GEOMETRIC THINKING IN A *N*-SPACE

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Abstract

What does "thinking geometrically" mean in a n-dimensional vector space, with n>3?According to several authors, like Fischbein, or Harel, a productive reasoning in mathematics always relies upon an intuitively accepted model. A first possibility of model in dimension n is provided by the use of coordinates; the use of that model can be called "analytic thinking". But other possibilities exist, related with synthetic geometry; in that case the expression "geometric thinking" applies. I will display some of these possibilities here, and show on a particular example how they can intervene in students problem solving processes in linear algebra.

Résumé

Quel sens peut avoir l'expression "pensée géométrique" dans un espace vectoriel de dimension *n>3* ? Selon différents auteurs, comme Fischbein, ou Harel, un raisonnement productif en mathématiques est toujours basé sur un modèle intuitivement acceptable. En dimension *n*, l'emploi de coordonnées fournit un premier modèle, dont l'emploi peut être qualifié de "pensée analytique". Mais il existe d'autres possibilités, liées à la géométrie synthétique ; dans ce cas on peut parler de "pensée géométrique". Nous allons présenter ici certaines de ces possibilités, et montrer sur un exemple comment elles peuvent être employées par des étudiants dans leur résolution d'un problème d'algèbre linéaire.

1.Introduction

When students encounter linear algebra for the first time, they find it very abstract, and disconnected from what they have learned before (Dorier 2000). Geometry is often mentioned by teachers as a way to help the students in their learning of linear algebra, because it seems to provide an opportunity to introduce concepts, that will be then generalised, within a more concrete context. The actual facts are more intricate; there is no natural generalisation process leading from geometry to linear algebra (Gueudet-Chartier 2000).

However, when the linear algebra concepts are formed, geometry can intervene in problem solving processes.

In his analysis of the genesis of linear algebra, Dorier (2000) distinguishes its analytical origin, linked with the solving of linear systems, and its geometrical origin, related with synthetic geometry. Both sides: analytic, and synthetic, are still present within modern linear algebra. It suggests the possibility of existence of a similar distinction in the processes of solving linear algebra problems.

The existence of analytical processes is obvious. Is it really possible to develop solving processes linked with synthetic geometry? And do the students use such processes? I provide here answers to these questions.

I first justify (section 2.1) the need for a "concrete" model in linear algebra. I explain then (section 2.2) the precise meaning that I confer here to the expression "geometric thinking", and I display possibilities of geometric thinking in linear algebra.

In section 3, I study the example of a particular problem, and the solving processes actually developed by the students.

2. Linear algebra and geometric thinking

2.1 The need for a model

According to Fischbein (1987) the reasoning activity in mathematics always rely upon conceptions that seem certain to the person producing that reasoning. Even when dealing with very abstract structures, the mathematician (or the student) acts in accordance with a credible reality. For Fischbein, "credible" is synonymous with "behaviourally meaningful".

That credible reality will often be provided by the use of a model. Fischbein defines a model as follows : *"A system B represents a model of system A if, on the basis of a certain isomorphism, a description or solution produced in terms of A may be reflected consistently in terms of B and vice-versa"(Fischbein, 1987, p.121).*

A model can be extramathematical, for example when drawings are used. But it can also stem from a mathematical theory, if the concepts involved have the appearance of a credible reality in the eyes of the student, or mathematician, who uses them.

Harel's researches about linear algebra lead to results very similar to Fischbein's theory.

Harel posits three principles for the learning of linear algebra. The *Concreteness Principle* is specially relevant here: *"For students to abstract a mathematical structure from a given model of that structure, the elements of that model must be conceptual entities in the student's eyes: that is to say, the student has mental procedures that can take these objects as inputs."(Harel, 2000, p.180)*

That principle is inspired by Piaget's and Greeno's work. Harel shows in particular that geometry can provide a model helping to construct linear algebra concepts, if the geometrical concepts are for the students mental entities on which mental operations can be performed.

Harel studies mostly the construction of the linear algebra concepts by the students. But models will also intervene in the problem solving processes of students already familiar with linear algebra concepts. Their reasoning must indeed be grounded on a credible reality, as Fischbein states it.

The notion of model permits to confer a precise meaning to the expression “geometric thinking” in linear algebra; I present it in the next subsection.

2.2 “Geometric thinking” in dimension *n*

I only study here the case of finite dimensional vector spaces, with an inner product.

Because these spaces are isomorphic to $\mathbb{R}^n$, a first possibility of concrete model is provided by the use of coordinates. A problem proposed within $\mathbb{R}^n$ can be first formulated and solved in $\mathbb{R}^2$ and/or $\mathbb{R}^3$, and then generalised to dimension n. That analytical process can be completely disconnected from any kind of geometry, for example to prove properties of matrices, or to compute determinants. It can also be associated with a geometric interpretation of the problem in the plane, or in the 3-space; but also in that case, I will call it “analytic thinking”.

Other possibilities exist, directly linked with synthetic geometry. A first step towards linear algebra was made by Leibniz when he tried to develop a geometrical calculus, opposed to analytic geometry. Using analogies between synthetic geometry and functions spaces has been determining in the genesis of linear algebra (see Dorier 2000). How can synthetic geometry intervene in modern linear algebra?

I will term here "geometric thinking", in the solving of linear algebra problems, the use of a geometric model, that can thus be associated with drawings, and involves no coordinates. The model is limited to dimension 1, 2 or 3; it can stem from traditional Euclidean geometry, vector geometry, or even from linear algebra.

So "geometric thinking" can intervene when solving a problem in dimension $n>3$, if a model restricted to dimension 2 or 3 can help the solving process.

An analysis of university textbooks indicates two main procedures (because that analysis was intended to ground a questionnaire for students at the University of Rennes, I retained for it the books officially recommended to these students during their two first university years: Grifone 1990, Guinin 1993, Liret&Martinet 1997 and 1999).

- *Decreasing the dimension*: in many problems, central elements of a solution take place in a 2 or 3-dimensional well chosen space. For example, to prove that the union of two subspaces is not a subspace if one of the subspaces is not included in the other, it is sufficient to consider two vectors, one in each subspace, that do not belong to the second subspace. Their sum does not belong to the union, thus it is not a vector space. The important point happens in a plane; it can be associated with a drawing, or a mental picture.

- *Increasing the dimension*: some linear algebra problems in dimension n can be proved by induction, or at least require the understanding of the process involved in moving up from n to $(n+1)$. The associated solving process can in such cases be grounded on the understanding of the process leading from dimension 1 to dimension 2, or from 2 to 3. A typical example is provided by the Gram-Schmidt orthonormalisation process.

Both cases correspond to "expert processes" (Robert 1998), commonly developed by the mathematicians.

But do students actually use such models in their solving processes? We will see in next section, on a particular example, that the answer is positive.

3. Geometric thinking in students' problem solving processes

During the second semester of the academic year 2000-2001, I observed a six-weeks long linear algebra course[1] for second year students. It focused mainly on quadratic forms, and vector spaces with an inner product (all the students had learned elementary linear algebra during their first year).

After these observations, I met the teacher, and then eight students for individual interviews. I have chosen to encounter four students who were following that course for the first time, and four students who had the same course the preceding year with another teacher. I also retained students with various levels (according to their results to an examination they had two weeks before). The interviews were recorded and transcripted. A short description of the whole students' questionnaire is given in the appendix.

It included in particular the following exercise:

"Find the length of a diagonal of a cube with edges of length 1 in $\mathbb{R}^n$."

During all the interviews, I have chosen to intervene as little as possible in order to avoid influencing the students solving processes. For that particular exercise, the only hint I proposed was to indicate where does the diagonal of a cube in dimension 3 lie, in case they drew such a cube, and needed such a hint.

I present here an analysis of the students' solving processes.

[1] That teaching consisted weekly in two lectures (1h15 each) and two tutorials (2h each). I observed all the lectures, and some of the tutorials.

Presentation of the exercise

A first difficulty when solving this exercise is linked with the geometrical vocabulary: these students never met "cubes" and "diagonals" in $\mathbb{R}^n$. It could embarrass some of them, and prevent them from solving the exercise, even if it was possible to answer without a clear insight of the n-cube's significance.

Two main ways of solving that exercise were accessible to these students.

The first one is the analytic solution. One of the diagonals of the cube is a vector *OA*, were *O* has coordinates (0,0...0) and *A* has coordinates (1,1 ...1). Thus the length of the diagonal is $OA = \sqrt{n}$.

Studying the problem for *n=2* or *n=3* can be naturally helpful in that case; it allows to discover the coordinates of *A* for these values of *n*, and that result can be immediately generalised.

The second solution is more interesting for our purpose, because it can be interpreted as "geometric thinking", instead of the "analytic thinking" of the first case. It uses an induction process. For *n=2*, the length of the diagonal is $\sqrt{2}$. For *n=3*, it is $\sqrt{3}$, it can therefore be induced that the length of the *(n-1)*-cube's diagonal is $\sqrt{(n-1)}$ The diagonal of the *n*-cube is the sum of a diagonal of the *(n-1)*-cube and of a unit vector, orthogonal to it. According to Pythagorean Theorem, the length of the diagonal is thus $\sqrt{n}$ (this is the vector version; the diagonals can also be interpreted as the sides of a right-angled triangle). That solution requires the study of the case *n=2*, but also of the process that leads from *n=2* to *n=3*. That process provides indeed the key of the induction; the associated drawing can display the diagonal of a side, and a unit vector orthogonal to it. Such a reasoning involves geometric thinking. The induction can also stay more or less analytic, as we will see it in the next paragraph.

Students' solving processes

Considering the students' answers and solving processes leads to distinguish three groups among the eight students that I interviewed.

Group 1: The obstacle of geometrical vocabulary (Students A, B, C, D).

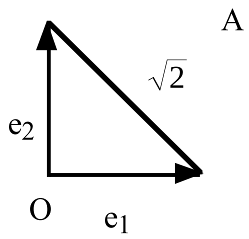
These students did not overcome the obstacle of vocabulary. They were not able to confer a meaning to the term "cube" in $\mathbb{R}^n$.

Student *A*: "For *n=3*, it's a cube... For *n*? I do not see what a cube can be! For *n=3*, the diagonal of a face is $\sqrt{2}$; it gives $\sqrt{3}$... But for dimension *n*..."

These students solved the cases *n=2* and *n=3*, but were unable to identify a generalisable process. They remained trapped in the geometric context, and were unable to develop a geometric thinking in higher dimensions.

Group 2: Using coordinates (Student E).

There is only one student in that "group". His reasoning process is very different from all the others. He started with dimension 2, and immediately made the associated drawing, figuring only two vectors and two points O and A, and not the complete square.



He made the same kind of drawing in dimension 3, deduces that the length of the diagonal is √3 is that case, and then that the coordinates of A will always be (1, 1, 1..), thus the length of the diagonal would be $\sqrt{n}$.

That student thinks in terms of linear algebra: he draws vectors, considers a basis; that allows him to use coordinates, and then the generalisation appears as a natural process. Although drawings are central in his solving process, his reasoning is more analytical than geometric.

Group 3: Induction process (Students F, G, H).

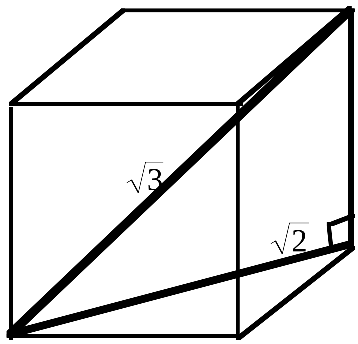
These three students started by calculating √2 for *n=2* (they first drew a square); then they drew a cube, and used Pythagorean theorem, quoting it or not, to compute √3 for n=3. They all claimed then that the general result was $\sqrt{n}$, using a kind of induction process, but none of them produced a rigorous proof.

Students *G* and *H* base their reasoning on geometrical statements, while student *F* grounded is answer only on the aspect of the result ("it is √2 for *n=2*, √3 for n=3, so it must be √n").

I give here the example of student *H* solving process. After calculating √2 for *n=2*, and using it to deduce √3 for *n=3*, he said:

Student *H*: "I would say then there are 1, and √3, and it is orthogonal, so it gives √4 = *2*, but I'm not sure, I dot not see it clearly... But I think it works, because √3 is the diagonal, and the last edge is orthogonal to it. So it is actually 2 for $n=4$. And it is always the same, so you obtain $\sqrt{}\sqrt{}\sqrt{}(((1 + 1)^2 + ... + 1)^2 + 1) = \sqrt{n}$."

He first felt embarrassed because he was missing a picture figuring exactly the *n*-cube. But his drawing of the 3-cube displayed in particular a right-angled triangle formed by the diagonal of a face, an edge of the cube, and the corresponding diagonal of the cube:



Because of that drawing, he focused on the orthogonality of the diagonal of a face and an edge of the cube. Even if he did not explicitly interpret the diagonal of the 3-cube as the diagonal of a "face" of the 4-cube, he used it that way; that allowed him to compute the result for $n=4$. He did not try to provide any further geometrical interpretation, but immediately generalized his result for any value of *n*.

In his course, the teacher spent a lot of time presenting $\mathbb{R}^2$ and $\mathbb{R}^3$, and gave many examples in $\mathbb{R}^2$ and $\mathbb{R}^3$ when presenting more general results. When I asked him about the possible use of $\mathbb{R}^2$ and $\mathbb{R}^3$ for the study of $\mathbb{R}^n$, his assumption was:

"For quadratic forms, everything already happens in dimension 3. The important point is to be able to move from dimension 2 to dimension 3."

In the particular case I presented here, the observations confirm that statement. The four students who did not identify the process that led from $n=2$ to $n=3$ failed. The four others found the result, and two of them really developed a reasoning involving "geometric thinking".

The observations related in this section result from a clinical study, about a precise task, conducted with a small number of students. However, the behaviours described indicate more general phenomena.

4.Conclusion

Fischbein, and Harel, state that “concrete” models are needed to learn linear algebra. The study of $\mathbb{R}^2$ and $\mathbb{R}^3$ with coordinates provides such a model. But other possibilities exist.

Some students actually use, when possible, "geometric thinking" to solve linear algebra problems; it seems to be really helpful for them. Proposing several tasks, to groups of students of several levels (Gueudet-Chartier 2000), indicates that it is only done by a minority of students. Why? Let us reconsider the two procedures presented in section 2.2.

- *Decreasing the dimension:* It is a familiar process for mathematicians to think in a well chosen 2-dimensional space (and it is then possible to help the reasoning with a drawing). The actual n-dimensional object can not be represented; but it is always possible to cut it along a plane, and represent the obtained section.
- *Increasing the dimension***:** Moving up from dimension $(n-1)$ to dimension n is a complex process. It is in particular necessary to interpret the space of dimension $(n-1)$ as an hyperplane of the n-space. Mathematicians are quite used to such processes; they consider the first space as an hyperplane, and then add a supplementary line to it, to obtain the whole n-space. It is a difficult process for students, that requires to be familiar enough with subspaces.

These possibilities of using geometric models have never been explicitly mentioned in the courses I observed, although the teacher himself used them. Students were encouraged to use induction with coordinates; but “geometric thinking” was not taught to them. The teacher seemed to expect that the students develop it spontaneously. Some students succeed in it, but not all of them. There is no evidence that “geometric thinking” can be taught. However, it would be interesting to design a linear algebra course including a detailed presentation of geometric models, and of their use in the solving of n-dimensional problems; it would allow us to observe the consequences of such a design on the students reasoning processes.

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Appendix : short description of the whole student's questionnaire

Part 1: The following two exercises were proposed in an examination the students sitted two weeks before. During the interview, we discussed their answers, and specially the drawing they produced (or not).

Exercise 1.1: *Let q be the quadratic form defined in* $I\!R^2$ *as* $q(x) = (2x_1 + 3x_2)^2$. *Write the corresponding matrix, compute its rank, and draw its isotropic cone.*

Exercise 1.2: *Let q be the quadratic form defined in* $I\!R^3$ *as* $q(x) = 3x_1^2 + 2x_2^2 - x_3^2$. *Give its rank, and draw its isotropic cone . Is q a positive form?*

Part 2: In the part 2, I proposed three exercises that the students did not encounter in the tutorials.

Exercise 2.1: *Four drawings (in a plane with a system of axes) were proposed. The task was to decide whether the represented sets can be the kernel, or the isotropic cone of a well chosen quadratic form.*

Exercise 2.2: *Let q be a quadratic form on* $I\!R_2[t]$, *and {P, Q, R} an orthonormal basisof* $I\!R_2[t]$ *for q. Compute, if possible, q(P+Q).*

Exercise 2.3: *Find the length of a diagonal of a cube with edges of length 1 in* $I\!R^n$

Part 3 : In the part 3, I asked questions related with the course and the tutorials, and linked with the use of drawings.

For example, the teacher was used to represent the elements of a vector space sometimes as vectors, and sometimes as points. He never explained the reasons for his choice of one or another representation.

He also used drawings (with arrows) to illustrate properties in vector spaces of polynomials.

So I investigated the students' interpretations of these pictures, and their own use of such drawings.