1972:" Cut programs of study fictitious topics". Would be "fictitious subject", the content broadcast by the NMM? Would be inappropriate to technical education?

In the first half of 1970's, despite the strong tendency to follow faithfully the textbook, some mathematics' teachers of ETFPR started developing their own material to work with students, such as "Geometry of Space Material" (document 15) [10].

The exercises in first worksheets had not any relation with technical matters because there was a culture of integration between the areas. According to the interviewed, the teaching of mathematics was not aimed at career academies: No, it was generic. At the time, from 1969 until 1974, it didn't have a very great integration between the teachers of general education and culture specific; they worked half apart (E3).

In 1970's, with the support and encouragement of the Department Mathematics Coordinator, the production of teaching material itself was improved and marked in a more intensive way the culture of ETFPR. This initiative was not alone, it was occurring in several federal technical schools in Brazil. In ETFPR, this initiative was consolidated in 1980's and resulted in a collection mathematics books directed to the Technical Education.

FINAL CONSIDERATIONS

The study indicates some aspects emphasized by NMM, as the theory of sets, the axiomatization, the new mathematical language, laden with symbolism, seemed incompatible with the needs of the students training in a technical school in the 1960's and 1970's.

Concerned to offer a "practical education", required by technical training, an ETFPR not prioritized the teaching of modern mathematics in their courses, at the top of the

movement. The testimonies show that there was non-official insertion of "some" ideas of NMM and this can be evidenced by the few traces of Modern Math, in documents found in the school.

The study shows that only from 1970, some contents of New Math were introduced in the course of school, and that means textbook from 1980, Mathematics teachers ETFPR started the preparation of a Mathematics textbook collection, putting an old idea to feature a "practice" to discipline by proposing a specific methodology able to articulate the rationale, graphic interpretations, problems applying physics problems and technical subjects. The weak presence of New Math in ETFPR, far from setting itself as a resistance from teachers to the ideals of the movement, indicates that in decades in question, a ETFPR wanted to amalgamate a difference in their school culture, slowly making a "creative consumption" of textbooks, strong responsible for the insertion of New Math in Brazilian schools.

NOTES

1. Today is called Federal Technological University of Parana (UTFPR). Use the name Federal Technical School of Parana (ETFPR), like this named because most of the period defined in the study, namely the 1960's and 1970's.
2. Bulletin of CBAI. Brazilian-American Commission of Industrial Education. Educational program of cooperation maintained by the governments of Brazil and the United States Research and Training of Teachers. Vol. XIV, n.5, 1960e, 16p.
3. Document 6: Diaries of the course belonging class of 1967 and 1972. - Archive of General UTFPR. 1967 to 1972.
4. The name of the interviewees E1, E2 and E3 was not revealed at their request.
5. KNESEBECK, Ricardo Luis ex-student, ex-teacher of physics, ex-director of the Federal Technical School of Parana. (Interview granted to Gilson Leandro San Mateo - NUDHI / UTFPR. Curitiba, 16/17 May 1995).
6. Document 11: Content to be established in 1969, 1969, 17p.
7. Document 12: Pilot Program for the school course prepared by GEEM in 1968. Sao Paulo: GEEM, 1968, 5p.
8. The first edition of this book is the year of 1973. In this study found was the eighth edition, published in 1980.
9. Document 4: The educational system of the Federal Technical School of Parana produced by the Education Department through the coordination of the Didactic ETFPR.
10. Document 15: Geometry of Space Material made by teachers of ETFPR. Library of UTFPR, s / date.

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HISTORY, HERITAGE, AND THE UK MATHEMATICS CLASSROOM

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Abstract

Since 1989 the UK mathematics curriculum has been dominated by a culture of testing ‘core skills’. From September 2008, a new curriculum places the history of mathematics as one of its “Key Concepts” which is now a statutory right¹ for all pupils. While the curriculum has changed, there has been virtually no relevant training for teachers, and while the testing regime remains in place, there seems little chance that pupils will obtain their entitlement. This paper examines the problem of teachers’ scant knowledge of history of mathematics and proposes a new approach to introducing relevant materials together with a pedagogy which capitalises on recent research, to introduce the heritage of mathematics into our curriculum.

1. THE NEW ENGLISH CURRICULUM

The first chapter of Fauvel and van Maanen (2000) considered the political context of the history of mathematics in school curricula. At that time, the UK curriculum² underwent radical changes, which produced a curriculum based on ‘core skills’ with modularised³ lessons that enshrined traditional beliefs about ‘levels’ of knowledge and portrayed school mathematics as a collection of disparate topics rarely connected in any sensible way. Textbook design followed the topics, and test papers became *de facto* part of the curriculum, setting the norms for the new culture. The emphasis on utilitarianism and examination results produced little serious engagement with substantial mathematical thinking⁴. The latest Inspectors’ report on our secondary schools shows that, as a consequence, too many pupils are taught formulas that they do not understand, and cannot apply:

“The fundamental issue for teachers is how better to develop pupils’ mathematical understanding. Too often, pupils are expected to remember methods, rules and facts without grasping the underpinning concepts, making connections with earlier learning and other topics, and making sense of the mathematics so that they can use it independently.” (Ofsted 2008: 5)

In contrast, the most recent version of the curriculum states that for the 11 to 16 age group, “Recognising the rich historical and cultural roots of mathematics” is one of its “Key Concepts” (QCA 2007)⁵.

¹ A ‘statutory right’ means that *by Law, all pupils at primary and secondary level* have the right to be taught about the “rich historical and cultural roots of mathematics”.

² The UK mathematics curriculum applies to England and Wales. Due to government devolution Scotland and Northern Ireland have different curricula, regulations and examination systems.

³ Modules purport to be convenient ‘packages of knowledge’ within the curriculum, with a well defined and limited range of knowledge. They are consequently easy to ‘teach’ and easy to pass.

⁴ There are, of course, a number of exceptional teachers who have overcome these difficulties.

⁵ *The Key Concepts* are: Competence, Creativity, Applications and Implications, Critical Understanding, and the *Key Processes* are: Representing, Analysing, Interpreting and Evaluating, Communicating and Reflecting. Applied to all pupils from age 11 to 16 (Key Stage 3 to Key Stage 4).

For the last fifteen years very few secondary school teachers have had the chance to discover the kind of contributions that history of mathematics could make to pupils' learning, and with the pressures of 'teaching to the test' it seems doubtful whether history of mathematics will make any impression in our classrooms while the examination structure remains the same⁶. So, what would 'recognising the rich historical and cultural roots of mathematics' mean in practical terms for our teachers?

Recently, colleagues have renewed their call for history of mathematics to be taken seriously as an essential part of the mathematics curriculum. Radford et. al. (2007) argue that an important sense of meaning lies within the cultural-epistemic conception of the history of mathematics:

"The very possibility of learning rests on our capability of immersing ourselves –in idiosyncratic, critical and reflective ways– in the conceptual historical riches deposited in, and continuously modified by, social practices. ... Classroom emergent knowledge is rather something encompassed by the Gadamerian link between past and present. *And it is precisely here, in the unravelling and understanding of this link, which is the topos or place of Meaning, that the history of mathematics has much to offer to mathematics education.*" (2007: 108) (italics mine)

In the terms described above, history stands in opposition to the utilitarian demands of the old curriculum, but having put history of mathematics into the curriculum, the government organization, QCA⁷ have now revealed the pressing problems of resources and training. Changes need to happen not only in the classroom but also, and more importantly, in teacher training. So, *how can we provide material from the history of mathematics that can be integrated in a meaningful and effective way into the everyday activities of the classroom?*

2. NOT HISTORY BUT HERITAGE

Ivor Grattan-Guinness (2004) has made an important distinction between the History and the *Heritage* of mathematics. History focuses on the detail, cultural context, negative influences, anomalies, and so on, in order to provide evidence, so far as we are able to tell, of what happened and how it happened. *Heritage*, on the other hand, address the question "how did we get here?" where previous ideas are seen in terms of contemporary explanations and similarities are sought.

⁶ Recently, the government has decided to abandon the tests at KS3 (age 14), and plans have been published to include 'Interpretation and Analysis' of problems as part of the assessments from 2010.

⁷ The Qualifications and Curriculum Authority, the Government sponsored body set up to maintain and develop the national curriculum and associated assessments, tests and examinations.

“The distinction between the history and the heritage of [an idea] clearly involves its relation to its prehistory and its posthistory. The historian may well try to spot the historical *foresight* - or maybe lack of foresight - of his historical figures, By contrast, the inheritor may seek historical *perspective* and hindsight about the ways notions actually seemed to have developed.” and “...heritage suggests that the foundations of a mathematical theory are laid down as the platform upon which it is built, whereas history shows foundations are dug down, and not necessarily into firm territory.” (2004:168; 171)

The interpretation of Euclid’s work as ‘geometrical algebra’ has since shown to be quite misguided⁸ as *history*, but as *heritage* is quite legitimate because it is the form in which some of the Arabs interpreted the Elements when they were creating algebra.

We have to be careful. Deterministically constructed heritage conveys the impression that the progress of ideas shows mathematics simply as a cumulative discipline. But, while mathematics does build on past achievements, and while we make stories about the links between the mathematics of the past to the present, *the mathematics of the past is not the same as the mathematics of now*. As Mathematics Educators we have a means of passing on our Heritage by bringing the links between the content we find in the curriculum to the attention of teachers and students. In this way it becomes possible to describe significant ideas in the history of mathematics in terms that teachers can use and pupils can understand without making impossible demands on their historical capability or on curriculum teaching time.

3. PROJECT AIMS AND OUTLINE

The project I describe is just beginning. It has arisen from the experiences of myself and other colleagues in presenting ‘episodes’ from the history of mathematics in workshop form to both teachers and pupils, so that interesting and worthwhile problems arise from interpretations of the historical context. Response to these classes has been encouraging, and has prompted wider experimentation. Some twenty teachers and teacher trainers around the country have joined an informal on-line workshop to experiment with the available materials and suggest ideas for the classroom presentation of current material and new topics to be explored. Initially, the principal interest is in providing Secondary teachers with materials for professional development that start from some of the important ideas in the existing curriculum, and to open up the possibilities

⁸ Typically, this is done with Euclid II,4 and described as ‘completing the square’, but see the examples in Katz (2008)

of developing the concepts involved by finding ‘historical antecedents’ to support the connections between and motivations for these ideas and the possible links between them. Exactly what form this material may take is still under consideration⁹. Some examples can be found on the NRICH website¹⁰ where themed historical ‘episodes’ are available with notes and pedagogical questions for teachers and pupils to explore.

While the web episodes are currently chronologically arranged, a more general idea is to produce a series of ‘concept maps’ that are intended to provide a topographical view of the significant features of a particular mathematical landscape¹¹(Burke & Papadimitriou 2002). A map can be examined and used from ‘inside-out’ and from ‘outside-in’, from following particular trails of thought to obtaining a broader overview of historical development. The ‘unravelling and understanding’ of the links between ideas, is the *topos* that Radford and our colleagues (quoted above) are talking about. The idea of a map is important here; it is intended to be a guide to how ideas might be connected, not a deterministically constructed list of events. In contrast, most curriculum activities are presented to teachers as a *narrative*, a list of topics to teach in a particular order, and often restricted to some imagined ‘levels of competence’ of the pupils. A map is there for teachers to have the freedom to make their own narrative. They have the responsibility for producing lessons, and it is up to them what parts of the map they want to use, and how they approach the pedagogical problems of dealing with the curriculum in their own classroom. The map can throw light on certain problems, it can suggest different approaches to teaching, it can help to generate didactical questions, but in the end it is there to be used or not, appropriately. The intention here is to develop ways in which the teacher, starting from a particular point in the standard curriculum, will be able to link a conceptual area with important developments in the history of mathematics through the use of ‘idealised’ historical problems and *canonical situations*¹² as part of the *Heritage* of mathematics. There is, of course, a considerable literature of historical and pedagogical material to draw on. The practical task is to find appropriate ways in which to link the source material with the curriculum opportunities.

⁹ Today, many options present themselves: texts, posters, PowerPoint, DVD are all possibilities.

¹⁰ The NRICH site is part of the UK Millennium Mathematics Project.

¹¹ I am indebted to my colleague Jeremy Burke for the use of this idea in his research, and for our conversation on 15 November 2008. I make no claims that such a map is (or even could be) ‘complete’.

¹² By a *canonical situation* I mean a diagram, or a way of setting out a problem or process which is developable, has potential to represent more than one idea, and is presented to students to encourage potential links between apparently different areas of mathematics. See the Appendix for an example.

4. METHODOLOGY AND PEDAGOGICAL APPROACH.

Since the English curriculum now focuses more on what we call the ‘process’ aspects of learning mathematics, it may now become easier to incorporate the teaching of the ‘key concepts’ in such a way as to enable the history to emerge from the discussion of *canonical situations* (be they images, texts, or conceptual problems) introduced by the teacher. This approach also has the advantage of being able to link different areas of a standard curriculum, thereby enabling pupils to see connections between parts of mathematics that have been concealed by the traditions of official curriculum organisation. When the text-books and exercises are arranged so that their chapter headings conform to the same organisation as the curriculum, it is most unlikely that pupils will gain any idea that different areas of mathematics are connected at all. In this pedagogical strategy we are concerned with the *dynamics of production of the pupils’ ideas* stimulated by episodes from the history of mathematics retold in heritage form. In principle, this is not new. I am advocating a methodology that is already available, which can bring mathematics education and the teaching of history of mathematics together. The principles are well-established, and the use of examples as a focus for discussion and exploration has been a tradition in teaching for many years. However, as Sierpinska (1994) has recognised:

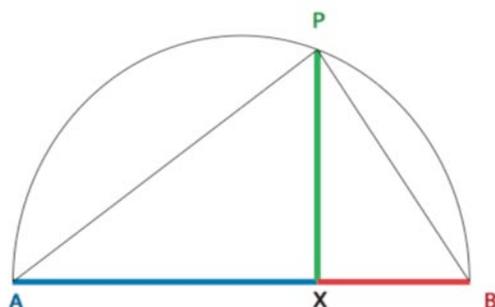
“Pedagogues, of course, think of paradigmatic examples of instances that can best explain a rule, or a method, or a concept. The learner is also looking for such paradigmatic examples as he or she is learning something new. The problem is, however, that before you have a grasp of a whole domain of knowledge you are learning, you are unable to tell a paradigmatic example from a non-paradigmatic one.” (1994; 88-89)

This problem is always present in the classroom, but there are many ways in which we try to alleviate the situation. Grosholz (2005) has demonstrated the role of ‘constructive ambiguity’ in Galileo’s discussion of free fall,¹³ and shows that ambiguity can play a constructive part in mathematics since it leads in this case to reading a particular diagram in more than one way. Galileo’s argument was put forward in terms of proportions, geometrical figures, numbers, and natural language. He was then able to exploit Euclidean results and the arithmetical pattern of the diagram, but in reading the intervals as infinitesimals he led the participants heuristically to his analysis of accelerated motion. The use of ambiguity in mathematical heuristic is still alive today. Changing the

¹³ Galileo (1638) *Discorsi e Dimostrazioni Matematiche* Day 3, Theorem 1, Proposition 1 (Dover edition p.173).

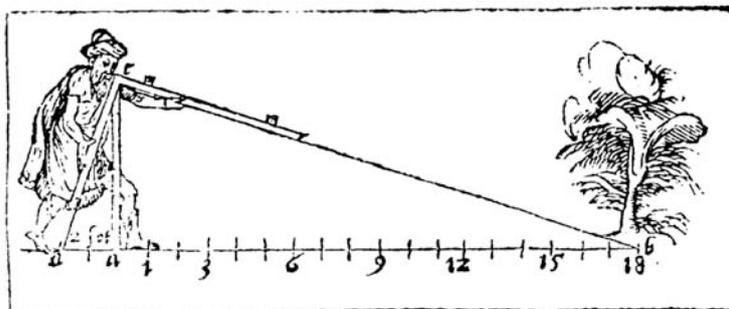
mathematical context by conceptualising new objects and the processes we use to deal with them, changes the ways in which arguments can be understood. This kind of ambiguity has been shown to provide useful material for classroom discussion.

The use of *canonical situations* is important in this context. A diagram can be interpreted in a number of ways, and this is where conceptualising new objects and new relationships becomes possible.



This diagram from Euclid VI, 13 is an example of a *canonical situation*. It describes a mean proportional between AX and XB and is fundamental to ideas of ratio and proportion. Pupils who know the properties of similar triangles can deduce that $AX \cdot XB = PX^2$.

Starting from the properties of right-angled triangles, elementary knowledge of ratio and proportion and its early practical applications to measurement of all kinds of heights and distances can be developed.



Using dynamic geometry, it is easy to show how the product of the segments produces a square, and thus we have entry to the diagrams used by Viete and Descartes for demonstrating their quadratic solutions.

In reading texts, Barbin (2008), has shown how text considered as a message to an audience can motivate a discussion about the intention and meaning of the author, and how it can be used as a means of encouraging pupils to consider the ways it could be interpreted and understood.

There is no sure way of posing problems or offering examples, but once done, then the learner's response has to be respected and managed carefully. We have become used to the principles of heuristic teaching, but Brent Davis claims that heuristic listening is also important:

“Heuristic Listening is more negotiatory, engaging, messy, involving the hearer and the heard in a shared project [which] is an imaginative participation in the formation and transformation of experience through an ongoing interpretation of the

taken-for-granted and the prejudices that frame perceptions and actions.” (Davis, 1996: 53)

When we engage in mathematical problems we inevitably construct our own examples to help us illustrate the ideas involved, and use these examples as material for personal contemplation or discussion amongst our peers. If we do this as adult mathematicians, why should it be different for pupils? Why is it not possible to develop this idea of self-construction in the classroom?

In England, there has been a tradition of producing materials for teachers and pupils that focuses on an individual’s learning process and encourages active engagement in, and discussion of mathematical problems¹⁴. Recent examples like Watson and Mason (1998) and Swan (2006) encapsulate this tradition and provide practical guidance to help teachers develop pupils’ powers of constructing mathematics for themselves in the classroom:

“Our interest is in using mathematical questions as prompts and devices for promoting students in thinking mathematically, and thus becoming better at learning and doing mathematics. ... We hope our work will show how *higher order mathematical thinking* can be provoked and promoted as an integral part of teaching and learning school mathematics...” (Watson & Mason 1998: 4)

Such publications display ideas for situations that are generic and offer ways for teachers of promoting ‘Learner Generated Examples’ *applicable at all stages* of teaching and learning mathematics. The materials are prepared to promote the kinds of activities that focus on ambiguity, raise doubts about interpretations, and encourage the learner (and the teacher) to develop a security with mathematical ideas that enables them to engage in intelligent questioning and active discussion of the problems concerned. A number of teachers are already engaged in this pedagogy that raises pupils’ learning above mere acquisition of skills, and helps pupils to develop their own cognitive tools and achieve a higher order of mathematical activity.

5. THE MATERIALS: BRIEF DESCRIPTION AND EXAMPLES.

¹⁴ This kind of material was introduced by the Association of Teachers of Mathematics, and has been its enduring hallmark. It is the result of a tradition of collaborative research and writing where texts and other materials have developed a particular type of pedagogical practice by offering examples of classroom work which require discussion, involve heuristic forms of reasoning, analogy and inference, and encourage the learner to create and verify their own examples.

Completing the Square is one of the drafts that has been used in a number of classrooms¹⁵ and covers is a traditional area of the curriculum showing some of the connections between the stages to the solution of quadratic equations. It comprises a series of links from one period to another, stressing the transformation of the ideas from simple surveying to ‘cut and paste’ problems in Mesopotamia, and more sophisticated procedures of ‘dissection and re-arrangement’ in India and China, and how the problems were transposed and represented within the more abstract ideals of classical geometry in Greece. The conceptual blending of different traditions by the Arabs in the 9th and 10th century introduced algebraic concepts which found their way into Europe and resulted in the attempts to find solutions of different types of equations. The materials provide plenty of opportunities to discuss the development of geometrical and number concepts and the way these were represented in text and diagram form (ratios, proportions, integers, fractions, rationals non-rationals and eventually ‘imaginary’ numbers). Key ideas like geometric visualisation and the different forms of representation, appropriate notation, and whether a particular procedure is ‘allowed’ in a given context, can be discussed, and show how finding representations for ‘impossible’ numbers like $\sqrt{3}$ or π can have a liberating effect in allowing new ideas to flourish. And, of course, there is the ever-present idea of ‘infinity’ to be explored. The material has been gathered from published research and expert analysis¹⁶ to identify and characterise significant moments in the evolution of particular ideas. In these examples we have not only translations into ‘modern’ language, but something of the pedagogical interpretations, so that these might be brought into the modern classroom and used in creative ways. The material is designed so that it can be used in ‘episodes’ in the normal course of teaching in school. Included are notes and references to the historical background, and ‘pedagogical notes’ aimed to help teachers raise questions and see where the material can be used in their classroom. In this way, selections can also be used as a basis for teachers’ professional development both in the historical and mathematical sense. There are optional entry (and exit) points to the material that allow considerable flexibility in its use. These ‘episodes’ apply to particular topics (or lack of them) in the English mathematics curriculum, and are

¹⁵ This material has been used in whole or in part, with various groups of pupils from age 10 to 18, with teachers, teacher trainers, and with graduate teacher trainees. I gratefully acknowledge their feedback, which has been most useful.

¹⁶ For example, over the years I have been able to access the specialised work of many researchers on Ancient, Classical, Mediaeval and Renaissance mathematics. Now we can find substantial examples of much of the ancient mathematical material collected and specially written up in Katz (2007).

each recognised as an interpretation of a particular context in our heritage. The historical process can be described *in terms communicable to a modern school audience* and furthermore, the teaching is specifically designed to focus on the pupils' *contemplation and discussion of the problems*, and engagement in a dialogue with the material. Using the pedagogy described above, we have a real chance to recognise “the rich historical and cultural roots of mathematics” in our classrooms.

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UK Web Based Documents

a) The NRIC site is at <http://nrich.maths.org/public/>

b) Pedagogical sources:

Changes in Mathematics Teaching Project: <http://www.cmtf.co.uk/>

Deep Progress in Mathematics

<http://atm.org.uk/reviews/books/deepprogressinmathematics.html>

c) Government documents:

QCA Website www.qca.org Maths_KS3_PoS.pdf and QCA Maths_KS4_PoS.pdf

<http://curriculum.qca.org.uk/key-stages-3-and-4/subjects/mathematics/index.aspx>

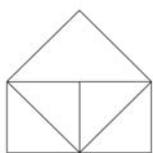
Ofsted UK (September 2008) Mathematics: Understanding the Score

Download review of mathematics led by Sir Peter Williams published June 2008

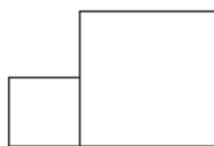
<http://www.ofsted.gov.uk/Ofsted-home/Publications-and-research/Documents-by-type/Thematic-reports/Mathematics-understanding-the-score>

COMPLETING THE SQUARE (Some Samples)

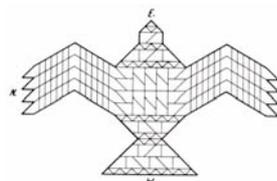
1. Indian Area Methods.



(a)



(b)

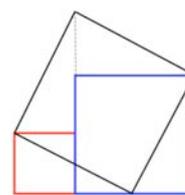


(c)

These diagrams and are inspired by practical Altar Building rules from the Sulbasutras, (15th - 5th Centuries BCE), (c) is the 'Kite Altar' still used in Kerala.

Challenge: It is easy to see how the combined areas of two equal squares can be found (a); with only a rope for measuring and drawing arcs, what about the combined area of (b)? Allow time for experiment and discussion of pupils' procedures. Ask pupils if they can find any more solutions. Does it work for any size of squares?

Canonical Activity: Use square dot-lattice paper to draw squares with a dot at each corner and no dots on the edge. Find areas using the smallest square as the unit. Discuss methods of dissecting the squares to find equivalent areas and how these may be combined. Display diagram (d) and discuss 'transformation of areas'.



(d)

Explore the visual dynamic of diagram with software; extend to rectangles and other shapes; identify basic properties and justify procedures.

Link with ideas from Mesopotamian mathematics and Euclid Book II.

2. The 'Babylonian Algorithm'.

A number game: "I am thinking of two numbers, their sum is 7 and their product 12, what are the numbers?" Extend with increasing pairs of sum and product numbers, encourage pupils to discover the original numbers. Pupils to challenge each other, share results, and find a way of writing instructions or developing a notation.

Introduce a standard algorithm: 'Take half of 7, square it, subtract 12 from this square and find the square root of the result, then add and subtract this square root from half of 7.' Use this to test other pairs. If it works for integers, try it with simple fractions. This algorithm originates in Mesopotamia and variations of it are found in Al-Khwarizmi, Fibonnaci, Cardano and others.

Extensions what happens when the pairs are 7, 11 and 7,13? These simple variations give non-rational ($\sqrt{5}$) and complex results ($\sqrt{-3}$) respectively.

Note 1: I see no problem in introducing quite young pupils to ideas like this. The process of 'following the algorithm' with simple numbers allows pupils to arrive at results which mirror in the discovery of these 'impossible' numbers.

Note 2: In this context, we also have the opportunity of introducing an iterative solution method for finding square roots, linked to the famous Old Babylonian tablet YBC7289. Discussion about the number that when multiplied by itself can produce 2, can lead to pupils' experimenting and developing their own methods of 'trial and error'. This is also one of the important opportunities to contemplate how we can manage and understand an infinite process.

Note 3: Finding a suitable notation is an important part of mathematical history and communication. In most cases in school mathematics notation is given unmotivated to pupils. Situations where pupils are challenged to communicate ideas to their peers through such examples provide opportunities for exploiting historical analogy.

INTRODUCTION OF AN HISTORICAL AND ANTHROPOLOGICAL PERSPECTIVE IN MATHEMATICS: AN EXAMPLE IN SECONDARY SCHOOL IN FRANCE

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Abstract: To introduce an anthropological and historical perspective in mathematics from middle school is a challenge that we have tried to face for several years. We first present what we mean with “an anthropological and historical perspective in mathematics”, our theoretical and didactical references, and our motivations for choosing the theme of irrationality. In the second part, we will present elements of three experimentations carried out with grade 8 (age 13-14) and grade 10 (age 15-16) pupils.

Key words: History of mathematics – Anthropological approach – Didactics of mathematics – Epistemology - Irrationality

I. MOTIVATIONS

In France, attempts to introduce an historical perspective in mathematics have been developed for several years, in particular, but not only, through the IREM Commission on History and Epistemology of Mathematics³. Some historical elements are also often introduced in textbooks (but most often without taking mathematical considerations into account). Beyond this, a crucial issue in a didactic perspective is the way it is possible to articulate historical elements with mathematical knowledge in teaching at the various levels of the curriculum. To approach historical texts mathematically most often necessitates an important effort to understand them, and the possibility of putting these texts in relation to the mathematical content for teaching is difficult and far from an evident choice, due in particular to the fact that the modern concepts are more efficient for solving the related problems. This could explain the rather common choice of limiting the introduction of history to informative aspects aiming mostly to motivate the students. Although this aspect should not be neglected, because it could allow us to modify the common representation of mathematics as timeless knowledge, it does not take into account the potential contribution of History of Mathematics for the learning of Mathematics itself. With Bkouche (2000), we consider that an historical perspective in the teaching of sciences « can be inserted less as a motivation than a *problematization* » in the following meaning: “Epistemology of problems aims to analyse how the problems that lead humanity to elaborate this mode of knowledge that we name scientific knowledge have modeled the theories invented in order to solve these problems”⁴.

II. THEORETICAL BACKGROUND

II.1. Anthropological foundations of mathematics

In continuity with Tardy (1997), we have chosen to situate the historical perspective in the field of Anthropology. Chevillard (1991) considers that Didactics of mathematics is the “headland of the anthropological continent in the mathematics universe”, that specifies its place in the field of Anthropology. In this perspective, he mainly studies the didactic transposition, i.e. the transformation undergone by mathematical knowledge when it is taught and used. For him, “present epistemology” studies the question of knowledge production while he considers Epistemology in the broader sense of Anthropology of knowledge.

In this paper, we refer to the sense of “present epistemology”, including anthropological considerations, according to Kilani (1992) that Anthropology searches for relations between local knowledge or specific discourses on cultures to global knowledge or general discourse on humanity.

II.2. Genetic psychology and Anthropology

Genetic Psychology elaborated by Piaget questions Anthropology. In opposition to Piaget, present Anthropology does not consider hierarchy among different stages. The stages that Piaget has distinguished (practical intelligence; subjective, egocentric, symbolic or operative thought) cut across the questions of Anthropology on the relationship between culture and thought, leading to debate around myth and rationality, magic and science and the way to pass from one aspect to another. Anthropology states that operative and symbolic thoughts have different purposes; that they do not exclude each other, coexisting in a singular person as well as in a given society⁵. Moreover, it could be thought that Imagination as well as reason could play a role in scientific discoveries (Kilani, 1992)

Following Vergnaud, we can add that in mathematics activity, these different modes of thought are necessary and complementary.

“Explicit concepts and theorems only form the visible part of the iceberg of conceptualisation: without the hidden part formed by operative invariants, this visible part would be nothing. Reciprocally, we are unable to talk about operative invariant integrated in Schemas without the categories of explicit knowledge: propositions, propositional functions, objects, arguments.” (Vergnaud, 1991, p.145)ⁱⁱⁱ

II.3. The epistemological model of « milieu » in the Theory of Didactical Situations (Bloch, 2002)

✓ About the concept of milieu

The concept of « milieu » plays an important role in the Theory of Didactical Situations (Brousseau, 1997). Several authors have reworked and developed this concept, which was one of the themes of The 11th Didactic Summer School in France in 2001. From our perspective, the models of *milieu* presented in this frame by Bloch is particularly enlightening. In the introduction to her course⁷, Bloch indicates:

“In this course, we aim to attempt a clarification of some fundamental concepts of Theory of Didactical Situations, and for this purpose to propose a reorganisation of the models of

milieu of this theory to predict and analyse teaching phenomena. It is clearly an elaboration aiming to classify the theoretical elements related to the *milieu* according with their functionality (from knowledge; from experiment; from contingency)” (Bloch, 2002, p.2)^{iv}.

This led her to propose the three following models: the *epistemological milieu* that concerns the cultural knowledge and their organisation, and the fundamental situations - the *experimental a priori milieu*, that concerns the researcher’s work preparing for the relevant teaching situations, and the *milieu for the contingency* concerning the effective realisation of these situations. In this section we focus on the epistemological model.

✓ **About fundamental situations**

For Brousseau, a fundamental situation for a given body of knowledge ought to permit the generation of a family of situations characterised by a set of relationships between student and milieu permitting the establishment of an adequate relationship to this knowledge.

✓ **The need of a model of epistemological *milieu***

To give a definition of what could be an adequate relationship to a given body of knowledge is not as easy as it might appear at first sight. It is the task of a researcher who attempts to elaborate a model of epistemological milieu:

“Such a model (written M_{IT}) is elaborated taking in account the cultural mathematical knowledge, but is not restricted to it. To elaborate *milieus* consists in grouping problems that do not necessarily strictly obey the knowledge organisation, thus a conjunction of mathematical, epistemological, and referential practices is necessary. I will also add the identification of knowing. Thus, one has to take into account not only problems for which this knowledge is functional, but also the relationship between these problems, and as far as it is possible, the related knowing (possible actions, intuitions, personal and cultural references) that the student could be able to actualise in the situation. “ (Bloch, 2002, p.5)

Our ambition, in this research, was not to elaborate a fundamental situation for a given notion (for us the notion of irrational number), but to attempt to enrich the net of relevant problems for the learning of this notion, leaning on a study (non exhaustive) of « the historical genesis of the knowledge concerning this concept and its ancient or contemporaneous occurrences, its functionalities in mathematics... » (Op. cit. p.7) as well as its links with other fields of human activity (philosophy; sociology; history; psychology; didactics ...), all links that have to be taken into account in the elaboration of an epistemological milieu as defined above. This permits us to investigate the way to elaborate the milieu for a teaching situation aiming to integrate this historical genesis and this anthropological perspective. In other words, how to make possible that historical or cultural references, beyond their function of motivation, contribute in a

genuine way to the teacher's project of the elaboration by students of knowing coherent and consistent with the involved knowledge. We will give further some elements that we have identified in this research.

III. AN EXAMPLE IN SECONDARY SCHOOL: IRRATIONALITY

III.1. Preliminary: a logical point of view

In a major work of Analytic Philosophy^v, the philosopher and logician Quine support the thesis that attributing a pre logical mentality to natives is wrong; in particular, rather than considering that they have contradictory beliefs, we have better to bet on an inadequate translation, or in a domestic situation^{vi}, on a linguistic disagreement. In other words, the irrationality or the incoherence of humans is less probable than a non adequate interpretation by the observer of the provided indicators. We have shown (Durand-Guerrier, 1996) an example of the domestic version in mathematics education in order to lift a suspicion of incoherence that might bear on students' responses^{vii}. Matters concerning contradiction, rationality and irrationality are subjects of study for logicians, either those attempting to elaborate systems accepting contradictory propositions, due to the fact that such propositions are everywhere in ordinary life (e.g. Da Costa, 1977), or those developing theories taking in account simultaneously syntactic, semantic and pragmatic considerations in natural languages^{viii}. In this perspective, the Model Theory developed by Tarski (1936) offers a relevant theoretical framework to deal with the questions of necessity and contingency, and to treat apparent contradictions (Durand-Guerrier, 2006, 2008).

The project of Granger (1998) is « to consider the sense and the role of irrational in some human works, in some major creations of human spirit, and particularly in sciences. »^{ix} (Op.cit. p.10). From an author who has devoted his work to description, analysis and promotion of what is rational in human thought, this is not an apology of irrationality, but the testimony of an inscription in « the perspective of an open and dynamic rationality, in order to recognise and delimitate the role of what is positive in irrational. » (Op.cit. p.10). Indeed, Granger considered that « the irrationality, eminently polymorphic, draws in hollows, so saying, the form of rationality (...), and always supposes, at least for analysis, a representation of what it is opposing with. » (Op.cit. p.9)

Accordingly, these short insights show that the crucial opposition in number theory between rational and irrational number, articulated by the opposition between coherence and contradiction, is a candidate for our exploration.

III.2 Our research hypotheses

Two main hypotheses are structuring our work. The first one is that the problematic of the articulations between various modes of thought, in particular the relationship between Science and Myth, Rationality and Beliefs, is relevant for the study of anthropological fundamentals of mathematics. The second one is that, through the intermediary of the genesis of mathematical knowledge, we will be able to achieve an

anthropological mode of/way of thinking concerning mathematics and its links with the various modes of human thoughts.

III.3 The inscription of Irrationality in our investigation

The term Irrational (in Greek: *alogen*) has two main significations. First, it means « without a common measure; that cannot be measured as a quotient of two integers ». Second, it means « that is unable to insure the coherence of discourse; illogical ». For Granger (1998) the encounter of irrational numbers in Greece was an example of what he named « the irrational as an obstacle, starting point of the conquest of rationality anew ». This leads to two partly philosophical questions: what does the obstacle really consist of? How can we come to its resolution? Arzac (1987) claims that the encounter with Irrationality is at the origin of the transformation of mathematics in hypothetical deductive system. Of course, it is clear that the confrontation of Irrationality by itself is not sufficient to create anew the conditions of the apparition of the proof, but this invites us to turn toward an interdisciplinary approach to rigor, that we have modestly done in our work. If students of grade 8 or 10 are not *a priori* able to overcome the epistemological obstacle¹⁴ (indeed, it would be necessary to work along two axes: Euclidean Theory of magnitudes; and a real number construction), our weaker hypothesis is that the confrontation of students with a mathematical or an interdisciplinary work about Irrationality could permit them to approach the question of the nature of this obstacle.

IV. OUR DIDACTIC INVESTIGATION

IV.1. General conditions for a didactical situation in our perspective

In coherence with our theoretical exploration, we propose conditions^x that a didactical situation dedicated to the introduction of an historical and anthropological perspective for a given body of knowledge in mathematics in secondary school ought to fulfil.

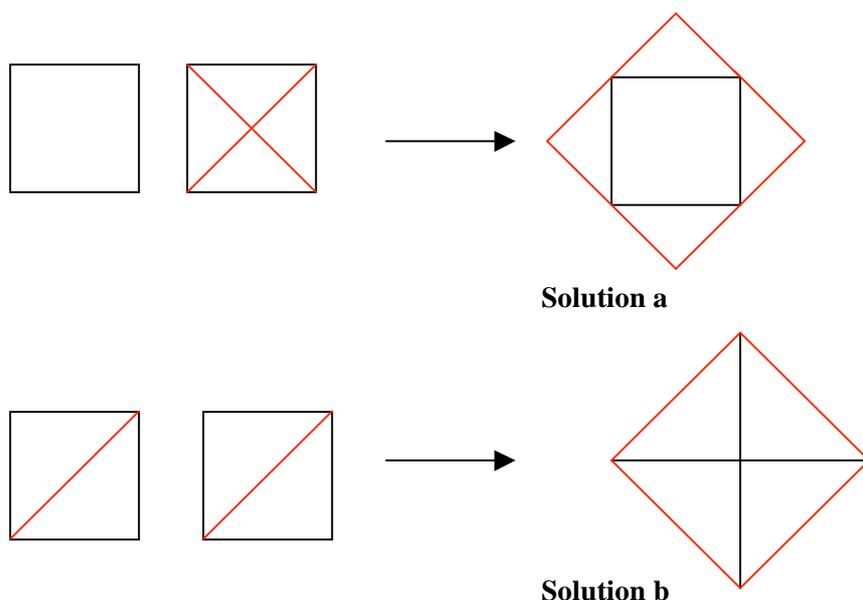
1. The situation is based on a moment well identified in the genealogy of this knowledge.
2. The situation permits us to question the formidable efficacy of mathematics to act in the real world.
3. The situation fulfils the minimal conditions of a problem situation, in particular favouring framework changes (Douady, 1986)^{xi}.
4. The milieu is rich enough to provide retroactions permitting to go forward in the situation and conditions for an intern validation.
5. From the situation, a contradiction between *a priori* beliefs and constraints from reality would emerge.
6. The situation permits us to end up in an institutionalisation of the concept involved in coherence with the curriculum, and of the specific contribution of mathematics to a more general problematic, linked most often to Human and Social Sciences.

IV.2. Brief description of the experiment in grade 8

This experiment took place in December 2000 and January 2001, in an interdisciplinary project. It comprised four sessions in History course (on the 18th century); five sessions in French (Literature) course, on the theme of rational and

irrational; and four sessions of mathematics that we describe below.

- First session: construction of a square from a pair of superposable squares with sides of 10 cm, using a minimal number of cuttings with scissors; elaboration of a proof that the figure is actually a square.



- Second session: synthesis of the proofs elaborated in the first session; investigation in order to determine the area of the big square.
- Third session: enlightening of the fact that the length of the side of the big square is not a decimal number. Emergence of the following question: is it a rational number?
- Fourth session: elaboration of a proof that $\sqrt{2}$ is not a rational number. Information about the circumstances of this discovery; historical and anthropological aspects; links with what had been done in History and French courses.

In April 2001, an evaluation was made through a role-playing game (Pythagoras' Trial) organised by the three teachers involved in the experiment.

IV.3. Some results of the experiment in grade 8

The interdisciplinary work has permitted us to make the links explicit, although the students did not always perceive them. Concerning mathematics, it is necessary to find a balance between levels of difficulty on the one hand and interest and relevance of the problem on the other hand. This is the case in general for problem situations, but here due to the conceptual ambition it is more acute. Teachers do not wish their students to face difficulties; but the contents, although they do not really exceed the programmes, mobilize cognitive capacities hardly required in the ordinary school mathematical work. However, the effective experiment allows us to reveal that most students appreciated this type of problem and were able to provide rich and relevant arguments.

Students have dealt with the following mathematical notions: area of a square by cutting out shapes; property of areas to be additive; units; recognition of equality of two squares constructed by two different methods; calculations with decimal numbers, and rational numbers; interrogation of the results given by a calculator. Moreover, they have developed argumentation and deductive reasoning in geometry (for example, justify that a figure is a square), and in the numerical field (it is impossible that the square of a decimal / a rational number be equal to 2). Notice that the last proof is that one using the possible digits of the numerator and the denominator, and *reductio ad absurdum* (or infinite descent).

The analyses of the evaluation (Pythagoras' trial) on the one hand, and of three interviews with students on the other hand, give us *a posteriori* information. The development of the trial seems to indicate that students have understood the arguments; have discussed together, but did not have enough time for a right appropriation of the working of a trial. Here are some arguments developed by students: "If the diagonal of the square is neither an integer, nor a decimal, nor a rational, he (Pythagoras) has not invented it, for this length existed." / "The accusation: it is serious not to reveal this discovery, it is a lost of time -The defence: he will not have been believed. -The accusation: but he had explication! In the end he will be believed; he had a theorem." / "If he revealed the irrational numbers, his whole previous theory would have been wrong. -these numbers are frightening - to say these numbers would have caused the end of the world ; it would have disturbed everything." (this student makes a distinction between ordinary people and scientists). / "When he (Pythagoras) said everything is number, he was not lying because at that time, he did not know about the existence of irrational numbers."

The students interviewed remembered precisely what had been done in the four sessions of mathematics. The link between Irrationality in Mathematics and in French and /or History courses is not done by all of them, but one of them summarized it saying "when we see the superstitions of humans, the sects, it may disrupt the world, and the number too, it may disrupt the world. There is a small link, but it is different."

This project provides an alternative to the aspect of "tools" generally devoted to mathematics. Although this aspect of "tools" is quite relevant, many teachers perceived it as a reduction of what mathematics really is. This project shows that school mathematics can also play its role, beside others disciplines, in the elaboration of elements of human culture, beyond the strictly technical aspects, that an excessive recourse to algorithms tends to reduce it to.

IV.4. Brief description of the experiment in grade 10

The experiment by an experienced teacher, took place in 2002-2003, and in 2006-2007 by a prospective teacher in the context of the professional dissertation in the Teacher Training Institute (IUFM) in Lyon. It comprised of five sessions:

- First session: Introduction of the problem of incommensurability through the following problem: given a square ABCD, is it possible to find a unit measuring both the side and the diagonal of the square; you may use calculator but not the key of square root. Students worked first in small groups; a square of side 12 cm had been provided; the synthesis was collective in the whole class.
- Second session: Working on the link between Incommensurability and GCD (Euclid Algorithm) in the whole class.
- Third session: Proof of the incommensurability of the diagonal and the side of a given square, by *reductio ad absurdum* in the geometric framework.
- Fourth session: Irrationality of $\sqrt{2}$; approximation by rational numbers.
- Fifth session: work on texts and documents; making of posters.

IV.5. Some results of the experiment in grade 10

In grade 10, the teachers considered that the first four sessions were rich for the following reasons. 1. They give a meaning and a legitimacy to proof, as said a teacher. « Indeed, some students have difficulties to understand the necessity of proof. When we propose a proof for a problem for which they know the result, they do not understand why they are proving. Here, a debate rose at the first session. Some were convinced of incommensurability of the side and the diagonal of the square, but others were not. The objective of the proof was to convince, to argue. Let us notice the role of *reductio ad absurdum* in the third session; however it is not involved *a priori* in the numerical field to prove irrationality, but in the geometrical situation that permits to prove this incommensurability; moreover this incommensurability has been studied experimentally in the first session (in a geometrical or numerical field, according with the process used by students), that permits us to pose the problem in a better way »; 2. “They make links between the numerical and geometrical fields. Some notions allowing solving the problem have got signification for students as GCD or Euclid algorithm.” / 3. “These sessions have permitted an evolution of the vision that students had of mathematics: « we have discovered the fact that the construction of mathematics did not occur in a linear way but through ruptures ». So the students could change their mind that mathematics “*vont de soi*”. As sometimes mathematicians face difficulties to apprehend some notions, students realise that their own difficulties were normal.” / 4. “They allow various mathematical notions to be revised: GCD – Euclidean Algorithm – Pythagoras theorem – rational number ...” / 5. “All students have been involved in this work (at school as well as for homework), and interested whatever their level.”

CONCLUSION

We consider we have given some evidence (in an *existential* sense) that it is possible in grade 8 and 10 in France to do interdisciplinary work, structured around a mathematical notion, for which a study, even of the partial historical genesis allows us to show the anthropological dimension in the sense we have defined above. Irrationality appears as a paradigmatic theme, or even an *ad hoc* theme, of what we

aim to develop. That other notions could permit such a work remains for us an open question, but it seems to us that it would be possible to find candidates towards themes common to mathematicians and philosophers, sociologists, historians, without forgetting artists; themes like propositions; infinity; emptiness; space-time; paradox; truth; necessity; transcendence....

We are aware that more work has to be done, particularly in defining relevant characteristics of the didactical situations in order to reach our learning objectives on the one hand, in identifying potential institutional “niches” depending on the curriculum on the other hand.

Another question concerns the way to elaborate and share with teachers situations aiming to integrate the historical genesis and the anthropological perspective for a given theme.

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Annex

Didactical conditions	Implementation in 4 ^{ème} (Grade 8)	Implementation in 2 nd (Grade 10)
Genealogy	The problem of the duplication of a square; Pythagoras' discovery of $\sqrt{2}$	Incommensurability of the diagonal of a square with its side; Euclid's algorithm about GCD.
Efficiency	Effective production of a square the area of which being 2 from a square with an area of 1	
Frames	Numerical/Geometrical	Arithmetical/geometrical Geometrical/numerical
“Milieu”	Effective realisations; success checking	Construction of a decreasing series of squares
Contradictions	“to double the area, you must double the side”	We know how to fix a measure to any measure of length
Institutionalisation	The length of the side of the square of area 2 is not a decimal number	Incommensurability of the diagonal of a square with its side; Irrationality of the number square of 2
Connections with human sciences	History, Philosophy (Menon's dialog) Arts,	Respective positions and roles of rational and irrational numbers; questioning about the meaning of “to exist”

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ⁱⁱⁱ Our translation

^{iv} Our translation

^v Quine (1960) Word and object

^{vi} That means our co speaker

^{vii} Durand-Guerrier (1996) pp. 276-280

^{viii} The use of such a perspective in primary and lower secondary education can be found in Durand-Guerrier & al (2006)

^{ix} Our translation

^x You can see a table about them in the annex

^{xixi} Framework changes refer to *Jeux de cadres*: framework is here to be taken in its usual meaning (algebraic, arithmetical, geometrical...); such changes are supposed to favour research process in problem solving and evolution of students' conceptions.

THE IMPLEMENTATION OF THE HISTORY OF MATHEMATICS IN THE NEW CURRICULUM AND TEXTBOOKS IN GREEK SECONDARY EDUCATION

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The official textbooks for the teaching of mathematics in the Greek high school (7th-9th grades) include a lot of historical material, following the guidelines of the new curriculum. However, their use is questionable because of serious historical errors, obscurities, or omissions. We support this conclusion by some examples, suggest alternative ways to use this material, and outline a deeper and more demanding implementation of the history of mathematics in the context of cross-curricular teaching activities.

Keywords: historical snippet, mathematics curriculum, cross-curricular, original sources, junior high school.

1. INTRODUCTION

In the last two decades, there is an internationally increasing interest in introducing a historical dimension in mathematics education (ME), both in didactical research and in educational policy, curriculum design and textbook content. This is reflected in the appearance of several publications, the organization of conferences, especially in the context of the HPM Study Group (e.g. Fauvel & van Maanen 2000, Siu & Tzanakis 2004, Katz & Michalowicz 2005, Schubring 2006, Furinghetti *et al* 2006, 2007, Barbin *et al* 2008). In Greece, there has always been an active interest in this area, as early as the late '80s, mainly in didactical research (Fauvel & van Maanen 2000 §11.8, Kastanis & Kritikos 1991, Thomaidis *et al* 2006, Chasapis 2002, 2006) and occasionally in the inclusion of short historical comments in school textbooks. Possibly, the influence of active researchers and educators' work in this area, made officials of the Ministry of Education more attentive to what international research and practice suggests on the role of the History of Mathematics (HM) in ME. Thus, for the first time in Greece the (new 2002) mathematics curriculum for compulsory education (Pedagogical Institute 2002) includes so extensive references to a historical dimension in ME, varying from the specific teaching objectives, to the didactical methodology and the textbook content, e.g. (Pedagogical Institute 2002 pp.311, 367-369; our translation):

Special objectives: “..... to reveal the virtue of mathematics (historical evolution of mathematical tools, symbols and notions).”

Didactical methodology: “... It is important to provide students with “safety valves” in the pursuit of knowledge; namely, students should be given the possibility to approach a notion in a variety of ways, i.e.: (a) By means of several different representations (using symbols, graphs, tables, geometrical figures); (b) In an interdisciplinary way; (c) With reference to the HM (the HM is a field rich of ideas to

approach a notion didactically).”

Didactical material: “... Moreover, reference to the great historical moments that step by step have determined the development of mathematics should be included in the mathematics textbooks, so that the student becomes aware of the genesis of the ideas, which is a prerequisite for grasping each subject. It is not necessary that the historical notes appear separately at the end of each §. (If required), they can also be (briefly) presented, at intermediate parts of the text.”

Though these guidelines follow what didactical research suggests on the role HM can play in ME, their actual classroom implementation is not satisfactory: the authors¹ have tried to follow these guidelines, incorporating in the new mathematics textbooks a great deal of material from the HM in the form of historical notes and associated activities. These notes and activities (called *historical snippets*; Fauvel & van Maanen 2000, ch.7) have different format and colors from the main text and usually contain pictures. Here we examine critically the validity of this material and its relevance to the curriculum, by means of specific examples and suggest other ways to integrate the HM in teaching, taking into account modern trends in this direction.

2. THE HISTORICAL TEXTBOOK MATERIAL & ITS RELEVANCE TO THE CURRICULUM

The quotations from the mathematics curriculum in §1 directly connect the use of the HM with a central issue of teaching and learning: how to pursue and grasp knowledge. Thus historical snippets in the textbooks should not be limited to factual information, but contribute to understanding the notions to be taught (Fauvel & van Maanen 2000, §7.4.1); they should provide ideas and material to organize teaching and motivate students to learn. Therefore, they should meet two reasonable requirements: (a) to be mathematically and historically correct; (b) to serve the objectives of the teaching units in which they are incorporated.

Unfortunately, in many cases the historical snippets in the new high school textbooks violate these requirements; the authors’ insistence on restricting the historical material to (often inaccurate and contradictory) biographical information, is a typical case. In general this material is presented in an informal style, inserted in separate boxes in the text, usually emphasizing historical facts, rather than the mathematical exposition. In some cases it also includes related activities (cf. Fauvel & van Maanen 2000, §7.4.1). Table 1 gives a summary of the historical material in the new textbooks:

Table 1

Grade	Number of historical snippets	Percentage of textbook pages covered	Percentage of snippets which include activities	Comments in the teachers book
7	21	11/220 = 5%	5/11 = 45,5%	Some comments on the HM
8	9	6/230 = 2.6%	0/6 = 0%	2 additional activities are recommended
9	5	5/240 = 2.1%	2/5 = 40%	10 additional comments covering 12 of the 100 pages (1 activity recommended as an interdisciplinary activity).

We illustrate this material and its weaknesses by means of indicative examples, mainly from the 7th grade textbook (Vandoulakis et al 2007, Vlamos et al 2007)².

Example 1: factual information; no mathematics involved

In the 7th grade textbook, the authors cite 3 contradictory lifetimes of Euclid giving contradictory results: p.26: 330-275BC; p.147: 300-275BC; p.182: 330-270BC, ignoring that the only existing valid historical source on this point, is an extract from *Proclus' Commentary* on Book I of Euclid's *Elements* with no possibility to specify exact dates. In addition to historical confusion, this note does not serve any of the purposes of introducing HM in teaching as detailed in the new curriculum (cf. §4 below).

Example 2: factual information; reference to mathematical & scientific results

In a separate box of the same textbook (p.29), brief information is given on Eratosthenes' life and some of his scientific achievements (e.g. the measurement of the earth's circumference), claiming that: Eratosthenes lived from 276BC to 197BC; from 235BC and for 40 years he was director of Alexandria's famous library; at the age of 82, he committed suicide because he became blind. These data are contradictory, however: Since $276-197=79$ and $235-40=195$, he lived 3 years less than the age at which he died, and directed Alexandria's library for two years after his death! This note could include interesting activities in accordance to the regulations of the new curriculum (e.g. the simplicity of the measurement method of the earth's circumference), but being restricted to simply assert the results, it is mystifying, rather than enlightening!

Example 3: fiction, mathematical results and a related mathematical activity

Occasionally, the historical narrative is fictitious. In the 7th grade textbook, historical accuracy is sacrificed in favor of a controversial story, aiming to dramatize an episode from Gauss' childhood (p.75, our translation):

“Sometimes a simple thought of a man is more worthwhile than the whole world's gold. With some clever ideas battles are gained, monumental pieces of work are done, people become famous and at the same time, science is developed, technology evolves, history is shaped and life changes. Just an example is the “smart addition” that Gauss (Karl Friedrich Gauss 1777-1850) had thought of in a small German village, around 1789, when he started learning about numbers and arithmetical operations in his first year at school. When the teacher asked his students to calculate the sum $1+2+3+\dots+98+99+100$, little Gauss had found it before the others even started. Then, he wrote on the blackboard:

$$(1+100)+(2+99)+(3+98)+\dots+(48+50)+(50+51)=101+101+101+\dots+101+101=101\cdot 50=5,050$$

Try to calculate in Gauss' way the sum $1+2+3+\dots+998+999+1000$ and measure the time needed. How long would it have taken if calculated it in the normal way?”

However, (a) Braunschweig, Gauss' native place, was a political and cultural center, capital of a ducat, with about 20.000 residents in the late 18th century, not a village; (b) given that Gauss had been characterized as a mathematics “child-prodigy” from the age of 3, how is it possible that he began learning arithmetical operations in 1789, at the age of 12? Gauss entered the Volksschule (elementary school) in 1784, the Gymnasium in 1788 and the Collegium in 1792 (Wussing & Arnold 1978, p.318); (c) Gauss died in 1855, not 1850!

More importantly, this note makes an extreme statement, suggesting that mathematical progress is due to a few geniuses, not a collaborative enterprise in

which personal skill is harmoniously combined with preceding achievements of the scientific community at the right moment. Thus, it implicitly gives a distorted view of history, which, considered didactically, is expected to discourage rather than engage students in mathematical activities in the classroom. Hence, this example shows lack of relevance of the textbook's historical material with the curriculum objective "to provide students with 'safety valves' in the pursuit of knowledge".

Example 4: historical snippets with historically motivated mathematical activity

In the same textbook there is the following activity (p.75, our translation):

ACTIVITY: On a gravestone the following problem is inscribed, whose solution gives the age of the great ancient Greek mathematician Diophantus:

*"This tomb holds Diophantus. Ah, how great a marvel! The tomb tells scientifically the measure of his life. God granted him to be a boy for the sixth part of his life, and adding a twelfth part to this, he clothed his cheeks with down; He lit him the light of wedlock after a seventh part, and five years after his marriage He granted him a son. Alas! Late-born wretched child; after attaining the measure of half his father's life, chill Fate took him. After consoling his grief by this science of numbers for four years he ended his life."*³

But where lies this gravestone? We do know that this story appears in the *Palatine Anthology*, of the Byzantine era, with no other reliable evidence for it. This activity, included in the chapter on "Equations and Problems", is not accompanied by any query, except mentioning in the teacher's book that (p.53, our translation):

"A. 4.2. Problem Solving: *Indicative design of the material of this unit. 1 teaching hour.*

The suggested activity aims to understand: The notions used in problems, their solutions, as well as, the solution process followed [*Answer: Diophantus lived for 74 years*]"

If this requires the formulation of an equation for Diophantus' age x , then the epigram implies: $\frac{x}{6} + \frac{x}{12} + \frac{x}{7} + 5 + \frac{x}{2} + 4 = x \Leftrightarrow x = 84$

However, 7th graders are not able to formulate and solve this equation, since solving such equations is taught in the 8th grade! Hence, this historical note is related neither to the mathematics of the textbook unit in which it is included, nor to the cognitive level of the students to whom it is addressed.

This epigram appears in an introductory note in the 8th grade textbook's chapter on "Equations and inequalities" with the following comments (Vlamos et al 2007, p.120, our translation):

"...From his [Diophantus'] 13 pieces of work only 10 had been found (6 in Greek manuscripts and 4 in Arabic translation). The most famous of his works is the "Arithmetika" (6 books). It is the most ancient Greek work in which for the first time a variable is used in problem solving...When he died, ...his students composed a riddle and wrote it on his grave, upon his request. Here is Diophantus Epigram..."

According to Diophantus' own statement, *Arithmetika* were divided into 13 "books"; 6 have been preserved in the Greek original and 4 in Arabic translation of the 9th century discovered in the 1960's. We also know another of Diophantus' works - "On polygonal numbers" – only fragments of which survive. Hence, the textbook confuses the 13 books

of “*Arithmetika*” and the total number of his works.

3. SOME CONCLUSIONS

All examples in §2 concern historical errors (there are still more, reinforcing the bad flavor got from the textbooks’ historical snippets) that nevertheless, could easily be corrected in a new textbook edition, though it is strange that they have not been avoided. It seems as if they were hurriedly written, mainly aiming to satisfy the relevant term of the announcement of the textbook writing competition and not to introduce a historical dimension in teaching.

The main characteristic of this historical material is the large amount of information and the rich illustrations, without however some methodological hints of how to benefit didactically from it. Though, the corresponding suggestions and instructions in the teacher’s book in general emphasize the positive contribution of the HM, the way this could be realized is left to the initiative and ideas of the teacher, with reference to the relevant bibliography. E.g., the teacher’s book for the 7th grade mentions that:

“In some sections, there are historical notes, which intend to stimulate the student interest and love for Mathematics and to inform them on the historical development of mathematical thinking. Their use in teaching depends on the initiative and the ideas developed by the teachers” (Vandoulakis et al 2007, p.31, our translation)

In the teacher’s book for the 9th grade this issue is detailed more:

“In some units there are topics from the HM intended to give the description of the problem that has been posed and the presentation of the conceptual tools applied to solve them. These topics, with the accompanying questions, aim to exploit the HM in the best possible way. Integrating the HM in teaching has become the subject of systematic studies at an international level. The positive contribution of the HM is corroborated in three groups of arguments: (a) It stimulates students’ interest and contributes to the development of a positive attitude towards mathematics. (b) It reveals and stresses the human nature of the mathematical activity throughout history. (c) It contributes to the understanding of mathematical concepts and problems, revealing not only the context and circumstances in which they originated, but also the conditions of their development.

These topics [from the HM and the accompanying questions], together with those points raised in the teacher’s book, should not be considered as complete studies; it is for this reason that references to the literature are given for those teachers and students who will have a special interest.” (Argyrakis et al 2007, pp.10-11, our translation)

Remark: Points (a)-(c) form part of the arguments for integrating HM in ME, put forward more systematically in Fauvel & van Maanen 2000, §7.2 (particularly §§(a1), (c1), (d1).

Introducing a historical dimension in the teaching of mathematics, based on teachers’ interest, initiative and ideas, needs extra teaching time. But, apart from the usual obligation to cover the school material (a very difficult problem in itself!), teachers have also to cope with the innovations of the new curriculum, like group-cooperative teaching based on learning activities, or an interdisciplinary approach to mathematics. Hence, introducing a historical dimension in ME to the benefit of both teachers and students,

requires additional support in the form of detailed guidelines (e.g. examples serving to illustrate how history could be integrated into teaching), extensive references for further reading and availability of relevant resources. Unfortunately, existing resources are limited (Fauvel & van Maanen 2000, p.212). In addition, from the evidence here, it is clear that the material of the new textbooks is not the most appropriate and valid guide in this direction. Therefore, high school mathematics teachers are not given any real motivation to take up the initiative to benefit from the new textbooks' historical material. In the next section, we examine whether the available historical snippets (after being corrected) can contribute positively to the teaching of high school mathematics.

4. USING HISTORICAL SNIPPETS IN CROSS-CURRICULAR ACTIVITIES

The errors in the historical notes of §2 indicate that integrating the HM in ME is a demanding activity, presuming, not only mathematical knowledge and the ability to approach, read and interpret the historical sources, but also to cross-check facts, to conclude and narrate. This seems to suggest cross-curricular activities as a privileged framework in this connection. Fortunately, such activities form an integral part of the new curricula and high school textbooks in Greece, an example being the determination of Euclid's lifetime: As mentioned in §2, the only valid historical source on this point comes from Proclus, who lived in the 5th century A.D. In his *Commentary* on the 1st Book of Euclid's Elements, he writes:

"[Euclid] lived in the time of Ptolemy the First, for Archimedes, who lived after the time of the first Ptolemy mentions Euclid. It is also reported that Ptolemy once asked Euclid if there was not a shorter road to geometry than through the Elements, and Euclid answered that there was no royal road to geometry. He was therefore later than Plato's group, but earlier than Eratosthenes and Archimedes, for these two men were contemporaries, as Eratosthenes somewhere says." (Morrow 1970, pp.56-57)

This is a nice extract for an activity, combining mathematics, history and language (for Greek students). Translating the ancient text into modern Greek, collecting information for the persons involved, studying more the historical period in which they lived, could be a student activity to provide material for further discussion in the classroom, leading to the following conclusion:

We know that Ptolemy the 1st, a general of Alexander the Great had been the satrap of Egypt from 323 to 305 B.C., and its king from 304 to 283, and Archimedes lived from 287 to 212 BC. Proclus cites the dialogue of Euclid with Ptolemy the 1st and says that he was older than Archimedes. Therefore, Euclid's period of activity is very close to 300 BC.

This activity has interesting didactical extensions and could lead to insightful discussions on the concept of mathematical proof: The method and logical arguments leading, from historical sources to the above conclusion, can be paralleled to those used to justify a general mathematical result from definitions, axioms and others previously proven. Hints can also be given for those characteristics of theoretical geometry that led Ptolemy to ask Euclid for a "short" learning path to it. Similarly, ancient texts on Eratosthenes' life and work could be used, with emphasis on the measurement of the earth's circumference (Thomaidis & Poulos 2006, p.110).

Cross-curricular activities could be also disconnected from conventional teaching

and be realized more efficiently in parallel school events, like the formation of a group of students, who, under the teachers' supervision and help, read mathematical works. E.g., studying Tent's book (2006) could be pedagogically and didactically more efficient results than the note on Gauss in § 2.

5. ANCIENT GREEK MATHEMATICAL TEXTS IN THE TEACHING OF EUCLIDEAN GEOMETRY IN HIGH SCHOOL: A CROSS-CURRICULAR APPROACH

We present some elements of a deeper and more demanding approach to integrate the HM in teaching mathematics, than the use of historical snippets; namely the use of original texts in carefully designed worksheets, implemented in cross-curricular activities (Fauvel & van Maanen 2000, ch.9).

We developed a cross-curricular activity in 4 classes of 10th-graders (15-16 year old students; 25 girls and 25 boys in total), for 2-hour sessions in which the teachers of mathematics, ancient Greek language and history were involved with alternating interventions. To this end excerpts from Euclid's *Elements* and Proclus' *Commentary*, have been used to construct 4 worksheets, each one of which was used in a 2-hour classroom session. They concern: (a) Euclid, Proclus and Pappus' different proofs of the equality of an isosceles triangle's angles; (b) the construction of an angle's bisector; (c) the triangle inequality for the sides of a triangle; (d) the sum of the angles of a triangle.

This activity aimed to (i) integrate original texts in a cross-curricular teaching of Euclidean Geometry in the 10th grade; (ii) to create a new didactical environment and accordingly explore the realization of specific teaching aims; "initiation in mathematical proof", and "development of critical thinking". More specifically, by the chosen excerpts and the questions addressed to the students, we sought to examine whether the students (i) share the criticism of the ancient philosophers against Euclid, (ii) understand the expediency of giving different proofs for the same geometrical proposition, particularly for obvious properties of geometric figures (as Proclus did while defending Euclid) and (iii) understand the expediency of mathematical proof in general. Under the teachers' supervision, students analyzed ancient texts mathematically, linguistically and historically, with focus on formulating corresponding questions emerging from this analysis and the classroom discussion of students' point of view on them.

The worksheets' structure was: (a) Ancient Greek mathematical text; (b) Request to read and translate the text; (c) Questions on the text: 2 to 3; (d) Homework: 1 or 2 assignments.

Remarks: (1) Three of the worksheets contained 2 excerpts, with this structure for each excerpt; the fourth included 4 excerpts. We outline this approach for worksheet No1. (2) The discussions in the classroom were videotaped. Students' answers below refer to questions raised in the classroom (Q1-Q3 below) and come from the analysis of videotapes and the teachers' hand-notes.

Worksheet No1

Excerpts: (i) Euclid "*Elements*" Book I, *prop.V*: equality of the basis angles of an isosceles triangle (Heath 1956, pp.251-252). (ii) Proclus' "*Commentary*", §§248, 250: Alternative

proofs of this proposition by Proclus and Pappus (Morrow 1970, pp.193-195).

Questions: Find: (1) the corresponding theorem in the geometry textbook
(2) similarities & differences between Euclid's and the textbook's proofs.

Homework: (1) Translate the ancient text keeping to Euclid's spirit as close as possible (e.g. avoid terminology and notation not used by Euclid).

(2) Get information on Euclid and his *Elements* from encyclopedias or other resources.

(3) Translate Proclus' text to modern Greek.

(4) Find similarities and differences among Euclid, Proclus and Pappus' proofs.

(5) Try to explain why all ancient proofs are different from that in the textbook⁴.

Classroom discussion on the following questions:

Q1. In your opinion, why did Euclid give a complicated proof?

Q2. Why did the ancients avoid using the bisector of the angle at the top vertex? How it can be ensured that the usual construction (by ruler and compass) of the bisector of an angle, does indeed bisect the angle?

Q3. Comment on Proclus' and Pappus' proofs.

Some of students' responses

On Q1, Q2:

(i) Euclid wanted to impress his readers, because when scientists do complicated things, their authority increases.

(ii) Euclid wanted to show how to use the triangles' equality criteria.

(iii) Euclid wants a theoretical, not a practical proof. Bisecting an angle is a practical issue and is not accurate. This construction is naïve, possible for all people, because it is like folding in two a piece of paper.

(iv) Euclid could not draw the bisector accurately; he could not prove that the two angles are equal. The bisector concept had not been discovered yet.

(v) Euclid wanted to exploit that particular proof in order to prove other properties that exist in that particular figure.

On Q3 (for Pappus' proof):

(i) It looks like proofs that we gave at the elementary school.

(ii) It is a proof appropriate for babies(!)⁵

(iii) It is more difficult; it requires more thinking (more probable to make a mistake).

(iv) It is adapted to practice, whereas, Proclus' and Euclid's proofs have elements of logic and scientific reasoning.

Remarks on methodological issues concerning cross-curricular activities:

(1) This cross-curricular approach helped to face important issues concerning translation & interpretation and placed original texts in the appropriate historical context.

(2) The original texts and the translation process led to etymological comments on the origin, meaning and accurateness of mathematical terminology.

(3) The clarity and conciseness of ancient Greek mathematical language was revealed by connecting two apparently disjoint disciplines; ancient Greek language and mathematics.

Some results: The remarks, and the analysis of the classroom discussion stimulated

by the study of the other three worksheets suggests:

- (a) Studying original texts created a new didactical environment, in which students actively participated in the classroom discourse and exhibited a positive attitude towards the subject, which never happens in conventional geometry teaching (this was particularly clear in the critical discussions on worksheet No3 on the triangle inequality and Stoics' objections reported by Proclus, that tried to ridicule Euclid).
- (b) Students' commented that this activity led them to a more global understanding of what Euclidean geometry really is (e.g. see answers (ii) and (v) to Q2).
- (c) The variety and mutual incompatibility of students' answers produced by studying original texts, reveal factors that influence the understanding of metamathematical concepts, like the concept of proof (e.g. compare answers to Q3; (i) & (ii) to (iii)).
- (d) Critical thinking requires both the technical ability to formulate particular proofs, and more general abilities to globally conceive notions, to formulate correct assertions etc (e.g. see answers (iii) to Q3 and (iv) to Q2).
- (e) The requirements for studying original texts, link the didactical aims of learning specific pieces of mathematics, with wider pedagogical aims of ME: raising metamathematical issues, access to philosophical & epistemological concepts, links to the historical & cultural tradition etc (e.g. see answers (i), (iii) and (iv) to Q2).

¹In Greece, there is only one textbook per subject in each grade of primary or secondary education, imposed by state regulation as a result of a public competition for writing these textbooks.

²In Greece, grades 1 to 9 constitute compulsory education: the elementary school (grades 1-6; students 6-12 year-old) and the "gymnasium" (junior high-school, grades 7-9, students 13-15 year-old). There are essentially no historical aspects in the elementary school textbooks; hence we restrict the discussion to junior high school.

³ See Cuomo 200, p.245.

⁴ In the textbook, the angle at the top vertex is bisected and the two resulting triangles are shown to be equal.

⁵ In Pappus' proof an isosceles triangle is turned and the resulting triangle is shown to be equal to the initial one.

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